PI MU EPSILON

JOURNAL

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PI MU EPSILON JOURNAL

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Editorial correspondence, including manuscripts, chapter reports, and news items should be sent to Underwood Dudley, Mathematics Department, DePauw University, Greencastle, Indiana 46135 (e-mail, dudley@depauw.edu). Students submitting manuscripts should give their school and date of graduation. Others should provide their affiliation, academic or otherwise.

Problems for solution and solutions to problems should be sent directly to Clayton W. Dodge, Mathematics Department, 5752 Neville Hall, University of Maine, Orono, Maine 04469.

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WHO GETS THE WASHERS? A PROBLEM IN DIVIDING RESOURCES

Rychard J. Bouwens
Hope College

Suppose that a college has a stock of washing machines and two dormitories with different numbers of residents. It wishes to divide the number of washers between the two dormitories so as to minimize the total waiting time for all the residents. One solution would be to divide the washers in proportion to the number of residents in each dormitory. It may come as a surprise that this is optimal only when washers are in short supply and the sizes of the dormitory are similar. The goal of this paper is to demonstrate this and find the optimal solution.

We will do this by constructing a model for the use of the washing machines by the residents from which the total waiting time can be calculated. The optimal distribution of washers can then be found numerically.

We make several assumptions. First, that the probabilities for washing machine use are time independent. (Actually, it is important only that the probabilities are time independent during the busiest parts of the day.) Second, that the probability that a resident will begin to use a washer in a time interval At is $\alpha\Delta t$ for some constant a. Third, that the probability that a resident will finish using a washer in the interval At is βAt for a constant β.

Let P_i denote the probability that i washers are being used provided that $i \le z$, where z is the number of washers in the dormitory. If $i > z$, P_i will denote the probability that $i - z$ people are waiting to use the washers.

The change in the probability P_0 that no washers are being used during an interval At depends on the probability that the only resident using the washers will leave, $P_1(\beta At)$, and the probability that a resident will come to use the washers when none are in use, $P_0(\alpha\Delta t)$. Since probabilities are assumed to be time-independent, the change in P_0 is zero, so $\Delta P_0 = -\alpha P_0\Delta t + \beta P_1 At = 0$ or $-\alpha P_0 + \beta P_1 = 0$. Similarly, the change in P_1 over At comes from the probabilities that

a resident will arrive when no washers are in use,
either of two residents will finish washing when two washers are in use,
a resident will arrive when one washer is in use, and
a resident will finish when one washer is in use:

$$\Delta P_1 = \alpha P_0 \Delta t + (2\beta)P_2 \Delta t - \alpha P_1 \Delta t - \beta P_1 \Delta t = 0.$$

Thus

$$\alpha P_0 + (-\alpha - \beta)P_1 + 2\beta P_2 = 0.$$

Using the same reasoning, we have in general, where $i < z$,

$$\alpha P_{i-1} + (-i\beta - \alpha)P_i + (i+1)\beta P_{i+1} = 0.$$

For $i \geq z$, the terms $(i+1)\beta$ and $i\beta$ must be modified since there are only z washers. Hence, for $i \geq z$,

$$\alpha P_{i-1} + (-z\beta - \alpha)P_i + z\beta P_{i+1} = 0.$$

Since it would be difficult to solve the preceding recursion relations explicitly, we will proceed to numerical work. Suppose that the average length of time that washers are used is *30* minutes. Then the decay rate β is $1/(0.5) = 2\ hr^{-1}$. Let $a = yn$, where n is the number of residents in a dormitory, so y is the probability that a resident will **begin** washing in an interval At. If we assume that residents visit the washing room two times a week and that there are *100* hours in a during which people wash their clothes, then we estimate that $\gamma = 21100 = 0.02\ hr^{-1}$.

Based on these parameters, we can calculate the values of P_i in terms of P_0 using the recursion relations. Requiring that the probabilities sum to one determines P_0. We will let $P_i(n, w)$ be the distribution of P_i's for a dormitory with n residents and w washers. For example, the values of $P_i(150, 4)$ are

i	0	1	2	3	4	5	6
P_i	.22	.33	.25	.12	.05	.02	.01

When $i > 4$, people are waiting in line to use the washers.

We now consider how to distribute w washers between two dormitories with populations n_1 and n_2 respectively. The total waiting time is proportional to the expected number of people who will be waiting to use the washers at any time. The expected number of residents waiting to use the washers for both dormitories is

$$\sum_{i=w_1+1}^{\infty} (i - w_1)P_i(n_1, w_1) + \sum_{i=w_2+1}^{\infty} (i - w_2)P_i(n_2, w_2)$$

where w_1 and w_2 are the number of washers in the first and second dormitories respectively.

We minimized the above expression for some values of n_1, n_2, and w by checking all possible distributions. For each minimization, $n_1 = 50$. The optimal distribution of washers is given in Table *1* for some values of w and n_2.

Residents in second dorm, n_2	Total number of washers, w		
	8	14	20
50	4 : 4	7 : 7	10 : 10
100	3 : 5	5 : 9	9 : 11
150	3 : 5	5 : 9	8 : 12
200	2 : 6	4 : 10	7 : 13
300	2 : 6	4 : 10	6 : 14
400	2 : 6	3 : 11	5 : 15
500	1 : 7	3 : 11	5 : 15

Table 1. Optimal distribution of washers for $n_1 = 50$.

Based on the notion that resources should be allocated in proportion to need, one might expect the optimal washer ratio to follow the **population** ratio quite closely. However, even for a small number of washers, the optimal washer ration lies below the population ratio of the dormitories. Furthermore, as the number of washers increases, the optimal washer ratio. lies even further below the population ratio of the dormitories. The optimal* washer ratio between two dormitories is generally much closer to one than is the population ratio.

Finally, we note that this model could also be used for other things, such as dividing **parking** space between two sites. Instead of having people use washers, they would use parking spaces. Of course, the assumptions underlying the model must be satisfied, and it seems as if they are for the parking model. The distribution of cars in a **parking** lot is approximately time-independent during the busiest parts of the day. Furthermore, people arrive at random times, occupy one space, and leave randomly. It would be interesting to see if actual distributions of washers or **parking** spaces, or any other resource to which the model could be applied, agree with the **optimums** determined by the model.

Rychard Bouwens graduated from Hope College in 1994 with degrees in mathematics, physics, and chemistry. He plans to study theoretical physics in graduate school. This paper was written as part of a seminar conducted by Dr. Timothy **J.** Pennings.

Chapter Report

The FLORIDA EPSILON Chapter (University of South Florida) held ten meetings in 1993-94 in conjunction with the student chapter of the Mathematical Association of America, reports Professor **FREDRIC** ZERLA. Included were talks by new faculty members, the presidents of the Chapter (Michael **Pippin—"Geometric** inequalities") and the president of the MAA student **Chrystal** Brandon—"Plato and mathematics", and visitors (Professor Vilmos Totik, of the Hungarian Academy of **Sciences—"Why** can we not decompose the square into an odd number of triangles of equal area?"). In addition, a meeting was devoted to mathematical socializing as the officers posed problems and invited the members to try to stump each other with mathematical puzzles. Twelve new members were initiated.

F_n AND L_n CANNOT HAVE THE SAME INITIAL DIGIT

Piero ***Filipponi***
Fondazione Ugo Bordoni

Let F_n ($F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ if n a 2) and L_n ($L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ if n a 2) denote the n-th element of the **well-**celebrated Fibonacci and Lucas sequences, respectively.

The closed-form expressions (the so-called Binet forms) for these elements are

$$(1) \qquad F_n = \frac{a^n - b^n}{\sqrt{5}} \quad \text{and} \quad L_n = a^n + b^n,$$

where a $= 1 - b = -1/b = (1 + \sqrt{5})/2$ is the positive root of the equation $z^2 - z - 1 = 0$.

The first few terms in the sequences are

F_n	0	1	1	2	3	5	8	13	21	34	55
L_n	2	1	3	4	7	11	18	29	47	76	123

Because of the periodicity of the Fibonacci and Lucas sequences reduced modulo an integer k (k = 10, in our case), it can be seen that F_n and L_n (expressed in base 10) have the same final digit if and only if n $= 10h + 1$ or $10h + 6$ (h = 0 , 1, 2, ...). The detailed proof of this is beyond the scope of this note, whose aim is to establish a property of the Fibonacci and Lucas numbers that seems to have passed unnoticed in spite of its simplicity: F_n and L_n cannot have the same initial digit for $n \geq 2$.

To show this we need two lemmas. We will let $I(x)$ denote the initial digit of a real positive number x a 1 (expressed in base 10) and $D(x)$ the number of digits of the integral part of x.

LEMMA 1. $I(L_n) = I(F_n\sqrt{5})$ for n a 5.

Proof. From **(1)**, we have $F_n\sqrt{5} = L_n - 2b^n$. Since $|2b^n| < 1$ (for n a 2), and $D(L_n)$ (for n a **5**), the statement clearly holds.

LEMMA 2. $I(x\sqrt{5}) \neq I(x)$.

Proof. There are two cases. If $D(x\sqrt{5}) = D(x)$ then, since $\sqrt{5} > 2$, the statement holds true. If $D(x\sqrt{5}) = D(x) + 1$ then, since $\sqrt{5} < 3$, we have $I(x\sqrt{5}) = 1$ or 2. If $I(x)$ were to equal to 1 or 2, then we should have $D(x\sqrt{5}) = D(x)$, a contradiction! The statement follows necessarily.

THEOREM. $I(F_n) \neq I(L_n)$ for $n \geq 2$.

Proof. By inspection, we see that $I(F_n) \neq I(L_n)$ for $n = 2, 3$, and 4. By Lemma 2, we have $I(F_n\sqrt{5}) \neq I(F_n)$ whence, by Lemma 1, $I(L_n) \neq I(F_n)$ for $n \geq 5$.

We conclude this note by challenging the reader to prove the identity

$$3 \leq I(F_n) + I(L_n) \leq 13 \quad (n \geq 2)$$

and to find the smallest n such that $I(F_n) + I(L_n) = 13$.

Acknowledgement

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Piero Filipponi is currently a senior researcher at the "Ugo Bordoni" Foundation in Rome. Besides being a member of **Unione Matematica Italiana** ***and the*** **Fibonacci Association,** ***he is a member of the*** **American Mathematical Society** ***and a reviewer for*** **Mathematical Reviews.** ***He is the author of more than seventy papers, most of which deal with second-order recurring sequences and their generalizations.***

Chapter Report

The **NEW YORK ALPHA EPSILON** Chapter (Siena College) was installed in May of 1993 with thirteen charter members. **LAURIE SCHLENKERMANN** was elected president of the chapter and she reports that members of the chapter participated in the first Hudson River Undergraduate Mathematics Conference and also tutored first through eighth grade students. Eight new members were initiated in May of 1994.

COUNTING TYPES OF SUBSETS OF LATTICES

Jared Grigsby
Hendrix College

In his textbook ***Applied Combinatorics,*** Fred Roberts presents the classic sick **tree/well** tree problem to determine the number of ways that a row of consecutive sick trees can be placed within a row of well trees. In this paper, we present Roberts' discussion of this problem and then extend the ideas to two-dimensional forests and consider some variations.

In the sick **tree/well** tree problem we are given a row of n trees and we want to determine the number of ways that j adjacent sick trees can appear among the n trees as shown in the following diagram, where the open circles represent sick trees.

$$\bullet \quad \circ \; \circ \; \dots \; \circ \; \bullet \; \bullet \; \dots \; \bullet$$

To count the number of ways that this can occur, we can think of the line of j adjacent sick trees as one object and count the number of ways to place that one object among the remaining well trees. If j trees are sick, then $n - j$ trees are well. This leaves $n - j + 1$ places in which to place the line of sick trees. So there are $\binom{n-j+1}{1} = n - j + 1$ ways to place the j adjacent sick trees among the **remaining** well trees.

The question arises: given a row of n trees with j sick trees, what is the probability that those j trees will appear adjacent to each other? To answer this, we first count the number of ways that the j sick trees can appear adjacent to one another in a row of n trees and divide by the number of ways to place j sick trees anywhere among the n trees. We have already counted the number of ways to place j adjacent sick trees in a row of n trees, and the number of ways to place j sick trees anywhere among the n trees is given by $\binom{n}{j}$. Therefore the probability that j sick trees will appear adjacent to each other is $(n - j + 1)\Big/\binom{n}{j}$.

For example, in a row of nine trees the probability that three sick trees will appear adjacent to each other is $(9-3+1)\Big/\binom{9}{3} = 7/84 = .083$. Since this is small, it is evidence that the disease is contagious.

In two dimensions, we begin with an $m \times n$ array of trees and we want to place a $k \times j$ **subarray** of sick trees within it as shown:

$$\begin{array}{cccccc}
\bullet & \bullet & \dots & \bullet & \bullet & \bullet \\
\bullet & \circ & \dots & \circ & \bullet & \bullet \\
 & & \dots & & & \\
\bullet & \circ & \dots & \circ & \bullet & \bullet \\
\bullet & \bullet & \dots & \bullet & \bullet & \bullet \\
\bullet & \bullet & \dots & \bullet & \bullet & \bullet
\end{array}$$

Counting the number of ways to place the **subarray** within the array involves two steps: counting the number of ways to place the k rows within the m rows, and counting the number of ways to place the j columns within the n columns. Since the k rows are adjacent to each other, they can be considered as one object. There are $m-k$ other rows, so there are $m-k+1$ ways to place the k rows among the m rows. Similarly, there are $n-j+1$ ways to place the j columns among the n columns. **Thus** there are $(m-k+1)(n-j+1)$ ways to place a $k \times j$ **subarray** into an $m \times n$ array.

Again, we can ask what is the probability that kj sick trees will appear in a $k \times j$ **subarray** within an $m \times n$ array of mn trees. Since the total number of ways to place kj sick trees among the nm trees is $\binom{mn}{kj}$, the probability is

$$\frac{(m-k+1)(n-j+1)}{\binom{mn}{kj}}$$

For example, if we have a 6×9 array of trees and know that twelve of them are diseased, then the probability that the diseased trees appear in a 3×4 **subarray** is $(6-3+1)(9-4+1)/\binom{54}{12} = 6.997 \times 10^{-11}$. Thus it is almost certain that the disease is contagious.

A more realistic model would have a **subarray** of diseased trees where there are holes where certain trees have not contracted the disease. We call

this a removal configuration. To count the number of ways this configuration can occur, we place a $d \times w$ **subarray** into an $m \times n$ array where up to $T-1$ vertices, where $T = \min\{d, w\}$, are removed (not diseased). The restriction on T is necessary to insure that an entire row or column is not removed. We have already calculated the number of ways to place the **subarray** in the larger array: $(m-d+1)(n-w+1)$. The number of ways to remove up to $T-1$ vertices is $\binom{dw}{i}$, so the total number of removal configurations is the product of those **two quantities.**

For example, suppose that we again have a 6×9 array and want to determine the probability that it contains a removal configuration of size 4×5, as in the following diagram.

$$\begin{array}{ccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet \\
\bullet & \circ & \circ & \circ & \bullet & \circ & \bullet & \bullet & \bullet \\
\bullet & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet \\
\bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

The number of ways that the **subarray** can be placed within the array is $(6-4+1)(9-5+1) = 15$, and the number of ways that up to three vertices can be removed is $\sum_{i=0}^{3}\binom{20}{i} = 1351$. So, the total number of 4×5 removal configurations is 20265. The total number of ways of placing 20, 19, 18, or 17 diseased trees is

$$W = \binom{54}{20} + \binom{54}{19} + \binom{54}{18} + \binom{54}{17},$$

so the probability that the trees will appear in a removal configuration is $20265/W = 3.12 \times 10^{-11}$. Again, the **probability** is very small that this configuration will appear by chance.

Removal configurations can be generalized to higher dimensions where we have a $w_1 \times w_2 \times \dots \times w_n$ **subarray** placed in a $k_1 \times k_2 \times \dots \times k_n$ array with up to $\min_{1 \le i \le n} \{w_1 w_2 \dots w_{i-1} w_{i+1} \dots w_n\} - 1$ vertices removed. The ideas are the same, though it is no longer appropriate to speak of trees.

Another type of configuration, the *shift* configuration, is perhaps more realistic than the removal **configuration** because more nondiseased trees can be omitted from the **subarray** while the diseased configuration still remains dense. A shift configuration is a subset of a $d \times w$ **subarray** within an $m \times n$ array that is obtained as follows:

1) Choose a $(d-1) \times (w-1)$ subarray, but do not place it on the rightmost column or along the bottom row.

2) Choose a subset of the rows and columns of the subarray. To avoid double-counting full arrays, we are allowed to choose no rows or no columns, but we can choose no more than $d-2$ rows or $w-2$ columns.

3) Shift each row chosen to the right one place.

4) Shift each column down one place.

For example, if we have a 3×4 **subarray** in a 6×9 array, with its upper left-hand element in row 2 and column 2 of the large array, and we choose rows 1 and 2 and column 3, the shift configuration that results is

$$\begin{array}{ccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \circ & \bullet & \circ & \circ & \bullet & \bullet & \bullet \\
\bullet & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array} .$$

To count the number of shift configurations, we

1) Count the number of ways to place the $(d-1) \times (w-1)$ **subarray** within the $m \times n$ array, but not placing it on the right-hand side or at the bottom, and count the number of ways to shift the chosen rows and columns.

2) Count the number of ways to place a **subarray** at the bottom of the array and shift rows

3) Count the number of ways to place a **subarray** on the right-hand side of the array and shift columns

4) Count the number of ways to place a **subarray** in the bottom right-hand comer of the array.

The result is

$$S = (m-d+1)(n-w+1)\sum_{i=0}^{d-2}\sum_{j=0}^{w-2}\binom{d-1}{i}\binom{w-1}{i}$$

$$+ (m-d+1)\sum_{i=0}^{w-2}\binom{w-1}{i} + (n-w+1)\sum_{i=0}^{d-2}\binom{d-1}{i} + 1 .$$

Again, we are interested in the probability that given $(d-1)(w-1)$ sick trees that they will appear in a shift configuration. This is

$$\frac{S}{\binom{mn}{(d-1)(w-1)}} .$$

These configurations can be applicable to other situations where the probability of clumping is important.

Jared Grigsby prepared this paper while a senior at Hendrix College, under the direction of Dr. Dwayne Collins. He is presently a graduate student at Clemson University.

