PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2012. Solutions identified as by students are given preference.*

Problems for Solution.

1253. *Proposed by Peter Linnell, Virginia Polytechnic University, Blacksburg, VA.*

This was problem 5 on the annual VPI Regional College Math Contest, 1995.

Let $\mathbb{R}$ denote the real numbers, and let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a map with the property that $x > y$ implies $(\theta(x))^3 > \theta(y)$. Prove that $\theta(x) > -1$ for all x, and that $0 \le \theta(x) \le 1$ for at most one value of x.

1254. *Proposed by Thomas Moore, Bridgewater State University, Bridgewater, MA.*

The numbers $9^n \pm 2$ are both primes for $n = 1$ and $n = 2$, yielding the prime pairs $7, 11$ and $79, 83$. Nevertheless, prove that (a) $9^n - 2$ is composite for infinitely many positive integers n and also (b) $9^n + 2$ is composite for infinitely many n. And (c) resolve the claims that both the numbers are simultaneously composite for infinitely many n as well as (d) are never both prime numbers for $n > 2$.

1255. *Proposed by Abdullah Özbekler, Atilim University, Ankara, TURKEY.*

Let I and G be the points of intersection of the bisectors and medians of a triangle ABC with sides a, b and c, respectively. Prove that

$$|IG|^2 = \frac{1}{9}\left\{2(b^2+c^2) - 3(b+c)(b+c-a) - a^2 + 9bc(b+c-a)/(a+b+c)\right\}.$$

1256. *Proposed by Hongwei Chen, Christopher Newport University, Newport News, VA.*

Let

$$h(i) = \sum_{j=0}^{i} \frac{1}{2j+1} = 1 + \frac{1}{3} + \cdots + \frac{1}{2i+1}.$$

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Show that, for all nonnegative integers k,

$$\sum_{i=0}^{k} \frac{1}{2(k-i)+1}\, h(i) = 2 \sum_{i=0}^{k} \frac{1}{2i+2}\, h(i).$$

1257. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.*

Prove that if $x, y, z > 0$, then

$$\frac{(x+y+z)^3 - 2(x+y+z)(x^2+y^2+z^2)}{xyz} \leq 9.$$

1258. *Proposed by Arthur Holshouser, Charlotte, NC.*

Let n be a given positive integer. If $P\left(x\right), Q\left(x\right)$ are polynomials of degrees at most n, we say that $P\left(x\right) \sim Q\left(x\right)$ if $\left(cx+d\right)^n \cdot P\left(\frac{ax+b}{cx+d}\right) = Q\left(x\right)$ for some (a, b, c, d) that satisfies $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Show that $\sim$ is an equivalence relation on the collection of all polynomials of degrees at most n.

1259. *Proposed by Mihaly Bencze, Brasov, Romania.*

Suppose a and b are real numbers with $0 < a < b$. Let $G_1 = \{x \mid -a+b < x < a+b\}$, and $G_2 = \{x \mid -1 < x < 1\}$. Define two operations $\perp$ and $\top$ as follows.

1. $x \perp y = \frac{bxy+(a^2-b^2)(x+y-b)}{xy-b(x+y)+a^2+b^2} \forall x, y \in G_1$.
2. $x \top y = \left(\frac{x^{1/(2n+1)}+y^{1/(2n+1)}}{1+(xy)^{1/(2n+1)}}\right)^{2n+1} \forall x, y \in G_2$.

Prove the following.

1. $(G_1, \perp)$ and $(G_2, \top)$ are abelian groups.
2. $(G_1, \perp) \cong (G_2, \top)$.

1260. *Proposed by Paul Bruckman, Nanaimo, British Columbia.*

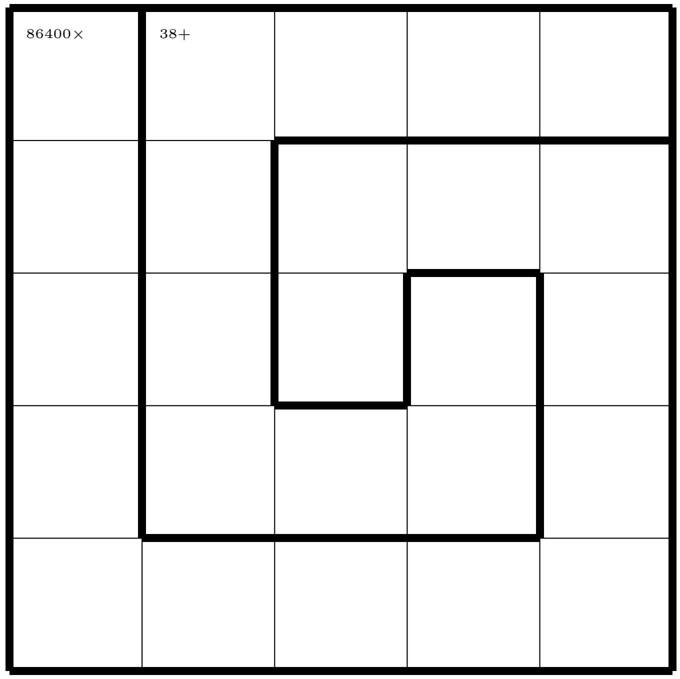
Prove the following identity, valid for $n = 0, 1, 2, \ldots$:

$$\sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-k}{2k} 4^k 3^{n-3k} = \frac{1}{9}\left(4^{n+1} + 6n + 5\right).$$

1261. *Proposed by The Editors.*

This problem appeared on the 2011 Lower Michigan Mathematics Competition.

In the puzzle KenKen®, the numbers in each heavily outlined set of squares, called cages, must combine (in any order) to produce the target number in the top corner of the cage using the mathematical operation indicated. A number can be repeated within a cage as long as it is not in the same row or column. In the 5×5 puzzle below, each of the digits 1 through 5 must appear in each row and column. A solution must include a proof of uniqueness.



1262. *Proposed by Mike Pinter, Belmont University, Nashville, TN.*

In connection with the most recent Super Bowl outcome, consider the following Base 16 (hexadecimal) alphametic:

$$SUPER$$

$$\underline{+BOWL}$$

$$GIANTS$$

Find a solution to the alphametic.

1263. *Proposed by Dr. Richard Stephens, Columbus State University, Columbus, GA.*

Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients and $a \neq 0$. Prove or disprove that $p(x)$ has a real root of order 2 if and only if $(27a^2d - 9bc - b^3)^2 = 4(b^2 - 3ac)^3 \neq 0$.