PI MU EPSILON JOURNAL

VOLUME 10 SPRING 1995 NUMBER 2

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PI MU EPSILON JOURNAL

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Editorial correspondence, including manuscripts, chapter reports, and news items should be sent to Underwood Dudley, Mathematics Department, DePauw University, Greencastle, Indiana 46135 (e-mail, dudley@depauw.edu). Students submitting manuscripts should give their school and date of graduation. Others should provide their affiliation, academic or otherwise.

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SIERPINSKI *n*-GONS

Steven Schlicker and Kevin Dennis
Grand Valley State U. and Michigan State U.

Introduction. The growing interest in chaos and fractal geometry has created a new field of mathematics that can be explored by faculty and undergraduates alike. "Sierpinski triangles" and "Koch's curves" have become common phrases in many mathematics departments across the country. In this paper we review some basic ideas from fractal geometry and generalize the construction of the Sierpinski triangle to what we will call Sierpinski polygons.

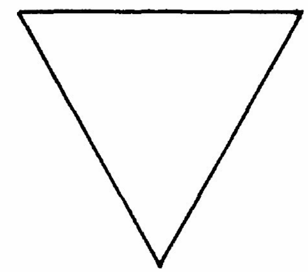

Figure 1

The Sierpinski Triangle. In fractal geometry, the well known Sierpinski triangle can be constructed as a limit of sets as follows. We begin with three points, x_1, x_2, and x_3 that fonn the vertices of an equilateral triangle A_0.

For i = 1, 2, or 3, let $x_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$. Let $\Re$ represent the set of real numbers and let $\Re^2$ be the real plane. For i = 1, 2, or 3, we define $\omega_i : \Re^2 \to \Re^2$ by

$$\omega_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a_i \\ b_i \end{bmatrix}.$$

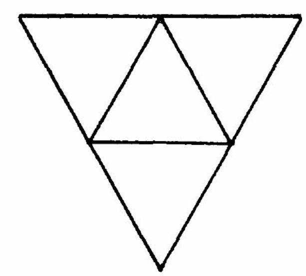

Figure 2

Then ω_i, when applied to A_0, contracts A_0 by a factor of two and then translates the image of A_0 so that the ith vertices of A,, and the image of A_0 coincide. Define $A_{1,i}$ to be $\omega_i(A_0)$. Then $A_{1,i}$ is the set of all points halfway between any point in A_0 and x_i, or $A_{1,i}$ is a triangle half the size of the original translated to the ith vertex of the original. Let $A_1 = \bigcup_{i=1}^{3} A_{1,i}$. A_0 and A_1 are shown in Figures 1 and 2, respectively. We can continue this procedure, replacing A_0 with A_1. For $i = 1, 2$, or 3, let $A_{2,i} = \omega_i(A_1)$ and let $A_2 = \bigcup_{i=1}^{3} A_{2,i}$. A_2 is pictured in Figure 3. Again, we can continue this procedure, each time replacing A_i with A_{i+1}. A_4 and A_8 are shown in Figures 4 and 5.

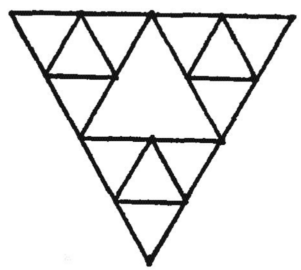

Figure 3

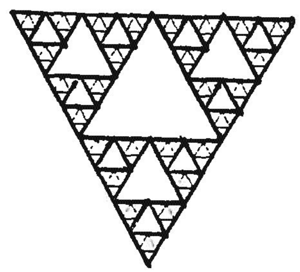

Figure 4

If we take the limit as $i \to \infty$, the resulting figure is the Sierpinski triangle. This algorithm for building the **Sierpinski** triangle is called the deterministic algorithm.

Classification. It is natural to ask what would happen if, in using the deterministic algorithm, we cut the distances by a factor of 3, 4, or 10, instead of 2. That is, what would happen if, for $r > 0$, we defined

$$\omega_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{r}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{r-1}{r}\begin{bmatrix} a_i \\ b_i \end{bmatrix},$$

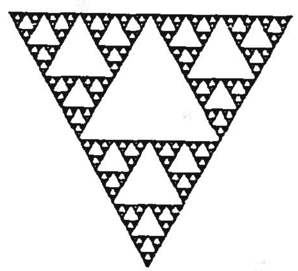

Figure 5

let $A_{m,i}(r) = \omega_i(A_{m-1}(r))$ and let $A_{m+1}(r) = \bigcup_{i=1}^{3} A_{m,r}(r)$? As earlier, $A_{1,i}$ is a triangle translated to the **ith** vertex whose sides have length $1/r$ of the lengths of the sides of the original triangle. If Figures 6 and 7 we see $A_1(1.5)$ and $A_4(1.5)$ and in Figures **8** and 9 we see $A_1(3)$ and $A_4(3)$.

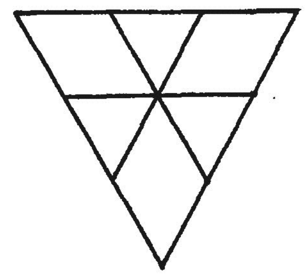

Figure 6

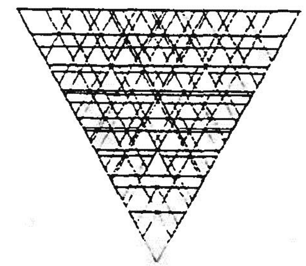

Figure 7

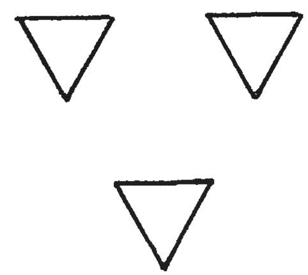

Figure 8

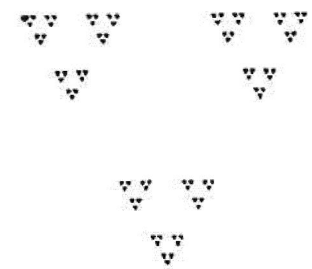

Figure 9

It is easy to see that if $r > 2$, then the resulting A_m consist of a collection of 3^m disjoint triangles. **In** this case we say that the A_m are totally disconnected. If $0 < r < 2$ and $r \neq 1$, A is a collection of 3^m intersecting triangles. In this case we say that the A_m are overlapping. However, if $r = 2$ we have seen that A is a collection of 3^m triangles that intersect only at the edges. In this case we say that the A_m are just touching. It seems that the most aesthetically pleasing situation is when $r = 2$, where the triangles are just touching.

The Deterministic Algorithm Applied to Regular n-gons. There is no reason why we should restrict ourselves to looking at only three points. Let $v_1, v_2, \dots, v_n$ be the vertices of a regular n-gon A_0. For $i \in \{1, 2, \dots, n\}$ let $v_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$, and, for $r > 0$, let $\delta_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{r}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{r-1}{r}\begin{bmatrix} a_i \\ b_i \end{bmatrix}$, $A_{m,i}(r) = \delta_i(A_{m-1}(r))$, and $A_{m+1}(r) = \bigcup_{i=1}^{n} A_{m,i}(r)$. Again, $A_{1,i}$, is an n-gon translated to the ith vertex whose sides have length $1/r$ of the lengths of the sides of the original n-gon. Some examples of $A_i(r)$ can be seen in Figures 10, 11, and 12.

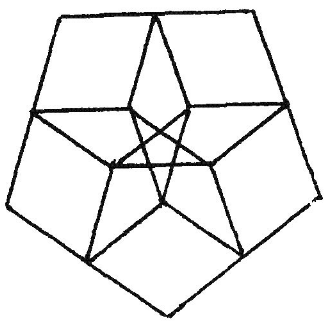

Figure 10

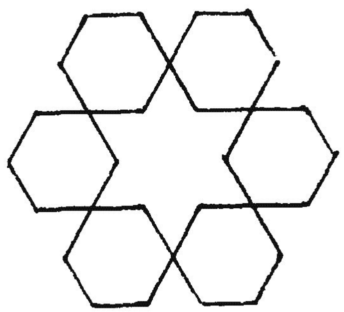

Figure 11

Again, for some values of r, $A_m(r)$ consists of overlapping n-gons, and for others totally disconnected n-gons. This raises the question of which value of r makes the constituent n-gons just touching.

Since $(\Re^2, d)$ is a complete metric space, where d is the euclidean metric on $\Re^2$, the sequence $\{A_m(r)\}$ has a limit in $\Re^2$. (See [1] for details.) For the just touching r we will call $SP_n = \lim_{m\to\infty} A_m(r)$ a regular Sierpinski n-gon. (The figures included in this paper were obtained by beginning with the regular n-gon with vertices

$$v_i = (\cos(\alpha_0 + 2\pi i/n), \sin(\alpha_0 + 2\pi i/n))$$

for i = 1, 2, ..., n, where $\alpha_0 = \pi/2 - \pi/n$.)

Regular Sierpinski n-gons. Given a regular n-gon, construct line segments from the center to each vertex. The radian measure of each central angle is clearly $2\pi/n$. Each of the triangles determined by the center and two adjacent vertices is isosceles, so the measure of the base angles of each of these triangles is $(n-2)\pi/2n$. Consequently, the measure of a vertex angle of a regular n-gon is $(n-2)\pi/n$.

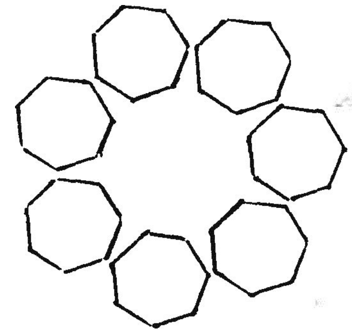

Figure 12

PROPOSITION 1. For $n \geq 5$, the value of r that determines a regular Sierpinski n-gon is

$$r_n = 2\left(1 + \sum_{k=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2k\pi}{n}\right)\right),$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to **x**.

Proof. Let $n \geq 5$ be an integer. Without loss of generality, we can consider a particular n-gon. Using our previous notation, we will call the $A_{m,i}(r)$ the constituent n-gons of $A_m(r)$. Since the length of any side of

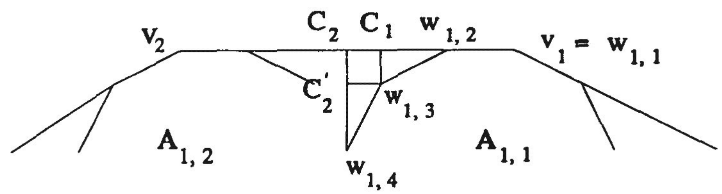

Figure 13

one of the constituent n-gons of $A_m(r)$ is $1/r$ times the length of any side of one of the constituent n-gons of $A_{m-1}(r)$, for a given m all of the n-gons making up $A_m(r)$ are congruent regular n-gons. Let d_0 be the length of one side of A_0 and let d_m be the length of one side of any constituent n-gon of $A_m(r)$. To find the value of r that makes the $A_{m,i}(r)$ just touching, all we need do is find the ratio d_0/d_1.

In Figure 13 we see a picture of a portion of $A_{1,1}$ and $A_{1,2}$. Label the vertices of $A_{1,1}$ as $w_{1,1}, w_{1,2}, \ldots, w_{1,n}$ starting with $w_{1,1} = v_1$ and proceeding counterclockwise. In Figure 13 we see the sides v_1v_2 and v_nv_1 of A_0 and sides $w_{1,1}w_{1,2}$, $w_{1,2}w_{1,3}$, and $w_{1,3}w_{1,4}$ of $A_{1,1}$. Here, $|v_1v_2| = d_0$ and $|w_{1,i}w_{1,i+1}| = d_1$ for each i. Construct a line from $w_{1,3}$ perpendicular to v_1v_2. Label the point of intersection C_1. Now, $m(\angle w_{1,3}w_{1,2}w_{1,1}) = (n-2)\pi/n$, so $|w_{1,2}C_1| = d_1\cos(2\pi/n)$. Since the sum of the measures of the angles in a right triangle is π radians, it follows that $m(\angle w_{1,2}w_{1,3}C_1) = (n-4)\pi/2n$. Now construct a line from $w_{1,4}$ perpendicular to v_1v_2 and call the point of intersection C_2. The points C_1, C_2, and $w_{1,3}$ form three vertices of a rectangle. Label the fourth vertex C_2'. Construct rectangle $C_2C_2'C_1w_{1,3}$. Since

$$m(\angle w_{1,4}w_{1,3}C_2') + m(\angle C_2'w_{1,3}C_1) + m(\angle w_{1,2}w_{1,3}C_1) + m(\angle w_{1,4}w_{1,3}w_{1,2}) = 2\pi,$$

$m(\angle w_{1,4}w_{1,3}C_2') = 4\pi/n$. So $m(\angle w_{1,3}w_{1,4}C_2') = (n-8)\pi/2n$. Then

$$|C_1C_2| = |w_{1,3}C_2'| = d_1\cos(4\pi/n).$$

We can continue this process inductively, at the nth stage obtaining an angle $\angle w_{1,t+1}w_{1,t+2}C_t'$ with measure $(n-4t)\pi/2n$, as long as $n \geq 4t$.

By the division algorithm we can find an integer k so that $n = 4k + r$, where $0 \leq r < 4$. So we can continue the construction up to the **kth** step. If $r = 0$, then $n = 4k$. In this situation, the kth side of $A_{1,1}$ in this progression coincides with a corresponding side of $A_{2,1}$. If $r > 0$, then $n > 4k$. In this situation $A_{1,1}$ and $A_{2,1}$ intersect at a vertex. Now we can see that if the $A_{m,i}(r)$ are just touching then, by symmetry, the sum of the lengths of $w_{1,1}w_{1,2}$, $w_{1,2}C_1$, and $\sum_{i=1}^{k} d_1\cos(2i\pi/n)$ will be half of d_0. So

$$\frac{d_0}{2} = |w_{1,1}w_{1,2}| + |w_{1,2}C_1| + \sum_{i=2}^{k} |C_{i-1}C_i| = d_1 + \sum_{i=1}^{k} d_1\cos\left(\frac{2i\pi}{n}\right).$$

Therefore, the contractivity factor necessary to obtain just touching $A_{m,i}(r)$ is

$$\frac{d_0}{d_1} = 2\left(1 + \sum_{i=1}^{k} \cos\left(\frac{2i\pi}{n}\right)\right).$$

Since $k = [n/4]$ our proof is complete.

The sequence $\{r_n\}$. **What** can be said about the sequence of distances

$$r_n = 2\left(1 + \sum_{i=1}^{[n/4]} \cos\left(\frac{2\pi i}{n}\right)\right)?$$

Intuitively, as n increases the polygons are approaching circles so we would expect that the numbers $1/r_n$ would converge to 0. It is easy to see that this is so.

PROPOSITION 2. $\{r_n\}$, as defined above, diverges to infinity.

Proof. It suffices to show that $\sum_{i=1}^{[n/4]} \cos(2\pi i/n)$ diverges to infinity. Let $n \geq 5$ and let $f(k) = \cos(2\pi k/n) - k/n$ for $k \in [0, n/6]$. Then $f'(k) = -(1/n)(2\pi\sin(2\pi k/n) + 1)$, so f is a decreasing function of k on $[0, n/6]$. Because $f(n/6) > 0$, $f(k) > 0$ on $[0, n/6]$. It follows then that $\cos(2\pi k/n) > k/n$ for $k \in [0, \pi/6]$. So

$$\sum_{i=1}^{[n/4]} \cos\left(\frac{2i\pi}{n}\right) > \sum_{i=1}^{[n/6]} \cos\left(\frac{2i\pi}{n}\right) > \sum_{i=1}^{[n/6]} \frac{i}{n} = \frac{1}{2n}\left[\frac{n}{6}\right]\left(\left[\frac{n}{6}\right]+1\right) > \frac{n-6}{72}$$

which diverges to infinity.

Fractal Dimension. The Sierpinski polygons we have been discussing are all examples of a wider class of objects known as fractals. Every fractal has a number associated with it, the fractal dimension, that **determines,** in some sense, how much of the underlying space it occupies. In this section we see that the **Sierpinski *n*-gons** arc really attractors of iterated function systems and we determine the fractal dimension of each of the Sierpinski ***n*-**gons. All definitions in this section can be found in Michael **Barnsley's** book Fractals Everywhere [1]. We begin with a discussion of iterated function systems. **As** earlier, let SP_n be the Sierpinski n-gon.

For a given n, in constructing SP_n we used n contraction mappings of the form $\delta_i(z) = (1/r_n)I_2z + u_i$, for $z, u_i \in \Re^2$, where I_2 is the 2×2 identity matrix. This set of mappings forms what is called an iterated function system (IFS) on $\Re^2$ and is denoted $\{\Re^2; \delta_1, \delta_2, \ldots, \delta_n\}$. We next need to view SP_n as the attractor of this IFS.

The attractor of an IFS $\{\Re^2; \omega_1, \omega_2, \ldots, \omega_N\}$ is found as follows. Let

$B \subset \Re^2$ be a compact set. Let $W(B) = \bigcup_{i=1}^{N} \omega_i(B)$. It turns out that W is a contraction mapping on the metric space of all non-empty compact subsets of $\Re^2$ with the Hausdorff metric. As such, W has a unique fixed point A in $\Re^2$. In other words, there is a non-empty compact subset A of $\Re^2$ so that $W(A) = \bigcup_{i=1}^{N} \omega_i(A) = A$. Another way to think of A is that A $= \lim_{i\to\infty} W^{oi}(B)$ for any compact subset $B \subset \Re^2$. The set A is called the attractor of the IFS. In our situation, we chose $B = A_0$ to be a regular n-gon. We then constructed sets $A_1(r)$, $A_2(r), \dots, A_m(r), \dots$. In following this construction of attractors, for each i the set $A_i(r)$ is equal to $W^{oi}(A_0)$. The attractor of the constructed IFS is then the set we are calling SP_n.

Next we give the definition of fractal dimension in $\Re^2$.

Let A be a non-empty subset of $\Re^2$. For each $\varepsilon > 0$ let $n(A, \varepsilon)$ denote the smallest number of closed balls of radius ε needed to cover A. If

$$D(A) = \lim_{\varepsilon\to 0} \frac{\ln(n(A,\varepsilon))}{\ln(1/\varepsilon)}$$

exists, then $D = D(A)$ is the fractal dimension of A.

In Fractals Everywhere there is a wonderful theorem [Theorem 3, p. 184] that allows us to determine easily the fractal dimensions of the Sierpinski n-gons. We state it for $\Re^2$ but it holds in all dimensions. A complete proof can be found in [2], [4], or [5].

THEOREM 1. Let $\{\Re^2; \omega_1, \omega_2, \dots, \omega_n\}$ be a just touching hyperbolic iterated function system and let A be its attractor. Suppose ω_k is a similitude of scaling factor s_k for each $k \in \{1, 2, \dots, N\}$. Then $D(A)$, the fractal dimension of A, is the unique solution to $\sum_{k=1}^{N} |s_k|^{D(A)} = 1$, $D(A) \in [0, 2]$.

Proposition 3. The fractal dimension of a Sierpinski n-gon is $\ln(n)/\ln(r_n)$, where $r_n = 2\left(1 + \sum_{k=1}^{[n/4]} \cos(2k\pi/n)\right)$.

Proof. Earlier we showed that SP_n is the attractor of a just touching iterated function system to which the contraction mappings $\delta_1, \delta_2, \dots, \delta_n$ all had the same contractivity factor $1/r_n$. Then, by Theorem 1, $1 = n\,|1/r_n|^{D(SP_n)}$. As a result, $D(SP_n) = \ln(n)/\ln(r_n)$.

At this point it seems natural to ask what happens to the sequence $\{D(SP_n)\}$ as $n\to\infty$. As mentioned earlier, as n increases, the polygons we start with are approaching circles. Intuitively, then, we would expect that, as $n\to\infty$, $D(SP_n)$ should approach the fractal dimension of a circle, which is 1. This is, in fact, exactly what happens.

Corollary. $\lim_{n\to\infty} D(SP_n) = 1$.

We omit the proof.

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This paper is the result of a senior project completed by Kevin Dennis and supervised by Steve *Schlicker* while both were at Luther College. Kevin is now at Michigan State, working on his doctorate in the field of analysis. Steve is now at Grand Valley State University.