Chapter Report

Professor JOHN **PETRO,** corresponding secretary for the MICHIGAN EPSILON Chapter (Western Michigan University), reports a very full round of activities for the past year. There was a pizza party, a used book sale which netted $700 to help support the activities of the chapter, and the chapter's annual service project consisting of a half-day program for high school students involving mathematical problem-solving and a broad range of short talks. Forty-nine new members were initiated. Nine invited talks were jointly sponsored by the chapter and the **Kalamazoo Area** Student Chapter of the MAA. Speakers included Philip **Hanlon,** Underwood Dudley, Joan **Hutchinson,** John Ewing, Peter **Hilton,** Jean Pedersen, **Estela Gavosto,** and Evelyn Hart. The chapter received funds to partially support these activities through student assessment fees and the Honors College.

Greedy Is Not Always Safe

Eric P. Kamprath
Carthage College

A personnel officer is considering a pool of one hundred applicants for a position. The officer would naturally like to hire the best applicant. We will assume that before interviews there is no way to know the ability range of the applicants and, because they will appear in random order, no way of knowing when the best applicant will be interviewed. What should the officer do?

The conditions are

1) All of the applicants can be ranked in quality from 1 (best) to 100.

2) The rank of an applicant relative to those already interviewed can be determined after the applicant is interviewed. The absolute rank of an applicant cannot be determined.

3) The position must be offered immediately following an interview since otherwise the applicant will leave and accept a job with a competitor.

4) If the position has not been filled by the 100th interview, the last applicant must be hired.

Traditionally, the strategy proposed has been to interview some number of applicants without hiring any, in order to determine a standard. Starting with applicant n, the next applicant who exhibits ability above this standard is hired. If none exceeds the standard, then the last applicant is hired. We will call this strategy n. For example, strategy 25 uses the first 24 applicants to determine the standard, and the first of applicants 25 through 100 to exceed the standard is hired.

The problem is to determine the optimal value of n, by applying some criterion. A natural criterion is to choose n so as to maximize the probability of hiring the best applicant. We will call this the greedy strategy. We will show that the greedy strategy may not be the best strategy after all.

To do this, we will use the

THEOREM. Given positive integers m, n, k with $n + k \le m + 1$, let $f(m, n, k)$ denote the probability of obtaining an object of rank k from a

random permutation of m objects by using strategy n. Then

$$f(m,n,k) = \begin{cases} 1/m & \text{whenever } n = 1 \\ \displaystyle\sum_{p=n}^{m} \frac{n-1}{m(p-1)} & \text{whenever } n > 1 \text{ and } k = 1 \\ \displaystyle\frac{n-1}{m(m-1)} + \sum_{p=n}^{m+1-k} \frac{(n-1)(m-p)!(m-k)!}{(p-1)(m+1-p-k)!\,m!} & \text{otherwise.} \end{cases}$$

Proof. We successfully obtain the object of **rank** k using strategy n whenever the random permutation of m objects exhibits all of the following qualities:

1) The object of **rank** k must not be contained among the first $n - 1$ objects.

2) If object k is in **position *p*,** then the object ranking best of the first $p - 1$ objects must be contained among the first $n - 1$ objects.

3) All objects of better **rank** than k must follow object k in the permutation.

Clearly, when strategy 1 is used, there is no standard and the object obtained is in the first position in the permutation. Hence $f(m, 1, k) = 1/m$. We will suppose hereafter that $n > 1$.

Object k is found in any position with a probability of $1/m$. If object k is in position p ($1 \le p \le m$) then the object ranking best of $p - 1$ objects must be contained among the first $n - 1$ objects. Thus condition 2 is fulfilled with probability $(n - 1)/(p - 1)$.

When $k = 1$, condition 3 is not a consideration because no other object has a better **rank** than object 1. Since by condition 1 p cannot be less than or equal to $n - 1$, the lower limit on p is n. Considering all possible positions p of object 1, the probability of obtaining object 1 is

$$f(m, n, 1) = \sum_{p=n}^{m} \frac{1}{m}\frac{n-1}{p-1}.$$

When $k > 1$ condition 3 must be taken into account. The $k - 1$ objects ranking better than object k must be contained in the $m - p$ positions following object k in the ordering. The probability that a permutation fulfills condition 3, then, is

$$\frac{\binom{m-p}{k-1}}{\binom{m-1}{k-1}} = \frac{(m-p)!\,(m-k)!}{(m+1-p-k)!\,(m-1)!}.$$

Condition 3 also dictates the upper limit of p because the last position p which fulfills condition 3 is that for which only objects of rank better than k follow k. These objects would occupy the last $k-1$ positions, which are positions $m-k+2$ through m. The upper limit of p is thus $m-(k-1)$. Because we must consider all possible positions p of object k, the probability that the permutation fulfills all of conditions 1 through 3 is

$$\sum_{p=n}^{m-k+1} \frac{1}{m}\,\frac{n-1}{p-1}\,\frac{(m-p)!\,(m-k)!}{(m+1-p-k)!\,(m-1)!}$$

We must also consider the case in which object k is obtained by default. The conditions for this are

1) Object k must appear in position m.

2) The object of rank 1 must appear in one of the first $n-1$ positions.

Object k appears in position m with probability $1/m$. The probability that object 1 appears in the first $n-1$ of the remaining $m-1$ positions is $(n-1)/(m-1)$. Thus the probability that object k is obtained by default is $(n-1)/m(m-1)$. This default factor must be added to the expressions above to give the correct probability. However, position m is not considered when $n = 1$, and the $k = 1$ expression above already considers the case $p = m$. This completes the proof.

By calculating $f(100, n, 1)$ for $1 \le n \le 100$, the greedy strategy is strategy **38—that** is, interview and reject the first 37 candidates and hire the first candidate beginning with the 38th who ranks higher than all of the previous candidates. The probability of hiring the best candidate using this strategy is 0.3710.

However, with the probabilities in the Theorem it is possible to calculate the expected rank of the candidate hired using strategy n. For strategy 38, it is 19.65. If instead of the greedy strategy, the personnel officer decides to select the strategy that makes the expected rank of the candidate selected the highest (the safe strategy), strategy 10 is the one to pick. The expected rank for this strategy is 9.40. The safe strategy produces, on the average, a

considerably better result than the greedy strategy. Which one to select may depend on whether the hiring process is going to take place many times, or only once.

By **plotting** the expected value of the rank of the candidate **selected by** the greedy and safe strategies for various values of m, it appears that the greedy strategy gives an expected value that is proportional to m while the safe strategy gives approximately $\sqrt{m}$. More work would be needed to determine if the appearances reflect reality.

Eric ***Kamprath,*** a native of ***Milwaukee,*** Wisconsin, graduated from Carthage College in 1994 with majors in mathematics and chemistry. He plans to enter a graduate program in ***matematics*** or computer science.

Chapter Reports

Members of the **CONNECTICUT GAMMA** Chapter (Fairfield University) assisted in coordinating the activities for **MathCounts,** a mathematics contest for junior high school students. Professor JOAN **WYZKOWSKI WEISS** also reports that three members of the chapter, Jennifer **Bacik,** Shannon **Latham,** and **Jody** Panchak were recognized for outstanding achievement in mathematics at the University's annual awards ceremony.

The **GEORGIA** BETA Chapter (Georgia Institute of Technology), reports Professor JAMES M. **OSBORN,** presented awards to two seniors majoring in applied mathematics, William Garrison and Christopher **Spruell,** who maintained at least a 3.70 grade-point average in mathematics courses.

Professor **CHRISTOPHER** LEARY reports that the major activity of the **NEW** YORK OMEGA Chapter (St. Bonaventure University) continues to be the, popular Mathematics Forum, co-sponsored with the MAA student chapter. Last year there were nine talks by students, faculty members, and visitors.

A Curious Equivalence Relation

James M. *Cargal*
Troy State University in *Mongomery*

The following equivalence relation is curious because it is elementary but not obvious or (very) well known. It may be useful as a source of examples.

We define the relation $\sim$ on the positive integers by

$$x \sim y \text{ if and only if } xy \text{ is a perfect square.}$$

Reflexivity and symmetry are immediate. The surprising thing is that transitivity also holds. That is, if $x \sim y$ and $y \sim z$ then $x \sim z$. This is easily proven: since $x \sim y$ and $y \sim z$ then xy and yz are squares, so their product, xy^2z, is a square. So, $xz = (xy^2z)/y^2$ is a square divided by a square and is therefore also a square.

Like all equivalence relations, $\sim$ partitions the set that it operates on, in this case the positive integers, into equivalence classes. It can be seen that the smallest element in a class is a square-free integer (one that has no square factors) and each square-free integer is the smallest element of its own class. If we define $sf(n)$ to be n with all square factors divided out (so, for example, $sf(12) = 3$ and $sf(200) = 2$) then the smallest element of n's class is $sf(n)$. Note that if n is a square then $sf(n) = 1$, so the first class under this relation is the class of square integers.

Let us denote, as is customary, the equivalence class that contains n by [n]. For example, the class of squares is denoted by [1]. The classes are well-defined under multiplication. If $x, y \in [n]$ and $u, v \in [m]$ then $[xu] = [yv]$ because $xuyv = (xy)(uv)$, the product of two squares, is itself a square. So, we can write $[m][n] = [mn]$.

It follows that [1], the class of squares, is an identity element for the multiplicative algebra of equivalence classes. Also, each class is its own inverse: $[n][n] = [n^2] = [1]$. Lastly, associativity and commutativity are inherited from the positive integers. Therefore, the equivalence classes constitute an abelian group under multiplication.

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The Death of Astrology

A survey of all the mathematicians appearing in the Biographical Dictionary *of Mathematicians* (Scribner's, New York, 1970-91) whose names begin with the letters from N to Z discloses the following distribution of month of birth. (For those mathematicians whose birth date is known, that is. The birth certificate of Pythagoras, for instance, is lost.)

Jan	Feb	Mar	**Apr**	May	Jun	Jul	Aug	Sep
22	23	18	26	14	20	8	31	18
			Oct	Nov	**Dec**			
			18	24	18			

There are 240 mathematicians in all, and so 20 births are to be expected in each month. Any student of statistics can calculate the value of χ^2 and see that it is not significant at the 5% level. The distribution of births by season is even more uniform: {63, 58, 59, 60} for {Win, Spr, Sum, Fal}. The value of χ^2 for that distribution is so small as to raise the suspicion that the agreement with the expected 60 per season is too good to be true. But that is how the birth dates come out, as anyone can check. So much for astrology!

But—we see that fifty-seven mathematicians were born in months that begin with *A*. Supposing that births are uniformly distributed, the number born in *A* months is approximately a normal random variable with mean $240(1/6) = 40$ and variance $240(1/6)(5/6) = 10013$. Fifty-seven is 2.9 standard deviations away from the mean! The probability of that happening by chance is so close to zero as to make no difference. There is something about *A* months that attracts mathematicians. Astrology is back again. ▪

Or is it? Has some lying with statistics been going on here?

THE SECRET SANTA PROBLEM

Michael J. Reske
Carthage College

There are twenty people in a class who participate in a "Secret Santa" party. Each person chooses someone else's name out of a hat and that person becomes their Secret Santa partner for whom they secretly buy a gift. Afterwards, the twenty people get together and stand next to their Secret Santa partner. There may be as few as one large cycle of twenty or as many as ten cycles of two. We will find the expected number of cycles for a group of n people.

Formulating the problem in terms of graph theory, we will find the expected number of cycles on a randomly generated n-vertex graph of a certain type. Suppose that we have a randomly generated graph, not necessarily connected, on n vertices where exactly one directed edge enters every vertex and exactly one directed edge leaves every vertex. We will refer to such graphs as Santa graphs, or S-graphs for short. We want to find the expected number of cycles in an *S*-graph.

THEOREM 1. Let $f(n, k)$ denote the number of S-graphs with n vertices that have exactly k cycles. Then

$$f(n, 1) = (n-1)f(n-1, 1),$$

$$f(n, k) = (n-1)f(n-1, k) + f(n-2, k-1), \text{ if } k > 1,$$

with $f(2, 1) = 1$ and $f(n, k) = 0$ when $n = 2$ or 3 and $k \geq 2$.

Proof. To find the number of n-vertex S-graphs with k cycles we will consider two cases. Each will correspond to a way of building a n-vertex *S*-graph from smaller S-graphs. In the first case, where the nth vertex is added to an $n-1$-vertex *S*-graph having k cycles, the nth vertex can be inserted on any of $n-1$ edges. This results in $f(n-1, k)$ cycles being formed. In the second case, the nth point is joined with any of the $n-1$ vertices to form a cycle of two vertices. The remaining $n-2$ vertices can be arranged in any of $f(n-2, k-1)$ ways. Thus $f(n, k) = (n-1)f(n-1, k) + f(n-2, k-1)$.

There follows a table showing the values of $f(n, k)$ for small values of

n and $F(n)$, the total number of S-graphs with n vertices.

		Number of cycles, k		
n	$F(n)$	1	2	3
2	1	1	0	0
3	2	2	0	0
4	9	6	3	0
5	44	24	20	0
6	265	120	130	15
7	1854	720	924	210

$f(n, k)$

Since $F(n) = \sum_k f(n, k)$, Theorem 1 gives

$$F(n) = (n-1)[F(n-1) + F(n-2)], \quad F(2) = 1, \quad F(3) = 2.$$

The expected number of cycles, $E(n)$, in an *S*-graph is

$$E(n) = \sum_{k=1}^{n} \frac{f(n, k)}{F(n)} k.$$

The next theorem gives a recursion relation for $E(n)$.

THEOREM 2. $E(2) = E(3) = 1$, and for $n \geq 4$,

$$E(n) = \frac{n-1}{F(n)}[F(n-1)E(n-1) + F(n-2)E(n-2) + F(n-2)].$$

Proof. Using the recursion relation for $F(n)$ and Theorem 1, we have

$$E(n) = \frac{1}{F(n)} \sum_{k=2}^{n} f(n, k)k + \frac{1}{F(n)} f(n, 1)$$

$$= \frac{1}{F(n)} \sum_{k=2}^{n} (n-1)[f(n-1, k) + f(n-2, k-1)]k + \frac{1}{F(n)} f(n, 1)$$

$$= \frac{F(n-1)(n-1)}{F(n)} \sum_{k=2}^{n} \frac{f(n-1,k)k}{F(n-1)} + \frac{n-1}{F(n)} \sum_{k=2}^{n} f(n-2,k-1)k + \frac{f(n,1)}{F(n)}$$

$$= \frac{F(n-1)(n-1)}{F(n)} \left[\sum_{k=1}^{n} \frac{f(n-1,k)k}{F(n-1)} \right] - \frac{(n-1)f(n-1,1)}{F(n)} + \frac{n-1}{F(n)} \sum_{k=2}^{n} f(n-2,k-1)k + \frac{(n-1)f(n-1,1)}{F(n)}$$

$$= \frac{F(n-1)(n-1)}{F(n)} E(n-1) + \frac{n-1}{F(n)} \sum_{k=1}^{n-1} f(n-2,k)(k+1)$$

$$= \frac{F(n-1)(n-1)}{F(n)} E(n-1) + \frac{n-1}{F(n)} \sum_{k=1}^{n} f(n-2,k)k + \frac{n-1}{F(n)} \sum_{k=1}^{n} f(n-2,k)$$

$$= \frac{F(n-1)(n-1)}{F(n)} E(n-1) + \frac{F(n-2)(n-1)}{F(n)} \sum_{k=1}^{n} \frac{f(n-2,k)k}{F(n-2)} + \frac{n-1}{F(n)} \sum_{k=1}^{n} f(n-2,k)$$

$$= \frac{F(n-1)(n-1)}{F(n)} E(n-1) + \frac{F(n-2)(n-1)}{F(n)} E(n-2) + \frac{n-1}{F(n)} F(n-2)$$

$$= \frac{n-1}{F(n)} [F(n-1)E(n-1) + F(n-2)E(n-2) + F(n-2)].$$

Calculations show that $E(n)$ increases slowly with n, reaching only 4.721 when $n = 170$. A plot of $E(n)$ against n leads to the conjecture that $E(n)$ is proportional to $\ln n$. This could be verified if $f(n, k)$ could be expressed in closed form.

Michael Reske prepared this paper while a junior at Carthage College, taking a course in the theory of probability.

Language and Mathematics

Professor I. J. GOOD (Virginia Polytechnic Institute and State University) writes, I hope with some irony,

> The worst linguistic error frequently perpetrated is to write "denoted x^n where "denoted by x" is correct. People don't say "the ball was kicked Tom" when they mean either (i) "the ball was kicked by Tom" or (ii) "the ball was kicked, Tom". For examples of this horrible error see $\pi\mu\epsilon J$ 9 (1994), p. 655, line 2, and p. 663, line 8. The editor has the primary responsibility!

Mea culpa, I guess. However, this may be one of those battles, like the one against improperly using "hopefully" instead of, as in the first sentence, the correct "I hope", that may have been lost and is therefore no longer worth fighting. Language changes, even if to the purists among us it seems to be continually on the decline.

Readers are invited to comment on any other misuses of mathematical language to which they are sensitive. For example, I find "math" hard to abide. The word is a proud one, mathematics, and deserves to be given in full. People don't talk about phys, or hist, or Eng. Yes, "mathematics" has all of four syllables, but life is not so rushed, nor are we so incapable, that we cannot come out with all of them. Besides, "math" already has a meaning—a mowing, whence "aftermath"—and we don't want confusion to arise, do we?

A Note on an Exponential Equation

Rex H. Wu
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In [1], Norman **Schaumberger** provided a positive integer solution to

$$x^{11} = y^4 + z^7 + w^9$$

with $x = 3^{(10!+1)/11}$, $y = 3^{10!/4}$, $z = 3^{10!/7}$, and $w = 3^{10!/9}$. In general,

$$(x_0, x_1, x_2, \dots, x_n) = \left(n^{((k_0-1)!+1)/k_0}, n^{(k_0-1)!/k_1}, n^{(k_0-1)!/k_2}, \dots, n^{(k_0-1)!/k_n}\right)$$

is a solution to

$$x_0^{k_0} = x_1^{k_1} + x_2^{k_2} + \dots + x_n^{k_n}$$

provided k_0 is prime and $k_i \mid (k_0 - 1)!$ for $i = 1, 2, \dots, n$. The reason for this is

$$\text{(1)} \qquad \left(m^{(q+1)/k_0}\right)^{k_0} = m^{q+1} = m \cdot m^q = \sum_{i=1}^{m} m^q = \sum_{i=1}^{m} \left(m^{q/k_i}\right)^{k_i}:$$

since k_0 is a prime, if we choose $q = (k_0 - 1)!$ then from Wilson's Theorem $(q + 1)/k_0$ is an integer, and $k_i \mid q$ for each i.

