PI MU EPSILON Journal

VOLUME 1 NUMBER 1, 2, 3 REPRINT

CONTENTS

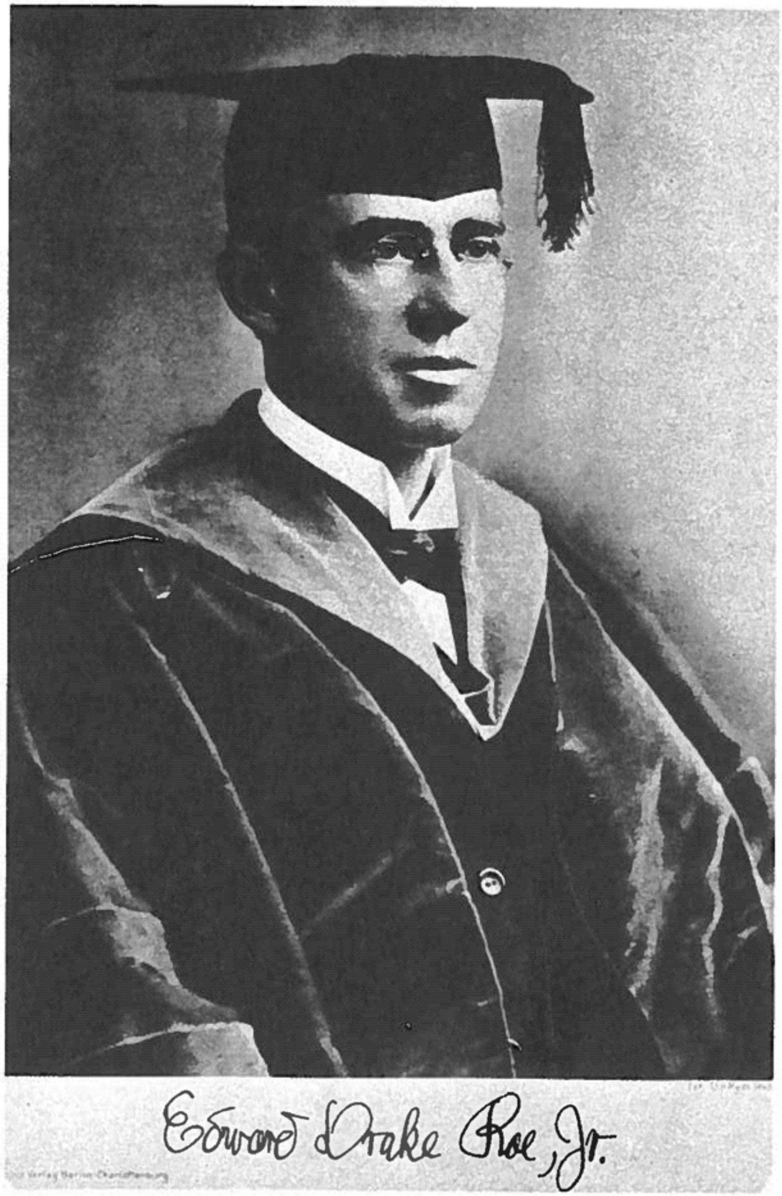
Pi Mu Epsilon Journal, volume 1:

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

—CONTENTS—

VOLUME 1 REPRINT



FOUNDER OF THE PI MU EPSILON FRATERNITY
INCORPORATED AT ALBANY, N.Y., **MAY,** 1914

BIOGRAPHY

PROFESSOR EDWARD DRAKE ROE, Jr.

By May Naramore Harwood,*
Syracuse University

When Pi Mu Epsilon was founded at Syracuse University, the founder envisaged that, when it became a national organization with chapters in the leading universities and colleges of America, there would be both a desire and a need for the fraternity to have its own publication to knit the chapters more closely and to provide an outlet for papers not otherwise likely to be published. Now that this part of the vision is being fulfilled it is fitting that a portion of the first issue of the publication be given to a sketch of the life and works of the man to whom the vision was given - Professor Edward Drake Roe, Jr. Professor Roe never doubted that Pi Mu Epsilon, once started, would flourish and the present large number of its widely distributed chapters would be no surprise to him.

Edward Drake Roe, Jr. was born at Elmira, N. Y. on the fourth of January, 1859, son of Edward Drake Roe, landowner, and his wife Eleanor Jane Roe, née Frost. He was educated at the Elmira Academy and at Syracuse, Harvard and Erlangen universities where he studied under such well known teachers as French, Haven, Bennett, Coddington, Smalley, Brown, Comfort, O. W. Holmes, H. P. Bowditch, T. Dwight, Fitz, Wood, Hills, C. S. Minot, Bowen, Lovering, Royce, Byerly, B. O. Pierce, J. M. Pierce, Simon, Falckenberg, Widermann, Nöther and Gordan. He obtained the degrees of A.B. (Syracuse 1880 and Harvard 1885) and M.A.

*Charter member.

(Harvard 1886). He was associate professor of mathematics at Oberlin College from 1892 to 1899. Oberlin granted him a leave of absence for three years (1897–1899 inclusive) during which leave he went to the University of Erlangen, Bavaria, where he won his doctorate in 1898. He then returned to America and in 1901 was awarded the John Raymond French Chair of Mathematics at Syracuse University and some years later was appointed director of the Holden Observatory at Syracuse.

Professor Roe's interest in astronomy was so great that in 1906 he erected on his own grounds a private observatory equipped with an Alvan Clark equatorial telescope, a superb 6-1/2 inch refractor with an objective by Ludin, a driving clock by Kandler, a micrometer by Gaertner, and several other smaller instruments. His observatory was considered one of the best equipped private observatories in the country. After his death, this observatory with all the equipment was given to Harvard College. In it he had found his recreation; besides making many observations of the sun, moon, planets, and comets, he discovered and measured 75 double stars.

Professor Roe so loved the observatory on the Syracuse campus that he held many of his graduate classes there. In a class in quarternions, the three students knew one another as Alpha, Beta and Gamma and the twinkle in Professor Roe's eyes showed that he understood that a fourth Greek letter was often added. A well-filled key ring gave evidence of his care of the observatory and he was quick to catch the implication when a student expressed astonishment on finding an unlocked compartment containing crayon and dust-cloths.

In 1904, Professor Roe attended the Third International Congress of Mathematicians and toured the German universities.

Professor Roe was the author of many scientific articles on mathematics, astronomy and philosophy. He was co-author of a text book in trigonometry and one in algebra. He was the founder and director-general of the honorary mathematical fraternity, Pi Mu Epsilon, and was a member of Phi Beta Kappa, Sigma Xi, Pi Kappa Phi, and Delta

Kappa Epsilon. He was a fellow of the American Association for the Advancement of Science, a member of the American Mathematical Society, the founder and president of the Syracuse Astronomical Society, a member of the Deutsche Mathematiker Vereinigung, Circolo Matematico di Palermo and Société Astronomique de France.

In appearance, Professor Roe was outstanding – tall, erect, spare and handsome; he dressed conservatively, usually in gray with ties of Harvard red. Coming across the campus in winter, in his long dark fur lined coat, a tall fur cap on his iron gray hair, his green felt book bag under his arm, and his rugged face reddened by the wind and snow, he was a goodly sight. The fur cap evidently was a treasure. On removal, when the snow and dampness were carefully brushed from its plushy surface, it was meticulously folded and placed in a desk compartment reserved for its concealment and protection. In this desk, too, during the early years of Pi Mu Epsilon, was stored the seal of the fraternity and on proper occasions this seal was used by Professor Roe himself to stamp the certificates of membership. Other parts of his desk, like desks of many scholars, were piled with all sorts of papers, letters, notices, class notes, catalogues and manuscripts – seemingly in hopeless confusion; but Professor Roe's fingers went to any desired object like steel to a magnet. His hands were most attractive. They were strong capable hands with long sensitive fingers, on one of which he wore an unusually fine and brilliant diamond ring. Difficult it was to decide whether the ring adorned the hand or the hand enhanced the beauty of the gem.

By his first wife Professor Roe had one daughter, Mrs. E. H. Gaggin of Syracuse. Some years after the death of his first wife he married Josephine Robinson, teacher of mathematics at Berea College in Kentucky. She loved the science of mathematics as did her husband and earned her doctorate at Syracuse in that field.

Professor Roe was for twenty-nine years a teacher of mathematics at Syracuse University. Throughout his years of service as a teacher, he always stood for the highest ideals of intellectual honesty and scientific achievement. He stood for high scholarship, thorough scientific study and research.

He impressed all who knew him as a scholar and teacher with a deep thirst for knowledge and an earnest desire to impart it with perfection. He worked with untiring patience in mathematics and its allied science astronomy in the University and in the community.

Professor Roe was the embodiment of character. He was a devoted teacher, a deep thinker, a philosopher and an earnest Christian. He died quite suddenly at his home on December 11, 1929. In his death Syracuse University and the scientific world suffered a distinct loss.

The following is a list of the published works of Professor Edward Drake Roe, Jr.

"The Probability of Freedom: A Critique of Spinoza's Demonstration of Necessity," Bibliotheca Sacra, Vol. LI, No. 204, 1894

"Trigonometry for Schools and Colleges" (with Professor F. Anderegg), Ginn and Co., 1898

"Two Developments," American Mathematical Monthly, Vol. IV, No. 3, 1897

"On the Circular Points at Infinity," American Mathematical Monthly, Vol. IV, No. 5, 1897

"Note on Integral and Integro-Geometric Series," Annals of Mathematics, Vol. II, No. 6, 1897

"Note on a Formula of Symmetric Functions," American Mathematical Monthly, Vol. V, No. 6-7, 1898

"Die Entwickelung der Sylvester'schen Determinante nach Normalformen," Leipzig, B. G. Teubner, 1898

"On Symmetric Functions," American Mathematical Monthly, Vol. VI, Nos. 1-6, 1899

"On the Transcendental Form of the Resultant," American Mathematical Monthly, Vol. VII, No. 3, 1900

"On a Formula of Interpolation," American Mathematical Monthly, Vol. VIII, No. 1, 1901

"On Symmetric Functions," American Journal of Mathematics, Vol. XXV, No. 1, 1903

"Note on a Partial Differential Equation," Annals of Mathematics, Vol. 4 (second series), 1902/03

"Selected Chapters in Algebra" (with Dr. W. G. Bullard, Syracuse), E. C. Johnson, 1903

"On the Convergence and Divergence of Some Particular Series," Syracuse University Bulletin, Series IV, No. 1, 1903

"On the Coefficients in the Product of an Alternant and a Symmetric Function," Transactions of the American Mathematical Society, Vol. 5, No. 2, 1904

"On the Coefficients in the Quotient of Two Alternants," Ibid., Vol. 6, No. 1, 1905

"On Complete Symmetric Functions," American Mathematical Monthly, Vol, XI, Nos. 8-10, 1904

"The Effect of Wind Forces on an Observatory Dome," Popular Astronomy, Vol. XIV, No. 6, 1906

"On Mathematics, Secondary Education," Bulletin 31, Albany, N. Y., 1906.

"On Sunset at Sea, Great Sun Spots, and the Transit of Mercury," Popular Astronomy, Vols. XIV, XV, XVI, 1906/08

"Fifteen Communications to the English Mechanic and World of Science," London, Astronomical Observations and Criticisms, Vols. LXXXVII-XC, 1907/09

"Selected Chapters in Algebra" (with Drs. Metzler and Bullard), Longmans, Green and Co., 1907

"Wind Pressure on an Observatory Dome," Popular Astronomy, Vol. XVI, No. 7, 1908

"Observations and Measures of Double Stars," Astronomische Nachrichten, Kiel, No. 4259, 1908

"Some Observations of Double Stars," Popular Astronomy, Vol. XVII, No. 2, 1909

"Measure of Double Stars," Astronomische Nachrichten, No. 4338, 1909

"Some Thoughts on Space," The Mathematics Teacher, Syracuse and Lancaster, Vol. II, No. 1, 1909

"On the Extension of the Exponential Theorem," American Mathematical Monthly, Vol. XVI, Nos. 6-7, 1909

"College Algebra" (with Drs. W. H. Metzler and W. G. Bullard), Longmans, Green and Co., 1908

"New Double Stars," Astronomical Journal, Albany, N. Y., No. 611, 1910

"Achromatic and Apochromatic. Comparative Tests," Popular Astronomy, Vol. XVIII, No. 4, 1910

