

Problem Department

Ashley Ahlin Harold Reiter*

April 14, 2008

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk () preceding a problem number indicates that the proposer did not submit a solution.*

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to hbreiter@email.uncc.edu. Electronic submissions using L^AT_EX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2008. Solutions identified as by students are given preference.

Problems for Solution

1171

Proposed by S.C. Locke, Florida Atlantic University, Boca Raton, FL

For any integer q , $q \geq 2$, let $k(q)$ denote the smallest positive integer m such that there is a monic polynomial $p(n)$ with integer coefficients and which is divisible by q for every integer n . For example, $24 \mid n^3(n^2 - 1)$ for every integer n and, hence, $k(24) \leq 5$. Determine, with proof, the value of $k(q)$ for each integer q , $q \geq 2$.

1172

Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH

Let $a, b, c, d \geq 1$ be natural numbers and let $S = \sum_{m=1}^{\infty} \frac{1}{(am+b)(cm+d)}$. Find

*University of North Carolina Charlotte

the sum

$$S(a, b, c, d) = \sum_{n=1}^{\infty} (-1)^n \left(S - \sum_{m=1}^n \frac{1}{(am+b)(cm+d)} \right).$$

1173

Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington PA

ABC is a triangle. A perpendicular BD to BA is constructed such that $BD = r \cdot BA$. Similarly, a perpendicular CE to CA is constructed such that $CE = r \cdot AC$. Find the locus of midpoints of DE for all values of the parameter r .

1174

Proposed by Tom Moore, Bridgewater State College, Bridgewater, MA

The integer 99 has the property that $9 \cdot 9 + (9 + 9) = 99$. Find all the positive integers N (base 10) with the property that N equals the sum of the product of its digits and the sum of its digits.

1175

Proposed by Proposed by Zokhrab Mustafaev, Victor Dontsov, Evgeni Maevski, University of Houston-Clear Lake, Houston, TX

It is known that numbers 14529, 15197, 20541, 38911, 59619 are multiples of 167. Without actually calculating, prove that the determinant of the 5×5

matrix A is also a multiple of 167, where $A = \begin{pmatrix} 1 & 4 & 5 & 2 & 9 \\ 1 & 5 & 1 & 9 & 7 \\ 2 & 0 & 5 & 4 & 1 \\ 3 & 8 & 9 & 1 & 1 \\ 5 & 9 & 6 & 1 & 9 \end{pmatrix}$

1176

Proposed by Jim Jamison, the University of Memphis, Memphis, TN

Let $\{a_1, a_2, a_3, \dots\}$ be any sequence of real (or complex) numbers. Define

$$\rho_n := \frac{a_1 + \dots + a_n}{a_{n+1} + \dots + a_{2n}}.$$

Observe that if we consider the sequence of odd integers $\{1, 3, 5, \dots\}$ then $\rho_1 = \rho_2 = \rho_3 = \dots = \frac{1}{3}$. Define ρ to be the constant ratio, i.e. if $\rho_1 = \rho_2 = \rho_3 = \dots =$

constant, then $\rho_1 = \rho_2 = \dots = \rho$. Hence we ask for each nonzero ρ , does there exist a sequence with the property

$$\rho_1 = \rho_2 = \rho_3 = \dots = \rho? \tag{1}$$

1177

Proposed by Arthur Holshouser, Charlotte, NC

Suppose $n \geq 4$ lines in the plane intersect each other in $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct points. A *quadrilateral set* is a set S having the following properties.

1. S has 4 points as members, and
2. These 4 points can be labeled $\{A, B, C, D\}$ in such a way that A, B are colinear, B, C are colinear, C, D are colinear, D, A are colinear, A, C are not colinear and B, D are not colinear.

How many quadrilateral sets are there?

1178

Proposed by Paolo Perfetti, Dipartimento di matematica, Università degli Studi di Roma "Tor Vergata", Rome, Italy

Let $[a]$ the integer part of a and $\{a\} = a - [a]$. Evaluate

$$\int_0^1 \int_0^1 \frac{\left\{ \frac{x}{y} \right\}}{\left[\frac{x}{y} \right] + 1} dx dy - \int_{x=0}^1 \int_{y=0}^x \ln \left[\frac{x}{y} \right] dy dx.$$

1179

Proposed by Cecil Rousseau, University of Memphis

In a MONTHLY problem posed by Freeman J. Dyson¹, the reader was asked to prove that

$$\sum_{m=1}^N \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2},$$

given N numbers a_m satisfying the N equations

$$\sum_{m=1}^N \frac{a_m}{m+n} = \frac{4}{2n+1}, \quad n = 1, 2, \dots, N.$$

Now the reader is asked to determine a_1, a_2, \dots, a_N given the same system.

¹F. J. Dyson, Problem 4389, *Amer. Math. Monthly* **57** (1950), pp. 188–189. Solutions by E. Trost and G. Szegö, *Amer. Math. Monthly* **58** (1951), pp. 640–641.

1180

Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain

Let x be a positive real number. Prove that

$$\frac{[x]}{3x + \{x\}} + \frac{\{x\}}{3x + [x]} > \frac{4}{15},$$

where $[x]$ and $\{x\}$ represents the integer and fractional parts of x respectively.

1181

Proposed by Brian Bradie, Christopher Newport University, Newport News, VA

In *The Problem Department* of the Fall 2007 issue of this journal, readers were challenged to find a closed form expression for the trigonometric sum

$$\cos^{2n} 1^\circ + \cos^{2n} 2^\circ + \cdots + \cos^{2n} 89^\circ,$$

where n is a positive integer. Here, the challenge is to find a closed form expression for the trigonometric sum

$$\cos^{2n+1} 1^\circ + \cos^{2n+1} 2^\circ + \cdots + \cos^{2n+1} 89^\circ,$$

for $n \geq 0$.

1182

Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland

Let a, b, c, d, e be decimal digits satisfying $\overline{abc} \cdot a = \overline{bda}$ and $\overline{bda} \cdot a = \overline{cdde}$. What is $\overline{cdde} \cdot a$? Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the eighth of several such problems we plan for this column.

1183

Proposed by Mohammad K. Azarian, University of Evansville

Evaluate the indefinite integral

$$\int \frac{\sqrt{1-x^2} - x}{x^3 - x^2 - x + 1 - \sqrt{1-x^2} + x\sqrt{1-x^2}} dx.$$