In this note, we will use this idea to develop solutions to

$$\text{(2)} \qquad a_0 x_0^{k_0} = \sum_{i=1}^{n} x_i^{k_i}.$$

We do not require that the k_i be distinct. To avoid trivial cases, we will take $n > 1$.

THEOREM 1. If $a_0 \mid n$ and $\gcd(k_0, k_i) = 1$ for $i = 1, 2, \dots, n$ then (2) has a solution in positive integers.

Proof. It suffices to show that there is a q such that $k_0 \mid (q+1)$ and $k_i \mid q$ for $i = 1, 2, \dots, n$. Then we can apply (1).

Let $M = \operatorname{lcm}(k_0, k_1, \dots, k_n)$. Then there exists an integer x such that $q = Mx$ and $k_0 \mid (Mx + 1)$. This is so because $\gcd(k_0, k_i) = 1$, which implies $\gcd(k_0, M) = 1$, and $Mx + 1 \equiv 0 \pmod{k_0}$ has a solution if $\gcd(k_0, M) = 1$.

Next, apply (1) and let $m = n/a_0$. Then m is an integer and

$$a_0 \cdot \left[(n/a_0)^{(q+1)/k_0}\right]^{k_0} = a_0 \cdot m^{q+1} = a_0 \cdot m \cdot m^q = n \cdot m^q$$
$$= \sum_{i=1}^{n} m^q = \sum_{i=1}^{n} \left(m^{q/k_i}\right)^{k_i} = \sum_{i=1}^{n} \left[(n/a_0)^{q/k_i}\right]^{k_i}.$$

Thus

$$(x_0, x_1, x_2, \dots, x_n) = \left((n/a_0)^{(q+1)/k_0}, (n/a_0)^{q/k_1}, (n/a_0)^{q/k_2}, \dots, (n/a_0)^{q/k_n}\right)$$

is a solution.

For example, let us find a solution to $x^{11} = y^4 + z^7 + w^9$ using the theorem. Here $a_0 = 1$ and $n = 3$. Since $1 \mid 3$ and $\gcd(11, 4) = \gcd(11, 7) = \gcd(11, 9) = 1$, there exists a solution. We have $m = 3$ and $M = \operatorname{lcm}(4, 7, 9) = 252$. Solving $252v + 1 \equiv 0 \pmod{11}$ gives $v \equiv 1 \pmod{11}$. So, taking $q = 252$ would give a solution, specifically $(x, y, z, w) = (3^{23}, 3^{63}, 3^{36}, 3^{28})$.

While $\gcd(k_0, k_i) = 1$ and $a_0 \mid n$ are sufficient conditions for (2) to have a solution, they are not necessary. There may be solutions if $\gcd(k_0, k_i) \neq 1$ for some (but not all) i. The next theorem determines some.

THEOREM 2. If $a_0 = t^s$, $n = r^s$, $t \mid r$, and $\gcd(k_0, M) \mid s$ where $M = \operatorname{lcm}(k_1, k_2, \dots, k_n)$ and c is an integer, then (2) has a solution in positive integers.

Proof. **Since** $\gcd(k_0, M) \mid (ck_0 - s)$, the congruence $Mx + s \equiv 0$ (mod An) has a solution. Now let $q = Mx$. Then

$$(x_0, x_1, x_2, \dots, x_n) = \left((r/t)^{(q+s)/k_0}, (r/t)^{q/k_1}, (r/t)^{q/k_2}, \dots, (r/t)^{q/k_n}\right)$$

is a solution, as can be seen from

$$a_0\left[(r/t)^{(q+s)/k_0}\right]^{k_0} = a_0(r/t)^{q+s} = t^s(r/t)^s(r/t)^q = r^s(r/t)^q$$

$$= n(r/t)^q = \sum_{i=1}^{n}\left[(r/t)^{q/k_i}\right]^{k_i}.$$

For example, to **find** a solution to $x_0^{14} = x_1^2 + 2x_2^5 + x_3^6$, we have $n = 4 = 2^2$, $M = \mathrm{lcm}(2, 5, 6,) = 30$, $\gcd(14, 30) = 2$ and $2 \mid (14 - 2)$, so the equation has a solution. Solve $30v + 2 \equiv 0 \pmod{14}$ to obtain $v \equiv 6$ or 13 (mod 14). If we take $x = 6$, then $q = 180$ and a solution **is** $(x_0, x_1, x_2, x_3) = (2^{15}, 2^{90}, 2^{56}, 2^{30})$.

There are many difficult questions that can be asked about exponential equations. The two theorems do not provide all solutions to an equation even if their conditions are met. Is there an algorithm that can generate more or even all solutions? Can we determine when (2) does not have any solutions? Obviously, the theorems fail to find solutions for **certain** equations. For instance, $x^2 = y^3 + z^4$ has a solution, namely $(x, y, z) = (3, 2, 1)$. A special case of the equation is $x_0^{k_0} = x_1^{k_1} + x_2^{k_2}$ with jig, k_1, and $k_2 > 2$. Can we conclude that if this equation has a solution then $\gcd(k_0, k_1) = \gcd(k_0, k_2) = 1$? (Theorem 2 fails to give a counterexample for this hypothesis.) If the above were true, **Fermat's** Last Theorem would be a corollary.

Acknowledgement. I wish to thank Susan **Hom** and Shi-Feng Lu for reviewing this paper.

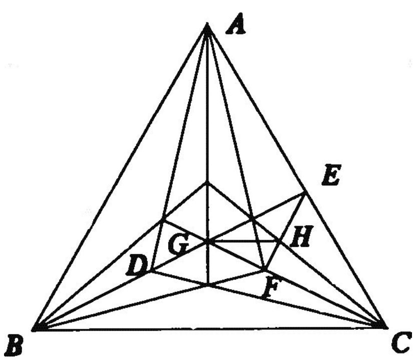
Reference

1. Norman Schaumberger, Quickie **Q795,** Mathematics Magazine 49 (1992) #4, 266, 273.

Rex Wu received his bachelor's degree in mathematics from New York University. His favorite subjects are algebra and number theory.

A Triangle

In the figure on the right, the angles at A, B, and C in the equilateral triangle ABC has been bisected twice. **ANDREW CUSUMANO** challenges you to show that EF is perpendicular to DC and that GH is parallel to BC.



A FABLE OF TWO COORDINATE STATES

Ellen Oliver
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Years ago, in lands far away, there were two countries known as the Cartesian State and the Argand State. Now the Cartesian State was governed by the Cartesian or rectangular coordinate system while the Argand state was governed by the Argand or complex coordinate state.

It so happened that from each of these states a scholar journeyed to meet with many other scholars from other states to hear leading professors and to discuss and learn from each other in Seminars and Discussion Sessions. That year, meetings were held in the great city of Know More and each scholar who came hoped to gain new knowledge and understanding.

During one of the Discussion Sessions Dr. Real and Dr. Imaginary began to compare their two systems of government. They soon found that their two systems were alike in some details. They each used two perpendicular number lines which intersected at a point called the origin and used the values $x = 0$ and $y = 0$ to name its location. These perpendicular lines were called the axes. All values to the right of the vertical axis were considered to be positive values of x and all values above the horizonal axis were considered to be positive values of y.

Dr. Imaginary was now very interested in finding what was different about their two systems. Dr. Real said that only real numbers could be represented on the Cartesian plane. He gave an example of the ordered pair (x, y) and how it would be located. The example he gave was that of the point $(1, 3)$. The value of $x = 1$ is represented by a vertical line which is one unit to the right of the vertical axis and the value of $y = 3$ is represented by the horizontal line which is three units above the horizontal axis. The point $(1, 3)$ is then the point of the intersection of these two lines.

Dr. Imaginary said that he thought that there was a relationship between this example and the complex number $z = x + iy$. Using the same ordered pairs that Dr. Real had used he wrote $z = 1 + 3i$. In order to show this point on his system he labeled the horizontal axis as the x axis with the real units of x as Dr. Real had done and the vertical axis was labeled the i axis.

This difference from they axis now came as Dr. Imaginary changed the units label to be units of yi value instead of regular units in the reals. Then he located $z = 1 + 3i$ to be the intersection of the vertical line one unit to the right of the vertical axis and the horizontal line $3i$ which is three units above the horizontal axis. Both scholars were excited to find that for the ordered pair of real numbers $(1, 3)$ and for the same ordered pair of the complex number $z = 1 + 3i$ that the location would be the same if one of the two coordinate systems were to be superimposed on the other.

Since both these scholars were interested in the behavior of conies they began to look into any relationship which might exist between their two systems concerning these topics. For the short time they had left they decided to limit their discussion to those conics whose vertices or centers were located at the origin.

Dr. Real graphed a circle using the unit circle $x^2 + y^2 = 1$. Immediately Dr. Imaginary spoke up and said, "But how about those x values excluded from he domain of the real for the circle you have graphed?" He then proceeded to use those excluded values of x and the imaginary values of y which he found to graph his results on the Argand coordinate system. When he had finished they discovered that the graph was a hyperbola whose vertices intersected the x axis at the same point that the circle intersected it when one system was superimposed on the other. Then Dr. Real graphed the hyperbola $x^2 - y^2 = 1$ on the Cartesian coordinate system and Dr. Imaginary found that for the excluded of values of x that his graph was that of a unit circle with the intersection of the x axis at the same point as that of the hyperbola that Dr. Real had graphed. Then after more study they found a similar relationship between the ellipse and the hyperbola.

When they studied the parabola, they discovered a very interesting result. For any given parabola with its vertex on the vertical axis in the Cartesian coordinate system there was a mirror image across the axis in the Argand system.

There was no time left for further study on any of these topics of mutual interest, so these two learned men left the city of Know More. Before leaving, they made plans to continue their studies and meet for discussions at the meeting to be held in the city of Further Knowledge during the next year. Their plans for study were to investigate what happens to the relationships between the conies of the two systems when a conic is* translated from the origin or rotated away from the horizontal axis. We can be sure of a most interesting meeting between these two scholars when they meet next year in the city of Further Knowledge.

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One Derivative is Plenty

Professor **JAMES CHEW** of North Carolina A & T State University presents the following proof that a differentiable function of a complex variable has infinitely many derivatives. This may be news to some readers of the Journal, who should then read the proof carefully.

Let $f(z) = U(x, y) + iV(x, y)$ be a differentiable function of a complex variable $z = x + iy$. Then

$$f'(z) = U_x + iV_x = V_y - iU_y.$$

The equations $U_x = V_y$ and $-U_y = V_x$ are **called** the Cauchy-Riemann equations. The first formula for $f'(z)$ is the limit of the expression for the derivative as $\Delta z \to 0$ horizontally, while in the **second formula** $\Delta z \to 0$ vertically. Conversely, if the Cauchy-Riemann equations hold, and the four first partial derivatives of U and V are continuous, then $f'(z)$ exists.

Now let $g(z) = P + iQ$ where $P = U_x$ and $Q = V_x$. The horizontal approach for the difference-quotient expression for $g'(z)$ is $P_x + iQ_x$ while the vertical approach answer is $Q_y - iP_y$. Hence to show that $g'(z)$ exists we need to show that $P_x = Q_y$ and $P_y = -Q_x$. But $P_x = (U_x)_x = (V_y)_x = (V_x)_y = Q_y$. We show $P_y = -Q_x$ using the other Cauchy-Riemann equation, $U_y = -V_x$.

Hence $g'(z) = f''(z)$ exists and, by induction, all order derivatives of $f(z)$ exist!

ON THE NUMBER OF INVERTIBLE MATRICES OVER GALOIS RINGS

Beth Miller
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Recently, Lancaster [2] determined the number of 2×2 invertible matrices over Z_{p^e} and showed that the probability of choosing a 2×2 invertible matrix over Z_{p^e} is equal to the probability of choosing a 2×2 invertible matrix over Z_p. He also conjectured that the result is true for $n \times n$ matrices over Z_{p^e}.

In this note we will prove a generalized version of Lancaster's conjecture. We will show that the probability of choosing an $n \times n$ invertible matrix over the Galois ring $GR(p^e, m)$ is equal to the probability of choosing an $n \times n$ invertible matrix over the finite field $F_{p^m} = GR(p, m)$.

For a prime p, let $GR(p^e, m)$ denote the Galois ring of order p^{em} which **can be** obtained as a Galois extension of Z_{p^e} of degree m. Thus $GR(p^e, 1) = Z_{p^e}$ and $GR(p, m) = F_{p^m}$. For example, to construct the ring $GR(3^2, 2)$, we first find an irreducible polynomial over the field Z_3, say $x^2 + 1$, and then we construct the quotient ring $Z_9[x]/\langle x^2 + 1\rangle$; i. e.,

$$GR(3^2, 2) = Z_9[x]/\langle x^2 + 1\rangle = \{a + bi \mid a, b \in Z_9 \text{ and } i^2 + 1 = 0\}.$$

We also see that the function β defined by

$$\beta((3a_1 + a_0) + (3b_1 + b_0)i) = a_0 + b_0 i$$

for all a_1, a_0, b_1, and b_0 in Z_3 is a **homomorphism** from the ring $GR(3^2, 2)$ onto the field $GR(3, 2)$.

More generally, if $f(x)$ denotes an irreducible polynomial of degree m over Z_p, then

$$GR(p^e, m) = \{A_0 + A_1 i + \dots + A_{m-1} i^{m-1} \mid A_k \in Z_{p^e}, 0 \le k \le m - 1\}.$$

Further, if $A_k = a_{k0} + a_{k1}p + a_{k,e-1}p^{e-1}$ with $a_{kr} \in Z_p$ for $0 \le k \le m - 1$

and $0 \le r \le e - 1$, then the function β defined by

$$\beta(A_0 + A_1 i + \dots A_{m-1} i^{m-1}) = a_{00} + a_{10}\, i + \dots + a_{m-1,0}\, i^{m-1}$$

is a **homomorphism** from the ring $GR(p^e, m)$ onto the field $GR(p, m)$.

Further details concerning Galois rings can be found in **[1].**

Now we are ready for our result.

THEOREM. Let p be a prime and $GR(p^e, m)$ be the Galois ring of order p^{em}. Then the probability of choosing an $n \times n$ invertible matrix over the ring $GR(p^e, m)$ is equal to the probability of choosing an $n \times n$ invertible matrix over the field $F_{p^m} = G(p, m)$.

Proof. Let $M_{n\times n}(p^e, m)$ denote the ring of $n \times n$ matrices over the ring $GR(p^e, m)$ and let $M^*_{n\times n}(p^e, m)$ denote its corresponding group of units. Then

$$C = (b_{ij}) \in M^*_{n\times n}(p^e, m) \iff \det(C)\text{â,¬GR*(pe,}$$
$$\iff \beta(\det(C)) \in GR^*(p, m)$$
$$\iff \det(\beta(b_{ij})) \in GR^*(p, m),$$

where $GR^*(p^e, m)$ and $GR(p, m)$ denote the units of $GR(p^e, m)$ and $GR(p, m)$ respectively. Hence, if

$$b_{ij} = c_{ij0} + c_{ij1}p + \dots + c_{ij,e-1}p^{e-1} \text{ with } c_{ijk} \in GR(p, m),$$

then

$$C = (b_{ij}) \in M^*_{n\times n}(p^e, m) \iff (c_{ij0}) \in M^*_{n\times n}(p, m).$$

Thus,

$$|M^*_{n\times n}(p^e, m)| = |M^*_{n\times n}(p, m)|\, p^{m(e-1)n\times n}$$

and

$$\frac{|M^*_{n\times n}(p^e, m)|}{|M^*_{n\times n}(p, m)|} = p^{m(e-1)n\times n} = \frac{|M_{n\times n}(p^e, m)|}{|M_{n\times n}(p, m)|},$$

which is the result.

References

1. McDonald, B. R., Finite Rings with Identity, Marcel **Dekker,** New York, 1974.

2. Lancaster, M. J., On the number of invertible matrices over Z_{p^e}, this Journal, 9 (1989-94) #7, 440-445.

Beth Miller prepared this note during her senior year at ***Penn*** State University under the supervision ***of Professor*** Javier ***Gomez-Calderon.*** She is now a graduate student at the University ***of*** Pittsburgh.

Mathematics in Literature

From ***Smilla's*** Sense of Snow, by Peter **Høeg,** translated by **Tiina** Nunnally **(Farrar,** Strauss, and Giroux, New York, **1993),** contributed by Professor DONALD **CROWE,** University of Wisconsin, Madison:

Cantor illustrated the concept of infinity for his students by telling them that there was once a man who had a hotel with an infinite number of rooms, and the hotel was fully occupied. Then one more guest arrived. So the owner moved the guest in room number 1 into room number 2; the guest in room number 2 into number 3; the guest in 3 into room 4, and so on. In that way room number 1 became vacant for the new guest.

What delights me about this story is that everyone involved, the guest and the owner, accept it as perfectly natural to **carry** out an infinite number of operations so that one guest can have peace and quiet in a room of his own. That is a great tribute to solitude.

Long Chains of Primes

Thomas Koshy
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In 1772, Euler discovered the celebrated prime-generating polynomial

$$E(x) = x^2 - x + 41$$

that yields distinct primes for $1 \le x \le 40$ [1, vol. 1, 420-421]. The largest prime in the Euler chain is $E(40) = 1601$ and the smallest is $E(1) = 41$. In 1899, Escott observed that

$$ES(x) = E(x - 39) = x^2 - 79x + 1601$$

is a prime for $-40 \le x \le 39$. Although the Escott polynomial yields a chain of eighty primes, they are the Euler primes, each repeated twice. In 1982, Higgins [2] found a polynomial,

$$H(x) = 9x^2 - 231x + 1523,$$

that produces 40 distinct primes, some of which are different from the Euler primes, the largest being $H(39) = 6203$.

Although it is known that no quadratic polynomial can do better than the Euler polynomial in giving a chain of primes, in 1983 Schram [3] devised a variation, $S(x)$, that generates a chain of 176 primes consisting of the forty distinct Euler primes with repetitions. To define $S(x)$, let d denote the sum of the digits of x, with the same sign as x. Since $x \equiv d \pmod 3$, $x + 2d \equiv 0 \pmod 3$. That is, $(x + 2d)/3$ is an integer. Then

$$S(x) = E\left(\frac{x+2d}{3}\right) = \left(\frac{x+2d}{3}\right)^2 - \frac{x+2d}{3} + 41$$

will generate a prime for $-88 \le x \le 87$, a total of 176 primes.

