PI MU EPSILON JOURNAL

VOLUME 10 FALL 1995 NUMBER 3

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PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF THE

NATIONAL HONORARY MATHEMATICS SOCIETY

Editorial correspondence, including manuscripts, chapter reports, and news items should be sent to Underwood Dudley, Mathematics Department, DePauw University, Greencastle, Indiana 46135 (e-mail address dudley@depauw.edu). Students submitting manuscripts should give their school and date of graduation. Others should provide their affiliation, academic or otherwise.

Problems for solution and solutions to problems should be sent directly to Clayton W. Dodge, Mathematics Department, 5752 Neville Hall, University of Maine, Orono, Maine 04469.

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A CORRESPONDENCE ON TELESCOPING SERIES

Dan Kalman and John Mathews
The American University and California State University, Fullerton

November 23

Dear Dan,

It was good to see you at the MAA Section meeting last weekend. I meant to tell you about a problem I have been working on, which I think might interest you. I got interested in it one day while fooling around with Matheniatica. I think that programs like Matheniatica offer real opportunities for students to make their own niatheniatical discoveries, and was working on one possible direction for exploration—telescoping series. You know the standard example?

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

and all the terms cancel except the first so the sum is I. I was looking for generalizations. One obvious sum to consider is

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}.$$

Proceeding as in the first example, I used a partial fractions decomposition tor the summand. Mathematica has a built-in function for this, you know, so it is really effortless to see what will happen. Anyway, here is what came out:

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2} \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} \right)$$

As before, the sum telescopes. To simplify the notation. multiply both sides

by 2. Then the first several terms of the sum reveal the following pattern:

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)(k+2)} = \begin{array}{l} 1/1 + 1/2 + 1/3 + 1/4 + \cdots \\ \quad - 2/2 - 2/3 - 2/4 - \cdots \\ \quad\quad + 1/3 + 1/4 + \cdots \end{array}$$

So all but three terms are consumed by the collapse of the telescope, leaving

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)(k+2)} = \frac{1}{2}.$$

Encouraged by this success, I went onto $\sum_{k=1}^{\infty} 1/k(k+1)(k+2)(k+3)$ and a few more in the same pattern. Each time Mathematica gave me a partial fractions decomposition that turned out to telescope. All of the sums fit a nice pattern, and suggest a general identity:

$$\sum_{k=1}^{\infty} \frac{m!}{k(k+1)(k+2)\cdots(k+m)} = \frac{1}{m}. \tag{1}$$

Have you ever seen that before? Any ideas on how to prove it? The partial fractions decompositions themselves fit a nice pattern. After multiplying the summand by m!, it looks like this:

$$\frac{m!}{k(k+1)(k+2)\cdots(k+m)} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{k+j}.$$

I don't have a proof that either pattern holds in general, although they hold in every case I checked. Maybe the partial fractions decomposition identity could be used to prove the other one. Any ideas? The first identity (1) seems like such a nice result, I would sure like to see a proof.

John

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December 5

Dear John,

Thanks for a very interesting letter. I have never seen your identity before, but 1 can't imagine that it is new. It is too natural a generalization of telescoping series, and too pretty an extension, not to have been

discovered before. My first thought was that it should be easy to prove, too, but I have changed my opinion! Of course, the most direct way to prove the conjecture is to carry out the telescoping operation in the general case. This requires two steps. First, we need to establish the pattern you found for the partial fraction decomposition of the summand

$$a_k = \frac{m!}{k(k+1)(k+2)\cdots(k+m)}$$

The second step is to carry out the telescoping operation on the sum. It is easy to verify that the terms do eventually all cancel, leaving a finite number of initial terms. To complete the proof, we need to show that these initial terms sum to $1/m$. I worked on that part for a bit with no success. And since I had no success, I didn't bother to work on the partial fractions part.

I also thought briefly about induction on m, but could find no way to get at the induction step.

Then I thought about generating functions. You know how they work? The basic idea is this. If you have a series Σa_k you turn it into a power series $\Sigma a_k x^k$. Then it is a function of x and you can use methods of analysis on it, like differentiation. If you can work out a closed form representation of the function, then plugging in $x = 1$ will give the sum of the original series. Neat, huh? In some sense you make the original sum infinitely harder because you transform it from a single sum to an uncountably infinite number of sums. But if it works, you get not only the sum you wanted, hut infinitely many others, too. Wilf has a beautiful little book on the subject, called *Generatingfunctionology* [7].

Anyway, I did make some progress on a generating functions proof of your identity. Let

$$f_m(t) = (-1)^{m+1} \sum_{k=1}^{\infty} \frac{(1-t)^{m+k}}{k(k+1)(k+2)\cdots(k+m)}$$

Then if you differentiate m times you get the series expansion for the natural logarithm. That is,

$$\left[\frac{d}{dt}\right]^m f_m(t) = \ln t.$$

Also, f_m and its first m derivatives all vanish at $t = 1$. So just integrate the

natural log from 1 to t, then integrate the result from 1 to t, then integrate that from 1 to t, and so on. After m steps you will have a formula for f_m as a function of t. I think that if I could work out the first few steps, maybe I could see a pattern that can be established by induction, but I keep making errors. If I was smart I would use Mathematica like you, but somehow I can't get motivated to sit down and mess with it. Anyway, if I could somehow get a formula for f_m, then all we need to do is show that

$$f_m(0) = (-1)^{m+1}\frac{1}{m\,m!}.$$

Well, actually we need to compute that as a limit for t decreasing to 0 because 0 is at the boundary of the circle of convergence of the power series. So there are a few twists and turns, but it might lead to a proof eventually.

It is a good problem, John. Thanks for sharing it with me. I have already spent more time on it then I ought. I wish I could just get a proof, then I could leave it alone.

dan

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December 9

Dear John,

Just a quick footnote to the last letter. I have nearly completed the proof using generating functions. All I need to do is prove one simple identity:

$$(2)\qquad -\sum_{k=1}^{m}\frac{(-1)^k}{k}\binom{m}{k} = \sum_{k=1}^{m}\frac{1}{k}$$

That is interesting in itself, don't you think? I never saw anything like it before. Did you?

dan

****

December 12

Dear Dan,

SUCCESS! If you were right in your last letter we have a proof! At any rate I was able to prove your identity (2). I want to see the details of the generating function argument.

I do like your identity by the way, and the proof is really very easy. Let

$$A(m) = \sum_{k=1}^{m}\frac{(-1)^{k-1}}{k}\binom{m}{k}.$$

Next use the identity

$$\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$$

to replace the binomial coefficient in the definition of $A(m)$. Actually, you can only make the replacement in the first $m-1$ terms of the sum, since the identity doesn't hold for $k = m$. So split off the final term, and then use the identity. That gives

$$A(m) = \sum_{k=1}^{m-1}\frac{(-1)^{k-1}}{k}\binom{m-1}{k} + \sum_{k=1}^{m-1}\frac{(-1)^{k-1}}{k}\binom{m-1}{k-1} + (-1)^{m-1}\frac{1}{m}.$$

Now look—the first sum is just $A(m-1)$. Also,

$$\frac{1}{k}\binom{m-1}{k-1} = \frac{1}{m}\binom{m}{k}.$$

Substituting these leads to

$$A(m) = A(m-1) + (-1)^{m-1}\frac{1}{m} + \frac{1}{m}\sum_{k=1}^{m-1}(-1)^{k-1}\binom{m}{k}.$$

The sum at right simplifies using the binomial expansion of $(1-1)^m$ to

give $(1/m)(1 + (-1)^m)$, which combines with the other $1/m$ term to give just 1lm. This shows that the first difference of the sequence $A(m)$ is $1/m$. Since A(1) = 1, your identity is established.

I still want to work on the general case of the partial fractions decomposition, if I can find some time to get to it. As soon as I get somewhere, I will let you know.

John

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December 20

Dear John,

Thanks for the proof of that last identity. Yes, I have rechecked everything, and I am confident now that we have a proof. But what a monstrosity it is! I refuse to believe it has to be that hard. I had to define a whole family of generating functions, indexed on m, and then resort to an induction on m to get it to work out. **Yukk!** And there is something else that bothers me. The whole point of using a generating function in the first place is to avoid an induction argument. Somehow, the properties of analytic functions carry out the induction for you. So I was thinking that there ought to be a bivariate generating function argument. I tried to concoct a power series in x and y so that taking partials with respect to y and evaluating at 1 would give the parameterized **univariate** power series of the induction argument. I thought this would avoid the induction on m, just leaving me with some PDE to solve. Unfortunately, I couldn't get it to work out.

Where that leaves us is with a proof of your identity, but not a very nice one. There must be a better approach. If we can find it, maybe we should give a talk about all this at the MAA meeting in March? One thing we definitely should do first though is try to find somewhere that the identity has already appeared in print.

I guess you are about done with your semester now. That should give you a little more free time. That is one advantage that your academic position has over mine in industry. I can take some time off at Christmas, but it all counts against my annual allotment of vacation time. I will probably only take off a few days this Christmas, so I can save up for a big

family vacation next summer. On the other hand, I am mighty glad not to have any finals to grade!

dan

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March 4

Dear Dan,

Sorry not to write for so long. I got real busy around the holidays and then the time just seemed to fly past! Then, seeing you at the section meeting last Saturday made me think about the problem again. As I mentioned, I did succeed in proving the general partial fractions decomposition pattern for $m!/k(k+1)(k+2)\cdots(k+m)$. It is a nice little induction argument. Clearly the pattern is valid for m = 1. So assume the m − 1 case:

$$\frac{(m-1)!}{k(k+1)(k+2)\cdots(k+m-1)} = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \frac{1}{k+j}$$

Multiply both sides by $m/(k+m)$ and obtain

$$\frac{m!}{k(k+1)(k+2)\cdots(k+m)} = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \frac{m}{(k+j)(k+m)}$$

$$= \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \frac{m}{m-j} \left[\frac{1}{k+j} - \frac{1}{k+m} \right]$$

$$= \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \left[\frac{1}{k+j} - \frac{1}{k+m} \right]$$

$$= \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \frac{1}{k+j} - \frac{1}{k+m} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j}.$$

The second sum is just the binomial expansion for $(1-1)^m$ lacking only the final term. Since the complete sum is 0, the sum of the initial terms must be the negative of the final term. This gives

$$\frac{(m-1)!}{k(k+1)(k+2)\cdots(k+m-1)} = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \frac{1}{k+j} + (-1)^m \frac{1}{k+m}.$$

Observing that the final term is just the with term of the sum now completes the proof.

I have been looking through back issues of Mathematics Magazine, the Monthly, and the *AMATYC* Journal, but so far have not seen the identity. I am sure that you are right about it being previously known, but it would be interesting to see a different proof.

John

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March 23

Dear John,

I appreciated getting the proof of that partial fractions identity. Interesting how the binomial expansion of $(1-1)^k$ popped up in both the proofs, isn't it? I have been trying to find a better proof for your main identity (1), but so far with no success. The corporation's library is pretty limited as far as mathematics goes, and most of the holdings are on permanent loan in people's offices. I think I heard once that the library building isn't big enough to shelve all the holdings. Anyway, I really haven't gotten around to doing any library research. Instead, I have been trying to come up with some different ways of looking at the identity. Actually, I have found a couple of pretty interesting reformulations, but nothing that has led to a proof so far.

One idea is to introduce a binomial coefficient into the sum by multiplying the numerator and denominator of the summand by $(k-1)!$. The identity can then be rearranged in a few steps:

$$\sum_{k=1}^{\infty} \frac{m!(k-1)!}{(k+m)!} = \frac{1}{m}$$

$$\frac{1}{m+1} \sum_{k=1}^{\infty} \frac{(m+1)!(k-1)!}{(k+m)!} = \frac{1}{m}$$

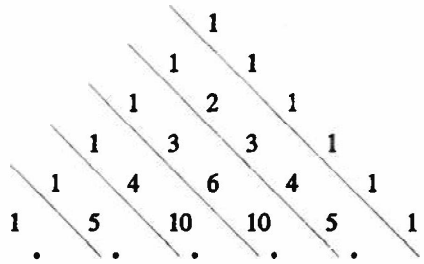
$$\sum_{k=1}^{\infty} \binom{m+k}{k-1}^{-1} = \frac{m+1}{m},$$

$$\sum_{k=0}^{\infty} \binom{m+k+1}{k}^{-1} = \frac{m+1}{m},$$

(3)

$$\sum_{k=0}^{\infty} \binom{n+k}{k}^{-1} = \frac{n}{n-1}.$$

For the last transformation, I substituted n for $m+1$. In the final form, you can see that the terms of the sum are the reciprocals of the entries on one diagonal of Pascal's triangle. Partition the triangle into diagonal lines as illustrated on the right and call the sum of the reciprocals of the entries in a line a reciprocal sum. The first two reciprocal sums clearly diverge. The remaining sums are given by your identity. Is there a simple combinatorial argument that can be used to derive the identity? I found none. It would be nice to find a proof that connects in a neat way with this Pascal's triangle interpretation.



Another idea was suggested by Art Benjamin. I stopped to see him on a recent visit to Harvey Mudd. Don Goldberg (from Occidental) was there too, so I mentioned your identity to them. Art took one look at the form (3) with the binomial coefficients and immediately started talking about probability. He pointed out that reciprocals of binomial coefficients often show up as probabilities. With an appropriate model, it might be possible to view the infinite sum in the identity as an expectation or as some computation associated with probability. If so, a proof might be constructed by showing in an alternate fashion that this computation should have the value at the right side of the identity. Art, Don, and I played around with it a bit, but never got anywhere. Later, on a visit to Northridge, I mentioned Art's idea to Mark Shilling who knows all about probability and

stats. Mark seemed pretty sure that Art's approach should work out, so maybe he will come up with an alternate proof. But so far I haven't heard anything more from him.

