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Introduction. The growing interest in chaos and fractal geometry has created a new field of mathematics that can be explored by faculty and undergraduates alike. "Sierpinski triangles" and "Koch's curves" have become common phrases in many mathematics departments across the country. In this paper we review some basic ideas from fractal geometry and generalize the construction of the Sierpinski triangle to what we will call Sierpinski polygons.

The Sierpinski Triangle. In fractal geometry, the well known Sierpinski triangle can be constructed as a limit of sets as follows. We begin with three points, \( x_1, x_2, \) and \( x_3 \) that form the vertices of an equilateral triangle \( A_0. \)

For \( i = 1, 2, \) or \( 3, \) let \( x_i = \left[ \begin{array}{c} a_i \\ b_i \end{array} \right]. \) Let \( \mathbb{R} \) represent the set of real numbers and let \( \mathbb{R}^2 \) be the real plane. For \( i = 1, 2, \) or \( 3, \) we define \( \omega_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by

\[
\omega_i \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} x \\ y \end{array} \right) + \frac{1}{2} \left[ \begin{array}{c} a_i \\ b_i \end{array} \right].
\]

Then \( \omega_i \), when applied to \( A_0 \), contracts \( A_0 \) by a factor of two and then translates the image of \( A_0 \) so that the \( i \)th vertices of \( A_0 \) coincide. Define \( A_{1,i} \) to be \( \omega_i(A_0) \). Then \( A_{1,i} \) is the set of all points for solution and solutions to problems should be sent directly to Clayton W. Dodge, Mathematics Department, 5752 Neville Hall, University of Maine, Orono, Maine 04469.

The Pi Mu Epsilon Journal is published at DePauw University twice a year—Fall and Spring. One volume consists of five years (ten issues) beginning with the Fall 19n9 or Fall 19(n + 1)4 issue, \( n = 4, 5, \ldots, 8. \)
halfway between any point in $A_0$ and $x_i$, or $A_{1,i}$ is a triangle half the size of the original translated to the $i$th vertex of the original. Let $A_1 = \bigcup_{i=1}^3 A_{1,i}$, $A_0$ and $A_1$ are shown in Figures 1 and 2, respectively. We can continue this procedure, replacing $A_0$ with $A_1$. For $i = 1, 2, 3$, let $A_{2,i} = \omega_i(A_1)$ and let $A_2 = \bigcup_{i=1}^3 A_{2,i}$. $A_2$ is pictured in Figure 3. Again, we can continue this procedure, each time replacing $A_i$ with $A_{i+1}$. $A_4$ and $A_8$ are shown in Figures 4 and 5.

If we take the limit as $i \to \infty$, the resulting figure is the Sierpinski triangle. This algorithm for building the Sierpinski triangle is called the deterministic algorithm.

Classification. It is natural to ask what would happen if, in using the deterministic algorithm, we cut the distances by a factor of 3, 4, or 10, instead of 2. That is, what would happen if, for $r > 0$, we defined

$$\omega_t \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{r} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{r-1}{r} \begin{bmatrix} a_t \\ b_t \end{bmatrix},$$

let $A_{m,i}(r) = \omega_i(A_{m-1}(r))$ and let $A_{m+1}(r) = \bigcup_{i=1}^3 A_{m,i}(r)$? As earlier, $A_{1,i}$ is a triangle translated to the $i$th vertex whose sides have length $1/r$ of the lengths of the sides of the original triangle. If Figures 6 and 7 we see $A_1(1.5)$ and $A_4(1.5)$ and in Figures 8 and 9 we see $A_1(3)$ and $A_4(3)$.
touching. It seems that the most aesthetically pleasing situation is when $r = 2$, where the triangles are just touching.

The Deterministic Algorithm Applied to Regular $n$-gons. There is no reason why we should restrict ourselves to looking at only three points. Let $v_1, v_2, \ldots, v_n$ be the vertices of a regular $n$-gon $A_0$. For $i \in \{1, 2, \ldots, n\}$ let $v_i = [a_i/b_i]$, and, for $r > 0$, let $\delta_i([x/y]) = [x/y] + (r - 1)[a_i/b_i], A_m, i(r) = \delta_i(A_m - 1(r)), \text{and } A_{m+1}(r) = \bigcup_{i=1}^{n} A_{m, i}(r)$. Again, $A_0$ is an $n$-gon translated to the $i$th vertex whose sides have length $1/r$ of the lengths of the sides of the original $n$-gon. Some examples of $A_i(r)$ can be seen in Figures 10, 11, and 12.

Figure 10

Figure 11

Again, for some values of $r$, $A_m(r)$ consists of overlapping $n$-gons, and for others totally disconnected $n$-gons. This raises the question of which value of $r$ makes the constituent $n$-gons just touching.

Since $(\mathbb{R}^2, d)$ is a complete metric space, where $d$ is the euclidean metric on $\mathbb{R}^2$, the sequence $\{A_m(r)\}$ has a limit in $\mathbb{R}^2$. (See [1] for details.) For the just touching $r$ we will call $SP_n = \lim_{m \to \infty} A_m(r)$ a regular Sierpinski $n$-gon. (The figures included in this paper were obtained by beginning with the regular $n$-gon with vertices $v_i = (\cos(\alpha_0 + 2\pi i/n), \sin(\alpha_0 + 2\pi i/n))$ for $i = 1, 2, \ldots, n$, where $\alpha_0 = \pi/2 - \pi/n$.)
side of $A_0$ and let $d_m$ be the length of one side of any constituent n-gon of $A_m(r)$. To find the value of $r$ that makes the $A_{m,i}(r)$ just touching, all we need do is find the ratio $d_0/d_1$.

In Figure 13 we see a picture of a portion of $A_{1,1}$ and $A_{1,2}$. Label the vertices of $A_{1,1}$ as $w_{1,1}, w_{1,2}, \ldots, w_{1,n}$ starting with $w_{1,1} = v_1$ and proceeding counterclockwise. In Figure 13 we see the sides $v_1 v_2$ and $v_2 v_1$ of $A_0$ and sides $w_{1,1} w_{1,2}, w_{1,2} w_{1,3},$ and $w_{1,3} w_{1,4}$ of $A_{1,1}$. Here, $|v_1 v_2| = d_0$ and $|w_{i,1} w_{i+1}| = d_1$ for each $i$. Construct a line from $w_{1,3}$ perpendicular to $v_1 v_2$. Label the point of intersection $C_1$. Now, $m(L w_{1,3} w_{1,2} w_{1,1}) = (n - 2)\pi/n$, so $|w_{1,2} C_1| = d_1 \cos(2\pi/n)$. Since the sum of the measures of the angles in a right triangle is $\pi$ radians, it follows that $m(L w_{1,2} w_{1,1} C_1) = (n - 4)\pi/2n$. Now construct a line from $w_{1,4}$ perpendicular to $v_1 v_2$ and call the point of intersection $C_2$. The points $C_1, C_2$, and $w_{1,3}$ form three vertices of a rectangle. Label the fourth vertex $C_2'$. Construct rectangle $C_2 C_2' C_1 w_{1,3}$. Since

$$m(\angle w_{1,4} w_{1,1} C_2') + m(\angle C_2' w_{1,1} C_1) + m(\angle w_{1,2} w_{1,3} C_1)$$

$$= m(\angle w_{1,4} w_{1,1} C_2') = 4\pi/n.$$ So $m(L w_{1,3} w_{1,4} C_2') = (n - 8)\pi/2n.$ Then $|C_1 C_2| = |w_{1,3} C_2'| = d_1 \cos(4\pi/n)$.

We can continue this process inductively, at the $n$th stage obtaining an angle $\angle w_{1,i+1} w_{1,i} C_{i}'$ with measure $(n - 4\pi)/2n$, $n - 4\pi/2n$, as long as $n \geq 4$. By the division algorithm we can find an integer $k$ so that $n = 4k + r$, where $0 \leq r < 4$. So we can continue the construction up to the $k$th step. If $r = 0$, then $n = 4k$. In this situation, the $k$th side of $A_{1,1}$ in this progression coincides with a corresponding side of $A_{2,1}$. If $r > 0$, then $n > 4k$. In this situation $A_{1,1}$ and $A_{2,1}$ intersect at a vertex. Now we can see that if the $A_{m,i}(r)$ are just touching then, by symmetry, the sum of the lengths of $w_{1,1} w_{1,2}, w_{1,2} C_1$, and $\sum_{i=1}^{k} d_i \cos(2\pi/n)$ will be half of $d_0$. So

$$d_0/2 = |w_{1,1} w_{1,2}| + |w_{1,2} C_1| + \sum_{i=2}^{k} |C_{i-1} C_i| = d_1 + \sum_{i=1}^{k} d_i \cos(2\pi/n).$$

Therefore, the contractivity factor necessary to obtain just touching $A_{m,i}(r)$ is

$$d_0/2 = 2\left(1 + \sum_{i=1}^{k} \cos\left(\frac{2\pi}{n}\right)\right).$$

Since $k = \lfloor n/4 \rfloor$ our proof is complete.

The sequence $\{r_n\}$. What can be said about the sequence of distances $r_n = 2\left(1 + \sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2\pi}{n}\right)\right)$?

Intuitively, as $n$ increases the polygons are approaching circles so we would expect that the numbers $1/r_n$ would converge to 0. It is easy to see that this is so.

**Proposition 2.** $\{r_n\}$, as defined above, diverges to infinity.

Proof. It suffices to show that $\sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2\pi}{n}\right)$ diverges to infinity. Let $n \geq 5$ and let $f(k) = \cos(2\pi k/n) - k/n$ for $k \in [0, n/6]$. Then $f(k) = -(1/n)(2\pi k/n + 1)$, so $f$ is a decreasing function of $k$ on $[0, n/6]$. Because $f(n/6) > 0$, $f(k) > 0$ on $[0, n/6]$. It follows then that $\cos(2\pi k/n) > 1/n$ for $k \in [0, n/6]$. So

$$\sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2\pi}{n}\right) > \sum_{i=1}^{\lfloor n/6 \rfloor} \cos\left(\frac{2\pi}{n}\right) > \sum_{i=1}^{\lfloor n/6 \rfloor} \frac{1}{2n/6} \left[\frac{n}{6}\right] + 1 > \frac{n - 6}{72}$$

which diverges to infinity.

Fractal Dimension. The Sierpinski polygons we have been discussing are all examples of a wider class of objects known as fractals. Every fractal has a number associated with it, the fractal dimension, that determines, in some sense, how much of the underlying space it occupies. In this section we see that the **Sierpinski n-gons** are really attractors of iterated function systems and we determine the fractal dimension of each of the Sierpinski $n$-gons. All definitions in this section can be found in Michael Barnsley's book Fractals Everywhere [1]. We begin with a discussion of iterated function systems. As earlier, let $SP_n$ be the Sierpinski $n$-gon.

For a given $n$, in constructing $SP_n$ we used $n$ contraction mappings of the form $\delta_i(z) = (1/r_n)I_2 z + u_i$, for $z, u_i \in \mathbb{R}^2$, where $I_2$ is the $2 \times 2$ identity matrix. This set of mappings forms what is called an iterated function system (IFS) on $\mathbb{R}^2$ and is denoted $\{\mathbb{R}^2; \delta_1, \delta_2, \ldots, \delta_n\}$. We next need to view $SP_n$ as the attractor of this IFS.

The attractor of an IFS $\{\mathbb{R}^2; \omega_1, \omega_2, \ldots, \omega_N\}$ is found as follows. Let
Let $W(B) = \bigcup_{i=1}^{N} \omega_i(B)$. It turns out that $W$ is a contraction mapping on the metric space of all non-empty compact subsets of $\mathbb{R}^2$ with the Hausdorff metric. As such, $W$ has a unique fixed point $A$ in $\mathbb{R}^2$. In other words, there is a non-empty compact subset $A$ of $\mathbb{R}^2$ so that $W(A) = \bigcup_{i=1}^{N} \omega_i(A) = A$. Another way to think of $A$ is that $A = \lim_{i \to \infty} W^i(B)$ for any compact subset $B \subseteq \mathbb{R}^2$. The set $A$ is called the attractor of the IFS. In our situation, we chose $B = A_0$ to be a regular $n$-gon. We then constructed sets $A_i(r)$, $A_2(r)$, ..., $A_m(r)$, ... In following this construction of attractors, for each $i$ the set $A_i(r)$ is equal to $W^{R_i}(A_0)$. The attractor of the constructed IFS is then the set we are calling $SP_n$.

Next we give the definition of fractal dimension in $\mathbb{R}^2$.

Let $A$ be a non-empty subset of $\mathbb{R}^2$. For each $\varepsilon > 0$ let $n(A, \varepsilon)$ denote the smallest number of closed balls of radius $\varepsilon$ needed to cover $A$. If

$$D(A) = \lim_{\varepsilon \to 0} \frac{\ln(n(A, \varepsilon))}{\ln(1/\varepsilon)}$$

exists, then $D = D(A)$ is the fractal dimension of $A$.

In Fractals Everywhere there is a wonderful theorem [Theorem 3, p. 184] that allows us to determine easily the fractal dimensions of the Sierpinski $n$-gons. We state it for $\mathbb{R}^2$ but it holds in all dimensions. A complete proof can be found in [2], [4], or [5].

**Theorem 1.** Let $\{\mathbb{R}^2; \omega_1, \omega_2, \ldots, \omega_n\}$ be a just touching hyperbolic iterated function system and let $A$ be its attractor. Suppose $\omega_k$ is a similitude of scaling factor $s_k$ for each $k \in \{1, 2, \ldots, n\}$. Then $D(A)$, the fractal dimension of $A$, is the unique solution to $\sum_{k=1}^{n} |s_k|^D(A) = 1$, $D(A) \in [0, 2]$.

**Proposition 3.** The fractal dimension of a Sierpinski $n$-gon is $\ln(n)/\ln(r_n)$, where $r_n = 2(1 + \sum_{k=1}^{[n/4]} \cos(2k\pi/n))$.

Proof. Earlier we showed that $SP_n$ is the attractor of a just touching iterated function system to which the contraction mappings $\delta_1, \delta_2, \ldots, \delta_6$ all had the same contractivity factor $1/r_n$. Then, by Theorem 1, $1 = n|1/r_n|^D(SP_n)$. As a result, $D(SP_n) = \ln(n)/\ln(r_n)$.

At this point it seems natural to ask what happens to the sequence $\{D(SP_n)\}$ as $n \to \infty$. As mentioned earlier, as $n$ increases, the polygons we start with are approaching circles. Intuitively, then, we would expect that, as $n \to \infty$, $D(SP_n)$ should approach the fractal dimension of a circle, which is 1. This is, in fact, exactly what happens.

**Corollary.** $\lim_{n \to \infty} D(SP_n) = 1$.

We omit the proof.

**References**


This paper is the result of a senior project completed by Kevin Dennis and supervised by Steve Schlicker while both were at Luther College. Kevin is now at Michigan State, working on his doctorate in the field of analysis. Steve is now at Grand Valley State University.
THE AM-GM INEQUALITY VIA ONE OF ITS CONSEQUENCES

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The arithmetic mean-geometric mean inequality states that if \( a_1, a_2, \ldots, a_n \) are positive real numbers, then

\[
A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n} = G_n
\]

with equality if and only if \( a_1 = a_2 = \ldots = a_n \).

Putting \( a_1 = a_2 = \ldots = a_r = x \) and \( a_{r+1} = a_{r+2} = \ldots = a_n = a_r \), (1) becomes

\[
\frac{r}{n} x + \frac{n-r}{n} a_r \geq x^{r/n} a_r^{(n-r)/n}.
\]

(2) can be used to derive the basic inequalities of Hölder and Minkowski (see, for example, [1], pp. 67-71). It is easy to verify that (2) still holds for the more general hypothesis that the coefficients of \( x \) and \( a_n \) are positive reals whose sum is 1. (See, for example, [2], pp. 21-22.) However, for our purpose we use (2) with \( r = n - 1 \). Thus

\[
\frac{n-1}{n} x + \frac{1}{n} a_n \geq x^{(n-1)/n} a_n^{1/n}.
\]

Using basic properties of the derivative we establish (3) without recourse to (1). It is then a simple matter to obtain (1) inductively.

If \( x > 0 \), then

\[
f(x) = \frac{n-1}{n} x - x^{(n-1)/n} a_n^{1/n}
\]

has an absolute minimum at \( x = a_n \), because

\[
f'(x) = \frac{n-1}{n} - \frac{n-1}{n} x^{-1/n} a_n^{1/x}.
\]

vanishes if and only if \( x = a_n \), and

\[
f''(x) = \frac{n-1}{n} \left( \frac{n-1}{n} \right) x^{(-1-n)/n} a_n^{1/n}
\]

is positive. (3) now follows from the observation that

\[
\frac{n-1}{n} x - x^{(n-1)/n} a_n^{1/n} \geq \frac{n-1}{n} a_n - a_n^{(n-1)/n} a_n^{1/n} \geq -\frac{1}{n} a_n
\]

with equality if and only if \( x = a_n \).

