PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF

THE HONORARY MATHEMATICAL FRATERNITY

VOLUME 1 NUMBER 4

CONTENTS

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NON-LINEAR VIBRATIONS

M. L. Cartwright, Cambridge University

During the last ten years a subject called Non-Linear Vibrations or sometimes Non-Linear Mechanics has received considerable publicity in the mathematical world. Since a very large part of mechanics including the theory of the motion of the planets involves non-linear differential equations and lies outside the subject to which I refer, it is preferable to speak of non-linear vibrations.

The non-linear theory of vibrations concerns vibrations represented by non-linear ordinary differential equations; the essential mathematical difference from the linear theory is that while solutions of a linear differential equation can be added to give a more general solution this is not so with non-linear equations. The theory includes the solution of the simple pendulum equation by means of elliptic integrals which was well known in the 18th century, but Duffing (1918) is perhaps the first exponent of the modern theory, and van der Pol (1920) the next with more outstanding contributions to it, the first in connection with mechanical vibrations and the second in connection with electrical oscillations. The essential feature of the modern theory is the study of equations in which the non-linear terms, although perhaps small, give rise to features of the solutions which are qualitatively different from those of similar linear equations. For instance the equation

$$\ddot{x} + x = 0 \tag{1}$$

has a periodic solution $x = A\cos(t + \alpha)$ whose period is independent of the values of x and $\dot{x}$ at $t = 0$, while the amplitude A depends on these initial values, but the solutions of van der Pol's equation

$$\ddot{x} + k(1 - x^2)\dot{x} + x = 0 \tag{2}$$

when k is quite small tend to $x = 2\cos(t + \alpha)$ approximately as $t \to \infty$ whatever the initial values of x and $\dot{x}$ may be. It is easy to verify that if (2) has a solution approximately of the form $x = A\cos(t + \alpha)$, A must be 2, by substituting in the corresponding energy equation. This is obtained by multiplying by $\dot{x}$ and integrating from $t = 0$ to t, and we have

$$\tfrac{1}{2}(\dot{x}^2 - A^2\sin^2\alpha) - k\int_0^t (1-x^2)\dot{x}^2 dt + \tfrac{1}{2}(x^2 - A^2\cos^2\alpha) = 0. \quad (3)$$

Now then $t = 2\pi$, $x = A\cos\alpha$ and $\dot{x} = -A\sin\alpha$ again, approximately, and is

$$\int_0^{2\pi} (1 - A^2\cos^2(t+\alpha))\, A^2\sin^2(t+\alpha)\, dt = \pi A^2(1 - \frac{A^2}{4}) = 0.$$

Duffing however was concerned with resonance phenomena associated with an equation of the form

$$\ddot{x} + x + \beta x^3 = F\cos\lambda t.$$

It was already known that the period of the free oscillation with $F = 0$ decreased as β increased, and that for the linear equation with $\beta = 0$, the amplitude $\frac{F}{1-\lambda^2}$ of the forced oscillation

$$x = \frac{F}{1-\lambda^2}\cos\lambda t$$

increases in the same way as λ^2 tends to 1 from either side, but experimental results suggested that if $\beta > 0$, the resonance curves were unsymmetrical about $\lambda^2 = 1$. Duffing showed that for $\beta > 0$ they increase as λ^2 increases, to a point of resonance beyond 1, and after resonance drop sharply, whereas when λ^2 decreases the rise does not take place at the same point. Both these qualitative results were established by considering equations in which the non-linear terms were small compared with the others by trying a solution of the form $x = A\cos(\lambda t + \alpha)$.

The mechanical applications were developed by various authors including the Russian authors Kryloff and Bogoliuboff, Vlasov and others. The physical problems concern

such things as the rolling of ships, steering devices and vibrations of a gage for investigating explosive blast. The vibrations are usually undesirable ones, the mathematician is required to explain why they exist so that they can be eliminated. On the other hand the electrical, and more particularly the radio problems, concern highly desirable oscillations; moreover the very great rapidity of the oscillations, and the radio engineer's ability to vary the system while it is still oscillating, make it possible to observe a number of phenomena which could scarcely have been expected. The physical results of Appleton, van der Pol and van der Mark obtained for systems represented by the equation

$$\ddot{x} - k(1-x^2)\dot{x} + x = pk\cos\lambda t, \quad (4)$$

or similar equations, directed the attention of the pure mathematicians at a later stage into the most fruitful fields in the attempt to explain how to maintain oscillations and improve their stability and regularity.

Let us compare (4) with the linear equation

$$\ddot{x} + 2a\dot{x} + x = pk\cos\lambda t$$

which has a solution of the form

$$x = Ae^{-at}\cos(\omega t + \alpha) + pk\frac{(1-\lambda^2)\cos\lambda t + 2a\lambda\sin\lambda t}{(1-\lambda^2)^2 + 4a^2\lambda^2}$$

$$= \varphi(t) + \psi(t), \quad -1 < a < 1, \quad \omega^2 = 1 - a^2.$$

If $a > 0$, the free oscillation $\varphi(t)$ exists for finite t unless x and $\dot{x}$ have one particular pair of initial values, but $\varphi(t) \to 0$ as $t \to \infty$. If $a < 0$, the amplitude of the free oscillation tends to infinity with t, unless $A = 0$. Further as $\lambda^2 \to 1$ the amplitude of the forced oscillation increases and if $a = 0$ it tends to infinity. But in the case of certain physical systems represented by (4) the free oscillation can be detected by ear for all t, provided that λ^2 is not too near 1, and its amplitude does not increase beyond a certain point as $t \to \infty$, on the other hand if λ^2 approaches 1 the free oscillations vanish entirely at a certain point and the forced oscillations increase up to a certain point and no further.

There are many techniques for dealing with (4) when k is small, and we may begin by following van der Pol and putting

$$x = b_1 \cos \lambda t + b_2 \sin \lambda t$$

Since k is small and λ^2 near 1, it seems probable that b_1 and b_2 vary slowly, and that the higher harmonies may be neglected in the first approximation. With these assumptions we obtain after some detailed calculations equations of the form

$$\dot{b}_1 = B_1(b_1, b_2),$$
$$\dot{b}_2 = B_2(b_1, b_2). \qquad (6)$$

The original treatment of the particular equations obtained in this way explained some, but not all, of the observed phenomena; the Russians, especially Andronov and Witt made further progress by applying the topological methods of Poincaré to equations of the form (6), and so developed a beautiful general theory of curves in the (b_1, b_2) plane represented by (6). They also justified the approximations used by van der Pol and Appleton by appealing to Fatou's results. At the same time Kryloff and Bogoliuboff developed the analytical theory of nearly-linear equations representing oscillating systems.

Let us now return to (4) with p = 0 and k large; van der Pol considered systems represented by such equations and found graphical solutions of (4) with p=0 for k=.1,1 and 10, as curves in the phase plane (x,y) where y = $\dot{x}$. Liénard and others developed a technique which became popular later for dealing with equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

in the phase plane.

This takes us to about 1938, and up to this time most of the work had been done by physicists, and mostly by experimental physicists, although some of the Russians developed the theory independently of its applications, and some of the French work included purely mathematical treatments. In 1938 the British Department of Scientific and Industrial Research issued a memorandum to the London Mathematical

Society appealing for the assistance of pure mathematicians in solving a certain type of non-linear differential equation arising in radio problems and giving a few references including one to a summary by van der Pol in the Proceedings of the Institute of Radio Engineers in 1934. Certain suggestions in this summary caught my attention, and Littlewood and I began our collaboration. Soon after this Levinson began to write on the pure theory, handling it partly by the methods of Liénard and partly by skillful application of classical results from other fields. Naval research during the war aroused more interest in the subject, and the American Office of Naval Research has sponsored much work in this field including translations of Russian work. Besides reading the published works of van der Pol, Littlewood and I have had correspondence with him and Appleton and others, and we have found much interesting work in the literature; but, so far as Littlewood and myself are concerned, by far the most fruitful of our investigations have been those inspired by van der Pol's suggestion implied in his summary and developed further in a letter that the equation (4) with k large could have two stable periodic solutions one of period $(2n-1)2\pi/\lambda$, where n is a positive integer which may be as large as k, and the other of period $(2n+1)2\pi/\lambda$. This has led us from the physical experiments of van der Pol and van der Mark via Levinson's topological treatment of forced oscillations into the lofty realms of pure topology and pathological curves described as indecomposable continua.