This can be improved upon. Since $x \equiv d \pmod 9$, we have $x + 8d \equiv 0 \pmod 9$ and hence $(x + 8d)/9$ is an integer. If

$$K(x) = E\left(\frac{x+8d}{9}\right) = \left(\frac{x+8d}{9}\right)^2 - \left(\frac{x+8d}{9}\right) + 41$$

then it will be found that $K(x)$ generates a chain of primes of length 336 for

$-167 \le x \le 168$. Of course, the primes are the Euler primes with repetitions. There may not be any simple formula that produces a longer chain of primes.

References

1. Dickson, L. E., *History of the Theory of Numbers,* reprinted by Chelsea, New York, 1971.
2. Higgins, O., Another long string of primes, *Journal of Recreational Mathematics 14* (1981-82) #3, 185.
3. Schram, J. M., A string of 176 primes, *Journal of Recreational Mathematics 15* (1982-83) #3, 168-169.

Thomas Koshy did his undergraduate work in India and received his Ph. D. degree from Boston University. Since 1971 he has been on the faculty of Framingham State College where he is currently chairman of the mathematics department. He is working on a discrete mathematics text.

Chapter Report

Ms. MARLA EASON, president of the Arkansas Beta Chapter (Hendrix College) reports that the Chapter sent students to speak at conferences in Terre Haute, Kalamazoo, and Cincinnati, as well as at the MAA section meeting in Searcy and the Hendrix-Rhodes-Sewanee conference in Memphis. The Chapter held several meetings, sponsored a monthly problem-solving contest, and, after the initiation of new members, attended an Arkansas Travelers baseball game.

COMPARISON OF QUEUEING SYSTEMS

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In the standard one-server queue, the server is idle when there are no customers. The server could then be doing other work, or could take a break. In this note, we investigate an intermittent one-server queue, where the server works until the queue length is **zero** and then takes a break for time d. If at this time the queue length is still zero, the server takes another break for time d.

Let T be a random variable assuming values on $\{t \mid t > 0\}$. T is ***exponentially distributed*** if T has the "lack of memory" **property,** that is $P(T > t + s \mid T > t) = P(T > s)$. It can be shown that the lack of memory property implies that $P(T > t) = e^{-\lambda t}$ for t a 0 and some A. The expected value of T is $1/\lambda$.

We assume that the times between the arrival of customers are independent random variables, each exponentially distributed with mean $1/\lambda$. The queue is thus a Poisson process with parameter A. If $N(t)$ denotes the number of customers in the queue at time t, then

$$P(N(t) = j) = \frac{(\lambda t)^j e^{-\lambda t}}{j!}, \quad j = 0, 1, 2, \ldots .$$

(See, for example, [1, p. *3261.)* We also assume that the service times are independent of each other and exponentially distributed with parameter θ. Hence the mean service time is $1/\theta$.

We will calculate the expected time between breaks, the expected length of the queue, and the expected waiting time for a customer for various values of d. To get the time between breaks, suppose that the server takes a break of length d. During that time, since A customers arrive on the average per unit of time in a Poisson process, Ad can be expected to be in line when the server returns. Since the expected service time is $1/\theta$, the expected time to serve the Ad customers is $(\lambda d)/\theta$. During this $d(\lambda/\theta)$ time, an additional $d(\lambda/\theta)\lambda$ customers join the queue. The time to serve these customers is

$[d(\lambda/\theta)\lambda] = d(\lambda/\theta)^2$. And so on: the expected time from the beginning of the last break until all customers are served and the next break begins is

$$d + d(\lambda/\theta) + d(\lambda/\theta)^2 + \ldots = d\left(\frac{1}{1 - \lambda/\theta}\right) = \frac{d\theta}{\theta - \lambda}.$$

It is known that the expected length of the queue for the standard model is $\lambda/(\theta - A)$ [1, p. **337].** Since there is no corresponding formula for the intermittent-server model, we developed an algorithm that simulates the expected length. The following table shows the variation of the expected length as the serving rate is varied *(A* is held constant at *1)* for different values of the break times. Each data point represents the average of 20,000 simulations. The expected length for the standard model (d = 0) is included for comparison.

θ	Expected length of queue for d =						
	6	5	4	3	2	1	0
2	3.99	3.54	3.04	2.51	1.99	1.45	1.00
3	3.44	3.06	2.43	2.07	1.51	1.02	0.50
4	3.34	2.81	2.35	1.88	1.38	0.84	0.33
6	3.21	2.70	2.18	1.70	1.19	0.69	0.20
8	3.13	2.63	2.16	1.64	1.13	0.65	0.14
10	3.12	2.62	2.11	1.63	1.10	0.61	0.11

The average waiting time for a customer in the standard model can be calculated by adding the expected time to serve the expected length of line to the time it takes to serve the customer: $[\lambda/(\theta - \lambda)](1/\theta) + 1/\theta = 1/(\theta - A)$. With appropriate alterations in our previous algorithm we can simulate the waiting time for the intermittent-server model. The table on the next page shows the variation of the expected waiting time of a customer as the serving rate is varied *(A* is held constant at *1)* for various values of d.

To answer the question of which type of queue is better, it would be necessary to calculate the costs and benefits of having, on the one hand, less idle server time, and on the other, longer average queues and waiting times.

θ	Expected waiting time for d =						
	6	5	4	3	2	1	0
2	2.48	2.24	2.00	1.74	1.47	1.16	1.00
3	1.79	1.64	1.44	1.26	1.04	0.78	0.50
4	1.51	1.37	1.23	1.08	0.90	0.66	0.33
6	1.38	1.17	1.06	0.94	0.78	0.55	0.20
8	1.17	1.07	0.98	0.87	0.72	0.50	0.14
10	1.06	1.01	0.93	0.83	0.69	0.47	0.11

Reference

1. Solomon, Frederick, Probability and Stochastic Processes Prentice-Hall, New York, 1987.

Sumitava Chatterjee is a native of Calcutta, India. This paper is a result of his senior year mathematics research project. Frederick Solomon is the head of the mathematics department at Warren Wilson College.

A NOTE ON INTEGRALS INVOLVING MULTIPLE ROOTS

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R. **Euler [1]** gave a method for solving difference equations when the roots of the characteristic polynomial are equal. His idea suggested the following theorem, which can be applied as shown in the examples.

THEOREM. If R is a linear operator and p and q continuous functions with

$$R\left(\frac{p(x)}{q(x)}\right) = f(x) \quad \text{and} \quad R\left(\frac{p(x)}{q(x)+h}\right) = u(x,h)$$

then

$$R\left(\frac{p(x)}{(q(x))^2}\right) = -\lim_{h\to 0}\frac{\partial}{\partial h}u(x,h).$$

Proof. We have

$$R\left(\frac{p(x)}{(q(x))^2}\right) = \lim_{h\to 0} R\left(\frac{p(x)}{(q(x)(q(x)+h)}\right)$$

$$= \lim_{h\to 0} R\left(\frac{1}{h}\left[\frac{p(x)}{q(x)} - \frac{p(x)}{(q(x)+h)}\right]\right)$$

$$= \lim_{h\to 0}\frac{f(x)-u(x,h)}{h}.$$

Applying **L'Hôpital's** Rule to the quantity on the right yields the result.

For example, let us take R to be antidifferentiation, $p(x) = 1$, and $q(x) = 1 + x^2$ and use the theorem to evaluate $\int 1/(1+x^2)^2\,dx$. Here,

$$u(x,h) = \int \frac{1}{x^2+1+h}\,dx = \frac{1}{\sqrt{1+h}}\tan^{-1}\frac{x}{\sqrt{1+h}}$$

(from the well-known formula $\int dx/(x^2+a^2) = (1/a)\tan^{-1}(x/a)$) so

$$\int \frac{1}{(x^2+1)^2}\,dx = -\lim_{h\to 0}\frac{\partial}{\partial h}u(x,h) = \frac{1}{2}\left[\frac{x}{x^2+1} + \tan^{-1}x\right].$$

For a second example, we will **find** the inverse **Laplace** transform of $s/(s^2+a^2)^2$. Recall that the **Laplace** transform of a function $f(t)$ is defined by $\mathcal{L}f(t) = \int_0^\infty f(t)e^{-st}\,dt = F(s)$ and the inverse **transform** $\mathcal{L}^{-1}F(s) = f(t)$ is a linear operator. Also, St $\cos(at) = s/(s^2+a^2)$. Here we have

$$u(t,h) = \mathcal{L}^{-1}\frac{s}{s^2+a^2+h} = \cos\sqrt{a^2+h}\,t.$$

Therefore,

$$\text{St}^{-1}\frac{s}{(s^2+a^2)^2} = \lim_{h\to 0}\frac{t}{2\sqrt{a^2+h}}\sin\sqrt{a^2+h}\,t = \frac{t}{2a}\sin at.$$

The usual textbook procedure for solving this type of problem uses the convolution integral which involves further integration, completely eliminated by this technique.

It is possible to extend the idea to show that

$$R\left(\frac{p(x)}{((q(x))^3}\right) = \lim_{h\to 0}\frac{1}{2}\frac{\partial^2}{\partial h^2}u(x,h).$$

Reference

1. Euler, Russell, A note on a difference equation, this Journal, 9 (1989-94) #8, 530-531.

M. *A.* Khan is Deputy Director of the Research Design and Standards Organisation in *Lucknow*. *His* last paper in this Journal appeared in the Fall, 1993 issue.

PARALLEL CURVES AT INFINITY

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Let $\beta(t) = (x(t), y(t))$ denote a smooth curve in R^2. We will say that a curve β_r is ***r-parallel*** to β if

$$\beta_r(t) = \beta(t) - rN(t)$$

where $N(t)$ denotes the counterclockwise unit normal vector of β at the point $(x(t), y(t))$; i. e.,

$$N(t) = \frac{(-y'(t), x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}.$$

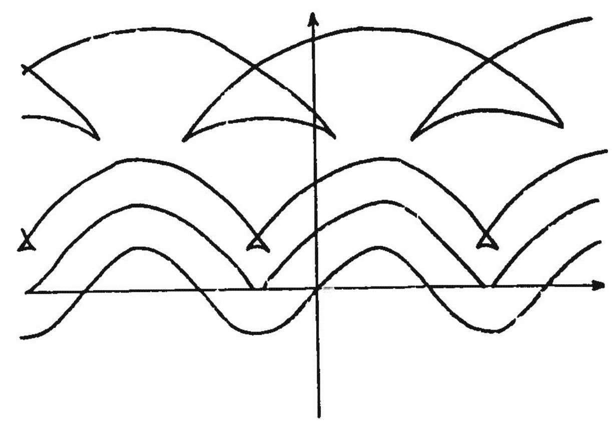

Figure 1

Curves r-parallel to y = sin x

Figure 1 shows some r-parallel curves to $y = \sin x$. More information about r-parallel curves can be found in Do **Carmo** [1, p. **47]** and Pita-Ruiz **[2]**. In **[2]**, Pita-Ruiz considers several basic properties (curvature, length, simplicity, ...) of r-parallel curves and then gives a detailed study of **r-parallel curves** to parabolas and ellipses.

In this note we will show that at infinity all r-parallel **curves** are **circles.** We will prove

THEOREM. Let $\beta(t)$ be a smooth curve in R^2 with r-parallel curve $\beta_r(t)$. Let $\beta_\infty(t) = \lim_{r\to\infty} \beta_r(t)$. Then $|\beta_\infty(t)| = C$ for some constant C.

Proof. **First,** we differentiate

$$x_r(t) = x(t) - \frac{ry'(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{1/2}}$$

and

$$y_r(t) = y(t) + \frac{rx'(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{1/2}}$$

to get, after **simplification,**

$$x_r' = x' + \frac{(x''y' - x'y'')rx'}{\left((x')^2 + (y')^2\right)^{3/2}}$$

and

$$y_r' = y' + \frac{(x''y' - x'y'')ry'}{\left((x')^2 + (y')^2\right)^{3/2}}$$

Hence,

$$\lim_{r\to\infty} \frac{y_r'}{x_r'} = \frac{y' + \dfrac{(x''y' - x'y'')ry'}{((x')^2 + (y')^2)^{3/2}}}{x' + \dfrac{(x''y' - x'y'')rx'}{((x')^2 + (y')^2)^{3/2}}} = \frac{y'}{x'}.$$

On the other hand,

$$\lim_{r\to\infty} \frac{y_r}{x_r} = \lim_{r\to\infty} \frac{y + rx'((x')^2 + (y')^2)^{-1/2}}{x - ry'((x')^2 + (y')^2)^{-1/2}} = -\frac{x'}{y'}.$$

Hence,

$$\frac{y_\infty'}{x_\infty'} = -\frac{x_\infty}{y_\infty} \quad \text{or} \quad x_\infty x_\infty' + y_\infty y_\infty' = 0 .$$

Therefore,

$$x_\infty^2 + y_\infty^2 = C$$

for some constant C, and the proof is **complete.**

References

1. Do Carmo, **Manfredo** P., Differential Geometry and Surfaces, **Prentice-**Hall, 1976.

2. Pita-Ruiz, Claudio, **Curvas** paraletas. Submitted.

The authors prepared this paper under the supervision of Professor Javier ***Gomez-Calderon*** while they were sophomores ***at Penn*** State ***University—New Kensington*** campus.

CHAPTER REPORTS, LETTERS, ETC.

Letter to the Editor

In "A note on isomorphic factor groups" in this Journal (9 (1989-94) **#10, 676-677**), K. Muthuvel gave an example of an infinite group G with normal subgroups H and K such that $G/H \cong G/K$ but H is not isomorphic to K. It may be of interest to note that an example with finite groups can be found in Problem 337 in Crux *Mathematicorum* (proposed by R. B. Killgrove, 9 (1983) #4, 113; solution by Curtis Cooper, 10 (1984) #7, 230, who credited K. R. McLean, "When isomorphic groups are not the same", Mathematical Gazette 57 (1973), 207-208). Let $G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = b^2 = 1$ and $ab = ba$, $H = \{1, a, a^2, a^3\}$, and $K = \{1, a^2, b, a^2b\}$. Then $G/H \cong G/K \cong C_2$ while $H \cong C_4$ and $K \cong C_2 \otimes C_2$. Also, if $H = \{1, a^2\}$ and $K = \{1, b\}$, then $H \cong K \cong C_2$ while $G/H \cong C_2 \otimes C_2$ and $G/K \cong C_4$.

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Chapter Reports

Professor **PREM** N. **BAJAJ,** faculty adviser to the KANSAS GAMMA Chapter (Wichita State University), reports that the chapter furnished volunteer tutors for help sessions in courses up to differential equations, sponsored the **Putnam** Examination, met (almost) biweekly for informal discussions on topics of mathematics and general education, and heard several talks (including "Mathematics as viewed by Plato", "Applications of spherical triangles", "A queueing model related to the ballot problem", and "Tilings of the plan—euclidean and non-euclidean"). Twenty-five new members were initiated.

The INDIANA **EPSILON** Chapter (Saint Mary's College) participated in several service activities in 1993-94, reports Professor **JOANNE SNOW.**

Members of the chapter represented the mathematics department at the college's Fall Day on campus for prospective students, presented a program at two eighth-grade mathematics classes at a South Bend middle school, and assisted with the administration of the Indiana High School **Mathematics** Contest. At the Christmas Bazaar, the chapter sold personalized picture frames as a fund-raising activity and made a gift of $75 to the mathematics department to be used towards the purchase of Math Horizons.

Pythagorean Theorem Proved

XUMING CHEN (University of Alabama, **Tuscaloosa)** provides yet another proof of the Pythagorean Theorem. The area of a right triangle with sides a and b and hypotenuse c is $ab/2$. On the other hand, the area of any triangle is $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$. When we expand this and set the two expressions equal, we have

$$\frac{ab}{2} = \frac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$$

Simplify this and it **turns** into $(a^2 + b^2 - c^2)^2 = 0$, which is the Pythagorean Theorem.

Two questions arise. The first is, are proofs of the Pythagorean Theorem that do not use the idea of area better than those that do not, and if so why? The second is, what happens when you equate $(ab\cos\theta)/2$, the area of any triangle (where θ is the angle included between the sides with lengths a and b) to the other expression for area and start doing algebra?

Letters to the Editor

There follow some identities. The first two are familiar, but the next four were new to me.

$1\cdot(1 + 2 + \cdots + n) = n(n+1)/2.$

$2\cdot(1^3 + 2^3 + \cdots + n^3) = n^2(n+1)^2/2.$

$3\cdot(1^5 + 2^5 + \cdots + n^5) + 1\cdot(1^3 + 2^3 + \cdots n^3) = n^3(n+1)^3/2.$

$4\cdot(1^7 + 2^7 + \cdots + n^7) + 4\cdot(1^5 + 3^5 + \cdots + n^5) = n^4(n+1)^4/2.$

$$5\cdot(1^9 + 2^9 + \cdots + n^9) + 10\cdot(1^7 + 2^7 + \cdots + n^7) + 1\cdot(1^5 + 2^5 + \cdots + n^5) = n^5(n+1)^5/2.$$

$$6\cdot(1^{11} + 2^{11} + \cdots + n^{11}) + 20\cdot(1^9 + 2^9 + \cdots + n^9) + 6\cdot(1^7 + 2^7 + \cdots + n^7) = n^6(n+1)^6/2.$$

The pattern for the coefficients is now apparent:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

and so on.

I would be interested in knowing whether any reader has seen these relations before.

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What were called "generalized random walks" in [1] are better called 'trinomial random walks" as in [2] where the relationship to **Legendre** polynomials is shown. When the probabilities of steps 1, 0, and –1 depend on the present position there is a connection with continued fractions **[3].** For random walks in n dimensions see [4] which generalizes [5].

References

1. Neal, D. K. and Maynard, L. (1994). Generalized random walks: areas and lengths, $\pi\mu\epsilon J$ 9, 654-661.

2. Good, I. J. (1958). **Legendre** polynomials and trinomial random walks. ***Proc.*** Cam. Philos. ***Soc.*** 54, 39-42.

3. Good, I. J. (1958). Random walks and analytic continued fractions. Proc. Cam. Philos. ***Soc.*** 54, 43-47.

4. Foster, F. G. and Good, I. J. (1953). On a generalization of Pdlya's random-walk theorem. Q. **J.** Math. Oxford **(2)4,** 120-126.