In a few weeks the kids will be out of school for their spring break. We have plans to go up to Santa Barbara to visit Linda's folks for a few days at Easter. While I am up there, I am supposed to visit the UC campus to check things out for the fall MAA section meeting. I also plan to take the opportunity to nose around in the library there, to see what I can find. If anything turns up, I will let you know.

dan

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May 25

Dear John,

I have all kinds of news to report from my visit to UC Santa Barbara. The main item is that I found your identity in print, and a nice neat proof. I even found my little binomial identity (2) in one reference **[5]**. But I am getting ahead of the story. Abraham Ungar is visiting us for a few days from **Fargo** (where he is a math prof at North Dakota State). He came along on the library trip, and when I told him what I was after he knew just where to look: an encyclopedia of sums, products, and integrals **[2]** that he enjoys browsing in. So I met with immediate success, of a sort. In [2] I found as equation 3 the following:

$$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q][p+kq]\cdots[p+(k+r)q]} = \frac{1}{(r+1)q}\left[\frac{1}{p(p+q)\cdots(p+rq)} - \frac{1}{(p+nq)\cdots(p+(n+r)q)}\right].$$

With $p = q = 1$ and r = m − 1, this particularizes to

$$(4)\quad \sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)} = \frac{1}{m}\left[\frac{1}{m!} - \frac{1}{(1+n)(2+n)\cdots(m+n)}\right]$$

which obviously implies your identity. However, no proof was given. It

did give a reference for the identity (in fact it has references for every identity it contains). The reference for our equation is an earlier book of tables **[1]** dating from 1922. There I found an even more general form of the identity, another reference, but still no proof. In another book **[4]**, the special case of (4) for m = 3 appears as equation 115. Again there is no proof, but a reference to **[3]**, which is a textbook from 1895. Just for fun, I looked that textbook up in the card catalog, and would you believe it, the Santa Barbara library had a copy, just a few aisles from where I was standing. There is something really intriguing and exciting about playing detective in this way. When I held that book, almost 100 years old, in my hands and looked down at the same identity you discovered, I got goose bumps. Honestly!

Anyway, the main thing I got out of all those references was a new perspective: finite sums! Where we began by considering infinite sums, the references I unearthed all made use of finite sums. The advantage of a finite sum is that it permits induction on the number of terms. Indeed, (4) is easily established by induction on n, as follows. For n = 1, each side of (4) is readily seen to equal $1/(m+1)!$. So suppose the identity holds up to n, and consider a sum of n + 1 terms:

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)\cdots(k+m)} = \sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)} + \frac{1}{(1+n)(2+n)\cdots(m+1+n)}.$$

Substitute the right side of (4) in the sum above to obtain

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)\cdots(k+m)} = \frac{1}{m}\left[\frac{1}{m!} - \frac{1}{(1+n)(2+n)\cdots(m+n)}\right] + \frac{1}{(1+n)(2+n)\cdots(m+1+n)}.$$

Now make a few rearrangements.

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)\cdots(k+m)} =$$

$$\frac{1}{m}\left[\frac{1}{m!} - \frac{1}{(1+n)(2+n)\cdots(m+n)} + \frac{m}{(1+n)(2+n)\cdots(m+1+n)}\right]$$
$$= \frac{1}{m}\left[\frac{1}{m!} - \frac{m+n+1-m}{(1+n)(2+n)\cdots(m+n)(m+n+1)}\right]$$
$$= \frac{1}{m}\left[\frac{1}{m!} - \frac{1}{(2+n)\cdots(m+n)(m+n+1)}\right]$$
$$= \frac{1}{m}\left[\frac{1}{m!} - \frac{1}{(1+n+1)\cdots(m+n+1)}\right].$$

This shows that the identity is valid for n + 1 and completes the proof. Simple! I would never have thought of the idea of trying to approach your identity in terms of finite sums, but it is a lesson I am not likely to forget soon!

Well, John, I guess that is about the end of the trail. I don't think you could ask for a simpler proof, and now we know that the identity has been known for a good long time. It was great fun! Thanks again for sharing it with me.

dan

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June 5

Dear Dan,

I enjoyed your letter and proof, but don't lay the problem to rest just yet. I can top your discoveries. I was browsing through some back issues of the College Math Journal when one of the articles caught my eye [6]. There, the special case of our identity corresponding to m = 2 is handled as a telescoping sum. Unlike our initial approach, where the telescope involves three term cancellations, in [6] there are only two terms. The example shown there generalizes in an obvious way to give this proof.

Observe that for any positive m,

$$\frac{1}{k(k+1)\cdots(k+m)} = \frac{1/m}{k(k+1)\cdots(k+m-1)} - \frac{1/m}{(k+1)(k+2)\cdots(k+m)}.$$

This clearly telescopes and you can see by inspection that the sum for k running from 1 to infinity is 1/(m m!) . That proves the identity!

This proof really brings us full circle, for it's an obvious and direct generalization of the telescoping series I started with. I just headed off in the wrong direction with that partial fractions stuff. You might say we looked the wrong way through the telescope!

On the other hand, as you said, it was lots of fun. And if I had done it right at the start, there would have been no partial fractions identity, no binomial identity, no need to use Mathematica, and you would have missed out on the joys of getting dusty up in the Santa Barbara library.

In fact, we had so much fun with the subject, I wonder if we ought to try and write it up somehow. Do you suppose there is a form we could put it in that would capture some of the experience of doing it? Do you suppose anyone would care to read about it?

John

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Dan Kalman was originally attracted to college mathematics by its lack of laboratories. He was an applied mathematician at the Aerospace Corporation and now teaches at the American University. John *Mathews* earned his doctorate at Michigan State University and has written texts on complex variables and numerical methods.

Chapter Reports

The NEW YORK OMEGA Chapter (St. Bonaventure University) had as its major activity, Professor **Francis Leary** reports, its popular Mathematics Forum. Fifteen talks were presented last year, mostly by students, including one on "The mathematics of coyotes, roadrunners, and ants". The Chapter's graduating vice-president, Heather Lecceardone, won the department's Mathematics Medal.

The MICHIGAN ZETA chapter (University of Michigan—Dearborn) cosponsored a student-faculty mixer which was attended by most of the faculty, nine alumni, and more than fifty students. **Professor John Frederick Fink** says that is the best attendance ever at such an event. The Chapter inducted eighteen new members last year.

The CONNECTICUT GAMMA Chaper (Fairfield University) sponsored its annual High School Math Bowl, similar to the College Bowl. Professor **Joan Wyzkoski Weiss** reports that eight teams from local high schools participated. At the spring initiation ceremony, nineteen new members were initiated and Carole Lacapagne of the U. S. Department of Education spoke on "The prime number connection: bow number theory helps secure vital data."

A WEIGHTED AM-GM-HM INEQUALITY

Ayoub *B.* Ayoub
Pennsylvania State *U., Ogontz Campus*

The familiar arithmetic mean-geometric mean inequality,

$$\frac{a+b}{2} \geq \sqrt{ab},$$

holds with more general weights **[1]:**

$$m_1 a + m_2 b \geq a^{m_1} b^{m_2}$$

where a, b > 0, $m_1 + m_2 = 1$, and $m_1, m_2 \geq 0$. We will modify this inequality to permit negative weights, then extend it to include the harmonic mean. To that end, we will first prove the weighted AM-GM inequality using the natural logarithm function. If we consider the points A: (a, ln a) and *B:* (b, ln b) on the graph of y = lnx, then the point C that divides *AB* in the ratio $m_2 : m_1$ will be

$$\left(\frac{m_1 a + m_2 b}{m_1 + m_2}, \frac{m_1 \ln a + m_2 \ln b}{m_1 + m_2} \right).$$

However, if $m_1 + m_2 = 1$, then C will have the coordinates

$$(m_1 a + m_2 b,\ m_1 \ln a + m_2 \ln b).$$

If, in addition, we assume that $m_1, m_2 > 0$, then C divides *AB* internally (see Figure 1). Since the graph of y = ln *x* is concave down ($y'' = -1/x^2 < 0$), the point D: $(m_1 a + m_2 b,\ \ln(m_1 a + m_2 b))$ lies vertically above C. It is obvious that C will coincide with *A* or *B* if $m_2 = 0$ or $m_1 = 0$, respectively. Thus, $m_1, m_2 \geq 0$ implies that

$$\ln(m_1 a + m_2 b) \geq m_1 \ln a + m_2 \ln b.$$

Since the function y = ln *x* is increasing ($y' = 1/x > 0$), then

$$m_1 a + m_2 b \geq a^{m_1} b^{m_2}.$$

One may ask, what if m_1 or m_2 is negative? In both cases C will divide AB externally and consequently C will be above D. See Figure 1, where C' and C'' illustrate this case. Thus,

$$\ln(m_1 a + m_2 b) < m_1 \ln a + m_2 \ln b.$$

But then,

$$m_1 a + m_2 b < a^{m_1} b^{m_2}$$

where $m_1 m_2 < 0$, $m_1 + m_2 = 1$, and a, b, $m_1 a + m_2 b > 0$. If, for example, $m_1 = 312$, $m_2 = -112$, a = 9, and b = 4, we get $2312 < 2712$. Now we may combine the three cases in which m_1 and m_2 are both positive, or one of them is zero, or one is positive and the other is negative, as follows:

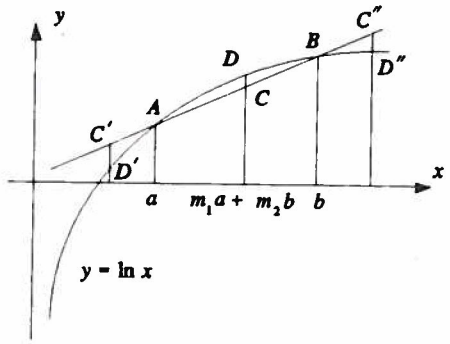

Figure 1

If a, b, $m_1 a + m_2 b > 0$ and $m_1 + m_2 = 1$ then

$$m_1 a + m_2 b \gtrless a^{m_1} b^{m_2} \text{ according to } m_1 m_2 \gtrless 0.$$

If we multiply this inequality by $a^{m_1} b^{m_2}/(m_1 a + m_2 b)$, it becomes

$$a^{m_1} b^{m_2} \gtrless \frac{a^{2m_1} b^{2m_2}}{m_1 a + m_2 b}.$$

Combining this inequality with the previous one, we get the weighted AM-GM-HM double inequality:

$$m_1 a + m_2 b \gtrless a^{m_1} b^{m_2} \gtrless \frac{a^{2m_1} b^{2m_2}}{m_1 a + m_2 b}$$

according to $m_1 m_2 \lessgtr 0$, where $m_1 + m_2 = 1$ and a, b, $m_1 a + m_2 b > 0$.

If we set $m_1 = m_2 = 1/2$, we get the standard AM-GM-HM inequality

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$

where a, b > 0.

Reference

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H R I E D T N R A Q U G E A Q H U D I S

K I E L S O D Q E C D U H R G O N R A O N O N U F O D

F L Z Q N I J L O G H ? I L H R I E D T U N

R O W A S I Q E B B D A H O N O D D ?

—Suggested by Robert C. Gebhardt

ON THE GIRTHS OF REGULAR PLANAR GRAPHS

Masakazu Nihei
Fujishiro High School

One of the most fascinating yet mysterious classes of graphs are the cages. We introduce a planar version, classify them. and use this information to present another proof of the fact that there are exactly five Platinic solids.

We begin with a few definitions. The degree of a vertex v in a graph G is the number of edges of G incident with v. A graph in which every vertex has the same degree is called a regular graph; if every vertex has degree k, the graph is called a k-regular graph. The cardinality of the vertex set of G is called the order of G and is denoted by p. while the cardinality of its edge set is the size of G and is denoted by q. The length of the shortest cycle in a graph G that contains cycles is called the girth of G and is denoted by $g(G)$ or g.

Let us consider the k-regular graphs with girth g. The minimal order of a k-regular graph with girth g is denoted by $f(k, g)$, and the k-regular graphs of girth g and order $f(k,g)$ are called (k, g)-*cages*. For example, $f(3,4) = 6$ and $f(3,5) = 10$. The $(3,4)$-cage and $(3, 5)$-cage are unique and nonplanar, [1, 236-2391, [2, 34-43]. They are shown below.

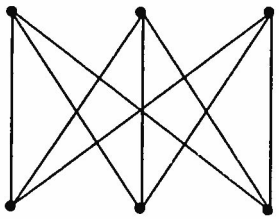

Fig. 1: $(3,4)$-cage

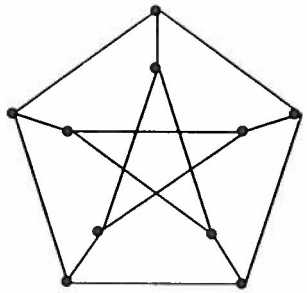

Fig. 2: $(3,5)$-cage

A planar. k-regular, graph of girth g of minimum order will be called a (k,g)-*page*. So every planar (k,g)-cage is a (k,g)-page. but the converse

is not true.

We will first determine the girths of all planar k-regular graphs for $k = 4, 5$.

THEOREM 1. If G is a connected planar 4-regular or 5-regular graph, then $g(G) = 3$.

Proof. Let p, q, and s denote the order, size, and number of faces of G. Then we have

(1) $$kp = 2q \quad (k = 4, 5)$$

and

(2) $$p - q + s = 2$$

by Euler's formula.

Let the distinct lengths of the boundaries of the faces of G be denoted by

$$g = g_0, g_1, \cdots, g_m \quad (g_0 \le g_i,\ i = 1, 2, \cdots, m).$$

Suppose that there are s_i faces with boundary of length g_i. Then we have

(3) $$\sum_{i=0}^{m} s_i g_i = 2q.$$

From (1) and (2) we have

(4) $$\sum_{i=0}^{m} s_i = 2 + q(k-2)/k,$$

and from (3) we also have

(5) $$0 < g \sum_{i=0}^{m} s_i \le \sum_{i=0}^{m} s_i g_i = 2q.$$

Hence we obtain

(6) $$g \le 2qk/(2 + q(k-2)) < 2qk/q(k-2)) = 2k/(k-2)$$

by (4) and (5).