Mathematical Literacy in the Media

A math student, for instance, might send e-mail to sequences @research.att.com, and in the body of the message "look up 4 9 16 25 36." The student would receive a quick e-mail response explaining the sequence is the start of an infinite series of square roots, and that the next numbers are 49, 64 and so on.—Newsweek 125 (1995) #2 (January 9), 10.

THE AM-GM INEQUALITY VIA ONE OF ITS CONSEQUENCES

Norman Schaumberger
Hofstra University

The arithmetic mean-geometric mean inequality states that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(1) \qquad A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n} = G_n$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Putting $a_1 = a_2 = \ldots = a_r = x$ and $a_{r+1} = a_{r+2} = \ldots a_n = a_n$, (1) becomes

$$(2) \qquad \frac{r}{n}x + \frac{n-r}{n}a_n \geq x^{r/n} a_n^{(n-r)/n}.$$

(2) can be used to derive the basic inequalities of Hölder and Minkowski (see, for example, [1], pp. 67-71). It is easy to verify that (2) still holds for the more general hypothesis that the coefficients of x and a_n are positive reals whose sum is 1. (See, for example, [2], pp. 21-22.) However, for our purpose we use (2) with $r = n - 1$. Thus

$$(3) \qquad \frac{n-1}{n}x + \frac{1}{n}a_n \geq x^{(n-1)/n} a_n^{1/n}.$$

Using basic properties of the derivative we establish (3) without recourse to (1). It is then a simple matter to obtain (1) inductively.

If $x > 0$, then

$$f(x) = \frac{n-1}{n}x - x^{(n-1)/n} a_n^{1/n}$$

has an absolute minimum at $x = a_n$, because

$$f'(x) = \frac{n-1}{n} - \frac{n-1}{n}x^{-1/n} a_n^{1/n}$$

vanishes if and only if $x = a_n$, and

$$f''(x) = \left(-\frac{1}{n}\right)\left(-\frac{n-1}{n}\right)x^{(-1-n)/n} a_n^{1/n}$$

is positive. **(3)** now follows from the observation that

$$\frac{n-1}{n}x - x^{(n-1)/n} a_n^{1/n} \geq \frac{n-1}{n}a_n - a_n^{(n-1)/n} a_n^{1/n} = -\frac{1}{n}a_n$$

with equality if and only if $x = a_n$.

Putting $x = A_{n-1}$, **(3)** can be written as

$$(4) \qquad A_n = \frac{n-1}{n}A_{n-1} + \frac{1}{n}a_n \geq A_{n-1}^{(n-1)/n} a_n^{1/n}$$

with equality if and only if $A_{n-1} = a_n$.

Now using the fact that $A_2 \geq G_2$ and assuming that $A_{n-1} \geq G_{n-1}$ with equality if and only if $a_1 = a_2 = \ldots = a_{n-1}$ it follows that

$$A_{n-1}^{(n-1)/n} a_n^{1/n} \geq G_{n-1}^{(n-1)/n} a_n^{1/n} = G_n$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

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Norman ***Schuamberger's*** last contribution to this Journal appeared in the Spring 1994 issue.

YOUNG'S LATTICES

Carissa Hurst
Hendrix College

The focus of this paper is finding relationships between partitions and their Young's lattices. We begin with a review of partitions, lattices, rank, and Hasse diagrams. Apartition of the natural number N is a finite sequence of natural numbers $n_1, n_2, \dots, n$ in non-increasing order such that $\sum n_i = N$. The numbers n_i are calledparts of the partition. One representation of partitions is called a Ferrer's diagram. In a Ferrer's diagram the partition is displayed in a matrix form where the ith row has n_i squares. For example, Figure 1 shows the Ferrer's diagram for the partition {5, 4, 1, 1} of 11. The squares will be referred to by ordered pairs, where (i, *j*) denotes the square in the ith row and jth column. There is no simple formula for the number of partitions of N. One recursion formula for the total number B_N of partitions **of** N **is** $B_N = \sum_{j=0}^{N-1} \binom{N-1}{j} B_j$. **See [1]** for details.

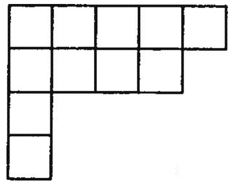

Figure 1

A partially ordered set (poset) that has the property that any two elements x and y have a least upper bound and a greatest lower bound is called a lattice. In particular, a poset that has a maximum element and a greatest lower bound for any two elements is a lattice **[1].**

For any partition P, let B be the set of all partitions whose Ferrer's diagram is contained in the Ferrer's diagram of P. If we order them by containment of their Ferrer's diagram, the resulting poset is called a Young's lattice. Young's lattices are named for Alfred Young, who was **born** in 1873 and graduated from Cambridge in 1895. His main interest was in quantitative substitutional analysis **[2].**

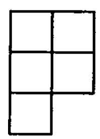

Figure 2

For example, consider the partition {2, 2, 1} with Ferrer's diagram in

Figure 2. The partitions whose Ferrer's diagrams are contained in this partition are in Figure 3. Ordering these by containment we obtain the lattice in Figure 4. This lattice is the Young's lattice.

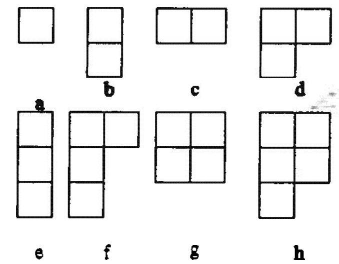

Figure 3

In a Hasse diagram the elements of a lattice are represented as points. Edges connect points using the following rule: if *x*, *y* are points with $x < y$, then an edge is drawn between them and we position *y* above *x*. Edges implied by transitivity are suppressed. We use a Hasse diagram to represent the lattice in the diagram.

The rank of a point in a Hasse diagram is a measure of how far the point is from the minimum, or bottom, element. The bottom element is assigned a rank of one. The lattice in the example has elements of ranks one through five.

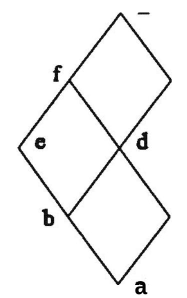

Figure 4

PROPOSITION. If A is a partition in a Young's lattice, then the rank of A is the number of squares in a Ferrer's diagram of A.

Proof. The result follows from the observation that given a Ferrer's diagram in the lattice, the only way to move down the poset is to remove squares from the Ferrer's diagram. Because of transitivity, the Ferrer's diagrams directly below the given one are those found by removing a single square.

We will now classify some Young's lattices.

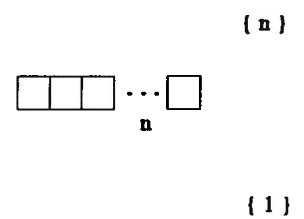

Figure 5

THEOREM 1. If a partition has the form (n), with a Ferrer's diagram as in Figure 5, then the corresponding Young's lattice is a single chain, as in Figure 5.

Proof. The proof is by induction on n. The result clearly holds for $n = 1$ and $n = 2$. Suppose the theorem is true for $n = k$. If we remove the rightmost square from a partition $(k + 1)$, we have the partition (k) whose lattice is a single chain. Since lattices are formed by containment and the two **Ferrer's** diagrams differ only be a single square, the Young's lattice for the larger partition is a chain with an extra vertex $\{k + 1\}$ above the vertex $\{k\}$ connected by an edge.

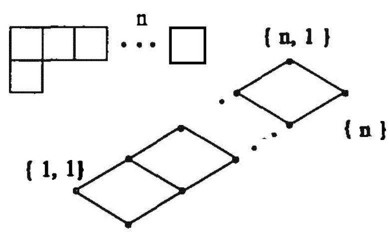

Figure 6

It follows similarly that the Young's lattice for the partition (1, 1, ... , 1) is a single chain.

THEOREM 2. The Young's lattice for the partition $(n, 1)$ is as in Figure 6.

Proof. Again we use induction. It is easy to see that the Young's lattice for (2, 1) has the correct form. Suppose that the theorem is true for $(k, 1)$ and consider the partition $(k + 1, 1)$.

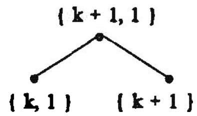

Figure 7

The only square that we can remove and still have a frame is the square at the end of the first row or the square on the second row. Removing the square at the end of the first row leaves a partition $(k, 1)$ to which the induction assumption applies. Removing the square in the second row leaves a partition $(k + 1)$ which we proved earlier has a Young's lattice consisting of a single chain. Therefore, the Young's lattice for $(k + 1, 1)$ will have one vertex of rank $k + 2$ and two vertices of rank $k + 1$ with the form in Figure 7. Using the induction assumption and the fact that the (k) chain is a sub-lattice of the $(k, 1)$ lattice, we obtain the desired form for the $(k + 1, 1)$ lattice.

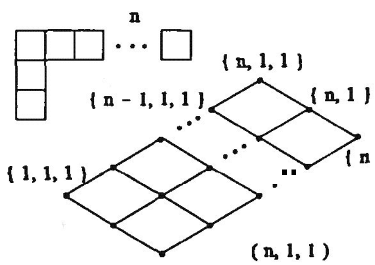

Figure 8

The same reasoning shows that the Young's lattice for the partition $(n, 1, 1)$ is as in Figure 8. This extends to partitions of the form $(n, 1, 1, \cdots, 1)$ in the obvious manner.

The next most complicated partition is (n, 2).

THEOREM 3. If a partition has the form (n, 2) where $n > 2$ with a **Ferrer's** diagram as in Figure 9, then its Young's lattice will be as in Figure 9.

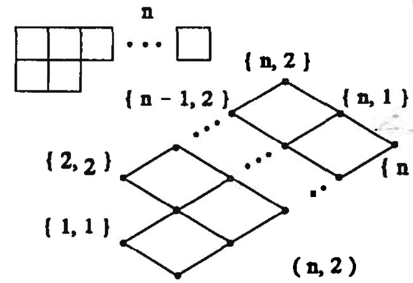

Figure 9

Proof. The proof is again by induction. The result holds for the partition (3, 2) since its Young's lattice (Figure 10) has the proper form. The induction argument proceeds as before since the only squares which can be removed leaving a frame are the square at the end of the first row and the square in the (2, 2) position. Both are partitions whose lattice are already known.

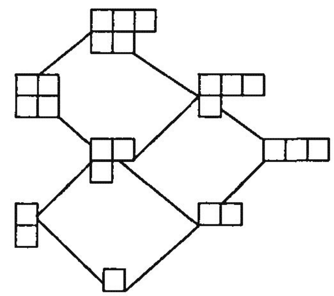

Figure 10

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1. **Bogart,** Kenneth P., Introductory ***Combinatorics,*** Harcourt Brace Jovanovich, **1990.**

2. Cohn, P. M. and Roger Lyndon, The Representation Theory of the Symmetric Group, Addison-Wesley, **1981.**

Carissa Hurst prepared this paper while a senior at ***Hendrix*** College, under the direction of Dr. David Sutherland. She is presently a graduate student in statistics at the University of Arkansas in ***Fayetteville.***

DIVISIBILITY TESTS FOR PRIMES GREATER THAN 5

Phil Plummer
Portland State University

Many papers have been written giving divisibility tests for integers. This note does not contain any new result, but it gathers previous tests in one place and shows how to generate any number of new ones.

Divisibility tests for 2, 3, 4, 5, 8, and 10 are taught as early as elementary school but tests for arbitrary prime numbers are not given. Here is a test for divisibility by 7. Take the number to be tested and double its last digit. Subtract this from the number with its last digit removed. If 7 divides this new number, then 7 divides the original. For example, $7 \mid 294$ since $7 \mid (29 - 2\cdot 4) = 21$. Alternatively, one can multiply the last digit by 5 and add the result: $7 \mid 294$ since $7 \mid (29 + 5\cdot 4) = 49$.

Consider the number 51, a multiple of 17. Does $17 \mid 51$ because $17 \mid (5 - 5\cdot 1) = 0$? In other words, does 5 work for 17 the same way that 2 works for 7 ? The answer is "yes". The proof is as follows. Let $x = 10a + b$ and $r = a - 5b$. We have

$$x + 7r = (10a + b) + 7(a - 5b) = 17a - 34b.$$

If $17 \mid r$, then $17 \mid 7r$. Since $17 \mid (x + 7r)$ we have that $17 \mid x$.

By generalizing this procedure we can prove that a test can devised for any prime and we can find the constant ti. However, if we had a 50-digit number to be tested for divisibility by 7, removing only one digit at a time would be time-consuming indeed. But if we removed ten digits for each iteration it would cut the calculation time needed immensely. A test can be given where the number of digits removed at each stage, y, can be chosen arbitrarily:

THEOREM 1. Given a **prime** p and $x = 10^y a + b$, let $r = a + nb$ where n is the solution to $10^y n \equiv 1 \pmod p$. If $p \mid r$ then $p \mid x$.

Proof. Let n' be such that $nn' \equiv 1 \pmod p$. Note that $10^y nn' \equiv n'$ (mod p) so $10^y \equiv n' \pmod p$. Thus, both $10^y - n'$ and $1 - nn'$ are $\equiv 0$ (mod p). We have

$$x + n'r = 10^y a + b + n'(a - nb)$$
$$= (10^y - n')a + b(1 - nn') \equiv 0 + 0 \pmod p.$$

If $p \mid r$, then $p \mid n'r$. Since $p \mid x + n'r$ we have that $p \mid x$.

In exactly the same way we could prove

THEOREM 2. Given a prime p and $x = 10^y a + b$, let $r = a - mb$, where m is the solution to $10^y m \equiv -1 \pmod p$. If $p \mid r$ then $p \mid x$.

There is a connection between m and n.

COROLLARY. $m = p - n$.

Proof. Since $10^y n \equiv 1 \pmod p$ and $10^y m \equiv -1 \pmod p$, we have $10^y(n + m) \equiv 0 \pmod p$, **and so** $n + m \equiv 0 \pmod p$. Since m and n are both between 0 and p, the corollary follows.

For an example, let us take $x = 28{,}842$, $p = 23$, and $y = 2$. To determine n we solve $10^2 n \equiv 1 \pmod{23}$. This is $8n \equiv 1 \pmod{23}$, so $n = 3$. Thus, $23 \mid 28{,}842$ if $23 \mid (288 + 3\cdot 42) = 414$. Does $23 \mid 414$? It will if $23 \mid (4 + 3\cdot 14) = 46$. Since it does, it follows that $23 \mid 28{,}842$.

The table on the next page gives, for primes $p < 100$, the value of n for $y = 1, 2, \ldots, 12$. The values of n are periodic, with period equal to the order of 10 (mod p).

If we let n_y denote the value of n for y, then we have

THEOREM 3. $n_{y+1} \equiv n_1 n_y \pmod p$.

Proof. $10^{y+1} n_{y+1} \equiv 1 \equiv 1\cdot 1 \equiv (10 n_1)(10^y n_y) \pmod p$.

This recursive property allows the generation of large tables very quickly in a spreadsheet program without the problem of **roundoff** error. For efficient tests, small values of n (or $m = p - n$) can be quickly determined.

Phil Plummer received his B. S. degree in mathematics and physics at Portland State University and is currently finishing work toward *his* M. S. degree in mathematics. He wishes to thank his ***high-school*** mathematics teacher, Mr. Wayne Wheeler of Springfield (Oregon) High School, ***for*** teaching him that mathematics could be fun.

y (number of digits removed)

	1	2	3	4	5	6	7	8	9	10	11	12
3	1	1	1	1	1	1	1	1	1	1	1	1
7	5	4	6	2	3	1	5	4	6	2	3	1
11	10	1	10	1	10	1	10	1	10	1	10	1
13	4	3	12	9	10	1	4	3	12	9	10	1
17	12	8	11	13	3	2	7	16	5	9	6	4
19	2	4	8	16	13	7	14	9	18	17	15	11
23	7	3	21	9	17	4	5	12	15	13	22	16
29	3	9	27	23	11	4	12	7	21	5	15	16
31	28	9	4	19	5	16	14	20	2	25	18	8
37	26	10	1	26	10	1	26	10	1	26	10	1
41	37	16	18	10	1	37	16	18	10	1	37	16
43	13	40	4	9	31	16	36	38	21	15	23	41
47	33	8	29	17	44	42	23	7	43	9	15	25
53	16	44	15	28	24	13	49	42	36	46	47	10
59	6	36	39	57	47	46	40	4	24	26	38	51
61	55	36	28	15	32	52	54	42	53	48	17	20
67	47	65	40	4	54	59	26	16	15	35	37	64
71	64	49	12	58	20	2	57	27	24	45	40	4
73	22	46	63	72	51	27	10	1	22	46	63	72
79	8	64	38	67	62	22	18	65	46	52	21	10
83	25	44	21	27	11	26	69	65	48	38	37	12
89	9	81	17	64	42	22	20	22	18	73	34	39
97	68	65	55	54	83	18	60	6	20	2	39	33

n (multiplier for the last y digits)

A PROOF OF PASCAL'S HEXAGON THEOREM USING ABRIDGED NOTATION

Margaret Maxfield
Louisiana Tech University

In a recent article about Pascal's hexagon theorem for a circle, Jan van Yzeren [2] credited H. Guggenheimer with a previous **proof.** However, Professor Guggenheimer [1] explained that the proof had in fact been taught him in his 11th grade Descriptive Geometry class in **Basel,** Switzerland. The following proof, which I learned as a college junior, used to be called a proof by "abridged notation". In it, a linear form is represented by a single letter, and forms are combined to make second-degree expressions that will stand for conic sections.

For example, if $\alpha = x + 2y + 1$ and $\beta = x - 2y + 2$, then $\alpha = 0$ and $\beta = 1$ are equations of lines, $\alpha\beta = 0$ is the equation of a pair of intersecting lines (a degenerate conic—asymptotes only), and $\alpha\beta = 1$ is the equation of a nondegenerate hyperbola.

Any equation of the form $\alpha + k\beta = 0$ represents a line (it is linear in form) that passes through the intersection of the lines $\alpha = 0$ and $\beta = 0$ since substitution of the coordinates of the point of intersection make both α and β take on the value 0. We use a similar strategy for conic sections. If $S = 0$ and $T = 0$ are conies, then they are of second degree in x and y. Then for any nonzero constant k, $S + kT = 0$ is of second degree so it represents a conic. Since the point where S and T intersect has coordinates that satisfy both $S = 0$ and $T = 0$, the conic $S + kT = 0$ passes through the points of intersection.

PASCAL'S THEOREM: If a closed hexagon is inscribed in a conic section, then the three points of intersection of its opposite sides are collinear.

(If the two opposite sides are parallel, their point of intersection is taken to be the point at infinity. The conic section may be degenerate, and the hexagon need not be convex.)

Proof. Let the six vertices of the inscribed hexagon be labelled in order A, B, C, D, E, F. Let the sides be labelled a , β, y, 6, E, ζ (see Figure **1),** where, for instance a = $a_1x + a_2y + a_3$ and a = 0 is the equation of the line AB. Continuing, β = 0 is the equation of the line through BC and so on. It is to be proved that the intersections P of side a and side 6, Q of side β and E, and R of y and ζ are **collinear.**

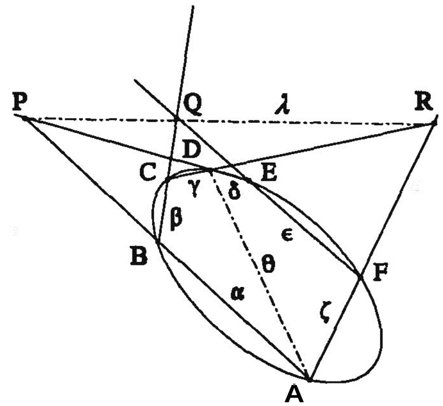

Figure 1

Construct a seventh line, 9, through A and D. Let S_1 stand for the **second-degree** sum of products

$$ay + r\beta\theta.$$

For any value of r, $S_1 = 0$ is the equation of a conic section that circumscribes the quadrilateral ABCD since, for instance, C lies on both β and y. Similarly, if

$$S_2 = \delta\zeta + s\varepsilon\theta$$

then $S_2 = 0$ is the equation of a conic section that circumscribes the quadrilateral DEFA. Constants r, s, and t exist for which $S_1 \equiv tS_2$ a C, where C is the given conic section circumscribing the hexagon. From $S_1 \equiv tS_2$, it follows that

$$\theta(r\beta - st\varepsilon) \equiv t\delta\zeta - ay.$$

The equation rβ − $st\varepsilon$ = 0 represents a line A that passes through the intersection of lines **β** and E (Pascal point Q), so the left member of the identity is the degenerate conic section composed of lines 9 and A. This is the same degenerate conic section as that on the right, including line 9 through A and D and a second line that passes through the intersection of lines a and 6 (Pascal point P) and the intersection of lines y and ζ (Pascal point R). All **three** Pascal **points lie** on line A, and the theorem is proved.