5. Pdlya, G. (1921). **Über** eine Aufgabe der Wahrscheinlichkeitsrechnung betrefend die **Ihrfahrt** in **Strassennetz,** Math. ***Annalen*** 84, 149-160.

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Editorial

I hope that readers will not be annoyed by my taking up space in the Journal with this bit of personal opinion. Writing editorials is traditionally one of the perquisites of an editor.

What is it that brings people to mathematics? I will answer that question, to my satisfaction at least, and point out some implications for the teaching of mathematics.

Let us pull some biographies of mathematicians off the shelf and see what they have to say on the subject. Lucidly, such information always come near the beginning of a biography so the search for it is not difficult. Here is an excerpt from an autobiography, that of Paul Halmos [2, p. **25]:**

> Analytic geometry was great. It began with a description of Descartes' great victory, the insight that made algebra out of geometry and vice versa. The conic sections were defined three ways: as plane sections of cones, in terms of foci and directrices, and by quadratic equations. The conies had eccentricities and latera recta (and we were expected to remember that that's the plural of **latus** rectum). There were also **lemniscates** and **limaçons,** and most phenomena had three-dimensional versions (but that got short shrift near the end of the course). **...** I thought it was all great stuff and in my letters home I wrote enthusiastically about my mathematics course; it was a beauty, I said.

That passage contains a highly significant absence, which will be pointed out later.

Constance Reid on David **Hilbert** [3, p. 6]:

> Already he had found the school subject which was perfectly suited to his mind and a source of inexhaustible delight. He said later that mathematics **first** appealed to him because it was **"bequem"—*easy,*** effortless. It required no memorization. He could always figure it out again for himself.

S. M. Ulam on himself [5, pp. 20-21]:

> I had mathematical curiosity very early. My father had in his library a wonderful series of German paperback **books—*Reklam,*** they were called. One was Euler's Algebra. I looked at it when I was perhaps ten or eleven, and it gave me a mysterious feeling. The symbols looked like magic signs; I wondered whether one day I could understand them. This probably contributed to the development of my mathematical curiosity. I discovered by myself how to solve quadratic equations. ... In general, the mathematics classes did not satisfy me. They were dry, and I did not like to have to memorize certain formal procedures. I preferred reading on my own. ... I also read a book by the mathematician Hugo **Steinhaus** entitled What *is* and What Is Not Mathematics and a Polish translation of **Poincaré's** wonderful La Science et ***l'Hypothèse,*** La Science et la ***Méthode,*** La Valeur de Science, and his ***Dernière Pensées.*** Their literary quality, not to mention the science, was admirable. **Poincaré** molded portions of my scientific thinking.

Bruce Berndt on Hans Rademacher [1]:

> At the age of 18 he had a curiosity for mathematics, the natural sciences, foreign languages, and philosophy. It was to the latter area that Rademacher devoted his primary initial attention. However, he enjoyed the lectures of two of Felix **Klein's** assistants, **Erich** Hecke and Hermann Weyl, and eventually he turned to mathematics.

Norbert Wiener on himself [6, pp. 22-23]:

> His course [G. H. Hardy's] was a delight to me. My previous adventures into higher mathematics had not been completely satisfying, because I sensed gaps in many of the proofs which I was unwilling to disregard-and correctly too, as it later turned out, for the gaps were really there and they should have disturbed not only me but my former teachers. Hardy, however, led me through the complicated logic of higher mathematics with such clarity and in such detail that he resolved these difficulties as we came to them and gave me a real sense of what is necessary for a mathematical proof.

Finally, Constance Reid on **Jerzy** Neyman [4, p. **21]:**

> It was after this incident that Professor C. K. Russyan, a Polish professor whose lectures on the theory of functions he was attending, spoke to him about a new integral from France-quite different from the classical integral of **Riemann.** Because such prerequisites as set theory were not offered at the university, Russyan had not been able to treat the Lebesgue integral in his lectures. He now suggested that Neyman investigate it on his own. ...
>
> Neyman looks in his library for **Henri** Lebesgue's ***Leçons sur l'intégration*** *et* la recherche des ***fonctions*** primitives and quickly locates it. ...
>
> From the first page, the ***Leçons*** enthralled him. When he is asked for other recollections of this period of his life, he squirms a little in his chair, embarrassed by his inability to produce any.
>
> 'The war," he apologizes. "Wiped out everything. And then there was Mr. Lebesgue ... "
>
> But it seems to have been "Mr. Lebesgue" who wiped out everything, even the war, as far as Neyman was concerned. The year 1914-15, which saw defeat after defeat for the Russian army on its western front, was a year of intense self-education for the young man. His first task was to familiarize himself with set theory, and he found that subject fascinating. Especially intriguing was **Ernst Zermelo's** axiom of choice. He remembers how he walked the icy streets of Kharkov trying to explain the concept to his friend Leo Hirschvald.

I will not bore you with any more excerpts, though I could find many more. (And they are not boring: the history of mathematics is fascinating, as can be the biographies of mathematicians. It is too bad that there are not more such biographies, but the general public is not very interested in them nor, for reasons that could be gone into in another editorial, are many mathematicians.) The thread that runs through the excerpts, those given above as well as those that could be cited, is that it is the subject *of* mathematics that is the attraction.

The subject: the matter, the ideas, the glory, the beauty, the excitement, the exhilaration—mathematics! Mathematics is splendid, and that is all there is to it. It is the most marvelous and magnificent activity the human mind carries out. I will not go so far as to say that doing mathematics is the purpose of the human race, but I think I could make a case for that, at least as good a case as is made for many other purported purposes.

The implication for teaching is that it is the subject that is important and not the person who teaches it. Note that Halmos does not even mention whoever it was who was in charge of his analytic geometry course. It was all great stuff, he said. Not the class, not the teacher, it-the subject. Hilbert found the subject easy: no memorization was needed, you can always figure things out. Ulam didn't like his mathematics classes much—too much memorization-but books showed enabled him to find out about it. Rademacher and Wiener heard lectures, lectures so good that the subject came through. Neyman also: **Lebesgue's** book showed him what it was about.

So, clearly, the first duty of a teacher of mathematics is to be clear, to allow what is important, namely the subject of mathematics, to shine through. Teachers of mathematics are best when they are most transparent. If the blackboard can be read through them, so much the better. Lectures, clear lectures, have persuaded people to go into mathematics.

Unfortunately, clarity is not valued as highly as it might be in these days of student evaluations. Students in general cannot distinguish how well the subject is being presented. How could they, since they have never seen it before? My doctoral supervisor, William J. **LeVeque,** once taught a class that I attended during which he proved a theorem whose proof I knew already. His proof was absolutely brilliant: different, better, and clearer than any I had seen before. It was a stunning performance. I was amazed, but the rest of the class took it as a matter of course. They might even have written on their student evaluations, "This teacher is no good, all he does is lecture and he doesn't always let us out on time." But there were no student evaluations then.

In these days, caring for students as people seems to be more important than being a transparent transmitter of the wonders of mathematics. While it is certainly nicer to be cared about as a person than not, there does not seem to be much relation between caring and transmitting. I have encountered many students who will praise their former mathematics teachers to the skies—wonderful people, they really cared about their students-but who

will demonstrate, conclusively, that they did not learn even a fraction of what they were supposed to learn. Are there lawyers who really care about their clients but who lose every case, and still get clients? Maybe there are. There is something to be said for caring more about mathematics than about feelings, but these are not the days to say such things.

Using graphing calculators, writing mathematical autobiographies, **working** in groups, doing projects on computers and writing up laboratory reports—these things are, I guess, all very well. They may do good. If, however, they get in the way, if they clog up the channel carrying mathematics to the learner, then they are not good. The teacher of mathematics, more than anything else, should be clear. Do I make myself clear?

References

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2. Halmos, Paul, I Want to be a Mathematician, Springer-Verlag, New York, 1985.
3. Reid, Constance, Hilbert, Springer-Verlag, New York, 1970.
4. Reid, Constance, ***Neyman—from*** Life, Springer-Verlag, 1982.
5. Ulam, S. M., Adventures ***of*** a Mathematician, **Scribner's,** New York, **1976.**
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The Applications of Mathematics

The following is from Don Quixote de la ***Mancha,*** by Miguel de Cervantes. The Don is speaking on the subject of knight-errantry:

"It is a science," said Don Quixote, "that comprehends in itself all or most of the sciences in the world, for he who professes it must be a jurist, and must know the rules of justice, distributive and equitable, so as to give to each one what belongs to him and is due to him. He must be a theologian, so as to be able to give a clear and distinctive reason for the Christian faith he professes, wherever it may be asked of him. He must be a physician, and above all a herbalist, so as in wastes and solitudes to know the herbs that have the property of healing wounds, for a knight-errant must not go looking for some one to cure him at every step. He must be an astronomer, so as to know by the stars how many hours of the night have passed, and what clime and quarter of the world he is in. He must know mathematics, for at every turn some occasion for them will present itself to him; and, putting it aside that he must be adorned with all the virtues, cardinal and theological, to come down to minor particulars, he must, I say, be able to swim as well as Nicholas or Nicolao the Fish could ...

Did you notice that the Don was able to give specific uses for all his sciences until he came to mathematics, where he lamely trailed off? Was this due to Quixote's addle-headedness? Or was it the best that Cervantes could do? Of course, things have changed since the days of Cervantes (1547-1616). Or have they?

Mathematics Education in the Nineteenth Century

From War and Peace, Book 1, Chapter 25, by Leo Tolstoy (. 328-1910).

The princess gave a wrong answer.

'Well now, isn't she a fool!" shouted the prince, pushing the book aside and turning sharply away; but rising immediately, he paced up and down, lightly touched his daughter's hair and sat down again. He drew up his chair and continued to explain.

"This won't do, Princess; it won't do," said he, when Princess Mary, having taken and closed the exercise book with the next day's lesson, was about to leave: "Mathematics are most important, madam! I don't want to have you like our silly ladies. Get used to it and you'll like it," and he patted her cheek. "It will drive all the nonsense out of your head."

Of course, such things would not be said nowadays.

Mathacrostics

Solution to Mathacrostic 38, by Theodor Kaufman (Spring, 1994).

Words:

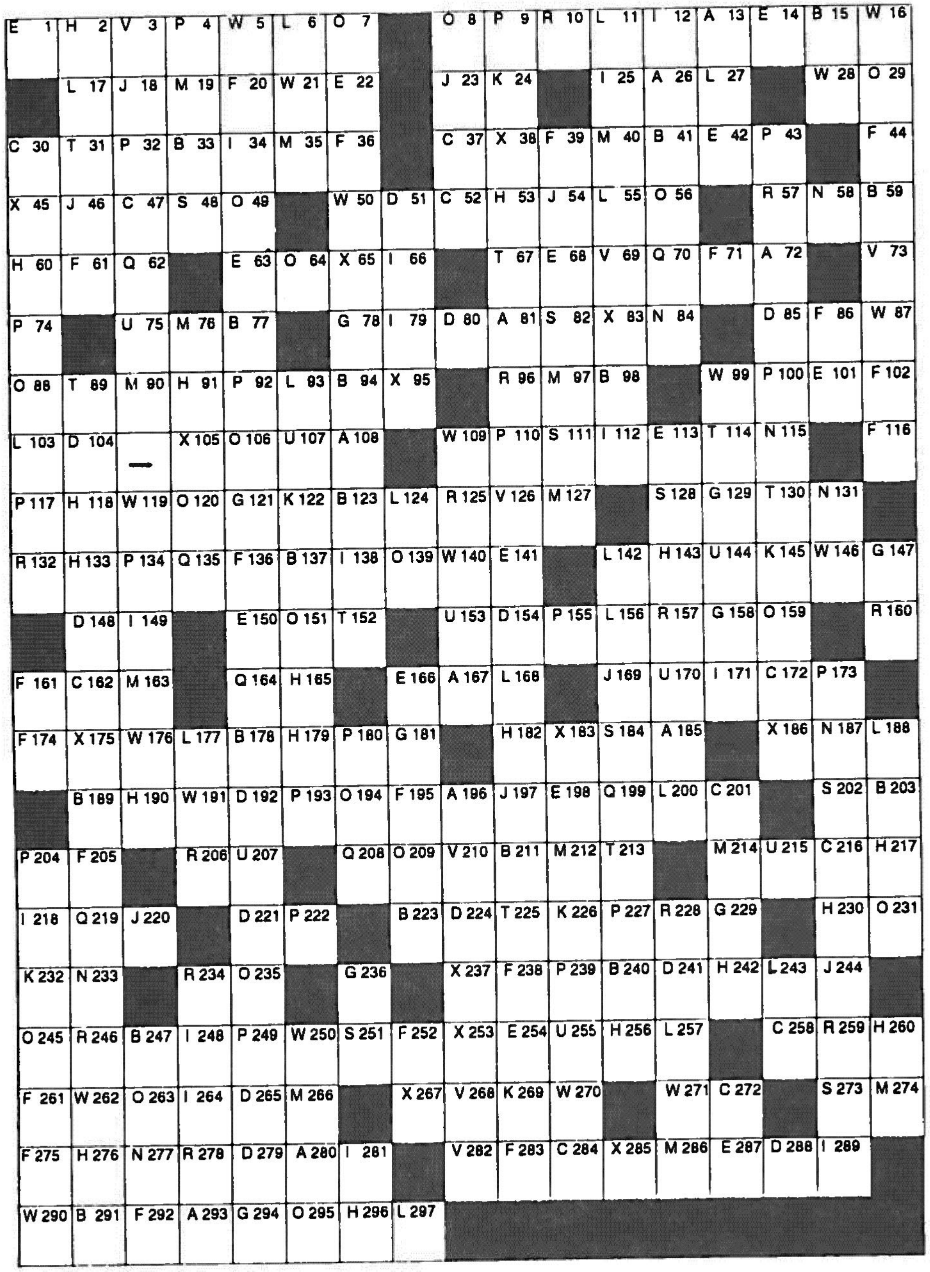
A. tac-au-tac	K. linear	U. itineracy
B. utter loss	L. dispatches	V. Napier's bones
C. friar's cap	M. integral equation	W. first bass
D. tophetic	N. scission	X. ovals of Cassini
E. Euclid's algorithm	O. pseudosphere	Y. radius vector
	P. lithochromy	Z. Marsh test
F. vinculum	Q. abysmal	*. ably
G. interpolate	R. yurt	#. thither
H. sinner	S. ophiouride	@. intertwist
I. unending decimal	T. fractal	+. obstetrics
J. afterthought		§. nuke

Author and title: Tufte, Visual Display *of Information.*

Quotation: The use of abstract, non-representational pictures to show numbers is a surprisingly recent invention, perhaps due to the diversity of visual-artistic, empirical-statistical, and mathematical skills required. This occurred long after such triumphs of ingenuity as logarithms, cartesian coordinates, the calculus, and the basics of probability theory.

Solvers: THOMAS BANCHOFF, Brown University, JEANETTE BICKLEY, St. Louis Community College, PAUL S. BRUCKMAN, Highwood, Illinois, CHARLES R. DIMMINIE, St. Bonaventure University, VICTOR G. FESER, University of Mary, ROBERT FORSBERG, Lexington, Massachusetts, META HARRSEN, Durham, North Carolina, HENRY LIEBERMAN, Waban, Massachusetts, CHARLOTTE MAINES, Rochester, New York, DON PFAFF, University of Nevada—Reno, NAOMI SHAPIRO, Piscataway, New Jersey, and STEPHANIE SLOYAN, Georgian Court College. Late solution to #37 by VICTOR G. FESER, University of Mary.

The puzzle was marred by a large number of errors, some serious, for which the editor was wholly responsible and for which he apologizes. Mathacrostic 39, constructed by ROBERT FORSBERG, follows on the next three pages. It has been more carefully checked, so its number of errors may be asymptotically closer to zero. To be listed as a solver, send your solution. to the editor.



A. His filter design features a **passband** with ripples of equal amplitude
280 81 167 108 293 185 72 26 196 13

B. A polyhedron with thirty-two faces
223 59 41 240 33 98 137 211 77 189
123 178 203 247 291 94 15

C. Name borne by a large number of Oriental kings, soldiers, and statesmen
47 52 272 258 162 30 201 216 172 284
37

D. Location on a curve where it does not cross itself and there is a smoothly turning tangent (2 wds)
221 80 51 288 224 279 192 104 154 148
85 265 241

E. One of **Georg** Cantor's alephs (2 wds)
166 42 254 1 22 287 14 101 198 150
68 141 113 63

F. Greek mathematician, 5th cent. B.C., banished for revealing that $\sqrt{2}$ is irrational (3 wds)
136 252 39 283 161 205 71 174 292 20
275 102 44 36 116 238 86 195
61 261

G. Family of Phanariot Greeks, active in the liberation of Greece
181 78 147 129 294 236 158 229 121

H. A puzzle for the mathematical world since 1637 (3 wds)
165 2 143 296 242 230 217 256 91 53
182 60 190 276 179 118 133 260

I. The distance an airplane travels for each revolution of its propeller (2 wds)
79 149 171 281 112 25 264 138 248 34
12 66 289 218

J. Expresses verbal encouragement (2 wds)
18 23 46 244 220 169 54 197

K. Useful fiber obtained from an African palm tree
122 269 24 145 226 232

L. Of them you can say that $P(A \cap B) = P(A)\,P(B)$ (2 wds)	93 156 142 188 177 168 55 17 27 6
	103 257 11 200 243 124 297
M. A yam-making machine (2 wds)	163 40 286 97 76 35 127 266 214 90
	19 274 212
N. _________ of Newton, a curve with equation $xy = ax^3 + bx^2 + cx + d$, $a \neq 0$.	115 187 58 277 131 233 84
O. Percy Bridgman's field (3 wds)	151 106 159 231 88 263 295 49 56 209
	29 139 120 64 235 7 245 194 8
P. Phenomenon relating current, magnetic field, and temperature in a metal strip (2 wds)	155 43 4 100 239 227 74 173 193 9
	204 110 180 134 222 249 117 32 92
Q. Person of European/Indian ancestry	62 219 208 135 70 199 164
R. Arab poet, 860-940, *Iqd al-Farid* (3 wds)	157 234 246 96 206 57 10 259 160 132
	278 228 125
S. Controversial, ineffective cure for cancer touted some time back	202 48 184 251 111 273 128 82
T. Metaphorical expressions used in Skaldic poetry	130 152 213 114 89 31 67 225
U. Mountain range in Asiatic Russia	207 144 255 215 75 107 153 170
V. It makes DOS-type computers more like Macintoshes	268 73 69 210 126 3 282
W. "He looked again and saw it was _______" (Lewis Carroll, *Sylvie and Bruno*) (4 wds)	271 109 50 140 99 270 290 5 87 146
	250 16 262 119 21 191 176 28
X. Chinese city, Shanhsi province (old sp.)	175 186 83 253 285 183 38 267 105 237
	45 65 95

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1995.