Putting \( x = A_{n-1} \), (3) can be written as

\[
A_n = \frac{n-1}{n} A_{n-1} + \frac{1}{n} a_n \geq A_n^{(n-1)/n} a_n^{1/n}
\]

with equality if and only if \( A_{n-1} = a_n \).

Now using the fact that \( A_2 \geq G_2 \) and assuming that \( A_{n-1} \geq G_{n-1} \) with equality if and only if \( a_1 = a_2 = \ldots = a_n \) it follows that

\[
A_n^{(n-1)/n} a_n^{1/n} \geq G_n^{(n-1)/n} a_n^{1/n} = G_n
\]

with equality if and only if \( a_1 = a_2 = \ldots = a_n \).

References


Norman Schuamberger's last contribution to this Journal appeared in the Spring 1994 issue.
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The focus of this paper is finding relationships between partitions and their Young's lattices. We begin with a review of partitions, lattices, rank, and Hasse diagrams. A partition of the natural number \(N\) is a finite sequence of natural numbers \(n_1, n_2, \ldots, n\) in non-increasing order such that \(n_i = N\). The numbers \(n_i\) are called parts of the partition. One representation of partitions is called a Ferrer's diagram. In a Ferrer's diagram the partition is displayed in a matrix form where the \(i\)th row has \(n_i\) squares. For example, Figure 1 shows the Ferrer's diagram for the partition \(\{5, 4, 1\}\) of 11. The squares will be referred to by ordered pairs, where \((i, j)\) denotes the square in the \(i\)th row and \(j\)th column. There is no simple formula for the number of partitions of \(N\). One recursion formula for the total number \(B_N\) of partitions of \(N\) is

\[
B_N = \sum_{j=0}^{N-1} \left( N - 1 \right) B_j.
\]

See [1] for details.

A partially ordered set (poset) that has the property that any two elements \(x\) and \(y\) have a least upper bound and a greatest lower bound is called a lattice. In particular, a poset that has a maximum element and a greatest lower bound for any two elements is a lattice [1].

For any partition \(P\), let \(B\) be the set of all partitions whose Ferrer's diagram is contained in the Ferrer's diagram of \(P\). If we order them by containment of their Ferrer's diagram, the resulting poset is called a Young's lattice. Young's lattices are named for Alfred Young, who was born in 1873 and graduated from Cambridge in 1895. His main interest was in quantitative substitutional analysis [2].

For example, consider the partition \(\{2, 2, 1\}\) with Ferrer's diagram in Figure 2. The partitions whose Ferrer's diagrams are contained in this partition are in Figure 3. Ordering these by containment we obtain the lattice in Figure 4. This lattice is the Young's lattice.

In a Hasse diagram the elements of a lattice are represented as points. Edges connect points using the following rule: if \(x, y\) are points with \(x < y\), then an edge is drawn between them and we position \(y\) above \(x\). Edges implied by transitivity are suppressed. We use a Hasse diagram to represent the lattice in the diagram.

The rank of a point in a Hasse diagram is a measure of how far the point is from the minimum, or bottom, element. The bottom element is assigned a rank of one. The lattice in the example has elements of ranks one through five.

**Proposition.** If \(A\) is a partition in a Young's lattice, then the rank of \(A\) is the number of squares in a Ferrer's diagram of \(A\).

**Proof.** The result follows from the observation that given a Ferrer's diagram in the lattice, the only way to move down the poset is to remove squares from the Ferrer's diagram. Because of transitivity, the Ferrer's diagrams directly below the given one are those found by removing a single square.

We will now classify some Young's lattices.

**Theorem 1.** If a partition has the form \((n)\), with a Ferrer's diagram as in Figure 5, then the corresponding Young's lattice is a single chain, as in Figure 5.

**Proof.** The proof is by induction on \(n\). The result clearly holds for \(n = 1\) and \(n = 2\). Suppose the theorem is true for \(n = k\). If we remove the
rightmost square from a partition \((k + 1, 1)\), we have the partition \((k)\) whose lattice is a single chain. Since lattices are formed by containment and the two Ferrer's diagrams differ only be a single square, the Young's lattice for the larger partition is a chain with an extra vertex \(\{k + 1\}\) above the vertex \(\{k\}\) connected by an edge.

It follows similarly that the Young's lattice for the partition \((1, 1, \ldots, 1)\) is a single chain.

**Theorem 2.** The Young's lattice for the partition \((n, 1)\) is as in Figure 6.

**Proof.** Again we use induction. It is easy to see that the Young's lattice for \((2, 1)\) has the correct form. Suppose that the theorem is true for \((k, 1)\) and consider the partition \((k + 1, 1)\).

The only square that we can remove and still have a frame is the square at the end of the first row or the square on the second row. Removing the square at the end of the first row leaves a partition \((k, 1)\) to which the induction assumption applies. Removing the square in the second row leaves a partition \((k + 1)\) which we proved earlier has a Young's lattice consisting of a single chain. Therefore, the Young's lattice for \((k + 1, 1)\) will have one vertex of rank \(k + 2\) and two vertices of rank \(k + 1\) with the form in Figure 7. Using the induction assumption and the fact that the \((k)\) chain is a sub-lattice of the \((k, 1)\) lattice, we obtain the desired form for the \((k + 1, 1)\) lattice.

The next most complicated partition is \((n, 2)\).

**Theorem 3.** If a partition has the form \((n, 2)\) where \(n > 2\) with a Ferrer's diagram as in Figure 9, then its Young's lattice will be as in Figure 9.

**Proof.** The proof is again by induction. The result holds for the partition \((3, 2)\) since its Young's lattice (Figure 10) has the proper form. The induction argument proceeds as before since the only squares which can be removed leaving a frame are the square at the end of the first row and the square in the \((2, 2)\) position. Both are partitions whose lattice are already known.

**References**


Carissa Hurst prepared this paper while a senior at Hendrix College, under the direction of Dr. David Sutherland. She is presently a graduate student in statistics at the University of Arkansas in Fayetteville.
DIVISIBILITY TESTS FOR PRIMES GREATER THAN 5

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Many papers have been written giving divisibility tests for integers. This note does not contain any new result, but it gathers previous tests in one place and shows how to generate any number of new ones.

Divisibility tests for 2, 3, 4, 5, 8, and 10 are taught as early as elementary school but tests for arbitrary prime numbers are not given. Here is a test for divisibility by 7. Take the number to be tested and double its last digit. Subtract this from the number with its last digit removed. If 7 divides this new number, then 7 divides the original. For example, 7 \( | \) 294 since 7 \( | \) (29 + 5·4) = 49. Alternatively, one can multiply the last digit by 5 and add the result: 7 \( | \) 294 since 7 \( | \) (29 + 5·4) = 49.

Consider the number 51, a multiple of 17. Does 17 \( | \) 51 \( \text{mod} \) 17? In other words, does 5 work for 17 the same way that 10 does for 9? The answer is "yes". The proof is as follows. Let \( x = 10a + b \) and \( r = a - 5b \). We have

\[
x + 7r = (10a + b) + 7(a - 5b) = 17a - 34b.
\]

If 17 \( | \) r, then 17 \( | \) 7r. Since 17 \( | \) (x + 7r), we have that 17 \( | \) x.

By generalizing this procedure we can prove that a test can devised for any prime and we can find the constant \( t \). However, if we had a 50-digit number to be tested for divisibility by 7, removing only one digit at a time would be time-consuming indeed. But if we removed ten digits for each iteration it would cut the calculation time needed immensely.

A test can be given where the number of digits removed at each stage, \( y \), can be chosen arbitrarily:

**THEOREM 1.** Given a prime \( p \) and \( x = 10^ya + b \), let \( r = a - m \cdot b \) where \( m \) is the solution to \( 10^ym = 1 \) \( \text{mod} \) \( p \). If \( p \mid r \) then \( p \mid x \).

**Proof.** Let \( n' \) be such that \( nn' = 1 \) \( \text{mod} \) \( p \). Note that \( 10^ynn' = n' \) \( \text{mod} \) \( p \) so \( 10^ya + n' \) \( \text{mod} \) \( p \). Thus, both \( 10^ya - n' \) and \( 1 - nn' \) are \( a \) \( \text{mod} \) \( p \). We have

\[
x + n'r = 10^ya + b + n'(a - nb)
= (10^y - n')a + b(1 - nn') = 0 + 0 \text{ \( \text{mod} \) } p.
\]

If \( p \mid r \), then \( p \mid n'r \). Since \( p \mid x + n'r \) we have that \( p \mid x \).

In exactly the same way we could prove

**THEOREM 2.** Given a prime \( p \) and \( x = 10^ya + b \), let \( r = a - m \cdot b \), where \( m \) is the solution to \( 10^ym = -1 \) \( \text{mod} \) \( p \). If \( p \mid r \) then \( p \mid x \).

There is a connection between \( m \) and \( n \).

**COROLLARY.** \( m = p - n \).

**Proof.** Since \( 10^yn = 1 \) \( \text{mod} \) \( p \) and \( 10^ym = -1 \) \( \text{mod} \) \( p \), we have \( 10^y(n + m) = 0 \) \( \text{mod} \) \( p \), and so \( n + m = 0 \) \( \text{mod} \) \( p \). Since \( m = p - n \), the corollary follows.

For an example, let us take \( x = 28,842 \), \( p = 23 \), and \( y = 2 \). To determine if 23 divides \( x \), the value of \( n \) for \( y = 1, 2, ..., 12 \). The values of \( n \) are periodic, with period equal to the order of 10 \( \text{mod} \) \( p \).

If we let \( n_y \) denote the value of \( n \) for \( y \), then we have

**THEOREM 3.** \( n_{y+1} = n_1n_y \) \( \text{mod} \) \( p \).

**Proof.** \( 10^yn_{y+1} = 1 \cdot 1 \cdot (10n_1)(10n_y) \) \( \text{mod} \) \( p \).

This recursive property allows the generation of large tables very quickly in a spreadsheet program without the problem of roundoff error. For efficient tests, small values of \( n \) (or \( m = p - n \)) can be quickly determined.

Phil Plummer received his B. S. degree in mathematics and physics at Portland State University and is currently finishing work toward his M. S. degree in mathematics. He wishes to thank his high-school mathematics teacher, Mr. Wayne Wheelers of Springfield (Oregon) High School, for teaching him that mathematics could be fun.
In a recent article about Pascal's hexagon theorem for a circle, Jan van Yzeren [2] credited H. Guggenheimer with a previous proof. However, Professor Guggenheimer explained that the proof had in fact been taught him in his 11th grade Descriptive Geometry class in Basel, Switzerland. The following proof, which I learned as a college junior, used to be called a proof by "abridged notation". In it, a linear form is represented by a single letter, and forms are combined to make second-degree expressions that will stand for conic sections.

For example, if \( a = x + 2y + 1 \) and \( \beta = x - 2y + 2 \), then \( a = 0 \) and \( \beta = 1 \) are equations of lines, \( a\beta = 0 \) is the equation of a pair of intersecting lines (a degenerate conic— asymptotes only), and \( a\beta = 1 \) is the equation of a nondegenerate hyperbola.

Any equation of the form \( a + k\beta = 0 \) represents a line (it is linear in form) that passes through the intersection of the lines \( a = 0 \) and \( \beta = 0 \) since substitution of the coordinates of the point of intersection make both \( a \) and \( \beta \) take on the value 0. We use a similar strategy for conic sections. If \( S = 0 \) and \( T = 0 \) are conics, then they are of second degree in \( x \) and \( y \). Then for any nonzero constant \( k \), \( S + kT = 0 \) is of second degree so it represents a conic. Since the point where \( S \) and \( T \) intersect has coordinates that satisfy both \( S = 0 \) and \( T = 0 \), the conic \( S + kT = 0 \) passes through the points of intersection.

PASCAL'S THEOREM: If a closed hexagon is inscribed in a conic section, then the three points of intersection of its opposite sides are collinear.

(If the two opposite sides are parallel, their point of intersection is taken to be the point at infinity. The conic section may be degenerate, and the
Proof. Let the six vertices of the inscribed hexagon be labelled in order A, B, C, D, E, F. Let the sides be labelled \(a, \beta, y, 6, \varepsilon, \zeta\) (see Figure 1), where, for instance \(a = a_1x + a_2y + a_3\) and \(\alpha = 0\) is the equation of the line \(AB\). Continuing, \(\beta = 0\) is the equation of the line through \(BC\) and so on. It is to be proved that the intersections \(P\) of side \(a\) and side \(6\), \(Q\) of side \(\beta\) and \(\varepsilon\), and \(R\) of \(y\) and \(\zeta\) are collinear.

Construct a seventh line, \(g\), through \(A\) and \(D\). Let \(S_1\) stand for the second-degree sum of products

\[ay + r\beta\theta.\]

For any value of \(r\), \(S_1 = 0\) is the equation of a conic section that circumscribes the quadrilateral \(ABCD\) since, for instance, \(C\) lies on both \(\beta\) and \(y\). Similarly, if

\[S_2 = \delta\xi + s\varepsilon\theta,\]

then \(S_2 = 0\) is the equation of a conic section that circumscribes the quadrilateral \(DEFA\). Constants \(r, s, t\) exist for which \(S_1 = tS_2\), where \(C\) is the given conic section circumscribing the hexagon. From \(S_1 = tS_2\), it follows that

\[\theta(r\beta - s\varepsilon) = t\delta\xi - ay.\]

The equation \(r\beta - s\varepsilon = 0\) represents a line \(A\) that passes through the intersection of lines \(\beta\) and \(\varepsilon\) (Pascal point Q), so the left member of the identity is the degenerate conic section composed of lines \(9\) and \(A\). This is the same degenerate conic section as that on the right, including line \(9\) through \(A\) and \(D\) and a second line that passes through the intersection of lines \(a\) and \(6\) (Pascal point P) and the intersection of lines \(y\) and \(\zeta\) (Pascal point R). All three Pascal points lie on line \(A\), and the theorem is proved.

Two of the vertices may be brought into coincidence, in which case the side joining them becomes the tangent to the conic at the double point. When the circumscribing conic section degenerates to straight lines, the special case is the Theorem of Pappus: Let three vertices, say \(A, B, C\) on one line \(L\) of the degenerate conic and three vertices \(A, B, C\) on the other line \(L\) be connected in any order that alternates the two sets of three, such as \(ABACCA\) or \(ACBAC\), forming a closed (not convex) hexagon, as in Figure 2. Then the intersections of opposite sides of the hexagon (side 1 with 4, 2 with 5, and 3 with 6) are collinear. The six Pascal lines from the six different hexagons that can be formed pass three by three through two points; that is, the six Pascal lines divide into two concurrent sets of three.

An inscribed hexagon has sides that connect pairs of vertices on the conic. The three points of intersection of pairs of opposite sides are collinear. Using duality, we replace each line by point, immediately obtaining Brianchon's Theorem: A circumscribed hexagon has vertices that connect pairs of lines "on" the conic (that is, it has vertices that are the intersections of tangents to the conic). The three lines connecting pairs of opposite vertices are concurrent.

A way of constructing points on a conic section can be derived from Pascal's Theorem. Five points determine a conic. To construct a sixth point given \(A, B, C, D, E\), let \(AB\) and \(DE\) intersect at the Pascal point \(P\). Draw a line \(x\) through \(E\) passing through the general region where the sixth point \(F\) is desired. (See Figure 3.) The intersection of line \(x\) with \(BC\) is a second Pascal point \(Q\). Complete the Pascal line \(PQ\). Its intersection with \(CD\) is the third Pascal point \(R\), and line \(AR\) intersects line \(x\) in the desired point \(F\).
on the conic. Further points are obtained by rotating the position of the line \(x\), as to \(x'\), and repeating the construction to get another Pascal line, \(PQ'R'\), and another point on the conic, \(F'\). This construction has actually been used in fairing in smooth curved outlines for aircraft design.

How many hexagons can be drawn using a given set of six vertices on a conic? There are 6! permutations, but the starting point is arbitrary for a closed hexagon, and reversing the order does not change the hexagon, so there are \(6!/(6-2) = 60\) hexagons. The 60 corresponding Pascal lines pass 3 by 3 through 20 Steiner points, which lie 4 by 4 on 15 Steiner-Pliicker lines. Also, the 60 Pascal lines pass 3 by 3 through 60 Kirkman points, which lie 3 by 3 on 20 Cayley-Salmon lines other than the Pascal lines.

Although conic sections are only one degree more advanced than straight lines, they have a rich analytic geometry.