If k is 4 or 5, then $2k/(k-2)$ is 4 or 1013, so we have $g \le 3$. On the other hand, it is clear that $g \ge 3$. This completes the proof.

The girth of a 3-regular graph need not be 3, however. In fact. the girths

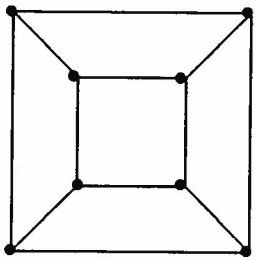

Fig. 3: Girth 4

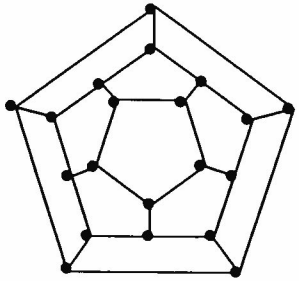

Fig. 4: Girth 5

of the graphs in Figures 3 and 4 are 4 and 5, respectively.

Since no planar k-regular graphs exist for $k \geq 6$, we need consider only $2 \leq k \leq 5$. If $k = 4$ or 5, then $g = 3$ by Theorem 1. When $k = 2$, a $(2, g)$-page is the cycle whose length is g. If the graph G is a $(5,3)$-page, then G must satisfy

$$p - q + s = 2, \quad 3s \leq 2q, \text{ and } 5p = 2q.$$

From this it is easy to see that the graph of Figure 5 is a $(5,3)$-page, and the graph of Figure 6 is a $(4,3)$-page.

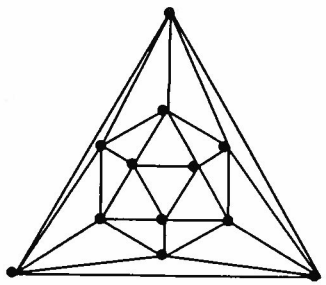

Fig. 5: A $(5, 3)$-page

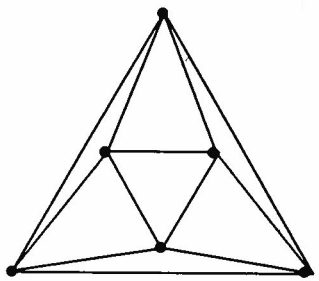

Fig. 6: A $(4,3)$-page

Therefore it remains only to investigate k = 3.

THEOREM 2. If $g(G) > 6$, then a $(3, g)$-page does not exist.

Proof. Let G be a $(3, g)$-page of order p and size q. The for a $(3, g)$-page. we have $g(q - p + 2) \leq 2q$ by Euler's formula. Hence we obtain

$$q \leq g(p - 2)/(g - 2). \tag{7}$$

Since G is a 3-regular graph, we also have

$$3p = 2q. \tag{8}$$

Hence, with (7) and (8) we have

$$p(g - 6) \leq -4g \tag{9}$$

Thus $g < 6$.

PROPOSITION. Let G_g be a $(3, g)$-page of order p_g and size q_g $(g = 3, 4, 5)$. Then $(p_3, q_3) = (4, 6)$, $(p_4, q_4) = (8, 12)$, and $(p_5, q_5) = (20, 30)$.

Proof. We may deal only with $g = 4$, since the other cases are similar. Putting $g = 4$ in (7), we have $q \leq 2(p - 2)$. Therefore we obtain $p \leq 8$ and $q_4 = 12$ by (8).

From this proposition it is easy to check that the graphs of Figures 7, 3, and 4 are a $(3,3)$-page, a $(3,4)$-page, and a $(3,5)$-page, respectively.

A regular polyhedron is a polyhedron whose faces are bounded by congruent regular polygons and whose polyhedral angles are congruent. Then every regular polyhedron P is associated with a regular connected planar graph $G(P)$ whose vertices and edges are the vertices and edges of P.

If $G(P)$ is a k-regular graph with girth g, then the order of $G(P)$ becomes minimal in such graphs since P is a regular polyhedron. So, $G(P)$ becomes a (k, g)-page.

When $g \geq 3$, it is clear that the number of different types of (k, g)-pages is only five by our previous results. This shows that the number of regular polyhedra is at most five. On the other hand, we can construct five regular polyhedra from the graphs of Figures 3-7 (the cube, dodecahedron, icosahedron, octahedron, and tetrahedron, respectively). We therefore have the well-known theorem

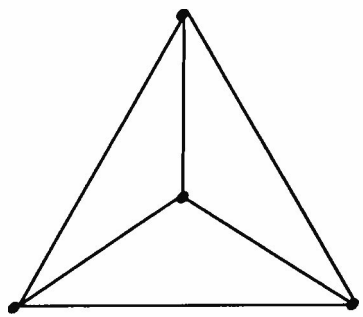

Fig. 7: A $(3, 3)$-page

Theorem 3. There are exactly five regular polyhedra

Acknowledgment: The author would like to thank the refcrce for helpful suggestions.

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2. Chartrand, G., and L. Lcsniak. *Graphs and Digraphs,* Wadsworth, 2nd cd 1986.

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Hurry! Time's A-wasting!

Ordinarily, only the physicist or the mathematician can hope to cntcr early middle age having niadc a scholarly mark; indccd. for such a scientist to glidc into the thirtics without distinction can bc causc for dcspair—or a job in university administration.

—David Remnick, The devil problem, *New Yorker* **71** (1995) #6 (April 13), 54-65, page 54.

The most common fcmalc first namc among subscribers to the *Journal* is Jennifer. The most common malc first namc is Michael Tlic most common last name is Smith, but neither Jennifer Smith nor Michael Smith is on the subscription rolls

PREDICTION IN INSURANCE

Mark Bonsall
Conrad M. Siegel, Inc.

Insurance companies are interested in predicting the future. Not for individual policyholders, but for groups of them, so that they can predict as accurately as possible the loss that will result from the risk they are insuring. To do their analyses, insurance actuaries use many applications of mathematics, sonie very advanced, but they also use simple regression, a technique studied in basic statistics classes. The purpose of this note is to give an example of its use, showing that there is an actual application of undergraduate mathematics outside the classroom.

In linear regression, we wish to find the equation of a line, $Y_{fit} = \alpha + \beta X$, which minimizes the sum of the squared deviations between the y-coordinates of the data points and the corresponding y-coordinates on the fitted curve, i. e.

$$SS = \sum (Y_{data} - Y_{fit})^2 = \sum (Y_{data} - \alpha - \beta X)^2 .$$

To do this, we calculate $\partial SS/\partial \alpha$ and $\partial SS/\partial \beta$, set them equal to zero, and solve for α and β to get

$$\beta = \frac{\Sigma XY - \bar{X}\Sigma Y}{\Sigma X^2 - \bar{X}\Sigma X}, \quad \alpha = \bar{Y} - \beta \bar{X}.$$

Table 1 gives actual data for a casualty insurance company for 1987-1993. The figures in the loss column are estimates of the total amount that will eventually be paid because of claims made in the year (the number of which appear in the third column). Exposures, the numbers in the second column, are, roughly, the number of items being insured. The frequency in the fourth column is the result of dividing the number of claims by the exposure. Severity, in the fifth column is the loss per claim, and the pure premium in the last column is the amount that each policyholder would have to pay to just cover the losses—loss divided by number of claims.

Ycar	Loss	Exposures	Claims	Severity	Pure Premium
1987	23,875,471	38,836	1,555	15,536	615
1988	22,951,051	35,965	1,566	14,657	638
1989	24,446,385	33,647	1,438	16,996	727
1990	29,372,493	32,673	1,453	20,218	899
1991	31,741,929	31,667	1,542	20,591	1,002
1992	32,032,910	30,470	1,507	21,620	1,051
1993	33,242,092	27,574	1,390	23,911	1,206

Table 1

Of course, the insurance conipany would like to bc able to extrapolate the pure premium into the future so as to bc able to set its rates appropriately. Applying linear regression, we would get the prediction in Tablc 2.

Year	88	89	90	91	92	93	94	95
Prcni.	673	767	867	972	1082	1198	1319	1445

Tablc 2

However, linear regression niay not bc the best model. If we assume that the premium tends to incrcasc a constant pcrccntagc each year, then the best fit would bc an exponential function,

$$Y_{fit} = \alpha e^{\beta X}$$

There is no difficulty in finding tlic parameters, since

$$\ln(Y_{fit}) = \beta X + \ln(\alpha),$$

which is a linear function $W = \theta + \beta X$, with $W = \ln(Y)$ and $\theta = \ln(\alpha)$. This gives tlic predictions in Tablc 3

Year	88	89	90	91	92	93	94	95
Prem.	671	756	852	960	1081	1218	1373	1547

Table 3

As could be expected the exponential fit gives larger predictions, since exponential curves bend upward while lines are straight. But they arc more likely to be accurate. Answering the question of how accurate would involve finding the variance of the predicted values. This would involve the variances of the observed values, which can only bc estiniated. This is one of the reasons that, in spite of all that actuaries and mathematics can do. the future can still be surprising.

Mark Bonsall wrote this paper while a student at Moravian College He is now an assistant actuary with a consulting firm in Harrisburg, Pennsylvania.

Chapter Reports

Profcssor Betty Mayfield reports that tlic speaker at tlic induction cercniony of the MARYLAND DELTA Chapter (Hood College) was Profcssor Cora Sadosky of Howard University, tlic immediate past president of the Association for Wonicn in Mathematics. She spoke, appropriately, on "Women in niathcniatics".

Profcssor Prcni N. Bajaj has retired as advisor to tlic KANSAS GAMMA Chapter (Wichita State University) reports tlic new advisor, Profcssor Andrew Acker. During Profcssor Bajaj's twelve-year tenure as advisor. 307 new members were initiated, 86 lectures or presentations were organized. 1 students gave papers at state or regional meetings, and 6 students presented papers at national Pi Mu Epsilon meetings. Profcssor Acker notes . chapter is going to miss his active involvement The same is true organization as a whole

AN APPLICATION OF PARTITIONS TO THE FACTORIZATION OF POLYNOMIALS OVER FINITE FIELDS

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University of Kentucky and Skidmore College

In this paper, partitions of natural numbers are used to count the irreducible polynomials of degree n over a finite field. This apparently little known application of partitions is described in detail in Section II. Section I is a brief introduction to partitions. These results were developed as part of the first author's senior mathematics thesis under the direction of the second author.

I) Introduction. Let n be a natural number. A partition of n is a finite set π of natural numbers (possibly with repetitions) whose sum is n. We sometimes write $\pi \vdash n$ to indicate that π is a partition of n. For example if $\pi = \{4, 3, 3, 1, 1, 1\}$, then $\pi \vdash 13$. Two partitions are considered equal if they have the same entries or *parts*, regardless of the order of those parts. For convenience, partitions are frequently written with their parts in nonincreasing order: $\pi = \{p_1, p_2, \ldots, p_m\}$ where $p_1 \geq p_2 \geq \cdots \geq p_m$, as in the above example.

For each i $(1 \leq i \leq n)$, the number of times i occurs as a part of the partition π is called the multiplicity of i in π, and is denoted by $m_i(\pi)$ or more simply by π_i (so π_i is the cardinality of $\{p_k \mid p_k = i\}$). This leads to an alternate notation for partitions where π is denoted by $[1^{\pi_1}, 2^{\pi_2}, \ldots, n^{\pi_n}]$ with entries of multiplicity zero omitted. Thus the above partition π of 13 can also be written as $[1^3, 3^2, 4]$, suppressing the superscripts equal to 1. We shall refer to the number of parts $\ell(u)$ of π as its length and the number of distinct parts $d(\pi)$ as its depth. In the above example $\pi = [1^3, 3^2, 4]$, we have $\ell(\pi) = 6$ and $d(\pi) = 3$. It is clear that the length of any partition is the sum of the multiplicities of its parts:

$$\ell(\pi) = \sum_{i=1}^{n} \pi_i \quad \text{if } \pi \vdash n. \tag{1}$$

Let P stand for the set of all partitions. Let $P_n = \{\gamma \in P \mid \gamma \vdash n\}$ be the set of partitions of n, so $P = \bigcup_{n=1}^{\infty} P_n$ (disjoint union). It will be convenient to allow the number 0 to be a part of a partition. In fact, although we will omit 0 when writing a partition, we will follow the convention of assuming that 0 is a part of any partition, of multiplicity 1, although it contributes nothing to the length or the depth of the partition. Thus $m_0(\pi) = \pi_0 = 1$ for all $\pi \in P_n$, while the sum in (1) is not adjusted to begin at i = 0. Furthermore, in order to treat the formulas appearing in Section II uniformly, it will also be convenient to assume there is precisely one partition of 0 (which has length 0 and depth 0), namely [0], which we adjoin to P.

The *partition function* is defined by $p(n) = |P_n|$ (cardinality of P_n). For example $p(5) = 7$, since

$$P_5 = \{[5], [1, 4], [2, 3], [1^2, 3], [1, 2^2], [1^3, 2], [1^5]\}.$$

One may compute $p(n)$ for small values of n by hand, although this process soon becomes tedious since $p(n)$ grows rather quickly. For instance, $p(10) = 42$, $p(20) = 627$, and $p(100) = 190{,}569{,}292$. While there is an exact formula for $p(n)$ (see [7]), it is rather complicated and many calculations of $p(n)$ rely instead on some kind of recursion.