"Double Star Measures," Journal Astronomique, Tome I, No. 4, 1910
"Measures of Double Stars," Astronomische Nachrichten, No. 4381, 1910
"New Double Stars," Popular Astronomy, Vol. **XVIII,** No. 6, 1910
"New Double Stars, and Double Star Work," Popular Astronomy, Vol. XVIII, No. 9, 1910
"A Generalized Definition of Limit," The Mathematics Teacher, Syracuse and Lancaster, Vol. **III,** No. 1, 1910
"New Double Stars," Astronomische Nachrichten, No. 4467, 1911
"Suggestions for a New Theory of Comets" (with Professor W. P. Graham), Astronomische Nachrichten, No. 4466, 1911
"A New Invariantive Function" Jahresbericht Deutschen, Mat. Ver., Leipsig, Band 20, No. 9-10, 1911
"New Double Stars," Astronomische Nachrichten, Kiel, No. 4544, 1911
"New Double Stars," Astronomische Nachrichten, Kiel, No. 4762, 1914
"Book Review," The Mathematics Teacher, March 1914
"The Conjunction of Jupiter and Venus," Astronomische Nachrichten, No. 4814,1915
"Lectures on Determinants. A brief course," Syracuse, 1915
"Lectures on Selected Topics in the Calculus," Syracuse, 1916
"New Double Stars and Measures of Double Stars," Astronomische Nachrichten, Kiel, February 1916
"Direct Micrometrical Observations of the Sun, Eclipse of June 8, 1918," Astronomical Journal, No. 744, 1918
"Measures of Double Stars," Astronomical Journal, No. 748, 1919
"New Double Stars," Astronomical Journal, No. 745, 1919
"Direct Micrometrical Observations of the Sun," Astronomical Journal, No. 769, 1920
"Recent Auroras and Sun Spots," Science, Vol. 15, No. 1324, 1920
"Review of Space and Time in Contemporary Physics; an introduction to the relativity theory by Moritz Schlick," Mathematics Teacher, Vol. 2, 1920

"Conservative Appreciation of Michelson's Interferometer," Popular Astronomy, Vol. 25, No. 9, 1921
"Henry Allen Peck Collection of Astronomical Works," Revised from Daily Orange, June 12, 1922
'Occultation du m e Satellite de Jupiter par la Terre," L'Astronomie, Vol. 37, No. 10. 1923
"L'Observations Roe," L'Astronomie, Vol. 38, No. 4, 1924
"Direct Micrometrical Observation of the Sun. Exact formulas," Astronomical Journal, No. 869, 1926

THE EARLIEST DAYS OF PI MU EPSILON

By Floyd Fiske Decker,*
Syracuse University

The need for securing a more responsive audience for the presentation of student mathematical papers at Syracuse University, well recognized by the faculty members, was made clear to the undergraduates by Professor Edward Drake Roe, Jr., early in the academic year 1913-14. He based his suggestions on the fact that the Mathematical Club, which would complete ten years of activity at the end of that year, should look forward to an even more active second decade. The club members accepted the challenge and **turned prompt**ly to the consideration of the problems involved. The Alpha chapter of Pi Mu Epsilon emerged.

The club meetings had consisted of two parts - the presentation of one or two mathematical papers and an aftermath of refreshments and social activities. There were no intellectual requirements for membership, and by 1913 it became evident that too many members were not really interested in mathematics. The more responsible members realized that action was necessary. After an academic year packed with suggestions, reports, deliberations, agreements and decisions, the fraternity was organized.

At the regular meeting of the club on November 17,1913, a committee was authorized to arrange for a celebration to commemorate its tenth birthday. It was no coincidence that at the same meeting a second committee, with Professor Roe as chairman, was established to consider and report on a possible revision of the constitution of the club. Here Professor Roe had the opportunity to start his campaign for the

*Charter member.

establishment of a fraternity of exactly the kind he had in mind. Under his skillful leadership most of his ideas prevailed.

Following some meetings of the committee and a special meeting attended by the executive committee and representatives from the faculty, a report was made to the club on the eighth of December. The report suggested, in outline, four alternative steps to strengthen the organization:

1. To establish scholarship requirements for membership in the club.
2. To reorganize the club as a professional fraternity.
3. To continue the club without change and to organize an honorary fraternity.
4. To provide within the club for the recognition of student performance by 1) barring freshmen, 2) accepting sophomores on probation, and 3) electing to full membership only those upper-classmen who establish satisfactory competence.

Alternative 2 was adopted and the committee was directed to draw up a skeleton constitution, - a result very pleasing to Professor Roe, as it was a first step toward his objective.

The committee again reported on the occasion of the regular meeting of the club on March 2, 1914. The report was not made to the club, however, but to a convention of those present since it was not yet known whether or not all members of the club would be permitted to join the fraternity if it should be established. The committee offered a constitution containing 13 articles and it was considered article by article.

The first article to be adopted was article 2 which read:

> "The first and primary aim of this fraternity shall be scholarship in all subjects and particularly in mathematics for the individual members and, secondly, the advancement of the science of mathematics and, lastly, the mutual mathematical and personal advancement of its members."

Next article 3 was revised and adopted in the following form:

"The charter members of the fraternity shall consist of those present members of the mathematical club of Syracuse University who have met all the requirements prescribed by the constitution of this fraternity and whose names follow."

A revision of article 4 was adopted reading as follows:

"Members of the faculty, graduate students and also those undergraduate students who are taking the equivalent of major or minor work in mathematics shall be eligible to membership in the fraternity.')

With the revision of articles 12 and 13, articles 5 - 13 were adopted. Of these article 5 is significant in that it established the policy that no one was to become a member automatically by meeting the formal qualifications for membership. Election was necessary. Moreover by-law 2 stated that the affirmative vote of three-fourths of the voting members present was necessary for election. The presentation of article 5 divided the house into two nearly equal sets, the affirmative majority being due perhaps to the experience of some of the undergraduates with their social fraternities.

Articles 6 - 13 provide for the usual officers, the scholarship committee and amendments to the constitution.

Article 1, which gives the fraternity a name, was the last to be adopted (March 23). Several designations were discussed including:

1. EII (The promotion of scholarship)
2. **EIIM** (The promotion of scholarship and mathematics)
3. **IIΦM** (Loving disciples of mathematics)
4. **AII** (Efficiency in all things)
5. MPB (Mathematics the foundation of mental powers. These letters happened to be the first letters of the last names of three of the professors in the department.)

It was voted that combination 2 be adopted but that the order be changed to IIME if that order were consistent with the above meaning, as it was in fact found to be.

At this meeting the by-laws were adopted.

At the convention of April 27 officers for 1914-15 were elected, the choice of Professor Roe for director being an obvious recognition of the masterly way in which he had

presided over the deliberations of the group during the preceding months. At the same election I was given the opportunity of working with the group as vice-director. The offices of secretary and treasurer went to undergraduates.

Beginning in 1916 the highest official honor going to an undergraduate was the chairmanship of the committee on scholarship, a position which went to the senior major with highest average. The other two student posts on the scholarship committee were filled on the basis of grades except that both sexes were to be represented.

At the first regular meeting of the fraternity on October 5, 1914, the following scholastic standards for eligibility were established:

	Mathematics Average	General Average
Sophomore	82	75
Junior	80	72

At the first fall meeting of 1915, the requirements were raised 3 points in each bracket. On later occasions these were further raised until it became necessary to re-establish the Mathematical Club as an organization to meet the needs of majors and minors who were unable to qualify for election to the fraternity.

Furthermore a person once elected to membership could not rest on his record. He must maintain a satisfactory scholastic record, for at the meeting on March 15, 1915, a new rule placed on the inactive list those whose general or mathematical average fell below 70. Three members met that fate immediately. They found they were not dealing with an honorary fraternity.

A point of procedure is indicated in the minutes of the meeting of October 8, 1917. "Seventeen names were presented of students meeting the scholastic standard for election to membership. Thirteen of these were certified as meeting the requirement of majoring or minoring in mathematics or planning to do so. The student members of the scholarship committee were directed to contact the remaining four and learn of their intentions regarding the continuance of the study of mathematics. . . . Three of the four were found eligible and elected."

After the academic year 1914-15, the fraternity was definitely a functioning organization, most of the decisions having been made on such items as formal initiation ritual, die, seal, badge, colors, flower and shield.

At Professor Roe's suggestion a forward-looking charter was obtained, under the laws of the State of New York. The charter authorized, for example, the owning and operating of fraternity houses.

The stimulating effect of the organization on the position of mathematics on the campus and the vitalizing of contacts between the faculty and potential and actual majors and minors soon attracted the attention of other departments of our university and led to what is sometimes designated as "the sincerest form of flattery" – the founding of similar organizations in other departments.

After a four-year development on our campus we were highly gratified to witness the spread of the fraternity to other institutions, so that we became the Alpha chapter of a national organization of which we are very proud. We were delighted at the recognition accorded Professor Roe, as the first director-general.

We are deeply appreciative of the many ways in which the fraternity has been improved during the 45 years of its existence by many representative mathematicians throughout the land, but the surviving members of that Syracuse group of 1913-14 will understand my pleasure in reliving those days.

GREETINGS FROM THE DIRECTOR-GENERAL

Last December a meeting of the representatives of the chapters of Pi Mu Epsilon was held in Pomerene Hall of the Ohio State University in Columbus. At this meeting the opinion was advanced that our fraternity is now large enough and important enough to warrant the support of an official journal. This suggestion was greeted with considerable enthusiasm, and it was voted to appropriate a small sum of money for the publication of the first issue of such a journal. Whether or not there will be subsequent issues is a matter for the chapters to decide.

The fraternity is singularly fortunate in that Professor Ruth Stokes has consented to serve as editor and Mr. Howard C. Bennett as business manager. Both are from the New York Alpha Chapter and it seems particularly appropriate that the first home of our journal should be with the mother chapter at Syracuse.

It seems as if some unifying agent such as an official journal would benefit the fraternity. The forty-six chapters are independent organizations with the privilege of running their own affairs pretty much as they please, which situation is as it should be, since conditions are different at different universities. But there seems to be no good reason why each chapter should not be cognizant of what the other chapters are doing, their programs, prize competitions and other activities, as well as their problems and frustrations and how they meet them.

I am convinced that Pi Mu Epsilon has an important mission in American university life in pointing out to competent students the beauty and utility of mathematics and its related sciences, and in insisting that mathematics be accorded its rightful place in the curriculum. Among the sections of the American Association for the Advancement of Science, mathematics has always been designated as Section A. Surely we can defend the thesis that mathematics deserves an important place in the education of every competent man and woman. By so doing we may fulfil the pledge which we made when we were initiated into the fraternity.

I wish to take this opportunity of congratulating the Editor and the Business Manager upon the fine magazine which they have produced, and of extending my best wishes to each chapter and to each member of our fraternity. May Pi Mu Epsilon continue to be a potent force for genuine progress in American education.

Cyrus **Colton MacDuffee**

A CERTAIN PROPERTY OF CONTINUOUS FUNCTIONS

By Melvin Hausner
Brooklyn College

This article considers a certain type of periodicity of functions. To be definite, we restrict ourselves to continuous functions $f(x)$ defined in the interval $0 \leqq x \leqq 1$. Given such a function we ask if it ever repeats itself, i.e. if there is an $h > 0$ and an x_1 such that $f(x_1) = f(x_1 + h)$. Is there a definite $h > 0$ such for all $f(x)$ there will be an x such that $f(x) = f(x + h)$? The answer is obviously no, for any increasing function never repeats itself. To eliminate this possibility let us further stipulate that $f(0) = f(1) = 0$ for all our functions $f(x)$.

We now phrase the question formally. Consider the class of all continuous functions $f(x)$ defined in $0 \leqq x \leqq 1$ such that $f(0) = f(1) = 0$. Henceforth, unless we say otherwise, we are restricted to this class of functions. We will say that $f(x)$ has a subperiod h, where $0 < h \leqq 1$, when there is an x_1 such that $0 \leqq x_1 < x_1 + h \leqq 1$ and $f(x_1) = f(x_1 + h)$. We ask: Are there any numbers h which are subperiods of all functions? *Of* course $h = 1$ is one such number due to our restrictions on our functions. (Take $x_1 = 0$.) Are there any, other than $h = 1$? The answer is yes. In this article we prove that all of the numbers 1, $1/2$, $1/3, \ldots, 1/n, \ldots$ are subperiods of all functions we consider and that no other number is a subperiod of all functions.

The proof will be in two parts. In part I, we show that $1/n$, where n is a positive integer, is a subperiod of any function. In part II we show that if a number k is not of this type, then k is not a subperiod of all functions.

I. Since the number 1 is a subperiod of all functions as stated before, we may assume that we are given a number $1/n$ with $n > 1$. Let us assume that there is a function $f(x)$

such that $1/n$ is not a subperiod of $f(x)$. This means that for any x such that $0 \leqq x \leqq 1-1/n$, $f(x) \neq f(x + 1/n)$. This will lead to a contradiction and the first part will be proven. First $f(1/n) = f(0 + 1/n) \neq f(0) = 0$. We may assume that $f(1/n) > 0$. (If $f(1/n) < 0$, we would consider the function $-f(x)$). Now $f(2/n) = f(1/n + 1/n) \neq f(1/n)$. We wish to show that $f(2/n) > f(1/n)$. If not, consider the case $f(2/n) < f(1/n)$. This will lead to a contradiction. For define $g(x) = f(x + 1/n) - f(x)$, where $g(x)$ is a continuous function in the interval $0 \leqq x \leqq 1/n$. Secondly $g(0) = f(1/n) - f(0) > 0$ and $g(1/n) = f(2/n) - f(1/n) < 0$ by our assumption. Thus $g(x)$ is positive at $x = 0$ and negative at $x = 1/n$. Because $g(x)$ is continuous in $0 \leqq x \leqq 1/n$ there is an x_1 between 0 and $1/n$ such that $g(x_1) = 0$. Thus we have $0 = g(x_1) = f(x_1 + 1/n) - f(x_1)$ or $f(x_1) = f(x_1 + 1/n)$ which contradicts our original assumption on $f(x)$. Thus $f(2/n) > f(1/n)$. In a similar way we may prove that $f(3/n) > f(2/n)$ and by induction that $f((i + 1)/n) > f(i/n)$ for $i = 0, 1, 2, \ldots, n-1$. Thus we have $0 < f(1/n) < f(2/n) < \ldots < f(n/n) = f(1) = 0$ yielding an obvious contradiction. Thus we have proven that there is no function which does not have $1/n$ for a subperiod or that every function has $1/n$ as a subperiod.