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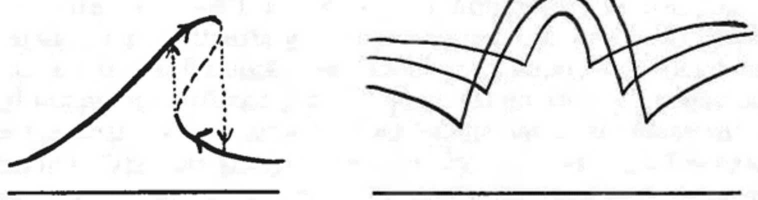

Fig. 1. Some non-linear resonance curves. The one on the left arises in connection with Duffing's equation and the ones on the right in connection with equation (4) when k is small and λ near 1.

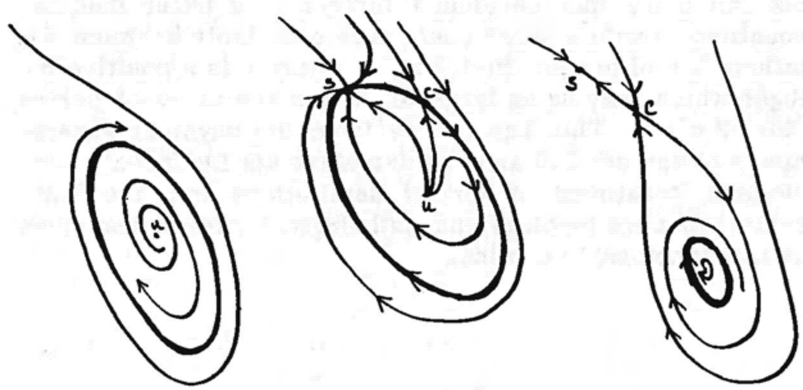

Fig. 2. Some typical curves in the (b_1, b_2) plane which arise in connection with (4) when k is small and λ near 1. The thick curve in the figures on the left and right are called limit cycles' and correspond to stable solutions for which b_1 and b_2 vary slowly. The points **u, c, s,** correspond to solutions with period $2\pi/\lambda$ for which b_1 and b_2 are fixed, s is stable.

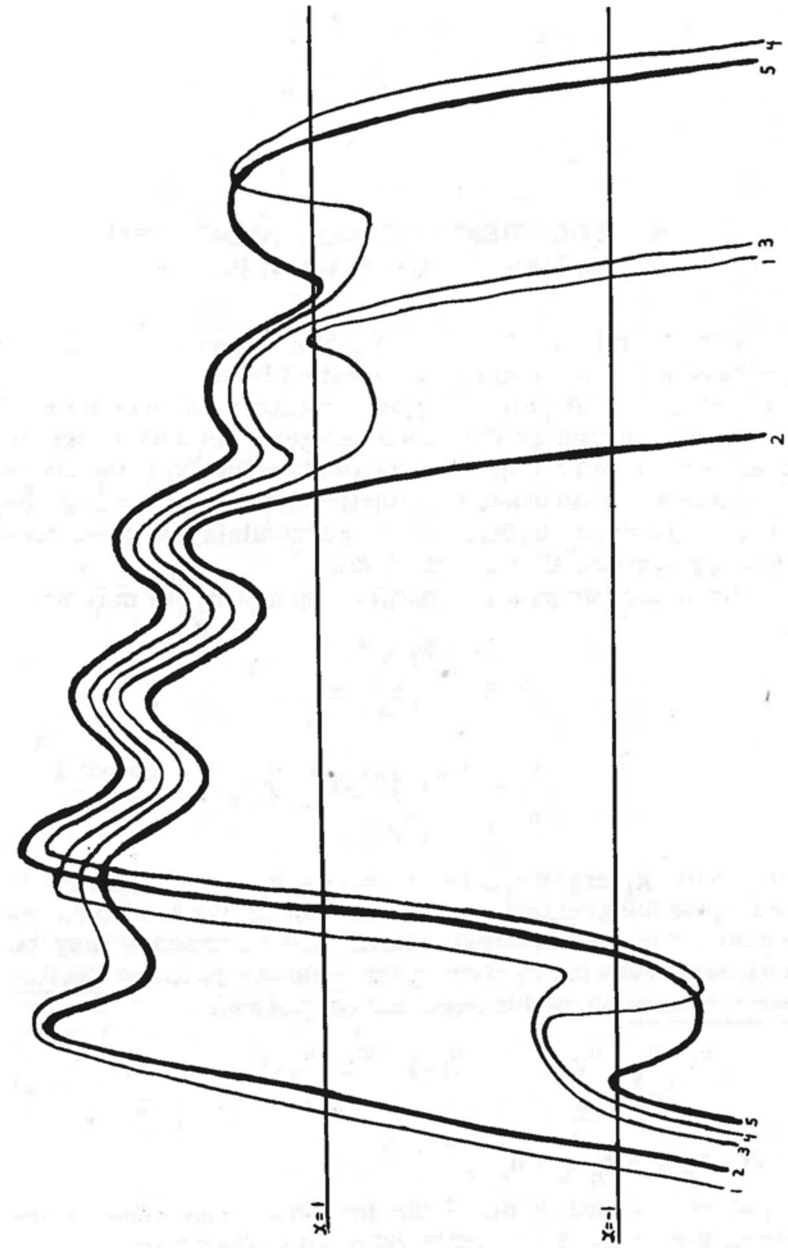

Fig. 3. Some typical solutions of (4) with k large and $0 < p < \frac{2}{3}$. The thick curves 2 and 5 are solutions with periods $14\pi/\lambda$ and $18\pi/\lambda$. The others either converge to these or are unstable; 1 might be an unstable solution with period $16\pi/\lambda$.

THE EUCLIDEAN DIVISION ALGORITHM

B. E. Meserve, University of Illinois

The Euclidean division algorithm gives a systematic process for determining the greatest common divisor of two integers and expressing this greatest common divisor linearly in terms of the given integers. In this paper we consider a synthetic method for performing both the above operations, an additional synthetic method for finding the greatest common divisor of two polynomials and some further applications of these methods.

Given any two positive integers n_0 and n_1 we may write

$$\begin{aligned} n_0 &= g_1 n_1 + n_2 \\ n_1 &= g_2 n_2 + n_3 \\ &\ldots\ldots\ldots \\ n_{r-2} &= g_{r-1} n_{r-1} + n_r \\ n_{r-1} &= g_r n_r \end{aligned} \tag{1}$$

where the g_j are integers, $0 < n_{i+1} < n_i$ $(i = 1, 2, \ldots r - 1)$ and n_r is the greatest common divisor of the two given integers. The arithmetical details in this process may be noticeably reduced by writing the numbers n_j in the Euclidean sequence above the sequence of quotients

$$\begin{array}{cccccccccc} n_0 & n_1 & n_2 & \ldots & n_{k-1} & n_k & n_{k+1} & \ldots & n_r & 0 \\ & g_1 & g_2 & \ldots & & g_k & & \ldots & g_r & \end{array} \tag{2}$$

where $n_{k-1} = g_k n_k + n_{k+1}$.

When a relationship of the form $n_r = Pn_0 + Qn_1$ is desired, the array (2) is easily extended to four rows

$$\begin{array}{cccccccc} n_0 & n_1 & n_2 & n_3 & n_4 & \ldots & n_r & 0 \\ & g_1 & g_2 & g_3 & g_4 & \ldots & g_r & \\ & 1 & -g_1 & A & A & \ldots & A_r & \\ & & 1 & -g_2 & B_4 & \ldots & B_r & \end{array} \tag{3}$$

where $A_1 = 1$, $A_2 = -g_1$, $B_1 = 0$, $B_2 = 1$, $B_3 = -g_2$ and in general

$$\begin{aligned} A_{i+1} &= A_{i-1} - g_i A_i \\ B_{i+1} &= B_{i-1} - g_i B_i \end{aligned} \qquad (i = 2, 3, \ldots, r - 1).$$

Then

$$n_j = B_j n_0 + A_j n_1 \qquad (j = 1, 2, \ldots, r)$$

and in particular

$$n_r = B_r n_0 + A_r n_1. \tag{4}$$

These formulas may be quickly verified by induction upon j. If $n_{k-1} = B_{k-1}n_0 + A_{k-1}n_1$ and $n_k = B_k n_0 + A_k n_1$, then from (2) $n_{k+1} = n_{k-1} - g_k n_k = (B_{k-1} - g_k B_k)\, n_0 + (A_{k-1} - g_k A_k) n_1 = B_{k+1} n_0 + A_{k+1} n_1$.