The partition function and partitions in general have a long history, and they arise in many diverse situations. The grade school student may meet them in simple counting exercises such as "how many ways are there to make change for a dollar without pennies?", which is really the question of how many partitions of 100 are there with each part equal to 5, 10, 25, or 50. In a similar manner one can rephrase questions such as "how many different ways are there to roll a 12 with exactly three dice?" or "what is the largest number of Chicken McNuggets that you cannot order exactly if they come in packages of 6, 9, and 20?". The reader should have no trouble seeing that these questions and many others like them are really questions about counting partitions with certain restrictions on the parts. While these questions can be answered easily by ad hoc methods, a very satisfying

uniform method exists which is sometimes covered in an introductory course in discrete mathematics. Originally due to Euler, it is based on multiplying together certain power series (e.g., see [4], section 19.3) known as generating *functions*.

Like the above examples, many of the most interesting problems related to partitions involve counting partitions with restrictions on the parts. However, just as $p(n)$ itself is elusive, it is frequently impossible to do this directly with the added restrictions. As a consequence, many counting arguments focus on showing that one set of restricted partitions is equinumerous with a different set of restricted partitions without actually counting either set. For example, it is a well known result that the number of partitions of n into at most k parts is the same as the number of partitions of n into parts which are at most k. As a concrete example consider the case $n = 6$, k = 3. The partitions of 6 into parts which are at most 3 are

$$\{[1^6], [1^4, 2], [1^2, 2^2], [2^3], [1^3, 3], [1, 2, 3], [3^2]\}$$

and the partitions of 6 into at most 3 parts are

$$\{[6], [1, 5], [2, 4], [3^2], [1^2, 4], [1, 2, 3], [2^3]\}$$

There are the same number of partitions in each set, namely seven.

The interested reader will find an elegant proof of this assertion (and many others like it) using a graphical device known as the Ferrers diagram of a partition (invented by N. M. Ferrers and later popularized by J. J. Sylvester) in [1].

In addition to these counting problems there are many celebrated applications of partitions in other areas of mathematics. Two of our favorites in group theory are the connection between partitions of n and the conjugacy classes in the symmetric group S_n and the classification theorem of finite Abelian groups. Since these applications can be found in any good introduction to abstract algebra ([6], for example), we will not dwell on them. At a somewhat deeper level there are also some beautiful applications to the representation theory of S_n (see [8]). Partitions of a natural number can also be used to extend the chain rule of calculus to higher derivatives, leading to the well-known Bell polynomials (see [3]), among other things. The interested reader can find some applications of this in [12]. We will

presently describe an application of partitions to undergraduate mathematics which does not seem to be as widely known.

11) Counting Irreducible **Polynomials** Over Finite Fields.

Let F be a field and let $F[X]$ be the ring of polynomials with coefficients in F. Then $F[X]$ is a unique factorization domain, and the nonzero constant polynomials form the group of units (see [5], [6], or [10]). Thus, every polynomial of degree 1 or larger factors into a product of a nonzero constant and monic irreducible polynomials in a unique way (up to the order of the factors), where *monic* means the leading coefficient is 1. Thus in many ways this ring behaves like the ring Z of integers. In particular, the (monic) irreducible polynomials play the role of the (positive) prime numbers in Z.

Therefore, one could ask questions about the distribution of irreducible monic polynomials which are analogous to questions about the distribution of primes in Z. Because $F[X]$ is an integral domain, the degree of $p(X)q(X)$ is the sum of the degrees of $p(X)$ and $q(X)$. By induction on m, the following is true:

$$\deg\left(\prod_{i=1}^{m} p_i(X)\right) = \sum_{i=1}^{m} \deg(p_i(X)). \tag{2}$$

It follows that every (monic) polynomial of degree one is irreducible. Such a polynomial has the form $X + a$ for some $a \in F$, so if F is infinite, the ring $F[X]$ has an infinite number of irreducible polynomials just as Z has an infinite number of primes. (In case F is algebraically closed, these are the only monic irreducible polynomials.) It turns out that even if F is a finite field, there are still infinitely many irreducible polynomials (see page 274 of [5] for the case $F = Z_p$, the field of integers modulo the prime p), although there can be only a finite number of any fixed degree n. This raises the question of finding the number of irreducible polynomials of each degree in case F is finite.

Let $N_F(n)$ stand for the number of irreducible monic polynomials of degree n over the field F. In case F is a finite field with q elements (where $q = p^r$, p is the characteristic of F), we will also use the standard notation $N_q(n)$. Our method for computing $N_F(n)$ is a recursive method based on (2). It is a generalization of exercise C, page 255 of [10], except that there

it is only carried out for n = 2 and n = 3. When this is attempted for larger values of n, partitions enter the picture. Indeed, if $p(X)$ is a monic polynomial of degree n, it factors uniquely as a product of monic irreducible polynomials $p(X) = \Pi_{i=1}^{m} p_i(X)$. So if we let $d_i = \deg(p_i(X))$, then (2) implies that the set of degrees $\{d_1, d_2, \cdots, d_m\}$ is a partition of n. Furthermore, $p(X)$ is itself irreducible only if the partition so obtained is [n], otherwise each $d_i < n$. A typical monic polynomial of degree n has the form

$$X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_1X + a_0,$$

with the a_i's in F. Let F be finite from now on, with $q = |F|$. Now there are exactly q choices for each of the n a_i's, so there are q^n **monic** polynomials of degree n altogether. Our goal is to count the number of reducible polynomials of degree n (provided $N_F(d_i)$ for $d_i < n$ has already been computed), and subtract this number from q^n to find $N_F(d_i)$.

First we consider some preliminary information regarding partitions. We remind the reader that the conventions about 0 from Section I are still in effect. We now introduce an operation on P. Given a partition $\pi \vdash n$, recall that the length $\ell(\pi)$ is the total number of (nonzero) parts and the depth $d(\pi)$ is the number of distinct (nonzero) parts. From (1) it follows that the set of multiplicities $\delta(\pi) = \{\pi_1, \pi_2, \cdots, \pi_n\}$ is a partition of $I(\pi)$, which we call the derived partition of π. Observe that $\delta(\pi)$ has length equal to the depth of π (many of the multiplicities π_i are 0). For example, if $\pi = [1^3, 2, 4^2, 5^3]$, then $\pi \vdash 28$ and there are 9 parts, but only 4 of them are distinct, so $I(\pi) = 9$ and $d(\pi) = 4$. The multiplicities are {3, **1, 0,** 2, 3}, leading to $\delta(\pi) = [1, 2, 3^2]$, a partition of 9 of length 4. We further illustrate this with the table on the next page.

Next, suppose that $\pi \in P_n$; $\pi = \{p_1, p_2, \cdots, p_m\}$. Then we remind the reader that the expression $n! \Big/ \Pi_{i=1}^{m} p_i!$ is an integer, known as a multinomial coefficient, and often written as $\begin{bmatrix} n \\ p_1, p_2, \cdots, p_m \end{bmatrix}$. We will further abbreviate this to simply $\binom{n}{\pi}$, writing the name of the partition on the bottom. When m = 2, we will follow the usual convention of writing the binomial coefficient as $\begin{bmatrix} n \\ p_i \end{bmatrix}$ rather than $\begin{bmatrix} n \\ p_1\ p_2 \end{bmatrix}$ or $\binom{n}{\pi}$.

Derived partitions for P_4

π	π_i i: 1,2,3,4	$\ell(\pi)$	$\delta(\pi)$	$d(\pi) = \ell(\delta(\pi))$
4	0,0,0,1	1	1	1
3,1	1,0,1,0	2	1, 1	2
2,2	0,2,0,0	2	2	1
2,1,1	2,1,0,0	3	2, 1	2
1,1,1,1	4,0,0,0	4	4	1

We are now prepared to prove the following:

THEOREM 1: Let F be a finite field with $|F| = q$. Then the total number of monic polynomials of degree n with coefficients in F is given by:

$$(3) \qquad q^n = \sum_{\pi \in P_n} \left[\prod_{j=1}^{n} \left[\sum_{\beta \in P_{\pi_j}} \begin{bmatrix} N_q(j) \\ \ell(\beta) \end{bmatrix} \cdot \begin{bmatrix} \ell(\beta) \\ \delta(\beta) \end{bmatrix} \right] \right]$$

Proof: We have observed above that the left side of (3) is correct. We now count the monic polynomials a different way to see that the right side is also correct. Let $p(X)$ be one, and let $p(X) = \Pi_{i=1}^{m} p_i(X)$ be its factorization into irreducible monic polynomials. Let π be the degree set $\{d_1, d_2, \cdots, d_m\}$ (listed with multiplicities). Since $F[X]$ is a commutative ring, the d_i's may be listed in any order, so as noted above, π belongs to P_n. Let j be an integer with $1 \le j \le n$. So there are π_j of the irreducible factors $p_i(X)$ with degree j. If N_j represents the number of ways of choosing these factors of degree j, then the multiplication principle of counting yields that there are $\Pi_{j=1}^{n} N_j$ ways of choosing all the factors together, so this accounts for the product in (3).

It remains to show that

$$N_j = \sum_{\beta \in P_{\pi_j}} \begin{bmatrix} N_q(j) \\ \ell(\beta) \end{bmatrix} \cdot \binom{\ell(\beta)}{\delta(\beta)}.$$

First observe that for a given value of j, it may be the case that none of the $p_i(X)$'s have degree j, so that particular factor N_j should have a value of 1 But if there are no factors of degree j, then $\pi_j = 0$ and so the innermost sum runs over the index set $P_0 = \{[0]\}$. Thus there is only one summand corresponding to the partition [0] of 0. By our conventions about zeros, both the length and the depth of the partition [0] are 0, so that both the binomial and the multinomial coefficient have the value 1, as desired.

So now consider the case where $\pi_j > 0$, so there is at least one of the $p_i(X)$'s with degree j. Now there are exactly $N_q(j)$ monic irreducible polynomials of degree j to choose from, and we must choose exactly π_j of them, possibly with repetitions, for the $p_i(X)$'s of degree j. The possibility of repetitions complicates matters, so we break the problem into two steps. First, select the distinct factors, and second select their multiplicities to add up to π_j, the total number of factors of degree j. Since their multiplicities add up to π_j (and again the order of the factors is irrelevant), the set of such multiplicities $\beta = \{\mu_1, \mu_2, \cdots, \mu_r\}$ forms a partition of π_j of length $\ell(\beta) = d$ equal to the number of distinct factors. Conversely, every partition of π_j accounts for a possible set of multiplicities for the factors of degree j. This is why the innermost sum runs over P_{π_j}.

Now given a partition β of π_j, since $d = \ell(\beta)$ is the number of distinct factors, the binomial coefficient $\begin{bmatrix} N_q(j) \\ d \end{bmatrix}$ counts all the possible sets of distinct factors from among the $N_q(j)$ which are available. For each such set, the multinomial coefficient $\begin{pmatrix} d \\ \beta_1 \beta_2 \cdots \beta_{\pi_j} \end{pmatrix}$ counts the number of ways of assigning the given multiplicities to that particular set of factors, where β_k is the multiplicity of k as a part of β. (That is, β_k is the number of distinct factors (of degree j) which have multiplicity k.) Rut observe that the set $\{\beta_1, \beta_2, \cdots, \beta_{\pi_j}\}$ is nothing more than the derived partition $\delta(\beta)$ of 13. Thus N_j has the desired form and this completes the proof of the theorem.

COROLLARY 1: Let F be a finite field with $|F| = q$. Then the number

$N_q(n)$ of monic irreducible polynomials of degree n can be computed recursively from the formula

$$(4) \quad N_q(n) = q^n - \sum_{\substack{\pi \in P_n \\ \ell(\pi) > 1}} \left[\prod_{j=1}^{n} \left[\sum_{\beta \in P_{\pi_j}} \begin{bmatrix} N_q(j) \\ \ell(\beta) \end{bmatrix} \cdot \binom{\ell(\beta)}{\delta(\beta)} \right] \right].$$

Proof. Formula (4) follows immediately from (3) because the only partition in P_n of length 1 is [n], which corresponds to the case of $p(X)$ being irreducible in Theorem 1. It is recursive because $\ell(v) > 1$ implies that $\pi_n = 0$, whence the only $N_q(j)$'s which appear in the right hand side are those for which j < n.

Some examples may help to clarify what we have done. First, we show a specific example of the counting technique used in the proof of Theorem 1. 9Suppose that $p(X)$ is a degree 24 polynomial, with factorization $P(X) = \prod_{i=1}^{\prime} p_i(x)$, with one linear factor, three quadratic factors, four cubic factors, and one quintic factor. This corresponds to the partition $\pi = [1, 2^3, 3^4, 5]$ of 24 in the outermost sum of (3). There are 24 factors N_j in the product, but since $\pi_4 = 0$ and $\pi_j = 0$ for $6 \le j \le 24$, most of these factors have the value 1. By definition there are $N_q(1) = q$ ways to choose the linear factor and $N_q(5)$ ways to choose the quintic factor. Consider next the quadratic factors. Since $\pi_2 = 3$, the index set for the innermost sum of (3) is $P_3 = \{[1^3], [1, 2], [3]\}$.