Two of the vertices may be brought into coincidence, in which case **the** side joining **them** becomes the tangent to the conic at the double point. **When the circumscribing** conic section degenerates to straight **lines**, the special case is the **Theorem** of **Pappus:** Let three **vertices**, say A, ***B***, C on one line L of the degenerate conic and three vertices A, B, C on the other line L be connected in any order that alternates the two sets of three, such as ***ABBACCA*** or ***ACBBCAA***, forming a closed (not convex) hexagon, as in Figure 2. Then the **intersections** of opposite sides of the hexagon (side 1 with 4, 2 with 5, and 3 with 6) are **collinear.** The six Pascal lines from the six different hexagons that can be **formed** pass three by three through two points; that is, the six Pascal lines divide into two concurrent sets of three.

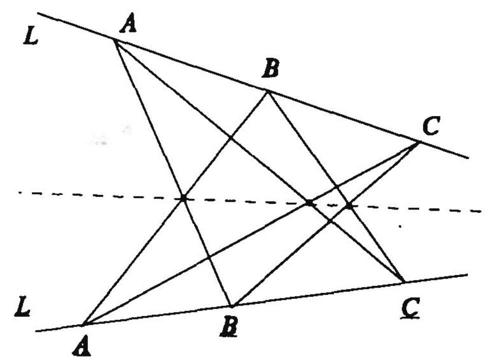

Figure 2

An inscribed hexagon has sides that connect pairs of vertices on the conic. The three points of intersection of pairs of opposite sides are *collinear.* Using duality, we replace each line by point, immediately obtaining **Brianchon's** Theorem: A circumscribed hexagon has vertices that connect pairs of lines "on" the conic (that is, it has vertices that are the intersections of tangents to the conic). The three lines connecting pairs of opposite vertices arc concurrent.

A way of constructing points on a conic section can be derived from Pascal's **Theorem.** Five points determine a conic. To construct a sixth point given ***A,*** B, C, D, ***E,*** let AB and DE intersect at the Pascal point **P.** Draw a line x through E passing through the general region where the sixth point F is desired. (See Figure 3.) The intersection of line x with BC is a second Pascal point Q. Complete the Pascal line PQ. Its intersection with CD is the third Pascal point R, and line AR intersects line x in the desired point F on the conic. Further points are obtained by rotating the position of the line x, as to x', and repeating the construction to get another Pascal line, $PQ'R'$, and another point on the conic, F' This construction has actually been used in fairing in smooth curved outlines for aircraft design.

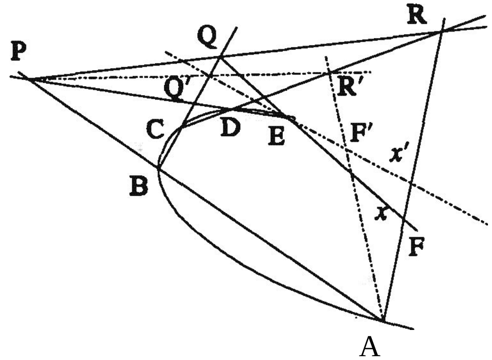

Figure 3

How many hexagons can be drawn using a given set of six vertices on a conic? There are **6!** permutations, but the starting point is arbitrary for a closed hexagon, and reversing the order does not change the hexagon, so there are $6!/(6\cdot 2) = 60$ hexagons. The 60 corresponding Pascal lines pass 3 by 3 through 20 Steiner points, which lie 4 by 4 on 15 Steiner-Pliicker lines. Also, the 60 Pascal lines pass 3 by 3 through 60 **Kirkman** points, which lie 3 by 3 on 20 Cayley-Salmon lines other than the Pascal lines.

Although conic sections arc only one degree more advanced than straight lines, they have a rich analytic geometry.

References

1. Guggenheimer, H., Pascal's Theorem, ***American Mathematical Monthly*** **101** (1994) **#5, 4**16.

2. van Yzeren, Jan, A simple proof of Pascal's Hexagon Theorem, ***American Mathematical Monthly*** **100** (1993) **#1**0, **930-93**1.

In response to a request for a few words about the author, Professor Maxfield wrote

Among all the beautiful proofs in mathematics, this abridged notation proof of Pascal's theorem is my favorite. Isn't it elegant? For many of us, mathematics is a branch of esthetics, certainly more art than science. I teach statistics, which is kind of a branch of epistemology—"How can we know?". From students' difficulties in learning elementary statistics, I suspect that they have a severe problem with semiotics, a problem that has cursed their efforts in arithmetic and algebra all their lives. Are you a semiotician?

A Novel Sequence

From ***The Theory of Everything,*** by Lisa Grunwald (Knopf, New York, 1991, page 142):

> Alexander opened his eyes and tried to find a pattern in the row of Linda's flowerpots. He'd always been good at sequences. If one, ten, three, nine, five, eight, seven, seven, nine, and six are the first elements in a group, what number continues the sequence? Easy. Five.

It certainly is, and the next term is equally easy, is it not?

That was all right, but I think that in general novelists ought to stay away from mathematics. E. g., on page 117 we find

> Physicists and mathematicians played a similar game in their heads, called sphere packing: How many spheres could fit inside one sphere? How many circles could fit inside one circle?

Anybody here ever played the sphere packing game? I never have: have I missed out on all the fun?

Then on page 144,

> Zeno lived around Aristotle's time, and like Aristotle, he enjoyed stirring things up.

Would we say that Euler lived around **Hilbert's** time? I don't think so, though the separation-around 150 years—is the same. Perhaps time intervals foreshorten from a long way away.

A Diophantine Equation

Efraim Berkovich
Georgetown University

During a course in number theory, we solved the diophantine equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{14}.$$

The solution given in class and in the teacher's edition of the textbook was rather long and time-consuming. In this note I give a straightforward method of solving this problem, and any diophantine equation

$$\frac{1}{ax} + \frac{1}{by} = \frac{1}{c}$$

where a, b, and c are integers. The solution involves only factoring an integer.

One method of solving this equation would be to set

$$c(ax + by) = abxy = A$$

and thus reduce the problem to solving the quadratic equation

$$z^2 - \frac{\lambda z}{c} + \lambda = 0$$

of one variable. This method, however, requires that we then find values of A which give integer values to the square root of the discriminant.

A simpler method is to make the substitutions $s = c - ax$ and $t = c - by$. The equation becomes

$$\frac{1}{c - s} + \frac{1}{c - t} = \frac{1}{c}$$

$$c(c - s) + c(c - t) = (c - s)(c - t)$$

which simplifies to

$$c^2 = St.$$

We now need integers s and t which satisfy $c^2 = st$. So, s and t have the same sign and are factors of c^2. Factoring yields all possible values-of the integer pairs (s, t). Taking each (s, t) pair, we have a solution (x, y) if $x = (c - s)/a$ and $y = (c - t)/b$ are integer valued. In this manner we will find all the solutions of the original equation because any (x, y) which satisfy the original diophantine equation will necessarily satisfy $c^2 = st$.

This procedure requires nothing more complicated than factoring c^2 and provides an upper bound to the number of solutions. If c^2 has ξ factors, then there are at most $25 - 1$ solutions: the number of different factors and their negatives, excluding $s = c$ and $t = c$ since that would give $x = 0$ and $y = 0$. The actual number of solutions would be reduced by the conditions that $c - s$ and $c - t$ be divisible by a and b, respectively.

As an example, let us consider $1/x + 1/y = 14$. Since the equation is symmetric in x and y, when we satisfy the condition $c^2 = st$, it is sufficient to consider the pairs (s, t) where $s \geq t$. The pairs are (196, 1), (98, 2), (49, 4), (28, 7), (–1, –196), (–2, –98), (–4, –49), (–7, –28), and (–14, –14). The corresponding solutions (x, y) with $x \leq y$ are therefore (–182, 13), (–84, 12), (–35, 10), (–14, 7), (15, 210), (16, 112), (18, 63), (21, 42), and (28, 28).

This technique also allows us to make some observations about solutions of the equation. For example, if c is prime and $a = b = 1$, then we know that there are exactly three solutions.

Efraim Berkovich was graduated from Georgetown University in May 1994 with a B. S. degree in mathematics. He hopes to pursue graduate studies in electrical engineering.

All stiff regularity (such as borders on mathematical regularity) is inherently repugnant to taste, in that the contemplation of it affords us no lasting entertainment. —Immanuel Kant, Critique of Judgement.

Group Generators and Subgroup Lattices

Scott M. Wagner
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This paper will examine one of the connections between combinatorics and algebra. We will use Philip Hall's Eulerian function to count the number of generating sets of size n for the twenty-eight groups of order fifteen or less. Recall that a generating set of a group is a subset of the group such that every element in the group can be expressed as a product of one or more elements of the generating set. For example, the cyclic group of order 4, $\{e, a, a^2, a^3\}$ has two generating sets of size 1: $\{a\}$ and $\{a^3\}$.

The set of all subgroups of a group G under subset inclusion forms a lattice where the greatest lower bound of two subgroups is their intersection, and the least upper bound of two subgroups is the smallest subgroup of G containing both of them. Hasse subgroup diagrams are diagrams of these lattices. The subgroups are represented by points, and if A and B are subgroups with $A \subseteq B$ then the points representing A and B are connected with an edge, with B positioned above A. The subgroup $\{e\}$ is at the bottom of the lattice, and the group itself is at the top. Examples of Hasse subgroup diagrams for groups of low order can be found in [1].

In 1935 Philip Hall described a method for counting the number of ways of generating the group of symmetries of the icosahedron from a given number of elements [2]. One outcome of his method was the development of a generalized Eulerian function which can be used to count the number of generating sets of a given size for any finite group. The method does not determine the actual generating sets, but only their number.

The function is defined by

$$\varphi_n(G) = \sum_{H \leq G} \mu(H, G) \binom{|H|}{n}$$

where G is the finite group, n is the number of elements in the generating set, H is a subgroup in the Hasse diagram, $|H|$ is the order of H, and μ is defined recursively by $\mu(G, G) = 1$ and, for each subgroup H of G,

$$\sum_{K: H \leq K} \mu(K, G) = 0.$$

So, μ determines a coefficient for each subgroup of G. The top subgroup of the lattice, G itself, has coefficient 1. The coefficient of any other subgroup in the lattice is the integer that, added to the coefficients of every subgroup above it in the lattice, gives a sum of zero.

For example, let us calculate $\varphi_1(C_4)$, where C_4 is the cyclic group of order 4. The Hasse diagram for C_4 and the values of $\mu(H, C_4)$ for the two subgroups of C_4 are shown in Figure 1. Applying Hall's formula, we see $\varphi_1(C_4)$

$$= 1\binom{4}{1} - 1\binom{2}{1} + 0\binom{1}{1}$$

$= 4 - 2 + 0 = 2$, giving the number of one-element generators of C_4.

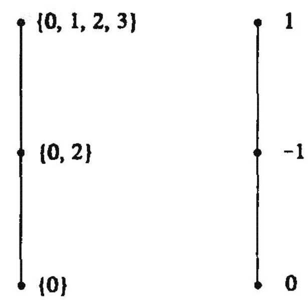


Figure 1

Now let us consider D_3, the group of symmetries of an equilateral triangle. The group has three rotations, e, r, r^2, and three reflections, s, rs, r^2s. The Hasse subgroup diagram and the values of μ are shown in Figure 2, where $A = D_3 = \{e, r, r^2, s, rs, r^2s\}$, $B = \{e, s\}$, $C = \{e, rs\}$, D $= \{e, r^2s\}$, $F = \{e, r, r^2\}$, and G $= \{e\}$. Since cyclic

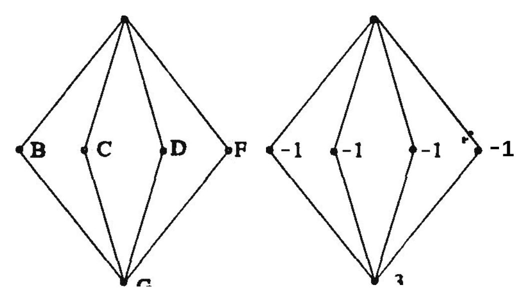


Figure 2

groups are the only groups with one-element generating sets, we know that $\varphi_1(D_3) = 0$, which is verified by the formula:

$$\varphi_1(D_3) = 1\binom{6}{1} - 3\binom{2}{1} - 1\binom{3}{1} + 3\binom{1}{1} = 6 - 6 - 3 + 3 = 0.$$

Moreover, $\varphi_2(D_3)$ is

$$\varphi_2(D_3) = 1\binom{6}{2} - 3\binom{2}{2} - 1\binom{3}{2} = 15 - 3 - 3 = 9.$$

To see why that is correct, note that the first term in the sum, $1\binom{6}{2} = 15$, counts the number of two-element subsets of D_3. From these we must exclude those that generate a proper subgroup of D_3. A two-element subset that does not generate D_3 must generate one of the two subgroups of order two or the unique subgroup of order three. The term $3\binom{2}{2}$ eliminates those that generate the three two-element subgroups while the term $1\binom{3}{3}$ eliminates the three two-element sets that generate the subgroup of order three. This can be verified in another way by noting that of the $\binom{5}{2} = 10$ two-element subsets not containing the identity, only $\{r, r^2\}$ generates a proper subgroup of D_3.

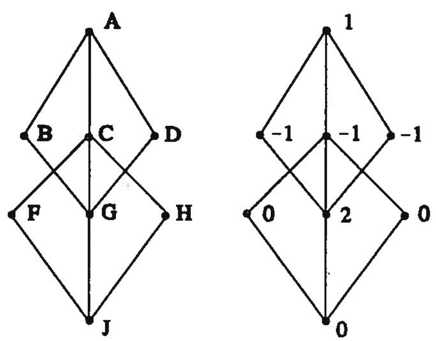


Figure 3

We will give three more examples and a table giving values of φ_n for $n = 1, 2, 3$ for some finite groups. This may serve as a source of examples.

The Hasse diagram and μ values for

$$C_2 \times C_4 = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)\}$$

appear in Figure 3, where $A = C_2 \times C_4$, $J = \{(0, 0)\}$,

$B = \{(0, 0), (1, 1), (0, 2), (1, 2)\}$,

$C = \{(0, 0), (1, 0), (0, 2), (1, 2)\}$,

$D = \{(0, 0), (0, 1), (0, 2), (0, 3)\}$,

$F = \{(0, 0), (1, 0)\}$, $G = \{(0, 0), (0, 2)\}$, and

$H = \{(0, 0), (1, 2)\}$.

Since this group is not cyclic, the number of two element generators is

$$\varphi_2(C_2 \times C_4) = 1\binom{8}{2} - 3\binom{4}{2} + 2\binom{2}{2} = 28 - 18 + 2 = 12.$$

We will look next at A_4, the alternating group of degree four. Its members are all the even permutations in S_4, the symmetric group of degree four. The Hasse subgroup diagram and the μ values are in Figure 4, where A = A_4, L = $\{e\}$, B = $\{e, (12)(34), (13)(24), (14)(23)\}$,

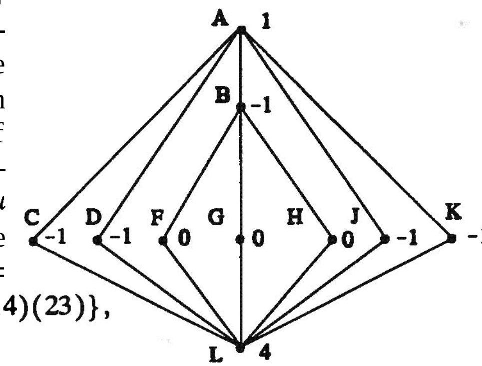


Figure 4

$C = \{e, (123), (132)\}$,

$D = \{e, (124), (142)\}$,

$F = \{e, (12)(34)\}$,

$G = \{e, (13)(24)\}$, $H = \{e, (14)(23)\}$,

$J = \{e, (234)(243)\}$, and $K = \{e, (134)(143)\}$.

So, we find that the number of two element generators is

$$\varphi_2(A_4) = \binom{12}{2} - \binom{4}{2} - 4\binom{3}{2} = 48.$$

The final group we will examine is a dicyclic group. A dicyclic group, G, can be represented by the following group presentation:

$$G = \{(x, y) \mid x^{2n} = 1, x^{2n}y^{-2} = 1, y^{-1}xyx = 1\}.$$

We will consider T, the dicyclic group of order 12, where $n = 3$. The elements of T are

$$e, x, x^2, x^3, x^4, x^5, y, y^3, xy, yx^2, x^2y, \text{and } y^3x^2.$$

The Hasse subgroup diagram and the μ values are in Figure 5, **where** A = T, J $=\{e\}$, and B = ,
$\{e, x^3, yx^2, y^3x$
$C = \{e, y, x^3, y^3\}$,
$D = \{e, x^3, xy, x^2y\}$,
$F = \{e, x, x^2, x^3, x^4, x^5\}$,
$G = \{e, x^3\}$, and
$H = \{e, x^2, x^4\}$.

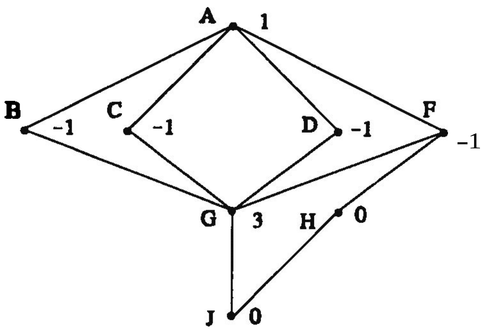


Figure 5

Once again, since this group is not cyclic, we will find the number of two-element generating sets:

$$\phi_2(T) = \binom{12}{2} - 3\binom{4}{2} - \binom{6}{2} + 3\binom{2}{2} = 36.$$

The table below gives the number of one, two, and three element generators for twenty-seven groups.

Group	φ_1	φ_2	φ_3
C_1	1	-	-
C_2	1	1	-
C_3	1	3	1
C_4	2	5	3
C_5	4	10	20
C_6	2	11	19
C_7	6	28	35
C_8	4	22	52
C_9	6	33	84
C_{10}	4	34	90
C_{11}	10	55	165
C_{12}	4	46	196
C_{13}	12	78	286
C_{14}	6	69	329
C_{15}	8	92	444

Group	φ_1	φ_2	φ_3
D_3	0	9	19
D_4	0	12	52
D_5	0	30	110
D_6	0	18	150
D_7	0	63	329
$C_2 \times C_2$	0	3	4
$C_2 \times C_4$	0	12	44
$C_3 \times C_3$	0	24	80
$C_2 \times C_2 \times C_2$	0	0	44
$C_2 \times C_2 \times C_3$	0	18	156
A_4	0	48	212
T	0	36	188

References

1. Butt, Melanie, Automorphism groups of Hasse subgroup lattices for groups of low order, this Journal 9 (1989-94) #1, 2-8.
2. Hall, Philip, The **Eulerian** functions of a group, Quarterly ***Journal*** of Mathematics 7 (1936), **143-151.**

Scott Wagner wrote this paper while a senior mathematics major at ***Hendrix*** College, with project advisor David Sutherland. He wishes to thank Joseph Kung and Cynthia ***Sewall*** for the idea for the project, and the referee for valuable suggestions.