References


In response to a request for a few words about the author, Professor Maxfield wrote:

Among all the beautiful proofs in mathematics, this abridged notation proof of Pascal's theorem is my favorite. Isn't it elegant? For many of us, mathematics is a branch of esthetics, certainly more art than science. I teach statistics, which is kind of a branch of epistemology—'How can we know?'. From students' difficulties in learning elementary statistics, I suspect that they have a severe problem with semiotics, a problem that has cursed their efforts in arithmetic and algebra all their lives. Are you a semiotician?

A Novel Sequence


Alexander opened his eyes and tried to find a pattern in the row of Linda's flowerpots. He'd always been good at sequences. If one, ten, three, nine, five, eight, seven, seven, nine, and six are the first elements in a group, what number continues the sequence?

It certainly is, and the next term is equally easy, is it not? That was all right, but I think that in general novelists ought to stay away from mathematics. E. g., on page 117 we find

Physicists and mathematicians played a similar game in their heads, called sphere packing: How many spheres could fit inside one sphere? How many circles could fit inside one circle?

Anybody here ever played the sphere packing game? I never have: have I missed out on all the fun?

Then on page 144,

Zeno lived around Aristotle's time, and like Aristotle, he enjoyed stirring things up.

Would we say that Euler lived around Hilbert's time? I don't think so, though the separation—around 150 years—is the same. Perhaps time intervals foreshorten from a long way away.
A DIOPHANTINE EQUATION

Efraim Berkovich
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During a course in number theory, we solved the diophantine equation
\[ \frac{1}{x} + \frac{1}{y} = \frac{1}{14}. \]
The solution given in class and in the teacher's edition of the textbook was rather long and time-consuming. In this note I give a straightforward method of solving this problem, and any diophantine equation
\[ \frac{1}{ax} + \frac{1}{by} = \frac{1}{c} \]
where a, b, and c are integers. The solution involves only factoring an integer.

One method of solving this equation would be to set
\[ c(ax + by) = abxy = A \]
and thus reduce the problem to solving the quadratic equation
\[ z^2 - \frac{\lambda z}{c} + \lambda = 0 \]
of one variable. This method, however, requires that we then find values of A which give integer values to the square root of the discriminant.

A simpler method is to make the substitutions \( s = c - ax \) and \( t = c - by \). The equation becomes
\[ \frac{1}{c - s} + \frac{1}{c - t} = \frac{1}{c} \]
\[ c(c - s) + c(c - t) = (c - s)(c - t) \]
which simplifies to
\[ c^2 = st. \]

We now need integers \( s \) and \( t \) which satisfy \( c^2 = st \). So, \( s \) and \( t \) have the same sign and are factors of \( c^2 \). Factoring yields all possible values of the integer pairs \((s, t)\). Taking each \((s, t)\) pair, we have a solution \((x, y)\) if \( x = (c - s)/a \) and \( y = (c - t)/b \) are integer valued. In this manner we will find all the solutions of the original equation because any \((x, y)\) which satisfy the original diophantine equation will necessarily satisfy \( c^2 = st \).

This procedure requires nothing more complicated than factoring \( c^2 \) and provides an upper bound to the number of solutions. If \( c^2 \) has \( \xi \) factors, then there are at most \( 2\xi - 1 \) solutions: the number of different factors and their negatives, excluding \( s = c \) and \( t = c \) since that would give \( x = 0 \) and \( y = 0 \). The actual number of solutions would be reduced by the conditions that \( c - s \) and \( c - t \) be divisible by \( a \) and \( b \), respectively.

As an example, let us consider \( \frac{1}{x} + \frac{1}{y} = 14 \). Since the equation is symmetric in \( x \) and \( y \), when we satisfy the condition \( c^2 = st \), it is sufficient to consider the pairs \((s, t)\) where \( s \geq t \). The pairs are \((196, 1), (98, 2), (49, 4), (28, 7), (-1, -196), (-2, -98), (-4, -49), (-7, -28), and (-14, -14)\). The corresponding solutions \((x, y)\) with \( x \leq y \) are therefore \((-182, 13), (-84, 12), (-35, 10), (-14, 7), (15, 210), (16, 112), (18, 63), (21, 42), and (28, 28)\).

This technique also allows us to make some observations about solutions of the equation. For example, if \( c \) is prime and \( a = b = 1 \), then we know that there are exactly three solutions.

Efraim Berkovich was graduated from Georgetown University in May 1994 with a B. S. degree in mathematics. He hopes to pursue graduate studies in electrical engineering.

All stiff regularity (such as borders on mathematical regularity) is inherently repugnant to taste, in that the contemplation of it affords us no lasting entertainment. —Immanuel Kant, Critique of Judgement.
GROUP GENERATORS AND SUBGROUP LATTICES

Scott M. Wagner
Hendrix College

This paper will examine one of the connections between combinatorics and algebra. We will use Philip Hall's Eulerian function to count the number of generating sets of size $n$ for the twenty-eight groups of order fifteen or less. Recall that a generating set of a group is a subset of the group such that every element in the group can be expressed as a product of one or more elements of the generating set. For example, the cyclic group of order 4, \( \{e, a, a^2, a^3\} \) has two generating sets of size 1: \( \{a\} \) and \( \{a^3\} \).

The set of all subgroups of a group $G$ under subset inclusion forms a lattice where the greatest lower bound of two subgroups is their intersection, and the least upper bound of two subgroups is the smallest subgroup of $G$ containing both of them. Hasse subgroup diagrams are diagrams of these lattices. The subgroups are represented by points, and if $A$ and $B$ are subgroups with $A \subseteq B$ then the points representing $A$ and $B$ are connected with an edge, with $B$ positioned above $A$. The subgroup $\{e\}$ is at the bottom of the lattice, and the group itself is at the top. Examples of Hasse subgroup diagrams for groups of low order can be found in [1].

In 1935 Philip Hall described a method for counting the number of ways of generating the group of symmetries of the icosahedron from a given number of elements [2]. One outcome of his method was the development of a generalized Eulerian function which can be used to count the number of generating sets of a given size for any finite group. The method does not determine the actual generating sets, but only their number.

The function is defined by

$$
\varphi_n(G) = \sum_{H \leq G} \mu(H, G) \left( \frac{|H|}{n} \right)
$$

where $G$ is the finite group, $n$ is the number of elements in the generating set, $H$ is a subgroup in the Hasse diagram, $|H|$ is the order of $H$, and $\mu$ is defined recursively by $\mu(G, G) = 1$ and, for each subgroup $H$ of $G$,

$$
\sum_{K : H \leq K} \mu(K, G) = 0.
$$

So, $\mu$ determines a coefficient for each subgroup of $G$. The top subgroup of the lattice, $G$ itself, has coefficient 1. The coefficient of any other subgroup in the lattice is the integer that, added to the coefficients of every subgroup above it in the lattice, gives a sum of zero.

For example, let us calculate $\varphi_1(C_4)$, where $C_4$ is the cyclic group of order 4. The Hasse diagram for $C_4$ and the values of $\mu(H, C_4)$ for the two subgroups of $C_4$ are shown in Figure 1. Applying Hall's formula, we see $\varphi_1(C_4) = 1 \left( \begin{array}{c} 4 \\ 1 \end{array} \right) - 1 \left( \begin{array}{c} 2 \\ 1 \end{array} \right) + 0 \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 4 - 2 + 0 = 2$, giving the number of one-element generators of $C_4$.

Now let us consider $D_3$, the group of symmetries of an equilateral triangle. The Hasse subgroup diagram and the values of $\mu$ are shown in Figure 2, where $A = \{e, r, r^2, s, rs, r^2s\}$, $B = \{e, s\}$, $C = \{e, rs\}$, $D = \{e, r^2s\}$, $F = \{e, r, r^2\}$, and $G = \{e\}$. Since cyclic
groups are the only groups with one-element generating sets, we know that
\( \varphi_1(D_3) = 0 \), which is verified by the formula:

\[
\varphi_1(D_3) = 1 \binom{6}{1} - 3 \binom{2}{1} - 1 \binom{3}{1} + 3 \binom{1}{1} = 6 - 6 - 3 + 3 = 0.
\]

Moreover, \( \varphi_2(D_3) \) is

\[
\varphi_2(D_3) = 1 \binom{6}{2} - 3 \binom{2}{2} - 1 \binom{3}{2} = 15 - 3 - 3 = 9.
\]

To see why that is correct, note that the first term in the sum, \( 1 \binom{6}{2} = 15 \), counts the number of two-element subsets of \( D_3 \). From these we must exclude those that generate a proper subgroup of \( D_3 \). A two-element subset that does not generate \( D_3 \) must generate one of the two subgroups of order two or the unique subgroup of order three. The term \( 3 \binom{2}{2} \) eliminates those that generate the three two-element subgroups while the term \( 1 \binom{3}{2} \) eliminates the three two-element sets that generate the subgroup of order three. This can be verified in another way by noting that of the \( \binom{5}{2} = 10 \) two-element subsets not containing the identity, only \( \{ r, r^2 \} \) generates a proper subgroup of \( D_3 \).

We will give three more examples and a table giving values of \( \varphi_n \) for \( n = 1, 2, 3 \) for some finite groups. This may serve as a source of examples.

The Hasse diagram and \( \mu \) values for

\[
C_2 \times C_4 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (1, 3)\}
\]
appear in Figure 3, where \( A = C_2 \times C_4 \), \( J = \{(0, 0)\} \).

\( B = \{(0, 0), (1, 1), (0, 2), (1, 2)\} \),

\( C = \{(0, 0), (1, 0), (0, 2), (1, 2)\} \),

\( D = \{(0, 0), (0, 1), (0, 2), (0, 3)\} \),

\( F = \{(0, 0), (1, 0)\} \), \( G = \{(0, 0), (0, 2)\} \), and

\( H = \{(0, 0), (1, 2)\} \).

Since this group is not cyclic, the number of two element generators is

\[
\varphi_2(C_2 \times C_4) = 1 \binom{8}{2} - 3 \binom{4}{2} + 2 \binom{2}{2} = 28 - 18 + 2 = 12.
\]

We will look next at \( A_4 \), the alternating group of degree four. Its members are all the even permutations in \( S_4 \), the symmetric group of degree four. The Hasse subgroup diagram and the \( \mu \) values are in Figure 4, where

\( A = A_4 \), \( L = \{ e \} \), \( B = \{ e, (12)(34), (13)(24), (14)(23) \} \),

\( C = \{ e, (123), (132) \} \),

\( D = \{ e, (124), (142) \} \),

\( F = \{ e, (12)(34) \} \),

\( G = \{ e, (13)(24) \} \),

\( J = \{ e, (234)(243) \} \), and

\( H = \{ e, (14)(23) \} \), \( K = \{ e, (134)(143) \} \).

So, we find that the number of two element generators is

\[
\varphi_2(A_4) = \binom{12}{2} - \binom{4}{2} - 4 \binom{3}{2} = 48.
\]

The final group we will examine is a dicyclic group. A dicyclic group, \( G \), can be represented by the following group presentation:

\[
G = \{(x, y) \mid x^{2n} = 1, x^{2n}y^{-2} = 1, y^{-1}xyx = 1\}.
\]

We will consider \( T \), the dicyclic group of order 12, where \( n = 3 \). The elements of \( T \) are

\[
e, x, x^2, x^3, x^4, x^5, y, y^3, xy, yx^2, x^2y, \text{ and } y^3x^2.
\]
The Hasse subgroup diagram and the \( p \) values are in Figure S, where \( A = T \), \( J = \{ e \} \), and \( B = \{ e, x^3, yx^2, y^3x^2 \} \), \( C = \{ e, y, x^3, y^3 \} \), \( D = \{ e, x^3, xy, x^2y \} \), \( F = \{ e, x, x^2, x^3, x^4, x^5 \} \), \( G = \{ e, x^3 \} \), and \( H = \{ e, x^2, x^4 \} \).

Once again, since this group is not cyclic, we will find the number of two-element generating sets:

\[
\phi_2(T) = \binom{12}{2} - 3 \binom{4}{2} - \binom{6}{2} + \binom{2}{2} = 36.
\]

The table below gives the number of one, two, and three element generators for twenty-seven groups.

<table>
<thead>
<tr>
<th>Group</th>
<th>( \varphi_1 )</th>
<th>( \varphi_2 )</th>
<th>( \varphi_3 )</th>
<th>Group</th>
<th>( \varphi_1 )</th>
<th>( \varphi_2 )</th>
<th>( \varphi_3 )</th>
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<td>( C_1 )</td>
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<td>-</td>
<td>( D_3 )</td>
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<td>19</td>
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<td>-</td>
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<td>1</td>
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<td>110</td>
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<td>5</td>
<td>3</td>
<td>( D_6 )</td>
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<td>150</td>
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<td>20</td>
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<tr>
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<tr>
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<td>( T )</td>
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References

1. Butt, Melanie, Automorphism groups of Hasse subgroup lattices for groups of low order, this Journal 9 (1989-94) #1, 2-8.

Scott Wagner wrote this paper while a senior mathematics major at Hendrix College, with project advisor David Sutherland. He wishes to thank Joseph Kung and Cynthia Sewall for the idea for the project, and the referee for valuable suggestions.

All Statements Are Not Correct

On the important subject of language, Professor James Chew (North Carolina A & T State University) writes

One of my biggest peeves in English mis-usage has to do with negations of statements. For example, "All athletes are not basketball players". We all know what is intended is "Not all athletes are basketball players". I would be willing to let this matter slide were it not for the fact that even mathematicians who should know better, since they are the presumed guardians of clear and logical thinking, commit this error regularly. In the Transactions of the American Mathematical Society, that pinnacle of mathematical journals, whose contents are so sublime that I wish I could understand .05 of one percent of them, I found "All topological spaces are not Hausdorff". This is FALSE since the real line (with its usual topology) is a topological space which is Hausdorff.
An nth-order linear homogeneous difference equation with constant coefficients has the form

\[ a_n y_{n+k} + a_{n-1} y_{n+k-1} + \ldots + a_1 y_{k+1} + a_0 y_k = 0 \]

where \( a \neq 0 \). When seeking a solution of (1), students are frequently asked under what conditions, if any, will

\[ y_n = \lambda^n \]

be a nontrivial solution? This approach is used in several standard textbooks. It is sometimes motivated by examining the general form of the solution to a first-order linear difference equation. Substituting (2) into (1) and simplifying yields the characteristic equation

\[ a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0. \]

This method is probably adequate motivation for students who have not studied differential equations. However, those who have do not expect to see solutions of the form (2). They know that solutions of an nth-order linear homogeneous differential equation with constant coefficients,

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \]

have the form \( y = e^{\lambda x} \) where the \( \lambda \)'s are distinct roots of the same characteristic equation (3). Such students would see the similarities between difference equations and differential equations if they were asked under what conditions, if any, will

\[ y_n = e^{\lambda n} \]

be a nontrivial solution of (1)? Substituting (4) into (1) yields

\[ a_n e^{\lambda(n+k)} + a_{n-1} e^{\lambda(n+k-1)} + \ldots + a_1 e^{\lambda(k+1)} + a_0 e^{\lambda k} = 0 \]

and so

\[ \frac{a_n e^{\lambda n} + a_{n-1} e^{\lambda(n-1)} + \ldots + a_1 e^{\lambda} + a_0}{2a_2} = 0. \]

As an illustration, let \( n = 2 \). Then (5) becomes

\[ a_2 e^{2\lambda} + a_1 e^\lambda + a_0 = 0 \]

and so

\[ e^\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2} \]

Hence, (4) becomes

\[ y_n = (e^{\lambda n}) = \left(\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}\right)^n \]

For the case when \( n = 2 \) and the characteristic equation has a repeated root, the form of the general solution is motivated in [1].

Finally, it can be seen that (3) and (5) have the same form by letting \( t = e^\lambda \) in (5).

Reference


Russell Euler is a frequent contributor to this and other journals.

I don't take no stock in mathematics, anyway. —Mark Twain, Huckleberry Finn, Chapter 4.
A Test for Affineness

Christopher Kohl
James Madison University

The purpose of this paper is to present a characterization of affine spaces and a necessary condition for a function to be an affine map. The ideas for these theorems arose from a pair of problem sets in a linear algebra textbook [1]. Specifically, the reader is given either a subset of a vector space or a function and asked to determine whether or not it is affine. An examination of the first problem set led to an observation that was generalized into a theorem (Theorem 1 below), which states a condition under which a subset of a vector space is not an affine subspace. Similarly, an examination of the second problem led to a theorem (Theorem 2 below) which states an assertion about the image of an affine map.

DEFINITION 1. A subset \( U \) of a vector space \( V \) is called an affine subspace of \( V \) provided that, for some subspace \( W \) of \( V \) and some fixed vector \( b \) in \( V \), \( U = \{ w + b \mid w \in W \} \).

DEFINITION 2. A function \( g : U \to V \) is called an affine map provided that, for some linear map \( f : U \to V \) and some fixed element \( b \) of \( V \), \( g(u) = f(u) + b \) for all \( u \) in \( U \).