The partition $\beta = [1^3]$ corresponds to choosing three distinct (quadratic) factors, since $\ell(\beta) = 3$. Since there are only three quadratic factors altogether, each of these three necessarily occurs with multiplicity 1 (the parts of β), so there is only one way to assign these multiplicities. Observe that $\delta([1^3])$ is the partition [3], since $\beta_1 = 3$, $\beta_2 = 0$, $\beta_3 = 0$. Thus the number of ways to choose 3 distinct quadratic factors is

$$\begin{bmatrix} N_q(2) \\ \ell(\beta) \end{bmatrix} \cdot \binom{\ell(\beta)}{\delta(\beta)} = \begin{bmatrix} N_q(2) \\ 3 \end{bmatrix} \cdot \binom{3}{3, 0, 0} = \begin{bmatrix} N_q(2) \\ 3 \end{bmatrix} \cdot 1.$$

The partition $\beta = [1, 2]$ corresponds to choosing two distinct (quadratic) factors, since $\ell(\beta) = 2$. One of them will have multiplicity 1 and one will have multiplicity 2 (the parts of β), and clearly there are exactly two ways to make that assignment. Observe that the derived

partition in this case is $\delta(\beta) = [1^2]$, since $\beta_1 = \beta_2 = 1$ and $\beta_3 = 0$. Thus the number of ways to choose two distinct factors with one repeated twice is

$$\begin{bmatrix} N_q(2) \\ \ell(\beta) \end{bmatrix} \cdot \binom{\ell(\beta)}{\delta(\beta)} = \begin{bmatrix} N_q(2) \\ 2 \end{bmatrix} \cdot \binom{2}{1,1,0} = \begin{bmatrix} N_q(2) \\ 2 \end{bmatrix} \cdot 2.$$

Finally the partition [3] of length 1 corresponds to choosing exactly one (quadratic) factor and using it three times (there is only one way to assign the multiplicity). In this case the derived partition is [1] as $\beta_1 = \beta_2 = 0$ and $\beta_3 = 1$. Thus

$$\begin{bmatrix} N_q(2) \\ \ell(\beta) \end{bmatrix} \cdot \binom{\ell(\beta)}{\delta(\beta)} = \begin{bmatrix} N_q(2) \\ 1 \end{bmatrix} \cdot \binom{1}{0,0,1} = \begin{bmatrix} N_q(2) \\ 1 \end{bmatrix} \cdot 1.$$

This shows that in this case

$$N_2 = \begin{bmatrix} N_q(2) \\ 3 \end{bmatrix} \cdot 1 + \begin{bmatrix} N_q(2) \\ 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} N_q(2) \\ 1 \end{bmatrix} \cdot 1.$$

Similarly, for the $\pi_3 = 4$ cubic factors, one may compute N_3 as a sum over P_4. Using the table above of the derived partitions for all $\pi \vdash 4$, one obtains $N_3 =$

$$\begin{bmatrix} N_q(3) \\ 4 \end{bmatrix} \cdot 1 + \begin{bmatrix} N_q(3) \\ 3 \end{bmatrix} \cdot 3 + \begin{bmatrix} N_q(3) \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} N_q(3) \\ 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} N_q(3) \\ 1 \end{bmatrix} \cdot 1.$$

Therefore, the total number of degree 24 monic polynomials which factor into monic irreducible polynomials with degrees corresponding to the partition $\pi = [1, 2^3, 3^4, 5]$ is

$$\prod_{j=1}^{24} N_j = N_1 \cdot N_2 \cdot N_3 \cdot 1 \cdot N_5 \cdot \cdots \cdot 1 =$$

$$\begin{bmatrix} N_q(1) \\ 1 \end{bmatrix} \cdot \left[\begin{bmatrix} N_q(2) \\ 3 \end{bmatrix} + \begin{bmatrix} N_q(2) \\ 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} N_q(2) \\ 1 \end{bmatrix} \right] \cdot$$

$$\left[\left[\begin{bmatrix} N_q(3) \\ 4 \end{bmatrix} + \begin{bmatrix} N_q(3) \\ 3 \end{bmatrix} \cdot 3 + \begin{bmatrix} N_q(3) \\ 2 \end{bmatrix} \cdot 3 + \begin{bmatrix} N_q(3) \\ 1 \end{bmatrix} \right] \cdot \begin{bmatrix} N_q(5) \\ 1 \end{bmatrix} \right.$$

Of course, there are many other partitions of 24 to consider! Let us close

with an example of the corollary. Let $F = Z_p$ so that $q = p$, and let $n = 5$. Then (3) becomes

$$p^5 = \begin{bmatrix} N_p(5) \\ 1 \end{bmatrix} + \begin{bmatrix} N_p(4) \\ 1 \end{bmatrix} \cdot \begin{bmatrix} N_p(1) \\ 1 \end{bmatrix} +$$

$$\begin{bmatrix} N_p(3) \\ 1 \end{bmatrix} \cdot \begin{bmatrix} N_p(2) \\ 1 \end{bmatrix} +$$

$$\begin{bmatrix} N_p(3) \\ 1 \end{bmatrix} \cdot \left[\begin{bmatrix} N_p(1) \\ 1 \end{bmatrix} + \begin{bmatrix} N_p(1) \\ 2 \end{bmatrix} \right] +$$

$$\left[\begin{bmatrix} N_p(2) \\ 1 \end{bmatrix} + \begin{bmatrix} N_p(2) \\ 2 \end{bmatrix} \right] \cdot \begin{bmatrix} N_p(1) \\ 1 \end{bmatrix} +$$

$$\begin{bmatrix} N_p(2) \\ 1 \end{bmatrix} \cdot \left[\begin{bmatrix} N_p(1) \\ 3 \end{bmatrix} + \begin{bmatrix} N_p(1) \\ 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} N_p(1) \\ 1 \end{bmatrix} \right] +$$

$$\begin{bmatrix} N_p(1) \\ 5 \end{bmatrix} + \begin{bmatrix} N_p(1) \\ 4 \end{bmatrix} \cdot 4 + \begin{bmatrix} N_p(1) \\ 3 \end{bmatrix} \cdot 3 + \begin{bmatrix} N_p(1) \\ 3 \end{bmatrix} \cdot 3$$

$$\begin{bmatrix} N_p(1) \\ 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} N_p(1) \\ 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} N_p(1) \\ 1 \end{bmatrix}.$$

Notice that the number $N_p(5)$ occurs only in the first term, which corresponds to the sole partition [5] of length 1 in P_5. Specialize to the case $p = 3$. Then $N_3(1) = 3$, $N_3(2) = 3$, $N_3(3) = 8$, and $N_3(4) = 18$, as can be verified by using (3). Alternatively, one could look them up in a table (such as what appears in [9]), or use the approach outlined in [11]. Then the above expression for p^5 simplifies to

$$3^5 = \begin{bmatrix} N_3(5) \\ 1 \end{bmatrix} + \binom{18}{1} \cdot \binom{3}{1} + \binom{8}{1} \cdot \binom{3}{1} +$$

$$\binom{8}{1} \cdot \left[\binom{3}{1} + \binom{3}{2} \right]$$

$$+ \left[\binom{3}{1} + \binom{3}{2} \right] \cdot \binom{3}{1} + \binom{3}{1} \cdot \left[\binom{3}{3} + \binom{3}{2} \cdot 2 + \binom{3}{1} \right]$$

$$+ \binom{3}{5} + \binom{3}{4} \cdot 4 + \binom{3}{3} \cdot 3 + \binom{3}{3} \cdot 3 + \binom{3}{2} \cdot 2 + \binom{3}{2} \cdot 2 + \binom{3}{1}.$$

or, evaluating the binomial coefficients,

$243 = N_3(5) + 54 + 24 + 48 + 18 + 30 + 0 + 0 + 3 + 3 + 6 + 6 + 3$, which yields $N_3(5) = 243 - 195 = 48$. This checks with Table C, page 555 of [9].

Readers familiar with combinatorial arguments may have noticed a somewhat more direct way to compute N_j. Indeed, the binomial coefficient $\left[\begin{matrix} n+s-1 \\ n \end{matrix}\right]$ counts "combinations with repetition", i. e., it counts the number of ways of selecting (when order is irrelevant) n objects from a set of s objects where there is no restriction on the number of times a particular object may be repeated in the selection (see theorem 4.2 of [4], for example).

Now, there are precisely $N_q(j)$ factors of degree j available, and π_j of them must be selected (with possible repetitions, and without regard for order), so $N_j = \left[\begin{matrix} N_q(j) + \pi_j - 1 \\ \pi_j \end{matrix}\right]$. Thus, we have the following simplified version of (3)

$$(5) \qquad q^n = \sum_{\pi \in P_n} \left[\prod_{j=1}^{n} \left[\begin{matrix} N_q(j) + \pi_j - 1 \\ \pi_j \end{matrix}\right] \right].$$

Applying (5) to the case $p = q = 3$; $n = 5$ yields

$$3^5 = \left[\begin{matrix} N_3(5) \\ 1 \end{matrix}\right] + \left[\begin{matrix} N_3(4) \\ 1 \end{matrix}\right] \cdot \left[\begin{matrix} N_3(1) \\ 1 \end{matrix}\right] + \left[\begin{matrix} N_3(3) \\ 1 \end{matrix}\right] \cdot \left[\begin{matrix} N_3(2) \\ 1 \end{matrix}\right] + \left[\begin{matrix} N_3(3) \\ 1 \end{matrix}\right] \cdot \left[\begin{matrix} N_3(1)+1 \\ 1 \end{matrix}\right] + \left[\begin{matrix} N_3(2)+1 \\ 2 \end{matrix}\right] \cdot \left[\begin{matrix} N_3(1) \\ 1 \end{matrix}\right] + \left[\begin{matrix} N_3(2) \\ 1 \end{matrix}\right] \cdot \left[\begin{matrix} N_3(1)+2 \\ 3 \end{matrix}\right] + \left[\begin{matrix} N_3(1)+4 \\ 5 \end{matrix}\right]$$

or, evaluating the binomial coefficients

$$243 = N_3(5) + 54 + 24 + 48 + 18 + 30 + 21,$$

which of course leads to the same value 48 for $N_3(5)$ computed above but is somewhat less tedious. While (5) is clearly more efficient than (3), the reason for giving both versions is to point out the connections with [2]. Indeed, one might distinguish the cases of factoring a quartic polynomial into either two distinct quadratics or one quadratic factor which is repeated by the notation "22" vs. "2^2". With this notation, it is clear that the 17 terms in the sum above obtained from (3) correspond to the 17 "factorization patterns" of 5: 5, 41, 32, 31^2, 311, 2^21, 221, 2111, 21^21, 21^3, 11111, 1^2111, 1^311, 1^21^21, 1^41, 1^31^2, 1^5. Thus, (3) will always reduce to a sum over the factorization patterns of n, while (5) will be a sum over the partitions of n. In [2], it is shown that the number of factorization patterns of n can be obtained by counting the partitions with "$d(a)$ copies of a". From our results, it is clear that each summand in (3) corresponds to a different factorization pattern of n, so it is immediate that one may obtain the number of factorization patterns of n by replacing each summand of the form

$$\left[\begin{matrix} N_q(j) \\ \ell(\beta) \end{matrix}\right] \cdot \left(\begin{matrix} \ell(\beta) \\ \delta(\beta) \end{matrix}\right)$$

by a 1. We obtain

COROLLARY 2 ([2], Lemma 2.1): Let $F(n)$ stand for the number of factorization patterns of n. Then

$$F(n) = \sum_{\pi \in P_n} \prod_{j=1}^{n} \left[\sum_{\beta \in P_{\pi_j}} 1 \right] = \sum_{\pi \in P_n} \prod_{j=1}^{n} p(\pi_j).$$

This illustrates the recursive approach to computing $N_q(n)$ for any finite field F. However, in case $F = Z_p$, it certainly is not the quickest approach to this problem. The approach in [11] or [9, p. 91-93] based on Möbius inversion is much more efficient. Nevertheless, we hope this simple application conveys to the reader something of the ubiquity and the beauty of partitions of natural numbers. The reader with a further interest in the subject of partitions might consult [1] for more information.

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A CHARACTERIZATION OF QUADRATFREI LUCAS PSEUDOPRIMES

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Consider the following property of certain positive integers n:

(1) $$L_n \equiv 1 \pmod{n},$$

where $\{L_n\}$ is the sequence of Lucas numbers, $L_{n+1} = L_n + L_{n-1}$, $L_0 = 2$, $L_1 = 1$. It is well-known that (1) is satisfied for all prime n. If (1) is satisfied for some composite n, then n is called a Lucas *pseudoprime* (or LPP). Let V denote the set of LPP's.

Some of the known properties of the LPP's have been discussed in [1]-[4]. Among them are that all LPP's are odd (the smallest is 705), there are infinitely many, and all known LPP's are square-free, or *quadratfrei* (q. *f.*). P. Filipponi has compiled a list of the 4438 LPP's less than 2^{32} (without factorizations) and one of the 852 LPP's less than 10^8 with factorizations, all q. f. ([5], [6]). The author acknowledges his debt to Filipponi for graciously making these tables available.

On the basis of the admittedly skimpy numerical evidence of the tables. it is tempting to make the following conjecture:

CONJECTURE 1: All Lucas pseudoprimes are quadratfrei.

The author has shown [4] that Conjecture 1 is equivalent to

CONJECTURE 2: $Z(p^2) = pZ(p)$ for all primes p that divide some LPP.

Here $Z(p)$ is the Fibonacci entry-point of p, that is, the smallest positive integer m such that $p \mid F_m$. (F_m denotes the mth Fibonacci number.)

It seems very likely (see [4]) that all primes p divide some LPP, but this has not been established. The validity of this assertion would allow us to replace Conjecture 2 by the stronger statement

CONJECTURE 3: $Z(p^2) = pZ(p)$ for all primes p.

In correspondence [7], J. **Lagarias** expressed his doubts that Conjecture 3 is valid, even though it has been verified for all $p < 10^9$ by H. C. Williams **[8].** Using a heuristic argument, **Lagarias** surmises that the number of $p \leq x$ such that $Z(p^2) = Z(p)$ (such being the negation of Conjecture 3) is $O(\log\log x)$. If **Lagarias** is correct, this would of course invalidate all three conjectures, under the assumption that every p divides some LPP. Until this question is resolved, we must allow for the possibility that there may exist **LPP's** that are not q. f.

The aim of this paper is to characterize q. f. **LPP's** in a manner distinct from the definition in (1). As we will see, our characterization will facilitate numerical computations since it involves much smaller numbers than those involved in (1). We require a preliminary definition, along with some relevant results. As they are easily derived (or found elsewhere in the literature), they are given here without proof.