II. Now suppose that $0 < a < 1$ and that $a \neq 1/n$ for any integer n. We need show that a is not a subperiod of all functions which we consider. Equivalently, we need only produce one function $f(x)$ such that for all x, $0 \leqq x \leqq 1-a$, $f(x) \neq f(x + a)$. Then a will not be a subperiod of all functions. To this end, we go through a construction of such a function.

Define k to be the greatest integer such that $ka < 1$. (Such k exist and furthermore $k \geqq 1$.) Then $(k + 1)a \geqq 1$. But by our assumption on a the equality cannot occur and we thus have $ka < 1 < (k + 1)a$ or $0 < 1 - ka < a$. Call $p = 1 - ka$.

We will now construct the function $f(x)$. Graphically, consider the three points $A(0, 0)$, $B(p, -k)$ and $C(a, 1)$. By drawing the straight lines AB and BC, we construct a function $f(x)$ defined in $0 \leqq x < a$. (We surely do not have to give the explicit formula for $f(x)$!) Now extend $f(x)$ by stipulating that for all x we must have $f(x + a) = 1 + f(x)$. We leave it as an easy exercise to show that there is one and only one function satisfying the conditions of extension. (In succession define $f(x)$ in $a \leqq x < 2a$, $2a \leqq x < 3a$, etc. and similarly for negative x.) Continuity of the extended $f(x)$ need only be verified at the integral multiples of a, and is easily done. By induction it is also rather easy to show that $f(x + na) = n + f(x)$ for all values of x. What of $f(1)$? The answer is $f(1) = f(p + ka) = k + f(p) = k + (-k) = 0$. And so the $f(x)$ defined only in the interval $0 \leqq x \leqq 1$ is a function of the type considered, since by definition $f(0) = 0$. This $f(x)$ cannot have a subperiod a, because for all x we have $f(x + a) = 1 + f(x) \neq f(x)$. The theorem is thus completely proved.

QUOTATION: "Of all the reciprocals of integers, the one that I best like is $1/0$ for it is a titan amongst midgets" -

Ascribed to Sam Linial, CCNY, '49

NOTE ON THE PRODUCT OF POWER SUMS*

By J. S. Frame,
Michigan State College

1. Introduction. The fact that the square of the sum of the first n integers is equal to the sum of their cubes is but one of a number of interesting relationships between the products of power sums. For example, the fourth power of the sum of the first n integers is equal to the average of the sum of fifth powers and the sum of seventh powers.

In general, let $S_p(n)$, or just S_p, denote the sum of the p^{th} powers of the first n integers. It is known that $S_p(n)$ is a polynomial in n of degree p+1, with coefficients involving the Bernoulli numbers B_k, such that $S_p(0) = 0$, $S_p(1) = 1$. It is natural to expect that a product $S_p(n)S_q(n)$ of two polynomials of respective degrees p + 1 and q + 1 in n should be expressible as a linear combination of p + q + 1 such polynomials of degrees less than or equal to p 1 2. However, the expression for the product $S_p(n)S_q(n)$, for $p \geqq q$, actually involves only $[p/2] + 1$ polynomials $S_k(n)$ all having subscripts of the same parity, and having numerical coefficients whose sum is one. The coefficient of S_{p+q+1} is $[1/(p+1)] + [1/(q+1)]$. For example:

$$(1)\quad S_1^2 = S_3,\quad S_2^2 = \frac{2}{3}S_5 + \frac{1}{3}S_3,\quad S_3^2 = \frac{1}{2}S_7 + \frac{1}{2}S_5,$$

$$S_4^2 = \frac{2}{5}S_9 + \frac{2}{3}S_7 - \frac{1}{15}S_5;$$

$$(2)\quad S_1S_2 = \frac{5}{6}S_4 + \frac{1}{6}S_2,\quad S_1S_3 = \frac{3}{4}S_5 + \frac{1}{4}S_3,$$

$$S_2S_3 = \frac{7}{12}S_6 + \frac{5}{12}S_4.$$

*Paper presented in April, 1949, at the meeting of the Michigan Section of the Mathematical Association of America at Detroit.

2. Polynomials which represent power sums. A simple derivation of the polynomials $S_p(n)$ is obtained by means of generating functions. The coefficient of $t^p/p!$ in

$$(3)\quad G(n,t) = e^t + e^{2t} + \ldots + e^{(n-1)t} + e^{nt}$$

is precisely $S_p(n)$. Hence, summing the geometric series on the right of (3) we have

$$(4)\quad \sum_{p=0}^{\infty} S_p(n)t^p/p! = (e^{nt} - 1)/(1 - e^{-t}) = G(n,t).$$

We are led to the two series expansions

$$(5)\quad \frac{t}{2}\coth\frac{t}{2} = \left[\frac{t}{1-e^{-t}} - \frac{t}{2}\right] = 1 + \frac{B_1}{2!}t^2 - \frac{B_2}{4!}t^4 + \frac{B_3}{6!}t^6 - \frac{B_4}{4!}t^8 + \ldots,$$

$$(6)\quad e^{nt} - 1 = (nt) + \frac{(nt)^2}{2!} + \frac{(nt)^3}{3!} + \ldots$$

The product of the two series (5) and (6) gives the following modification of (4).

$$(7)\quad (e^{nt}-1)\frac{t}{2}\coth\frac{t}{2} = \sum_{p=0}^{\infty}\left[S_p(n) - \frac{1}{2}n^p\right]\frac{t^{p+1}}{p!} + \frac{t}{2} = G(n,t) - \frac{1}{2}t(e^{nt}-1).$$

Comparing coefficients of t^{p+1} on both sides of (7) we have

$$(8)\quad S_p(n) - \frac{1}{2}n^p = p!\left[\frac{n^{p+1}}{(p+1)!} + \frac{B_1 n^{p-1}}{2!(p-3)!} - \frac{B_2 n^{p-3}}{4!(p-3)!} + \ldots + \frac{(-1)^{k-1}B_k n^{p+1-2k}}{2k!(p+1-2k)!} + \ldots\right],$$

where the value $k = (p+1)/2$ is omitted. This may be written

$$(9)\quad F_p(n) = S_p(n) - \frac{1}{2}n^p = \frac{n^{p+1}}{p+1} + \sum_{k=1}^{[p/2]}(-1)^{k-1}\frac{B_k}{2k}\left(2k^p - 1\right)n^{p+1-2k},$$

where

$$(10)\quad B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},\ B_5 = \frac{5}{66},\ \ldots$$

3. Products of power sums. Using (9) we write

$$(11)\quad S_p(n) = F_p(n) + \frac{1}{2}n^p,\quad S_p(n-1) = S_p(n) - n^p = F_p(n) - \frac{1}{2}n^p.$$

Then the product $S_p(n)\ S_q(n)$ can be obtained by summing a telescoping series of differences.

$$(12)\quad S_p(n)\ S_q(n) = \sum_{m=1}^{n}\left[S_p(m)\ S_q(m) - S_p(m-1)\ S_q(m-1)\right]$$

$$= \sum_{m=1}^{n}\left[\left(F_p(m) + \frac{1}{2}m^p\right)\left(F_q(m) + \frac{1}{2}m^q\right) - \left(F_p(m) - \frac{1}{2}m^p\right)\left(F_q(m) - \frac{1}{2}m^q\right)\right].$$

After simplifying and applying (9) we have

$$(13)\quad S_p(n)S_q(n) = \sum_{m=1}^{n} m^q F_p(m) + \sum_{m=1}^{n} m^p F_q(m)$$

$$= \sum_{m=1}^{n}\left[\frac{m^{p+q+1}}{p+1} + \sum_{k=1}^{[p/2]}(-1)^{k-1}\frac{B_k}{2k}\binom{p}{2k-1}m^{p+q+1-2k}\right]$$

$$+ \sum_{m=1}^{n}\left[\frac{m^{p+q+1}}{q+1} + \sum_{k=1}^{[q/2]}(-1)^{k-1}\frac{B_k}{2k}\binom{q}{2k-1}m^{p+q+1-2k}\right].$$

$$(14)\quad S_p(n)S_q(n) = \left(\frac{1}{p+1} + \frac{1}{q+1}\right)S_{p+q+1}(n)$$

$$+ \sum_{k=1}^{[p/2]}(-1)^{k-1}\frac{B_k}{2k}\binom{p}{2k-1}S_{p+q+1-2k}(n)$$

$$+ \sum_{k=1}^{[q/2]}(-1)^{k-1}\frac{B_k}{2k}\binom{q}{2k-1}S_{p+q+1-2k}(n).$$

The coefficients of $S_{p+q+1-2k}$ may be combined to give a total of $1 + [p/2]$ terms if $p \geqq q$, with alternating signs after the second term. In particular for $p = q$ we have

$$(15)\quad S_p^2(n) = \frac{2}{p+1}S_{2p+1}(n) + \sum_{k=1}^{[p/2]}(-1)^{k-1}\frac{B_k}{k}\binom{p}{2k-1}S_{2p+1-2k}(n)$$

We note that since $S_p(1) = 1$, the sum of the coefficients of the various $S_k(n)$ on the right of (14) or (15) is equal to 1. The relations in (1) and (2) are special cases of (15) and (14).

Successive application of (14) gives a means of expressing a product of $S_k(n)$ with any number of factors as a linear combination of these power sums. For example:

$$(16)\quad S_1^4 = \frac{1}{2}(S_7 + S_5);\quad S_2^3 = \frac{1}{12}(4S_8 + 7S_6 + S_4).$$

PROBLEM DEPARTMENT

This department welcomes problems believed to be new, and, as a rule, demanding no greater ability in problem solving than that of the average member of the fraternity, but occasionally we shall publish problems that should challenge the ability of the superior undergraduate or graduate student. Solutions of these problems should be submitted on separate, signed sheets within three months after publication. Address all communications concerning problems to the Editor.

1. Proposed by the Problem Editor

Prove the following construction for finding the radius of a circumference. With any point O on the circumference as center and any convenient radius describe an arc PQR, cutting the given circumference in P and Q. With Q as center and the same radius describe an arc OR cutting PQR in R, R being inside the circumference. Join P and R, cutting the given circumference in L. Then LR is the radius of the circumference. (This is known as Swale's construction, and is probably the simplest solution of the problem yet discovered.)

2. Proposed by the Problem Editor

Instrumental errors may often be made to react against themselves and automatically disappear. A simple example is the process of "double weighing," in which the effect of inequality in the arms of the balance is removed by weighing with the object first on one pan, then on the other, and taking an appropriate mean. Several texts say that the arithmetic mean should be taken. Show that this is not true, and find the correct mean.

3. Proposed by the Problem Editor

The lengths of the sides of a triangle are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$. Find the area of the triangle.

4. Proposed by the Problem Editor

Towards the bottom of page 268 of Cajori's "A History of Mathematics" (1926) we find the following statements: "Napoleon proposed to the French mathematicians the problem, to divide the circumference of a circle into four equal parts by the compasses only. Mascheroni does this by applying the radius three times to the circumference; he obtains the arcs AB, BC, CD; then AD is a diameter; the rest is obvious." Show how the "obvious" part of the problem may be accomplished.

5. Proposed by P. M. Anselone, College of Puget Sound

If $$f(M,N) = M!(N+1)\sum_{i=0}^{M}(M+N-i)!/(M-i)!,$$

show that $f(M,N) = f(N,M)$.

6. Proposed by C. W. Trigg, Los Angeles City College

Starting with a straight edge, closed compasses, and two straight line segments a and b, construct the harmonic mean of a and b in the least number of operations. Changing the opening of the compasses, drawing a circle or the arc of a circle, and drawing a straight line are each considered an operation.

7. Proposed by Arthur Rosenfeld, N. Y. C.

Prove that $$\sum_{j=0}^{R}\binom{Q}{j}\binom{Q-j}{R-j}=\binom{Q}{R}2^R.$$

8. Proposed by R. T. Hood, University of Wisconsin

Consider the stereographic projection of a sphere onto a plane tangent to it at its south pole S, the center of projection being the north pole N. Prove that every great circle on the sphere not passing through N is mapped into a circle whose center is on the line through N which is perpendicular to the plane of the great circle.

9. Proposed by the Problem Editor

If the bases of a prismatoid are equal in area, then so are the sections equidistant from the midsection.