THEOREM 1. Let \( S \) be a subset of a vector space \( V \). If there is a fixed vector in \( V \), call it \( p \), so that \( 0 \in S + p \) and \( S + p \) is not a subspace of \( V \), then \( S \) is not an affine subspace of \( V \).

Proof. Suppose \( S \) is an affine subspace of \( V \) and that \( p \) is a fixed vector in \( V \) so that \( 0 \in S + p \). It follows from Definition 1 that there is a subspace \( U \) of \( V \), call it \( W \), and there is a fixed vector in \( V \), call it \( b \), so that \( S = \{ m + b \mid m \in W \} \). Thus,

\[
S + p = \{ m + b + p \mid m \in W \}.
\]

Since \( 0 \in S + p \), it follows that there is an element of \( W \), call it \( r \), so that \( 0 = r + b + p \). Thus, we can write \( r = -(b + p) \), which implies that \( b + p \in W \). Now we show that \( S + p \) is a subspace of \( V \).

Let \( x_0, x_1 \in S + p \) and \( c \in \mathbb{R} \). It follows that there are elements of \( W \), call them \( t_0 \) and \( t_1 \), so that

\[
x_0 = t_0 + b + p \quad \text{and} \quad x_1 = t_1 + b + p.
\]

Therefore, we can write

\[
x_0 + x_1 = (t_0 + t_1 + b + p) + b + p.
\]

Since \( t_0, t_1, (b + p) \in W \), it follows that \( (t_0 + t_1 + b + p) \in W \). Thus, \( S + p \) is closed under addition.

Also,

\[
cx_0 = c(t_0 + b + p) = ct_0 + (c - 1)(b + p) + (b + p).
\]

Since \( c, c - 1 \in \mathbb{R} \) and \( t_0, (b + p) \in W \), it follows that \( ct_0 + (c - 1)(b + p) \in W \). Thus, \( S + p \) is closed under scalar multiplication. And so, by [1, Theorem 4.2], \( S + p \) is a subspace of \( V \).

THEOREM 2. The image of an affine map \( f : U \to V \) between vector spaces \( U \) and \( V \) is an affine subspace of \( V \).

Proof. Let \( f : U \to V \) be an affine map between vector spaces \( U \) and \( V \), and let \( M \) denote the image off. By Definition 2 there is a linear map, call it \( g : U \to V \), and a fixed element of \( V \), call it \( b \), so that \( f(u) = g(u) + b \) for all \( u \) in \( U \). Since \( g \) is a linear map it follows from [1, Theorem 5.5] that the image of \( g \), call it \( T \), is a subspace of \( V \). Since \( f(u) = g(u) + b \) for all \( u \) in \( U \), we can write \( M = T + b \). And so, by Definition 1, \( M \) is an affine subspace of \( B \).

The problems that motivated the ideas for Theorem 1 deal exclusively with two- and three-dimensional vector spaces. For example, given the set \( T = \{(x, \sin x + 3) \mid x \in \mathbb{R} \} \), determine whether \( T \) is a subspace of \( \mathbb{R}^2 \), an affine subspace of \( \mathbb{R}^2 \), or neither. After examining this problem and formulating Theorem 1, it was observed that the geometry of \( T \) could have been used to conclude that \( T \) is not an affine subspace of \( \mathbb{R}^2 \). This results from the tractable geometry of \( \mathbb{R}^2 \). Since a subspace of \( \mathbb{R}^2 \) has a geometric realization as either the origin, a line through the origin, or the plane itself, it follows that an affine subspace of \( \mathbb{R}^2 \) must have a geometric realization as either a point, a line, or the plane itself. Therefore, given a subset of \( \mathbb{R}^2 \) and asked to determine whether or not it is affine, one can examine the geometry...
of the subset and make conclusions accordingly. Consider the above example dealing with the subset $T$ of $R^2$. Since the geometric realization of $T$ in $R^2$ is not a line, a point, or the plane itself, one can conclude that $T$ is not an affine subspace. Now consider the vector space consisting of all $12 \times 15$ matrices. The geometry of such a vector space is certainly much more complex than that of $R^2$. Thus, given a subset of such a vector space, one cannot rely on its geometry to test for affineness. Its algebraic structure must be considered and that is where the utility of Theorem 1 is realized. There will follow an application of Theorem 1 where the geometry of the underlying vector space is too complex to use as a tool in testing for affineness.

Inspection of Theorem 2 reveals that it can be restated as follows: if the image of a function from a vector space $U$ to a vector space $V$ is not an affine subspace of $V$, then the underlying function is not an affine map. Clearly, Theorem 2 is a direct outgrowth of Theorem 1. Because of this relationship between the two theorems, the ideas for employing the geometry of a given subset can be carried over to Theorem 2.

Let $M_{nn}$ denote the vector space of all $n \times n$ matrices with real number entries. Consider the set

$$S = \{A \in M_{nn} \mid \det(A) \neq 0\}.$$ 

By definition, $S$ is the set of all nonsingular matrices of size $n \times n$. By employing Theorem 1, we will show that $S$ does not constitute an affine subspace of $M_{nn}$. From the definition of $S$, it follows that $S$ is a subset of $M_{nn}$ and that $0 \notin S$, which implies that $S$ is not a subspace of $M_{nn}$. Since $I \in S$, it follows that $0 \notin (S + -I)$. Next, we verify that $(S + -I)$ is not a subspace of $M_{nn}$ by showing that it is not closed under addition.

Consider the $n \times n$ matrix $\frac{1}{2}I$. Since $\frac{1}{2}I \in S$, it follows that $(\frac{1}{2}I + -I) \in (S + -I)$. We can write $(\frac{1}{2}I + -I) + (\frac{1}{2}I + -I) = -I$. Since $0 \notin S$, it follows that $-I \notin (S + -I)$. Therefore, we can conclude that $(S + -I)$ is not closed under addition and it follows from [1, Theorem 4.2] that $(S + -I)$ is not a subspace of $M_{nn}$. Since $S$ is a subset of $M_{nn}$, $-I$ is a fixed vector in $M_{nn}$ so that $0 \in (S + -I)$, and $(S + -I)$ is not a subspace of $M_{nn}$, it follows from Theorem 1 that $S$ is not an affine subspace of $M_{nn}$.

Rex H. Wu (Brooklyn, New York) points out two applications of the pigeon-hole principle. If you ask people unfamiliar with the problem how many socks it is necessary to take from a drawer containing four red socks, five blue ones, six green, and seven white, you will get a variety of answers, some irrelevant (e.g., "Nobody wears green socks, except maybe on St. Patrick's day"). Try asking! All it can do is make you unpopular. A second application proves the theorem that there are now living two people who were born at exactly the same time, to the second. This follows from the calculation that there have been fewer than five billion seconds in the last 150 years and the fact that the world contains more than five billion people. Since there are only 31,536,000 seconds in a 365-day year and more births than that this year, there will be two such people born in 1995.

Xuming Chen (University of Alabama) gives a good reason why 1 is not counted among the primes. The valuable identity

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

would not hold if 1 was a prime.
Let $X$ be the upper half-plane $X = \{ (x, y) \in \mathbb{R}^2 : y \geq 0 \}$ and let $L$ be its boundary $L = \{ (x, 0) : x \in \mathbb{R} \}$ . The Niemytzki plane is the space $X$ whose basic sets are the open discs in $X \setminus L$ and the open discs in $X \setminus L$ tangent to $L$ together with the point of tangency. We will refer to the points on $L$ as being rational or irrational with the obvious correspondence. This space and the Sorgenfrey plane (i.e., the product space $S \times S$ where $S$ is the real line with the half-open interval topology) provide examples of a $T_3$ topological space which is not a $T_4$-space. While the Sorgenfrey plane has the merit of being the product of $T_A$-spaces, the Niemytzki plane lacks this. But it has another, namely that its topology is more familiar and easier to work with.

While it is easy to see that both spaces are $T_3$, the usual proofs showing that neither is a $T_A$-space use a cardinality or second category argument and are often omitted from elementary textbooks. In this note we present an alternate proof for the Niemytzki plane. The proof is simple, uses only the nested set theorem, and works just as well for the Sorgenfrey plane.

The Niemytzki plane can be envisioned by thinking of hot air balloons. The basic sets then become those balloons anchored at points of $L$ and those balloons floating above $L$. We refer to rational balloons as those balloons $B_q$ anchored at rational points $q$ on $L$. Similarly, we refer to irrational balloons $B_p$ anchored at irrational points $p$ on $L$.

Since every subset of $L$ is closed, the rationals $Q$ and irrationals $L \setminus Q$ form two disjoint closed subsets of $L$. If $X$ is normal, then there are disjoint open subsets $U$ and $V$ such that $U \cup Q$ and $V \supset L \setminus Q$. Thus, for each $q \in Q$ we may find a corresponding rational balloon $B_q$ with positive radius.
lying inside $U$ and for each $p \in L \setminus Q$ we may similarly find an irrational balloon $B_p \subset V$ also with positive radius.

Now the rational and irrational balloons lying in $U$ and $V$ respectively cannot overlap. Hence, for example, given a rational point $q \in Q$ and its corresponding rational balloon $B_q$ with radius $r_q$, all irrational balloons anchored in the shadow of $B_q$ must have a radius $\leq r_q/4$.

This is easily seen by taking the worst possible scenario, i.e., where the smaller balloon is anchored at the edge of the shadow of the larger balloon. Then the distance between anchors is exactly the radius of the larger balloon, say $r$. Allowing the smaller balloon to have as large a radius as possible, say $s$, so that the two balloons touch, simple geometry shows

$$(r + s)^2 = r^2 + (r - s)^2 \quad \text{or} \quad s = r/4.$$  

Now choose some rational balloon $B_0$ lying inside $U$. In $B_0$'s shadow, on the right side, choose irrational $B_1 \subset V$. On the left side in $B_1$'s shadow, choose rational $B_2 \subset U$. Continuing, we form a sequence $B_0, B_1, B_2, \ldots$ of rational and irrational balloons in $U$ and $V$ respectively such that each is anchored in the shadow of the previous balloon, and in such a way that the anchors alternate sides—that is, if $B_n$ lies to the right of $B_{n-1}$'s anchor, then $B_{n+1}$ lies to the left of $B_n$'s anchor.

This alternation of left, right, ... ensures a nested sequence of intervals $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq \ldots$ where $I_n$ is the interval between $B_n$'s and $B_{n+1}$'s anchors. Denoting $B_k$'s radius by $r_k$, we have $r_n \leq r_{n-1}/4$ and therefore $r_n \leq r_0/4^n$ for every $n \in \mathbb{N}$. By the nested set theorem, some point $x$ must lie in all the intervals $I$.

If $x$ is an irrational point of $L$, then there is some irrational balloon $B_x$ with positive radius $r_x$ such that $B_x \subset V$. But $B_x$ is anchored in the shadow of every rational balloon $B_{2n}$, $n \in \mathbb{N}$ in the sequence $B_0, B_1, B_2, \ldots$ and hence

$$r_x \leq r_{2n}/4 \leq r_0/4^{2n+1}.$$  

for every $n \in \mathbb{N}$. This says $r_x = 0$ which contradicts $B_x$ having a positive radius. A similar contradiction arises if $x$ is rational.

The author, a graduate student, thanks Professor Transue for the problem.

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**A New View of Goldbach's Conjecture**

Professor Simon Berkovich (George Washington University) notes that the famous Goldbach Conjecture, made in 1742, that

Every even integer greater than two is a sum of two primes.

can be stated in another form:

Every integer greater than one is the average of two primes.

He writes:

This has a physical interpretation. If we think of a wave spreading out from an integer, then, if Goldbach's Conjecture is true, it will simultaneously hit a lesser and greater prime number.

To me, this makes the conjecture (which is certainly true) even more mysterious.
Chapter Reports

Professor Gerald W. Young, reporting for the Ohio Nu Chapter (University of Akron), says that the department of mathematical sciences, with assistance from the chapter, awarded eighteen one-year memberships in five organizations (AMS, SIAM, ASA, MAA, ACM) and fourteen scholarships.

The Pennsylvania Omicron Chapter (Moravian College) sponsored the eighth annual Moravian College Student Mathematics Conference in February 1994, with 62 participants from 15 colleges and universities. The keynote speaker was Dr. Diane L. Souvaine, of the National Science Foundation, whose topic was "Geometric computations and applications." There were twelve undergraduate student speakers as well.

The Wisconsin Delta Chapter (St. Norbert College) held its Northeastern Wisconsin Math Seminar Series, with several guest speakers, hosted the annual Pi Mu Epsilon Regional Undergraduate Math Conference, and, with SNC’c Math Club, held the annual SNC High School Math Meet.

Professor James R. Weaver reports that the Florida Kappa chapter (University of West Florida) was addressed by the president of the university at its induction ceremony. The chapter also has subsidized members to attend meetings and assisted in the annual northwest Florida Mathcounts contest.

Letter to the Editor

In the Spring, 1994 issue (vol. 9, no. 10) of the Pi Mu Epsilon Journal, a problem on page 646 asks "What is the largest integer that in base-16 is an ordinary English word?" The suggested answer, on page 661, of defacaded, is certainly not an ordinary English word, if indeed it is an English word at all. (I did not find "to facade" as a verb in any dictionary at my disposal, which makes the longer construction highly implausible.) In any case, under similar rules of admissibility, I can top it. How do we describe a necklace made with ten glass spherules? Clearly, it is a decabeaded necklace! This converts from base-16 to the decimal value 956,884,233,709, if my hand-held calculator can be trusted. I should also point out that since decabeaded used only the first five letters of the alphabet, it could also be the representation of a number in base-15.

A more sweeping generalization of this problem is: for each of the $2^{26} - 1 = 67,108,863$ non-empty subsets of the 26 letters of the alphabet, what is the longest English word which can be formed using only letters from that subset? I particularly like the 8-letter word dedeeded (also a palindrome!) from the 2-letter subset \{d, e\}. There are only 325 subsets consisting of two letters, and unless one is a vowel and the other a consonant, the possibilities are very limited. This reduces us to 125 sets, where I have allowed y to be paired with either a vowel (as in yoyo) or a consonant (as in lyly).

Solomon W. Golomb
University of Southern California

Editorial note: This may be the last word on hexadecimal words. Since English is not an agglutinative language, the act of making a ten-bead necklace cannot be called a decabeaddeed, and describing the once-tenbeaded necklace as dedecabeaded is strained. In any event, the proper reaction to Professor Golomb’s linguistic virtuosity is the longest base-2 word: oooooo!

A Conjecture

Xuming Chen (University of Alabama, Tuscaloosa) conjectures that any prime is a sum of a prime and two squares, as $5 = 3 + 1^2 + 1^2$, $7 = 2 + 1^2 + 2^2$, $11 = 3 + 2^2 + 2^2$, $1997 = 1987 + 3^2 + 1^2$. This conjecture is very likely to be true and very likely to be unprovable, but it might be of interest to set a computer the task of determining in how many ways a prime $p$ can be so represented. I have no idea at what rate the number of representations would grow as $p$ increases.
Laplace Transforms

Professor James Chew (North Carolina A & T State University) offers another proof that, if the Laplace transform of \( f(t) \) is \( F(s) \) then the transform of \( \int_0^t f(\tau) \, d\tau \) is \( \frac{F(s)}{s} \). Suppose that \( f(t) \) has a formal power series, \( \sum_{n=0}^{\infty} a_n t^n \). Then \( \int_0^t f(\tau) \, d\tau = \frac{\sum_{n=0}^{\infty} a_n t^{n+1}}{n+1} \). Since \( \mathcal{L}\{t^n\} = \frac{k!}{s^{n+1}} \), we have

\[ \mathcal{L}\{\int_0^t f(\tau) \, d\tau\} = \sum_{n=0}^{\infty} a_n \frac{n}{n+1} = \frac{1}{s} \sum_{n=0}^{\infty} a_n \mathcal{L}\{t^n\} = \frac{F(s)}{s}. \]

A Triangle

In the last issue of the Journal (10 (1994-99) #1, 25), Andrew Cusumano asserted that, in the figure on the right, where the angles at A, B, and C in the equilateral triangle \( ABC \) have been bisected twice, \( EF \) is perpendicular to \( DC \) and that \( GH \) is parallel to \( BC \). Paul Bruckman (Edmonds, Washington) shows that this is correct:

We embed the figure in the complex plane, with points represented by complex numbers. Let

\[ (1) \quad A = 1, \quad B = a_1, \quad C = \omega^2, \quad \text{where} \quad a_1 = e^{2\pi i/3}. \]

Clearly, \( G = 0 \). Let \( J \) denote the intersection of the first quadrisectors (uppermost in the figure) of angles \( B \) and \( C \) (\( J \) lies on line \( AG \)). Since \( 1 + \omega + \omega^2 = 0 \) and \( E = (A + C)/2 \), then

\[ \text{for} \quad n \geq 0, \quad a_n = \frac{(-1)^n}{n+1}. \]

Let \( \mathcal{L}\{t^n\} = k! \frac{1}{s^{n+1}} \). Then

\[ \mathcal{L}\{\int_0^t f(\tau) \, d\tau\} = \sum_{n=0}^{\infty} a_n \frac{n}{n+1} = \frac{1}{s} \sum_{n=0}^{\infty} a_n \mathcal{L}\{t^n\} = \frac{F(s)}{s}. \]

To determine \( H \), we note that it is the intersection of \( EF \) with \( CJ \). Thus real \( r \) and \( s \) exist, with \( 0 < r < 1 \) and \( 0 < s < 1 \), such that

\[ H = (1-r)E + rF = (1-s)C + sJ. \]

We may substitute the expressions for \( C, E, F, \) and \( J \) in \( (1)-(3) \), substitute \((-1 + i\sqrt{3})/2 \) and \((-1 - i\sqrt{3})/2 \) for \( w \) and \( w^2 \) respectively, and equate real and imaginary parts. This yields a pair of equations in \( r \) and \( s \) which yield \( r = s = 1/\sqrt{3} \). Then substituting this in the expression for \( H \) yields \( H = -ie \) (note that \( J, H, F, \) and \( D \) lie on the circle \( |z| = c \)). Since \( H - G \) and \( C - B \) are pure imaginary, we see that \( GH \) is parallel to \( BC \).