If the computation of the A_i and B_i in (3) is carried one step further we obtain a check on our calculations from the relations

$$A_{r+1} = \pm\, n_0/n_r, \qquad B_{r+1} = \pm\, n_1/n_r.$$

These relations may be most easily proved using the theory of continued fractions [1] in which the successive convergents of the continued fraction expansion for n_0/n_1 are $-A_i/B_i$ $(i = 1, 2, \ldots, r + 1)$.

We thus have a synthetic method for using the Euclidean algorithm to find the greatest common divisor of any two integers and a relationship of the form (4). The following examples illustrate this method.

$$\begin{array}{cccccc}
54 & 32 & 22 & 10 & 2 & 0 \\
 & 1 & 1 & 2 & 5 & \\
 & 1 & -1 & 2 & -5 & \\
 & & 1 & -1 & 3 &
\end{array}$$

$$2 = 3\cdot 54 - 5\cdot 32$$

Similarly

$$\begin{array}{cccccccc}
147 & 64 & 19 & 7 & 5 & 2 & 1 & 0 \\
 & 2 & 3 & 2 & 1 & 2 & 2 & \\
 & 1 & -2 & 7 & -16 & 23 & -62 & \\
 & & 1 & -3 & 7 & -10 & 27 &
\end{array}$$

$$1 = 27\cdot 147 - 62\cdot 64.$$

Now let us consider the Euclidean algorithm in the ring of polynomials in one variable. For convenience the coefficients will be taken as real numbers, but this restriction is not always necessary. Given two polynomials

$$\begin{aligned} f_0 &= ax^m + a_1x^{m-1} + \ldots + a_n, \quad a \neq 0 \\ f_1 &= bx^n + b_1x^{n-1} + \quad \ldots + b_n, \quad b \neq 0 \end{aligned} \tag{5}$$

arrays similar to (1), (2) and (3) may be formed, giving their greatest common divisor and an identity of the form

$$f_r = Bf_0 + Af_1 \tag{6}$$

where the degree of the polynomial B is less than that of f_1 and the degree of A is less than that of f_0. The calculations here are much more tedious than those in the numerical case, and we shall consider only the sequence $f_0, f_1, f_2, \ldots, f_r$. Under the convention that f_r shall have leading coefficient $+1$, we may multiply any terms of the array corresponding to (1) by arbitrary constants without affecting f_r.

Given two polynomials f_0 and f_1 as in (5) we may choose the notation such that the degree of f_0 is greater than or equal to that of f_1, i.e., $m \geq n$. In order to obtain a polynomial f_2 satisfying $c_0f_0 = g_1f_1 + d_2f_2$ we must divide f_1 into f_0 and find the remainder. In other words we eliminate successively $x^m, x^{m-1}, \ldots, x^n$ between f_1 and f_0. These

eliminations may be performed by a sequence of computations of the form $f_1 \otimes f_0, f_1 \otimes^2 f_0, \ldots, f_1 \otimes^s f_0$ where the operation $\otimes$, cross-multiplication, is defined by

$$f_1 \otimes f_0 = -f_0 \otimes f_1 = bf_0 - ax^{m-n}f_1,$$

$$f_1 \otimes^k f_0 = f_1 \otimes [f_1 \otimes^{k-1} f_0], \; f_1 \otimes^1 f_0 = f_1 \otimes f_0.$$

The coefficients of $f_1 \otimes f_0$ may be easily computed from those of f_0 and f_1 as indicated in the third line of the array below. The process may also be very easily performed on a computing machine.

The above operation of cross-multiplication has been used by Routh [5] to obtain conditions for the stability of a linear dynamical system. More recently it has been used [2] to obtain the Sturm functions of a polynomial $f(x)$ under the unstated assumption that the degrees of the Sturm functions are consecutive integers. One purpose of this paper is to give a generalization of this process that may be used to find the Sturm functions of an arbitrary polynomial $f(x)$ with real coefficients. Both the special case treated in [2] and the general case will be discussed and illustrated by examples.

Using cross-multiplication and the fact that constant factors of any f_j's may be inserted or removed, the calculation of the f_j's may be reduced to the construction of an array. In particular for the special case in which the degrees of the f_j's turn out to be consecutive integers we may write the coefficients, including zeros for missing terms, of f_0 and f_1 on consecutive lines and fill out the following array.

$$\begin{array}{llllll}
f_0 & a & a_1 & a_2 & \ldots & a_{m-1} \quad a_m \\
f_1 & b & b_1 & b_2 & \ldots & b_n \\
c_1f_1 \otimes f_0 & ba_1-ab_1 & ba_2-ab_2 & ba_3-ab_3 & \ldots & ba_m \\
f_2 = c_2f_1 \otimes^2 f_0 & & & & &
\end{array}$$

$$c_3 f_2 \otimes f_1$$

$$f_3 = c_4 f_2 \otimes^2 f_1$$

$$\cdots\cdots\cdots\cdots$$

where the c's are non-zero constants.

For example, if $f_0 = x^4 - 24x^2 + 16x + 12$ and

$$f_1 = x^3 - 12x + 4,$$

we may form the array

$$\begin{array}{rrrrrr}
f_0 & 1 & 0 & -24 & 16 & 12 \\
f_1 & 1 & 0 & -12 & 4 & \\
f_1 \otimes f_0 & 0 & -12 & 12 & 12 & \\
(1/12)f_1 \otimes^2 f_0 = f_2 & -1 & 1 & 1 & & \\
f_2 \otimes f_1 & -1 & 11 & -4 & & \\
(1/5)f_2 \otimes^2 f_1 = f_3 & -2 & 1 & & & \\
f_3 \otimes f_2 & -1 & -2 & & & \\
(1/5)f_3 \otimes^2 f_2 = f_4 & 1 & & & &
\end{array}$$

Thus f_0 and f_1 are relatively prime and the sequence of polynomials is

$$f_0 = x^4 - 24x^2 + 16x + 12$$
$$f_1 = x^3 - 12x + 4$$
$$f_2 = -x^2 + x + 1$$
$$f_3 = -2x + 1$$
$$f_4 = 1$$

In general the method is essentially that of long division using detached coefficients. It may be formally represented in a single array by writing the coefficients of f_0 on the first row, those of f_1 on the second row, and taking $f_1 \otimes^j f_0$, $j = 1, 2, \ldots$ on successive rows until some row, say for $j = k$, is "shorter" than that of f_1 and has a non-zero leading

term. This row is then divided by b^k to obtain f_2 and the process repeated for f_1 and f_2. The division by b^k is essential only when the signs of the f's have significance, b is negative and k is odd.

Suppose $f_0 = x^6 - 3x^2 - 2$ and $f_1 = x^5 - x$, then we have

$$\begin{array}{rrrrrrrr}
f_0 & 1 & 0 & 0 & 0 & -3 & 0 & -2 \\
f_1 & 1 & 0 & 0 & 0 & -1 & 0 & \\
f_1 \otimes f_0 & 0 & 0 & 0 & -2 & 0 & -2 & \\
(1/2)f_1 \otimes^2 f_0 & 0 & 0 & -1 & 0 & -1 & & \\
(1/2)f_1 \otimes^3 f_0 & 0 & -1 & 0 & -1 & & & \\
f_2 = (1/2)f_1 \otimes^4 f_0 & -1 & 0 & -1 & & & & \\
f_2 \otimes f_1 & 0 & 1 & 0 & 1 & & & \\
f_2 \otimes^2 f_1 & 1 & 0 & 1 & & & & \\
f_2 \otimes^3 f_1 & 0 & 0 & & & & &
\end{array}$$

The zero leading coefficient of $f_1 \otimes^2 f_0$ indicates that the degrees of the polynomials f_j will not be consecutive integers. The row of zeros $f_2 \otimes^3 f_1$ indicates that $-f_2 = x^2 + 1$ is the greatest common divisor of f_0 and f_1.