10. Proposed by P. L. Chessin, Columbia University

Sum the series

$$a + (a + d)x + (a + 2d)x^2/2! + \dots + (a + nd)x^n! + \dots$$

11. Proposed by Frank Hawthorne, Hofstra College

A projectile in vacuo passes through two given points. Determine the locus of foci and of vertices of the parabolic trajectories.

Question: How many legs has a horse?
Answer: Twelve; two in front, two behind, two on each side, and one in each corner.

FLOYD FISKE DECKER*
1881 - 1949

THE death of Dr. Floyd Fiske Decker, November 28, 1949, brings to a close a life devoted to the teaching of mathematics. Born March 23, 1881, at Dennison, Texas, he spent his undergraduate years, 1897-1901, at Syracuse University where he received the degrees of **Ph.B.,** 1901, **Ph.M.,** 1905, **Ph.D.,** 1910.

Following some outside teaching, Dr. Decker returned to Syracuse University where he was a member of the mathematics faculty from 1904 to the time of his death.

He belonged to various honorary societies but was always most closely associated with those of the department of which he was a member. He was a member of the first mathematics club of the University and a charter member of Pi Mu Epsilon. At the founding of Pi Mu Epsilon, Dr. Decker was elected the first vice-director.

Aside from his work in mathematics he will be remembered especially for his work as the first Director of the Extension School, now known as University College. Plans for the Extension School occupied his mind during his waking hours from the beginning of the school until it became so large that a full-time director was necessary. Dr. Decker's efforts had already placed the school on a firm foundation.

Dr. Decker will be long remembered by the many members of the faculty with whom he was associated and by the many grateful students who passed through his classes, for he was a patient, thorough and conscientious teacher.

In 1910, he married Mary Makepeace, a member of the class of 1907, besides whom he leaves a daughter, Mrs. Elizabeth Corwin, a son, Robert Decker, and two grandchildren.

Louis Lindsey

*Dr. Decker was the author of the article "The Earliest Days of Pi Mu Epsilon" which appeared in the first issue of this Journal.

AN APPROXIMATION FOR THE DIP OF A CATENARY

By J. S. Frame
Michigan State College

Let a uniform inextensible cable or chain of length 2s (ft.) be hung between supports A, B at the same level and at distance AB = 2x (ft.) apart, where x is less than s. Then the dip or sag y (ft.) from the line AB to the lowest point O on the curve may be closely approximated by the rather simple formula

$$(1)\qquad y^2 \cong (s - x)(s + x/2).$$

The value of y computed from (1) is in error by less than one part in 117, for all values of x less than s.*

If ρ is the constant linear density of the catenary, and if $H = \rho a$ and $W = \rho s$ denote the horizontal and vertical components of force at a variable point P of the curve, and if we introduce a parameter u such that sinh u is the slope of the curve at P, then

$$(2)\qquad \sinh u = dy/dx = W/H = s/a.$$

$$(3)\qquad \cosh u = (1 + \sinh^2 u)^{1/2} = ds/dx.$$

After differentiating equation (2) and using (3), we have

$$(4)\qquad \cosh u\, du = ds/a = \cosh u\, dx/a.$$

We can express dx, dy and ds in terms of du by using (4), (2), and (3), and can then integrate to find x, y and s in terms of u, keeping a constant and noting that y = s = 0 when x = 0. Thus

$$(5)\qquad dx = a\,du,\quad dy = a \sinh u\, du,\quad ds = a \cosh u\, du.$$

$$(6)\qquad x = a u,\quad y = a(\cosh u - 1),\quad s = a \sinh u.$$

***Professor E. P. Starke has called my attention to another approximation for the dip of a catenary, published as the solution of a problem in the National Mathematics Magazine (Feb. 1945, p. 262, Problem 570) but valid only for small values of the ratio x/s.**

The relative error of the approximation (1) may now be written as a function F(u) of the parameter u. The parameter a, which was constant for a particular catenary, must be eliminated in order that the approximation should not involve the unknown location of the directrix (y = -a) of the catenary. We have

$$(7)\qquad F(u) = 1 - \frac{(s-x)(s+x/2)}{y^2} = \frac{(\sinh u - u)(\sinh u + u/2)}{(\cosh u - 1)^2}$$

$$= \frac{N(u)}{2(\cosh u - 1)^2},$$

where the numerator N(u) in (7) is given by

$$(8)\qquad N(u) = 2(\cosh u - 1)^2 - 2\sinh^2 u + u \sinh u + u^2$$

$$= u^2 + u \sinh u - 4(\cosh u - 1).$$

The fact that N(u) is non-negative may be seen by writing

$$(9)\qquad N(u) = u(\sinh u - u) - 4(\cosh u - 1 - u^2/2)$$

$$= \sum_{n=3}^{\infty} \frac{(2n-4)u^{2n}}{(2n)!} \geq 0.$$

Thus we conclude that the relative error function F(u) is positive for positive u, increases from 0 like $u^2/180$ for small u, and decreases towards 0 again like $(u-4)e^{-u}$ as u becomes infinite. To find the value of u for which F(u) is maximum, we set the numerator of the derivative of F(u) equal to 0. We then have

$$(10)\qquad 2(\cosh u - 1)^2 N'(u) - 4(\cosh u - 1)\sinh u\, N(u) = 0.$$

This equation may be transformed into the form

$$(11)\qquad \frac{2N(u)}{\cosh u - 1} - \frac{N'(u)}{\sinh u} = 0.$$

When the values of N(u) and its derivative N'(u) are substituted from (8) into (11), the condition for maximum F(u) becomes

$$(12)\qquad G(u) = \frac{2u^2}{\cosh u - 1} + u \coth u = 5.$$

Solving equation (12) requires special care in computation, since the function G(u) differs but little from 5 for small values of u. This is seen from the following table of values

(13)

u	0	1	2	3	4
G(u)	5.0000	4.9959	4.9705	4.9999855	5.2187

The positive value of u for which $G(u) = 5$, or $F'(u) = 0$, is found to be just greater than $u = 3.00$. For this particular parameter value the dip y is nearly 1.5 times as great as the distance $2x$ between supports. The maximum relative error $F(u)$ in y^2 is then found from accurate tables of hyperbolic functions to be

$$\text{(14)} \qquad \text{Max } F(u) = F(3.00+) = 0.017.$$

For y itself, as computed from (1), the relative error is half this amount, or about 0.0085. The computed value is always too small, since $F(u)$ is positive for positive u. Should an approximate formula for dip be desired which has four place accuracy but does not explicitly involve the hyperbolic functions, the relative error of y may be fitted by a rational function. We leave it to the reader to show that the dip (or sag) of a catenary is given accurately to four significant figures by the formula

$$\text{(15)} \qquad y = \sqrt{(s-x)(s+x/2)}\left(1 + \frac{0.85\,x(s-x)}{s(9s+40x)}\right).$$

The applied mathematician may perhaps wonder how formula (1) should be altered to allow for elastic stretching according to Hooke's Law. Let us replace the inextensible catenary by a flexible elastic cord of unstretched length 2L and weight 2W, and define $k = W/2E$, where E is Young's modulus. Then when the two ends are together the midpoint will hang down a distance $L(1 + k)$. If we again define sinh u to be the slope of the curve at a point P, the parametric equations for the coordinates (x,y) of the endpoint B and for the stretched arc OB = s turn out to be

$$\text{(16)} \qquad \frac{x}{L} = \frac{u + 2k}{\sinh u}, \quad \frac{y}{L} = \tanh\frac{u}{2} + k, \quad \frac{s}{L} = 1 + k\frac{u + \sinh u \cosh u}{2\sinh^2 u}$$

We leave it to the ingenuity of the reader to find a simple approximate relationship between the variables x/L, y/L and k, that does not involve the parameter u but does reduce to formula (1) when $k = 0$ and $L = s$.

THE NUMBER OF SOLUTIONS OF THE TOTIENT EQUATION $\varphi(x) = B$

By Richard V. Andree,
University of Oklahoma

The totient of Euler φ-function $\varphi(x)$ is defined for positive integers x as the number of positive integers less than or equal to x which are relatively prime to x.

A student of elementary number theory attempting to solve the equation $\varphi(x) = 24$, where $\varphi(x)$ is the Euler φ-function or totient, finds x = 39, 52, 56, 72, 78, 84,. . .and other values. The question arises, quite naturally, if there always exist only a finite number of solutions of $\varphi(x) = B$, or if, for some B at least, there may exist infinitely many such x. That there do not exist infinitely many solutions for all B is at once obvious from the cases where B = 3 and B = 1. The former has no solution while the latter has only the solutions x = 1 and x = 2.

The answer to this question may be obtained by specializing certain rather powerful theorems which the graduate student has at his disposal. The purpose of this note, however, is to give the answer in a form simple enough for the undergraduate or beginning graduate student taking number theory.

Theorem: Given a positive integer B, there exists a number N_B such that for all $x > N_B$, $\varphi(x) > B$.

If B = 1, take $N_B = 2$.

If $B > 1$, there exist only a finite number of primes $p_i \leqq B + 1$. Call these $p_1, p_2, \ldots, p_n$.

Since $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$ we have, at once,

$$1 = \varphi(p_i^0) \leqq \varphi(p_i) < \varphi(p_i^2) < \varphi(p_i^3) < \ldots < \varphi(p_i^k) < \ldots.$$

Thus there exist only a finite number of prime powers $p_i^{\gamma_i}$ such that

$$\varphi(p_i^{\gamma_i}) \leqq B.$$

Let $\alpha_i \geqq 0$ be the maximum exponent satisfying

$$\varphi(p_i^{\alpha_i}) \leqq B.$$

All powers of all primes, except the finite collection

$$p_1, p_1^2, p_1^3, \ldots, p_1^{\alpha_1}, p_2, p_2^2, \ldots, p_2^{\alpha_2}, p_3, \ldots, p_n^{\alpha_n}$$

have $\varphi(q^\delta) > B$, since if $q > B + 1$ and q prime, then

$$\varphi(q^\delta) = q^{\delta-1}(q-1) > B.$$

Let $N_B = \prod p_i^{\alpha_i}$. For every $x = \prod r_i^{\ell_i} > N_B$, at least one of the prime power factors $r_i^{\ell_i}$ must be such that $\varphi(r_i^{\ell_i}) > B$. Hence, since $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are relatively prime, we have

$$\varphi(x) > B \text{ for all } x > N_B = \prod p_i^{\alpha_i}.$$

The question proposed in the first paragraph is now answered in the following immediate corollary.

Corollary: There exist at most a finite number of solutions of $\varphi(x) = B$.

An additional problem of this type which the beginning student should be expected to work is "Show that, for all odd primes p, $\varphi(x) = p$ has no solution." The maverick nature of the prime 2 is again displayed.

ERRATA

The following errata in Volume 1, Number 1, have been called to the attention of the editors:

J. S. Frame, "Note on the Product of Power Sums." Page 19, line 13, formula (7), insert t in right member of equation so that it reads $tG(n,t)-1/2t(e^{nt}-1)$. Page 19, last line, formula (9), replace

$$(2k^p - 1) \text{ by } \binom{p}{2k-1}.$$

Problem 7. Proposed by Arthur Rosenfeld, N.Y.C.

Page 23, change $\binom{Q}{J}$ to $\binom{Q}{j}$.

Problem 10. Proposed by P. L. Chessin, Columbia University.

Page 24, change $(a + nd)x^n!$ to $(a + nd)\frac{x^n}{n!}$.

PROBLEM DEPARTMENT

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the master's degree. Solutions of these problems should be submitted on separate, signed sheets within five months after publication. Address all communications concerning problems to the Editor.

PROBLEMS FOR SOLUTION

12. Proposed by C. O. Oakley, Haverford College

(Note. This problem has appeared elsewhere, but because it is a timely problem it has been reprinted.)

The Census Enumerator Problem

A census enumerator goes to a house and notes the number of the house. The man who answers the door bell gives the enumerator his age and adds that there are three other members of his household. But he refuses to give the ages of the other three. Instead he says, "The product of their ages is 1296 and the sum is equal to the house number." The enumerator, after some computing, says, "I need some more information. Are you the oldest member of the household?" The man replied, "Yes." The enumerator now has enough information to compute the ages of the other three and so have you.

13. Proposed by W. R. Ransom, Tufts College

An electric clock has hour, minute and second hands turning about the same axis. Prove that they are never together except at twelve o'clock.

14. Proposed by C. W. Trigg, Los Angeles City College

1. How may a sealed envelope be folded into a rectangular parallelepiped if overlapping is permitted? 2. What is the maximum volume so obtainable in terms of the edges a and b of the envelope? 3. What must be the relative dimensions of the envelope in order to yield a cube? 4. What will be the volume of the cube?

15. Proposed by Lester G. Riggs, Evanston, Illinois

(Note. The proposer does not claim to have originated this problem.)

Russian Multiplication

Consider any two positive integers x_1 and y_1 and the table

y_1	y_2	$\cdots$	y_k
x_1	x_2	$\cdots$	x_k

The x and y values are formed according to the laws:

$$y_{n+1} = 2y_n,\quad x_{n+1} = \frac{x_n}{2} \text{ for } x_n \text{ even},\quad x_{n+1} = \frac{x_n - 1}{2} \text{ for } x_n \text{ odd},$$

$x_k = 1$ for $n = 1, 2, \ldots, k-1$.