Correction

820. [Fall 1993] *Proposed by William Moser, McGill University, Montreal, Quebec, Canada.*

Let $a_{n,k}$ $(0 \le k < n)$ denote the number of n-bit strings (sequences of 0's and 1's of length n) with exactly k occurrences of two consecutive 0's. Show that

$$a_{n,k} = \sum_{r=2k}^{n} \binom{r-k}{k}\binom{n-r+1}{r-k},$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ **if** $0 \le k \le n$ **and** $\binom{n}{k} = 0$ **otherwise.**

Editor's comment. **The problem *is* unclear as to how many pairs of zeros you count when there are three or more consecutive zeros. The proposer's intent was that three or more consecutive zeros are not allowed; consider strings where zeros appear (between ones) only singly or in pairs.**

There is no such restriction on the ones; any number of consecutive ones can appear any place. The lower limit on the sum ($r = 0$) was not wrong, but terms for $r < 2k$ vanish. The upper limit, which the editor had supplied, was changed from $n - 1$ to n to avoid errors for some small values of k. Finally, the editor inadvertently put $n - r - 1$ in place of $n - r + 1$ in the summation. Other than those errors, the problem was fine.

Problems for Solution

836. Proposed by the editor.

Solve this base ten holiday addition alphametic. Since the coming year 1995 is an odd year, you are asked to find that solution such that A is an odd digit.

$$\begin{array}{r} MANY \\ NEW \\ \underline{NEW} \\ YEARS \end{array}$$

837. ***Proposed by J. Sutherland Frame, Michigan State University, East Lansing, Michigan.***

Evaluate in closed form the integral

$$I = \int_{-a}^{a} \sqrt{a^2 - x^2}\, \ln|z - x|\, dx, \qquad |z| < a.$$

838. ***Proposed by Florentin Smarandache, Phoenix, Arizona.***

Let $d_n = p_{n+1} - p_n$, $n = 1, 2, 3, \ldots$, where p_n is the nth prime number. Find the nature of the series

$$\sum_{n=1}^{\infty} \frac{1}{d_n}.$$

839. ***Proposed by James Chew, North Carolina Agricultural and Technical State University, Greensboro, North Carolina.***

a) A ticket buyer chooses a number from 10 through 99 inclusive. A number is randomly picked as winner. If, for example, **63** is the winner, then each ticket number **63** that has been sold is awarded $\$A$. The reversal ticket number **36** is awarded $\$5$. That is, the second prize goes to any ticket

with both digits correct, but in the wrong order. The third prize of $\$C$ is paid to any ticket that contains at least one of the correct digits, **e.g. 33, 43,** 34, 65, 76, etc. A ticket can win only one prize and prizes are not shared. If you have bought 5 tickets numbered 63, you win **$5A.** Find the fair price for a ticket.

*b) Find the fair price for the game of part (a) if prizes are shared. That is, the ticket seller pays out a total of at most $\$(A + B + C)$ in winnings for any one game, $\$A$ is shared among all winning tickets (number **63**), if any. Then $\$B$ is shared among all holders of second prize tickets (number 36). Finally, all thud prize winners share the one amount $\$C$.

840. ***Proposed by Seung-Jin Bang, Seoul, Republic of Korea.***

Prove that, for $n \ge 2$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \ln n + \frac{n+1}{2n}.$$

841. ***Proposed by Seung-Jin Bang, Seoul, Republic of Korea.***

For given real constants a, b, and c, let $\{a\}$ be the sequence satisfying the recursion equation $na_n = aa_{n-1} + ba_{n-2}$ for $n > 1$, $a_0 = 0$, $a_1 = c$. Find the sum of the series

$$\sum_{n=0}^{\infty} a_n.$$

842. ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.***

Let x_i be a positive real number for $i = 1, 2, \ldots, n$. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)\left(\sum_{i=1}^{n} (x_i)^2\right)^{1/2} \ge n\sqrt{n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

843. ***Proposed by Bill Correll, Jr., student, Denison University, Granville, Ohio.***

Let $s(n)$ denote the sum of the binary digits of the positive integer n. Find a value for c so that

$$\sum_{n=1}^{c} \frac{1}{s(n)} = \frac{2342173}{5544}$$

*844. Proposed by Bill **Correll**, Jr., student, **Denison** University, Granville, Ohio.*

If F_n denotes the nth Fibonacci number ($F_1 = F_2 = 1$ and $F_{k+2} = F_k + F_{k+1}$ for k a positive integer), evaluate

$$\sum_{i=1}^{\infty} \frac{\binom{n}{k} F_n}{2^{n+k}}.$$

*845. Proposed by Russell Euler, Northwest Missouri State University, **Maryville,**, Missouri.*

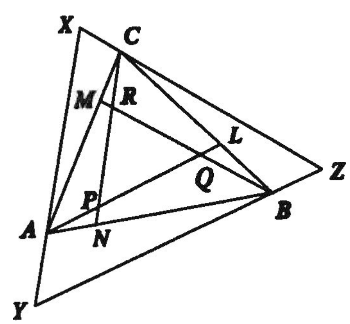
Let A, B, and C be subsets of $U = \{1, 2, 3, \ldots, m\}$. An ambitious student wants to prove that if $A \subseteq B$, then $A \cup (B \cap C) = (A \cup C) \cap B$ for all A, B, and C. Express in closed form the number of specific cases the student must consider.

*846. Proposed **by M.** A. Khan, **Lucknow, India.***

Let N, L, M be points on sides AB, BC, CA of a given triangle ABC such that

$$0 < \frac{AN}{AB} = \frac{BL}{BC} = \frac{CM}{CA} = k < 1.$$

Let AL meet CN at P and BM at Q, and let BM and CN meet at R. Draw lines parallel to CN through A, parallel to AL through B, and parallel *to* BM through C. Let XYZ be the triangle formed by these three new lines. Prove that:



a) Triangles ABC, PQR, and XYZ have a common centroid, and

b) If the areas of triangles PQR, ABC, and XYZ are in geometric progression, then $k = \sqrt{3} - 1$.

***847.** Proposed **by Dmitry** P. **Mavlo,** Moscow, Russia.*

From the SYMP-86 Entrance Examination: The midline of an isosceles trapezoid has length L and its acute angle is a. Determine the trapezoid's area, if it is known that a circle can be inscribed in the trapezoid-

*848. Proposed **by** Rex H. **Wu, SUNY** Health Science Center, Brooklyn, New **York.***

a) Given a non-trivial group (a group having more than one element) such that, if x, y are any members, then (i) $x \neq y$ implies $x^2 \neq y^2$ and (ii) $xy = y^2x^2$, prove the-group is **abelian** (commutative).

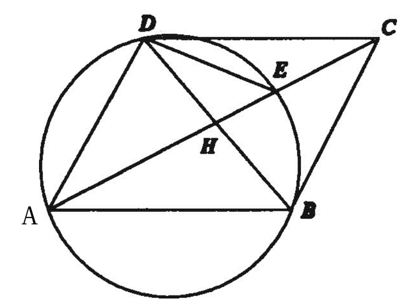
b) Prove part (a) if the term *group* is replaced by *semigroup*.

Solutions

780. [Spring 1992, Fall 1992, Fall **1993]** *Proposed by R. S. Luthar, University of Wisconsin Center, **Janesville,** Wisconsin.*

Let $ABCD$ be a parallelogram with $\angle A = 60°$. Let the circle through A, B, and D intersect AC at E. See the figure. Prove that $BD^2 + AB \cdot AD = AE \cdot AC$.

*Solution by John D. **Moores, Westbrook,** Maine.*



Recall that $BC = AD$, $\cos 60° = 1/2$ and $\cos 120° = -1/2$, and apply the law of cosines to triangle ABD and to triangle ABC, obtaining

$$BD^2 = AB^2 + AD^2 - AD \cdot AB.$$

and

$$AC^2 = AB^2 + AD^2 + AD \cdot AB.$$

Eliminating $AB^2 + AD^2$ between the two equations, obtain

(1) $$AC^2 = BD^2 + 2AD \cdot AB.$$

Let H denote the point of intersection of the chords AE and BD. Then

(2) $$AH \cdot EH = BH \cdot DH.$$

Note that $DH = BH = BD/2$, $AH = AC/2$, and therefore $EH = AE - ACE$. Substitute these values into Equation *(2)* to get

(3) $$AC^2 = 2AC \cdot AE - BD^2.$$

Finally, eliminate AC^2 between equations (1) and (3) to get the desired result

$$BD^2 + AB \cdot AD = AC \cdot AE.$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, *Edmonds*, WA, MARK EVANS, Louisville, KY, LEE LIAN KIM, Messiah College, *Grantham*, PA, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY College at Oneonta, YOSHINOBU MURAYOSHI (2 solutions), Okinawa, Japan, BOB PRIELIPP, University *of Wisconsin-Oshkosh*, HARRY SEDINGER, St. *Bonaventure* University, NY, PAUL D. SHOCKLEE, Memphis, TN, KENNETH M. WILKE, Topeka, KS, REX H. WU, Brooklyn, NY, SAMMY YU and JIMMY YU (2 solutions), University *of* South Dakota, Vermillwn, and the PROPOSER.

In the original figure *it* appeared that ABCD was a rhombus. BILL CORRELL, JR., *Denison* University, *Cincinnati*, OH, BARBARA J. LEHMAN, St. Peter's College, Jersey City, NJ, and DAVID INY, *Westinghouse* Electric Corporation, Baltimore, MD, solved the problem under that assumption. *Bruckman, Murayoshi*, and *Wilke* all pointed out the error *in* the original statement *of* the problem.

801. [Spring 1993, Spring **1994]** Proposed by Norman *Schaumberger*, Bronx Community College, Bronx, New York.

If a, b, and c are real numbers, then prove that

$$e^a(a-b) + e^b(b-c) + e^c(c-a) \geq 0$$
$$\geq e^a(c-a) + e^b(a-b) + e^c(b-c).$$

III. Solution by Murray S. *Klamkin*, University *of Alberta*, Edmonton, Alberta, Canada.

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be given non-increasing sequences of real numbers. Let $z_1, z_2, \ldots, z_n$ be any permutation of the $\{y_i\}$ sequence. Then it follows from a known **[1]** rearrangement inequality that

$$\text{(1)} \qquad x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq x_1z_1 + x_2z_2 + \cdots + x_nz_n$$
$$\geq x_1y_n + x_2y_{n-1} + \cdots + x_1y_1.$$

A physical intuitive example of these inequalities is loading one side of a seesaw with differently weighted people. To get the largest **turning** moment we put the lightest person closest to the fulcrum, the next lightest next, and so on, putting the heaviest person furthest out. To get the smallest turning moment, we **reverse** the order. Any other arrangement will produce an intermediate turning moment.

Since a ≥ b ≥ c implies that $e^a \geq e^b \geq e^c$, we apply Inequality (1) to get

$$ae^a + be^b + ce^c \geq a_1e^a + b_1e^b + c_1e^c \geq ce^a + be^b + ae^c,$$

where a_1, b_1, c_1, is any permutation of a, b, c. The given inequalities and others as well now follow.

Reference

1. G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, London, 1934, p. 261.

802. [Spring 19931 Proposed by Murray S. *Klamkin*, University *of* Alberta, Edmonton, Alberta, Canada.

Let a and b be positive real numbers. Determine the maximum value of

$$f(x) = (a-x)\left(x + \sqrt{x^2 - b^2}\right)$$

over all real x with $x^2 \geq b^2$. A non-calculus solution is requested.

II. Solution by the proposer.

First we assume that the maximum occurs for x > 0, so we can let $x = b\cosh\theta$ ($\theta \geq 0$). This substitution reduces the given function $f(x)$ to

$$\text{(1)} \qquad 2\left[\frac{1}{2}be^{\theta}\left(a - \frac{1}{2}be^{\theta}\right)\right] - \frac{1}{2}b^2.$$

Since the quantity in the brackets is of the form $u(a-u)$, which takes on its maximum when $u = a/2$, then Expression (1) takes its maximum $(a^2 - b^2)/2$ when $be^{\theta} = a$. This requires, however, that a ≥ b.

If b > a ≥ b/2, then the maximum of **(1)**, b(a − b), occurs at the smallest value of be^{θ}, when θ = 0.

If b/2 > a, we can get values of $f(x)$ greater than b(a − b) by taking x negative. Letting $x = -b\cosh\theta$ ($\theta \geq 0$), $f(x)$ now reduces to

$$\text{(2)} \qquad -be^{-\theta}\left(a + \frac{1}{2}be^{-\theta}\right) - \frac{1}{2}b^2.$$

Clearly Expression (2) is increasing and approaches $-b^2/2$ as $\theta \to \infty$. Since $-b^2/2 > b(a-b)$, there is no maximum for this case, only a **lim** sup.

810. [Fall **1993]** Proposed by Alan Wayne, *Holiday*, Florida.

In the following base eight multiplication, the digits of the two multipliers have been replaced in a one to one manner by letters:

$$(I)(CLUED) = 437152.$$

Restore the digits. Similarly replace 437152 to find out who might have said "I clued."

Solution by Paul S. ***Bruckman***, Everett, Washington.

With the aid of a TI-60 calculator, which readily converts between octal and decimal notation, we find that

$$(437152)_8 = (147050)_{10},$$

which factors into

$$2 \cdot 5^2 \cdot 17 \cdot 173.$$

Hence I must be either 2 or 5. If I = 2, then

$$(CLUED)_8 = (73525)_{10} = (217465)_8,$$

which is excluded since CLUED is a **5-digit** number in base 8. Thus I = 5 and hence

$$(CLUED)_8 = (29410)_{10} = (71342)_8.$$

We have the unique solution I = 5, CLUED = 71342, so 437152 = EUCLID. Euclid couldn't have said "I clued," however, since such a phrase would have been all English to him (as it is all Greek to me).

Also solved by CHARLES ASHBACHER, Cedar Rapids, *IA*, **SEUNG-JIN** BANG, Seoul, Korea, SCOTT H. BROWN, Auburn University. *AL*, **JAMES** E. CAMPBELL, ***Arnold***, MO, **CAVELAND MATH** GROUP, Western ***Kentucky*** University, Bowling Green, BILL **CORRELL**, JR., ***Denison*** University, Cincinnati, OH, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, *NJ*, MARK EVANS, Louisville, *KY*, VICTOR G. **FESER**, University of Mary, Bismarck, ND, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, *CA*, CARL **LIBIS**, Idaho State University, ***Pocatello***, YOSHINOBU **MURAYOSHI**, ***Okinawa***, Japan, BOB PRIELIPP, University of ***Wisconsin-Oshkosh***, **MOHAMMAD** P. SHAIKH, University of ***Missouri***, Columbia, DAVID S. SHOBE, New Haven, *CT*, SONNY W, University of Illinois, ***Urbana***, **KENNETH** M. WILKE, Topeka, *KS*, **REX** H. WU, *SUNY* Health Science Center, Brooklyn, *NY*, SAMMY YU and JIMMY YU,

University of South Dakota, ***Vermillion***, and the PROPOSER.

811. [Fall **1993**] Proposed by Tom Moore, Bridgewater State College, ***Bridgewater***, Massachusetts.

If $a < b < c$ are positive integers with $\gcd(a,b) = 1$ and $a^2 + b^2 = c^2$, then (a, b, c) is called a primitive Pythagorean *triple* (PPT). If both a and c are primes, then we shall call it a prime PPT (P^3T).

a) If (a, b, c) is a P^3T, deduce that $b = c - 1$.

b) Find all P^3Ts in which a and c are

i) twin primes.

ii) both Mersenne primes.

iii) both Fennat primes.

iv) one a Mersenne, the other a **Fermat** prime.

Solution by Bob ***Prielipp***, University of ***Wisconsin-Oshkosh***, ***Oshkosh***, Wisconsin.

a) If (a, b, c) **is** a P^3T, then $a^2 + b^2 = c^2$ where a is a prime number and $a < b < c$. Thus

$$a^2 = c^2 - b^2 = (c - b)(c + b).$$

This factorization of a^2 cannot be $a = c - b$ and $a = c + b$, so we must have $1 = c - b$ and $a^2 = c + b$. It follows that $b = c - 1$ and $b = (a^2 - 1)/2$. Hence

$$(a, b, c) = \left(a, \frac{a^2 - 1}{2}, \frac{a^2 + 1}{2}\right).$$

b) i) If a and c are twin primes, then

$$2 = c - a = \frac{a^2 + 1}{2} - a = \frac{(a - 1)^2}{2}$$

Thus $a = 3$ and $(3, 4, 5)$ is the only solution.

ii) If a and c are both Mersenne primes, then $a = 2^m - 1$ and $c = 2^n - 1$ where m and n are both prime numbers with $n > m \geq 2$. Hence

$$2^n - 2^m = c - a = \frac{(a - 1)^2}{2} = \frac{(2^m - 2)^2}{2},$$

making $2^{m-1}(2^{n-m} - 1) = (2^{m-1} - 1)^2$. This is impossible since the right side of this last equation is odd and the left side is even because $m \geq 2$, so there are no P^3Ts in which a and c are both Mersenne primes.

iii) If a and c are both Fennat primes, then $a = 2^{2^m} + 1$ and $c = 2^{2^n} + 1$ where n and m are integers with $n > m \geq 0$. Hence

$$2^{2^n} - 2^{2^m} = c - a = \frac{(a-1)^2}{2} = \frac{(2^{2^m})^2}{2},$$

making $2^{2^n - 2^m} - 1 = 2^{2^m - 1}$. It follows that $m = 0$ and $n = 1$. **Thus** (3, 4, **5)** is the only solution.

iv) If a is a Mersenne prime and c is a Fennat prime, then $a = 2^m - 1$ and $c = 2^{2^n} + 1$ where m is a prime and n is a positive integer because $c > a \geq 3$. Thus

$$2^{2^n} - 2^m + 2 = c - a = \frac{(a-1)^2}{2} = \frac{(2^m - 2)^2}{2},$$

making

$$2^{2^n - 1} - 2^{m-1} + 1 = (2^{m-1} - 1)^2.$$

Hence

$$2^{2^n - 1} = (2^{m-1} - 1)^2 + (2^{m-1} - 1) = (2^{m-1} - 1)2^{m-1}.$$

The only solution occurs for $m = 2$ and $n = 1$, that is, (3, 4, 5).

