

PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk () preceding a problem number indicates that the proposer did not submit a solution.*

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to hbreiter@unc.edu. Electronic submissions using L^AT_EX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2011. Solutions identified as by students are given preference. The current column is dedicated to the memory of our dear friend, mentor, problem poser and solver, and colleague Leo Schneider, who died while returning from a math conference in August. Leo was a consummate organizer. As secretary of Pi Mu Epsilon, he established a standard his successors will find difficult to emulate.

Problems for Solution.

1226. *Proposed by Jonathan Hulgan and Cecil Rousseau, University of Memphis, Memphis, TN.*

In part two of his justly famous *Textbook of Algebra*, G. Chrystal poses two intriguing problems of a similar nature. Suitably generalized, problem 2 on page 33 reads

Segments of length $1, 2, \dots, n$ are given. The number of triangles that can be built using three of these segments as edges is _____.

Problem 9 on page 34 reads

Out of n straight lines $1, 2, \dots, n$ inches long respectively, four can be chosen to form a pericyclic quadrilateral in _____ ways.

The answers have been excised. Without solving either of these two problems, prove that the answers are the same. A pericyclic quadrilateral is one that has an inscribed circle.

1227. *Proposed by Stanislav Molchanov, Department of Mathematics and Statistics, University of North Carolina Charlotte*

1. Let A be a four-digit positive integer. For what k is it always possible to append to A a k -digit integer N such that $A \cdot 10^k + N$ is a perfect square?
2. Again let A be a four-digit positive integer. For what k is it always possible to append to A a k -digit integer N such that $A \cdot 10^k + N$ is a perfect cube?
3. Prove that for any positive integers A and n , it is possible to find a t -digit integer N such that $A \cdot 10^t + N$ is an n^{th} power.

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1228. Proposed by Hongwei Chen, Christopher Newport University, Newport News, VA.

Let $a_0 = 0$ and $a_k > 0$ for all $1 \leq k \leq n$ with $\sum_{k=1}^n a_k = 1$. Prove that

$$1 \leq \sum_{k=1}^n \frac{a_k}{\sqrt{1 + a_0 + \cdots + a_{k-1}} \sqrt{a_k + \cdots + a_n}} \leq \frac{\pi}{2}.$$

1229. F. Hoffman and S.C. Locke, Florida Atlantic University, Boca Raton, FL.

For a non-zero integer s , let $h(s) = t$, where $2^t \mid s$ and $2^{t+1} \nmid s$. Let (a_1, \dots, a_n) be a sequence of non-zero integers, let $m = \max\{h(a_j) : 1 \leq j \leq n\}$, and let $Y = \{j : 1 \leq j \leq n, h(a_j) = m\}$.

Part (a). Suppose that $|Y|$ is odd and that $S = \sum_{j=1}^n \frac{1}{a_j} = \frac{p}{q}$, with $\gcd(p, q) = 1$. Prove that p is odd.

Part (b). Let $T = \sum_{j=k}^n \frac{1}{j} = \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n} = \frac{p}{q}$, with $\gcd(p, q) = 1$. Prove that p is odd.

1230. Proposed by Mohammad K. Azarian, University of Evansville, Indiana.

Let

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

be a polynomial of degree n ($n \geq 1$) where $a_n \neq 0$. If r_1, r_2, \dots, r_n are the roots of $P(x)$, then show that

$$\left(\sum_{i=1}^n r_i\right) \cdot \left(\sum_{i=1}^n \frac{1}{r_i}\right) \cdot \left(\prod_{i=1}^n r_i\right) = (-1)^n a_1 a_{n-1}.$$

1231. Proposed by Arthur L. Holshouser, Charlotte, NC.

A real number is called a Cantor number if it can be written in base 3 notation without the use of the digit 1. For example $1/3 = 0.1 = 0.0\bar{2}$ is a Cantor number. Prove that every real number is the sum of two Cantor numbers.

1232. Proposed by Paul Bruckman, Nanaimo, BC.

Given $0 < x < 1$, define $F(x)$ by $F(x) = \frac{1}{x} + \frac{1}{\ln(1-x)}$. Show that $\frac{1}{2} < F(x) < 1$. Show that F may be extended continuously to $[0, 1]$.

1232. Proposed by David Wells, Penn State New Kensington, Upper Burrell, PA.

Numbers r, s , and t are chosen independently and uniformly at random from the interval $(0, 1]$. Circles with radii r, s , and t are then constructed so that they are pairwise externally tangent. What is the probability that the circles can be enclosed in a triangle, each of whose sides is tangent to two of the circles?

1233. *Proposed by Perfetti Paolo, Dipartimento di matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy.*

Evaluate

$$\sum_{k=1}^{\infty} \frac{k(2k+1)H_{k-1}^2 + k(1+k)^2 H_{k-1} + (1+k)^2}{k^3(k+1)^2},$$

where $H_k = 1 + 1/2 + \dots + 1/k$, $H_0 = 0$.

1234. *Proposed by Sharon Sternadel Harada, Corvallis, Oregon and Leo Schneider¹, John Carroll University, Cleveland, Ohio.*

The streets in Mathville look like a grid formed by an x - y coordinate system. All the even-numbered east-west streets, $y = \dots, -2, 0, 2, 4, \dots$, are one-way east (the direction of increasing x), and all the odd-numbered east-west streets $y = \dots, -1, 1, 3, \dots$ are one-way west. Similarly, all the even-numbered north-south streets $x = \dots, -2, 0, 2, 4, \dots$, are one-way north (the direction of increasing y), and all the odd numbered north-south streets are one-way south. Compute the (Euclidean) area of the smallest convex quadrilateral that contains all the points (x, y) that can be reached by driving no more than n blocks, starting at the origin and traversing these one-way streets.

¹See the dedication at the top of the column