This synthetic method for determining the greatest common divisor of two polynomials is useful in finding the multiple roots of a polynomial, **Sturm's** functions, and the sequences of polynomials used in the Cauchy-Sylvester Theorem [3]. Sturm functions may be used to find the number of distinct roots of a polynomial $f(x)$ of any multiplicity k on any interval $a < x \leq b$ [6]. The **Cauchy-Sylvester** Theorem enables one to determine the number of distinct roots of a polynomial at which a second polynomial is positive. This is necessary in the discussion of the content of a polynomial inequality in one variable [4; 359] where the even roots of the polynomial which are minima of the curve must be distinguished from those which are maxima. For double roots the number of zero-minima on a segment $a < x \leq b$ is the number of double roots at which the second derivative is positive.

Sturm functions for any polynomial $f(x)$ with real coefficients may be obtained by (i) taking $f_1 = c^2f'$, (ii) finding a sequence $f, f_1, f_2, \ldots, f_r$ as above, except that only positive constants may be inserted, and (iii) taking as Sturm functions

$$f, f_1, -f_2, -f_3, f_4, f_5, \ldots$$

when $f_r = 1$, these functions divided by f_r when $f_r \neq 1$.

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PROBLEM DEPARTMENT

Edited by Leo Moser, Texas Technological College

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's degree. Solutions of these problems should be submitted on separate, signed sheets within five months after publication. Address all communications concerning problems to Leo Moser, Mathematics Department, Texas Technological College, Lubbock, Texas.

25. Proposed by Polly Tope, Institute for Hyper Study

A square has 4 lines of symmetry and a cube has 13. Derive a formula for the number of lines of symmetry of an n dimensional cube.

26. Proposed by Pedro A. Piza, San Juan, Puerto Rico

For positive integers n and c, let the number n:c be defined by the relation

$$[n:c] = \frac{2^{c-1}\binom{2n-c}{c-1}}{c}.$$

Show that the numbers [n:c] satisfy the recurrence relation

$$[n:c] = \frac{2(2n-c)}{c}\,[n-1:c-1] \qquad (1)$$

and the formula

$$\frac{2^{2n}-1}{2n+1} = \sum_{c=1}^{n} [n:c]. \qquad (2)$$

27. Proposed by Arthur B. Brown, Queens College

The number 3 can be expressed as a sum of one or more positive integers in 4 ways, namely as 3, **1+2, 2+1, 1+1+1.** Show that any positive integer n can be so expressed as a sum in 2^{n-1} ways.

28. Proposed by N. S. Mendelsohn, University of Manitoba

The isle of Pythagora, while very sparsely populated, is capable of supporting a population of thirty million. On the sixth day of the twenty-eighth anniversary of his accession to the throne, the king of the island called a meeting of his 496 advisers to decide how to celebrate the auspicious occasion. They decided to divide the regal jewels among the people of the land. All the people, including the king and the advisers, were lined up in single file, and the jewels were distributed as follows.

Starting with the second in line, each person was given one jewel. Starting with the fourth in line, each person was given two jewels. Starting with the sixth in line, each person was given three jewels. Starting with the eighth in line, each person was given four jewels, and so on.

The man at the extreme end of the line noticed that the number of jewels he received corresponded to his position in line.

How many people were there in Pythagora?

Where was the person who got the most jewels standing?

29. Proposed by Francis L. **Miksa,** Aurora, Illinois

For a given positive integer k, find integers m and n such that

$$1+2+3+\ldots+m = (m+k) + (m+k+1) + (m+k+2) + \ldots + n.$$

SOLUTIONS

1. Proposed by the Problem Editor

Prove the following construction for finding the radius of a circumference. With any point 0 on the circumference

as center and any convenient radius describe an arc PQR, cutting the given circumference in P and Q. With Q as center and the same radius describe an arc OR cutting PQR in R, R being inside the circumference. Join P and R, cutting the given circumference in L. Then LR is the radius of the circumference. (This is known as **Swale's** construction, and is probably the simplest solution of the problem yet discovered.)

Solution **by** Dine **Hwang,** University of California

Let C be the center of the circumference.

Clearly A **POR** is equilateral. Hence $\angle$ LCQ = 2 $\angle$ LPQ = $\angle$ ROQ = **60°.** Thus $\triangle$ LCQ is equilateral. To show that LR = CO, we shall prove that A LRQ $\cong$ $\triangle$ CQO. We already have LQ = CQ and RQ = OQ. Finally, $\angle$ RQL = $\angle$ **CQR + 60°** = $\angle$ RLQ which completes the proof.

13. Proposed by W. **R.** Ransom, Tufts College

An electric clock has hour, minute and second hands turning about the same axis. Prove that they are never together except at twelve o'clock.

Solution by the Proposer

Suppose they are together at h hours m minutes and s seconds, h < 12, m < 60 and s < 60. Designate positions on the dial by the number of minute spaces after **XII.** Then the hour hand will point to 5h + **m/60 + s/3600,** the minute hand to m + **s/60,** and the second hand to s. By equating the last two we get 60m = 59s. We may assume that h and m are integers so that the last equation implies that 59s is an integer. Since 59 is a prime, m must be 0 or a multiple of 59; but it cannot be as much as 59 without making s as much as 60. Hence m = 0, and so s = 0. From this it is clear that h = 0 also, so that the hands are together only at **XII** o'clock.

15. Proposed by Lester G. **Riggs,** Evanston, Illinois

Russian Multiplication.

Editorial Note:

This method of multiplication actually amounts to multiplying the numbers, using the base 2. Detailed explanations can be found in a number of books including: Mathematics and the Imagination, Kasner and **Newman,** pp. 167–168, and To Discover Mathematics, G. M. Merriman, p. 363.

17. Proposed by C. Stanley **Ogilvy,** Columbia University

All the coefficients in the expansion of $(a+b)^n$ except the first and last are divisible by n if and only if n is a prime.

Solution by the Proposer

Sufficiency. $\frac{n!}{r!\,(n-r)!} = j$, an integer. $n! = j\cdot r!\,(n-r)!$ Neither r! nor (n–r)! is divisible by n if n is prime. But n! is divisible by n. Hence j must be divisible by n.

Necessity. If n is composite, let k be its smallest prime factor. The (k + 1)st term of the expansion is

$$j = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

None of the factors $(n-1)\ldots(n-k+1)$ is divisible by k. Therefore to reduce j to an integer, k must be divided into n. The result is an integer:

$$j = \frac{(n/k)\,(n-1)\ldots(n-k+1)}{(k-1)!}$$

Now n does not divide this j, for none of $(n-1),\ldots,(n-k+1)$ contains k to start with, and division by the factors of $(k-1)!$ can never introduce k by composition because k is a prime.

20. Proposed by Murray S. Klamkin, Brooklyn Polytechnic Institute

Show that

$$\frac{b^b - a^a}{b(1+\ln b)} < \int_a^b x^{x-1}dx < \frac{b^b - a^a}{a(1+\ln b)}, \quad b > a > 1.$$

Solution by the Proposer

One easily verifies by differentiation that

$$\int x^x dx = x^x - \int x^x \ln x\, dx$$

hence

$$b^b - a^a - \ln b \int_a^b x^x dx < \int_a^b x^x dx < b^b - a^a - \ln a \int_a^b x^x dx$$

from which we obtain

$$\frac{b^b - a^a}{1+\ln b} < \int_a^b x^x dx < \frac{b^b - a^a}{1+\ln a}$$

which implies the required inequality.

21. Proposed by Mary **Anice** Seybold, North Central College

A Christmas Problem

In the following long division, each letter represents one and only one number and each number is represented throughout by the same letter. There are no unconventional arrangements of digits. To find my Christmas wish for readers of Pi Mu Epsilon Journal, establish the correspondence of letters to numbers and arrange them in the order 1, 2, 3, . . . , 0. Prove the solution is unique.

```
          F Y
J S L / F U M X
        F O F
          Y F X
          Y J M
            O A
```

Solution by Leonard Gibbs, King College, Tennessee

The Christmas wish expressed is "JOYFUL **XMAS**."

Proof: Clearly J = 1.