DEFINITION: Given an integer $m > 1$, the *Lucas* period (mod m), denoted by $\bar{k}(m)$, is the smallest positive integer e such that $L_{j+e} \equiv L_j$ (mod m) for all integers **j.**

PROPERTY 1: $\bar{k}(m) = \operatorname{lcm}\{\bar{k}(p^e) : p^e \| m\}$.

PROPERTY 2: $\bar{k}(m)$ is even for all $m > 2$; $\bar{k}(2) = 3$, $\bar{k}(5) = 4$.

LEMMA 1: $\bar{k}(m)$ is the smallest positive integer e such that $\alpha^e \equiv 1$ (mod m), where $\alpha = \frac{1}{2}(1 + \sqrt{5})$. That is, $\bar{k}(m) = \operatorname{ord}_m \alpha$.

LEMMA 2: If n is odd and p a prime # 2, 5, then $L_n \equiv 1$ (mod p) iff either

(a) $\alpha^{n-1} \equiv 1$ (modp) or (b) $\alpha^{n+1} \equiv -1$ (modp).

The next result is very important, and we consequently elevate it to the status of a theorem.

THEOREM 1: If $n \in V$, then for all $p \mid n$ either

(a*) $n \equiv 1 \pmod{\bar{k}(p)}$ or (b*) $n \equiv \frac{1}{2}\bar{k}(p) - 1 \pmod{\bar{k}(p)}$.

Proof: Suppose $n \in V$. Then $L_n \equiv 1$ (mod n), and so $L_n \equiv 1$ (mod p) for all $p \mid n$. If p # 5, the conclusion of Lemma 2 applies and there are two cases. If part (a) holds, then $\bar{k}(p) \mid n - 1$ by Lemma 1. Hence $n \equiv 1 \pmod{\bar{k}(p)}$, which is part (a*). If part (b) holds, then $\alpha^{n+1} \equiv -1$ (mod p) and so $\alpha^{2n+2} \equiv 1$ (mod p). Using Lemma 1, this implies $\bar{k}(p) \mid 2n + 2$ but $\bar{k}(p) \nmid n + 1$. Then $2n + 2 = (2s + 1)\bar{k}(p)$ for some integer s, or

$$n = s\bar{k}(p) + \tfrac{1}{2}\bar{k}(p) - 1.$$

Then

$$n \equiv \tfrac{1}{2}\bar{k}(p) - 1 \pmod{\bar{k}(p)},$$

which is part (b*). If $5 \mid n$, then $L_n \equiv 1$ (mod 5), which implies $n \equiv 1$ (mod 4), i. e., $n \equiv 1 \pmod{\bar{k}(p)}$. This completes the proof.

We now give our characterization of q. f. **LPP's,** which is our main theorem.

THEOREM 2: n is a q. f. LPP if and only if n is odd, composite, q. f. and, for all $p \mid n$, either

(a') $n \equiv 1 \pmod{\bar{k}(p)}$ or (b') $n \equiv \frac{1}{2}\bar{k}(p) - 1 \pmod{\bar{k}(p)}$.

Proof: If n is a q. f. LPP, then n is odd, composite, q. f., and the conclusions in (a') or (b') follow from Theorem 1. Thus, it remains only to show that if n is odd, composite, q. f. and if either (a') or (b') holds, then n is an LPP. There are two cases. In the first, if $p \mid n$ and $n \equiv 1 \pmod{\bar{k}(p)}$, then $L_n \equiv L_1 \equiv 1$ (mod p). In the second, if $p \mid n$ and $n \equiv \frac{1}{2}\bar{k}(p) - 1 \pmod{\bar{k}(p)}$, then $n + 1 = \frac{1}{2}r\bar{k}(p)$ for some odd integer r. Then

$$\alpha^{2n+2} \equiv \alpha^{r\bar{k}(p)} \equiv 1 \pmod{p},$$

using Lemma 1. Therefore, $\alpha^{n+1} \equiv \pm 1$ (mod p). Since $\bar{k}(p) \nmid n + 1$, we have $\alpha^{n+1} \not\equiv 1$ (mod p), which implies $\alpha^{n+1} \equiv -1 \pmod{p}$. Then $\alpha^n \equiv \beta$ (mod p), where $\beta = \frac{1}{2}(1 - \sqrt{5})$; likewise, $\beta^n \equiv \alpha \pmod{p}$. This implies that

$$L_n = \alpha^n + \beta^n \equiv \alpha + \beta = 1 \text{ (modp)}.$$

In either case, $L_n \equiv 1 \pmod{p}$ for all $p \mid n$. Since n is q. f., it follows that $L_n \equiv 1$ (mod n). Since n is composite, therefore it is a LPP. This completes the proof.

It is of interest to derive necessary conditions for $n \in V$ that are independent of the assumption that n be q. f. Suppose $n \in V$ and $p \nmid n$. Then either (a*) or (b*) of Theorem 1 holds. If (a*) holds, then clearly $n^2 \equiv 1 \pmod{\bar{k}(p)}$ for all such p. If (b*) holds, note that $\frac{1}{2}\bar{k}(p)$ 1

must be odd, since n is odd and $\bar{k}(p)$ is even. Thus $4 \mid \bar{k}(p)$. Squaring both sides of (b*) we obtain

$$n^2 \equiv \tfrac{1}{4}\left(\bar{k}(p)\right)^2 - \bar{k}(p) + 1 \pmod{\bar{k}(p)}$$

or $n^2 \equiv 1 \bmod \bar{k}(p))$ for all $p \mid n$. Our conclusion is the following corollary of Theorem 1:

COROLLARY 1: If $n \in V$, then $n^2 \equiv 1 \pmod{\bar{k}(p)}$ for all $p \mid n$.

Using Property 1 of the Lucas period, we obtain similarly the following corollary of Theorem 2:

COROLLARY 2: If $n \in V$ and n is q. f., then $n^2 \equiv 1 \pmod{\bar{k}(n)}$.

Unfortunately, the converse of either corollary is false. If n is composite and if $n^2 \equiv 1 \pmod{\bar{k}(p)}$ for all $p \mid n$, it is not necessarily true that $n \in V$. The counterexamples less than 500 are n = 15, 105, 161, 195, **231,** 323, **341,** 377, 435, and 451; since all these are q. f. they are counterexamples of either corollary.

The hypothesis $n^2 \equiv 1 \pmod{\bar{k}(p)}$ does imply the weaker conclusion

$$L_{n^2} \equiv 1 \pmod{p} \text{ for all } p \mid n.$$

If in addition we restrict n to be q. f. this implies only $L_{n^2} \equiv 1 \pmod{n}$.

In conclusion, Theorem 2 could have some usefulness in testing for $n \in V$, provided n is q. f. and its factorization is known. Until Conjecture 1 is disposed of, we cannot completely characterize **LPP's,** but only q. f. **LPP's.** A priori, it might be the case that $L_n \equiv 1 \pmod{p}$ but $L_n \not\equiv 1 \pmod{p^2}$ for some p with $p^2 \mid n$, so that $n \notin V$.

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The Public Perception of Mathematicians

Mathematics is at the heart of the sciences. All of them require mathematical formulas to express their various truths. As the saying goes, the physicists defer only to the mathematicians, and the mathematicians defer only to God. (Though one would be hard-pressed to find a mathematician that modest.)

—Dick Teresi, review of Science Matters by R. M. **Hazen** and James Trefil, New York Times Book Review, February 3, 1991, pp. 7, 9.

ENUMERATING PARTITIONS

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Let $I(m)$ (where m is a positive integer) denote the number of partitions having m parts in which the kth part is less than or equal to $m - k + 1$. For example,

$I(1) = 1$: 1
$I(2) = 2$: 11, 21
$I(3) = 5$: 111, 211, 221, 311, 321
$I(4) = 14$: 1111, 2111, 2211, 2221, 3111, 3211, 3221, 3311,3321,4111,4211,4221,4311,4321.

In this note we will show that

$$I(m) = \frac{\binom{2m}{m}}{m+1}.$$

the well-known Catalan numbers.

Let $I_k(m)$ denote the number of partitions counted in $I(m)$ having largest part k. For example,

$$I_1(4) = 1,\ I_2(4) = 3,\ I_3(4) = 5,\ I_4(4) = 5,$$

We will get a formula for $I_k(m)$. Clearly, $I_1(m) = 1$, the only partition being $111\cdots1$ (m 1s). For $I_2(m)$ there are $m - 1$ places where the rightmost 2 can be placed, so $I_2(m) = m - 1$. To count $I_3(m)$ note that we get a suitable partition by placing a 3 on the left of any partition into $m - 1$ parts with largest part 3, so

$$I_3(m) = I_1(m-1) + I_2(m-1) + I_3(m-1)$$

Applying this to the last term, we have

$$I_3(m) = I_1(m-1) + I_2(m-1) + I_1(m-2) + I_2(m-2) + I_3(m-2).$$

Continuing this process and using the values for $I_1(m)$ and $I_2(m)$ we get

$$I_3(m) = 1 + \binom{m-2}{1} + 1 + \binom{m-3}{1} + \cdots + 1 + \binom{2}{1} + I_3(3).$$

Since $I_3(m) = 2$. we have, since the sum contains m - 3 1s,

$$I_3(m) = (m-3) + [(m-2) + (m-3) + \cdots + 2] + 2$$

$$= \binom{m}{2} - \binom{m}{0}.$$

In a similar manner, starting with

$$I_4(m) = I_1(m-1) + I_2(m-1) + I_3(m-1) + I_4(m-1)$$

and using the formula for $I_3(m)$, we get

$$I_4(m) = \binom{m+1}{3} - \binom{m+1}{1}.$$

Then, using mathematical induction, we can show that for $t \geq 3$,

$$I_t(m) = \binom{m+t-3}{t-1} - \binom{m+t-3}{t-3}$$

Since $I(m) = \sum_{t=1}^{m} I_t(m)$, using the last formula and the identity

$$\sum_{i=0}^{n} \binom{r+1}{i} = \binom{r+n-1}{n},$$

we find that, for $m \geq 3$,

$$I(m) = \binom{2m-2}{m-1} - \binom{2m-2}{m-3}.$$

It is not hard to show that this is equivalent to

$$I(m) = \frac{\binom{2m}{m}}{m+1}$$

There are other questions that could be asked about similar partitions. For example, let $O(m)$ denote the number of partitions with m parts in which the kth part is less than or equal to $m - k + 1$ and all parts are odd. For example, $O(5) = 7$ because of the partitions

53311 53111 51111 33311 33111 31111 11111.

I conjecture that, for $m \geq 2$,

$$O(2m) = \frac{\binom{3m-1}{m} - \binom{3m-1}{m-2}}{m+1}$$

and

$$O(2m+1) = \frac{\binom{3m}{m+1} - \binom{3m}{m-1}}{m}$$

Rachele Dembowski did the work that led to this paper while a student at Seton Hall University, from which she was graduated in May. Her advisor was Professor Esther Guerin.

Xuming Chen (University of Alabama, Tuscaloosa) notes that it seems as if, given three consecutive odd primes, p_n, p_{n+1}, p_{n+2}, it is always the case that $p_n + p_{n+1} > p_{n+2}$ ($3 + 5 > 7$, $5 + 7 > 11$, ... , $99971 + 99989 > 99991$, ...) and wonders if this is easy or hard to prove. Is there any relation to Bertrand's Theorem that for any positive integer $n > 2$ there is a prime between n and $2n$?

AUTOMORPHISMS OF HASSE SUBGROUP DIAGRAMS FOR CYCLIC GROUPS

Lars Seme
Hendrix College

The work presented here extends that of Butt [1] and Woodard [4], who calculated automorphisms of Hasse subgroup diagrams, Butt for groups of small order and Woodard for the cyclic group $C_{p^m q^n}$, where p and q are prime and m and n are natural numbers. Here we extend their results to cover all finite cyclic groups. The theory of Hasse subgroup diagrams is not new; the definitive texts are Suzuki [3] and the recently published book [2] by Schmidt. Our results are a special case of Jones' theorem on classifying the isomorphism classes of Hasse subgroup diagrams associated to any finite group (see [3], p. 37, Theorem 4.5). However, the work in this paper was done independently of these references.

The ***Hasse subgroup diagram*** for a group G is the lattice of subgroups ordered by subgroup containment. The group is the top element of the lattice and the subgroup containing only the identity is the bottom element. Subgroup A is below subgroup B in the lattice if $A \subseteq B$. An edge connects A and B whenever there are no intermediate subgroups; thus edges implied by transitivity are suppressed.

An ***automorphism,*** φ, of a Hasse subgroup diagram, H, is a bijection from H to H that preserves or reverses order. An ***order-preserving*** automorphism has the property that for two lattice elements x and y, if $x \leq y$, then $\varphi(x) \leq \varphi(y)$. If $x \geq y$, then φ is an ***order-reversing*** automorphism. Finally, the ***identity*** automorphism is the bijection that fixes the elements of H.

Let $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$, where p_i is prime and n_i and j are positive integers, denote a finite cyclic group. Since the subgroups of a cyclic group are cyclic, we can give the subgroups the general form $\langle p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} \rangle$,

where k_i can take on any value between 0 and n_i and using the notation $< g >$ to denote the subgroup generated by g. A few simple calculations show that

$$< p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i+1} \cdots p_j^{k_j} > \subseteq < p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$$

and that

$$< p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i+1} \cdots p_j^{k_j} >$$

is directly below

$$< p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i} \cdots p_j^{k_j} >$$

in the Hasse subgroup diagram.

LEMMA 1. In the Hasse subgroup diagram of $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$ the subgroup $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$ has:

0 subgroups directly below if $k_i = n_i$ for i = 1 to j,

j subgroups directly below if $k_i \neq n_i$ for i = 1 to **j**,

j - m subgroups directly below if m is the number of times $k_i = n_i$ for i = 1 to j.