Prove that

$$x_1 y_1 = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

where the $p_1, p_2, \ldots, p_k$ are such that $x_{p_1}, x_{p_2}, \cdots, x_{p_k} = 1$ are all of the odd x values.

Examples:

(a)

7	~~14~~	~~28~~	56
9	~~4~~	~~2~~	1

; $9 \cdot 7 = 7 + 56 = 63.$

(b)

~~13~~	~~26~~	52	~~104~~	208
~~20~~	~~10~~	5	~~2~~	1

; $20 \cdot 13 = 52 + 208 = 260.$

(c)

56	112	~~224~~	448	~~896~~	1792	~~3584~~	7168
171	85	~~42~~	21	~~10~~	5	~~2~~	1

;

$171 \cdot 56 = 56 + 112 + 448 + 1792 + 7168 = 9576.$

16. Proposed by W. J. Jenkins, Livingston, Alabama

Given a circle and two exterior points not in a straight line with the center. Construct a circle passing through these two points and dividing the given circle into two equal arcs.

17. Proposed by C. Stanley Ogilvy, Columbia University

All the coefficients in the expansion of $(a+b)^n$ except the first and the last are divisible by n if and only if n is a prime.

(Note. The proposer gives the complete proof and remarks that the sufficiency proof is given in another connection in Van der Waerden, page 93 of the English translation; but, he adds, he has not yet found in print the necessity part of the proof.)

18. Proposed by Lindley J. Burton, Bryn Mawr College

Points A_1, B_1, C_1, are chosen on the sides BC, CA, AB of triangle ABC such that $AC_1 = 1/2 C_1B$, $BA_1 = 1/2 A_1C$, $CB_1 = 1/2 B_1A$. The lines AA_1, BB_1, CC_1 determine a triangle $A_2B_2C_2$. Show that the area of $A_2B_2C_2$ is one seventh the area of ABC.

19. Proposed by E. P. Starke, Rutgers University

Given two polynomials of degree n,

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

$$g(x) \equiv b_0x^n + b_1x^{n-1} + \cdots + b_n,$$

and $n+1$ arbitrary constants $k_1, \ldots, k_{n+1}$, not necessarily distinct; if

$f(k_1) = g(k_1),\ f'(k_2) = g'(k_2),\ f''(k_3) = g''(k_3),\ \ldots,$

$f^{(n)}(k_{n+1}) = g^{(n)}(k_{n+1})$, then $f(x) = g(x)$ identically.

20. Proposed by Murray S. Klamkin, Brooklyn Polytechnic Institute

Show that

$$\frac{b^b - a^a}{b(1 + \ln b)} < \int_a^b x^{x-1}\,dx < \frac{b^b - a^a}{a(1 + \ln b)}, \quad b > a > 1.$$

SOLUTIONS

2. Proposed by the Problem Editor

Instrumental errors may often be made to react against themselves and automatically disappear. A simple example is the process of "double weighing," in which the effect of inequality in the arms of the balance is removed by weighing with the object first on one pan, then on the other, and taking an appropriate mean. Several texts say that the arithmetic mean should be taken. Show that this is not true, and find the correct mean.

Solution by Lawrence Bennett, Brooklyn College

The weight to be found is W. We shall consider two cases. In the first, the weight W, at a distance d_1 from the fulcrum, is balanced by a standard weight S_1 at a distance d_2 from the fulcrum; in the second, W, at a distance d_2 from the fulcrum, is balanced by a second standard weight S_2 at a distance d_1 from the fulcrum, where S_2 will be different from S_1 if $d_1 \# d_2$.

By the law of the lever, for the first case,

$$d_1 W = d_2 S_1,$$

and

$$d_1 S_2 = d_2 W$$

for the second. This gives

$$W = \frac{d_2}{d_1} S_1 \quad \text{and} \quad W = \frac{d_1}{d_2} S_2.$$

Eliminating W, we get

$$S_1 = \frac{d_1^2}{d_2^2} S_2.$$

The geometric mean will then give:

$$\sqrt{S_1 S_2} = \sqrt{\left(\frac{d_1^2}{d_2^2} S_2\right)(S_2)} = \frac{d_1}{d_2} S_2 = W.$$

The arithmetic mean is

$$\frac{S_1 + S_2}{2} = \frac{\dfrac{d_1^2}{d_2^2} S_2 + S_2}{2} = \frac{d_1}{d_2} S_2 \left[\frac{d_1}{2d_2} + \frac{d_2}{2d_1}\right].$$

Suppose $\dfrac{d_2}{d_1} = a$, then

$$\frac{S_1 + S_2}{2} = \frac{d_1}{d_2} S_2 \left[\frac{1}{2a} + \frac{a}{2}\right] = W \left[\frac{1}{2a} + \frac{a}{2}\right]$$

$$= W \left[\frac{a^2 + 1}{2a}\right].$$

Then the arithmetic mean will equal W only if

$$\frac{a^2 + 1}{2a} = 1,$$

$$a^2 - 2a + 1 = 0,$$
$$a = 1,$$

which gives $d_1 = d_2$.

In other words, the arithmetic mean is correct only in the case of exactly equal arms.

Also solved by C. W. Trigg, Los Angeles City College.

3. Proposed by the Problem Editor

The lengths of the sides of a triangle are the roots of the cubic equation $ax^3 + bx2 + cx + d = 0$. Find the area of the triangle.

Solution by Lawrence Bennett, Brooklyn College

The area of the triangle is

$$A = \sqrt{s(s-x_1)(s-x_2)(s-x_3)}\ ,$$

where

$$s = \frac{x_1 + x_2 + x_3}{2}$$

and x_1, x_2, x_3 are the roots of the cubic $ax^3 + bx^2 + cx + d = 0$.

$$A = \sqrt{s^4 - s^3(x_1+x_2+x_3) + s^2(x_1x_2+x_2x_3+x_3x_1) - s(x_1x_2x_3)}$$

But
$$s = \frac{x_1 + x_2 + x_3}{2} = -\frac{b}{2a},$$

and
$$x_1 + x_2 + x_3 = -\frac{b}{a},$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a},$$

$$x_1x_2x_3 = -\frac{d}{a},$$

$$A = \sqrt{\left(\frac{-b}{2a}\right)^4 - \left(\frac{-b}{2a}\right)^3\left(\frac{-b}{a}\right) + \left(\frac{-b}{2a}\right)^2\left(\frac{c}{a}\right) - \left(\frac{-b}{2a}\right)\left(\frac{-d}{a}\right)}$$

$$= \sqrt{\frac{b^4}{16a^4} - \frac{b^4}{8a^4} + \frac{b^2c}{4a^3} - \frac{bd}{2a^2}}$$

$$= \frac{1}{4a^2}\sqrt{-b^4 + 4ab^2c - 8a^2bd}\ .$$

Also solved by C. W. Trigg who lists the following references.

(1) School Science and Mathematics, Nov. 1936, vol. 36, p. 915. It is shown there, also, that the altitudes of the triangle are the roots of $8a^2R^3X^3 - 4acR^2X^2 + 2bdRX - d^2 = 0$, where R is the circumradius.

(2) The American Mathematical Monthly, May 1925, vol. 32, p. 266. It is shown that if the roots of a cubic equation considered as lengths are to form a triangle then the discriminant of the cubic must be positive and b and $(4abc - 8a^2d - b3)$ must have like signs, $a > 0$.

4. Proposed by the Problem Editor

Towards the bottom of page 268 of Cajori's "A History of Mathematics" (1926) we find the following statements: "Napoleon proposed to the French mathematicians the problem, to divide the circumference of a circle into four equal parts by the compasses only. Mascheroni does this by applying the radius three times to the circumference; he obtains the arcs AB, BC, CD; then AD is a diameter; the rest is obvious." Show how the "obvious" part of the problem may be accomplished.

Solution by John A. Dyer, University of Alabama

Napoleon's problem may be stated as follows: By use of compasses only find the side of a square inscribed in a given circle O, radius R, given also the consecutive points A, B, C, D on the circumference such that AB = BC = CD = R, and AD a diameter.

Now, with A as center and radius AC, draw circle A. Similarly, with D as center and radius DB, draw circle D. Circles A and D intersect at points, say E and E'. Then OE is the required side of the inscribed square.

7. Proposed by Arthur Rosenfeld, N.Y.C.

Prove that $$\sum_{j=0}^{R} \binom{Q}{j}\binom{Q-j}{R-j} = \binom{Q}{R} 2^R.$$

Solution by Ben A. Green, University of Alabama

We have by definition

$$\sum_{j=0}^{R} \binom{Q}{j}\binom{Q-j}{R-j} = \sum_{j=0}^{R} \left\{\frac{Q!}{j!(Q-j)!} \cdot \frac{(Q-j)!}{(R-j)!(Q-R)!}\right\} .$$

Since (Q-j)! divides out, and $\frac{Q!}{(Q-R)!}$ is independent of the index of summation, we may write this as

$$\frac{Q!}{(Q-R)!} \sum_{j=0}^{R} \frac{1}{j!(R-j)!} .$$

Multiplying numerator and denominator by R! gives

$$\frac{Q!}{R!(Q-R)!} \sum_{j=0}^{R} \frac{R!}{j!(R-j)!},$$ which is notationally,

$$\binom{Q}{R} \sum_{j=0}^{R} \binom{R}{j} .$$

However, in the binomial expansion

$$(a+b)^R = \sum_{j=0}^{R} \binom{R}{j} a^{R-j} b^j, \quad \text{place } a = b = 1$$

and get $$\sum_{j=0}^{R} \binom{R}{j} = (1+1)^R = 2^{R*}$$

Thus we have

$$\sum_{j=0}^{R} \binom{Q}{j}\binom{Q-j}{R-j} = \binom{Q}{R} 2^R .$$

*Note: Reference is made to the fact that $\sum_{j=0}^{R} \binom{R}{j} = 2^R$ in one form or another, in the following:

(1) Courant, Differential and Integral Calculus, Nordemann, New York, 1940, vol. 1, p. 28.
(2) Ferror, Convergence, Oxford Univ., London, 1945, Prob. 7.
(3) Fine, College Algebra, Ginn, Boston, 1904, p. 402.
(4) Hall and Knight, Higher Algebra, Macmillan, London, 1946, p. 146.
(5) Whitworth, Choice and Chance, Stechert, New York, 1942, p. 48.

Also solved by C. W. Trigg, Los Angeles City College, Sherman Babcock, Syracuse University, as well as the proposer, Arthur Rosenfeld, N. Y. C.

10. Proposed by P. L. Chessin, Columbia University

Sum the series

$$a + (a+d)x + (a+2d)x^2/2! + \ldots + (a+nd)x^n/n! + \ldots .$$

Solution by H. Pitt, University of Wisconsin

We denote the sum of the series by S, then show that $S = a + (a+d)x + (a+2d)x^2/2! + \ldots + (a+nd)x^n/n! + \ldots .$

By the ratio test, this series is absolutely convergent. Hence, rearranging the terms, we have

$$S = (a+ax+ax^2/2! + \ldots + ax^n/n! + \ldots) + (dx+2dx^2/2!+3dx^3/3!+ \ldots + ndx^n/n! + \ldots)$$

$$= a(1+x+x^2/2!+\ldots+x^n/n!+\ldots)+dx(1+x+x^2/2!+\ldots+x^{n-1}/(n-1)!+\ldots)$$

$$= ae^x + dxe^x$$

$$= (a+dx)e^x.$$

Also solved by Lawrnece Bennett, Brooklyn College, Sherman Babcock, Syracuse University, besides the proposer, P. L. **Chessin,** Columbia University.

* * *

To drive a car through the main gate of Syracuse University campus is a privilege reserved for the Chancellor, the Vice-Chancellor, the Deans, the Department Heads, and a few other important persons. This privilege is indicated by a "campus sticker" on the windshield.

Recently a student from the New York State College of Forestry with no sticker on his car drove up and said to the guard, "I've a whole box of logarithms in this car."

The guard with no hesitation replied, "O.K., go ahead."

* * *

The different branches of Arithmetic – Ambition, Distraction, Uglification, and Derision.

– The Mock Turtle (Alice in Wonderland)

EXCERPTS FROM LETTERS TO THE EDITORS

BEFORE and AFTER
Publication of the First Issue of the Journal

— BEFORE —

"... I appreciate very much being one of the authors selected for the new publication.

"I am very happy to hear that Pi Mu Epsilon is publishing such a journal, believing that it could greatly help students as well as stimulate them. I assume that the articles will be written by undergraduates, for the most part, and will not be 'too advanced.' ..."

"I wish Pi Mu Epsilon all good fortune in general, the new publication in particular."

Melvin Hausner
Oct. 25, '49 Princeton University

"The members to whom I have spoken personally think the publication is a splendid idea and share the hope that it will become a permanent institution."

W. S. Beckwith
Nov. 15, '49 University of Georgia

"This publication should be most helpful and interesting. Thank you for sponsoring the project."