Since the last digit in **F·L** is F, the possibilities for **F·L** are 2.6, **4·6, 5·3, 5·5, 5·7, 5·9** and **8·6.** If F were 5, M would be zero. This is impossible since X − M = A. If F were S, both L and M would have to be 6. If F were 2, the letter O would have to be zero or 1, but J = 1, and by its position in the remainder O is not equal to zero. Hence F = 4 and L = 6, from which M = 8 follows.

Now **F·S+2** can be at most 9. Hence S equals zero.

The multiplication **F·JSL** implies O = 2.

The product **Y·L** = 18 implies Y = 3; and the difference U − O = Y implies U = 5.

Finally, by the last subtraction in the division process **M > X so that X = 7 and A = 9.**

Thus the correspondence of letters to numbers is given **by**

J	O	Y	F	U	L	X	M	A	S
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**0**

Also solved by W. E. Chapman, C. R. Hicks, C. W. Trigg, William Wright and the proposer.

22. Proposed by Don A. Gorsline, University of Oklahoma

Each letter in the following addition problem represents a unique digit. Furthermore L^2 = L and M^2 = T. Establish the correspondence of letters to digits.

```
     N O T E
     P I M U
 E P S I L O N
 -------------
 E P I T O M E
```

Solution by C. R. Hicks, Syracuse University

```
     3 5 4 6
     1 9 2 7
 6 1 8 9 0 5 3
 -------------
 6 1 9 4 5 2 6
```

Also solved by W. E. Chapman, C. W. Trigg and W. Wright.

23. (Incorrectly given as 22.) Proposed by Roy Dubisch, Fresno State College

If in a triangle with sides a, b and c we have $c \geqq b$, $c \geqq a$, find k such that $c^2 = ka^2 + b^2$.

Solution by Howard Eves, Oregon State College.

By **Mollweide's** equations **(c.f.** Sparks and Rees, Trigonometry, pp. **159-160),**

$$\frac{c - b}{a} = \frac{\sin \frac{1}{2}(C - B)}{\cos \frac{1}{2}A}$$

and

$$\frac{c + b}{a} = \frac{\cos \frac{1}{2}(C - B)}{\sin \frac{1}{2}A}.$$

Therefore,

$$\frac{c^2 - b^2}{a^2} = \frac{2 \sin \frac{1}{2}(C - B) \cos \frac{1}{2}(C - B)}{2 \sin \frac{1}{2}A \cos \frac{1}{2}A} = \frac{\sin (C - B)}{\sin A} = k.$$

Also solved by W. E. Chapman, B. E. **Meserve** and the proposer.

24. Proposed by Paul J. Schillo, University of Buffalo

If θ_n is the angle opposite the side of length $4n^2$ in the integer right triangle with sides $4n^2$, $4n^4 - 1$, $4n^4 + 1$, where n is any positive integer, show that

$$\lim_{n \to \infty} \sum_{i=1}^{n} \theta_i \quad \text{is a right angle.}$$

Solution by G. E. **Raynor,** Lehigh University

From the conditions of the problem we have

$$\theta_n = \text{Arc tan } \frac{4n^2}{4n^4 - 1}$$

$$= \text{Arc tan } \frac{n^{\frac{1}{2}}}{1 - \frac{1}{4n^4}}$$

$$= \text{Arc tan } \frac{\frac{1}{2n^2} + \frac{1}{2n^2}}{1 - \left(\frac{1}{2n^2}\right)^2}$$

$$= 2 \text{ Arc tan } \frac{1}{2n^2}.$$

Now it is easily shown, by mathematical Induction, that

$$\sum_{i=1}^{n} \text{Arc tan } \frac{1}{2i^2} = \text{Arc tan } \frac{n}{n+1}.$$

Hence

$$\sum_{i=1}^{n} \theta_i = 2 \text{ Arc tan } \frac{n}{n+1}.$$

From this it follows at once that

$$\lim_{n \to \infty} \sum_{i=1}^{n} \theta_i = \frac{\pi}{2}.$$

Also solved by J. Price, T. **Levine,** William Chapman and the proposer.

EXCERPTS FROM LETTERS TO THE EDITORS

"Many thanks for the copy of the first issue of the PI MU EPSILON JOURNAL. You have done a good job. Our department head here, Dr. Z. L. Loflin, has read the Journal with great interest since he was a charter member of the chapter at Louisiana State University."

Ida Mae Heard
Southwestern Louisiana Institute

April 26, '50

"I thank you for sending me a copy of a recent issue of the new publication, PI MU EPSILON JOURNAL. There should be a definite place for a publication of this nature in our mathematical circles, and I wish for it much success."

W. L. Williams, Chairman
University of South Carolina

June 29, '50

"I appreciate the copies of the PI MU EPSILON JOURNAL, and I shall endeavor to get it on our library list."

Thomas L. Wade, Chairman
The Florida State University

July 17, '50

"Enclosed is a list of the initiates in.. .PI MU EPSILON during the past academic year. ...

It would be appreciated if the Journal could be mailed directly to the initiates. In addition, we wish to order 25 extra copies of the Journal for which you will naturally bill us; in fact, you may send us 25 copies of each issue until further notice.

We will solicit individual subscribers and you will hear from us shortly about it."

David Loev, Secretary
Pennsylvania Alpha Chapter

December 1, '50

"I am glad to see that the PI MU EPSILON JOURNAL has gotten off to a good start and that it is going strong. I think it is excellent in all departments. I want to congratulate you on a fine job and to wish you and the Journal continued success.

I am enclosing a solution to problem 24."

G. E. Raynor
Head of Dept. of Mathematics and Astronomy
Dec. 27, '50 Lehigh University

"As vice-president of the New York Gamma Chapter of PI MU EPSILON, permit me to congratulate you on the excellence of the PI MU EPSILON JOURNAL published under your direction. The members of our chapter look forward eagerly to its arrival each term.

Enclosed is a copy of the Math Mirror publication of 1950. I thought, perhaps, you would do us the honor of re-publishing one of the better articles.

We wish you success on the forthcoming journal."

Arthur Hausner
Editor-in-Chief, Math Mirror

ACKNOWLEDGMENTS

For complimentary copies of the following publications received at the office of The Pi Mu Epsilon Journal, the editor is very grateful:

THE PENTAGON, a Mathematics Magazine for Students, Fall 1950, published semi-annually by Kappa Mu Epsilon.

THE MISSOURI GAMMA NEWS, the News Magazine of the Missouri Gamma Chapter of Pi Mu Epsilon, July 1950.

MATH MIRROR, Spring 1950, published annually by the Mathematics Society of Brooklyn College.

JOURNAL OF THE YESHIVA COLLEGE MATH CLUB, Mu Alpha Theta, 1950.

Interfraternity Research and Advisory Council, March 1951, published by L. G. Balfour, Attleboro, Massachusetts.

S.17717.

THE SCIENCE LIBRARY
Science Museum, South Kensington,
LONDON, S.W.7.

6th January, 1951.

Madam,

I am directed to inform you that the Minister of Education has much pleasure in accepting the publications set out herein which you have been so good as to present to the National Collections in the Science Museum Library. I am to convey to you his best thanks for the gift.

I have the honour to be, Madam,
Your obedient Servant,

Director and Secretary.

Miss Ruth W. Stokes - Editor,
"Pi Mu Epsilon Journal",
15, Smith Hall,
Syracuse University,
Syracuse 10,
New York,
U.S.A.

Publications Received:

"PI MU EPSILON JOURNAL" Vol. I, Nos. 1-3.

REPORTS OF THE CHAPTERS

(Send reports to Ruth W. Stokes, 15 Smith Hall, Syracuse University, Syracuse 10, New York.)

EDITOR'S NOTE. According to Article VI, Section 3 of the Constitution: "The Secretary shall keep account of all meetings and transactions of the chapter and, before the close of the academic year, shall send to the Secretary-General and to the Director-General, an annual report of the chapter activities including programs of meetings, results of elections, etc." The Secretary-General now suggests that an additional copy of the annual report of each chapter be sent to the editor of the PI MU EPSILON JOURNAL. Besides the information listed above we are especially interested in learning what the chapters are doing by way of competitive examinations, medals, prizes and scholarships. These annual reports will be published in the chronological order in which they are received.