Proof. Suppose $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$ is a subgroup of $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$. If $k_i = n_i$, then our subgroup is the identity subgroup and can have no subgroups below it. Suppose $k_i \neq n_i$ for i = 1 to **j**. By adding 1 to any exponent in $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$ we obtain another subgroup in the form $< p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i+1} \cdots p_j^{k_j} >$. As noted above, this subgroup will be directly below $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$. Since there are **j** choices of exponents to increase by 1, there will be **j** distinct subgroups directly below $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$. Finally, suppose that there are m exponents such that $n_i = k_i$. We can add 1 only to those **j** - m exponents which are not n_i. Therefore, there are **j** - m subgroups directly below $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$.

LEMMA 2. In the Hasse subgroup diagram of $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$ the subgroup $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$ has:

0 subgroups directly above if $k_i = 0$ for i = 1 to j,

j subgroups directly above if $k_i \neq 0$ for i = 1 to j,

j - m subgroups directly above if m is the number of times $k_i = 0$ for $i = 1$ to j.

The proof is similar to that of Lemma 1.

The rank of a subgroup is its level or height in the Hasse subgroup diagram. We define the rank of the identity subgroup to be zero. The rank of a subgroup is then the number of lattice points passed following a continually ascending chain from the identity to the subgroup.

Figure 1 shows the Hasse subgroup diagram for the group C_{24}. Since $<4>$ is two lattice points above the identity subgroup, $<4>$ has rank 2. Figures 2 and 3 show C_{60}. Figure 2 shows the Hasse subgroup diagram and Figure 3 shows a two-dimensional representation to show the rank structure more clearly.

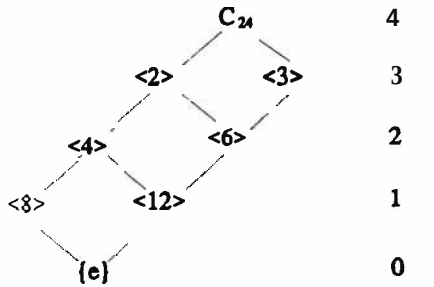


Figure 1

LEMMA 3. The rank of $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$ in the Hasse subgroup diagram of $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$ is $(n_1 + n_2 \cdots + n_j) - (k_1 + k_2 + \cdots + k_j)$.

Proof. Suppose $< p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} >$ is a subgroup of $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$. Using the fact that {e} can be written as $< p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j} >$, we create a chain by subtracting 1 from the exponent of p_1. We continue until the exponent of p_1 is k_1. We have now moved $n_1 - k_1$ lattice points (subgroups) up the diagram. Continuing this for each p_i, the chain is now $(n_1 + n_2 + \ldots$

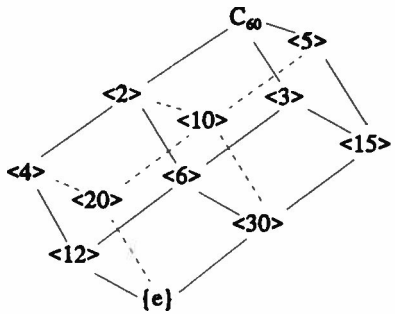


Figure 2

$+ n_j) - (k_1 + k_2 + \cdots + k_j)$ lattice points high, and that is the rank of the subgroup.

Clearly, the rank of $C_{p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}}$ is then $n_1 + n_2 + \cdots + n_j$. Looking at the examples, we see that C_{24} and C_{60} have rank 4.

THEOREM 1. The automorphism group of the Hasse subgroup diagram H for the group $C_{p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}}$, where p_i is prime and n is a positive integer, is isomorphic to $S_j \times C_2$.

Proof. Suppose we have the Hasse subgroup diagram, H, of the group $C_{p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}}$.

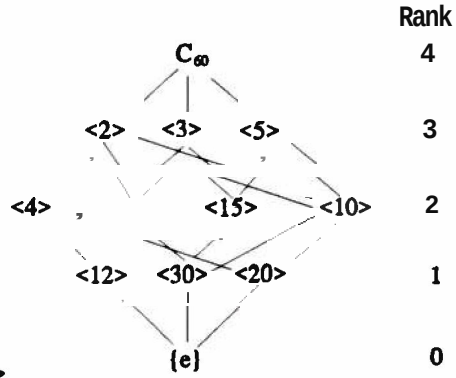


Figure 3

We first claim that the permutation group of any number of the primes creates a valid automorphism of H.

Define a function φ : H → H by $\varphi(A) = B$, where $A = \langle p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i} \cdots p_k^{m_k} \cdots p_j^{m_j} \rangle$ and $B = \langle p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \cdots p_i^{m_i} \cdots p_j^{m_j} \rangle$.

To show that φ is an automorphism, we must show that it is bijective and order-preserving. The bijective result follows immediately from the definition of φ. Function φ is order-preserving provided A has the same number of subgroups above (respectively below) it as does B, and furthermore φ maps the subgroups directly above (respectively below) A to those directly above (respectively below) B. In other words, if this is true, A can fit into B's slot in the lattice H. The proof of this follows from Lemmas 1 and 2, their proofs, and the definition of φ. The rest of the structure is clearly preserved since this transposition affects all subgroups, and hence φ is an automorphism. Recalling that transpositions can generate any permutation, we see that showing this function to be an automorphism shows that all permutations are automorphisms.

We now wish to show that there exist no other order-preserving

automorphisms. Suppose we have a function φ which preserves order. Since φ must preserve the rank of individual subgroups, φ must move subgroups around in the same rank. In addition, each chain of subgroups must be preserved since the ordering of H must be preserved. Now, any reordering of subgroups is accomplished by permuting the p_i's within each subgroup. This forces any attempt at reordering to be applied globally since φ cannot switch p_4 and p_7 in rank four and switch p_4 and p_6 in rank five, as this would result in chains being broken. Hence, any valid order-preserving automorphism must come from the permutation group S_n.

We also claim that there exists a reverse automorphism $\varphi_r : H \to H$ defined by

$$\varphi_r(A) = \langle p_1^{n-m_1} p_2^{n-m_2} \cdots p_i^{n-m_i} \cdots p_k^{n-m_k} \cdots p_j^{n-m_j} \rangle \equiv B,$$

where A is the same as above. (Geometrically, this is a flip followed by a 180° rotation about the axis joining the group and the identity subgroup, although this analogy falls short in more than three dimensions.)

By Lemma 3, A has rank $(jn) - (m_1 + m_2 + \cdots + m_j)$ and B has rank $(m_1 + m_2 + \cdots + m_j)$. By applying Lemmas 1 and 2, we see that A has the same number of subgroups above as B has below and vice versa. Also, any chain that originally liked A and B still exists, except that it has been turned upside down. Clearly, φ is bijective and so φ is an automorphism of our diagram.

Finally, these two automorphisms can be combined, yielding a total automorphism group isomorphic to $S_j \times C_2$.

We now turn to the general case.

THEOREM 2. The automorphisms of the Hasse subgroup diagram for $C_{p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}}$ form a group isomorphic to $S_a \times S_b \times \cdots S_f \times C_2$. where a is the number of times n_i = m, , for a given m_1, b is the number of $n_i = m_2$ for a given m_2, and so on. Disregard S_1 whenever it appears.

Proof. Consider the group $C_{p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}}$. Suppose two exponents, say n_i and n_k, are equal. Then permuting p_i and p_k is an automorphism by the proof of Theorem 1.

We also claim that if $n_i \neq n_k$, permuting p_i and p_k is not an isomorphism. Assume that $n_i < n_k$ and define a function φ : H → H as in Theorem 1. Consider the subgroup

$$< p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i} \cdots p_k^{n_i+1} \cdots p_j^{m_j} > \equiv A.$$

Applying φ to this subgroup produces

$$\varphi(A) = < p_1^{m_1} p_2^{m_2} \cdots p_k^{m_i} \cdots p_i^{n_i+1} \cdots p_j^{m_j} >$$

But φ is then not a function as $< p_1^{m_1} p_2^{m_2} \cdots p_k^{m_i} \cdots p_i^{n_i+1} \cdots p_j^{m_j} >$ is not an element of $C_{p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}}$ since $n_i < n_i + 1$. Therefore φ is not an automorphism.

As in the proof of Theorem 1, nothing other than permutations can be used as order-preserving elements in the automorphism group. We also see that the reverse automorphism as defined in the proof of Theorem 1 is still an automorphism.

Therefore, our automorphisms form a group automorphic to $S_a \times S_b \times \cdots \times S_f \times C_2$, where $a, b, \ldots, f$ are as defined above.

References

1. Butt, Melanie, Automorphism groups of Hasse subgroup diagrams for groups of low order, this *Journal* 9 (1989-94) #1, 2-8.
2. Schmidt, R., *Subgroup Lattices of Groups,* DeGruyter Expositions in Mathematics #14, 1994.
3. Suzuki, Michio, *Structure of a Group and the Structure of its Lattice of Subgroups,* Springer-Verlag, 1956.
4. Woodard, Dawn, Automorphisms of Hasse subgroup diagrams, unpublished.

When this paper was written, Lars Seme was a senior mathematics and physics major at Hendrix College. He would like to thank the referee for making helpful suggestions. David Sutherland served as his project advisor.

Solution to Mathacrostic 40, by Robert Forsberg (Spring, 1995).

Words:

A arnotto
B. exine
C. differentiable
D. dartle
E. ichneumon
F. neodymium
G. Giuseppe Peano
H. the venerable Bede
I. Owen Gwynedd
J. Nuvistor
K. phenolphthalein
L. Horvath
M. Yahwist
N. sieve of Eratosthenes
O. icotype
P. Christoffel
Q. aurora
R. light-nanosecond
S. scofflaw
T. Charles Lutwidge Dodgson
U. inactive
V. equivocation
W. nematocyst
X. cord
Y. effuse

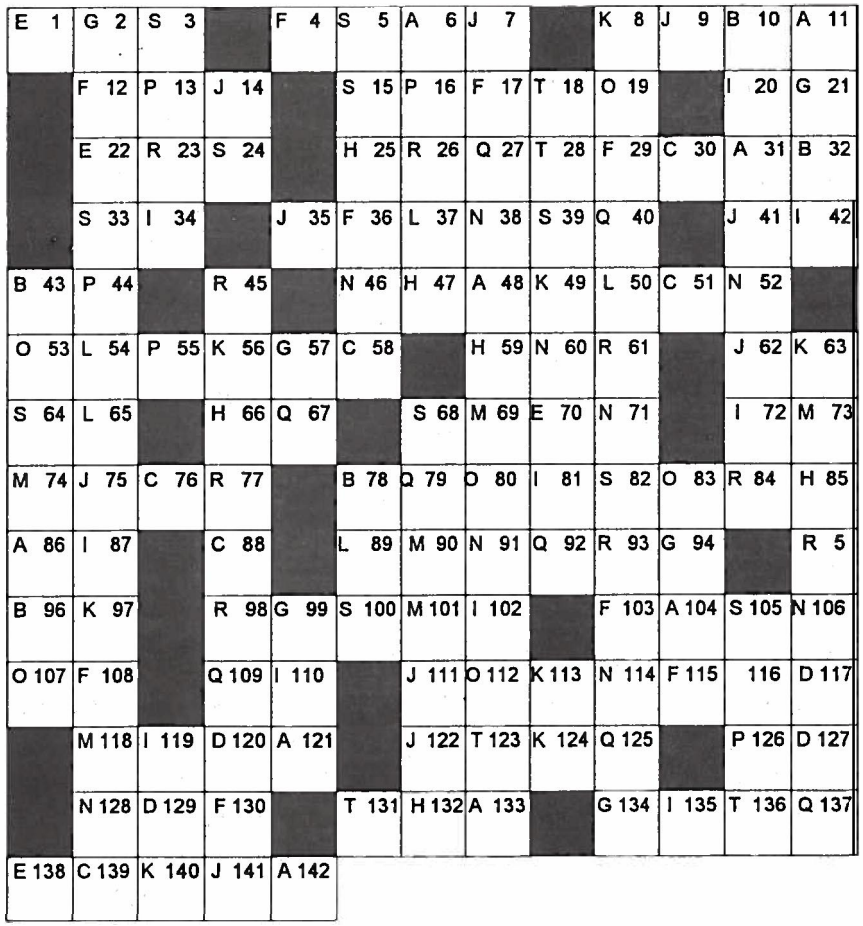
Author and title: A Eddington, *Physical Science.*

Quotation: Pure mathematicians, having learned by experience that the obvious is difficult to prove—and not always true—found it necessary to delve into the processes of reasoning. In so doing, they developed a powerful technique which has been welcomed for the advancement of logic generally.

Solvers: Thomas Banchoff, Jeanette Bickley, Barbara Buckley, Charles R. Diminnie, Thomas L. Drucker, Victor G. Feser, Richard C. Gebhardt, Henry S. Lieberman, Naomi Shapiro, and the proposer.

Two errors escaped both the proposer and the editor: several solvers noted that the G in word I was omitted and Naomi Shapiro points out that the definition of word Y should be "Spread out *without* a definite form." No solver identified the "powerful technique" referred to in the quotation.

Mathacrostic 41, by Corine Bickley appears on the next three pages. It has been some time since the directions for solving acrostics have been given, so they appear as well. To be listed as a solver, send your solation to the one of the two editors who is not Clayton Dodge.