If a is a Fennat prime and c is a Mersenne prime, then $a = 2^{2^m} + 1$ and $c = 2^n - 1$ where n is a prime number and m is a nonnegative integer with $n > 2^m$ because $c > a$. **Thus**

$$2^n - 2^{2^m} - 2 = c - a = \frac{(a-1)^2}{2} = \frac{(2^{2^m})^2}{2},$$

making

$2^{n-1} - 2^{2^m - 1} - 1$ (22--)2.

It follows that

$2^{n-1} - 1 = (2^{2^m - 1})^2 - 2^{2^m - 1} = 2^{2^m - 1}$(22--1 $+ 1)$,

which is impossible because the left side of the equation is odd and the right side is even. Hence there are no solutions to this part.

Also solved by PAUL S. BRUCKMAN, *Everett,* WA, JAMES E. CAMPBELL, *Arnold,* MO, BILL CORRELL, JR., *Denison University, Cincinnati,* OH, CHARLES *R.* **DIMINNIE,** *St. Bonaventure University, NY,*

VICTOR G. **FESER,** *University of Mary, Bismarck, ND,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania, RICHARD* I. **HESS,** *Rancho Palos Verdes, CA,* **CARL LIBIS,** *Idaho State University, Pocatello,* **HENRY** *S.* **LIEBERMAN,** *Waban,* MA, DAVID E. **MANES,** *SUNY College at Oneonta,* LAWRENCE SOMER, *Catholic University of America, Washington., D.C.,* **KENNETH** *M.* **WILKE,** *Topeka, KS,* REX H. WU, *SUNY Health Science Center, Brooklyn, NY, and the* PROPOSER.

812. [Fall **1993]** *Proposed by George P. Evanovich, Saint Peter's College, Jersey City, New Jersey.*

If $n \geq 2$ is a positive integer, prove that

$$\sum_{j=1}^{n} \cos\left(\frac{2j\pi}{n}\right) = \sum_{j=1}^{n} \sin\left(\frac{2j\pi}{n}\right) = 0.$$

I. Solution by John F. Putz, Alma College, Alma, Michigan.

Let $\left(\cos\frac{2j\pi}{n}, \sin\frac{2j\pi}{n}\right)$, $j = 1, 2, \ldots, n$, represent $n \geq 2$ unit forces acting on a particle. Since they are equally spaced around the unit circle centered at the particle, these forces are in equilibrium, and hence, the resultant is the **zero** vector. That is,

$$\sum_{j=1}^{n} \left(\cos\frac{2j\pi}{n}, \sin\frac{2j\pi}{n}\right) = (0, 0).$$

Therefore,

$$\sum_{j=1}^{n} \cos\frac{2j\pi}{n} = \sum_{j=1}^{n} \sin\frac{2j\pi}{n} = 0.$$

II. *Solution by Sammy Yu, age 14, and Jimmy Yu, age 12, special students at University of South Dakota, Vermillion, South Dakota.*

Let 9 = W n. **Then** $e^{in\theta} = e^{2\pi i} = 1$. Calculating the sum of the geometric series, we have

$$\sum_{j=1}^{n} e^{ij\theta} = \frac{e^{i\theta}(1 - e^{in\theta})}{1 - e^{i\theta}} = 0.$$

Since

$$\sum_{j=1}^{n} e^{ij\theta} = \sum_{j=1}^{n} (\cos j\theta + i \sin j\theta) = \sum_{j=1}^{n} \left(\cos\frac{2j\pi}{n} + i \sin\frac{2j\pi}{n}\right),$$

we obtain

$$\sum_{j=1}^{n} \cos \frac{2j\pi}{n} = \mathrm{Re}\left(\sum_{j=1}^{n} e^{ij\theta}\right) = 0$$

and

$$\sum_{j=1}^{n} \sin \frac{2j\pi}{n} = \mathrm{Im}\left(\sum_{j=1}^{n} e^{ij\theta}\right) = 0.$$

Also solved by SEUNG-JIN *BANG, Seoul, Korea, CHRISTOPHER N. BAUNACH (2 solutions). University of* ***Louisville,*** *KY PAUL S. BRUCKMAN, Everett, WA, BILL* **CORRELL,** JR., ***Denison*** *University, Cincinnati, OH, MIGUEL AMENGUAL COVAS,* ***Cala Figuera, Mallorca,*** *Spain, RUSSELL EULER, Northwest Missouri State University, Maryville, JOHN DOUGLAS* **FAIRES,** ***Iffley,*** *Oxford, England,* **JAYANTHI** *GANAPATHY, University of Wisconsin-Oshkosh,* **STEPHEN** *I.* **GENDLER,** *Clarion University of Pennsylvania, RICHARD* I. **HESS,** *Rancho* ***Palos Verdes,*** *CA, FRANCIS C. LEAKY,* Saint **Bonaventure** University, *NY,* CARL LIBIS, *Idaho State University,* ***Pocatello,*** *HENRY S. UEBERMAN,* ***Waban, MA,*** *DAVID E.* **MANES,** ***SUNY College*** *at* ***Oneonta,*** **YOSHINOBU MURAYOSHI,** *Okinawa, Japan, BOB* **PRIELIPP,** *University of* ***Wisconsin-Oshkosh,*** *HENRY* J. *RICARDO,* ***Medgar*** *Evers College, Brooklyn,* ***NY,*** **MOHAMMAD** *P. SHAIKH, University of Missouri, Columbia,* **KENNETH** *M. WILKE, Topeka, KS,* **REX** *H. WU, SUNY Health Science Center, Brooklyn,* ***NY,*** *and the PROPOSER. Most solvers used the fact that the sum of the n nth roots of a complex number is zero.*

813. [Fall **1993]** *Proposed by the late Jack* ***Garfunkel,*** *Flushing, New York.*

Given a triangle ABC with sides a, b, c and a **triangle** $A'B'C'$ with sides $(b+c)/2$, $(c+a)/2$, $(a+b)/2$. Prove that $r' \geq r$, where r and r' are the **inradii** of triangles ABC and $A'B'C'$ respectively.

I. *Solution by Sammy* ***Yu*** *and Jimmy* ***Yu,*** *special students at University of South Dakota,* ***Vermillion,*** *South Dakota.*

It is well known that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{(b+c-a)(c+a-b)(a+b-c)}{8s}}$$

In triangle $A'B'C'$, let the sides be $a' = (b+c)/2$, $b' = (c+a)/2$, and $c' = (a+b)/2$ and the **semiperimeter** $s' = (a'+b'+c')/2$. Then $s' = s$ and

$$r' = \sqrt{\frac{(s'-a')(s'-b')(s'-c')}{s}} = \sqrt{\frac{abc}{8s}}.$$

Therefore it is sufficient to prove that

$$(b+c-a)(c+a-b)(a+b-c) \leq abc.$$

Since we have

$$(b+c-a)(c+a-b) = c^2 - (a-b)^2 \leq c^2,$$
$$(c+a-b)(a+b-c) = a^2 - (b-c)^2 \leq a^2,$$
$$(a+b-c)(b+c-a) = b^2 - (c-a)^2 \leq b^2,$$

multiply these inequalities side for side and take the positive square root to get the desired inequality, with equality holding if and only if triangle ABC is equilateral.

II. *Comment* ***by*** *Murray S.* ***Klamkin,*** *University of Alberta, Edmonton,* ***Alberta,*** *Canada.*

This same problem by the same author has already appeared as problem number *4303* in *School Science and Mathematics* **91** *(1991) 390.* **A** more general inequality among others appears in my paper "Notes on Inequalities Involving Triangles or Tetrahedrons," ***Publ. Electrotehn.*** *Ser. Mat.* ***Fiz.*** *Univ.* ***Beograd,*** No. *330-337, 1970,* pp. *1-15. A* portion of this paper is quoted in the former reference.

Also solved ***by*** *SCOTT H. BROWN, Auburn University, AL, PAUL S.* **BRUCKMAN,** *Everett, WA, WILLIAM* **CHAU,** *New* ***York,*** *NY, BILL CORRELL, JR.,* ***Denison*** *University, Cincinnati, OH, MIGUEL AMENGUAL COVAS, Cala* ***Figuera, Mallorca,*** *Spain, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City,* ***NJ,*** *HENRY S.* **LIEBERMAN,** ***Waban,*** *MA, DAVID E.* **MANES,** *SUNY College at* ***Oneonta,*** **YOSHINOBU** *MURAYOSHI, Okinawa, Japan, BOB PRIELIPP, University of* ***Wisconsin-Oshkosh,*** *and the PROPOSER.*

814. [Fall **1993]** *Proposes* ***by*** *Nathan* ***Jaspen,*** *Stevens Institute of Technology,* ***Hoboken,*** *New Jersey*

For any decimal integer n, prove that n^5 and n end in the same digit, that n^6 and n^2 end in the same digit, that n^7 and n^3 end in the same digit, and so forth.

I. *Solution by Jimmy Yu, special student at University of South Dakota, **Vermillion,** South Dakota.*

It is sufficient to prove that 10 divides $n^{m+4} - n^m$, where m is a positive integer. We see that

$$n^{m+4} - n^m = n'''(n^2 + 1)(n^2 - 1) = n^m(n^2 - 4 + 5)(n^2 - 1)$$
$$= 5n'''(n^2 - 1) + n^{m-1}(n - 2)(n - 1)n(n + 1)(n + 2).$$

Since n^m and $n^2 - 1$ have opposite parities, then $5n'''(n^2 - 1) \equiv 0 \pmod{10}$. Because $(n - 2)(n - 1)n(n + 1)(n + 2)$ is a product of five consecutive **integers,** it if divisible by $5! = 120$. Consequently $n^{m+4} - n^m \equiv 0 \pmod{10}$ and the desired result follows.

II. *Solution by Alma College Problem Solving Group, Alma College, Alma, Michigan.*

The expression

$$n^{k+5} - n^{k+1} = n^k(n^5 - n) = n^k(n - 1)n(n + 1)(n^2 + 1),$$

where $n > 0$ and $k \ge 0$ are integers, is even because it has consecutive integer factors. By **Fermat's** Little Theorem, $n^5 - n \equiv 0 \pmod{5}$. Thus $n^{k+5} - n^{k+1}$ is a multiple of 10 and the theorem follows.

Also solved by AARDVARK PROBLEM SOLVING GROUP, *Trenton State College, **NJ,*** CHARLES ASHBACHER, *Cedar Rapids,* IA, **SEUNG-JIN** BANG, *Seoul, Korea,* **MARIO** R. BORDOGNA, *Allegheny College, **Meadville,** PA,* PAUL S. BRUCKMAN, ***Everett,** WA,* **CAVELAND MATH** GROUP, *Western Kentucky University, Bowling Green,* **JAMES** E. CAMPBELL, ***Arnold,** MO,* BILL CORRELL, **JR.,** ***Denison** University, **Cincinnati,** OH,* CHARLES R. **DIMINNIE,** *St. Bonaventure University, **NY,*** RUSSELL EULER, *Northwest Missouri State University, Maryville,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* **MARK** EVANS, *Louisville, **KY,*** VICTOR G. FESER, *University of Mary, Bismarck, ND,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD *I.* HESS, *Rancho **Palos** Verdes, **CA,*** FRANCIS C. **LEARY,** Saint Bonaventure University, **NY,** CARL **LIBIS,** *Idaho State University, Pocatello,* HENRY S. **LIEBERMAN,** *Waban,* MA, DAVID E. **MANES,** ***SUNY** College at **Oneonta,*** BOB **PRIELIPP,** *University of Wisconsin-Oshkosh,* HENRY **J.** RICARDO, ***Medgar** Evers College, Brooklyn, **NY,*** **MOHAMMAD** P. SHAIKH, *University of Missouri Columbia,* LAWRENCE **SOMER,** *Catholic **University of America,** Washington., D.C.,* KENNETH M. **WILKE,** *Topeka, **KS,*** REX *H.* **WU** *(2 **solutions),** SUNY Health Science Center, Brooklyn, **NY,*** SAMMY **YU,** *University of South Dakota, **Vermillion,** and the* PROPOSER.

815. [Fall *19931 Proposed **by** Bill **Correll,** Jr., Cincinnati, **Ohio.***

Let $[x]$ denote the greatest integer not exceeding x. Solve for x:

$$\left[\frac{x+1}{2}\right]\left[\frac{x+2}{3}\right]\left[\frac{x+3}{4}\right] = 819.$$

I. Solution by George P. Evanovich, Saint Peter's College, Jersey City, New Jersey.

Since $819 = (13)(9)(7)$, then $13 \le (x + 1)/2 < 14$, $9 \le (x + 2)/3 < 10$, and $7 \le (x + 3)/4 < 8$, or equivalently, $25 \le x < 27$.

II. *Solution by Paul S. Bruckman, Everett, Washington.*

Let

$$f(x) = \left[\frac{x+1}{2}\right]\left[\frac{x+2}{3}\right]\left[\frac{x+3}{4}\right].$$

For any 6, $0 < 6 < 1$, we find that $f(25 - 6) = (12)(8)(6) < 819$, $f(25) = (13)(9)(7) = 819$, $f(27 - 6) = (13)(9)(7) = 819$, and $f(27) = (14)(9)(7) > 819$. Note also that $f(x)$ is nondecreasing for all positive x. The equation $f(x) = 819$ cannot have negative solutions since $f(x) \le 0$ if $x < 0$. Finally, $f(0) = 0$. Therefore, the solution to the equation **is** $25 \le x < 27$.

Also solved by AARDVARK PROBLEM SOLVING GROUP, *Trenton State **College, NJ,** ALMA* COLLEGE PROBLEM SOLVING GROUP, MI, **JAMES** E. CAMPBELL, ***Arnold,** MO,* **CAVELAND MATH** GROUT, *Western Kentucky University, Bowling Green,* **MARK** EVANS, ***Louisville, KY,*** **VICTOR** G. FESER, *University of Mary, Bismarck, ND,* **JAYANTHI GANAPATHY,** *University of Wisconsin-Oshkosh,* STEPHEN **GOODMAN,** *University of Dayton, OH,* RICHARD *I.* HESS, *Rancho **Palos Verdes,** CA,* HENRY S. **LIEBERMAN,** *Waban,* MA, DAVID E. **MANES,** SUNY College at *Oneonta*, YOSHINOBU **MURAYOSHI**, Okinawa, Japan, **HENRY** J. RICARDO, Medgar *Evers* College, Brooklyn, *NY*, DAVID S. SHOBE, New Haven, *CT*, LAWRENCE SOMER, Catholic University of America, Washington., D.C., SONNY W, University of Illinois, *Urbana*, KENNETH M. WILKE, Topeka, *KS*, **REX** H. WU, *SUNY* Health Science Center, Brooklyn, *NY*, SAMMY YU and JIMMY YU, University of South Dakota, *Vermillion*, and the PROPOSER. Partial solutions by WILLIAM CHAU, New *York*, *NY*, CARL **LIBIS**, Idaho State University, Pocatello, and **MOHAMMAD** P. **SHAIKH**, University of Missouri, Columbia.

*816. [Fall **1993**] Proposed by Robert C. *Gebhardt*, Hopatcong, New Jersey.

a) From the integers 1, 2, 3, ..., n, a state **lottery** selects at random k numbers ($k < n$). A person who had previously chosen at random m of those k numbers ($m \le k$) is a winner.' Find the probability of being a winner.

b) The Tri-State Megabucks (Maine, New Hampshire, and Vermont) tickets cost $1 each. A participant selects m = 6 numbers out of n = 40 and is a winner if all six numbers match the k = 6 numbers the game selects. The winnings are paid in 20 equal annual installments. How large does the pot have to be before a ticket is worth $1?

I. Solution by Richard **I.** *Hess*, Rancho Palos Verdes, *California*.

a) There are $\binom{k}{m}$ ways to pick m of k numbers and $\binom{n}{m}$ ways to pick m of the n numbers. So the probability P of picking m of the k the state selected is $P = \binom{k}{m} \Big/ \binom{n}{m} = \dfrac{k!(n-m)!}{n!(k-m)!}$.

b) For n = 40 and m = k = 6, we get $P = 6!34!/40! = 1/3838380$. So, if the winnings W were paid immediately, then a ticket would be worth $1 when the pot reached $3,838,380. The present value PV of winnings paid in 20 equal installments (first installment immediately) is

$$PV = \frac{W}{20}[1 + (1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-19}]$$
$$= \frac{W}{20}\cdot\frac{1-(1+i)^{-20}}{1-(1+i)^{-1}} = \frac{W}{20}\cdot\frac{1+i-(1+i)^{-19}}{i}.$$

Now we must have $P \cdot PV = 1$ and the following table gives, for various

values of i, the value of W that yields PV = $3,838,380:

i	*W*	*i*	*W*
.04	$ 5,431,437	.08	$ 7,239,768
.05	$ 5,866,696	.10	$ 8,197,358
.06	$ 6,314,103	.12	$ 9,176,386

That is, when money is worth i rate of interest per year, then a ticket is worth $1 when the pot has reached W.

II. Comment on part (b) by Mark Evans, *Louisville*, Kentucky.

The question as worded cannot be answered because the size of the pot is just one of several variables. The following are variables that must be considered:

i = interest rate used to **reflect** the time value of money,
C = money **carried** over from previous no-winner periods,
S = new money for the current period,
$T(W)$ = tax on winnings per year, given there are W winners,
$TI(W)$ = tax rate on interest, given there are W winners,
$T1$ = tax on $1 bet,
$P1$ = probability the bettor wins, and
$P(W)$ = probability there are W winners including the bettor.

Now $P1 = \binom{40}{6}^{-1} = \dfrac{1}{3838380}$ and

$$P(W) \approx P1 \cdot P(W-1) \approx P1 \cdot \frac{e^{-S \cdot P1} \cdot (S \cdot P1)^{W-1}}{(W-1)!}$$

Note that Poisson is an excellent approximation for binomial in this case. Let $A = 1 + i[1 - TI(W)]$. Now we can write an expression for the expected gain E from a bet of $1:

$$E \approx \sum_{W=1}^{S} \frac{C+S}{W} \cdot P(W) \cdot \frac{1}{20} \cdot \frac{1-A^{-20}}{1-A^{-1}} \cdot (1 - T(W)) + T1 - 1.$$

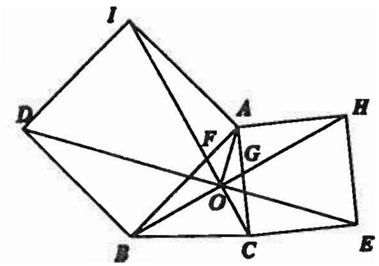
Practically, the terms of the summation are insignificant **after** the first several. The value of E is highly dependent on C and S and the ratio C/S.