All Statements Are Not Correct

On the important subject of language, Professor **James Chew** (North Carolina A & T State University) writes

> One of my biggest peeves in English mis-usage has to do with negations of statements. For example, "All athletes are not basketball players". We all know what is intended is "Not all athletes are basketball players". I would be willing to let this matter slide were it not for the fact that even mathematicians who should know better, since they arc the presumed guardians of **clear** and logical thinking, commit this error regularly. In the ***Transactions*** of the American Mathematical Society, that pinnacle of **mathematical** journals, whose contents are so sublime that 1 wish 1 could understand .05 of one percent of them, I found "All **topological** spaces are not **Hausdorff**". This is FALSE since the **real** line (**with** its usual topology) is a topological space which is **Hausdorff.**

MOTIVATING THE SOLUTION TO A DIFFERENCE EQUATION

Russell Euler
Northwest Missouri State University

An nth-order linear homogeneous difference equation with constant coefficients has the form

(1) $$a_n y_{n+k} + a_{n-1} y_{n+k-1} + \dots + a_1 y_{k+1} + a_0 y_k = 0$$

where $a \neq 0$. When seeking a solution of **(1),** students are frequently asked under what conditions, if any, will

(2) $$y_n = \lambda^n$$

be a nontrivial solution? This approach is used in several standard textbooks. It is sometimes motivated by examining the general form of the solution to a first-order linear difference equation. Substituting (2) into (1) and simplifying yields the characteristic equation

(3) $$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0.$$

This method is probably adequate motivation for students who have not studied differential equations. However, those who have do not expect to see solutions of the form (2). They know that solutions of an nth-order linear homogeneous differential equation with constant coefficients,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0,$$

have the form $y = e^{\lambda_i x}$ where the λ_i's are distinct roots of the same characteristic equation (3). Such students would see the similarities between difference equations and differential equations if they were asked under what conditions, if any, will

(4) $$y_n = e^{\lambda n}$$

be a nontrivial solution of **(1)?** Substituting (4) into (1) yields

$$a_n e^{\lambda(n+k)} + a_{n-1} e^{\lambda(n+k-1)} + \dots + a_1 e^{\lambda(k+1)} + a_0 e^{\lambda k} = 0$$

and so

(5) $$a_n e^{\lambda n} + a_{n-1} e^{\lambda(n-1)} + \dots + a_1 e^{A} + a_0 = 0.$$

As an illustration, let $n = 2$. Then (5) becomes

$$a_2 e^{2\lambda} + a_1 e^{\lambda} + a_0 = 0$$

and so

$$e^{\lambda} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$$

Hence, (4) becomes

$$y_n = (e^{\lambda})^n = \left(\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}\right)^n$$

For the case when $n = 2$ and the characteristic equation has a repeated root, the form of the general solution is motivated in **[1].**

Finally, it can be seen that (3) and (5) have the same form by letting $t = e^{A}$ in (5).

Reference

1. Euler, Russell, A note on a difference equation, this *Journal* 9 (1989-94) #S, 530-531.

Russell Euler is a frequent contributor to this and other journals.

I don't take no stock in mathematics, anyway.—Mark Twain, ***Huckleberry Finn,*** Chapter 4.

A Test for Affineness

Christopher Kohl
James Madison University

The purpose of this paper is to present a characterization of affine spaces and a necessary condition for a function to be an affine map. The ideas for these theorems arose from a pair of problem sets in a linear algebra textbook **[1]**. Specifically, the reader is given either a subset of a vector space or a function and asked to determine whether or not it is affine. An examination of the first problem set led to an observation that was generalized into a theorem (Theorem 1 below), which states a condition under which a subset of a vector space is not an affine subspace. Similarly, an examination of the second problem led to a theorem (Theorem 2 below) which states an assertion about the image of an affine map.

We assume that the reader is familiar with the fundamental definitions of a vector space, a subspace, and a linear map. These and the following definitions can be found in chapters 4 and 5 of **[1]**.

DEFINITION 1. A subset U of a vector space V is called an ***affine subspace*** of V provided that, for some **subspace** W of V and some fixed vector b in V, $U = \{w + b \mid w \in W\}$.

DEFINITION 2. A function $g : U \rightarrow V$ is called an ***affine map*** provided that, for some linear map $f : U \rightarrow V$ and some fixed element b of V, $g(\mathbf{u}) = f(\mathbf{u}) + b$ for all u in U.

THEOREM 1. Let S be a subset of a vector space V. If there is a fixed vector in V, call it p, so that $0 \in S + p$ and $S + p$ is not a **subspace** of V, then S is not an affine **subspace** of V.

Proof. Suppose S is an affine **subspace** of a vector space V and that p is a fixed vector in V so that $0 \in S + p$. It follows from Definition 1 that there is a **subspace** of V, call it W, and there is a fixed vector in V, call it b, so that $S = \{m + b \mid m \in W\}$. Thus,

$$S + \mathbf{p} = \{\mathbf{m} + \mathbf{b} + \mathbf{p} \mid \mathbf{m} \in W\}.$$

Since $0 \in S + p$, it follows that there is an element of W, call it r, so that

$0 = r + \mathbf{b} + p$. Thus, we can write $r = -(\mathbf{b} + p)$, which implies that $\mathbf{b} + p \in W$. Now we show that $S + p$ is a **subspace** of V.

Let $\mathbf{x}_0, \mathbf{x}_1 \in S + p$ and $c \in \mathbf{R}$. It follows that there are elements of W, call them $\mathbf{t}_0$ and $\mathbf{t}_1$, so that

$$\mathbf{x}_0 = \mathbf{t}_0 + \mathbf{b} + p \quad \text{and} \quad \mathbf{x}_1 = \mathbf{t}_1 + \mathbf{b} + p.$$

Therefore, we can write

$$\mathbf{x}_0 + \mathbf{x}_1 = (t_0 + \mathbf{t}_1 + \mathbf{b} + p) + \mathbf{b} + p.$$

Since $\mathbf{t}_0, \mathbf{t}_1, (\mathbf{b} + p) \in W$, it follows that $(\mathbf{t}_0 + \mathbf{t}_1 + b + p) \in W$. Thus, $S + p$ is closed under addition.

Also,

$$c\mathbf{x}_0 = c(\mathbf{t}_0 + b + p) = c\mathbf{t}_0 + (c - 1)(\mathbf{b} + p) + (b + p).$$

Since $c, c - 1 \in \mathbf{R}$ and $\mathbf{t}_0, (\mathbf{b} + p) \in W$, it follows that $c\mathbf{t}_0 + (c - 1)(\mathbf{b} + p) \in W$. Thus, $S + p$ is closed under scalar multiplication. And so, by **[1, Theorem 4.2]**, $S + p$ is a **subspace** of V.

THEOREM 2. The image of an affine map $f : U \rightarrow V$ between vector spaces U and V is an affine **subspace** of V.

Proof. Let $f : U \rightarrow V$ be an affine map between vector spaces U and V, and let M denote the image off. By Definition 2 there is a linear map, call it $g : U \rightarrow V$, and a fixed element of V, call it b, so that flu) = $g(\mathbf{u}) + b$ for all u in U. Since g is a linear **map** it follows from **[1,** Theorem 5.5] that the image of g, call it T, is a **subspace** of V. Since flu) = $g(\mathbf{u}) + b$ for all u in U, we can write $M = T + b$. And so, by Definition 1, M is an affine **subspace** of B.

The problems that motivated the ideas for Theorem 1 deal exclusively with two- and three-dimensional vector spaces. For example, given the set $T = \{(x, \sin x + 3) \mid x \in \mathbf{R}\}$, determine whether T is a **subspace** of $\mathbf{R}^2$, an affine **subspace** of $\mathbf{R}^2$, or neither. After examining this problem and formulating Theorem 1, it was observed that the geometry of T could have been used to conclude that T is not an affine subspace of $\mathbf{R}^2$. This results from the tractable geometry of $\mathbf{R}^2$. Since a subspace of $\mathbf{R}^2$ has a geometric realization as either the origin, a line through the origin, or the plane itself, it follows that an affine subspace of $\mathbf{R}^2$ must have a geometric realization as either a point, a line, or the plane itself. Therefore, given a subset of $\mathbf{R}^2$ and asked to determine whether or not it is affine, one can examine the geometry of the subset and make conclusions accordingly. Consider the above example dealing with the subset T of $\mathbf{R}^2$. Since the geometric realization of T in $\mathbf{R}^2$ is not a line, a point, or the plane itself, one can conclude that T is not an affine subspace. Now consider the vector space consisting of all 12 x 15 matrices. The geometry of such a vector space is certainly much more complex than that of $\mathbf{R}^2$. Thus, given a subset of such a vector space, one cannot rely on its geometry to test for affineness. Its algebraic structure must be considered and that is where the utility of Theorem 1 is realized. There will follow an application of Theorem 1 where the geometry of the underlying vector space is too complex to use as a tool in testing for affineness.

Inspection of Theorem 2 reveals that it can be restated as follows: if the image of a function from a vector space U to a vector space V is not an affine **subspace** of V, then the underlying function is not an affine map. Clearly, Theorem 2 is a direct outgrowth of Theorem 1. Because of this relationship between the two theorems, the ideas for employing the geometry of a given subset can be carried over to Theorem 2.

Let M_{nn} denote the vector space of all n x n matrices with real number entries. Consider the set

$$S = \{A \in M_{nn} \mid \det(A) \neq 0\}.$$

By definition, S is the set of all nonsingular matrices of size n x n. By employing Theorem 1, we will show that S does not constitute an affine **subspace** of M_{nn}. From the definition of S, it follows that S is a subset of M_{nn} and that $0 \notin S$, which implies that S is not a **subspace** of M_{nn}. Since $I \in S$, it follows that $0 \in (S + -I)$. Next, we verify that $(S + -I)$ is not a **subspace** of M_{nn} by showing that it is not closed under addition.

Consider the n x n matrix $\frac{1}{2}I$. Since $\frac{1}{2}I \in S$, it follows that $(\frac{1}{2}I + -I) \in (S + -I)$. We can write $(\frac{1}{2}I + -I) + (\frac{1}{2}I + -I) = -I$. Since $0 \notin S$, it follows that $-I \notin (S + -I)$. Therefore, we can conclude that $(S + -I)$ is not closed under addition and it follows from [1, Theorem 4.2] that $(S + -I)$ is not a **subspace** of M_{nn}. Since S is a subset of M_{nn}, $-I$ is a fixed vector in M_{nn} so that $0 \in (S + -I)$, and $(S + -I)$ is not a **subspace** of M_{nn}, it follows from Theorem 1 that S is not an affine **subspace** of M_{nn}.

Reference

1. Johnson, Robert S. and Thomas O. Vinson, ***Elementary Linear Algebra***, Harcourt Brace Jovanovich, 1987.

Chris Kohl graduated from James Madison University in May 1994 with a B. S. degree in mathematics. Currently, he is a software developer at Avtec Systems in Fairfax, Virginia. He would like to thank Dr. Carter Lyons for his guidance and support during the preparation of this paper.

Rex H. Wu (Brooklyn, New York) points out two applications of the pigeon-hole principle. If you ask people unfamiliar with the problem how many socks it is necessary to take from a drawer containing four red socks, five blue ones, six green, and seven white, you will get a variety of answers, some irrelevant (e. g., "***Nobody*** wears green socks, except maybe on St. Patricks's day"). Try asking! All it can do is make you unpopular. A second application proves the theorem that there are now living two people who were born at exactly the same time, to the second. This follows from the calculation that there have been fewer than five billion seconds in the last 150 years and the fact that the world contains more than five billion people. Since there are only 31,536,000 seconds in a 365-day year and more births than that this year, there will be two such people born in 1995.

Xuming Chen (University of Alabama) gives a good reason why 1 is not counted among the primes. The valuable identity

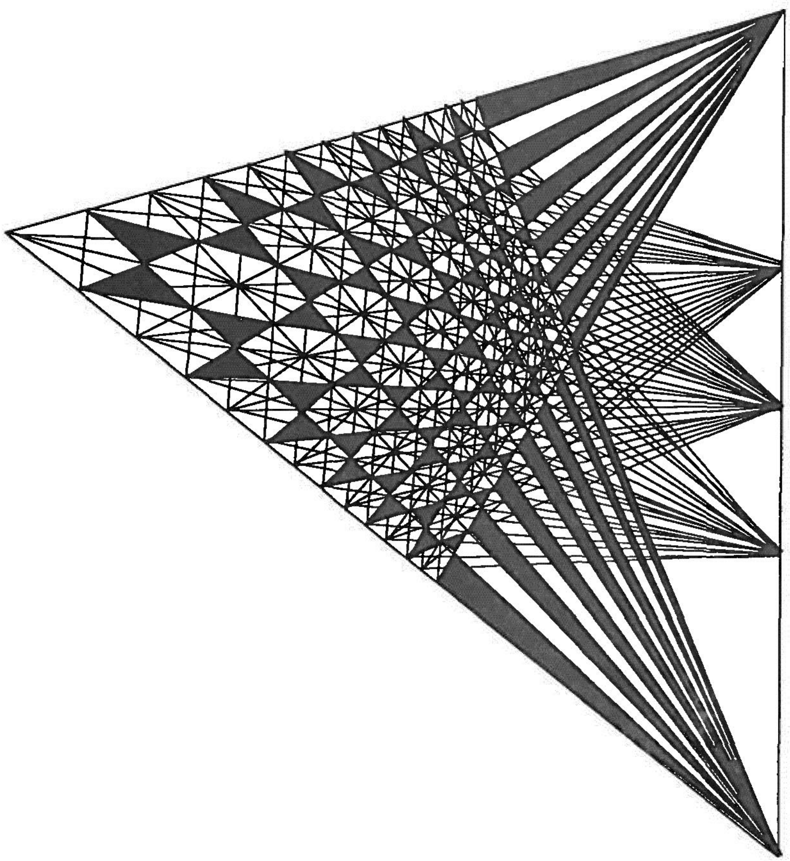
$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - (1/p)^s}, \quad s > 1$$

would not hold if 1 was a prime.

STARS AND STRIPES

A. Cusumano and *W. Topazio*
Great Neck, New York

(Symmetric projective geometry.)



ANOTHER PROOF THAT THE NIEMYTZKI PLANE IS NOT NORMAL

David H. Vetterlein
Auburn University

Let X be the upper half-plane $X = \{(x, y) \in \Re^2 : y \geq 0\}$ and let L be its boundary $L = \{(x, 0) : x \in \Re\}$. The Niemytzki plane is the space X whose basic sets are the open discs in $X \setminus L$ and the open discs in $X \setminus L$ tangent to L together with the point of tangency. We will refer to the points on L as being rational or irrational with the obvious correspondence. This space and the Sorgenfrey plane (i. e. the product space $S \times S$ where S is the real line with the half-open interval topology) provide examples of a T_3 topological space which is not a T_4-space. While the Sorgenfrey plane has the merit of being the product of T_4-spaces, the Niemytzki plane lacks this. But it has another, namely that its topology is more familiar and easier to work with.

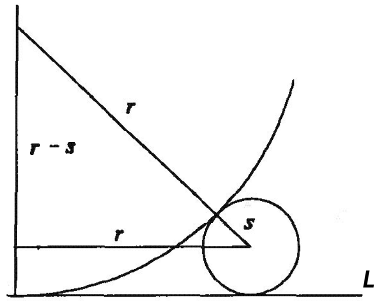
While it is easy to see that both spaces are T_3, the usual proofs showing that neither is a T_4-space use a cardinality or second category argument and are often omitted from elementary textbooks. In this note we present an alternate proof for the Niemytzki plane. The proof is simple, uses only the nested set theorem, and works just as well for the Sorgenfrey plane.

The Niemytzki plane can be envisioned by thinking of hot air balloons. The basic sets then become those balloons anchored at points of L and those balloons floating above L. We refer to rational balloons as those balloons B_q anchored at rational points q on L. Similarly, we refer to irrational balloons B_p anchored at irrational points p on L.

Since every subset of L is closed, the rationals Q and irrationals $L \setminus Q$ fonn two disjoint closed subsets of L. If X is normal, then there are disjoint open subsets U and V such that $U \supset Q$ and $V \supset L \setminus Q$. Thus, lor each $q \in Q$ we may find a corresponding rational balloon B_q with positive radius lying inside U and for each $p \in L \setminus Q$ we may similarly find an irrational balloon $B_p \subset V$ also with positive radius.

Now the rational and irrational balloons lying in U and V respectively cannot overlap. Hence, for example, given a rational point $q \in Q$ and its corresponding rational balloon B_q with radius r_q, all irrational balloons anchored in the shadow of B_q must have a radius $\leq r_q/4$.

This is easily seen by taking the worst possible scenario, i. e. where the smaller balloon is anchored at the edge of the shadow of the larger balloon. Then the distance between anchors is exactly the radius of the larger balloon, say r. Allowing the smaller balloon to have as large a radius as possible, say s, so that the two balloons touch, simple geometry shows



$$(r+s)^2 = r^2 + (r-s)^2 \quad \text{or} \quad s = r/4.$$

Now choose some rational balloon B_0 lying inside U. In B_0's shadow, on the right side, choose irrational $B_1 \subset V$. On the left side in B_1's shadow, choose rational $B_2 \subset U$. Continuing, we form a sequence $B_0, B_1, B_2, \cdots$ of rational and irrational balloons in U and V respectively such that each is anchored in the shadow of the previous balloon, and in such a way that the anchors alternate sides—that is, if B_n lies to the right of B_{n-1}'s anchor, then B_{n+1} lies to the left of B_n's anchor.

This alternation of left, right, ... ensures a nested sequence of intervals $I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$ where I_n is the interval between B_n's and B_{n+1}'s anchors. Denoting B_k's radius by r_k, we have $r_n \leq r_{n-1}/4$ and therefore $r_n \leq r_0/4^n$ for every $n \in N$. By the nested set theorem, some point x must lie in all the intervals I .

If x is an irrational point of L, then there is some irrational balloon B_x with positive radius r_x such that $B_x \subset V$. But B_x is anchored in the shadow of every rational balloon B_{2n}, $n \in N$ in the sequence $B_0, B_1, B_2, \cdots$ and hence

$$r_x \leq r_{2n}/4 \leq r_0/4^{2n+1}$$

for every $n \in N$. This says $r_x = 0$ which contradicts B_x having a positive radius. A similar contradiction arises if x is rational.

The author, a graduate student, thanks Professor *Transue for* the problem.

A New View of Goldbach's Conjecture

Professor **Simon Berkovich** (George Washington University) notes that the famous Goldbach Conjecture, made in 1742, that

> Every even integer greater than two is a sum of two primes.

can be stated in another form:

> **Every integer** greater than one is the average of **two primes.**

He writes:

> This has a physical interpretation. If we think of a wave spreading out from an integer, then, if **Goldbach's** Conjecture is true, it will simultaneously hit a lesser and greater prime number.
>
> To me, this makes the conjecture (which is certainly true) even more mysterious.

Chapter Reports, Letters, Etc.

Chapter Reports

Professor Gerald W. Young, reporting for the OHIO NU Chapter (University of Akron), says that the department of mathematical sciences, with assistance from the chapter, awarded eighteen one-year memberships in five organizations (AMS, SIAM, ASA, MAA, ACM) and fourteen scholarships.

The PENNSYLVANIA OMICRON Chapter (Moravian College) sponsored the eighth annual Moravian College Student Mathematics Conference in February 1994, with 62 participants from 15 colleges and universities. The keynote speaker was Dr. Diane L. Souvaine, of the National Science Foundation, whose topic was "Geometric computations and applications." There were twelve undergraduate student speakers as well.

The WISCONSIN DELTA Chapter (St. Norbert College) held its Northeastern Wisconsin Math Seminar Series, with several guest speakers, hosted the annual Pi Mu Epsilon Regional Undergraduate Math Conference, and, with SNC'c Math Club, held the annual SNC High School Math Meet.

Professor James R. Weaver reports that the FLORIDA KAPPA chapter (University of West Florida) was addressed by the president of the university at its induction ceremony. The chapter also has subsidized members to attend meetings and assisted in the annual northwest Florida Mathcounts contest.

Letter to the Editor

In the Spring, 1994 issue (vol. 9, no. 10) of the Pi Mu Epsilon Journal, a problem on page 646 asks "What is the largest integer that in base-16 is an ordinary English word?" The suggested answer, on page 661, of defacaded, is certainly not an ordinary English word, if indeed it is an English word at all. (I did not find "to facade" as a verb in any dictionary

at my disposal, which makes the longer construction highly implausible.) In any case, under similar rules of admissibility, I can top it. How do we describe a necklace made with ten glass spherules? Clearly, it is a decabeaded necklace! This converts from base-16 to the decimal value 956,884,233,709, if my hand-held calculator can be trusted. I should also point out that since decabeaded used only the first five letters of the alphabet, it could also be the representation of a number in base-15.