W. M. Whyburn, Cor. Sec'y.,
N. C. Beta and Dept. Head
Nov. 16, '49 University of North Carolina

"California Alpha Chapter of Pi Mu Epsilon is glad to hear of the Journal to be published."

Margaret B. Lehman, Cor. Sec'y.
California Alpha
Nov. 16, '49 University of California at Los Angeles

. ..I am sure that all members of the Fraternity are eagerly awaiting the appearance of this first issue of the Journal. I feel that it is a worthwhile project which will make the student members more fully aware of the national scope of the Fraternity. The sponsors are to be commended for supporting the project financially, and the Committee on Publication is to be congratulated for completing preparations for the first issue in so short a time."

H. S. Thurston, Secretary
Alabama Alpha
Nov. 17, '49 University of Alabama

"We are looking forward with interest to this first issue of the PI MU EPSILON JOURNAL."

Dorothy J. Christensen, Sec'y.-Treas.
Oregon Alpha
Nov. 23, '49 University of Oregon

"...The idea of bringing out a journal at this time is an excellent one. If there is anything that we can do to help you along in this venture let us know.

"Best wishes for the success of the publication."

William A. Golomski, Director
Wisconsin Alpha
Dec. 4, '49 Marquette University

-- AFTER —

"Congratulations on a very excellent Journal! I think that you and Mr. Bennett deserve a great deal of credit on getting out such an attractive magazine on so small an appropriation. I am sure that the Fraternity will be pleased with it."

C. C. MacDuffee, Director-General
Pi Mu Epsilon Fraternity
Dec. 20, '49 University of Wisconsin

"The new PI MU EPSILON JOURNAL has arrived and I can't resist writing to congratulate you upon its appearance. Ever since the discussion at the Columbus meetings last year many of us have been looking forward to this first issue."

Richard V. Andree
Ass't. Prof. of Math
Dec. 29, '49 University of Oklahoma

"Thank you very much for letting me have a copy of the PI MU EPSILON JOURNAL. I have read it with much interest. I am sure that it will equally interest other mathematicians including undergraduate members of the Fraternity."

Harry M. Gehman, Sec'y.-Treas. MAA
Jan. 3, '50 University of Buffalo

"Let me add to the many congratulations you must be receiving concerning the first issue of the PI MU EPSILON JOURNAL. The entire issue is excellent and so neat in appearance. It was also a pleasure to read the historical articles written by my good friends Professor Harwood and Professor Decker."

Howard Eves
Jan. 6, '50 Oregon State College

'I am delighted to have a PI MU EPSILON JOURNAL. I feel you are to be congratulated on the fine work you have done in connection with this."

Eric H. Faigle, Assoc. Dean
Jan. 9, '50 Syracuse University

"You and all the others who worked on the first number of Volume I of the PI MU EPSILON JOURNAL are to be congratulated on a fine job. It is grand in every respect and I enjoyed reading it. Best wishes for continued success."

Houston T. Karnes, Sec'y.
Louisiana Alpha
Jan. 10, '50 Louisiana State University

'The Michigan Alpha Chapter of Pi Mu Epsilon and I should like to congratulate you on the first issue of the PI MU EPSILON JOURNAL. It is indeed a fine job, and we are eagerly awaiting the publication of the next issue."

Lindbergh C. Vogt
Michigan State College
Jan. 15, '50

"Our library was pleased to receive v.1, #1 of PI MU EPSILON JOURNAL. We trust that the name of our library was placed on your mailing list to receive all future numbers as issued."

Frances Warner, Serials Librarian
Iowa State College
Jan. 25 '50

"Both you and Mr. Bennett are to be congratulated on the preparation and publication of the first issue of the PI MU EPSILON JOURNAL. The biography of Professor Roe will be welcomed by every member of Pi Mu Epsilon and the description of the earliest days of Pi Mu Epsilon by Professor Decker contained some very interesting material. You provided a good balance of other material too in the two additional articles, the problem department, the reports of the chapters' activities and prizes awarded, and the news notes. You have certainly accomplished the purpose proposed by the Council in providing a journal which should do much to service the various needs of our chapters."

E. H. C. Hildebrandt, Sec.-Treas.-Gen.
Pi Mu Epsilon Fraternity
Northwestern University
Feb. 2, '50

"I am writing to acknowledge and to thank you for the complimentary copy of Volume I, Number 1, of the PI MU EPSILON JOURNAL.

. . . I feel that there is a real need for a journal addressed to the interests of advanced undergraduates and candidates for the Master's degree, and that the PI MU EPSILON JOURNAL might find a real niche for itself in this field."

H. M. MacNeille, Executive Director
American Mathematical Society
Columbia University
March 24, '50

"Purchase Order, DUKE UNIVERSITY, Durham, N. C.
. . . PI MU EPSILON JOURNAL.

SUBSCRIPTION TO VOLUME I AND CONTINUATION. UPON EXPIRATION OF THIS SUBSCRIPTION PLEASE ENTER RENEWAL EACH YEAR UNTIL FURTHER NOTICE."

(signed)
B. E. Powell, for the
Duke University Library
April 5, '50

'I appreciate receiving the first issue of PI MU EPSILON JOURNAL. I enjoyed reading the Journal and I feel that the Committee in charge should be highly commended for their efforts.

"I would like very much to be put on your mailing list and to subscribe to the publication as soon as the subscription rates are set up."

Frances E. Walsh
Atchison, Kansas
April 15, '50

". . . I had intended a long time ago to write you and tell you what a good job I feel was done on the PI MU EPSILON JOURNAL. I think it will fill a real need and that it will be much appreciated by all who read it. I shall look forward to future issues. I hope you will also extend my congratulations to Mr. Bennett for the part he played in connection with the periodical. . . ."

S. S. Cairns
Head of Math. Dept.
University of Illinois
April 18, '50

LETTER FROM THE DIRECTOR GENERAL

Dear fellow members of Pi Mu Epsilon:

The fraternity begins another year under a slightly modified constitution, for six of the proposed amendments which were submitted to the chapters last year were approved. The only one which failed of approval was Proposal V, which would have required the Director to be a member of the faculty.

The purpose of the Councillors in making Proposal V was evidently not clear to the chapters, some of whom assumed that the proposal would take the management of the chapter out of the hands of the students. A little investigation now shows that Section 2 of Article VI of the constitution has been variously interpreted by the different chapters. Many chapters have a President, Secretary and Treasurer picked from among the students, while the Director is a faculty member who serves merely as a Faculty Adviser. This is the organization which Proposal V would have made standard. But many of our chapters have a Director, Vice-Director, Secretary and Treasurer, all of whom are students. If Proposal V had passed, it would have required such chapters to adopt the first type of chapter organization. It is scarcely conceivable that these chapters do not already have Faculty Advisers, for almost all colleges and universities require that every campus organization have a faculty sponsor. In other words, it was merely a matter of titles.

But the other six proposals, which were the important ones, were all accepted by the chapters. The first three were necessary to implement the Journal, and their acceptance constitutes approval by the chapters of the establishment of the Journal. Long may it prosper! There are still problems concerned with the financing of the Journal which

remain to be worked out, but with the support of the members these will be solved so that the cost of the Journal will remain minimal.

Proposal VII is an important one in that it clarifies and codifies the standards for admission, but actually it does not change present practice in a material way. It was the belief of the Council that the constitution laid undue stress on whether an institution called itself a university or a college, and the amendment allows the Council to judge a petitioning group on a more significant basis.

The year ahead of us will present many problems, but with courage and clear vision we shall continue to advance the cause of Mathematics in American institutions of learning.

Sincerely yours,

C. C. MacDuffee
Director General

WHY STUDY MATHEMATICS?

by Norman Miller*
Queen's University

(1) A belief in the value and importance of mathematics in the modern world;

(2) A belief that greater efficiency in the learning and teaching of the subject would result from a clearer understanding of its vocational uses and of its cultural values.

These headings from the introduction to a pamphlet **"Why Study Mathematics?"** indicate the point of view of the authors, Committee appointed in 1945 by the Canadian Mathematical Congress. This pamphlet, of which the present article is a condensation, is designed primarily for distribution to high school teachers of mathematics and vocational guidance, through whom it is hoped that the information will percolate to the students who are facing the important question of their choice of a career.

For students with a keen interest in the subject, a survey is made of those professions which draw heavily on mathematics. For students looking to professions less mathematical, an appraisal is made of the contribution which mathematics can make to their training. For students who will make little explicit use of mathematics, the subject nevertheless furnishes the basis for an appreciation of the scientific age in which we live. Finally, for all students, mathematics, taught for its human values, is a means of enlarging their mental and cultural horizons.

*A condensation of a pamphlet published by the Canadian Mathematical Congress and prepared by Professor Miller and others.

CAREERS WHICH REQUIRE AN EXPERT KNOWLEDGE OF MATHEMATICS

Research and teaching in mathematics

To young people today it is more obvious than in a previous generation that mathematics is necessary for many different kinds of positions in our modern society. Only a few years ago thousands of recruits in the armed services discovered that a basic knowledge of mathematics was required of them. Partly as a result of this, thousands of veterans chose to embark upon courses in which mathematics is a prime requirement.

One of the important lessons of history is that much of human progress has been associated with the discovery of new knowledge. Every day we see evidence that this process is still going on, as witness, for example, the efforts put forth to discover the cause and cure of baffling diseases.

It may not be so widely known that discoveries in mathematics are keeping pace with those in other fields. Far from being a dead or "finished" subject, mathematics is active and growing as rapidly as at any previous period in history. Scores of mathematical journals are published throughout the world for the purpose of making known the new discoveries. The subject has become so vast that each mathematician has to be content to specialize in some branch of the subject.

Research in mathematics proceeds from two incentives. In the first place, a mathematician, absorbed in his subject, may regard it simply as a creation of the mind, without concerning himself with possible applications of his work outside of mathematics. Such activity is often designated <u>pure mathematics</u>. The contrasting term, <u>applied mathematics</u>, suggests a slightly different point of view. The applied mathematician is concerned with adapting mathematics to the solution of problems in other subjects or with developing new mathematical theory for this purpose.

From whichever source his interest in mathematics arises, the mathematician arrives at new results by a combination of imagination, intuition, and logic. A flash of insight may be the genesis of his idea, which then requires patient elaboration with careful logic and with attention to the results of other workers in the field. The flashes of insight, it is hardly necessary to add, are likely to be the fruit of long study and reflection on the problem in hand.

Most of the research in mathematics is carried on by members of the staffs of universities, who fill a dual role in teaching and research. Increasingly, however, industrial corporations and departments of the government are establishing scientific research laboratories in which engineers, scientists, and, in smaller numbers, mathematicians are employed to deal "with problems met with in industry.

The extent to which mathematics remains vital and creative will depend largely on the efficiency and enthusiasm of the teachers in the secondary schools and universities. For young men and women with a keen interest in mathematics, a career devoted to informing the minds of the future scientific leaders as well as of other citizens, presents both a challenge and an opportunity for service of a high order.

Physics and Astronomy

Of all the sciences, physics and astronomy are most closely allied to mathematics. Experimentation, observation and mathematical investigation are the methods of the physicist and the astronomer. In some cases the mathematical investigation has been suggested by experimental results as, for example, in the mathematical explanation of the phenomena observed in the refraction of light. In other cases, experiments were devised as a result of mathematical research. A classical example of such research was the formulation of Maxwell's differential equations which, in giving rise to the discovery of electromagnetic waves, contained the germ of such spectacular developments as radio communication, radar, and television.

In astronomy a similar reciprocity holds between observation and mathematical study. The laws of the motion of the planets were formulated by Kepler in advance of their mathematical explanation, which was later supplied by Newton. On the other hand, mathematical calculations have sometimes furnished the inspiration for discoveries in astronomy. It was the calculations of two mathematicians, Adams

and Leverrier, based on the perturbations of Uranus, which indicated the existence and location of the planet Neptune.

A student should consider a career in the field of physics or astronomy only if he finds pleasure in the study of mathematics and science. A student for whom these subiects are a delight will find no more fascinating career and he need never fear that the profession will be overcrowded.

CAREERS WHICH REQUIRE A KNOWLEDGE OF SPECIAL FIELDS OF MATHEMATICS

Engineering

The importance of engineering in a national economy, especially in those areas in which pioneer conditions still prevail, will be apparent to everyone. The development of transportation by land, sea, and air, the harnessing of waterfalls to produce hydro-electric power, the development of natural resources of forest and mine, the design and operation of the machines required by industry, all these require the study and ingenuity of a large number of engineers.

Regarding the training of an engineer, a pamphlet issued by the Engineering Institute of Canada gives this advice:

> "The engineering courses at universities are not easy. It is necessary, therefore, that prospective engineering students should have a thorough preparation at school before entering such courses. They should have a good standing in their classes, especially in mathematics, and should obtain a cultural training which will broaden their outlook and fit them for intelligent citizenship."

Regarding the qualities most desirable in an engineer, we quote from the same pamphlet:

> "A general liking for things related to engineering is, of course, highly desirable. What are some other requirements? A good imagination is one, for engineers have to visualize, and must foresee what is involved in the operation of a proposed plant

or machine. Even more important, however, is a real ability to use mathematics, the engineer's indispensable tool."