Editorial note. While the problem could have been stated by Euclid, he could never have constructed a solution which involved complex numbers. Two questions arise: does a purely geometric proof exist, and, if one does, is it superior to one using is? The answer to the first surely is "yes", but the answer to the second is less clear. On the one hand, in the universe of mathematics we should be free to use whatever we need to solve a problem. On the other, many mathematicians have a feeling that there is something not quite right about using nineteenth-century mathematics to solve what could have been a second-century problem, and prefer to avoid such things. For example, Selberg and Erdős were acclaimed for proving the prime number theorem without using complex variables. Would Bach have written for electronic instruments, had he been able? Should his music be performed on them? Analogous questions, analogously difficult to answer. The art and
science of mathematical esthetics is in its infancy.

Verse

Ode to e

Paul S. Bruckman
Edmonds, Washington

I dedicate this rhyme and rhythm
To sing the natural logarithm.
That is, its base, great Euler’s e
Is lauded in this rhapsody.
In higher math and nature both
We learn that e expresses growth
Of special, exponential kind,
And e’s the limit, you will find,
As n grows larger by the hour,
Of something raised to the nth power,
That something being none other, then,
Than n + 1 all over n.
Another well-known fact, my dearies,
Is given by the endless series
Where each term’s the nth power of x
Divided (there’s no need for checks)
By n!, with such sum
Converging to e x, by gum,
Provided that we sum from naught;
The one who found this, we are taught,
Was Leibniz, and another Kraut,
Herr Lindemann, proved beyond doubt
That e’s a transcendental number
(although his proof induces slumber).
For elegance, Euler’s relation
Is unsurpassed; its declaration

Identities

In the last issue of the Journal (10 (1994-99) #1, 43-44), Kenneth Davenport gave the following identities.

1. \(1\cdot(1 + 2 + \ldots + n) = n(n + 1)/2\).
2. \((1 + 2^3 + \ldots + n^3) = n^2(n + 1)^3/2\).
3. \(3\cdot(1^5 + 2^5 + \ldots + n^5) + 1\cdot(1^3 + 2^3 + \ldots + n^3) = n^3(n + 1)^3/2\).
4. \(4\cdot(1^7 + 2^7 + \ldots + n^7) + 4\cdot(1^5 + 3^5 + \ldots + n^5) = n^4(n + 1)^4/2\).
5. \(5\cdot(1^9 + 2^9 + \ldots + n^9) + 10\cdot(1^7 + 2^7 + \ldots + n^7) + 1\cdot(1^5 + 2^5 + \ldots + n^5) = n^5(n + 1)^5/2\).
6. \(6\cdot(1^{11} + 2^{11} + \ldots + n^{11}) + 20\cdot(1^9 + 2^9 + \ldots + n^9) + 6\cdot(1^7 + 2^7 + \ldots + n^7) = n^6(n + 1)^6/2\).

They are explained by Odoardo Brugia and Piero Filipponi:

We were not aware of the cute identities reported by K. B. Davenport in this journal (10 (1994-99) #1, 43-44). They can be proved as follows.

For \(r, n, \) and \(k\) natural numbers, let us define \(f(r)\) to be the greatest integer not exceeding \((r - 1)/2\), \(S_n(k) = \sum_{j=1}^{\lfloor f(r) \rfloor} j^k\), and

\[
X(n, r) = \sum_{i=0}^{\lfloor f(r) \rfloor} \binom{r}{2i+1} S_n(2r - 2i - 1).
\]

The proof that \(X(n, r) = n^r(n + 1)^r/2\) follows:
\[ X(n, r) = \sum_{j=1}^{n} \left( \sum_{i=1}^{j} (2i+1) \right) j^{2r-2i-1} \]
\[ = \frac{1}{2} \left[ \sum_{j=1}^{n} j^r (j+1)^{y} - \sum_{j=1}^{n} j^r (j-1)^{y} \right]. \]

Put \( j - 1 = h \) in the second summation of the above identity, thus getting

\[ X(n, r) = \frac{1}{2} \left[ n^r (n+1)^{y} + \sum_{j=1}^{n-1} j^r (j+1)^{y} - \sum_{h=0}^{n-1} h^r (h+1)^{y} \right] \]
\[ = n^r (n+1)^{y}. \]

Odoardo Brugia and Piero Filipponi
Fondazione Ugo Bordoni
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I-00142 Rome, Italy

(Xuming Chen, of the University of Alabama, Tuscaloosa, provided a similar derivation.)

Trisecting the Paper

Augustus De Morgan (1806-1871), whose name is attached to De Morgan's Laws, after discussing a trisection of the angle wrote

There is one trisection which is of more importance than that of the angle. It is easy to get half the paper on which you write for margin; or a quarter; but very troublesome to get a third. Show us how, easily and certainly, to fold the paper into three, and you will be a real benefactor to society.


Though we no longer feel the nineteenth-century urge to fold paper into vertical thirds, should the need again arise John Kasbohm (Oak Park, Illinois) has a solution. He writes:

"With a sheet \( W \) wide and \( L \) long, \( L \geq 4 \sqrt{W} \), you can twice bi-fold the long right edge, making just little pinches to show where folds across the paper would be, because we wouldn't want deep creases all over the place. (See Figure 1.) This makes little dents at \( L/2 \) and \( 3L/4 \) on the edge. Now lift the lower right corner up and lay it across the face of the sheet, so that the dent at \( 3L/4 \) just touches the left hand edge and the top and right hand edges still meet at the upper right corner. (See Figure 2.) Again this time you don't have to make a full sharp crease, but just a little pinch at that corner. The \( L/2 \) dent is now two-thirds of the way across the sheet and you can fold the sheet vertically through that point. The finished sheet with the one-third margin is still in pretty good shape, especially if you have some origami practice. If \( L \) is too long you can use only an upper part of it.

'Of course the easiest way is to measure and divide, but I suppose that is not what De Morgan meant.'"

A difficulty is that \( 8\frac{1}{2} \times 11 \) paper is not quite long enough (by a third of an inch) for the method to work. One way around this is to make the sheet slightly longer, with a temporary paper-clipped extension. Another is...
to crease it in half vertically and apply the method to that half, thus getting one-sixth of the width which can quickly be doubled to get one-third. Though neither may be as easy as De Morgan would have wanted, I think that he would have been pleased with the solution.

**Editorial Comment**

(In case there is any doubt, the following comments do not necessarily reflect the opinions of any of the officers of Pi Mu Epsilon or, for that matter, any of its members. They contain the personal views of the editor and do not need to be taken seriously or, for that matter, read at all.)

A not-bad definition of mathematics is that it is the art and science of reasoning about and with quantities. The definition is **not** complete because there are branches of mathematics where there are **no** quantities—mathematical logic, for example—but whatever your definition is, it will have some mention of reason in it. Mathematics is **not** broad enough to include **all** of reason (lawyers reason too, sometimes) but reasoning is what it does. In fact, it was mathematics that taught the human race to reason, but that's another story.

The thing about reason is that it's so ... reasonable. "Socrates is a human. All humans are mortal. Therefore Socrates is mortal." Of course. It follows. It's logical. It's easy to understand. All you have to do is listen to reason, right?

Wrong. Teachers of mathematics spend thousands of hours trying to get students of mathematics to listen to reason, and they do not always succeed. Sometimes they do, of course, but **many** times they do not, as is shown by the number of appearances of \( \sqrt{a^2 + b^2} = a + b \) on tests (even on calculus tests), along with \( \sin 3x = 3 \sin x \) *(yes, that does make life easier, but it's not reasonable—sines don't get as large as 3)*, and even the horrible

\[
\frac{2x^2 + 3y^2}{x + y} = 2x + 3y
\]

(gotten by applying a corollary of the Law of Universal Cancellation: whenever a symbol appears in the numerator and the denominator of a fraction, it may be cancelled). It's usually possible to convince students of mathematics that those equalities are not reasonable, but sometimes it takes effort.

However, there is a class of people who deal in mathematics but who will not listen to reason no matter how much effort is spent, namely mathematical cranks. Mathematical cranks are people who think that they have done the impossible, like trisecting angles with straightedge and compass alone, or that they have done something that they have not, like prove Fermat's Last Theorem. They try to convince mathematicians, using reason, that they have done what they claim to have done, but reason is not of much use in convincing them that what they say is not correct.

Recently a crank wrote me,

If **EVERYBODY** loves mathematics, all the bad things on this planet will pass into history. I can edit a mathematics magazine because I know **EVERYTHING** about mathematics.

Shall I reason with him? He went on,

It is a pity that mathematics is unpopular as it is the worst subject in every school because every time a mathematics teacher gives his class mathematics homework, none of the students do their mathematics homework which is never encouraged by their parents and all mathematics teachers approve of students not doing their mathematics homework thus making mathematics a futile subject.

A possible reason for that will be given in the last paragraph.

You might think that an angle trisector would be convinced of his error if he was shown that his construction is not accurate. Not so. For example, there was the angle trisector who was shown that, using trigonometry, the angle he got when his construction was applied to a 60° angle was something like 19° 56' 42". He saw that, but he did not therefore conclude that his construction was wrong. The choice was between his trisection and trigonometry, so he concluded that trigonometry was wrong, and that all the trigonometry textbooks would have to be rewritten. How are you going to reason with him?

Or with the circle-squarer (circle-squarers construct with straightedge and compass alone squares with the same areas as circles, something as impossible as trisecting angles) who said, when asked how he knew that his construction was correct,

The author was supernaturally taught the exact measure of the
circle. All knowledge is revealed directly or indirectly, and the truths hereby presented are direct revelations and are due in confirmation of scriptural promises.

Reason is helpless against revelation.

Then there was the crank who considered the four-color theorem (that no map needs more than four colors if adjacent countries have different colors) and went it two better by claiming to have proved that two colors were enough. Reason was not enough to prove to him that he was not correct.

Cranks can get around anything. An angle trisector took care of the fact that the trisection was impossible by saying

The author resents the negative implications since, if everyone were to accept statements as valid, there would be very little progress.

Of course, no progress in the trisection is all we can get. The same trisector had been told that Wantzel had proved that the trisection was impossible in 1837, and his retort to that was

A mathematical proof is merely an established approximation, indicating a limitation of errors to a minimum applicable to each on hand to be solved, and from a point, or point, of reference as they appear.

I did not make that up. I could not make that up. Absolute non-sense is very hard to create. The trisector's point, whatever his words, was that the proof did not count. His trisection was right, and any reason to the contrary could be brushed aside. He would not listen to reason.

One circle-squarer who said that \( \pi \) was 2518 = 3.125 had a voluminous correspondence with a mathematician. The mathematician brought up the proof that \( \pi \) was irrational and that Archimedes had shown that \( \pi \) was greater than 3 \( \frac{10}{71} \) ( = 3.1408... > 3.125 ) more than two thousand years ago, but they meant nothing to the circle-squarer. Eventually he was backed into a corner and had to admit that the reason that \( \pi \) was 2518 was because he said so. He assumed that \( \pi = 2518 \).

If you dislike the term datum, then, by hypothesis, let 8 circumferences exactly equal 25 diameters.

He could do this because, he said

I think you will not dare to dispute my right to this hypothesis when I can prove by means of it that every other value of \( \pi \) will lead to the grossest absurdities.

Using exactly the same method, I can prove that \( 3^2 = 10 \). Here I go. By hypothesis, let \( 3^2 = 10 \). Then any other assumption, such as \( 3^2 = 9 \), leads to the grossest absurdities, in this case that \( 9 = 10 \).

You cannot reason with cranks. They will not follow the laws of logic, they will not listen to reason. Even students of mathematics have difficulty, sometimes, seeing things that are absolutely clear to their instructors and follow mathematically from things that have gone before. Students of mathematics sometimes do not even do their homework. Why is this?

The reason is clear. Mathematics is unnatural; mathematics is a perversion. Of course it is. What are you doing when you are doing mathematics? You are not doing something natural. You are sitting, hardly moving, your only bodily activity outside of some necessary breaths and heartbeats a few weak and flickering electric currents passing along the neurons of your brain. Occasionally you may move a hand to write a bit, but you then lapse again into motionlessness, eyes unfocussed, with only the electrons in your brain moving. You cannot do mathematics for very long, can you? The reason you cannot is that mathematics is not what your body, with its dexterous fingers and powerful legs, was made for. Our legs were made to run after our next meal, and our fingers to pick it apart after we catch it. Sitting and thinking is an unnatural activity, and we have not yet gotten used to it. What was it that last made you think, "Wow! I had a really good time!"? Wasn't it after some activity that was entirely mindless? Of course it was. Evolution has not yet progressed far enough for reasoning to be painless, much less enjoyable. It will take another hundred thousand years, or maybe a million. At present, the race finds it hard. Thinking, especially thinking about mathematics, is unnatural. However, mathematics is not a bad perversion. Keep listening to reason: the more we try to hear, the easier it will become.
Pentagonal Number Identities

Kenneth Davenport (Pittsburgh, Pennsylvania) points out a relationship between the generalized pentagonal numbers,

\[(3n \pm 1)n/2, \quad n = 1, 2, \ldots, 1, 2, 5, 7, 12, 15, 22, 26, \ldots\]

and partial sums of cubes:

\[
\frac{(2 - 1)^2}{2} = \frac{1^3}{1^3},
\]

\[
\frac{(5 - 2)^2}{8} = \frac{1^3 + 2^3}{2^3},
\]

\[
\frac{(7 - 5)^2}{3} = \frac{1^3 + 2^3 + 3^3}{3^3},
\]

\[
\frac{(12 - 7)^2}{16} = \frac{1^3 + 2^3 + 3^3 + 4^3}{4^3},
\]

\[
\frac{(15 - 12)^2}{5} = \frac{1^3 + 2^3 + 3^3 + 4^3 + 5^3}{5^3}.
\]

The denominators of the odd-ranked fractions are the odd numbers 1, 3, 5, ..., and the denominators of the even-ranked fractions are successive multiples of 8.

While it is a simple exercise in algebra to verify that the identities hold in general, they may be useful as an example, or as a curious connection between two sequences that on the face of it have nothing in common other than their geometric origin. And pentagons are two-dimensional, while cubes are three-dimensional.

Mathacrostics

Solution to Mathacrostic 39, by Robert Forsberg (Fall, 1994).

Words:

A. Tchebyshev  I. effective pitch  Q. mestizo
B. icosidodecahedron  J. roots for  R. Ibn Abd Rabbihi
C. Mithridates  K. raffia  S. laetrile
D. ordinary point  L. independent events  T. kennings
E. transfinite set  M. spinning frame  U. Yablonoi
F. Hippasus of Metapontum  N. trident  V. Windows
G. Ypsilanti  O. high pressure physics  W. a penny postage stamp
H. Fermat’s last theorem  P. Ettingshausen effect  Y. Yangchuanchan

Author and title: Timothy Ferris, The Milky Way.

Quotation: Newton's surviving drafts of the Principia support Thomas Edison’s dictum that genius is one percent inspiration and ninety-nine percent perspiration. Like Beethoven’s drafts of the opening bars of the Fifth Symphony, they are characterized less by sudden flashes of insight than by a constant, indefatigable hammering away at immediate, specific problems.

Solvers: THOMAS BANCHOFF, Brown University, JEANETTE BICKLEY, St. Louis Community College, PAUL S. BRUCKMAN, Highwood, Illinois, CHARLES R. DIMMINIE, St. Bonaventure University, VICTOR G. FESER, University of Mary, ROBERT C. GEBHARDT, Hopatcong, New Jersey, META HARRSEN, Durham, North Carolina, HENRY LIEBERMAN, Waban, Massachusetts, CHARLOTTE MAINES, Rochester, New York, DON PFAFF, University of Nevada—Reno, NAOMI SHAPIRO, Piscataway, New Jersey, STEPHANIE SLOYAN, Georgian Court College, and JOSEPH S. TESTEN, Mobile, Alabama. Late solution to #37 by VICTOR G. FESER, University of Mary.