A more sweeping generalization of this problem is: for each of the $2^{26} - 1 = 67{,}108{,}863$ non-empty subsets of the 26 letters of the alphabet, what is the longest English word which can be formed using only letters from that subset? I particularly like the 8-letter word dedeeded (also a palindrome!) from the 2-letter subset {d, e}. There are only 325 subsets consisting of two letters, and unless one is a vowel and the other a consonant, the possibilities are very limited. This reduces us to 125 sets, where I have allowed y to be paired with either a vowel (as in *yoyo*) or a consonant (as in lyly).

Solomon W. Golomb
University *of* Southern California

Editorial note: This may be the last word on hexadecimal words. Since English is not an agglutinative language, the act of making a ten-bead necklace cannot be called a decabeaddeed, and describing the once-tenbeaded necklace as *dedecabeaded* is strained. In any event, the proper reaction to Professor Golomb's linguistic virtuosity is the longest base-2 word: oooooo!

A Conjecture

Xuming Chen (University of Alabama, Tuscaloosa) conjectures that any prime is a sum of a prime and two squares, as $5 = 3 + 1^2 + 1^2$, $7 = 2 + 1^2 + 2^2$, $11 = 3 + 2^2 + 2^2$, $\cdots$, $1997 = 1987 + 3^2 + 1^2$. Thisconjecture is very likely to be true and very likely to be unprovable, but it might be of interest to set a computer the task of determining in how many ways a prime p can be so represented. I have no idea at what rate the number of representations would grow as p increases.

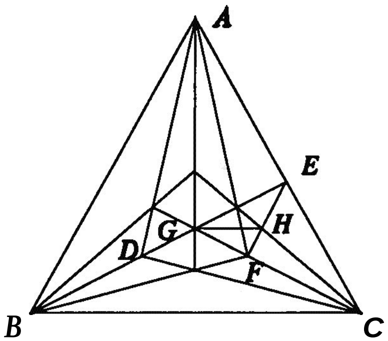
Laplace Transforms

Professor **James Chew** (North Carolina **A** & T State University) offers another proof that, if the **Laplace** transform of $f(t)$ is $F(s)$ then the transform of $\int_0^t f(\tau)\,d\tau$ is $F(s)/s$. Suppose that $f(t)$ has a formal power series, $\sum_0^\infty a_n t^n$. Then $\int_0^t f(\tau)\,d\tau = \sum_0^\infty a_n t^{n+1}/(n+1)$. Since $\mathfrak{L}\{t^k\} = k!/s^{k+1}$, we have

$$\mathfrak{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \sum_0^\infty a_n n!/s^{n+2} = (1/s)\sum_0^\infty a_n \mathfrak{L}\{t^n\} = F(s)/s.$$

A Triangle

In the last issue of the Journal (10 (1994-99) #1, **25**), **Andrew Cusumano** asserted that, in the figure on the right, where the angles at A, B, and C in the equilateral triangle ABC have been bisected twice, EF is perpendicular to DC and that GH is parallel to BC. **Paul Bruckman** (Edmonds, Washington) shows that this is correct:



We embed the figure in the complex plane, with points represented by complex numbers. Let

(1) $A = 1$, B =a), $C = \omega^2$, where a) = $e^{2\pi i/3}$.

Clearly, G = 0. Let **J** denote the intersection of the first quadrisectors (uppermost in the figure) of angles B and C (J lies on line AG). Since $1 + \omega + \omega^2 = 0$ and $E = (A + C)/2$, then

(2) $E = (1 + \omega^2)/2 = -\omega/2.$

Let 0 = $\pi/12$ and $c = (\sqrt{3} - 1)/2$. Note that $\angle DAE = 30$ and $\angle DEA = 60$, a right **angle.** Thus triangle **DEA** is an isosceles right triangle and $|DE| = |AE| = \sqrt{3}/2$. Since $|GE| = 1/2$, it follows that $|GD| = c$, and so D = ca). Similarly,

(3) $F = c\omega^2,\ J = c.$

Now

$$E - F = -\omega/2 - c\omega^2 = [\sqrt{3} - i(2\sqrt{3} - 3)]/4,$$

after simplification. Thus,

$$\arg(E - F) = 5\theta - (-\theta) = 6\theta = \pi/2,$$

which shows that EF extended is perpendicular to CD.

To determine H, we note that it is the intersection of EF with CJ. Thus real r and s exist, with $0 < r < 1$ and $0 < s < 1$, such that

$$H = (1 - r)E + rF = (1 - s)C + sJ.$$

We may substitute the expressions for C, E, F, and **J** in **(1)-(3),** substitute $(-1 + i\sqrt{3})/2$ and $(-1 - i\sqrt{3})/2$ for w and w^2 respectively, and equate real **and imaginary** parts. This yields a pair of equationsin r and s which yield $r = s = 1/\sqrt{3}$. Then substituting this in the expression for H yields $H = -ic$ (note that **J,** H, F, and D lie on the circle $|z| = c$). Since $H - G$ and $C - B$ are pure imaginary, we see that GH is parallel to BC.

Editorial note. While the problem could have been stated by Euclid, he could never have constructed a solution which involved complex numbers. Two questions arise: does a purely geometric proof exist, and, if one does, is it superior to one using is? The answer to the first surely is "yes", but the answer to the second is less clear. On the one hand, in the universe of mathematics we should be free to use whatever we need to solve a problem. On the other, many mathematicians have a feeling that there is something not quite right about using nineteenth-century mathematics to solve what could have been a second-century problem, and prefer to avoid such things. For example, Selberg and **Erdős** were acclaimed for proving the prime number theorem without using complex variables. Would **Bach** have written for electronic instruments, had he been able? Should his music be performed on them? Analogous questions, analogously difficult to answer. The art and science of mathematical esthetics is in its infancy.

Verse

Ode to e

Paul S. **Bruckman**
Edmonds, Washington

I dedicate this rhyme and rhythm
To sing the natural logarithm.
That is, its base, great Euler's e
Is lauded in this rhapsody.
In higher math and nature both
We learn that e expresses growth
Of special, exponential kind,
And e's the limit, you will find,
As n grows larger by the hour,
Of something raised to the nth power,
That something being none other, then,
Than n + 1 all over n.
Another well-known fact, my dearies,
Is given by the endless series
Where each term's the nth power of x
Divided (there's no need for checks)
By n!, with such sum
Converging to e^x, by gum,
Provided that we sum from naught;
The one who found this, we are taught,
Was Leibniz, and another Kraut,
Herr **Lindemann,** proved beyond doubt
That e's a transcendental number
(although his proof induces slumber).
For elegance, Euler's relation
Is unsurpassed; its declaration

Is this: that $e^{\pi i}$
+ 1 is 0—herein lie
Math's basic five constants (with zero),
Undreamt of by Euclid or Hero.
I could go on—it seems a crime—
But limits of both space and time
Restrict me and curtail my song:
Adieu e, you can do no wrong!
To readers of this little ditty
Who think it drivel—more's the pity!

Identities

In the last issue of the *Journal* (10 (1994-99) #1, 43-44), Kenneth Davenport gave the following identities.

$$1\cdot(1+2+\dots+n) = n(n+1)/2.$$

$$(1^3+2^3+\dots+n^3) = n^2(n+1)^2/2.$$

$$3\cdot(1^5+2^5+\dots+n^5) + 1\cdot(1^3+2^3+\dots n^3) = n^3(n+1)^3/2.$$

$$4\cdot(1^7+2^7+\dots+n^7) + 4\cdot(1^5+3^5+\dots+n^5) = n^4(n+1)^4/2.$$

$$5\cdot(1^9+2^9+\dots+n^9) + 10\cdot(1^7+2^7+\dots+n^7) + 1\cdot(1^5+2^5+\dots+n^5) = n^5(n+1)^5/2.$$

$$6\cdot(1^{11}+2^{11}+\dots+n^{11}) + 20\cdot(1^9+2^9+\dots+n^9) + 6\cdot(1^7+2^7+\dots+n^7) = n^6(n+1)^6/2.$$

They are explained by **Odoardo Brugia** and **Piero** Filipponi:

We were not aware of the cute identities reported by K. B. Davenport in this journal (10 (1994) #1, 43-44). They can be proved as follows.

For r, n, and k natural numbers, let us define $f(r)$ to be the greatest integer not exceeding $(r-1)/2$, $S_n(k) = \sum_{j=1}^{f(r)} j^k$, and

$$X(n,r) = \sum_{i=0}^{f(r)} \binom{r}{2i+1} S_n(2r-2i-1).$$

The proof that $X(n,r) = n^r(n+1)^r/2$ follows:

$$X(n, r) = \sum_{j=1}^{n} \sum_{i=1}^{f(r)} \binom{r}{2i+1} j^{2r-2i-1}$$

$$= \frac{1}{2}\left[\sum_{j=1}^{n} j^r(j+1)^r - \sum_{j=1}^{n} j^r(j-1)^r\right].$$

Put $j - 1 = h$ in the second summation of the above identity, thus getting

$$X(n, r) = \frac{1}{2}\left[n^r(n+1)^r + \sum_{j=1}^{n-1} j^r(j+1)^r - \sum_{h=1}^{n-1} h^r(h+1)^r\right]$$

$$= n^r(n+1)^r.$$

Odoardo *Brugia* and Piero *Filipponi*
Fondazione *Ugo* Bordoni
Via B. *Castiglione,* 59
1-00142 Rome, Italy

(Xuming Chen, of the University of Alabama, Tuscaloosa, provided a similar derivation.)

Trisecting the Paper

Augustus De Morgan **(1806-1871),** whose name is attached to De Morgan's Laws, after discussing a trisection of the angle wrote

> There is one trisection which is of more importance than that of the angle. It is easy to get half the paper on which you write for margin; or a quarter; but very troublesome to get a third. Show us how, easily and certainly, to fold the paper into three, and you will be a real benefactor to society.

(A Budget of Paradoxes, London, 1872; reprinted by Dover, New York, 1954; vol. 2, p. 15.)

Though we no longer feel the nineteenth-century urge to fold paper into vertical thirds, should the need again arise John **Kasbohm** (Oak Park, Illinois) has a solution. He writes:

"With a sheet W wide and L long, $L \geq 4$ W13, you can twice **bi-**fold the long right edge, making just little pinches to show where folds across the paper would be, because we wouldn't want deep creases all over the place. (See Figure 1.) This makes little dents at $L/2$ and $3L/4$ on the edge. Now lift the lower right comer up and lay it across the face of the sheet, so that the dent at $3L/4$ just touches the left hand edge and the top and right hand edges still meet at the upper right comer. (See Figure 2.) Again this time you don't have to make a full sharp crease, but just a little pinch at that comer. The L12 dent is now two-thirds of the way across the sheet and you can fold the sheet vertically through that point. The finished sheet with the one-third margin is still in pretty good shape, especially if you have some origami practice. If L is too long you can use only an upper part of it.

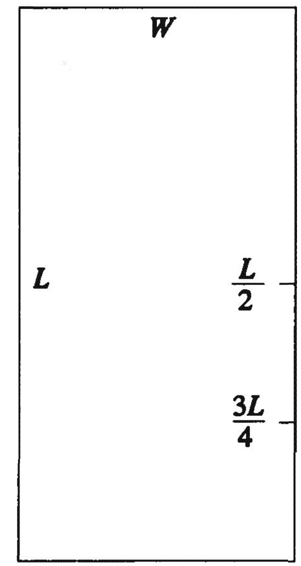


Figure 1

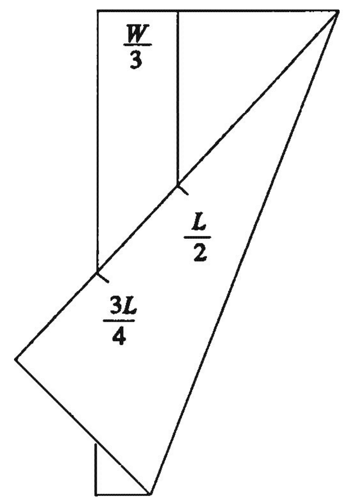


Figure 2

'Of course the easiest way is to measure and divide, but I suppose that is not what De Morgan meant."

A difficulty is that 8½ x 11 paper is not quite long enough (by a third of an inch) for the method to work. One way around this is to make the sheet slightly longer, with a temporary paper-clipped extension. Another is to crease it in half vertically and apply the method to that half, thus getting one-sixth of the width which can quickly be doubled to get one-third. Though neither may be as easy as De Morgan would have wanted, I think that he would have been pleased with the solution.

Editorial Comment

(In case there is any doubt, the following comments do not necessarily reflect the opinions of any of the officers of Pi Mu Epsilon or, for that matter, any of its members. They contain the personal views of the editor and do not need to be taken seriously or, for that matter, read at all.)

A not-bad definition of mathematics is that it is the art and science of reasoning about and with quantities. The definition is not complete because there are branches of mathematics where there are no quantities—mathematical logic, for example—but whatever your definition is, it will have some mention of reason in it. Mathematics is not broad enough to include ***all*** of reason (lawyers reason too, sometimes) but reasoning is what it does. In fact, it was mathematics that taught the human race to reason, but that's another story.

The thing about reason is that it's so ... reasonable. "Socrates is a human. All humans are mortal. Therefore Socrates is mortal." Of course. It ***follows.*** It's ***logical.*** It's easy to understand. All you have to do is listen to reason, right?

Wrong. Teachers of mathematics spend thousands of hours trying to get students of mathematics to listen to reason, and they do not always succeed. Sometimes they do, of course, but many times they do not, as is shown by the number of appearances of $\sqrt{(a^2 + b^2)} = a + b$ on tests (even on calculus tests), along with $\sin 3x = 3 \sin x$ (yes, that does make life easier, but it's not reasonable—sines don't get as large as 3), and even the horrible

$$\frac{2x^2 + 3y^2}{x + y} = 2x + 3y$$

(gotten by applying a corollary of the Law of Universal Cancellation: whenever a symbol appears in the numerator and the denominator of a fraction, it may be cancelled). It's usually possible to convince students of mathematics that those equalities are not reasonable, but sometimes it takes

effort.

However, there is a class of people who deal in mathematics but who will not listen to reason no matter how much effort is spent, namely mathematical cranks. Mathematical cranks are people who think that they have done the impossible, like trisecting angles with straightedge and compass alone, or that they have done something that they have not, like prove Fermat's Last Theorem. They try to convince mathematicians, using reason, that they have done what they claim to have done, but reason is not of much use in convincing them that what they say is not correct.

Recently a crank wrote me,

> If EVERYBODY loves mathematics, all the bad things on this planet will pass into history. I can edit a mathematics magazine because I know EVERYTHING about mathematics.

Shall I reason with him? He went on,

> It is a pity that mathematics is unpopular as it is the worst subject in every school because every time a mathematics teacher gives his class mathematics homework, none of the students do their mathematics homework which is never encouraged by their parents and all mathematics teachers approve of students not doing their mathematics homework thus making mathematics a futile subject.

A possible reason for that will be given in the last paragraph.

You might think that an angle trisector would be convinced of his error if he was shown that his construction is not accurate. Not so. For example, there was the angle trisector who was shown that, using trigonometry, the angle he got when his construction was applied to a 60° angle was something like 19° 56' 42". He saw that, but he did not therefore conclude that his construction was wrong. The choice was between his trisection and trigonometry, so he concluded that ***trigonometry*** was wrong, and that all the trigonometry textbooks would have to be rewritten. How are you going to reason with him?

Or with the circle-squarer (circle-squarers construct with straightedge and compass alone squares with the same areas as circles, something as impossible as trisecting angles) who said, when asked how he knew that his construction was correct,

> The author was supernaturally taught the exact measure of the

> circle. All knowledge is revealed directly or indirectly, and the truths hereby presented are direct revelations and are due in confirmation of scriptural promises.

Reason is helpless against revelation.

Then there was the crank who considered the four-color theorem (that no map needs more than four colors if adjacent countries have different colors) and went it two better by claiming to have proved that ***two*** colors were enough. Reason was not enough to prove to him that he was not correct.

Cranks can get around anything. An angle trisector took care of the fact that the trisection was impossible by saying

> The author resents the negative implications since, if everyone were to accept statements as valid, there would be very little progress.

Of course, no progress in the trisection is all we can get. The same trisector had been told that Wantzel had proved that the trisection was impossible in 1837, and his retort to that was

> A mathematical proof is merely an established approximation, indicating a limitation of errors to a minimum applicable to each on hand to be solved, and from a point, or point, of reference as they appear.

I did not make that up. I ***could not*** make that up. Absolute non-sense is very hard to create. The trisector's point, whatever his words, was that the proof did not count. His trisection was right, and any reason to the contrary could be brushed aside. He would not listen to reason.

One circle-squarer who said that π was $25/8 = 3.125$ had a voluminous correspondence with a mathematician. The mathematician brought up the proof that π was irrational and that Archimedes had shown that π was greater than $3\ 10/71$ ($= 3.1408... > 3.125$) more than two thousand years ago, but they meant nothing to the circle-squarer. Eventually he was backed into a corner and had to admit that the reason that π was $25/8$ was ***because he said so.*** He ***assumed*** that $\pi = 25/8$:

> If you dislike the term datum, then, by hypothesis, let 8 circumferences exactly equal 25 diameters.

He could do this because, he said

> I think you will not dare to dispute my right to this hypothesis when I can prove by means of it that every other value of π will lead to the grossest absurdities.

Using exactly the same method, I can prove that $3^2 = 10$. Here I go. By hypothesis, let $3^2 = 10$. Then any other assumption, such as $3^2 = 9$, leads to the grossest absurdities, in this case that $9 = 10$.

You cannot reason with cranks. They will not follow the laws of logic, they will not listen to reason. Even students of mathematics have difficulty, sometimes, seeing things that are absolutely clear to their instructors and follow mathematically from things that have gone before. Students of mathematics sometimes do not even do their homework. Why is this?

The reason is clear. Mathematics is unnatural; mathematics is a perversion. Of course it is. What are you doing when you are doing mathematics? You are not doing something natural. You are sitting, hardly moving, your only bodily activity outside of some necessary breaths and heartbeats a few weak and flickering electric currents passing along the neurons of your brain. Occasionally you may move a hand to write a bit, but you then lapse again into motionlessness, eyes unfocussed, with only the electrons in your brain moving. You cannot do mathematics for very long, can you? The reason you cannot is that mathematics is not what your body, with its dexterous fingers and powerful legs, was made for. Our legs were made to run after our next meal, and our fingers to pick it apart after we catch it. Sitting and thinking is an unnatural activity, and we have not yet gotten used to it. What was it that last made you think, "Wow! I had a really good time!"? Wasn't it after some activity that was entirely mindless? Of course it was. Evolution has not yet progressed far enough for reasoning to be painless, much less enjoyable. It will take another hundred thousand years, or maybe a million. At present, the race finds it hard. Thinking, especially thinking about mathematics, is unnatural. However, mathematics is not a bad perversion. Keep listening to reason: the more we try to hear, the easier it will become.

Pentagonal Number Identities

Kenneth Davenport (Pittsburgh, Pennsylvania) points out a relationship between the generalized pentagonal numbers,

1, 2, 5, 7, 12, 15, 22, 26, ... ,

$(3n \pm 1)n/2$, $n = 1, 2, \ldots$ and partial sums of cubes:

$$\frac{(2-1)^2}{2} = \frac{1^3}{1^3},$$

$$\frac{(5-2)^2}{8} = \frac{1^3+2^3}{2^3},$$

$$\frac{(7-5)^2}{3} = \frac{1^3+2^3+3^3}{3^3},$$

$$\frac{(12-7)^2}{16} = \frac{1^3+2^3+3^3+4^3}{4^3},$$

$$\frac{(15-12)^2}{5} = \frac{1^3+2^3+3^3+4^3+5^3}{5^3}.$$

The denominators of the odd-ranked fractions are the odd numbers 1, 3, 5, ... and the denominators of the even-ranked fractions are successive multiples of 8.

While it is a simple exercise in algebra to verify that the identities hold in general, they may be useful as an example, or as a curious connection between two sequences that on the face of it have nothing in common other than their geometric origin. And pentagons are two-dimensional, while cubes arc three-dimensional.

Mathacrostics

Solution to Mathacrostic 39, by Robert Forsberg (Fall, 1994).

