PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2008. Solutions identified as by students are given preference.*

Problems for Solution.

1146. *Proposed by Douglas Shafer, University of North Carolina Charlotte* This problem first appeared here in fall 2006. It has not yet been solved successfully. Because of a contribution from Professor Ali Amir-Moez, we are able to offer a $500 prize for the best undergraduate student solution to this problem.

Given six real constants a, b, c, d, e, and f, not all zero, a conic section $C:$ $ax^2+bxy+cy^2+dx+ey+f=0$ is determined. Since rescaling the six coefficients by a nonzero number does not change C, we may view (a,b,c,d,e,f) as lying in $S^5 \subset \mathbf{R}^6$. If (a,b,c,d,e,f) is selected based on a uniform distribution on S^5, what is the probability that C is an ellipse?

1159. *Proposed by S.C. Locke, Florida Atlantic University, Boca Raton, FL*

Suppose that

$$\log_x y + \log_y x = 11.$$

Evaluate

$$(\log_x y)^k + \left(\log_y x\right)^k,$$

for $k = 2, 3, 4, 5$.

1160. *Proposed by Leo Schneider, John Carroll University, University Heights, OH*

Construct a proof that e is an irrational number based on the Alternating Series Test.

1161. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$a^3 + 6\int_a^b x f(x^3)\,dx \ge b^3 + 3\int_a^b f^2(x^3)\,dx, \quad (a<b).$$

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1162. *Proposed by Paul S. Bruckman, Sointula, BC*

Given $M = M_k = 2^k, k = 1, 2, \ldots$, prove the congruence

$$3^M \equiv 1 + 52M (\text{mod } 64M), \text{for } k \geq 3.$$

Also, prove the "near" corollaries:

1. $3^M \equiv 1 + 20M (\text{mod } 32M)$, for $k \geq 2$.
2. $3^M \equiv 1 + 4M (\text{mod } 16M)$, for $k \geq 1$.
3. $4M \mid\mid (3^M - 1)$, for $k \geq 1$; that is, the largest exponent t such that 2^t divides $(3^M - 1)$ is $t = k + 2$.

1163. *Proposed by Stas Molchanov, University of North Carolina Charlotte*

A two-pan balance and 16 coins of different weights are given. What is the fewest number of usages of the balance needed to determine the heaviest coin, the second heaviest coin, and the third heaviest coin?

1164. *Proposed by Cecil Rousseau, University of Memphis*

Find a formula for evaluating the trigonometric sum

$$\cos^{2n} 1^\circ + \cos^{2n} 2^\circ + \cdots + \cos^{2n} 89^\circ,$$

where n is a positive integer. Your formula should demonstrate that each such sum is a rational number. For example,

$$\cos^{10} 1^\circ + \cos^{10} 2^\circ + \cdots + \cos^{10} 89^\circ = \frac{2771}{128}.$$

1165. *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

For positive integers n, k let $F(n, k)$ be the number of mappings of an n-element set into itself whose k^{th} iterate is the identity map (e.g. $F(3, 2) = 4$) – and let the number $F(4, 2) + F(8, 2) + F(8, 3)$ be nice and lucky and happy for you!! Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the seventh of several such problems we plan for this column.

1166. *Proposed by Peter A. Lindstrom, Batavia, NY*

Suppose that functions f, g, f', and g' are continuous over $[0, 1]$, $g(x) \neq 0$ for $x \in [0, 1]$, $f(0) = 0, g(0) = \pi, f(1) = 1004$, and $g(1) = 1$. Find the value of

$$\int_0^1 \frac{f(x)\, g'(x) \left\{g^2(x) - 1\right\} + f'(x)\, g(x) \left\{g^2(x) + 1\right\}}{g^2(x)} dx.$$

1167. *Proposed by Richard Armstrong, St. Louis Community College and Arthur Holshouser, Charlotte, NC*

Find necessary and sufficient conditions on positive integers a, b, c, and d such that

$$\prod_{n=1, n\neq a,b,c,d}^{\infty} \frac{(n-a)(n-b)}{(n-c)(n-d)}$$

converges.

1168. *Proposed by Sam Vandervelde, St. Lawrence University, Canton, NY*

Define the Fibonacci numbers as usual by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$, for $n \geqslant 2$. Determine the value of

$$\sum_{n=1}^{\infty} \frac{F_n}{n!}.$$

1169. *Proposed by James A. Sellers, Pennsylvania State University, University Park, PA*

A *composition* of the positive integer n is an ordered sequence of positive integers which sum to n. So, for example, $3+4+2$, $4+3+2$, and $2+2+2+3$ are different compositions of the number 9. Let $c_o(n)$ be the number of compositions of n where the last part is **odd**.

1. Find a Fibonacci-like recurrence satisfied by $c_o(n)$.
2. Use the above recurrence to find a closed formula for $c_o(n)$.

1170. *Proposed by Andy Niedermaier, University of California San Diego*

Consider a 10×10 grid of lights, each either on or off, which we denote using matrix notation $a_{i,j}$, where, for each $i = 1, 2, \ldots, 10$ and $j = 1, 2, \ldots, 10$, the entry in row i and column j is $a_{i,j}$ and its value is 0 or 1. We are allowed two types of *moves.* For each $1 \leq u \leq 8$ and $1 \leq v \leq 8$, we can change the status of all the lights $a_{i,j}$ for which both $u \leq i \leq u+2$ and $v \leq j \leq v+2$. This is called a *small block* move. The other type move is, for each $1 \leq u \leq 6$ and $1 \leq v \leq 6$, we can change the status of all the lights $a_{i,j}$ for which both $u \leq i \leq u+4$ and $v \leq j \leq v+4$. This is called a *large block* move. So essentially, we can change the status of all nine lights in each 3×3 subarray and of all the lights in each 5×5 subarray. Is it possible, beginning with the all on configuration, to achieve all possible on-off configurations of lights?