PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2010. Solutions identified as by students are given preference.*

Problems for Solution.

1214. *Proposed by William Gosnell, Amherst MA, and Herb Bailey, Rose-Hulman Institute of Technology, Terre Haute, IN.*

Tatyana loves track and trig. She runs one lap around a circular track with center O and radius r. Let T be her position on the track at any time and S her starting point. Let $\theta = \angle SOT$, $0 \le \theta \le 2\pi$ and d the distance she has run along the track from S to T. For fixed r, how many values of θ are there such that $d = \sec^2\theta$?

1215. *Tuan Le, Fairmont High School, Anaheim, CA.*

Let a, b, c be non-negative real numbers no two of which are zero. Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{3\sqrt[3]{abc}}{2(a+b+c)} \ge 2.$$

1216. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN.*

Evaluate the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \binom{2n}{2k}^{-1}.$$

1217. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate

$$\int_0^1 \left\{ (-1)^{\lfloor \frac{1}{x} \rfloor} \cdot \frac{1}{x} \right\} dx,$$

where $\lfloor a \rfloor$ denotes the **floor** of a and $\{a\} = a - \lfloor a \rfloor$ denotes the **fractional part** of a.

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1218. *Proposed by Mohammad K. Azarian, University of Evansville, Indiana.*

Find the following infinite sum

$$S = \frac{1}{1\cdot 2} - \frac{1}{1\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{5\cdot 7} + \frac{1}{7\cdot 8} + \frac{1}{9\cdot 10} - \frac{1}{9\cdot 11} + \frac{1}{11\cdot 12}$$
$$+ \frac{1}{13\cdot 14} - \frac{1}{13\cdot 15} + \frac{1}{15\cdot 16} + \frac{1}{17\cdot 18} - \frac{1}{17\cdot 19} + \frac{1}{19\cdot 20} + \frac{1}{21\cdot 22} - \frac{1}{21\cdot 23} + \ldots.$$

1219. *Proposed by Sam Vandervelde, St. Lawrence University, Canton, NY.*

A set of discs with radii 1, 3, 5, ..., 2009 are stacked on a peg from smallest on top to largest at bottom. Another set of discs having radii 2, 4, 6, ..., 2010 are similarly stacked on a second peg. A third peg is available but is initially empty. Let N be the fewest number of moves needed to transfer all 2010 discs to the empty peg, if each move consists of moving a single disc from the top of one stack onto another peg which is either empty or has only larger discs. (Following the familiar "Tower of Hanoi" rules.) Find, with proof, the number of 1's in the binary representation of N.

1220. *Proposed by Robert Gebhardt, Hopatcong, NJ.*

Find the area of the region bounded by the curve $x^{2n+1} + y^{2n+1} = (xy)^n$, $n = 1, 2, 3 \ldots$.

1221. *Proposed by Stas Molchanov, University of North Carolina Charlotte.*

Let p and q be different prime numbers greater than 2. Prove that $x^p + y^p = z^q$ has infinitely many solutions over the positive integers.

1222. *Proposed by Arthur Holshouser, Charlotte, NC and Ben Klein, Davidson, NC.*

A positive integer n bigger than 1 can be *split* into two positive integer summands, called its *offspring*, a and b, usually in several ways. Notice that 1 cannot be split. Starting with an integer n greater than 1, we can successively split n and its offspring to eventually arrive at all 1s. For a fixed real number θ, define the *value* of the split to be $V(a,b) = (\sin a\theta)(\sin b\theta)(\cos(a+b-1)\theta)$. Let n be a positive integer.

1. Prove that the number of splits must be exactly $n-1$.
2. Prove that the sum of the values of the $n-1$ splits does not depend on the sequence of splits and determine its constant value.

1223. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA.*

Show that the sum of any $2(2n+1)$ consecutive terms of the Fibonacci sequence is divisible by the $(2n+1)^{\text{st}}$ Lucas number.

1224. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove or disprove that

$$\sqrt[6]{\frac{a^3b^3c^3}{(-a+b+c)(a-b+c)(a+b-c)}} \geq \frac{2\sqrt{3}}{3}(d_a + d_b + d_c).$$