Of the attributes of a student from which one could predict success in an engineering career, it is generally agreed that the most outstanding is success in mathematics. It is therefore important for a prospective engineering student to obtain a good foundation in mathematics while in high school. If he finds mathematics difficult or uninteresting, he should not be misled by a fancied interest in gasoline engines or radios into thinking that he would be successful in an engineering course.

Closely related to engineering and requiring much the same preparation in mathematics is architecture. The radical changes in building design which are prevalent today point to the need of this science for a sound basis of mathematical theory.

Chemistry

Among the sciences chemistry may fairly claim to have the widest field of applications. Its evidences are all about us – in plastics, drugs, synthetic foods, cosmetics, and many other things. Most persons have visited one or more plants concerning with the processing of rubber, artificial silk, nylon, paper pulp, pigments, or metals.

Although, as a science, chemistry is not pervaded by mathematics to the same extent as is physics, nevertheless it makes constant use of mathematical techniques and certain branches of the subject require mathematics of an advanced character. Among these one may single out physical chemistry of which the name, suggesting a close relation with physics, implies that mathematics plays a prominent role.

Meteorology

The charting of atmospheric conditions has, in our modern age, become a matter of great importance. The subject of weather "probabilities" has emerged from the realm of

superstition and guessing to that of a science, meteorology, whose prestige is indicated by the fact that no aircraft takes off on a flight of any magnitude without the "all clear" signal from the meteorologist.

Although the development of aviation has been the greatest factor in directing attention to meteorology, its service is by no means confined to aviation. Almost everyone listens to the weather forecasts on his radio. The farmer and gardener are warned regarding early and late frosts, the fisherman regarding the prospect of gales, the lumberman regarding rains, and the forest conservationist regarding continued heat and dry weather.

Since meteorology is an application to atmospheric conditions of the principles of physics and mathematics, a fundamental training in mathematics is a prime requisite for a career in this field.

Statistics

The subject of Statistics has been well named "The Arithmetic of Human Welfare." From crude beginnings a century or so ago, when accurate social records were scarcely possible, this field of mathematics has developed rapidly, particularly since the First Great War.

Most fields of science and public welfare offer opportunities for statistical analysis. The industrial engineer interested in the improvement of the processes of production and distribution uses statistical methods. So do the agricultural and the biological scientists for studies in nutrition and disease, and in plant and animal genetics. Public information polls use technical methods in estimating public opinion. In the outfitting of armies and navies and in the vast number of records that must be kept of their personnel, statistical methods are called for. Public health services make use of statistical data and methods in the control of diseases. Statistical procedures are a standard part of modern research in education.

For able students of mathematics the field of mathematical statistics offers attractive opportunities which are likely to increase in number with the increasing use of modern mathematical procedures. Statisticians are now employed in business corporations, in large industries, and in various government services.

Actuarial **Work**

The work of an actuary is in the application of mathematics to problems involving death and survival, and other contingencies. Such problems arise in the theory of insurance, pension funds, and succession duties, and in studies of population growth. This explains why the chief employers of actuaries are the life insurance companies. Governments also have need of the services of actuaries, a need which increases with the introduction of unemployment insurance and other public welfare schemes.

Employers of actuaries report a shortage of well-trained men for this profession. In his apprenticeship period the actuary must expect to engage in much routine calculation. If, however, his interests and his abilities are broad, he may expect promotion to more responsible work. Many of the highest executive positions in insurance companies are held by men who were promoted from the actuarial departments.

CAREERS FOR WHICH MATHEMATICAL TRAINING IS IMPORTANT, THOUGH SUBSIDIARY

For careers in the fields mentioned below the necessary preparation in mathematics is less extensive than for those mentioned above. For most persons who engage in these occupations their mathematical requirements will be satisfied **by the** complete high school course in mathematics. On the other hand, it is of distinct advantage that these professions should include among their members some who are trained in more advanced mathematics.

In the business world of <u>Commerce</u> and <u>Finance</u> computational arithmetic and the mathematics of finance have constant application. An acquaintance with calculus and with statistics is increasingly desirable. For a career in <u>Public Accountancy</u> the mathematical requirements are much the same.

Men who have participated in the armed services will

need no reminder of the importance of mathematics in <u>Navigation —of Sea and Air</u> and in <u>Military and Naval Science.</u>

The natural and social sciences make increasing use of mathematics. While the <u>Biological</u> and <u>Geological sciences</u> are not yet considered exact sciences, there are branches of these subjects which use mathematical methods. Indeed it is becoming more and more important that a number of persons engaged in these sciences should have a specialized training in mathematics. <u>Economics</u> and <u>Social Sciences</u> make contact with mathematics in the realm of statistics and also in the solution of other problems which can be stated with mathematical exactness. In the investigation of some problems in economics, mathematics of the most advanced character has been brought into use.

The professions of <u>**Medicine,** Dentistry, Pharmacy, Nursing</u>, and <u>Law</u>, although they do not lean heavily on **mathe**matics, benefit much from its study. The ability to analyze a mathematical problem should, with proper teaching, make a contribution to the diagnosis of disease. Again, practice in the logic of mathematics should be a valuable training in the formulation of legal arguments.

FUNCTIONAL COMPETENCE IN MATHEMATICS

For citizens who do not enter the professions referred to above, an important question is that of the minimum **mathematical** needs for effective citizenship in a modern democratic society.

Some light is thrown on this question by recalling the fact that mathematics arose in response to human needs. As soon as men began asking the questions How much? and How many?, numbers were needed. Primitive man, to be sure, required little mathematical equipment; a few whole numbers satisfied his needs. The growth and increasing complexity of civilization opening vast new areas of science, was reflected in the development of mathematics. Today we are witnessing scientific achievements which would have been impossible without mathematics and which were unheard of one or two generations ago. Almost all citizens are now required to understand and to operate machines and instruments, an intelligent use of which requires a deeper knowledge of mathematics than our fathers needed.

During the war years, when thousands of citizens had to acquire mathematical information and skill in the interests of national defense, a careful study of minimum mathematical needs was made for the United States Army. Some 550 army jobs, most of which have civilian counterparts, were subsequently studied and their requirements used as a basis for defining minimum mathematical needs for citizenship. This definition of "functional competence in mathematics", which is partially outlined in the following paragraphs, is well worth study; for many students it is an answer to the question, "How much mathematics do I need to know?" It is true that there have been many competent citizens in the past with less mathematical knowledge than that outlined below. In the opinion of able experts, however, this outline indicates the knowledge of mathematics which will, in the future, be essential for effective participation in a modern democratic society. It will be observed that most of the topics to be mentioned are included in the curricula of high schools to the end of grade ten.

The phrase "functional competence in mathematics", which has gained some currency in present-day writing, implies an understanding of the subject which will make the person competent to make such use of mathematics as he is likely to need in the ordinary pursuits of life.

To begin with, every citizen should know the basic operations of arithmetic so that he can use whole numbers, common and decimal fractions, and per cents, with accuracy and ease. In these operations he should form the habit of estimating answers in advance and verifying them when found. Almost everyone needs to be able to make measurements in various units, to understand the possible errors in these measurements, and to use properly in computation numbers obtained from measurements. He should understand the distinction between positive and negative numbers and be able to use them.

Life situations furnish innumerable examples of quantities related so that the value of one depends on the value

of another: the amount of gasoline consumed depends on the speed of travel, the amount of postage depends on the weight of the parcel, the cost depends on the number of articles bought, and so on.

For dealing with problems arising in this area of relationships, mathematics has effective tools. Every citizen needs to be able to use tables of values, to draw and interpret graphs of various kinds, and to use simple formulas. This implies familiarity with the use of letters to represent numbers and with simple equations. The graph and the formula have become commonplace tools in everyday affairs. Newspapers and magazines make frequent use of graphs to illustrate the manner in which one quantity changes with another; graphs of price and wage increase and of changes in the cost of living index have become almost as familiar as the temperature charts used by nurses. Formulas, as well as being indispensable in any branch of technology, also play their part on the farm and in the home. They serve the farmer when he builds his silos and bins and the housewife when she uses her recipes or goes shopping; to say that many farmers and housewives do not use formulas is not to say that they do not need them. How does the capacity of a silo change with its height? With its diameter? In buying circular cookies how should the price change with the diameter? How much is saved in buying oranges by the dozen or by the crate instead of singly? If the recipe is designated to serve five, what adjustment makes it serve eight? How does the size of an egg depend on its length? In what way does a thick skin alter the value of an orange? Such questions provide a proper field for the use of formulas.

Even a superficial inspection of the structures and machines about him points to the fact that the competent citizen needs acquaintance with geometry. Angles, points, lines, triangles, rectangles, circles, prisms, cones, cylinders, and spheres are part of the experience of nearly everyone. The simple constructions with ruler and compasses of the more common plane figures of geometry are helpful in many trades and professions. Scale drawings, requiring a knowledge of ratio along with geometrical ideas, provide a meeting-ground for arithmetic, geometry, and algebra. Formulas for areas and volumes of common solids such as the cylinder and sphere, and the formula for Pythagoras for the right triangle should be included among the minimum needs for ordinary competence in mathematics

For other essentials for mathematical competence the reader is referred to the report of the Commission referred to above: Guidance Pamphlet in Mathematics for High School students.

THE CONTRIBUTION OF MATHEMATICS TO REASONING ABOUT OTHER THINGS

The question of whether the study of mathematics helps one to reason correctly about other things has wide implications for teachers of mathematics. It has, in the past, been answered both in the affirmative and in the negative and the answers have been used as arguments both for and against mathematics. As a result of experiments and observation of results, the smoke of controversy has gradually lifted and some positive conclusions can now be put forward.

Many investigations have been made of the possibility of transfer of training from reasoning in mathematics to reasoning in non-mathematical situations. Opinion today is that the likelihood of such transfer is slight or non-existent if no conscious effort is made to direct the mind of the student to extend the principles of logic acquired in mathematics to these other fields. But once the teacher is aware of the desired attainment, and once a constructive attempt is made to achieve the objective, satisfactory results may be expected.

An investigation of this question, together with experimental evidence, has been presented by H. P. Fawcett in the 13th yearbook of the National Council of Teachers of Mathematics. In this investigation, entitled The Nature of Proof, the following four conclusions were drawn, of which the third is particularly pertinent to our present discussion.

1) Mathematical method illustrated by a small number of theorems yields a control of the subject matter of geometry at least equal to that obtained from the usual course.

2) By following the procedures outlined, it is possible to improve the reflective thinking of secondary school pupils.

3) This improvement in the pupil's ability is general in character and transfers to a variety of situations.

4) If demonstrative geometry is presented in a formal manner, without being related to other situations, little improvement results in the reflective thinking of the pupils.

THE AESTHETIC AND CULTURAL ASPECTS OF MATHEMATICS

In the previous sections, many reasons have been shown for the study of mathematics. It remains to consider one other possibility, namely, that we might study it for no reason at all, except to obtain pleasure, stimulation, and insight into the world around us. The fact is that there are two sides to mathematics. It is, as we have already seen, a scientific method for solving problems in great variety, and it is a free flight of the imagination, taking these as a starting point and going far beyond them. Some particular problem presents itself and is solved; mathematicians then set out to collect and arrange as many related results as possible regardless of immediate usefulness. It quite often happens that after a period of time some new question arises, and that these other results are exactly what is needed for it; and this problem when solved raises others, and so the process continues.

This may be illustrated by a classical example. The ancient Greeks, having solved certain problems in land surveying, had their curiosity aroused regarding the properties of the straight line and circle. Having investigated these loci, they continued with the parabola, ellipse, hyperbola, and other curves, not dreaming that, centuries later, their work would be the foundation of new discoveries in astronomy, when Kepler (1609) showed that the ellipse is the curve followed by a planet as it goes around the sun. A closer study of this situation raised a host of new questions, which in turn gave rise to new branches of mathematics, and the process is still continuing at the present day.

Thus it happens that our modern civilization depends as much on mathematics as it does on chemistry, for example. The evidence of achievement in chemistry is all around us, so that anyone with normal curiosity would wish to study at least the elements of chemistry in order to have some idea of its processes, whether or not chemistry is thought of as a career. It is the same with mathematics but this requires further explanation, as it could not be guessed from the contents of high school mathematics alone. It is not the algebra and geometry of high school that affect the modern world so much as other branches of mathematics which are rooted in them.

Fundamentally mathematics is concerned with clear thinking rather than with solving specific problems. One feature of present-day mathematics is its concern with the study of axioms and their consequences and with the close scrutiny of fundamental ideas such as point, line, and number. On the imaginative side, mention should be made of recreational mathematics, which has to do with games, puzzles, and interesting questions of many kinds. They have formed the well-spring of much of the mathematical interest of the human race, from ancient times does to our own.

