PROBLEM DEPARTMENT

ASHLEY AHLIN* AND HAROLD REITER†

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by February 28, 2006. Solutions identified as by students are given preference.*

Problems for Solution.

1109. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN*

(a) Determine the value of $1 - \frac{2^5}{2!} + \frac{3^5}{3!} - \frac{4^5}{4!} + \cdots$.
(b) Generalize to

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^m}{n!}.$$

The sum of the generalized series can be expressed in terms of m as a finite sum of known numbers.

1110. *Proposed by David Wells, Penn State New Kensington, Upper Burrell, PA*

Let $p(x) = ax^2 + bx + c$ where a, b, and c are real numbers. Let $q(x) = p(p(x))$. Find necessary and sufficient conditions on a, b, and c such that the number of zeros of q is none, one, two, three, and four.

1111. *Proposed by Paul S. Bruckman, Sointula, BC, Canada*

(a) Let p be any odd prime and $S(m) = \sum_{k=1}^{p} k^m,\ m = 1, 2, \ldots$. Prove that $p|S(m)$ for any $m = 1, 2, \ldots, p-2, p$, and that $S(p-1) \equiv -1 (\text{mod } p)$.
(b) Let $\sigma(m)$ denote the elementary symmetric function of order m, on the numbers $1, 2, \ldots, p$. Prove that $p|\sigma(m),\ m = 1, 2, \ldots, p-2, p$, and $\sigma(p-1) \equiv -1 (\text{mod } p)$.

1112. *Proposed by Harry Sedinger, St. Bonaventure University, St. Bonaventure, NY*

Without using a calculator or computer, determine whether or not the positive integer $9753^{2468} + 3579^{8642} + 9357^{2468} + 9573^{8642}$ is a perfect square.

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1113. *Proposed by Robert C. Gebhardt, Hopatcong, NJ*

Find a closed form for each of the sums below, where the constant $k > 1$ (not necessarily an integer): $\sum_{n=1}^{\infty} \frac{n^p}{k^n}$ for $p = 1, 2, 3, 4$, and 5.

1114. *Proposed by Stas Molchanov, University of North Carolina Charlotte, Charlotte, NC*

Does there exist a measurable subset M of $[0, 1]$ whose intersection with each open interval (a, b), $0 < a < b < 1$, is $(b - a)/2$?

1115. *Proposed by Nick Hobson, London, England*

Is the number 2561280000000000000000000001 prime or composite? No calculators allowed, of course.

1116. *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

Call a number "nice" if it can be represented as the sum of fourth powers of three positive integers. Let q[1]=3, q[2]=18, q[3]=33, ... be the increasing sequence of all nice numbers – and let q[q[4]] be nice and lucky and happy for you! Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the second of several such problems we plan for this column.

1117. *Proposed by the editors*

Consider the $a \times b \times c$ rectangular box built from abc unit cubes, where a, b, and c are positive integers. How many paths of length $a + b + c$ are there from a fixed corner of the box to the corner farthest away along edges of the unit cubes that stay on the surface of the box?

1118. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA*

Prove that $\sin 9^\circ + \cos 9^\circ = \frac{1}{2}\sqrt{3 + \sqrt{5}}$.

1119. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $z \neq 1$ be a root of the equation $z^n - 1 = 0$. Prove that

$$\left|(n-2)z^2 - 2(n-1)z + n\right| \leq \frac{(n-1)(2n-1)}{6}|z-1|^3$$

1120. *Proposed by Arthur Holshouser, Charlotte, NC*

Let S denote an 81 element set with a binary operation $\circ$ satisfying

1. $a \circ a = a$ for all $a \in S$,
2. $a \circ b = b \circ a$ for all $a, b \in S$, and
3. $a \circ (b \circ a) = b$ for all $a, b \in S$.

Define a 3-element subset $\{x, y, z\}$ to be a SET if $x \circ y = z$. Next let A denote a 20-element SET-less subset of S and let $B = S \setminus A$. Find the number of SETs in B.

1121. *Proposed by Gus Mavrigian, Youngstown, OH*

Suppose angles α and β satisfy $\frac{\sin\alpha + \sin\beta}{\cos\beta - \cos\alpha} = \sqrt{3}$ and $(\cos\beta - \cos\alpha) \neq 0$. Show that $\sin(3\alpha) + \sin(3\beta) = 0$.

1122. *Proposed by Arthur Holshouser, Charlotte, NC*

A. $(S, \cdot)$ is a binary operator that satisfies $\forall x, y \in S, (xy)\,x = y$. This is what Donald Knuth calls a grope.
   (a) Is it provable that $\forall a, b \in S, \exists$ a unique $x \in S$ such that $xa = b$?
   (b) Is it provable that $\forall a, b \in S, \exists$ a unique $x \in S$ such that $ax = b$?

B. Now suppose that $\left(\overline{S}, \cdot\right)$ satisfies $\forall x, y \in \overline{S}, (xy)\,y = x$. Answer (a) and (b) for $\left(\overline{S}, \cdot\right)$ also.