Delta of New York, New York University

During the academic year 1949-1950 the following talks, or papers, were given:

"The Schillineer System in Music" by Mr. Rudolph Schramm

'Convex sets" by Professor John van Heijenoort

"Theory of relativity" by Professor Benesh Hoffman

"Computing machines" by Miss Eleanor Krawitz of the Watson Scientific Laboratories of Columbia University

"Mathematical applications to some sensory constructs" by Mr. Victor Twersky of the Institute for Mathematics and Mechanics

"An introduction to information theory" by Dr. B. MacMillan of the Bell Telephone Laboratory.

The annual PI MU EPSILON dinner took place at the Fifth Avenue Hotel. Professor Albert Hofstadter of the Philosophy Department, Washington Square College, spoke on "Mathematics and Philosophy."

The April meeting was given over to business the purpose of which was to discuss revision of the constitution and to elect new officers.

The following officers and committees were elected for the academic year 1950-1951: Director, Elaine Weiss; Vice-Director, Alvin Saperstein; Secretary-Treasurer, Mordecai Schwartz; Faculty Adviser and Permanent Secretary, Dr. Gottfried Noether. Executive Council: Dr. Noether, Professors Schoonmaker, Cooley and John; Edwin Spiegelthal, Leonard Sherry and Joan Friedman. Scholarship Committee: Professors Cooley and van Heijenoort, Evelyn Berezin, Eleanor Karasak and Mordecai Schwartz.

Gamma of New York, Brooklyn College

The nine meetings of the chapter during the academic year 1949-1950 included several business meetings for the election and installation of new members and officers, social meetings and those at which there was a guest speaker. Talks presented were:

"The concept of homotopy" by Professor Samuel Eilenberg

"The history and ideals of PI MU EPSILON" by Professor R. A. Johnson

"The National Science Foundation Bill" by Professor Melba Phillips

"Topology" by Professor Norman E. Steenrod of Princeton University

Lecture-demonstration of "Simple Simon, a miniature mechanical brain" by Mr. Edmund C. Berkeley. (Mr. Berkeley, a consultant mathematician, is the author of the recent book, "Giant Brains," which deals with the mathematics and mechanics of calculating machines.)

Officers are elected twice a year. Those for the Fall Term, 1949, were: Director, Professor Moses Richardson; President, Franc Wertheimer; Vice-president, Arienne Silverstein; Corresponding Secretary, Sidelle Shaiken; Financial Secretary, Abe Shenitzer. For the Spring Term: Director, Professor Moses Richardson; President, Arienne Silverstein; Vice-president, George Booth; Corresponding Secretary, Lawrence Bennett; Financial Secretary, Anatole Beck.

Officers elected for the Fall Term, 1950, are: Director, Professor Moses Richardson; President, Anatole Beck; Vice-President, Lawrence Bennett; Corresponding Secretary, Alvin Hausner; Financial Secretary, George Beck.

Alpha of New Hampshire, University of New Hampshire

The chapter held nine meetings during the year 1949–1950, including organizational and business meetings, initiation banquet and program meetings.

The speaker for the initiation banquet was Dean Seely who spoke on "Effects of humidity on the human organism" and he illustrated his lecture with lantern slides.

At the February meeting Professor Solt of the mathematics department spoke on astronomy.

At the April meeting Dr. Kuhlthau of the physics department spoke on "High Rotational Speeds, their Attainment and Application."

At the May meeting; officers for 1950–1951 were elected: Director, Dorothy Breynaert; Vice–Director, Joseph Lambert; Secretary–Treasurer, Evelyn Snow; Permanent Secretary, Sven Peterson.

Gamma of Missouri, St. Louis University

At the first 1949-1950 meeting of PI MU EPSILON, Mr. John Hagan was elected the new Vice-Director to succeed Mr. William Felling, and Miss Virginia Herre was re-elected Secretary-Treasurer.

The chapter held regular meetings during the year. Papers presented were:

"Seismic Ray Paths" by Mr. Carl Kisslinger

"Tabulating Machines" (followed by demonstration) by Dr. John D. Elder and Mr. S. A. Michalski, director of the tabulating department.

"Topological Concepts" by Mr. Robert Pohrer and Mr. Roman Gawkowski

"Life with Statistics" by Professor G. W. Snedecor, Professor of Statistics, Iowa State College.

Officers for the academic year 1950-1951, elected on November 4, are: Director, Joseph Santner; Vice-Director, Richard Kern; Secretary–Treasurer, Virginia Herre; Permanent Secretary, Dr. Francis Regan.

Gamma of Ohio, University of Toledo

The chapter held regular bi-monthly meetings during the year 1949–1950. Talks on the following topics were given:

"Indeterminate Equations of the First Degree" by Edward Ebert

"Lagrange" — a short talk by Robert Gates

"Life of Hamilton" – a short talk by Paul Brinker

"Computation and Transcendental Numbers" by Professor Edwin Godfrey of Defiance College

"An Application of Analytical Geometry to Astronomy" by Dr. Dustheimer.

Officers elected for 1950–1951 are: Director, Grace Cutler; Vice–Director, Martha Goodwin; Secretary, Anna Tom; Treasurer, Norman Krohn.

Alpha of Virginia, University of Richmond

The chapter held six regular meetings during the year 1949-1950, business and regular. Papers presented were:

"Mathematical Oddities" by Dr. E. R. Sleight

"Development of Our Number System" by H. V. Caldwell and P. H. Dalle Mura

"The Origin and Development of Systems of Numbers and Notation; The Evolution of Man's Ability to Count" by Dr. R. E. Gaines.

Dr. C. H. Wheeler III, Chairman of the Department, is the director of the chapter.

Alpha of New York, Syracuse University

The chapter held regular monthly meetings during the academic year 1949-1950. Papers presented were:

"Carl Gauss, His Life and Contributions to Mathematics" by Dr. Myron Rosskopf

"Dedekind and Cantor" by Dr. Donald E. Kibbey

"Philosophy and Mathematics" by Mr. Theodore Denise

"Elementary Geometrical Problems and Their Solutions" by Dr. Charles Loewner

"Gibbs and Vector Analysis" by Dr. Otway O'M. Pardee

"Fermat, His Life and His Times" by Dr. William P. Hotchkiss

"Interpretation of Fourier Integral in its Practical Applications" by Dr. Wilbur LePage

"Homeomorphism and Manifolds" by Dr. Arthur N. Milgram.

Officers for 1950–1951 are: Director, Robert Cooper; Vice-Director, Irving Ames; Recording Secretary, Donald Billet; Corresponding Secretary, Virginia Hamilton; Treasurer, Leon Blatt; Permanent Secretary and Librarian, Dr. Nancy Cole; Faculty Adviser, Mr. Howard C. Bennett.

Alpha of Wisconsin, Marquette University

The chapter held regular monthly meetings during the academic year 1949–1950, besides the initiation and Christmas party in December and the annual banquet in May. Papers presented were:

"Laplace Transforms - - Theory and Applications" by Father Frederick T. Daly, S. J.

"Some Applications of Matrices to the Theory of Equations" (See MAA MONTHLY, March 1950) by Dr. Cyrus C. MacDuffee

"Symbolic Logic" by Dr. L. V. Toralballa

"Actuarial Science" by Miss Ruth Salzmann

"Some Topics of Mathematical Logic" by Dr. L. V. Toralballa (a second talk on logic due to a request from the membership)

"IBM Calculators" by members of the staff of the Milwaukee Office of IBM

"Mathematical Biology" by Dr. John Z. Hearon of the University of Chicago.

Officers for 1949–1950 were: Director, William Golomski; Vice-Director, William Weideman; Corresponding Secretaries, Virginia Higgins and Rosemary Metz; Recording Secretary, Kathleen Murphy; Treasurer, Arthur Wellman; Librarian, Harry Hesse; Moderator, Dr. H. P. Pettit.

Officers for 1950–1951 are: Director, Dr. H. P. Pettit; President, Jerome Boknevitz; Vice-President, Norbert Bold; Corresponding Secretary, James Grundl; Recording Secretary, Father Anselm Nagy; Treasurer, Virginia Altenhoffen; Librarian, Ralph Beter.