Words:

A. Tchebyshev
B. icosidodecahedron
C. Mithridates
D. ordinary point
E. transfinite set
F. Hippasus of Metapontum
G. Ypsilanti
H. Fermat's last theorem
I. effective pitch
J. roots for
K. raffia
L. independent events
M. spinning frame
N. trident
O. high pressure physics
P. Ettingshausen effect
Q. mestizo
R. Ibn Abd Rabbihi
S. laetrile
T. kennings
U. Yablonoi
V. Windows
W. a penny postage stamp
Y. Yangchuanchan

Author and title: Timothy Ferris, The *Milky Way.*

Quotation: Newton's surviving drafts of the *Principia* support Thomas Edison's dictum that genius is one percent inspiration and ninety-nine percent perspiration. Like Beethoven's drafts of the opening bars of the Fifth Symphony, they are characterized less by sudden flashes of insight than by a constant, indefatigable hammering away at immediate, specific problems.

Solvers: THOMAS BANCHOFF, Brown University, JEANETTE BICKLEY, St. Louis Community College, PAUL S. BRUCKMAN, Highwood, Illinois, CHARLES R. DIMMINIE, St. Bonaventure University, VICTOR G. FESER, University of Mary, ROBERT C. GEBHARDT, Hopatcong, New Jersey, META HARRSEN, Durham, North Carolina, HENRY LIEBERMAN, Waban, Massachusetts, CHARLOTTE MAINES, Rochester, New York, DON PFAFF, University of Nevada—Reno, NAOMI SHAPIRO, Piscataway, New Jersey, STEPHANIE SLOYAN, Georgian Court College, and JOSEPH S. TESTEN, Mobile, Alabama. Late solution to #37 by VICTOR G. FESER, University of Mary.

Mathacrostic 40, constructed by ROBERT FORSBERG, follows on the next three pages. To be listed as a solver, send your solution to the editor.

K 1	F 2	H 3	N 4		F 5	K 6	T 7	N 8	H 9	W 10	C 11	U 12	R 13	V 14
P 15	D 16	G 17	J 18		R 19	T 20	H 21	N 22	U 23	G 24		K 25	T 26	M 27
A 28	V 29	I 30	X 31		H 32	W 33		C 34	B 35	O 36	H 37	T 38	K 39	N 40
R 41	P 42	V 43		K 44	E 45	Q 46	L 47		A 48	K 49	G 50		A 51	H 52
L 53	G 54	T 55	Y 56	N 57		U 58	G 59		T 60	M 61	P 62	Y 63	V 64	R 65
G 66	C 67	N 68		O 69	A 70		K 71	N 72	W 73	J 74	C 75		—	
U 76	T 77	I 78		E 79	V 80	W 81		L 82	R 83	M 84	W 85	O 86	T 87	
R 88	P 89	Q 90	B 91		—		C 92	I 93	T 94	H 95	D 96		U 97	M 98
	A 99	H 100	T 101	O 102	N 103	R 104	Q 105	L 106	I 107		J 108	R 109		I 110
Y 111	S 112	U 113	F 114		B 115	F 116	W 117	J 118		D 119	L 120	N 121		N 122
T 123	E 124	I 125	R 126	H 127	V 128	C 129	K 130	B 131	S 132		Q 133	C 134		N 135
H 136	G 137	E 138	Q 139		G 140	I 141	C 142		G 143	C 144	N 145	W 146	T 147	P 148
Y 149	K 150	M 151		P 152	Y 153		J 154	H 155	A 156	W 157	F 158	N 159	C 160	R 161
T 162		F 163	E 164		T 165	X 166		C 167	O 168	V 169	J 170	T 171		A 172
P 173	Y 174	M 175		T 176	I 177	N 178	U 179	D 180	S 181	G 182	H 183	F 184		V 185
	G 186	N 187	T 188	C 189	X 190	P 191	V 192	K 193		C 194	N 195	E 196	T 197	R 198
O 199	V 200	J 201	K 202		I 203	M 204	T 205	S 206	K 207		K 208	S 209	N 210	
C 211	G 212	N 213	I 214		S 215	D 216	T 217	U 218	L 219	F 220	P 221	H 222		S 223
V 224	D 225		H 226	L 227	B 228		G 229	T 230	V 231	N 232	K 233	X 234	R 235	E 236
H 237	W 238	P 239		R 240	S 241		P 242	E 243	R 244	J 245	O 246		I 247	E 248
K 249	W 250	Q 251	R 252	T 253	H 254	F 255								

A. A red-orange coloring material obtained from the seeds of ***Bixa orellana***
156 28 99 70 48 172 51

B. The outer of the two layers forming the wall of spores such as pollen
228 35 115 131 91

C. Describing a curve at a point where a tangent may be drawn
167 160 134 92 75 144 189 142 194 129
11 211 67 34

D. To thrust out repeatedly
96 16 225 119 180 216

E. Literally "Tracker"; an animal that hunts out crocodile eggs
138 196 45 164 248 124 236 243 79

F. An element
116 114 158 184 255 220 163 2 5

G. He discovered in 1890 a curve that could intersect every point in a plane (2 wds)
24 54 66 59 50 186 182 212 143 137
229 17 140

H. He produced the first **mathematical** works produced by an Englishman in England (3 wds)
226 136 183 21 155 95 100 3 127 32
254 37 52 237 222 9

I. Great figure in the resistance of Wales to the English, d. 1170 (2 wds)
93 141 30 214 203 107 125 177 78
110

J. A small **ceramic/metal** vacuum tube
170 201 74 245 18 108 118 154

K. A solution used as an **acid/** base indicator
1 208 202 249 130 25 71 49 44 207
6 193 150 39 233

L. Hungarian writer, 1760-1820, ***Hunnyas***
227 219 106 53 82 47 120

M. Author of the earliest sources of the Hexateuch
175 27 204 84 61 151 98

N. A method of finding prime numbers (3 wds)	210 22 121 178 213 187 122 4 72 232 135 145 57 68 8 195 159 40 103
O. Typical, accurately identified specimen of a species, but not a basis for a published description	199 246 168 69 86 36 102
P. Swiss mathematician, 1829-1900	42 173 89 15 148 239 152 191 62 221 242
Q. An asteroid of about 220km diameter	46 90 139 133 251 105
R. Equivalent to about 11.8 inches (hyph)	83 13 244 19 88 198 252 161 240 104 235 65 109 41 126
S. A word synthesized to mean: someone showing contempt for legalities	132 206 181 241 223 112 209 215
T. Author of The Dynamics of a Particle, 1865 (3 wds)	101 197 20 38 217 26 165 253 94 7 188 205 176 171 147 60 55 230 162 87 123 77
U. Out of use; retired from service	97 23 76 218 12 58 113 179
V. The entropy of the input to a communication channel when the output is known	43 200 192 169 231 224 14 185 128 64 80 29
W. An intracellular effector organelle in coelenterates	238 250 10 85 81 73 146 33 157 117
X. A unit of firewood volume	234 166 190 31
Y. Spread out in a definite form	174 153 63 56 149 111

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: **dodge@eauss.umemat.maine.edu.** *Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1995.*

Correction

844. [Fall 1994] *Proposed by Bill Correll, Jr., student, Denison University, Granville, Ohio.*

If F_n denotes the nth Fibonacci number ($F_1 = F_2 = 1$ and $F_{k+2} = F_k + F_{k+1}$ for k a positive integer), evaluate

$$\sum_{n=1}^{\infty} \binom{n}{k} F_n / 2^{n+k}.$$

(The index of the summation was incorrect.)

Problems for Solution

849. *Proposed by L. A. Bohr, Great Works, Maine.*

Solve this base 8 addition alphametic: *THIS* + *IS* = *WORK*.

***850.** *Proposed by Bill Correll, Jr., Denison University, Granville, Ohio.*

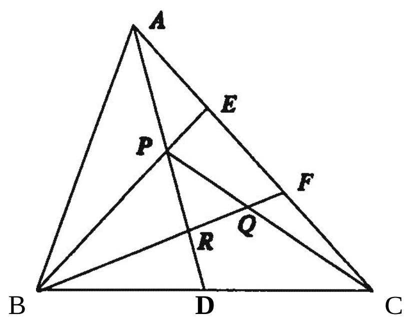
Clearly the following integral evaluation is incorrect. Find the flaw. By letting $u = -x$ we get that

$$I = \int \ln(e^x + 1)\,dx = -\int \ln(e^{-u} + 1)\,du = -\int \ln \frac{e^u + 1}{e^u}\,du$$
$$= -\int \ln(e^u + 1)\,du + \int \ln(e^u)\,du = -I + u^2/2 + C,$$

so that $I = x^2/4 + C'$. (See Problem 828.)

851. *Proposed by Bill Correll, Jr., Denison University, Granville, Ohio.*

In triangle ABC let Cevian AD bisect side BC and let Cevians BE and BF trisect side CA. Let AD intersect BE at P and BF at R, and let CP meet BF at Q. See the figure. If the area of triangle ABC is **1,** find the area of triangle PQR.



852. *Proposed by Rex H. Wu, Brooklyn, New York.*

Let E be a point inside square $ABCD$ with $BE = x$, $DE = y$, and $CE = z$. If $x^2 + y^2 = 2z^2$, find the area of $ABCD$ in terms of x, y, and z.

853. *Proposed by Charles Ashbacher, Cedar Rapids, Iowa.*

This problem was submitted by Vietnam for the **1990** International Mathematical Olympiad and has appeared in booklets overseas. If real numbers x a y a $z > 0$, then prove that

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

854. *Proposed by Jayanthi Ganapathy, University of Wisconsin at Oshkosh, Oshkosh, Wisconsin.*

Let a and b be two nonzero real numbers such that

$$a^3(3a^2 - 5ab + 3b^2) = b^3(5a^2 - 3ab + 5b^2).$$

Find the values of the expressions $(a^2 + b^2)/a^2$ and $(a^2 - b^2)/ab$.

855. *Proposed by Florentin Smarandache, Phoenix, Arizona.*

Prove that a square matrix of integers, having in each row and in each column a unique element not divisible by a given prime p, is nonsingular.

856. *Proposed by Paul S. Bruckman, Highwood, Illinois.*

Starting with a regular n-gon whose side is of unit length, snip off congruent isosceles triangles from each of its vertices, resulting in a regular $2n$-gon. Repeat the process indefinitely. Find the ratio of the area of the limiting circle to that of the original n-gon.

857. *Proposed by Andrew Cusumano, Great Neck, New York.*

Find all prime numbers whose reciprocals have repetends of exactly seven decimal places.

858. *Proposed by David Iny, Baltimore, Maryland.*

It is known that the rational numbers in the interval $[0, 1]$ can be enumerated. Let $\{r_k\}_{k=1}^{\infty}$ be such an enumeration and pick ε such that $0 < \varepsilon < 1$. Take an interval I_k of length $\varepsilon 2^{-k}$ centered on each r_k. Then the sum of all these interval lengths $\Sigma_{k=1}^{\infty} r_k = \varepsilon < 1$. Show how to find a real number in $[0, 1]$ and not contained in any of the intervals I_k.

859. *Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.*

Sum in closed form the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n+1}\binom{-1/2}{n}^2, \quad \text{where} \quad \binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.$$

860. Proposed by Richard I. Hess, Rancho Palos Verdes, California.

This problem originally appeared in a column by the Japanese problems columnist Nob **Yoshigahara.** Find the minimal positive integer n so that $2n + 1$ circles of unit diameter can be packed inside a 2 by n rectangle.

861. Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

Evaluate in **closed form** the sum

$$S(n,k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{2j}{k}.$$

Solutions

823. [Spring **1994]** *Proposed by Alan Wayne, Holiday, Florida.*

Find all solutions to the multiplication alphametic

$$(I)(DINE) = ENID.$$

That is, find the **form(s)** taken by all solutions in all bases.

I. ***Solution** by Paul S. **Bruckman, Highwood,** Illinois.*

Let b denote the base of the alphametic. We show that if $r \geq 3$, then a solution is provided by taking

$$b = r^2,\ D = r,\ I = r - 1,\ N = r^2 - r - 1, \text{ and } E = r^2 - r.$$

Then $E \cdot I = r^3 - 2r^2 + r = r^2(r - 2) + r$, so $E \cdot I \equiv D \pmod b$, with a carry to the next column of $r - 1$. Next we have that

$$N \cdot I + r - 2 = r^3 - 2r^2 + 1 + r - 2 = r^2(r - 2) + r - 1 \equiv I \pmod b)$$

with a carry of $r - 2$ again. Next $I^2 + r - 2 = r^2 - r - 1 = N$ with no carry. Finally, $D \cdot I = r^2 - r = E$. Therefore, we have *that* $(I)(DINE) = ENID$.

II. *Comment by Victor G. Fever, University of Mary, Bismarck, North Dakota.*

This problem is much trickier than it looks. I set up a little computer program to check successive bases. Solutions started to appear: bases 9, 16, 25, 36. Ah hah! There seems to be a pattern here. Right. The next solution was for base 39, and then base 47. I checked to base 200 and found the additional solutions for bases other than square numbers in the table that follows. There are fragments of patterns, but that is all I can find. It is exasperating to find, for example, that these five bases include two primes, one odd composite, one power of 2, and one composite with both odd and even factors.

base	D	I	N	E
39	1	16	37	23
47	4	9	43	37
109	1	45	107	63
128	16	5	26	80
200	5	11	128	55

III. *Comment by Richard I. Hess, Rancho Palos Verdes, California.*

If the base is b, then we must have that

$$I(Db^3 + Ib^2 + Nb + E) = Eb^3 + Nb^2 + Ib + D,$$

and hence

$$(E - ID)b^3 = (I^2 - N)b^2 + I(N - 1)b + IE - D$$

with $0 \leq D, I, N, E \leq b - 1, I > 1$, and $D \cdot E \neq 0$. If we have that $E = I \cdot D$, then

$$(N - I^2)b^2 = I(N - 1)b + D(I^2 - 1).$$

If also $N = I^2$, then $(Ib + D)(I^2 - 1) = 0$, which is impossible. Hence we have $N > I^2$, so $I < \sqrt{b}$. Further analysis might produce some results, but it appears tedious.

Also solved by VICTOR G. FESER, ***University** of Mary, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* **REX** H. **WU,** *Brooklyn, NY, and the* PROPOSER.

824. [Spring 1994] Proposed by Joel L. Brewer, *Palo* Alto, California.
Prove that there are no real integral solutions to the set of equations

$$(x^3 + 6x^2 - 159)y = 160,$$
$$(y^3 + 6y^2 - 159)z = 160,$$
$$(z^3 + 6z^2 - 159)x = 160.$$

You may not assume that a putative solution would possess any symmetry.

I. Composite of solutions submitted independently by Richard I. *Hess, Rancho* Palos Verdes, *California,* and Rex H. Wu, Brooklyn, New York.

We see that x, y, and z are factors of 160, as are also $x^3 + 6x^2 - 159$, $y^3 + 6y^2 - 159$, and $z^3 + 6z^2 - 159$. Now $|x^3 + 6x^2 - 159| \le 160$ for x $| \le 6$. Furthermore, $x^3 + 6x2 - 159$ is a factor of 160 only for x = 4, in which case $x^3 + 6x^2 - 159 = 1$. Similarly, $y = z = 4$. But then the stated products are each 4, not 160. Hence there is no solution.

II. Solution by Paul S. Bruckman, Highwood, Illinois.

Let $P(x) = x^3 + 6x^2 - 159$. By checking x = 0, 1, 2, 3, 4, we find that $P(x)$ ● 0 (mod 5) for all x. In any putative solution, it must be the case that the product of integers $P(x)$ and y is 160, so then y as 0 (mod 5). None of P(−5), $P(0)$, and P(5) is a divisor of 160, and if $|x| \ge 10$, $|P(x)| > 160$. Thus $P(x)$ cannot be a factor of 160 for any permissible x, and any putative solution is impossible.

III. Solution by *Kandasamy Muthuvel,* University of Wisconsin at Oshkosh, *Oshkosh,* Wisconsin.

From the third given equation, x divides 160. Then, from the first equation, we must have x divides $159y$. Since x and 159 are relatively prime, then x divides y. Similarly, y divides z and z divides x, so then x = y. Therefore,

$$(x^3 + 6x^2 - 159)x = 160.$$

Since x and 159 are relatively prime, then x and $x^3 + 6x^2 - 159$ are relatively prime. Thus the possible values for x are ±1, ±5, and ±32, but it is readily checked that none of them does yield a solution.

IV. Comment *by* the Proposer.

The following generalization is readily proved by the same methods. If

$l \ge 3$, m a 3, and n a 3 are integers, then there are no real integral solutions to the set of equations

$$(x^l + 6x^2 - 159)y = 160,$$
$$(y^m + 6y^2 - 159)z = 160,$$
$$(z^n + 6z^2 - 159)x = 160.$$

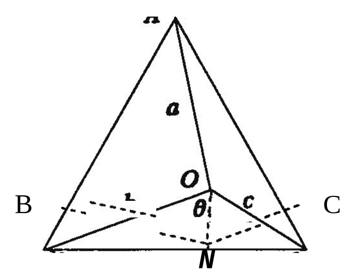
Also solved by SEUNG-JIN BANG, *Ajou* University, Suwon, Korea, BILL CORRELL, JR., *Denison* University, Granville, OH, VICTOR G. FESER, University of Mary, Bismarck, ND, STEPHEN I. GENDLER, Clarion University of Pennsylvania, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, *SUNY* College at Oneonta, DAVID S. SHOBE, New Haven, *CT,* and the PROPOSER.

825. [Spring 1994] Proposed by Leon *Bankoff,* Los Angeles, California:

Let O be a point inside the equilateral triangle ABC whose side is of length s. Let OA, OB, OC have lengths a, b, c respectively. Given the lengths a, b, c, find length s.

I. Solution by Rex H. Wu, Brooklyn, New York.

Let triangle ABC be oriented counterclockwise and rotate triangle AOC 60° about point C to triangle BNC as shown in the figure. Then AO = BN = a and $OC = NC = ON = c$ so $\angle NOC$ = 60°. Let $\theta = \angle BON$. Applying the law of cosines to triangles BON and BOC, we get



$$\cos 9 = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \cos(\theta + 60') = \frac{b^2 + c^2 - s^2}{2bc}.$$

From the left equation we get that

$$\sin 9 = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2bc}.$$

Now substitute these values into the equation

$$\cos(9 + 60°) = \cos\theta\cos 60° - \sin 9 \sin 60°$$

and then solve for s to get that

$$s = \sqrt{\frac{a^2+b^2+c^2}{2} + \frac{\sqrt{3}}{2}\sqrt{(2bc)^2 - (b^2+c^2-a^2)^2}}$$

Note that, although it does not appear so, the quantity inside the inner radical is *symmetric in a, b, and c. That is,*

$$(2bc)^2 - (b^2+c^2-a^2)^2 = (2ca)^2 - (c^2+a^2-b^2)^2$$
$$= (2ab)^2 - (a^2+b^2-c^2)^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - b^4 - c^4 - a^4.$$

The symmetric statement of the problem implies a symmetric solution. Ed.

II. *Comment by Seung-Jin Bang, Ajou University, Suwon, Korea.*

This problem is not new. See Problem 24.9, proposed by Russell Euler, in *Mathematical Spectrum* 1991192, vol. *24,* no. 3, p. 90 and 1992193, vol. 25, no. 1, p. 28.

III. *Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

This problem with solution appeared previously as number 39, *Crux Mathematicorum,* 1976, p. 7. An extension to a regular tetrahedron appeared as number 1087, *ibid,* 1987, p. 120. My solution there generalized the problem to show that for positive numbers a_0, a_1, ... , a_n, there exists a regular simplex S: $A_0A_1 \cdots A_n$ and a point P in its space such that $PA_i = a_i$, $i = 0, 1, \ldots, n$, if and only if

$$I = \{\Sigma a_i^2\}^2 - n\Sigma a_i^4 \ge 0.$$

The side length s of S is then given by

$$ns^2 = \sum a_i^2 \pm \sqrt{(n+1)I}$$

where the $\pm$ sign is chosen according to whether the point P is in the interior or exterior of S.