Mathacroistic 40, constructed by ROBERT FORSBERG, follows on the next three pages. To be listed as a solver, send your solution to the editor.
A. A red-orange coloring material obtained from the seeds of *Bixa orellana*

B. The outer of the two layers forming the wall of spores such as pollen

C. Describing a curve at a point where a tangent may be drawn

D. To thrust out repeatedly

E. Literally "Tracker"; an animal that hunts out crocodile eggs

F. An element

G. He discovered in 1890 a curve that could intersect every point in a plane (2 wds)

H. He produced the first mathematical works produced by an Englishman in England (3 wds)

I. Great figure in the resistance of Wales to the English, d. 1170 (2 wds)

J. A small ceramic/metal vacuum tube

K. A solution used as an acid/base indicator

L. Hungarian writer, 1760-1820, *Hunnyas*

M. Author of the earliest sources of the Hexateuch
N. A method of finding prime numbers (3 wds)

O. Typical, accurately identified specimen of a species, but not a basis for a published description

P. Swiss mathematician, 1829-1900

Q. An asteroid of about 220km diameter

R. Equivalent to about 11.8 inches (hyph)

S. A word synthesized to mean: someone showing contempt for legalities

T. Author of The Dynamics of a Particle, 1865 (3 wds)

U. Out of use; retired from service

V. The entropy of the input to a communication channel when the output is known

W. An intracellular effector organelle in coelenterates

X. A unit of firewood volume

Y. Spread out in a definite form

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@eauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1995.

Correction

844. [Fall 1994] Proposed by Bill Correll, Jr., student, Denison University, Granville, Ohio.
If \( F_n \) denotes the nth Fibonacci number (\( F_1 = F_2 = 1 \) and \( F_{k+2} = F_k + F_{k+1} \) for \( k \) a positive integer), evaluate

\[
\sum_{n=1}^{\infty} \binom{n}{k} \frac{F_n}{2^{n+k}}.
\]
(The index of the summation was incorrect.)

Problems for Solution

849. Proposed by L. A. Bohr, Great Works, Maine.
Solve this base 8 addition alphametic: \( THIS + IS = WORK. \)
Proposed by Bill Correll, Jr., Denison University, Granville, Ohio.

Clearly the following integral evaluation is incorrect. Find the flaw. By letting \( u = -x \) we get that

\[
I = \int \ln(e^x + 1) \, dx = -\int \ln(e^{-u} + 1) \, du = -\int \ln \left( \frac{e^u + 1}{e^u} \right) \, du
\]

so that \( I = x^2/4 + C' \). (See Problem 828.)

Proposed by Bill Correll, Jr., Denison University, Granville, Ohio.

In triangle \( ABC \) let Cevian \( AD \) bisect side \( BC \) and let Cevians \( BE \) and \( BF \) trisect side \( CA \). Let \( AD \) intersect \( BE \) at \( P \) and \( BF \) at \( R \), and let \( CP \) meet \( BF \) at \( Q \). See the figure. If the area of triangle \( ABC \) is 1, find the area of triangle \( PQR \).

Proposed by Rex H. Wu, Brooklyn, New York.

Let \( E \) be a point inside square \( ABCD \) with \( BE = x \), \( DE = y \), and \( CE = z \). If \( x^2 + y^2 = 2z^2 \), find the area of \( ABCD \) in terms of \( x \), \( y \), and \( z \).

Proposed by Charles Ashbacher, Cedar Rapids, Iowa.

This problem was submitted by Vietnam for the 1990 International Mathematical Olympiad and has appeared in booklets overseas. If real numbers \( x \), \( y \), \( z > 0 \), then prove that

\[
\frac{x^2 y}{z} + \frac{y^2 z}{x} + \frac{z^2 x}{y} \geq x^2 + y^2 + z^2.
\]

Proposed by Jayanthi Ganapathy, University of Wisconsin at Oshkosh, Oshkosh, Wisconsin.

Let \( a \) and \( b \) be two nonzero real numbers such that

\[
a^3(3a^2 - 5ab + 3b^2) = b^3(5a^2 - 3ab + 5b^2).
\]

Find the values of the expressions \( (a^2 + b^2)/a^2 \) and \( (a^2 - b^2)/ab \).

Proposed by Florentin Smarandache, Phoenix, Arizona.

Prove that a square matrix of integers, having in each row and in each column a unique element not divisible by a given prime \( p \), is nonsingular.

Proposed by Paul S. Bruckman, Highwood, Illinois.

Starting with a regular \( n \)-gon whose side is of unit length, snip off congruent isosceles triangles from each of its vertices, resulting in a regular \( 2n \)-gon. Repeat the process indefinitely. Find the ratio of the area of the limiting circle to that of the original \( n \)-gon.

Proposed by Andrew Casumano, Great Neck, New York.

Find all prime numbers whose reciprocals have repetends of exactly seven decimal places.

Proposed by David Iny, Baltimore, Maryland.

It is known that the rational numbers in the interval \( [0, 1] \) can be enumerated. Let \( \{r_k\}_{k=1}^\infty \) be such an enumeration and pick \( \varepsilon \) such that \( 0 < \varepsilon < 1 \). Take an interval \( I_k \) of length \( \varepsilon 2^{-k} \) centered on each \( r_k \). Then the sum of all these interval lengths \( \sum_{k=1}^\infty r_k = \varepsilon < 1 \). Show how to find a real number in \( [0, 1] \) and not contained in any of the intervals \( I_k \).

Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

Sum in closed form the series

\[
S = \sum_{n=1}^\infty \frac{1}{n + 1} \left( \frac{-1/2}{n} \right)^n, \quad \text{where } \left( \frac{m}{n} \right) = \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}.
\]
This problem originally appeared in a column by the Japanese problems columnist Nob Yoshigahara. Find the minimal positive integer \( n \) so that \( 2n + 1 \) circles of unit diameter can be packed inside a 2 by \( n \) rectangle.

Evaluate in closed form the sum
\[
S(n, k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{2j}{k}.
\]

Solutions

Find all solutions to the multiplication alphametic
\[
(I)(DINE) = ENID.
\]
That is, find the form(s) taken by all solutions in all bases.

I. Solution by Paul S. Bruckman, Highwood, Illinois.
Let \( b \) denote the base of the alphametic. We show that if \( r \geq 3 \), then a solution is provided by taking
\[
b = r^2, \quad D = r, \quad I = r - 1, \quad N = r^2 - r - 1, \quad \text{and} \quad E = r^2 - r.
\]
Then \( E \cdot I = r^3 - 2r^2 + r = r^2(r - 2) + r \), so \( E \cdot I \equiv D \pmod{b} \), with a carry to the next column of \( r - I \). Next we have that
\[
N \cdot I + r - 2 = r^3 - 2r^2 + r + r = r^2(r - 2) + r - 1 = I \pmod{b}
\]
with a carry of \( r - 2 \) again. Next \( I^2 \cdot r - 2 = r^2 - r - 1 = N \) with no carry. Finally, \( D \cdot I = r^2 - r = E \). Therefore, we have that \( (I)(DINE) \equiv ENID \pmod{b} \).

II. Comment by Victor G. Fever, University of Mary, Bismarck, North Dakota.
This problem is much trickier than it looks. I set up a little computer program to check successive bases. Solutions started to appear: bases 9, 16, 25, 36. Ah hah! There seems to be a pattern here. Right. The next solution was for base 39, and then base 47. I checked to base 200 and found the additional solutions for bases other than square numbers in the table that follows. There are fragments of patterns, but that is all I can find. It is exasperating to find, for example, that these five bases include two primes, one odd composite, one power of 2, and one composite with both odd and even factors.

<table>
<thead>
<tr>
<th>base</th>
<th>( D )</th>
<th>( I )</th>
<th>( N )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>1</td>
<td>16</td>
<td>37</td>
<td>23</td>
</tr>
<tr>
<td>47</td>
<td>4</td>
<td>9</td>
<td>43</td>
<td>37</td>
</tr>
<tr>
<td>109</td>
<td>1</td>
<td>45</td>
<td>107</td>
<td>63</td>
</tr>
<tr>
<td>128</td>
<td>16</td>
<td>5</td>
<td>26</td>
<td>80</td>
</tr>
<tr>
<td>200</td>
<td>5</td>
<td>11</td>
<td>128</td>
<td>55</td>
</tr>
</tbody>
</table>

III. Comment by Richard I. Hess, Rancho Palos Verdes, California.
If the base is \( b \), then we must have that
\[
(I)(Db^3 + Ib^2 + Nb + E) = Eb^3 + Nb^2 + Ib + D,
\]
and hence
\[
(E - ID)b^3 = (I^2 - N)b^2 + I(N - 1)b + IE - D
\]
with \( 0 \leq D, I, N, E \leq b - 1, I > 1, \) and \( D \cdot E \neq 0 \). If we have that \( E = I \cdot D \), then
\[
(N - I^2)b^2 = I(N - 1)b + D(I^2 - 1).
\]
If also \( N = I^2 \), then \((Ib \cdot D)(I^2 - 1) = 0\), which is impossible. Hence we have \( N > I^2 \), so \( I < \sqrt{b} \). Further analysis might produce some results, but it appears tedious.

Also solved by VICTOR G. FESER, University of Mary, Bismarck, ND, RICHARD I. HESS, Rancho Palos Verdes, CA, REX H. WU, Brooklyn, NY, and the PROPOSER.
Prove that there are no real integral solutions to the set of equations

\[
\begin{align*}
(x^3 + 6x^2 - 159)y &= 160, \\
(y^3 + 6y^2 - 159)z &= 160, \\
(z^3 + 6z^2 - 159)x &= 160.
\end{align*}
\]

You may not assume that a putative solution would possess any symmetry.


We see that \(x\), \(y\), and \(z\) are factors of 160, as are also \(x^3 + 6x^2 - 159\), \(y^3 + 6y^2 - 159\), and \(z^3 + 6z^2 - 159\). Now \(|x^3 + 6x^2 - 159| \leq 160\) for \(x \leq 6\). Furthermore, \(x^3 + 6x^2 - 159\) is a factor of 160 only for \(x = 4\), in which case \(x^3 + 6x^2 - 159 = 1\). Similarly, \(y = z = 4\). But then the stated products are each 4, not 160. Hence there is no solution.

II. Solution by Paul S. Bruckman, Highwood, Illinois.

Let \(P(x) = x^3 + 6x^2 - 159\). By checking \(x = 0, 1, 2, 3, 4\), we find that \(P(x) \equiv 0 \pmod{5}\) for all \(x\). In any putative solution, it must be the case that the product of integers \(P(x)\) and \(y\) is 160, so then \(y \equiv 0 \pmod{5}\). None of \(P(-5), P(0),\) and \(P(5)\) is a divisor of 160, and if \(|x| \geq 10, |P(x)| > 160\). Thus \(P(x)\) cannot be a factor of 160 for any permissible \(x\), and any putative solution is impossible.

III. Solution by Kandasamy Muthuvel, University of Wisconsin at Oshkosh, Oshkosh, Wisconsin.

From the third given equation, \(x\) divides 160. Then, from the first equation, we must have \(x\) divides 159. Since \(x\) and 159 are relatively prime, then \(x\) divides \(y\). Similarly, \(y\) divides \(z\) and \(z\) divides \(x\), so then \(x = y\). Therefore,

\[
(x^3 + 6x^2 - 159)x = 160.
\]

Since \(x\) and 159 are relatively prime, then \(x\) and \(x^3 + 6x^2 - 159\) are relatively prime. Thus the possible values for \(x\) are \(\pm 1, \pm 5,\) and \(\pm 32\), but it is readily checked that none of them does yield a solution.

IV. Comment by the Proposer.

The following generalization is readily proved by the same methods. If \(l \geq 3\), \(m\) a 3, and \(n\) a 3 are integers, then there are no real integral solutions to the set of equations

\[
\begin{align*}
(x^l + 6x^{l-1} - 159)y &= 160, \\
(y^m + 6y^{m-1} - 159)z &= 160, \\
(z^n + 6z^{n-1} - 159)x &= 160.
\end{align*}
\]

Also solved by SEUNG-JIN BANG, Ajou University, Suwon, Korea, BILL CORRELL, JR., Denison University, Granville, OH, VICTOR G. FESER, University of Mary, Bismarck, ND, STEPHEN I. GENDLER, Clarion University of Pennsylvania, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY College at Oneonta, DAVID S. SHOBE, New Haven, CT, and the PROPOSER.

825. [Spring 1994] Proposed by Leon Bankoff, Los Angeles, California:

Let \(O\) be a point inside the equilateral triangle \(ABC\) whose side is of length \(s\). Let \(OA, OB, OC\) have lengths \(a, b, c\) respectively. Given the lengths \(a, b, c\), find length \(s\).

I. Solution by Rex H. Wu, Brooklyn, New York.

Let \(\triangle ABC\) be oriented counterclockwise and rotate \(\triangle AOC\) \(60^\circ\) about point \(C\) to triangle \(BNC\) as shown in the figure. Then \(AO = BN = a\) and \(OC = NC = ON = c\) so \(\angle NOC = 60^\circ\). Let \(\theta = \angle BON\). Applying the law of cosines to triangles \(BON\) and \(BOC\), we get

\[
\cos \theta = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \cos(\theta + 60^\circ) = \frac{b^2 + c^2 - s^2}{2bc}.
\]

From the left equation we get that

\[
\sin \theta = \sqrt{\frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{2bc}}.
\]
Now substitute these values into the equation
\[
\cos (90°) = \cos \theta \cos 60° - \sin 9 \sin 60°
\]
and then solve for \(s\) to get that
\[
s = \sqrt{\frac{a^2 + b^2 + c^2}{2} + \sqrt{\frac{3}{2}(2bc)^2 - (b^2 + c^2 - a^2)^2}}
\]

Note that, although it does not appear so, the quantity inside the inner radical is symmetric in \(a\), \(b\), and \(c\). That is,
\[
(2bc)^2 - (b^2 + c^2 - a^2)^2 = (2ca)^2 - (c^2 + a^2 - b^2)^2
\]
\[
= (2ab)^2 - (a^2 + b^2 - c^2)^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - b^4 - c^4 - a^4.
\]
The symmetric statement of the problem implies a symmetric solution. Ed.

**Comment by Seung-Jin Bang, Ajou University, Suwon, Korea.**


**Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.**

This problem with solution appeared previously as number 39, Crux Mathematicorum, 1976, p. 7. An extension to a regular tetrahedron appeared as number 1087, ibid, 1987, p. 120. My solution there generalized the problem to show that for positive numbers \(a_1, a_2, \ldots, a_n\), there exists a regular simplex \(S: A_0A_1 \ldots A_n\) and a point \(P\) in its space such that \(PA_i = a_i\), \(i = 0, 1, \ldots, n\), if and only if
\[
I = \{\Sigma a_i^2\}^2 - n\Sigma a_i^4 \geq 0.
\]
The side length \(s\) of \(S\) is then given by
\[
ns^2 = \sum a_i^2 \pm \sqrt{(n+1)I}
\]
where the \(\pm\) sign is chosen according to whether the point \(P\) is in the interior or exterior of \(S\).

**III. Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.**

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\]
where the \(\pm\) sign is chosen according to whether the point \(P\) is in the interior or exterior of \(S\).
making \( L \) a large interval. There would be no such gap if one could always find another such prime \( q = (2r)^2 + 1 \) with \( 2r < p - 2r \). Unfortunately, according to Rosen, Elementary Number Theory, 2nd ed., p. 65, it is not even known whether there are infinitely many primes of the form \( x^2 + 1 \). For other primes of the form \( 4n + 1 \), there appears to be no rule to determine how far apart \( a \) and \( b \) are. All we know for sure is that \( L \) always contains \( (p - 1)/2 \). Most intervals \( I \) will contain more integers, but it appears we cannot guarantee that all the gaps will be filled. The table below shows that the theorem appears to be true. Note that the number of integers in the set \( I_p \) is equal to \( b_p - a_p \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( a_p )</th>
<th>( b_p )</th>
<th>( b_p - a_p )</th>
<th>( p )</th>
<th>( a_p )</th>
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</tr>
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<tr>
<td>5</td>
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<td>1</td>
<td>53</td>
<td>23</td>
<td>30</td>
<td>7</td>
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<td>13</td>
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<td>23</td>
<td>101</td>
<td>10</td>
<td>91</td>
<td>81</td>
</tr>
</tbody>
</table>

Editorial comment. It appears that the product \( P \) is a perfect square only for \( n = 3 \). Our proof is not complete, however, so the problem is open and further comments and, if possible, a full solution are invited.