Alpha of Kentucky, University of Kentucky

The activities of the chapter during the academic year 1949–1950 included nine regular meetings at which the following papers were given:

"Lunar Eclipses" by Miss Virginia Rhode

"An Unsolved Problem in Univalent Functions" by Dr. A. W. Goodman

"Summability of Certain Divergent Series" by Dr. V. F. Cowling

"What Is a Linear Algebra?" by Dr. J. A. Ward

"Tensor Method of Circuit Analysis" by Dr. N. B. Allison

"The Expected Value of Cost of Insurance" by Mr. E. C. Steele

"Symbolic Operators in Quantum Theory" by Dr. Marshall Haun.

Officers for 1949–1950 were: Director, Miss Elsie T. Church; Vice-Director, Mr. Winberly Royster; Secretary, Mr. Ruric E. Wheeler; Treasurer, Miss Sara Ripy; Librarian, Mrs. Lake C. Cooper; Permanent Corresponding Secretary, Dr. Paul P. Boyd.

Officers for 1950–1951 are: Director, Mr. Winberly Royster; Vice-Director, Mr. Donald C. Rose; Secretary, Mr. A. E. Foster; Treasurer, Mr. Sherman Vanaman; Librarian, Mr. Cordell B. Moore; Permanent Corresponding Secretary, Dr. Paul P. Boyd.

Alpha of North Carolina, Duke University

The chapter held five regular meetings, including business meetings and lectures, during the academic year 1949–1950.

Topics for the meetings were as follows:

"The Purposes of PI MU EPSILON" by Edwin Webb

"Mathematical Tricks" by John Putnam

"Lives of Unusual Mathematicians" by John Pierce

"The Theorem of the Mean" by Dr. J. J. Gergen.

Officers for the year 1949–1950 were: President, John Putnam; Vice-president, Edwin Webb; Secretary, Janet Henchie; Treasurer, Robert Gossett; Membership Chairman, Patricia Collins.

Officers for the year 1950–1951 are: President, Robert Gossett; Vice-president, Thomas Morris; Secretary, Mary Hooks; Treasurer, Robert Malone; Membership Chairman, Patricia Roberts.

Alpha of Arkansas, University of Arkansas

During the academic year 1949–1950 the following papers were read before the chapter:

"Cuba — The status of mathematics in its schools and opportunities for technical employment there" by Dr. B. H. Gundlach

"Report of the Christmas Meetings of the Mathematical Association of America and the American Mathematical Society" by Dr. D. P. Richardson

"Nuclear Physics for the Layman" by Dr. H. M. Schwartz

"Interesting Points about the Operation of a Friden Calculator" by Mr. R. H. McCarroll.

Officers for the year 1950-1951 are: President, Bill Spinelli; Vice-President, Eric Li; Secretary, Jess Olive; Treasurer, William Charles Robinson; Publicity Director, Bob Doyle; Faculty Adviser, Professor Stacy L. Hull and B. H. Gundlach.

Alpha of Pennsylvania, University of Pennsylvania

The following papers were presented to the Pennsylvania Alpha chapter during the academic year 1949-1950:

"Parallel Curves" by Professor C. B. Allendörfer

"Rational Numbers" by Professor Brinkman

"Sylvester's Identity" by Mr. Herman Zabronsky

"P-adic Numbers" by Dr. Emil Grosswald

"The Role of Electronic Computers in Mathematics" by Dr. Herbert Mitchell

"Logic of the Circuits of Electronic Digital Computers" by Professor George W. Patterson.

The officers for 1950-1951 are: Director, Dr. Richard D. Anderson; President, Mr. Herbert Gurk; Secretary, Mr. David Loev; Treasurer, Miss Adele Mildred Goss.

Beta of Wisconsin, University of Wisconsin

Papers presented before the chapter during the academic year 1949-1950 were:

"A Uniqueness Theorem" by Professor O. G. Owens

"Applications of Elliptic Functions to a Problem in Geometry" by Joseph Zemmer

"Application of Matrices to Linear Algebras" by Jack Goldhaber

"The Endomorphisms of a Finite Abelian Group" by Donald R. Morrison

"An Extension of the Hermite Canonical Form for a Matrix" by Leonard E. Fuller.

Officers for the year 1949-1950 were: Director, Professor R. E. Fullerton; President, Gerald P. Dinneen; Vice-President, Leonard E. Fuller; Secretary-Treasurer, Doris M. Efram.

Officers for the year 1950-1951 are: President, William F. Ames; Vice-President, Benjamin E. Mitchell; Secretary-Treasurer, Robert L. SanSoucie.

MEDALS, PRIZES AND SCHOLARSHIPS

EDITOR'S NOTE. Each chapter will undoubtedly be interested in learning what other chapters are doing along the line of prize competitions. So the editor makes the request that chapters offering prizes, scholarships, or other awards, write up their plans for such contests and submit them for publication in this journal.

"The fourth Annual Prize Essay Contest, open to undergraduate students only, was won by Miss Evelyn Agnes Murrill, a senior of Fontbonne College. The title of her paper was Pierre Fermat, His Life and Works. Miss Murrill was awarded a gold pin by Mr. Alois Lorenz, Assistant Professor of Mathematics, who conducted the contest. This award was made at the banquet."

—Missouri Gamma Chapter

"The annual Frumveller Mathematics Contest is for High School Seniors in Milwaukee County. The first prize is a $200 scholarship at Marquette University; the second, $25 cash; and the third, $15 cash."

—Wisconsin Alpha Chapter

"The Kentucky Alpha Chapter of PI MU EPSILON, cooperating with the Department of Mathematics each year selects the student who has done the most outstanding work in the department and nominates him for membership in the Mathematical Association of America. This year Mr. Carl Clinton Faith, a senior in the Department of Mathematics and Astonomy, was selected and his first year's dues in the Association were paid by this chapter."

—Kentucky Alpha Chapter

"The annual PI MU EPSILON freshman, first prize was won by F. Dale Flood, the second by R. C. Tafft. The PI MU EPSILON first prize in calculus was won by Chris E. Kuyatt, the second by Phillip Jones. Prizes of ten dollars for first place and five dollars for second were awarded in each contest."

—Nebraska Alpha Chapter

"The Kansas Gamma Chapter of PI MU EPSILON has established a scholarship at the University of Wichita known as the PI MU EPSILON Scholarship. This scholarship, amounting to the interest on $1,000, is to be awarded annually to an upper division student with a declared major in the field of mathematics. The student will be selected by the faculty committee in charge of scholarships, acting upon the recommendation of the mathematics department."

—C. B. Read, Dept. Head,
University of Wichita

"The DeCou prize of $25.00 was awarded to Eugene A. Maier, senior in mathematics, for his outstanding record and achievement in mathematics. This prize, named after a former head of the mathematics department, the late E. E. DeCou, is awarded annually to the junior or senior student making the most outstanding record in mathematics. The recipient is not necessarily a mathematics major."

—Oregon Alpha Chapter

"The L. C. Plant Award is awarded annually to the students who have in the past year contributed the most to mathematics through scholarship, interest in mathematics and help to the mathematics department.

The awards were presented on February 23, 1950, to George State, Martha Jean Kunkel and Charles Arthur Cassell, at the annual PI MU EPSILON banquet. Dr. J. S. Frame, department head, made the presentation."

—Michigan Alpha Chapter

At the first 1949-50 meeting of PI MU EPSILON an announcement was made to establish a ten thousand dollar endowment fund to honor the memory of the Reverend James E. Case, S. J., who

until his death last August, was head of the Mathematics Department of St. Louis University. The income of the fund is to support the "James E. Case Mathematics Lecture" to be delivered each spring.

—Missouri Gamma Chapter

EDITOR'S NOTE

Having survived the trials and perplexities of its formative stage extending over the past two academic years, the PI MU EPSILON JOURNAL is now an established publication. The editor wishes to express appreciation to the following persons who, in addition to the associate editors and the newly appointed editor of the problem department, have made possible any success of the first four numbers by giving so freely their many helpful suggestions, refereeing papers, preparing and proofreading the manuscript, packaging and mailing the Journals:

Jeanne Carrazzone, Nancy Cole, Edwin F. Gillette, May N. Harwood, E. H. C. Hildebrandt, Charles Loewner, C. C. MacDuffee, Norman Miller, Kathryn A. Morgan, Murray H. Protter, Winthrop H. Rice, the late R. G. D. Richardson, Paul C. Rosenbloom, J. M. Thomas and W. L. G. Williams.