Also solved by MIGUEL AMENGUAL COVAS, *Cala Figuera, Mallorca, Spain,* PAUL S. BRUCKMAN, *Highwood, IL,* XUMING CHEN, *University of Alabama, Tuscaloosa,* BILL CORRELL, JR., *Denison University, Granville, OH,* ROBERT DOWNES, *Mountain Lakes High School, NJ, RICHARD* I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban,* MA, DAVID E. MANES, *SUNY College at Oneonta,* SAMMY YU *and* JIMMY YU, *University of South Dakota, Vermillion, and the* PROPOSER.

Amengual Covas found the problem for (a, b, c) = (11, 14, 19) in Gaceta Matematica, Supplement, IV, 1952, p. 57.

826. [Spring 1994] *Proposed by M. A. Kahn, Lucknow, India.*

Prove or disprove that the product

$$P = \prod_{k=1}^{n}(1 + k^2)$$

is a perfect square only for $n = 3$ and for no other positive integer.

Partial solution adapted from that submitted by Paul S. Bruckman, Highwood, Illinois.

We write P_n in place of P for the stated product. Let p be any prime $\equiv 1 \pmod 4$. Then there exist exactly two distinct integers n, $2 \le n \le p - 2$, such that $n^2 + 1 \equiv 0 \pmod p$ and $n^2 + 1 \not\equiv 0 \pmod{p^2}$. Let these values be denoted by a_p and b_p. Note that $b_p = p - a_p$ and take, without loss of generality,

$$2 \le a_p \le (½)(p-1) < (½)(p+1) \le b_p \le p - 2.$$

Now we see that $p \mid P_n$ if and only if $n \ge a$ and that $p^2 \mid P_n$ if and only if $n \ge b_p$. Hence P_n cannot be a square number when

$$n \in I_p = \{n \in Z: a_p \le n < b_n\}.$$

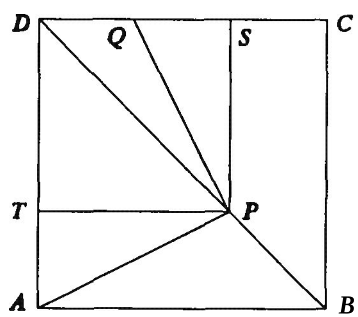
Now $P_1 = 2$ and $P_2 = 10$ are not squares, but $P_3 = 100 = 10^2$. We would like to show that the union S of all such sets I contains all integers greater than 3. Since there are infinitely many primes $p \equiv 1 \pmod 4$ and since each $b_p \ge p/2$, then S has no upper bound. Hence, all we must show is that there are no gaps between members of S. For any prime $p = (2r)^2 + 1$, such as 5, 17, 37, and 101, we have $a = 2r$ and $b_p = p - 2r$, making I_p a large interval. There would be no such gap if one could always find another such prime $q = (2t)^2 + 1$ with $2t \le p - 2r$. Unfortunately, according to Rosen, Elementary Number *Theory*, 2nd ed., p. 65, it is not even known whether there are infinitely many primes of the form $x^2 + 1$. For other primes of the form $4n + 1$, there appears to be no rule to determine how far apart a_p and b_p are. All we know for sure is that I_p always contains $(p - 1)/2$. Most intervals I_p will contain more integers, but it appears we cannot guarantee that all the gaps will be filled. The table below shows that the theorem appears to be true. Note that the number of integers in the set I_p is equal to $b_p - a_p$.

p	a_p	b_p	$b_p - a_p$	p	a_p	b_p	$b_p - a_p$
5	2	3	1	53	23	30	7
13	5	8	3	61	11	50	39
17	4	13	9	73	27	46	19
29	12	17	5	89	34	55	21
37	6	31	25	97	22	75	53
41	9	32	23	101	10	91	81

Editorial comment. *It* appears that the product P is a perfect square only for $n = 3$. Our proof is not complete, however, so the problem is open and *further* comments and, if possible, a full solution are invited.

827. [Spring **1994**] Proposed by Stanley Rabinowitz, *MathPro* Press, Westford, Massachusetts.

Let P be a point on diagonal BD of square ABCD and let Q be a point on side CD such that APQ is a right angle. Prove that AP = PQ.



I. Solution by Francine *Bankoff*, Beverly Hills, California.

Let S and T be the feet of perpendiculars dropped from P upon DC and DA respectively. Then triangles PSQ and PTA are similar since their corresponding sides are perpendicular. Since PS = PT because PSDT is a square, the triangles are also congruent, with PA = PQ.

II. Solution by Francine *Bankoff*, Beverly Hills, California.

Since QDA and APQ are both right angles, the points D, A, P, and Q lie on a circle whose diameter is AQ. It follows that chords QP and AP are equal, each measured by the equal **45°** angles QDP and PDA.

III. Comment by Leon *Bankoff*, Beverly Hills, California.

Although the diagram suggests that the point P be selected below the midpoint of DB, the stated proposal does not exclude a point above that midpoint. In that case the point Q falls on an extension of CD and solutions I and **II** remain valid with hardly any modification.

Also solved by MIGUEL AMENGUAL COVAS, Cala *Figuera,* Mallorca, Spain, SEUNG-JIN BANG, *Ajou* University, Suwon, Korea, SCOTT H. BROWN, Auburn University, *AL*, PAUL S. **BRUCKMAN,** Highwood, IL, XUMING CHEN (3 solutions), University of Alabama, Tuscaloosa, BILL CORRELL, JR., *Denison* University, Granville, OH, ANDREW CUSUMANO, Great Neck, *NY*, MARK EVANS, Louisville, *KY,* ROBERT C. GEBHARDT, *Hopatcong,* NJ, SOLOMON W. GOLOMB, University of Southern California, Los Angeles, MONICA GREENWOOD, St. Bonaventure University, *NY,* RICHARD I. **HESS,** Rancho *Palos* Verdes, *CA,* JOE HOWARD, New Mexico Highlands University, Las Vegas, ROSALIE JUNGREIS, James Madison High School No. Woodmere, *NY,* MURRAY S. **KLAMKIN,** University of Alberta, Canada, HENRY S. **LIEBERMAN** (2 solutions), Waban, MA, PETER A. **LINDSTROM,** North Lake College, Irving, *TX,* DAVID E. MANES, *SUNY* College at Oneonta, G. **MAVRIGIAN** (3 solutions), Youngstown State University, OH, YOSHINOBU MURAYOSHI, Okinawa, Japan, **KANDASAMY** MUTHUVEL, University of *Wisconsin-Oshkosh,* JOHN F. PUTZ, Alma College, MI, GEORGE W. RAINEY, Los Angeles, CA, DAVID S. **SHOBE,** New Haven, *CT,* ROMAN SZNAJDER, University of Maryland, Baltimore, L. J. **UPTON,** *Mississauga,* Ontario, Canada, REX H. WU (2 solutions), Brooklyn, *NY,* SAMMY YU and JIMMY YU, University of South Dakota, Vermillion, and the PROPOSER.

Other methods of solution included arguing that PA = PC and then showing that triangle PCQ is isosceles with apex P, applying the law of sines to triangles PQD and PAD, and placing the figure on the Cartesian plane with A at the origin, B at (1, 0) and D at (0, 1).

828. [Spring 1994] Proposed by Rex H. *Wu*, Brooklyn, New York.

Evaluate the integral

$$\int \ln(e^x + 1)\,dx.$$

Solution by Richard I. *Hess*, Rancho Palos Verdes, California.

"This integral cannot be expressed as a **finite** combination of elementary **functions"** – Gradshteyn and **Ryzhik** – 2.782.2.

For all b > 0, we have that

$$I = \int_0^b \ln(e^x + 1)\,dx = \int_0^b [x + \ln(1 + e^{-x})]\,dx$$
$$= \frac{b^2}{2} + \int_0^b (e^{-x} - \frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x} - \cdots)\,dx$$
$$= \frac{b^2}{2} + [-e^{-x} + \frac{1}{4}e^{-2x} - \frac{1}{9}e^{-3x} + \cdots]\Big|_0^b$$
$$= \frac{b^2}{2} + \sum_{k=1}^{\infty}(-1)^k \frac{e^{-kb} - 1}{k^2}.$$

If a < 0 then we have

$$I = \int_0^a \ln(e^x + 1)\,dx = \int_0^a \ln(1 + e^x)\,dx$$
$$= \int_0^a (e^x - \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} - \cdots)\,dx$$
$$= [e^x - \frac{1}{4}e^{2x} + \frac{1}{9}e^{3x} + \cdots]\Big|_0^a$$
$$= \sum_{k=1}^{\infty}(-1)^k \frac{1 - e^{ka}}{k^2}.$$

Hence the value of the indefinite integral depends upon whether $x > 0$ or $x < 0$. The value of any definite integral can be found by appropriately adding or subtracting one or both of the two forms above.

Also solved by SEUNG-JIN BANG, *Ajou* University, Suwon, Korea, PAUL S. BRUCKMAN, *Highwood*, IL, JOE HOWARD, New Mexico Highlands University, Las Vegas, PETER A. LINDSTROM, North Lake College, Irving, *TX*, DAVID E. MANES, *SUNY* College at Oneonta, STAN WAGON and JOAN HUTCHINSON, Macalester College, St. Paul, MN, and the PROPOSER.

829. [Spring 1994] Proposed by *Seung-Jin* Bang, *Ajou* University, Suwon, Republic of Korea.

Let f be a function such that f, f', ..., $f^{(n)}$ are all continuous, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ and $f^{(n)}(0) \neq 0$. Let

$$f_1(x) = \int_0^x f(t)\,dt \quad \text{and} \quad f_k(x) = \int_0^x f_{k-1}(t)\,dt \quad k = 2, 3, \ldots, n.$$

Evaluate the limit

$$\lim_{x\to 0} \frac{f_m(x)}{x^m f(x)}.$$

Solution by the Proposer.

Using **L'Hôpital's** rule, we have

$$\lim_{x\to 0}\frac{f(x)}{x^n} = \frac{f^{(n)}(0)}{n!} \quad \text{and} \quad \lim_{x\to 0}\frac{f_m(x)}{x^{n+m}} = \frac{f^{(n)}(0)}{(n+m)!}.$$

It follows that

$$\lim_{x\to 0}\frac{f_m(x)}{x^m f(x)} = \lim_{x\to 0}\frac{\dfrac{f_m(x)}{x^{n+m}}}{\dfrac{f(x)}{x^n}} = \frac{\dfrac{f^{(n)}(0)}{(n+m)!}}{\dfrac{f^{(n)}(0)}{n!}} = \frac{n!}{(n+m)!}.$$

Also solved *by* PAUL S. BRUCKMAN, *Highwood*, IL, MARK EVANS, Louisville, *KY*, RICHARD I. HESS, Rancho *Palos* Verdes, *CA*, CARL **LIBIS**, Idaho State University, *Pocatello*, HENRY S. LIEBERMAN, *Waban*, MA, and DAVID S. SHOBE, New Haven, CT.

830. [Spring **1994]** Proposed by David Iny, Baltimore, Maryland.
Let

$$w_k = \sum_{n=0}^{\infty} \frac{x^{4n+k-1}}{(4n+k-1)!} \qquad k = 1,\ 2,\ 3,\ 4.$$

a) Prove that $[(w_1 + w_3)^2 - (w_2 + w_4)^2][(w_1 - w_3)^2 + (w_2 - w_4)^2] = 1.$

b) Can you find similar identities with $p \geq 2$ for

$$w_k = \sum_{n=0}^{\infty} \frac{x^{pn+k-1}}{(pn+k-1)!},\ k = 1,\ 2,\ \ldots,\ p\,?$$

This problem is a generalization of a 1939 Putnam Exam problem, which considered the case of $p = 3$.

I. Solution to part (a) by Carl ***Libis,*** Idaho State University, Pocatello, Idaho.

Since $w_1 + w_3 = \cosh x$, $w_2 + w_4 = \sinh x$, $w_1 - w_3 = \cos x$, and $w_2 - w_4 = \sin x$, and since $w_1 + w_2 + w_3 + w_4 = e^x$ and $w_1 - w_2 + w_3 - w_4 = e^{-x}$, we have that

$$(w_1 + w_3)^2 - (w_2 + w_4)^2 = \cosh^2 x - \sinh^2 x = 1,$$

$$(w_1 - w_2)^2 + (w_2 - w_4)^2 = \cos^2 x + \sin^2 x = 1,$$

and

$$(w_1 + w_2 + w_3 + w_4)(w_1 - w_2 + w_3 - w_4) = e^x e^{-x} = 1.$$

II. Solution by Paul S. Bruckman, ***Highwood,*** Illinois.

We solve the problem for fixed $p \geq 2$. Then

(1) $$w_{k+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \delta_{n-k},\ \ k = 0,\ 1,\ \ldots,\ p-1\,.$$

and where

$$\delta_m = \begin{cases} 1 & \text{if } p \mid m, \\ 0 & \text{otherwise.} \end{cases}$$

If we let $\theta = \exp(2i\pi/p)$, then we may write

(2) $$\delta_m = \frac{1}{p} \sum_{j=0}^{p-1} \theta^{jm}.$$

We next look for a closed form for w_{k+1}. Thus

$$w_{k+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \frac{1}{p} \sum_{j=0}^{p-1} \theta^{j(n-k)}.$$

Since the series in (1) is absolutely convergent for all complex x, we **may** rearrange its terms in any order. Then

$$w_{k+1} = \frac{1}{p} \sum_{j=0}^{p-1} \theta^{-jk} \sum_{n=0}^{\infty} \frac{(x\theta^j)^n}{n!}$$

(3) $$= \frac{1}{p} \sum_{j=0}^{p-1} \theta^{-jk} \exp(x\theta^j),\quad k = 0,\ 1,\ \ldots,\ p-1.$$

We define, for $m = 0, 1, 2, \ldots, p-1$, a sum S_{m+1} and then use equation (3) to get

$$S_{m+1} = \sum_{k=0}^{p-1} \theta^{mk} w_{k+1} = \sum_{k=0}^{p-1} \theta^{mk} \cdot \frac{1}{p} \sum_{j=0}^{p-1} \theta^{-jk} \exp(x\theta^j)$$

$$= \sum_{j=0}^{p-1} \exp(x\theta^j) \cdot \frac{1}{p} \sum_{k=0}^{p-1} \theta^{k(m-j)} = \sum_{j=0}^{p-1} \exp(x\theta^j)\delta_{m-j},$$

and it follows that

(4) $$S_{m+1} = \exp(x\theta^m),\quad m = 0,\ 1,\ \ldots,\ p-1.$$

Note that

$$\sum_{m=0}^{p-1} \theta^m = 0$$

and consider the product

$$\prod_{m=0}^{p-1} S_{m+1} = \exp(x \sum_{m=0}^{p-1} \theta^m) = \exp(0) = 1.$$

This result provides us with the general identity satisfied by the w_k:

$$\prod_{m=0}^{p-1} \sum_{k=0}^{p-1} \theta^{mk} w_{k+1} = 1,$$

and equivalently,

$$(5) \qquad \prod_{m=0}^{p-1} (w_1 + \theta^m w_2 + \theta^{2m} w_3 + \cdots + \theta^{(p-1)m} w_p) = 1.$$

Some special cases follow. If $p = 2$ and hence $9 = -1$, then

$$(w_1 + w_2)(w_1 - w_2) = w_1^2 - w_2^2 = 1.$$

For $p = 3$, by repeated application of the relation $\theta^2 + 9 + 1 = 0$, the identity in (5) becomes

$$w_1^3 + w_2^3 + w_3^3 - 3w_1w_2w_3 = 1.$$

When $p = 4$, $9 = i$ and equation (5) becomes

$$(w_1 + w_2 + w_3 + w_4)\cdot(w_1 + iw_2 - w_3 - iw_4)\cdot$$
$$(w_1 - w_2 + w_3 - w_4)\cdot(w_1 - iw_2 - w_3 + iw_4)$$
$$= [(w_1 + w_3)^2 - (w_2 + w_4)^2][(w_1 - w_3)^2 + (w_2 - w_4)^2] = 1.$$

III. *Solution by Richard I.* ***Hess,*** *Rancho* ***Palos Verdes,*** *California.*

We take, for any given positive integer p and for $k = 1, 2, \ldots, p$,

$$w_k = \sum_{n=0}^{\infty} \frac{x^{pn+k-1}}{(pn+k-1)!}$$

and define $u_m = e^{2\pi i m/p}$. Suppose p is an odd number. Then

$$e^{ux} = 1 + ux + \frac{u^2x^2}{2!} + \frac{u^3x^3}{3!} + \cdots = w_1 + uw_2 + u^2w_3 + \cdots + u^{p-1}w_p$$

and

$$e^{-ux} = 1 - ux + \frac{u^2x^2}{2!} - \frac{u^3x^3}{3!} + \cdots = w_1 - uw_2 + u^2w_3 - \cdots + u^{p-1}w_p,$$

where u can be subscripted with $m = 1, 2, \ldots, p - 1$.

In the product

$$(1) \qquad 1 = e^{u_1x}e^{-u_1x}e^{u_2x}e^{-u_2x}\cdots e^{u_{p-1}x}e^{-u_{p-1}x}$$

pairs of factors can be split out, such as

$$e^{u_mx}e^{u_{p-m}x} = (w_1 + u_mw_2 + \cdots + u_m^{p-1}w_p)(w_1 + u_{p-m}w_2 + \cdots + u_{p-m}^{p-1}w_p)$$

$$= w_1^2 + w_2^2 + \cdots + w_p^2 + \sum_{1\le r<s\le p} w_rw_s(u_m^{r-1}u_{p-m}^{s-1} + u_m^{s-1}u_{p-m}^{r-1})$$

$$= w_1^2 + w_2^2 + \cdots + w_p^2\ 2 \sum_{1\le r<s\le p} w_rw_s \cos\frac{2\pi m(s-r)}{p} = e^{2x\cos(2\pi m/p)}.$$

Similarly

$$e^{-2x\cos(2m\pi/p)} = e^{-u_mx}e^{-u_{p-m}x}$$

$$= w_1^2 + \cdots + w_p^2 + 2\sum_{1\le r<s\le p} (-1)^{r+s} w_rw_s \cos\frac{2\pi m(s-r)}{P}$$

The product of these last two expressions is 1. There will be $(p - 1)/2$ such expressions for $m = 1, 2, \ldots, (p - 1)/2$.

Suppose now that p is an even number and define u_m as before. Then equation (1) still holds. Its factors get paired off as before except that $u_{p/2} = -1$ and

$$e^{u_{p/2}x} = e^{-x} = w_1 - w_2 + w_3 - \cdots - w_p$$

is paired with

$$e^{u_px} = e^x = w_1 + w_2 + w_3 + \cdots + w_p.$$

Then

$$e^{u_{p/2}x}e^{u_px} = (w_1 - w_2 + \cdots - w_p)(w_1 + w_2 + \cdots + w_p) = 1.$$

There are $p/2 - 1$ additional equations for $m = 1, 2, \ldots, p/2 - 1$,

$$\left[\sum w_r^2 + 2\sum_{r<s} w_rw_s \cos\frac{2\pi m(s-r)}{p}\right],$$

$$\left[\sum w_r^2 + 2\sum_{rcs} (-1)^{r+s} w_rw_s \cos\frac{2\pi m(s-r)}{P}\right] = 1.$$

When $m = p/4$, the two expressions in brackets are the same and each is equal to 1.

Also solved by the PROPOSER.