Let \( P \) be a point on diagonal BD of square ABCD and let Q be a point on side CD such that APQ is a right angle. Prove that AP = PQ.

I. Solution by Francine Bankoff, Beverly Hills, California.

Let S and T be the feet of perpendiculars dropped from P upon DC and DA respectively. Then triangles PSQ and PTA are similar since their corresponding sides are perpendicular. Since PS = PT because PSDT is a square, the triangles are also congruent, with PA = PQ.

II. Solution by Francine Bankoff, Beverly Hills, California.

Since QDA and APQ are both right angles, the points D, A, P, and Q lie on a circle whose diameter is AQ. It follows that chords QP and AP are equal, each measured by the equal 45° angles QDP and PDA.

III. Comment by Leon Bankoff, Beverly Hills, California.

Although the diagram suggests that the point P be selected below the midpoint of DB, the stated proposal does not exclude a point above that midpoint. In that case the point Q falls on an extension of CD and solutions I and II remain valid with hardly any modification.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain, SEUNG-JIN BANG, Ajou University, Suwon, Korea, SCOTT H. BROWN, Auburn University, AL, PAUL S. BRUCKMAN, Highwood, IL, XUMING CHEN (3 solutions), University of Alabama, Tuscaloosa, BILL CORRELL, JR., Denison University, Granville, OH, ANDREW CUSUMANO, Great Neck, NY, MARK EVANS, Louisville, KY, ROBERT C. GEBHARDT, Hopatcong, NJ, SOLOMON W. GOLOMB, University of Southern California, Los Angeles, MONICA GREENWOOD, St. Bonaventure University, NY, RICHARD I. HESS, Rancho Palos Verdes, CA, JOE HOWARD, New Mexico Highlands University, Las Vegas, ROSALIE J. JUNGREIS, James Madison High School No. Woodmere, NY, MURRAY S. KLAUKIN, University of Alberta, Canada, HENRY S. LIEBERMAN (2 solutions), Waban, MA, PETER A. LINDESTROM, North Lake College, Irving, TX, DAVID E. MANES, SUNY College at Oneonta, G. MAVRIGIAN (3 solutions), Youngstown State University, OH, YOSHINOBU MURAYOSHI, Okinawa, Japan, KANDASAMY MUTHUVEL, University of Wisconsin-Oshkosh, JOHN F. PUTZ, Alma College, MI, GEORGE W. RAINY, Los Angeles, CA, DAVID S. SHOBE, New Haven, CT, ROMAN SZNAJDER, University of Maryland, Baltimore, L. J. UPTON, Mississauga, Ontario, Canada, REX H. WU (2 solutions), Brooklyn, NY, SAMMY YU and JIMMY YU, University of South Dakota,
Vermillion, and the PROPOSER.

Other methods of solution included arguing that \( PA = PC \) and then showing that triangle \( PCQ \) is isosceles with apex \( P \), applying the law of sines to triangles \( PQD \) and \( PAD \), and placing the figure on the Cartesian plane with \( A \) at the origin, \( B \) at \((1, 0)\) and \( D \) at \((0, 1)\).

Evaluate the integral

\[
\int \ln(e^x + 1) \, dx.
\]

Solution by Richard I. Hess, Rancho Palos Verdes, California.
"This integral cannot be expressed as a finite combination of elementary functions" - Gradshteyn and Ryzhik - 2.782.2.

For all \( b > 0 \), we have that

\[
I = \int_0^b \ln(e^x + 1) \, dx = \int_0^b [x + \ln(1 + e^{-x})] \, dx
\]

\[
= \frac{b^2}{2} + \int_0^b (e^{-x} - \frac{1}{2} e^{-2x} + \frac{1}{3} e^{-3x} - \ldots) \, dx
\]

\[
= \frac{b^2}{2} + \left[-e^{-x} + \frac{1}{4} e^{-2x} - \frac{1}{9} e^{-3x} + \ldots\right]_0^b
\]

\[
= \frac{b^2}{2} + \sum_{k=1}^\infty (-1)^k \frac{e^{-kb} - 1}{k^2}.
\]

If \( a < 0 \) then we have

\[
I = \int_0^a \ln(e^x + 1) \, dx = \int_0^a \ln(1 + e^x) \, dx
\]

\[
= \int_0^a \left(e^x - \frac{1}{2} e^{2x} + \frac{1}{3} e^{3x} - \ldots\right) \, dx
\]

\[
= \left[e^x - \frac{1}{4} e^{2x} + \frac{1}{9} e^{3x} + \ldots\right]_0^a
\]

\[
= \sum_{k=1}^\infty (-1)^k \frac{1 - e^{ka}}{k^2}.
\]

Hence the value of the indefinite integral depends upon whether \( x > 0 \) or \( x < 0 \). The value of any definite integral can be found by appropriately adding or subtracting one or both of the two forms above.

Also solved by Seung-Jin Bang, Ajou University, Suwon, Korea, Paul S. Bruckman, Highwood, IL, Joe Howard, New Mexico Highlands University, Las Vegas, Peter A. Lindstrom, North Lake College, Irving, TX, David E. Manes, SUNY College at Oneonta, Stan Wagon and Joan Hutchinson, Macalester College, St. Paul, MN, and the PROPOSER.

Let \( f \) be a function such that \( f, f', \ldots, f^{(n)} \) are all continuous, \( f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0 \) and \( f^{(a)}(0) \neq 0 \). Let

\[
f_1(x) = \int_0^x f(t) \, dt \quad \text{and} \quad f_k(x) = \int_0^x f_{k-1}(t) \, dt \quad k = 2, 3, \ldots, n.
\]

Evaluate the limit

\[
\lim_{x \to 0} \frac{f_m(x)}{x^n f(x)}
\]

Solution by the Proposer.

Using L'Hôpital's rule, we have

\[
\lim_{x \to 0} \frac{f(x)}{x^n} = \frac{f^{(n)}(0)}{n!} \quad \text{and} \quad \lim_{x \to 0} \frac{f_m(x)}{x^n f(x)} = \frac{f^{(a)}(0)}{(n + m)!}.
\]

It follows that

\[
\lim_{x \to 0} \frac{f_m(x)}{x^n f(x)} = \lim_{x \to 0} \frac{f^{(a)}(0)}{(n + m)!} \implies \frac{n!}{(n + m)!}.
\]

Also solved by Paul S. Bruckman, Highwood, IL, Mark Evans, Louisville, KY, Richard I. Hess, Rancho Palos Verdes, CA, Carl Libisz, Idaho State University, Pocatello, Henry S. Lieberman, Waban, MA, and David S. Shobe, New Haven, CT.
830. [Spring 1994] Proposed by David Iny, Baltimore, Maryland.

Let

\[ w_k = \sum_{n=0}^{\infty} \frac{x^{4n+k-1}}{(4n+k-1)!} \quad k = 1, 2, 3, 4. \]

a) Prove that \( [(w_1 + w_3)^2 - (w_2 + w_4)^2][(w_1 - w_3)^2 + (w_2 - w_4)^2] = 1. \)

b) Can you find similar identities with \( p \geq 2 \) for

\[ w_k = \sum_{n=0}^{\infty} \frac{x^{pn+k-1}}{(pn+k-1)!}, \quad k = 1, 2, \ldots, p? \]

This problem is a generalization of a 1939 Putnam Exam problem, which considered the case of \( p = 3. \)

I. Solution to part (a) by Carl Libis, Idaho State University, Pocatello, Idaho.

Since \( w_1 + w_3 = \cosh x, \) \( w_2 + w_4 = \sinh x, \) \( w_1 - w_3 = \cos x, \) and \( w_2 - w_4 = \sin x, \) and since \( w_1 + w_2 + w_3 + w_4 = e^x \) and \( w_1 - w_2 + w_3 - w_4 = e^{-x}, \)

we have that

\[ (w_1 + w_3)^2 - (w_2 + w_4)^2 = \cosh^2 x - \sinh^2 x = 1, \]

\[ (w_1 - w_2)^2 + (w_2 - w_4)^2 = \cos^2 x + \sin^2 x = 1, \]

and

\[ (w_1 + w_2 + w_3 + w_4)(w_1 - w_2 + w_3 - w_4) = e^x e^{-x} = 1. \]

II. Solution by Paul S. Bruckman, Highwood, Illinois.

We solve the problem for fixed \( p \geq 2. \) Then

\[ w_{k+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \delta_{n-k}, \quad k = 0, 1, \ldots, p-1. \]

and where

\[ \delta_m = \begin{cases} 1 & \text{if } p \mid m, \\ 0 & \text{otherwise.} \end{cases} \]

If we let \( 0 = \exp(2ix/p), \) then we may write

\[ \delta_m = \frac{1}{p} \sum_{j=0}^{p-1} \theta^j. \]

We next look for a closed form for \( w_{k+1}. \) Thus

\[ w_{k+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{1}{p} \sum_{j=0}^{p-1} \theta^{jn-k}. \]

Since the series in (1) is absolutely convergent for all complex \( x, \) we may rearrange its terms in any order. Then

\[ w_{k+1} = \frac{1}{p} \sum_{j=0}^{p-1} \theta^j \exp(x\theta^j), \quad k = 0, 1, \ldots, p-1. \]

We define, for \( m = 0, 1, 2, \ldots, p-1, \) a sum \( S_{m+1} \) and then use equation (3) to get

\[ S_{m+1} = \sum_{k=0}^{p-1} \theta^m \delta_{k+1} = \sum_{k=0}^{p-1} \theta^m \frac{1}{p} \sum_{j=0}^{p-1} \theta^{jn-k} \exp(x\theta^j) \]

\[ = \sum_{j=0}^{p-1} \exp(x\theta^j) \cdot \frac{1}{p} \sum_{k=0}^{p-1} \theta^{(m-j)} = \sum_{j=0}^{p-1} \exp(x\theta^j) \delta_{m-j}, \]

and it follows that

\[ S_{m+1} = \exp(x\theta^m), \quad m = 0, 1, \ldots, p-1. \]

Note that

\[ \sum_{m=0}^{p-1} \theta^m = 0 \]

and consider the product

\[ \prod_{m=0}^{p-1} S_{m+1} = \exp(x \sum_{m=0}^{p-1} \theta^m) = \exp(0) = 1. \]

This result provides us with the general identity satisfied by the \( w_k: \)

\[ \prod_{m=0}^{p-1} \sum_{k=0}^{p-1} \theta^m w_{k+1} = 1, \]

and equivalently,
(5) \[
\prod_{m=0}^{p-1} (w_1 + \theta^m w_2 + \theta^{2m} w_3 + \cdots + \theta^{(p-1)m} w_p) = 1.
\]

Some special cases follow. If \( p = 2 \) and hence \( 9 = -1 \), then
\[
(w_1 + w_2)(w_1 - w_2) = w_1^2 - w_2^2 = 1.
\]

For \( p = 3 \), by repeated application of the relation \( \theta^2 + 9 + l = 0 \), the identity in (5) becomes
\[
w_1^3 + w_2^3 + w_3^3 - 3w_1 w_2 w_3 = 1.
\]

When \( p = 4, 9 = i \) and equation (5) becomes
\[
(w_1 + w_2 + w_3 + w_4)(w_1 - i w_2 - w_3 - i w_4)
\]
\[
= ((w_1 + w_3)^2 - (w_2 + w_4)^2)((w_1 + w_3)^2 + (w_2 + w_4)^2) = 1.
\]

III. Solution by Richard I. Hess, Rancho Palos Verdes, California.

We take, for any given positive integer \( p \) and for \( k = 1, 2, \ldots, p, \)
\[
w^k = \sum_{n=0}^{\infty} \frac{x^{pn+k-1}}{(pn+k-1)!}
\]
and define \( u_m = e^{2\pi i m / p} \). Suppose \( p \) is an odd number. Then
\[
e^{ux} = 1 + ux + \frac{u^2 x^2}{2!} + \frac{u^3 x^3}{3!} + \cdots = w_1 + u w_2 + u^2 w_3 + \cdots + u^{p-1} w_p,
\]
and
\[
e^{-ux} = 1 - ux + \frac{u^2 x^2}{2!} - \frac{u^3 x^3}{3!} + \cdots = w_1 - u w_2 + u^2 w_3 - \cdots - u^{p-1} w_p,
\]
where \( u \) can be subscripted with \( m = 1, 2, \ldots, p - 1 \).

In the product
\[
1 = e^{ux} e^{-u^2 x} e^{u^3 x} \cdots e^{u^{p-1} x} e^{-u^{p-1} x}
\]
pairs of factors can be split out, such as
\[
e^{ux} e^{-u^{p-1} x} = (w_1 + u m w_2 + \cdots + u^{p-1} m w_p)(w_1 + u_{p-m} w_2 + \cdots + u^{p-1} m w_p)
\]
\[
= w_1^2 + w_2^2 + \cdots + w_p^2 + \sum_{l \neq r \neq p \neq 1} w_r w_s (u_{m} - u_{m}^{p-1} - u_{m}^{p-1} - u_{m}^{p-1})
\]

Similarly
\[
e^{-2x \cos(2\pi m / p)} = e^{-w_{m} x} e^{-u_{m} x} e^{-w_{m} x}
\]
\[
= w_{m}^2 + \cdots + w_{p}^2 + 2 \sum_{l \neq r \neq p \neq 1} (-1)^{rs} w_r w_s \cos \frac{2\pi m(s - r)}{p}
\]
The product of these last two expressions is 1. There will be \( (p - 1)/2 \) such expressions for \( m = 1, 2, \ldots, (p - 1)/2 \).

Suppose now that \( p \) is an even number and define \( u_m \) as before. Then equation (1) still holds. Its factors get paired off as before except that \( u_{p/2} = -1 \) and
\[
e^{u_{p/2} x} = e^{-x} = w_1 - w_2 + w_3 - \cdots - w_p,
\]
is paired with
\[
e^{-u_{p/2} x} = e^{x} = w_1 - w_2 + w_3 - \cdots + w_p,
\]
Then
\[
e^{u_{p/2} x} e^{-u_{p/2} x} = (w_1 - w_2 + \cdots - w_p)(w_1 + w_2 + \cdots + w_p) = 1.
\]
There are \( p/2 - 1 \) additional equations for \( m = 1, 2, \ldots, p/2 - 1 \),
\[
\sum_{r \neq s} w_r^2 + 2 \sum_{r \neq s} (-1)^{rs} w_r w_s \cos \frac{2\pi m(s - r)}{p}
\]
\[
\left[ \sum_{r \neq s} w_r^2 + 2 \sum_{r \neq s} (-1)^{rs} w_r w_s \cos \frac{2\pi m(s - r)}{p} \right] = 1.
\]

When \( m = p/4 \), the two expressions in brackets are the same and each is equal to 1.

Also solved by the PROPOSER.


Solve exactly and completely
\[x^5 - 8x^3 + 18x^3 - 6x^2 - 12x + 1 = 0.\]
I. Solution by Henry' S. Lieberman, Waban, Massachusetts.

It seems reasonable that the stated polynomial might factor as

\[(x^2 + ax + 1)(x^4 + bx^3 + cx^2 + dx + 1),\]

where \(a, b, c,\) and \(d\) are integers. We multiply out this product and then equate its coefficients with the corresponding ones in the given polynomial to get that

\[
\begin{align*}
  a + b &= -8, \quad c + ah + 1 = 18, \\
  d + ac + b &= -6, \quad 1 + ad + c = -12, \\
  a + d &= 2.
\end{align*}
\]

These equations have the solution \(a = -3, b = -5, c = 2,\) and \(d = 5.\) Thus the given polynomial is equal to

\[(x^2 - 3x + 1)(x^4 - 5x^3 + 2x^2 + 5x + 1).\]

'Although the quartic factor can be factored similarly into the product of the two quadratic factors

\[(x^2 + ex - 1)(x^2 + fx - 1),\]

we shall follow Ferrari's solution as presented in *Higher Algebra* by Hall and Knight, pp. 483-484.