Particularly do we wish to thank June Hegendorfer for her enthusiastic cooperation and careful work in composition of the typescript before it went to the printers, Cushing-Malloy, Inc., whom also we wish to thank for an excellent job.

NEWS AND NOTICES

Space is not available for us to print in its entirety the very fine report of the Florida meeting, made by our retiring Secretary General but a resume is given here.

Evanston, Illinois
January 5, 1951

To All Chapters of PI MU EPSILON:

Greetings:

The annual PI MU EPSILON Breakfast was held on Friday morning, December 29, 1950, in the West Dining Room of the Cafeteria on the campus of the University of Florida, Gainesville, Florida. Professor Svend T. Gormsen (Ohio Alpha) of the University of Florida was in charge of local arrangements for the breakfast.

Sixteen chapters were represented by the following members: Alabama Alpha (University of Alabama) — *C. R. Ainsworth, *Wilbur T. Armstrong, F. A. Lewis, Mrs. F. A. Lewis, Volena Whaley; Arkansas Alpha (University of Arkansas) - *Eric T. C. Li; California Alpha (University of California at Los Angeles) —*Paul H. Daus; Delaware Alpha (University of Delaware) — *C. J. Rees, *Eleanor K. Rees; Georgia Alpha (University of Georgia) - *Owen H. Hoke, *T. A. Newton, *W. D. Peeples, Jr., Clinton Dyches, George O. Peters; Illinois Alpha (University of Illinois) - Beulah Armstrong; Illinois Beta (Northwestern University) - *Richard L. Goldberg, E. H. C. Hildebrandt; Kentucky Alpha (University of Kentucky) - *E. C. Church, *Sallie Pence, *D. E. South; Kansas Alpha (University of Kansas) - *Gilbert Ulmer; Kansas Gamma (University of Wichita) - *C. B. Read; Michigan Alpha (Michigan State College) - J. S. Frame; North Carolina Beta (University of North Carolina) - *Tullio J. Pignani; Nebraska Alpha (University of Nebraska) - C. C. Camp; New York Alpha (Syracuse University) - *O. Pardee, Freda Jones, Ruth W. Stokes; Pennsylvania Beta (Bucknell University) —Hazel M. Forquhar; Wisconsin Beta (University of Wisconsin) - *C. C. MacDuffee.

*Delegate

In addition, the following members located at universities without chapters were in attendance: D. B. Goodner (Illinois Alpha), at Florida State University; Svend T. Gormsen (Ohio Alpha), University of Florida; S. L. Jamison (California Alpha), Florida State University; D. R. Morrison (Wisconsin Beta), Tulane University; L. T. Ritner (California Alpha), Vanderbilt University; Edith R. Schneckenburger (New York Alpha), University of Buffalo; R. L. Wilson (Wisconsin Beta), University of Tennessee.

A short business meeting was held immediately following the breakfast. Each of the members present was introduced by Professor C. C. MacDuffee, Director-General, the presiding officer.

The report of the Nominating Committee, of which Professor Tomlinson Fort of the University of Georgia was chairman, was presented.... There were no further nominations made from the floor. The report of the Nominating Committee was approved. (Since this report was sent to each chapter, it is not printed here.)

...The Secretary-Treasurer General reported that the total number of members is now 18,489. As soon as the chapters at Howard University and at the University of Miami are installed, the total number of active chapters will be 50. The amount on hand in the treasury on December 1, 1950, was $4964.19.

Dr. Ruth Stokes, Editor of the PI MU EPSILON JOURNAL, reported on the various problems related to publications. She requested all delegates to encourage the members of their chapters to submit for publication in the Journal original problems and solutions of those published in our Problem Department and, also, copies of exceptional papers presented at their chapter meeting which might prove of especial interest to the members of the Fraternity.

...The meeting adjourned at 9:30 A. M.

E. H. C. Hildebrandt
Secretary-Treasurer General

*Delegate

New Chapters (Reported December 29, 1950, University of Florida).

At the meeting it was announced that the petitions for establishing chapters of PI MU EPSILON at the University of Miami and at Howard University had been approved by unanimous vote of the chapters. Since this announcement was made word has been received from Secretary-General Hildebrandt that "forty-one chapters voted on the petition of the University of Buffalo for a chapter of PI MU EPSILON and that the vote was unanimous that the petition be granted.

"The forty-ninth chapter of PI MU EPSILON will be the Florida Alpha chapter at the University of Miami. Professor Tomlinson Fort of the University of Georgia will officiate at the installation which will be held on March 21, 1951." The fiftieth chapter will be the District of Columbia Alpha at Howard University. The chapter at the University of Buffalo will be the fifty-first, and will doubtless be called the New York Zeta chapter. Temporary plans call for installing the chapter at Howard University on March 29 and the one at the University of Buffalo on April 2. Both of these will be installed by Director-General C. C. MacDuffee.

Recently, the editor received the following communication: "The editors of the MATH MIRROR would greatly appreciate your announcing that MATH MIRROR will be made available in limited numbers to departments of mathematics requesting them. Requests should be sent to: Anatole Beck, Care of Department of Mathematics, Brooklyn College, Brooklyn, New York, and should be timed to arrive between May 15 and June 1." This communication was signed by Anatole Beck, Business Manager of the MATH MIRROR.

Ballots for the election of officers for a three-year term beginning April 1, 1951, were supposed to be returned to the Secretary-General, E. H. C. Hildebrandt, not later than March 8. We are happy to announce the results of the 1951 balloting as a last minute news item, for just before this issue went to press, we received from Secretary Hildebrandt the following report on the recent election.

"Thirty-nine chapters returned their ballots for national officers of PI MU EPSILON. The following persons were elected for the three-year term beginning April 1, 1951.

Director General:
C. C. MacDuffee, University of Wisconsin, Madison, Wis.

Vice-Director General:
W. M. Whyburn, University of North Carolina, Chapel Hill, N. C.

Secretary-Treasurer General:
J. S. Frame, Michigan State College, East Lansing, Mich.

Councillors General:
S. S. Cairns, University of Illinois, Urbana, Illinois
Tomlinson Fort, University of Georgia, Athens, Georgia
Sophie McDonald, University of California, Berkeley, Cal.
Ruth W. Stokes, Syracuse University, Syracuse, New York.

Editor, PI MU EPSILON JOURNAL:
Ruth W. Stokes, Syracuse University, Syracuse, New York.

Business Manager, PI MU EPSILON JOURNAL:
Howard C. Bennett, Syracuse University, Syracuse, New York."

ERRATA

The following errata have been called to the attention of the editors.

Volume 1, Number 2; Page 58, line 5, the name is "Lawrence". Page 66, line 15, the word is "initiation".

Volume 1, Number 3: Page 103, 4th line from last, the word is "indeterminate", Page 104, 4th line, insert the symbol > between n and 4. Page 108, the problem proposed by Roy Dubisch should be numbered "23".

INITIATES

MICHIGAN ALPHA - Michigan State College

Academic Year, 1949-1950
(May 2, 1950)

Fuad Labib Abboud
Bruce L. Beeker
R. Douglas Behr
Donald E. Burrus
Raymond G. Covell
Carl N. Danielson
Donald L. Erhart
Edward T. Fellows
Uno W. Filpus
James Gates
Thomas Ginther
William R. **Guntrum**
Donald Ho
James Jursik
Walter Lander
William Little
Gordon Marine
Martin Mueller
Richard L. Nyquist
Charles E. Paul
Mary H. **Payne**
Arciel Roy
Roger **J.** Seibert
Charles C. **Sisler**
William E. Splinter
Charles L. Storm
Harmon D. Streiter

Academic Year, 1950-1951
(November 16, 1950)

George E. Archambault
C. William Axce
Reynold Berkey
Leonard T. Blouin
William **Bornstein**
Charles Robert Carlson
Ralph **Eaton**
Robert L. **Gault**
Barrett **Hargreaves**
Henrietta **Hartsuiker**
Ann Kapp
Ralph W. **Lankton**
Henry Leonard
George Linton
William F. McCowan
LaVerne R. **McHaney**
Clyde A. Morrison
Richard Moser
Charles Parker
Gerald Richards
Shirley Richards
Donald Sill
Walker M. Sowers
Salah Izzat Tahsim
William Tappe
William Thomas