If C is small, E < 0 for reasonable choices of interest and tax variables. Assuming i = 0.06, $T1$ = 0, and a typical tax structure, E = 0 when C = **$10,000,000** and S = **$4,000,000**, or if C = **$12,500,000** and S = $8,000,000.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, CHARLES ASHBACHER, Decisionmark, Cedar Rapids, *IA*, PAUL S. BRUCKMAN, Everett, WA, **JAMES** E. CAMPBELL, ***Arnold***, MO, **MARK** EVANS, ***Louisville***, Kentucky, **MOHAMMAD** P. **SHAIKH**, University of Missouri, Columbia, and DAVID S. SHOBE, New Haven, CT.

817. [Fall 19931 Proposed by Andrew ***Cusumano***, Great Neck, New ***York***.

In the accompanying figure squares ***CEHA*** and ***AIDB*** are erected externally on sides *CA* and AB of triangle ABC. Let BH meet IC at O and AC at G, and let ***CI*** meet AB at F.



a) Prove that points ***D***, O, and *E* are collinear.

b) Prove that angles HOE, EOC, AOH, and AOI are each **45°**.

c) If ACB is a right angle, then prove that ***E***, F, and G are collinear.

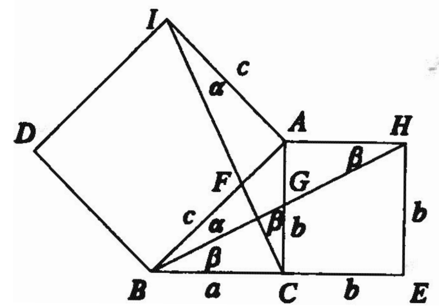
Find an "elegant" proof for parts (a) and (b), both of which are known to be true whether the squares are erected both externally or both internally (see The American Mathematical Monthly, problem **E831**, vol. 56, 1949, pp. 406-407). Part (c) is a delightful result that also should be known, but appears to be more difficult to prove.

Solution by Sammy Yu and Jimmy ***Yu***, special students at University of South Dakota, ***Vermillion***, South Dakota.

a) and b) By SAS, ΔBAH • MAC. Hence $\angle AHB = \angle ACI$ so that A, H, C, O are concyclic. This circle also passes through the fourth vertex E of the square AHEC. Consequently, LHOC = $\angle HAC = 90°$. Therefore, $IC \perp HB$. Also $\angle AOH = \angle ACH = 45°$, $\angle HOE = \angle HAE = 45°$, and $\angle EOC = \angle EAC = 45°$. Similarly, A, O, ***B***, ***D***, I are concyclic and $\angle AOI = 45°$. Result (b) follows. Now, since $\angle IOD = \angle EOC = 45°$, then ***D***, O, E are **collinear.** The desired result (a) thus follows.

c) Since ***ABAH*** = ΔIAC and $AH \perp$ BC, we let a = $\angle ABH = \angle AIC$

and we have $\beta = \angle ACI = \angle AHB = \angle HBE$. Let a, b, and c denote the lengths of BC, *CA*, and AB respectively. Then AF = c tan a,



$$FB = AB - AF = c - c\tan\alpha,$$

so that $AF/FB = (\tan\alpha)/(1 - \tan\alpha)$.

Now $\tan\beta = HE/BE = b/(a + b)$, and $\tan A = a/b$. Also

$$\tan\alpha = \tan(90° - A - \beta) = \cot(A + \beta) = \frac{1}{\tan(A + \beta)}$$

$$= \frac{1 - \tan A\tan\beta}{\tan A + \tan\beta} = \frac{b^2}{a^2 + ab + b^2}.$$

Therefore,

$$(1) \qquad \frac{AF}{FB} = \frac{b^2}{a(a + b)}.$$

From the figure we find that

$$(2) \qquad \frac{BE}{EC} = \frac{a + b}{b}.$$

Since $\Delta GBC \sim \Delta GHA$,

$$(3) \qquad \frac{CG}{GA} = \frac{BC}{AH} = \frac{a}{b}.$$

Since the product of the right sides of equations (1), (2), and (3) is 1, the desired result follows by the converse part of **Menelaus'** theorem applied to triangle ABC and Menelaus points ***E***, F, G.

Also solved by PAUL S. BRUCKMAN, Everett, ***WA***, **HENRY S. LIEBERMAN**, ***Waban***, ***MA***, **WILLIAM** H. PHRCE, Rangeley, ME, ***and the*** PROPOSER.

818. [Fall 1993]** Proposed by ***Dmitry P. Mavlo, Moscow, ***Russia***. **From the SYMP-86** Entrance Examination, solve the **inequality**

$$\frac{1}{x^3 - x} \le \frac{1}{|x|}.$$

Solution by Alma College Problem Solving Group, Alma College, Alma, Michigan.

Clearly we cannot have $x^3 - x = 0$, so x cannot be –1, 0, or 1. The inequality is satisfied when $x^3 - x < 0$, that is, when $x < -1$ or $0 < x < 1$. When $-1 < x < 0$, $|x| = -x$ and the given inequality reduces to $-x \le x^3 - x$, and to $0 \le x^3$, which is false in the stated interval. When $x > 1$, the given inequality reduces to $x \le x^3 - x$, $2x \le x^3$, which is true when $x \ge \sqrt{2}$. Hence the given inequality is true if and only if $x < -1$, $0 < x < 1$, or $x \ge \sqrt{2}$.

Also solved by AARDVARK PROBLEM SOLVING GROUP, Trenton State College, *NJ*, CHARLES ASHBACHER, Decisionmark, Cedar Rapids, *IA*, SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, JAMES E. CAMPBELL, Arnold, MO, **CAVELAND MATH** GROUP, Western Kentucky University, Bowling Green, BILL CORRELL, JR., *Denison* University, Cincinnati, OH, MASK. EVANS, Louisville, *KY*, **JAYANTHI GANAPATHY**, University of Wisconsin-Oshkosh, STEPHEN I. GENDLER, Clarion University of *Pennsylvania*, STEPHEN GOODMAN, University of Dayton, OH, RICHARD I. **HESS**, Rancho Palos Verdes, *CA*, HEATHER LECCEARDONE, St. *Bonaventure* University. *NY*, CARL LIBIS, Idaho State University, Pocatello, HENRY S. LIEBERMAN, Waban, MA, DAVID E. **MANES**, *SUNY* College at Oneonta, YOSHINOBU **MURAYOSHI**, Okinawa, Japan, MICHAEL R. **PINTER**, Belmont University, Nashville, TN, JENNIFER R. POWELL, *Hendrix* College, *Conway*, *AR*, BOB **PRIELIPP**, University of Wisconsin-Oshkosh, HENRY **J.** RICARDO, Medgar Evers College, Brooklyn, *NY*, **MOHAMMAD** P. SHAIKH, University of Missouri, Columbia, DAVID S. SHOBE, New Haven, *CT*, PATRICIA **WHALEN** and MICHAEL T. DANIELSON, Allegheny College, Meadville, PA, REX H. WU, SUNY Health Science Center, Brooklyn, *NY*, and SAMMY YU and JIMMY YU, University of South Dakota, Vermillion. Partial solution by SAM HOUSKER, Drake University, *Des Moines*, *IA*.

819. Proposed by Morris *Katz*, Macwahoc, Maine.

Evaluate the integral

$$\int \ln x \sin^{-1} x \, dx.$$

Solution by Henry J. Ricardo, Medgar Evers College, Brooklyn, New *York*.

Using integration by parts we find that

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

Clearly we must have $0 < x \le 1$ for the integral to be defined. Now use integration by parts with $u = \ln x$ and $dv = \sin^{-1} x \, dx$ on the given integral to get

$$\begin{aligned} I &= \int \ln x \sin^{-1} x \, dx \\ &= (x \sin^{-1} x + \sqrt{1 - x^2})\ln x - \int \sin^{-1} x \, dx - \int \frac{\sqrt{1 - x^2}}{x} \, dx \\ &= (x \sin^{-1} x + \sqrt{1 - x^2})(\ln x - 1) - \int \frac{\sqrt{1 - x^2}}{x} \, dx. \end{aligned}$$

This last integral can be evaluated by making the substitution $x = \sin\theta$, $0 < \theta \le \pi/2$:

$$\begin{aligned} \int \frac{\sqrt{1 - x^2}}{x} \, dx &= \int \frac{\cos^2\theta \, d\theta}{\sin\theta} = \int (\csc\theta - \sin\theta) \, d\theta \\ &= -\ln(\csc\theta + \cot\theta) + \cos\theta + C \\ &= -\ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) + \sqrt{1 - x^2} + C. \end{aligned}$$

It follows that

$$I = \left(x \sin^{-1} x + \sqrt{1 - x^2}\right)(\ln x - 1) + \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) - \sqrt{1 - x^2} + C.$$

Also solved by AARDVARK PROBLEM SOLVING GROUP, Trenton State College, *NJ*, RACHEL ABBEY, Alma College, MI, ALMA COLLEGE PROBLEM SOLVING GROUP, MI, ZIV A. ARIE and ROLAND A. ZOROTAC, Allegheny College, Meadville, PA, JASON **BANDLOW**, Alma College, *MI*, SEUNG-JIN BANG, Seoul, Korea, JEFF BEANE, Alma College, MI, SCOTT H. BROWN, Auburn University, *AL*, PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., ***Denison University, Cincinnati, OH***, **RUSSELL EULER**, ***Northwest Missouri State*** University, Maryville, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, **MARK** EVANS, Louisville, *KY*, **JAYANTHI**

GANAPATHY, University of Wisconsin-Oshkosh, LAURA GILBO, Alma College, MI, DEREK HANDZO, Alma College, MI, RICHARD I. HESS, Rancho Palos Verdes, CA, CHAD HUSBY, Alma College, MI, HEATHER LECCEARDONE, St. Bonaventure University, NY, CARL LIBIS, Idaho State University, Pocatello, HENRY S. LIEBERMAN, Waban, MA, PETER A. LINDSTROM, North Lake College, Irving, TX, DAVID E. MANES, SUNY College at Oneonta, WILLIAM D. MCINTOSH, Central Methodist College, Fayette, MO, KARYN MROCZKOWSKI, Alma College, MI, MICHAEL R. PINTER, Belmont University, Nashville, TN, HENRY J. RICARDO (second solution), STEVEN VANCE, Alma College, MI, REX H. WU, SUNY Health Science Center, Brooklyn, NY, SAMMY YU and JIMMY YU, University of South Dakota, Vermillion, and the PROPOSER.

821. [Fall 1993] Proposed by Zeev Barel, Hendrix College, Conway, Arkansas.

Problem B-2 at the fifty-second annual William Lowell Putnam Mathematical Competition (1991) stated: Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y) \text{ and } g(x+y) = f(x)g(y) - g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

In fact, one can do a little more under the same hypothesis. Prove that there exists a real number k such that $f(x) = \cos kx$ and $g(x) = \sin kx$ for all x.

I. Solution by David E. Manes, SUNY College at Oneonta, Oneonta, New York.

As in the published solution to Problem B-2 [1], begin by differentiating both sides of the two equations with respect to y, obtaining

$$f'(x+y) = f(x)f'(y) - g(x)g'(y) \text{ and } g'(x+y) = f(x)g'(y) - g(x)f'(y).$$

Setting y = 0 yields

$$f'(x) = f(x)f'(0) - g(x)g'(0) = -g'(0)g(x) \tag{1}$$

and

$$g'(x) = f(x)g'(0) - g(x)f'(0) = g'(0)f(x). \tag{2}$$

Note that f' and g' are both differentiable functions on $(-\infty, \infty)$ since f and g are. Let $g'(0) = k$. Then $k \neq 0$ since f and g are non-constant functions.

Now differentiate Equation (1) to get

$$f''(x) = -kg'(x) = -k^2 f(x),$$

where the second equality follows from Equation (2). This differential equation has a general solution

$$f(x) = C_1 \sin kx + C_2 \cos kx$$

for some constants C_1 and C_2. Since $f'(0) = 0$, we obtain $C_1 = 0$. Thus $f(x) = C_2 \cos kx$ and so, by (2),

$$g'(x) = kf(x) = C_2 k \cos kx.$$

Since $g'(0) = k$, it follows that $C_2 = 1$ and hence

$$g(x) = \sin kx + C_3$$

for some constant C_3. But $f'(x) = -kg(x)$ and $k \neq 0$ imply that $C_3 = 0$. Accordingly, there is a real number k, namely $k = g'(0)$, such that $f(x) = \cos kx$ and $g(x) = \sin kx$ for all x. Also, of course, the equation $(f(x))^2 + (g(x))^2 = 1$ is valid for all x.

II. Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

This is a known result [2] and holds under weaker conditions. It is also known [2] that even for the single equation

$$f(x-y) = f(x)f(y) + g(x)g(y),$$

the only non-constant continuous solutions are the same as above.

References

1. L. F. Klosinski, G. L. Alexanderson, and L. C. Larson, The Fifty-Second William Lowell Putnam Mathematical Competition, The American Mathematical Monthly, 99 (1992), 715-724.

2. J. Aczel, Lectures on Functional Equations and Their Applications, Academic Press, New York, 1966, 176-180.

Also solved by PAUL S. BRUCKMAN, Everett, WA, CHARLES R. DIMINNIE, St. Bonaventure University, NY, JAYANTHI GANAPATHY, University of Wisconsin-Oshkosh, HENRY S. LIEBERMAN, Waban, MA, and the PROPOSER.

Editorial comment. Unfortunately, neither your editor nor commercial word processors are perfect (although, of course, one of us *comes quite close). Hence the printed copy did not make it clear that $f'(0) = 0$ was intended. We have found that unless a tiny space is inserted between the f and the ', they overprint one another.*

822. [Fall 1993] *Proposed by Stanley **Rabinowitz, MathPro** Press, **Westford,** Massachusetts.*

If α is a root of the equation $x^5 + x - 1 = 0$, then find an equation that has $\alpha^4 + 1$ as a root.

I. *Solution by Francis C. Leary, Saint Bonaventure University, Saint Bonaventure, New York.*

Clearly, $\alpha \neq 0, 1$. Observe that $\alpha^5 + \alpha = \alpha(\alpha^4 + \alpha) = 1$, whence $\alpha = 1/(\alpha^4 + 1)$. Therefore,

$$\left(\frac{1}{\alpha^4 + 1}\right)^5 + \left(\frac{1}{\alpha^4 + 1}\right) - 1 = 0.$$

Clearing denominators yields $1 + (\alpha^4 + 1)^4 - (\alpha^4 + 1)^5 = 0$, so that $\alpha^4 + 1$ is a root of

$$x^5 - x^4 - 1 = 0.$$

This is a special case of the general result: α a **zero** if $f(x)$ implies that α^{-1} is a zero of $g(x) = x^n f(1/x)$, where n is the degree of f. The polynomial g is called the *reciprocal polynomial of f*. If $f(x) = a_n x^n + \cdots + a_1 x + a_0$, then $g(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$. See, for example, Theorem 36 in [1].

II. *Solution by Murray S. **Klamkin,** University **of Alberta,** Edmonton, Alberta, Canada.*

More generally, let α be a root of the equation $x^n P(x^m) - a = 0$, where P is a polynomial, and m, n are **nonnegative** integers. Then an equation which has $t = \alpha^m + \beta$ as a root is gotten from

$$x^{mn} P(x^m)^m = a^m,$$

(which is also satisfied by α) or

$$(t - \beta)^n P(t - \beta)^m = a^m.$$

III. *Solution by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Since an equation that has $\alpha^4 + 1$ as a root is $x - (\alpha^4 + 1) = 0$, the

problem was not posed correctly. Perhaps the proposer wanted the equation to be a polynomial with integer coefficients.

*IV. Solution by Bob **Prielipp,** University of **Wisconsin-Oshkosh, Oshkosh,** Wisconsin.*

Since $\alpha(\alpha^4 + 1) - 1 = 0$, then $\alpha x - 1 = 0$ has $\alpha^4 + 1$ as a root.

Reference

1. **Pless,** *Introduction to the Theory of Error Correcting Codes,* 2nd ed, **Wiley-Interscience,** New York, 1989.

Also solved by **AVRAHAM** ADLER, *Monsey, **NY,*** CHARLES ASHBACHER, ***Decisionmark,** Cedar Rapids, **IA,*** **SEUNG-JIN** BANG, *Seoul, Korea,* PAUL S. BRUCKMAN, *Everett, WA,* **CAVELAND** MATH GROUP, *Western **Kentucky** University, Bowling Green,* BILL CORRELL, JR., ***Denison** University, **Cincinnati,** OH,* MIGUEL **AMENGUAL** COVAS, ***Cala Figuera, Mallorca,** Spain,* GEORGE P. **EVANOVICH,** *Saint Peter's College, Jersey **City, NJ,*** **JAYANTHI GANAPATHY,** *University of **Wisconsin-**Oshkosh,* STEPHEN I. **GENDLER,** *Clarion University **of Pennsylvania,*** RICHARD I. **HESS,** *Rancho Palos **Verdes, CA,*** CARL **LIBIS,** *Idaho State University, Pocatello,* HENRY S. LIEBERMAN, *Waban, **MA,*** PETER A. **LINDSTROM,** *North Lake College Irving, **TX,*** DAVID **E. MANES,** ***SUNY** College at **Oneonta,*** THOMAS E. MOORE, *Bridgewater State College, MA,* **WILLIAM** H. PEIRCE, ***Rangeley,** ME,* MICHAEL R. **PINTER,** ***Belmont** University, Nashville, TN,* JENNIFER R. POWELL, ***Hendrix** College, **Conway,** AR,* **MOHAMMAD** P. **SHAIKH,** *University **of Missouri,** Columbia,* DAVID S. SHOBE, *New Haven, **CT,*** KENNETH **M. WILKE,** *Topeka, KS,* **REX** H. WU, ***SUNY Health** Science Center, Brooklyn, **NY,*** SAMMY **YU** *and* JIMMY **YU,** *University of South Dakota, Vermillwn, and the* PROPOSER. *One incorrect solution was received.*

Correction

Rex H. Wu pointed out that, in the solution to Problem 795 in the Fall 1993 issue, the open interval (–1, 1) at the top of page 634 should be, a closed interval **[–1, 1].**

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