To each side of

\[x^4 - 5x^3 + 2x^2 + 5x + 1 = 0\]

add \((Ax + B)^2,\) the quantities \(A\) and \(B\) being determined so as to make the left side a perfect square of the form \(x^2 - (5/2)x + k^2.\) That is, we want

\[
\begin{align*}
  (x^2 - (5/2)x + k^2) &= x^4 - 5x^3 + (25/4 + 2k)x^2 - 5kx + k^2 \\
  &= x^4 - 5x^3 + (2 + A^2)x^2 + (5 + 2AB)x + 1 + B^2 = (Ax + B)^2.
\end{align*}
\]

By comparing coefficients we get

\[2514 + 2k = 2 + A^2, \quad -5k = 5 + 2AB,\] and \(k^2 = 1 + B^2.\)

Eliminate \(A\) and \(B\) from these three equations by taking

\[(AB)^2 = (-1(5/2)k - 5/2)^2 = (1714 + 2k)(k^2 - 1).\]

The resulting cubic polynomial in \(k\) easily factors thus

\[4k^2 - 4k^2 - 29k - 2l = (k + 1)(2k + 3)(2k - 7) = 0.\]

Taking \(k = -1,\) we get \(B = 0\) and \(A = \pm 3/2,\) which gives rise to

\[x^2 - \frac{5}{2}x - 1 = \pm \frac{3}{2}x.\]

Thus the original polynomial factors into the product

\[(x^2 - 3x + 1)(x^2 - 4x - 1)(x^2 - x - 1),\]

which has the easily verified zeros

\[x = 2 \pm \frac{\sqrt{5}}{2}, \quad \frac{3 \pm \sqrt{5}}{2}.\]

The other two values of \(k, -312\) and \(712,\) similarly lead to factorizations which produce the same zeros.

Also solved by BILL CORRELL, JR., Denison University, Granville, OH, PATRICK COSTELLO, Eastern Kentucky University, Richmond, MARK EVANS, Louisville, KY, J. S. FRAMME, Michigan State University, Lansing, MARCIE GARDNER, TOM SYMONS, ARTHUR THOMASON, and RANDI KAY VEST, Hendrix College, Conway, AR, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, JOE HOWARD, New Mexico Highlands University, Las Vegas, BECKY LATCH, ANGELA JONES, and WADE WILLIAMS, Hendrix College, Conway, AR, CARL LIBIS, Idaho State University, Pocatello, PETER A. LINDSTROM, North Lake College, Irving, TX, DAVID E. MANES, SUNY College at Oneonta, YOSHINOBU MURAYOSHI, Okinawa, Japan, BOB PRIELIP, University of Wisconsin-Oshkosh, STAN WAGON and JOAN HUTCHINSON, Macalester College, St. Paul, MN, REX H. WU, Brooklyn, NY, SAMMY YU and JIMMY YU, University of South Dakota, Vermillion, and the PROPOSER.


The taxicab distance between points \((a, b)\) and \((c, d)\) is \(|a - c| + |b - d|\). Determine the circumference in taxicab space of the circle whose equation is \(x^2 + y^2 = 1.\)
I. Solution by Victor G. Feser, University of Mary, Bismarck North Dakota.

The circumference is 8. In taxicab geometry, the circle is a square, as in the figure. To confirm this fact, find equations of each of the four segments. For example, AB is given by $y = -x + 1$ on the interval $0 \leq x \leq 1$. Then a point on that segment is $(x, -x + 1)$ and its distance from $(0, 0)$ is given by

$$|x - 0| + |(-x + 1) - 0| = x + (-x + 1) = 1,$$

so it lies on the taxicab circle.

The length of segment AB is $|1 - 0| + |0 - 1| = 2$. Since each of the other three segments has the same length, the total circumference is 8.

II. Comment by Paul S. Bruckman, Highwood, Illinois.

Aha, Lindemann was wrong—you can square the circle after all. In taxicab geometry. Nor am I impressed with the Chudnowsky brothers’ achievement—I can compute $\pi$ to infinitely many decimal places, all of them zero! Now, for my next trick, I intend to show that $\infty$ is a rational number (in fact, I once actually met a rational mathematician!).

Also solved by SEUNG-JIN BANG, Ajou University, Suwon, Republic of Korea, PAUL S. BRUCKMAN, Highwood, IL, MARK EVANS, Louisville, KY, ROBERT C. GEBHARDT, Hopatcong, NJ, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, Waban, MA, DAVID S. SHOBE, New Haven, CT, ROMAN SZNAJDER, University of Maryland, Baltimore, MD, REX H. WU, Brooklyn, NY, and the PROPOSER.


Define a function $f$ by $f(0) = 1$ and

$$f(m) = \binom{n + m - 2}{m} + \binom{n + m - 3}{m - 1}.$$ 

Find the value of the sum $\sum_{m=0}^{k} f(m)$.

Solution by J. S. Frame, Michigan State University, East Lansing, MI.

Let us denote $f(m)$ by $f(m, n)$ to show its dependence on $n$. Then

$$f(0, n) = 1$$

and

$$f(m, n) = f(m, n + 1) - f(m - 1, n + 1)$$

for $m \geq 1$.

Thus the required sum telescopes and yields

$$\sum_{m=0}^{k} f(m, n) = f(0, n) + \sum_{m=1}^{k} [f(m, n + 1) - f(m - 1, n + 1)]$$

$$= f(0, n) + f(k, n + 1) - f(0, n + 1) = 1 + f(k, n + 1) - 1$$

$$= f(k, n + 1) = \binom{n + k - 1}{k} + \binom{n + k - 2}{k - 1} = \frac{n + 2k - 1}{k} \binom{n + k - 2}{k - 1}.$$

Also solved by PAUL S. BRUCKMAN, Highwood, IL, BILL CORRELL, JR., (who provided the final form for the solution), DENISON UNIVERSITY, Granville, OH, RICHARD I. HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, Waban, MA, PETER A. LINDSTROM, North Lake College, Irving, TX, PHYLLIS MAHAN, Eastern Kentucky University, Richmond, DAVID E. MANES, SUNY College at Oneonta, REX H. WU, Brooklyn, NY, and the PROPOSER.


Let $T$ and $T'$ denote two triangles with respective sides $(a, b, c)$ and $(a', b', c')$ where $a'^2 = 2a(s - a)$, $b'^2 = 2b(s - b)$, and $c'^2 = 2c(s - c)$. Prove that

(i) $s \geq s'$, (ii) $R \geq R'$, (iii) $r' \geq r$, and (iv) $F'ls^2 \geq Fls^2$.
where $s = (a + b + c)/2$ is the semiperimeter, $R$ the circumradius, $r$ the inradius, and $F$ the area of triangle ABC, and similarly for triangle $A'B'C'$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Given triangle $T$, triangle $T'$ exists from items (1) and (35) in [2] for the function $f(x) = \sqrt{x}$. (The converse is not true: given an arbitrary triangle $T''$, then $a$, $b$, and $c$ are not necessarily the sides of a triangle.) Next, we define $x = s - a$, $y = s - b$, and $z = s - c$. Then $x + y = c$, $y + z = a$, and $z + x = b$.

By the arithmetic-geometric mean inequality we get

$$2s' = a' + b' + c' = \sqrt{2x(y + z)} + \sqrt{2y(z + x)} + \sqrt{2z(x + y)}$$

$$= \frac{2x + (y + z)}{2} , \frac{2y + (z + x)}{2} + \frac{2z + (x + y)}{2}$$

$$= 2(x + y + z) = 2s.$$

from which inequality (i) follows.

Since we have

$$\cos A' = \frac{b'^2 + c'^2 - a'^2}{2b'c'} = \frac{2y(z + x) + 2x(y + z) + 2x(y + z)}{2\sqrt{2y(z + x) \cdot \sqrt{2x(y + z)}}}$$

$$= \frac{yz}{\sqrt{(z + x)(x + y)}} = \sqrt{\frac{yz}{(x + y)(z + x)}} = \frac{(s - b)(s - c)}{bc} = \frac{\sin A}{s}$$

then $A' = \pi/2 - A/2$ and similarly $B' = \pi/2 - B/2$ and $C' = \pi/2 - C/2$. Now

$$s = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = 4R \sin A' \sin B' \sin C'$$

$$= 4R \cdot 2 \sin A' \cdot 2 \sin B' \cdot 2 \sin C'$$

$$= 8R \sin A' \cdot \sin B' \cdot \sin C'$$

$$= 8R \sin A' \cdot \sin B' \cdot \sin C' \cdot s' \leq R \cdot s' \cdot \sin \frac{\pi}{2}$$

and inequality (ii) follows since, by item 2.12 in [1],

$$\sin \frac{A'}{2} \sin \frac{B'}{2} \sin \frac{C'}{2} \leq \frac{1}{8}.$$

By Heron's formula,

$$8F^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$$

$$= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

by just a bit of algebra. Similarly, placing a prime on each letter in this last equation and then replacing $a'^2$ by $2a(s - a)$, etc., and with just a bit more algebra, we get that

$$8F'^2 = \sum 2\cdot 2b(s - b)2c(s - c) - \sum 4a^2(s - a)^2$$

$$= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 8F^2.$$

Hence $F' = F$. Now $rs = F = F' = r's'$, so inequality (iii) follows from inequality (i).

Finally, from inequalities (i) and (iii) and $F = rs$, we get inequality (iv),

$$\frac{F'}{s'^2} = \frac{r'}{s} \cdot \frac{r}{s} = \frac{F}{s^2}.$$

References


Also solved by PAUL S. BRUCKMAN, Highwood, IL, and the PROPOSER.


Let $P(x)$ be a polynomial of degree $n \geq 2$ with real coefficients and whose leading three terms are $ax^n + bx^{n-1} + cx^{n-2}$. All remaining terms are
of degree \(n - 3\) or less. If \(b^2 \leq 2ac\), then prove that \(P(x)\) cannot have \(n\) distinct real roots. (This problem is a generalization of Problem 4 from the February 1990 issue of \textit{Problem Solving Newsletter} by Dr. Hugh Montgomery of the University of Michigan.)

I. Solution by Kandasamy \textit{Muthuvel}, University of Wisconsin at Oshkosh, Oshkosh, Wisconsin.

Suppose that \(P(x)\) has \(n\) distinct real zeros. Then by Rolle’s theorem, one can see that \(P'(x)\) must have \(n - 1\) distinct real zeros, and so forth, so that

\[
p^{(n-2)}(x) = \frac{(n-2)!}{2} [an(n-1)x^2 + 2b(n-1)x + 2c]
\]

has two distinct real zeros. Hence its discriminant

\[
4b^2(n-1)^2 - 8an(n-1)c = 4(n-1)[b^2(n-1) - 2acn] > 0.
\]

Thus we must have

\[
b^2 > \frac{2acn}{n-1} > 2ac,
\]

a stronger inequality than proposed.


Suppose \(P(x) = 0\) has \(n\) distinct real roots \(r_1, r_2, \ldots, r_n\). Then

\[
P(x) = a \prod_{i=1}^{n} (x - r_i) \quad \text{with} \quad a \neq 0.
\]

Then

\[
b = -a \sum_{i=1}^{n} r_i \quad \text{and} \quad c = a \sum_{1 \leq i < j \leq n} r_i r_j.
\]

Now

\[
b^2 - 2ac = a^2 \sum_{i=1}^{n} r_i^2.
\]

Now \((r_i - r_j)^2 > 0\) when \(i \neq j\), so then \(r_i^2 + r_j^2 > 2r_i r_j\) and hence

\[
(n-1)\sum_{i=1}^{n} r_i^2 > 2 \sum_{1 \leq i < j \leq n} r_i r_j,
\]

so

\[
b^2 - 2ac > \frac{2}{n-1} ac, \quad \text{and finally} \quad b^2 - 2ac \frac{n}{n-1} \leq 0.
\]

Thus, if \(b^2 - 2ac(n-1) \leq 0\), \(P(x)\) cannot have \(n\) distinct zeros. When \(n = 2\), this is the usual \(b^2 - 4ac \leq 0\) necessary and sufficient condition for \(P(x) = 0\) not to have distinct real roots.

Also solved by SEUNG-JIN BANG, Ajou University, Suwon, Korea, PAUL S. BRUCKMAN, Highwood, IL, BILL CORRELL, JR., Denison University, Granville, OH, RICHARD I. HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY College at Oneonta, DAVID S. SHOBE, New Haven, CT, REX H. WU, Brooklyn, NY, and the PROPOSERS.

Bruckman commented that it is proper to speak of the zeros of \(P\) (and not of the roots of \(P\)) as those values of \(x\) that are the roots of the equation \(P(x) = 0\). That is, an expression has zeros; an equation has roots.

How often have I told my students that same distinction! Yet I did overlook the error here. \textit{—Ed.}

A problemist is one of those heroes, Who, when he sees "roots" used for "zeros," \textit{Thinks} it a harsh grate Like nails scraping slate.
So he charges out like a rhinoceros. \textit{—Anon.}
The 1994 National Pi Mu Epsilon Meeting

The meeting took place in conjunction with the summer meeting of the Mathematical Association of America and the American Mathematical Society in Minneapolis, Minnesota, August 15-17, 1994.

There were thirty-three student papers delivered in four sessions:

- Analysis of covariance, by Mike Amend (Youngstown State University)
- Diagnostics in insurance: an application for simple regression, by Mark Bonsall (Moravian College)
- A Mellin-type integral transform method for solving Cauchy-Euler differential equations, by David A. Brown (Ithaca College)
- Aspects of orthonormalization in quantum information theory, by Bill Correll, Jr. (Denison University)
- The problems of scale in the hyperbolic world, by Andrew Douglass (Miami University)
- The construct matrix, by Jamie Downs (St. Norbert College)
- A practical application of Mutex, by Joseph Engel (St. Norbert College)
- Interface construction and 2-D fluid dynamics, by Geoffrey Gibbons (University of California)
- Three tools to help build term persistence assumptions, by Dale Hall (John Carroll University)
- The number theoretic properties of the dynamical system known as rigid rotation, by Allen G. Harbaugh (Boston University)
- The Haar wavelet and the dilation equation, by Kevin R. Hutson (Hendrix College)
- The numerical range of a matrix, by Dennis Keeler (Miami University)
- Beyond chaos, by Brian Kemery (SUNY College at Fredonia)
- The basic mathematics of production under uncertainty, by Nikolay K. Kolev (Moravian College)
- Tranquilizing Bigfoot, by Sondra C. Laird (University of West Florida)
- Magnetic monopoles in mathematical physics, by Erik Leder (Portland State University)

Finding probabilities with the Maple computer algebra system, by Joshua D. Levy (Hope College)
- Examples of negatively dependent random variables, by Teresa J. Murphy (Georgia State University)
- Quantum cryptography, by Kathryn Nyman (Carthage College)
- Nothing in moderation, everything in excess: a new weighted statistic on permutations, by Ann Marie Paulukonis (St. John's University, Minnesota)
- The Chinese Remainder Theorem and object oriented programming, by David M. Potts (Texas A & M University)
- A linear programming formulation to optimize sawmill operations, by Jerry Priddy (Youngstown State University)
- Bernoulli numbers in series summations, by Alan B. Shettel (Youngstown State University)
- Re-marking dice, by Kendra Sinopoli (Youngstown State University)
- Fault tolerance in parallel processing, by Jason M. Spangler (Youngstown State University)
- Using asymptotics to mathematically model macrosegregation in continuously cast steel slabs, by Carl Stitz (University of Akron)
- Existence theorems for 2-point boundary value problems, by Tishua Taylor (Spelman College)
- A discussion of simple continued fractions and their applications, by Christina T. Tsiaparas (Youngstown State University)
- Minimal Moebius strips, by Daniel L. Viar (University of Arkansas)
- On b, n-happiness sequences, by Sonny Vu (University of Illinois)
- Applications of fractals in geology and geophysics, by Lisa White (Youngstown State University)
- The combinatorics of semi-direct products of cyclic groups, by Jeb F. Willenbring (North Dakota State University)
- Having a ball with Pythagoras, by Adam J. Zeuske (St. Norbert College)

Five prizes of $100 each, for papers of unusual merit, were awarded to Andrew Douglass, Allen Harbaugh, Kathryn Nyman, Sonny Vu, and Jeb Willenbring.

The National Security Agency again awarded Pi Mu Epsilon a grant of $5000 for the support and encouragement of student speakers, and the American Mathematical Society contributed $1000 for prize awards.

The J. Sutherland Frame Lecture, delivered by Scoutmaster Mel
Slugbate, was entitled "Cheating your way to the knot merit badge". Some members of the audience detected a strong resemblance of the speaker to Professor Colin Adams of Williams College, but this may have been mere coincidence.

The Richard V. Andree Awards

The Richard V. Andree awards are given annually to the authors of the three papers written by students that have been judged by the officers and councilors of the Pi Mu Epsilon to be the best to have appeared in the Pi Mu Epsilon Journal in the past year.

Richard V. Andree was, until his death in 1987, Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon long and well in many capacities: as president, as secretary-treasurer, and as editor of the Journal.

The winner of the first prize for 1994 is Gina Aurello, for her paper "On the rearrangement of infinite series" (this Journal 9 (1989-94) #10, 641-646).

The second prize is awarded to Rychard Bouwens, for "Who gets the Washers?" (this Journal 10 (1994-99) #1, 1-4).

The third prize winner is Michael Reske, for "The secret Santa problem" (this Journal 10 (1994-99) #1, 18-21).

At the times the papers were written, Ms. Aurello was a student at Seton Hall University, Mr. Bouwens at Hope College, and Mr. Reske at Carthage College.

The officers and councilors of the Society congratulate the winners on their achievements and wish them well in their futures, whether or not they involve mathematics.
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