831. [Spring **1994**] *Proposed* by *Paul S. Bruckman,* ***Highwood,*** *Illinois.*

Solve exactly and completely

$$x^6 - 8x^5 + 18x^4 - 6x^3 - 12x^2 + 2x + 1 = 0.$$

I. *Solution by Henry' S. Lieberman, Waban, Massachusetts.*

It seems reasonable that the stated polynomial might factor as

$$(x^2 + ax + 1)(x^4 + bx^3 + cx^2 + dx + 1),$$

where $a, b, c,$ and d are integers. We multiply out this product and then equate its coefficients with the corresponding ones in the given polynomial to get that

$$a \ t\ b = -8, \qquad c + ah + 1 = 18,$$

$$d + ac + b = -6, \qquad 1 \ t\ a\ d + c = -12,$$

$$a + d = 2.$$

These equations have the solution $a = -3, b = -5, c = 2,$ and $d = 5.$ Thus the given polynomial is equal to

$$(x^2 - 3x + 1)(x^4 - 5x^3 + 2x^2 + 5x + 1).$$

'Although the quartic factor can be factored similarly into the product of the two quadratic factors

$$(x^2 + ex - 1)(x^2 + fx - 1),$$

we shall follow Ferrari's solution as presented in ***Higher Algebra*** by Hall and Knight, pp. *483-484.*

To each side of

$$x^4 - 5x^3 + 2x^2 + 5x + 1 = 0$$

add $(Ax + B)^2$, the quantities A and B being determined so as to make the left side a perfect square of the form $(x^2 - (5/2)x + k)^2$. That is, we want

$$(x^2 - (5/2)x + k)^2 = x^4 - 5x^3 + (25/4 + 2k)x^2 - 5kx + k^2$$

$$= x^4 - 5x^3 + (2 + A^2)x^2 + (5 + 2AB)x + 1 + B^2 = (Ax + B)^2.$$

By comparing coefficients we get

$$2514 + 2k = 2 + A^2, \quad -5k = 5 + 2AB, \text{ and } k^2 = 1 + B^2.$$

Eliminate A and B from these three equations by taking

$$(AB)^2 = (-(5/2)k - 5/2)^2 = (1714 + 2k)(k^2 - 1).$$

The resulting cubic polynomial in k easily factors thus

$$4k^3 - 4k^2 - 29k - 21 = (k + 1)(2k + 3)(2k - 7) = 0.$$

Taking $k = -1$, we get $B = 0$ and $A = \pm 3/2$, which gives rise to

$$x^2 - \frac{5}{2}x - 1 = \pm\frac{3}{2}x.$$

Thus the original polynomial factors into the product

$$(x^2 - 3x + 1)(x^2 - 4x - 1)(x^2 - x - 1),$$

which has the easily verified zeros

$$x = 2 \pm \sqrt{5}, \ \frac{1 \pm \sqrt{5}}{2}, \ \frac{3 \pm \sqrt{5}}{2}.$$

The other two values of k, *-312* and *712*, similarly lead to factorizations which produce the same zeros.

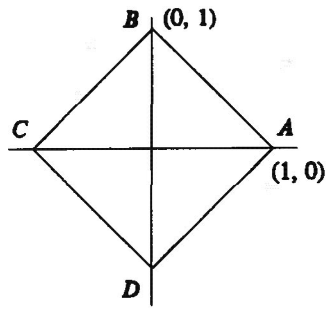
Also solved by BILL **CORRELL**, JR., ***Denison University, Granville, OH,*** PATRICK COSTELLO, ***Eastern Kentucky University, Richmond,*** **MARK** EVANS, ***Louisville, KY,*** J. S. FRAME, ***Michigan State University, Lansing,*** **MARCIE** GARDNER, TOM SYMONS, ARTHUR THOMASON, and **RANDI** KAY VEST, ***Hendrix College, Conway, AR,*** ROBERT C. GEBHARDT, ***Hopatcong, NJ,*** RICHARD I. **HESS**, ***Rancho Palos Verdes, CA,*** JOE HOWARD, New Mexico Highlands University, Las Vegas, BECKY LATCH, ANGELA JONES, ***and*** WADE WILLIAMS, ***Hendrix College, Conway, AR,*** CARL **LIBIS**, ***Idaho State University, Pocatello,*** PETER A. **LINDSTROM**, ***North Lake College, Irving, TX,*** DAVID *E.* MANES, ***SUNY College at Oneonta,*** **YOSHINOBU MURAYOSHI**, ***Okinawa, Japan,*** BOB **PRIELIPP**, ***University of Wisconsin-Oshkosh,*** STAN WAGON ***and*** JOAN HUTCHINSON, ***Macalester College, St. Paul, MN,*** **REX** H. **WU**, ***Brooklyn, NY,*** SAMMY **YU** ***and*** **JIMMY** YU, ***University of South Dakota, Vermillion, and the*** PROPOSER.

832. [Spring **1994]** ***Proposed by David Iny, Westinghouse Electric Corporation, Baltimore, Maryland.***

The ***taxicab distance*** between points (a, b) and (c, d) is $|a - c| + |b - d|$. Determine the circumference in taxicab space of the circle whose equation is $x^2 + y^2 = 1$.

I. ***Solution*** by Victor G. Feser, ***University*** of Mary, Bismarck *North* Dakota.

The circumference is 8. In taxicab geometry, the circle is a square, as in the **figure.** To confirm this fact, **find** equations of each of the **four** segments. For example, AB is given by $y = -x + 1$ on the interval $0 \le x \le 1$. Then a point on that segment is $(x, -x + 1)$ and its distance from $(0, 0)$ is given by



$$|x - 0| + |(-x + 1) - 0| = x + (-x + 1) = 1,$$

so it lies on the taxicab circle.

The length of segment AB is $|1 - 0| + |0 - 1| = 2$. Since each of the other three segments has the same **length,** the total circumference is 8.

II. Comment by Paul ***S. Bruckman, Highwood,*** Illinois.

Aha, **Lindemann** was wrong—you can square the circle after all. In taxicab geometry. Nor am I impressed with the Chudnowsky brothers' achievement—I can compute π to infinitely many decimal places, all of them zero! Now, for my next **trick,** I intend to show that ∞ is a rational number (in fact, I once actually met a rational mathematician!).

Also solved by SEUNG-JIN BANG, ***Ajou*** University, ***Suwon,*** Korea, PAUL S. BRUCKMAN, ***Highwood,*** IL, MARK EVANS, Louisville, ***KY,*** ROBERT C. GEBHARDT, ***Hopatcong,*** NJ, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, ***Rancho*** Palos Verdes, CA, HENRY S. LIEBERMAN, ***Waban,*** *MA,* DAVID S. SHOBE, New ***Haven,*** CT, ROMAN SZNAJDER, University of Maryland, Baltimore, MD, REX H. WU, Brooklyn, *NY,* and ***the*** PROPOSER.

833. [Spring 1994] *Proposed by Seung-Jin Bang, Ajou University, Suwon, Republic of Korea.*

Define a function f by $f(0) = 1$ and

$$f(m) = \binom{n+m-2}{m} + \binom{n+m-3}{m-1}.$$

Find the value of the sum $\sum_{m=0}^{k} f(m)$.

Solution by J. S. Frame, Michigan State University, East Lansing, MI.

Let us denote $f(m)$ by $f(m, n)$ to show its dependence on n. Then $f(0, n) = 1$ and

$$f(m, n) = f(m, n+1) - f(m-1, n+1) \quad \text{for } m \ge 1.$$

Thus the required sum telescopes and yields

$$\begin{aligned}\sum_{m=0}^{k} f(m, n) &= f(0, n) + \sum_{m=1}^{k} [f(m, n+1) - f(m-1, n+1)] \\ &= f(0, n) + f(k, n+1) - f(0, n+1) = 1 + f(k, n+1) - 1 \\ &= f(k, n+1) = \binom{n+k-1}{k} + \binom{n+k-2}{k-1} = \frac{n+2k-1}{k}\binom{n+k-2}{k-1}.\end{aligned}$$

Also solved by PAUL S. BRUCKMAN, ***Highwood, IL,*** BILL CORRELL, JR., ***(who provided the final*** form for ***the solution), Denison University, Granville,*** OH, RICHARD I. HESS, ***Rancho Palos*** Verdes, ***CA,*** HENRY S. LIEBERMAN, ***Waban, MA,*** PETER A. LINDSTROM, ***North*** Lake College, ***Irving, TX,*** PHYLLIS MAHAN, ***Eastern Kentucky University, Richmond,*** DAVID E. MANES, ***SUNY College*** at ***Oneonta,*** REX H. WU, ***Brooklyn, NY, and the*** PROPOSER.

834. [Spring **1994]** Proposed by Murray ***S. Klamkin, University*** of ***Alberta,*** Edmonton, ***Alberta,*** Canada.

Let T and T' denote two triangles with respective sides (a, b, c) and (a', b', c') where $a'^2 = 2a(s-a)$, $b'^2 = 2b(s-b)$, and $c'^2 = 2c(s-c)$. Prove that

(i) $s \ge s'$, (ii) $R \ge R'$, (iii) $r' \ge r$, and (iv) $F'/s'^2 \ge F/s^2$

where $s = (a + b + c)/2$ is the semiperimeter, R the circumradius, r the inradius, and F the area of triangle ABC, and similarly for triangle $A'B'C'$.

Solution by *Miguel Amengual* Covas, *Cala* Figuera, *Mallorca,* Spain.

Given triangle T, triangle T' exists from items (1) and (35) in [2] for the function $f(x) = \sqrt{x}$. (The converse is not true: given an arbitrary triangle T', then a, b, and c are not necessarily the sides of a triangle.) Next, we define $x = s - a$, $y = s - b$, and $z = s - c$. Then $x + y = c$, $y + z = a$, and $z + x = b$.

By the arithmetic-geometric mean inequality we get

$$\begin{aligned} 2s' &= a' + b' + c' = \sqrt{2x(y+z)} + \sqrt{2y(z+x)} + \sqrt{2z(x+y)} \\ &\le \frac{2x+(y+z)}{2} + \frac{2y+(z+x)}{2} + \frac{2z+(x+y)}{2} \\ &= 2(x+y+z) = 2s. \end{aligned}$$

from which inequality (i) follows.

Since we have

$$\cos A' = \frac{b'^2 + c'^2 - a'^2}{2b'c'} = \frac{2y(z+x) + 2z(x+y) - 2x(y+z)}{2\sqrt{2y(z+x)}\cdot\sqrt{2z(x+y)}}$$

$$= \frac{yz}{\sqrt{y(z+x)\cdot z(x+y)}} = \sqrt{\frac{yz}{(z+x)(x+y)}} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sin\frac{A}{2},$$

then $A' = \pi/2 - A/2$ and similarly $B' = \pi/2 - B/2$ and $C' = \pi/2 - C/2$. Now

$$\begin{aligned} s &= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R\sin A'\sin B'\sin C' \\ &= 4R\cdot 2\sin\frac{A'}{2}\cos\frac{A'}{2}\cdot 2\sin\frac{B'}{2}\cos\frac{B'}{2}\cdot 2\sin\frac{C'}{2}\cos\frac{C'}{2} \\ &= 8R\sin\frac{A'}{2}\sin\frac{B'}{2}\sin\frac{C'}{2}\cdot 4\cos\frac{A'}{2}\cos\frac{B'}{2}\cos\frac{C'}{2} \\ &= 8R\sin\frac{A'}{2}\sin\frac{B'}{2}\sin\frac{C'}{2}\cdot\frac{s'}{R'} \le R\cdot\frac{s'}{R'} \end{aligned}$$

and inequality (ii) follows since, by item 2.12 in [1],

$$\sin\frac{A'}{2}\sin\frac{B'}{2}\sin\frac{C'}{2} \le \frac{1}{8}.$$

By Heron's formula,

$$\begin{aligned} 8F^2 &= (a+b+c)(a+b-c)(b+c-a)(c+a-b) \\ &= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \end{aligned}$$

by just a bit of algebra. Similarly, placing a prime on each letter in this last equation and then replacing a'^2 by $2a(s - a)$, etc., and with just a bit more algebra, we get that

$$\begin{aligned} 8F'^2 &= \sum 2\cdot 2b(s-b)\cdot 2c(s-c) - \sum 4a^2(s-a)^2 \\ &= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 8F^2. \end{aligned}$$

Hence $F' = F$. Now $rs = F = F' = r's'$, so inequality (iii) follows from inequality (i).

Finally, from inequalities (i) and (iii) and $F = rs$, we get inequality (iv),

$$\frac{F'}{s'^2} = \frac{r'}{s'} \ge \frac{r}{s} = \frac{F}{s^2}.$$

References

1. O. Bottema et al, Geometric Inequalities, Wolters-Noordhoff, Groningen, 1968, p. 20.
2. D. S. Mitrinović, J. E. Pecarić, and V. Volenec, Recent Advances in Geometric Inequalities, Kluwer, 1989, p. 18-22.

Also solved by PAUL S. BRUCKMAN, *Highwood, IL*, and the PROPOSER.

835. [Spring 1994] Proposed by *the* Alma College Problem Solving Group, A*lma* College, A*lma,* Michigan.

Let $P(x)$ be a polynomial of degree $n \ge 2$ with real coefficients and whose leading three terms are $ax^n + bx^{n-1} + cx^{n-2}$. All remaining terms are of degree $n - 3$ or less. If $b^2 \le 2ac$, then prove that $P(x)$ cannot have n distinct real roots. (This problem is a generalization of Problem 4 from the February 1990 issue of *Problem Solving Newsletter* by Dr. Hugh Montgomery of the University of Michigan.)

I. *Solution by Kandasamy* ***Muthuvel****, University of Wisconsin at* ***Oshkosh****, Oshkosh,* ***Wisconsin.***

Suppose that $P(x)$ has n distinct real zeros. Then by **Rolle's** theorem, one can see that $P'(x)$ must have $n - 1$ distinct real zeros, and so forth, so that

$$p^{(n-2)}(x) = \frac{(n-2)!}{2}[an(n-1)x^2 + 2b(n-1)x + 2c]$$

has two distinct real zeros. Hence its discriminant

$$4b^2(n-1)^2 - 8an(n-1)c = 4(n-1)[b^2(n-1) - 2acn] > 0.$$

Thus we must have

$$b^2 > \frac{2acn}{n-1} > 2ac,$$

a stronger inequality than proposed.

II. *Solution by David* ***Iny****, Baltimore,* ***Maryland.***

Suppose $P(x) = 0$ has n distinct real roots $r_1, r_2, ..., r_n$. Then

$$P(x) = a\prod_{i=1}^{n}(x - r_i) \quad \text{with } a \neq 0.$$

Then

$$b = -a\sum_{i=1}^{n} r_i \quad \text{and} \quad c = a\sum_{1\le i<j\le n} r_i r_j.$$

Now

$$b^2 - 2ac = a^2\sum_{i=1}^{n} r_i^2.$$

Now $(r_i - r_j)^2 > 0$ when $i \neq j$, so then $r_i^2 + r_j^2 > 2r_i r_j$ and hence

$$(n-1)\sum_{i=1}^{n} r_i^2 > 2\sum_{1\le i<j\le n} r_i r_j,$$

so

$$b^2 - 2ac > \frac{2}{n-1}ac, \quad \text{and finally} \quad b^2 - 2ac\frac{n}{n-1} \le 0.$$

Thus, if $b^2 - 2nac/(n-1) \le 0$, $P(x)$ cannot have n distinct zeros. When $n = 2$, this is the usual $b^2 - 4ac \le 0$ necessary and sufficient condition for $P(x) = 0$ not to have distinct real roots.

Also *solved by* SEUNG-JIN BANG, ***Ajou*** *University,* ***Suwon,*** *Korea,* PAUL S. BRUCKMAN, *Highwood, IL,* **BILL** CORRELL, JR., ***Denison*** *University, Granville, OH,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban,* ***MA,*** DAVID E. MANES, ***SUNY College*** *at Oneonta,* DAVID S. SHOBE, *New Haven,* ***CT,*** **REX** H. **WU,** ***Brooklyn, NY, and*** *the* PROPOSERS.

Bruckman *commented that it is proper to speak of the zeros of P (and not of the roots of P) as those values of* ***x*** *that are the roots of the equation* ***P(x)*** *= 0. That is, an expression has zeros; an equation has roots.*

How ***often*** *have I told my students that same distinction! Yet I did overlook the error here. —****Ed.***

A problemist is one of those heros,
Who, when he sees "roots" used for "zeros,"
  Thinks it a harsh grate
  Like nails scraping slate.
So he charges out like a rhinoceros.

—Anon.

The 1994 National Pi Mu Epsilon Meeting

The meeting took place in conjunction with the summer meeting of the Mathematical Association of America and the American Mathematical Society in Minneapolis, Minnesota, August 15-17, 1994.

There were thirty-three student papers delivered in four sessions:

Analysis of covariance, by Mike Ameduri (Youngstown State University)

Diagnostics in insurance: an application for simple regression, by Mark Bonsall (Moravian College)

A Mellin-type integral transform method for solving Cauchy-Euler differential equations, by David A. Brown (Ithaca College)

Aspects of orthonormalization in quantum information theory, by Bill Correll, Jr. (Denison University)

The problems of scale in the hyperbolic world, by Andrew Douglass (Miami University)

The construct matrix, by Jamie Downs (St. Norbert College)

A practical application of Mutex, by Joseph Engel (St. Norbert College)

Interface construction and 2-D fluid dynamics, by Geoffrey Gibbons (University of California)

Three tools to help build term persistence assumptions, by Dale Hall (John Carroll University)

The number theoretic properties of the dynamical system known as rigid rotation, by Allen G. Harbaugh (Boston University)

The Haar wavelet and the dilation equation, by Kevin R. Hutson (Hendrix College)

The numerical range of a matrix, by Dennis Keeler (Miami University)

Beyond chaos, by Brian Kemery (SUNY College at Fredonia)

The basic mathematics of production under uncertainty, by Nikolay K. Kolev (Moravian College)

Tranquilizing Bigfoot, by Sondra C. Laird (University of West Florida)

Magnetic monopoles in mathematical physics, by Erik Leder (Portland State University)

Finding probabilities with the Maple computer algebra system, by Joshua D. Levy (Hope College)

Examples of negatively dependent randnm variables, by Teresa J. Murphy (Georgia State University)

Quantum cryptography, by Kathryn Nyman (Carthage College)

Nothing in moderation, everything in excess: a new weighted statistic on permutations, by Ann Marie Paulukonis (St. John's University, Minnesota)

The Chinese Remainder Theorem and object oriented programming, by David M. Potts (Texas A & M University)

A linear programming formulation to optimize sawmill operations, by Jerry Priddy (Youngstown State University)

Bernoulli numbers in series summations, by Alan B. Shettel (Youngstown State University)

Re-marking dice, by Kendra Sinopoli (Youngstown State University)

Fault tolerance in parallel processing, by Jason M. Spangler (Youngstown State University)

Using asymptotics to mathematically model macrosegregation in continuously cast steel slabs, by Carl Stitz (University of Akron)

Existence theorems for 2-point boundary value problems, by Tishua Taylor (Spelman College)

A discission of simple continued fractions and their applications, by Christina T. Tsiaparas (Youngstown State University)

Minimal Moebius strips, by Daniel L. Viar (University of Arkansas)

On b, *n*-happiness sequences, by Sonny Vu (University of Illinois)

Applications of fractals in geology and geophysics, by Lisa White (Youngstown State University)

The combinatorics of semi-direct products of cyclic groups, by Jeb F. Willenbring (North Dakota State University)

Having a ball with Pythagoras, by Adam J. Zeuske (St. Norbert College)

Five prizes of $100 each, for papers of unusual merit, were awarded to Andrew Douglass, Allen Harbaugh, Kathryn Nyman, Sonny Vu, and Jeb Willenbring.

The National Security Agency again awarded Pi Mu Epsilon a grant of $5000 for the support and encouragement of student speakers, and the American Mathematical Society contributed $1000 for prize awards.

The J. Sutherland Frame Lecture, delivered by Scoutmaster Mel .

Slugbate, was entitled "Cheating your way to the knot merit badge". Some members of the audience detected a strong resemblance of the speaker to Professor Colin **Adams** of Williams **College,** but this may have been mere coincidence.

The Richard V. hdree Awards

The Richard V. Andree awards are given annually to the authors of the three papers written by students that have been judged by the officers and councilors of the Pi Mu Epsilon to be the best to have appeared in the *Pi Mu Epsilon Journal* in the past year.

Richard V. Andree was, until his death in 1987, Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon long and well in many capacities: as president, as secretary-treasurer, and as editor of the *Journal.*

The winner of the first prize for 1994 is Gina Aurello, for her paper "On the rearrangement of infinite series" (this *Journal* 9 (1989-94) #10, 641-646).

The second prize is awarded to Rychard **Bouwens,** for "Who gets the Washers?" (this *Journal* 10 (1994-99) #1, 1-4).

The third prize winner is Michael **Reske,** for "The secret Santa problem" (this *Journal* 10 (1994-99) #1, 18-21).

At the times the papers were written, Ms. Aurello was a student at Seton Hall University, Mr. Bouwens at Hope College, and Mr. Reske at Carthage College.

The officers and councilors of the Society congratulate the winners on their achievements and wish them well in their futures, whether or not they involve mathematics.

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