Those who are engaged in developing mathematics in advance of current practical needs regard it as a free creation of the human mind, somewhat like music or literature, or even as a game or sport. Nobody stops in the middle of a football game to ask what is the practical value of football. In the same way, mathematics may be regarded as a game to occupy leisure hours, or as a field in which to display one's skill at discovering new results or new methods. It has already been noted that this game is being played vigorously at the present day, and in scores of journals throughout the world new results are published every month. Little thought is taken of practical applications, the only questions asked being "Is it true?" and "Is it interesting?".

It should be said finally that, whatever reason one has for studying mathematics, the most effective method is to study it for its own sake. A person who refused to study any topic in mathematics until he was shown a specific application for it in some other field would make very little progress.

From those who would be her devotees, mathematics demands much concentration on manipulative details, just as music demands finger exercises and practising of scales. The manipulation, however, is not the aim of mathematics any more than the playing of scales is the aim of music. In his mastery of mathematics, a student will often experience a thrill of imagination and an elevation of spirit not unlike those resulting from a mastery of music. A distinguished mathematician and philosopher, Bertrand Russell, has expressed much the same idea: "Mathematics possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, sublimely pure, and capable of a stern perfection such as only the greatest art can show".

A prominent official was asked to deliver an after-dinner speech at the banquet recently held in Cambridge, Mass., for the Mathematicians at the International Congress.

"What do you wish me to speak about?" he asked.

"About five minutes," was the answer.

ON THE SET OF LEGS OF A HORSE

by **Marlow** Sholander
Washington University

In a recent issue of this Journal [1] there appeared the question **"How** many legs has a horse?", and the answer **"Twelve;** two in front, two behind, two on each side, and one in each corner." The enumerator overlooks the existence of bottom legs, outside legs, and so on. It is the purpose of this paper to give the correct answer to this fundamental problem. We prove that the number is infinite; more precisely, that the set of legs has the power of the continuum.

This answer may come as a surprise to the casual observer but it was not unanticipated. There is some reason to believe that the originator of the word "equator" [from **"equus"** meaning "horse" and **"ater"** meaning "dark"] was attempting to announce this discovery in the subtle fashion peculiar to his period. Again, Stephen **Leacock** [2] writes of someone who jumped on his horse and rode madly off in all directions. This, of course, would be impossible with a finite-legged horse. One authority on horseplay [3] has even made an attempt at a proof. He starts with the assumption that the number of legs, y, of a horse is a non-decreasing function of its age, x (The variable x is, of course, determined by the inspection of the horse's teeth). He then seeks to show that y is unbounded in any finite interval of x. However his proofs invariably begin " **iven** a horse.. ." and he unfortunately overlooks that he has thus made x indeterminite since gift horses are not to be looked in the mouth.

[1] **Numbers** in brackets refer to the bibliography at the end of the paper.

We offer below several proofs of the fundamental **result.**[2] First let us note that the answer is not as unreasonable as it may first appear. Assume that the number of legs, n, of a horse is finite. From [1], we have n 4. Even the most skeptical will grant this (n) is an odd number of legs for a horse. But, in contradiction, we have that n is even since each horse has a plane of symmetry which contains no legs. If an argument of a direct type is desired, the reader will recall that n is infinite if **and only** if the set of legs can be placed in one-one correspondence with a proper subset of itself. But if each leg corresponds to itself, we have such a correspondence - if the set of legs of a horse is improper, how does one account for the disappearance of the horse-blanket?

Perhaps the simplest rigorous proof can be made by using the well-known schachprinzip or **"box** principle".

Proof I: Assume that the number of legs, n, is finite. As the horse remains standing, place his **legs** in n-1 boxes, so that at most one leg is found in each of the boxes. By the box principle, at least one leg is left over. But this is a contradiction since throughout the process each leg was left underneath.

Proof **II:** Suppose that there are some horses with a finite number and some with an infinite number of legs. This would clearly require the existence in every town of one each of two kinds of veterinarians, i. e., the existence of a paradox. Thus it is sufficient **to show** that at least one horse has an infinite number of legs. We prove more — that given any two horses, at least one **has an** infinite number of legs. Assume that all finite-legged horses have been colored red and all infinite-legged horses blue. Consider two horses. We may assume the first horse is red. If the second horse is blue, the theorem is proved. If the second horse is red, that's a horse of a different color. But in this case we have an immediate contradiction.

Proof **III:** We prove by induction that the number of legs, n, of a horse is greater than a positive integer m, for all m.

[2] **One** remark is in order. Here and there a wording may seem ambiguous. In these cases, the reader is to interpret the words in their horse sense.

By [1], this holds for $m = 1, 2, \ldots, 11$. Assume it holds for $m = k \geqq 11$, i.e., assume the number of legs exceeds k. If the reader grants this, he will surely grant that n exceeds $k + 1$. If, on the other hand, he feels this is too too many, we have $n = k + 2$. Thus, in either case, the statement holds for $m = k + 1$.

Once we grant that a horse had an infinite number of legs we suspect the set is uncountable for the simple reason that people who have tried to count the set usually obtain 4 as an answer. This suspicion is confirmed in the following proof that the set of legs has the cardinal number of the continuum.

Proof IV: By definition, **c**, the cardinal number of the continuum, is the number of points in a lion. We wish to prove $n = \mathbf{c}$. Let h be the number of points in a horse. Clearly $n \leqq h$ and, from the case of a hungry lion and a small horse, we see also that $h \leqq \mathbf{c}$. It remains to be proved that $c \leqq n$, i. e., that the points of a lion may be placed in one-many correspondence with the set of legs of a horse. But, trivially, we may have a one-many correspondence from a subset of the points of a lion to the set of legs of the horse. Regardless of which such correspondence we choose to consider, there is surely no point in a lion without a leg. Hence the correspondence considered is the one needed to complete the proof.

The author has as yet not had time to develop the innumerable consequence of this result. For one thing, it is clear that we have been misguided in shooting a horse when he breaks a leg since each leg has as its horsepower at most $\lim_{n \to \infty} 1/n$. It would be of interest also to generalize the result from horses to, say, the best cattle — those from which we get prime legs of beef. Those who make it a point to keep abreast of developments in foreign research inform the author that an Alexandrian geometer, Euclid, has already made progress in this direction. In closing, the author expresses regret that lack of space prevented his giving Proof V, perhaps the most interesting proof of all. This proof is based on an argument which makes use of the neigh-borhoods of a Horsedauf Space.

Bibliography

1. Pi Mu Epsilon Journal, vol. 1 (1949) p. 24.

2. Stephen **Leacock**, "Gertrude the Governess, or Simple Seventeen", The **Leacock** Roudabout, New York, 1946.

3. N. Baerbaki, "Die Theorie und Anwendungdes wiehernden **Gelächters**", Appendix **II.**

Note: N. Baerbaki is not to be confused with his distinguished French cousin N. Bourbaki. The former, in reply to a letter asking:

A. When his manuscript is to be printed,
B. If his last name is pronounced **"barebacky"**,
C. What his first name is,

cabled, from his present address in eastern Germany, as answer (presumably only to the third question) the single word **"Nichevo"**.

When the tall inductee presented himself for a physical examination and was asked his height by the examining sergeant the inductee promptly answered,

"I am five feet, seventeen and one half inches tall, Sir."

"You can't kid me, Slim. I know you are more than six feet," snapped the sergeant.

PROBLEM DEPARTMENT

This department welcomes problems believed to be new and. as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the master's degree. Solutions of these problems should be submitted on separate, signed sheets within five months after publication. Address all communications concerning problems to the Editor.

21. Proposed by Mary Anice Seybold, North Central College

A Christmas Problem

In the following long division, each letter represents one and only one number and each number is represented throughout by the same letter. There are no unconventional arrangements of digits. To find my Christmas wish for readers of Pi Mu Epsilon Journal, establish the correspondence of letters to numbers and arrange them in the order 1, 2, 3, . . . , 0. Prove the solution is unique.

```
          F Y
J S L / F U M X
        F O F
          Y F X
          Y J M
            O A
```


22. Proposed by Don A. Gorsline, University of Oklahoma

(Note. This problem appeared on page six of the Pi Mu Epsilon initiation banquet program of the Oklahoma Alpha chapter, and because it proved interesting on that occasion we are printing it here.)

Each letter in the following addition problem represents a unique digit. Furthermore $L^2 = L$ and $M^2 = T$. Establish the correspondence of letters to digits.

```
    N O T E
    P I M U
E P S I L O N
-------------
E P I T O M E
```

22. Proposed by Roy Dubisch, Fresno State College

If in a triangle with sides a, b and c we have $c \geqq b$, $c \geqq a$, find k such that $c^2 = ka^2 + b^2$.

24. Proposed by Paul J. Schillo, University of Buffalo

If θ_n is the angle opposite the side of length $4n^2$ in the integer right triangle with sides $4n^2$, $4n^4 - 1$, $4n^4 + 1$, where n is any positive integer, show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \theta_i \text{ is a right angle.}$$

SOLUTIONS

12. Proposed by C. O. Oakley, Haverford College

The Census Enumerator Problem

A census enumerator goes to a house and notes the number of the house. The man who answers the door bell gives the enumerator his age and adds that there are three other members of his household. But he refuses to give the ages of the other three. Instead he says, "The product of their ages is 1296 and the sum is equal to the house number." The enumerator, after some computing, says, "I need some more information. Are you the oldest member of the household?" The man replied, "Yes." The enumerator now has enough information to compute the ages of the other three and so have you.

Solution by C. S. **Goodrum,** Leo Moser and **J.** H. Wahab, University of North Carolina

The only factorizations of 1296 into three factors, having the same sum are 1-18-72 and 2-8-81. For the age of A, the man who answers the doorbell, to be a determining factor the other ages must be 1, 18 and 72.

The equations used in the solution of the problem are:

$$(1)\ abc = 1296 \qquad \text{and} \qquad (2)\ a + b + c = n$$

where n is the house number. Any solution in integers for these two equations is unique unless n = 91, in which case there are the two sets of factors as given above. This can be noted by determining all possible sets (a, b, c) satisfying (1).

Also solved by W. R. Ransom as well as the proposer C. O. **Oakley,**

19. Proposed by E. P. **Starke, Rutgers** University

Given two polynomials of degree n,

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \dots + a_n,$$

$$g(x) \equiv b_0x^n + b_1x^{n-1} + \dots + b_n,$$

and n + 1 arbitrary constants $k_1, \dots, k_{n+1}$, not necessarily distinct; if

$$f(k_1) = g(k_1), \quad f'(k_2) = g'(k_2), \quad f''(k_3) = g''(k_3), \dots,$$

$$f^{(n)}(k_{n+1}) = g^{(n)}(k_{n+1}), \quad \text{then } f(x) = g(x) \text{ identically.}$$

Solution by James **J.** Gehrig, University of Wisconsin

Differentiate both **f(x)** and **g(x)** n times:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n$$

$$f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \dots + 3a_{n-3}x^2 + 2a_{n-2}x + a_{n-1}$$

$$f''(x) = n(n-1)a_0x^{n-2} + \dots + 3\cdot 2a_{n-3}x + 2\cdot 1a_{n-2}$$

$$f'''(x) = n(n-1)(n-2)a_0x^{n-3} + \dots + 3\cdot 2\cdot 1a_{n-3}$$

$$\vdots$$

$$f^{(n-1)}(x) = n(n-1)(n-2)\dots 3\cdot 2a_0x + (n-1)!\,a_1$$

$$f^{(n)}(x) = n!a_0$$

Similarly,

$$g(x) = b_0x^n + b_1x^{n-1} + \dots + b_{n-2}x^2 + b_{n-1}x + b_n$$

$$g'(x) = nb_0x^{n-1} + \dots + 2b_{n-2}x + b_{n-1}$$

$$g''(x) = n(n-1)b_0x^{n-2} + \dots + 2!\,b_{n-2}$$

$$\vdots$$

$$g^{(n-1)}(x) = n(n-1)(n-2)\dots 3\cdot 2\,b_0x + (n-1)!\,b_1$$

$$g^{(n)}(x) = n!\,b_0.$$

By our hypothesis

$$f^{(n)}(k_{n+1}) = g^{(n)}(k_{n+1}); \quad \text{that is, } n!a_0 = n!b_0$$

$$\therefore \quad a_0 = b_0.$$

Also

$$f^{(n-1)}(k_n) = g^{(n-1)}(k_n), \quad \text{that is}$$

$$n(n-1)(n-2)\ldots 3\cdot 2a_0k_n+(n-1)!a_1=n(n-1)(n-2)\ldots 3\cdot 2a_0k_n+(n-1)!b_1$$

$$\therefore\ a_1 = b_1$$

Similarly

$$a_j = b_j, \quad j = 0, 1, 2, \ldots, n.$$

Hence we have shown that

$$f(x) = g(x) \quad \text{identically.}$$

Also solved by H. Pitt and S. Gartenhaus.

Have you seen this one?

$$(\tfrac{1}{2})^3 < (\tfrac{1}{2})^2.$$

Taking the logarithm to the base $\frac{1}{2}$ of each member of the above inequality, we write

$$3 \log_{\frac{1}{2}}(\tfrac{1}{2}) < 2 \log_{\frac{1}{2}}(\tfrac{1}{2}).$$

But $\log_b b = 1$. Therefore

$$3 < 2.$$