A. A kind of feldspar

B. Serial, for example

C. _____ rouser

D. Seen in the Grand Canyon

E. Urge

F. A method of ordering **(2 wds)**

G. Angels were heard there **(2 wds)**

H. Devised an algorithm for iterative solution of nonlinear differential equations

I. Force applied at a cross direction

J. x and y or a and β, for instance

K. Now and again (3 **wds)**

L. Not part

M. Compare

$\overline{11}\ \overline{133}\ \overline{31}\ \overline{48}\ \overline{142}\ \overline{86}\ \overline{104}\ \overline{121}\ \overline{6}$

$\overline{10}\ \overline{32}\ \overline{43}\ \overline{96}\ \overline{78}$

$\overline{76}\ \overline{88}\ \overline{139}\ \overline{30}\ \overline{51}\ \overline{58}$

$\overline{117}\ \overline{120}\ \overline{127}\ \overline{129}$

$\overline{22}\ \overline{1}\ \overline{70}\ \overline{138}$

$\overline{4}\ \overline{29}\ \overline{17}\ \overline{36}\ \overline{103}\ \overline{130}\ \overline{12}\ \overline{108}\ \overline{115}$

$\overline{21}\ \overline{134}\ \overline{94}\ \overline{99}\ \overline{57}\ \overline{2}$

$\overline{25}\ \overline{47}\ \overline{132}\ \overline{59}\ \overline{66}\ \overline{85}$

$\overline{72}\ \overline{119}\ \overline{135}\ \overline{42}\ \overline{110}\ \overline{102}\ \overline{20}\ \overline{81}\ \overline{116}$

$\overline{87}\ \overline{34}$

$\overline{7}\ \overline{35}\ \overline{141}\ \overline{75}\ \overline{41}\ \overline{14}\ \overline{122}\ \overline{62}\ \overline{9}$

$\overline{111}$

$\overline{140}\ \overline{49}\ \overline{8}\ \overline{124}\ \overline{56}\ \overline{63}\ \overline{113}\ \overline{97}$

$\overline{37}\ \overline{54}\ \overline{50}\ \overline{89}\ \overline{65}$

$\overline{90}\ \overline{73}\ \overline{74}\ \overline{69}\ \overline{118}\ \overline{101}$

N. _____ Rule, used for spatial orientation

114 128 106 71 46 60 38 91 52

O. A kind of signal, some analyses of which use spectral displays

83 80 107 19 53 112

P. Across the wide _____

126 16 44 55 13

Q. Haven't the _____ notion

67 109 92 137 27 79 40 125

R. French conference for two (3 wds)

23 61 98 77 45 93 84 95 26

S. Can be solved by QR or QL method, or by Householder reduction

82 33 64 68 105 15 39 24 5

3 100

T. _____ time

18 136 28 123 131

The mathacrostic is a keyed anagram. The 142 letters to be entered in the diagram in the numbered spaces will be identical with those in the 20 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution.

When completed, the initial letters of the words will give the name of an author and the title of a book; the completed diagram will be a quotation from that book.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying ***novel*** *and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

All communications should be addressed to C. W. Dodge, ***5752 Neville/Math,*** *University of Maine, Orono, ME* ***04469-5752.*** *E-mail:* ***dodge@gauss.umemat.maine.****edu. Pleasesubmit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly* ***identified*** *with name and address. Solutions to problems in this issue should be mailed by July 1,* ***19%.***

Problems for Solution

862. Proposed by Philip Tate, student, University of Maine, Orono, Maine.

"Solve this base ten addition alphametic."

"But it doesn't have a unique solution."

"It does if I give you the value of *T*."

"Never mind, I found it. Furthermore, it has a unique solution in base eight. Let me show it to you."

```
  DODGE
+ THE
-------
  GREAT
```

863. *Proposed by James Chew, North Carolina Agricultural and Technical State University, Greensboro, North Carolina.*

Here is a problem especially for undergraduates. Everyone is familiar with the story of the absent-minded professor who wears different colored socks on his feet. Suppose a month's supply of socks are in the clothes drier; specifically, let there be *n* pairs of socks in a drier containing only these socks.

a) Assume the socks are of n different colors. The professor draws socks one at a time from the drier without replacement, noting the color as he draws each sock. To get a pair of matching socks, at least 2 and at most n + 1 socks must be drawn. On average, how many socks would have to be drawn to get a matching pair?

b) Repeat part (a), assuming k different colors of socks: n_1 pairs of red socks, n_2 pairs of blue socks, etc., where $n_1 + n_2 + \ldots + n_k = n$.

864. Proposed by Charles Ashbacher, Geographic Decisions Systems, Cedar Rapids, Iowa.

On page 11 of the booklet Only Problems, Not Solutions! by Florentine Smarandache, there is the following problem.

> Let $a_1, a_2, \ldots, a$ be digits. Are there primes, on a base b, which contain the group of digits $\overline{a, \ldots a_m}$ into its writing? But n!? But n^n?

Prove that for any such sequence of digits $a_1, a_2, \ldots, a_m$, no matter how generated, there exists a prime such that the sequence is found in that prime.

865. Proposed by Miguel *Amengual Covas,* Mallorca, Spain.

Let *ABC* be a triangle with sides of lengths a, b, and c, semiperimeter s, and area K. Show that, if $\Sigma a(s - a) = 4K$, then the three circles centered at the vertices A, B, and C and of radii $s - a$, $s - b$, and $s - c$, respectively, are all tangent to the same straight line.

866. Proposed by J. Rodriguez, Sonora, Mexico.

For any nonzero integer n, the *Smarandache* junction is the smallest integer $S(n)$ such that $(S(n))!$ is divisible by n. Thus $S(12) = 4$ since 12 divides 4! but not 3!.

a) Find a strictly increasing infinite sequence of integers such that for any consecutive three of them the Smarandache function is neither increasing nor decreasing.

*b) Find the longest increasing sequence of integers on which the Smarandache function is strictly decreasing.

867. Proposed by Seung-Jin Bang, AJOU University, *Suwon,* Korea. Find the number of solutions (x, y, z, w) to the system

$$\begin{aligned} x + y + z + w &= 7 \\ x^2 + y^2 + z^2 + w^2 &= 15 \\ x^3 + y^3 + z^3 + w^3 &= 37 \\ xyzw &= 6. \end{aligned}$$

868. Proposed by William H. Peirce, *Delray* Beach, Florida.

1. Enter total amount of all social security benefits	1. ___S___
2. Enter one-half of line 1	2. _____
7. Enter your provisional income	7. ___P___
8. Enter $32,000 if married filing jointly	8. 32,000
9. Subtract line 8 from line 7. If zero or less, enter 0	9. _____
Is line 9 zero? If yes, enter 0 on line 18. If no, continue to line 10.	
10. Enter $12,000 if married filing jointly	10. 12,000
11. Subtract line 10 from line 9. If zero or less, enter 0	11. _____
12. Enter the smaller of line 9 or line 10	12. _____
13. Enter one-half of line 12	13. _____
14. Enter the smaller of line 2 or line 13	14. _____
15. Multiply line 11 by 0.85	15. _____
16. Add lines 14 and 15	16. _____
17. Multiply line 1 by 0.85	17. _____
18. Taxable social security benefits. Enter the smaller of line 16 or line 17	18. ___T___

Social Security Benefits Worksheet (somewhat simplified)

Computation of the taxable portion of social security benefits in 1994 is considerably more complicated than in past years, and the IRS has designed the 1994 accompanying worksheet to determine these taxable benefits. Let S be the total social security benefits on line 1, P the provisional income on line 7, and T the taxable benefits on line 18. For

married couples filing jointly, find T as a function of S and P. Exhibit the solution graphically by showing the function T for each pertinent region of the SP-plane, and give the boundary equations for each region. Assume $S > 0$ and $P >$ 32,000 and ignore their practical upper limits.

869. ***Proposed by Rasoul Behboudi, University of North Carolina, Charlotte, North Carolina.***

Consider an ellipse with center at O and with major and minor axes AB and CD respectively. Let E and F be points on segment OB so that $OE^2 + OF^2 = OB^2$. At E and F erect perpendiculars to cut arc BC at G and H respectively. Show that the areas of sectors OBH and OGC are equal. *See* Figure 1.

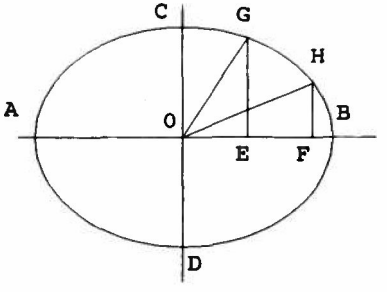


Figure 1. Problem **869.**

870. ***Proposed by Grattan P. Murphy, University of Maine, Orono, Maine.***

This proposal is based on a problem posed at a recent mathematics meeting and is intended especially for students. Without using machine calculation, that is, without actually finding the digits of the number, show that at least one digit occurs at least **6** times in the decimal representation of the number $(7^7)^7 \cdot 7^7 \cdot 77$.

871. ***Proposed by Miguel Amengual Covas, Mallorca, Spain.***

Let $ABCD$ be an isosceles trapezoid with major base BC. If the altitude AH is the mean proportional between the bases, then show that each side is the arithmetic mean of the bases, and show that the projection A P of the altitude on side AB is the harmonic mean of the bases. See Figure 2.

872. ***Proposed by Paul S. Bruckman, Edmonds, Washington.***

Given A_1, A_2, and A_3 are the angles of a triangle and 4 C k C 12, let $S_k = S_k(A_1, A_2, A_3) = \Sigma_{i=1}^{3} (k \cos A_i + \cos 2A_i)$, defined on the triangular plane region R: 0 C A, $< \pi$, 0 C A-, $< \pi$, 0 C A, + $A_2 < \pi$. Find the maximum value of S_k for all triangles.

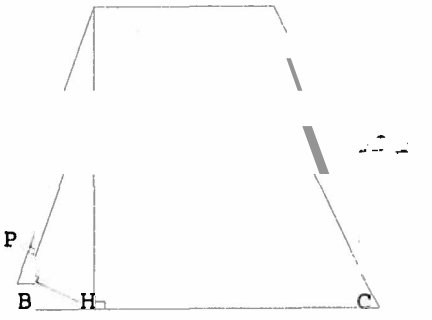


Figure 2. Problem 871.

873. ***Proposed by Mohammad K. Azarian, University of Evansville. Evansville, Indiana.***

For p and q positive real numbers and any positive integer m let

$$f(x) = \left[1 + x + \frac{x^m}{m!} + \frac{x^{m+1}}{m!}\right]^{pq} \left[1 + \frac{x}{p}\right]^{p^3} \exp\left[\frac{q^3 x}{q + x}\right],$$

where $x \geq 0$. Prove that

$$0 < \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \int_0^{\infty} f(x) \exp(-[(p + q)^2 + k^n]x)\, dx \leq 1.$$

874. ***Proposed by David Iny, Westinghouse Electric Corporation. Baltimore, Maryland.***

a) Given real numbers x_i and z_i for $1 \leq i \leq n$, prove that

$$n[\sum x_i^2 \sum z_i^2 - (\sum x_i z_i)^2] \geq (\sum x_i)^2(\sum z_i^2) + (\sum x_i^2)(\sum z_i)^2 - 2(\sum x_i)(\sum z_i)(\sum x_i z_i).$$

b) Determine a necessary and sufficient condition for equality.

Solutions

820. [Fall 1993, Fall 1994] Proposed by William Moser, *McGill* University, Montreal, Quebec, Canada.

Let $a_{n,k}$ $(0 \le k < n)$ denote the number of n-bit strings (sequences of 0's and 1's of length n) with exactly k occurrences of two consecutive 0's. Show that

$$a_{n,k} = \sum_{r=2k}^{n} \binom{r-k}{k}\binom{n-r+1}{r-k},$$

where $\dfrac{n!}{k!(n-k)!}$ if $0 \le k \le n$ and $\binom{n}{k} = 0$ otherwise.

Editor's comment. The problem is unclear as to how many pairs of zeros you count when there are three or more consecutive zeros. The proposer's intent was that three or more consecutive zeros are not allowed; consider strings where zeros appear (between ones) only singly or in pairs. There is no such restriction on the ones; any number of consecutive ones can appear any place.

I. Solution by the Proposer.

We use the well-known result that m like objects can be placed in q unlike boxes in $\binom{m+q-1}{q-1}$ ways. For r = 0, 1, 2, ..., we shall construct and count the n-bit strings with r zeros, n − r ones and exactly k occurrences of two consecutive zeros. We use the symbol Z to denote 00.

Place k Z's and r − 2k zeros in a row, which can be done in $\binom{r-k}{k}$ ways These r − k symbols in a row determine r − k + 1 boxes—one at each end and r − k − 1 between adjacent symbols. Into each in-between box place a 1. There remain n − r − (r − k − 1) = n − 2r + k + 1 ones, which we distribute into the r − k + 1 boxes without restriction. This distribution can be done in

$$\binom{(n-2r+k+1)+(r-k+1)-1}{(r-k+1)-1} = \binom{n-r+1}{r-k}$$

ways. We thus have

$$\binom{r-k}{k}\binom{n-r+1}{r-k}$$

n-bit sequences with r zeros, n − r ones, and exactly k 00's. Summing over r gives the desired result.

II. Comment by Paul S. Bruckman, *Edmonds*, Washington.

Of the a_0 strings having no occurrences of 00, let u_n and z_n denote those strings that end in 1 and 0 respectively. Clearly $a_{n,0} = u_n + z_n$. To form a string of n + 1 zeros and ones having no 00's one can append a 1 to the end of any such string of length n, or one can append a 0 to the end of any string of length n that ends in 1. That is, $u_{n+1} = a_{n,0}$ and $z_{n+1} = u_n = a_{n-1,0}$. It follows that $a_{n,0} = F_{n+2}$, where $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ are the Fibonacci numbers.

It may also be shown that the "row" sums of these coefficients, that is,

$$s_n = \sum_{k=0}^{[(n+1)/3]} a_{n,k},$$

where the brackets indicate the greatest integer function, may be expressed in terms of the Tribonacci numbers T_n, where $T_0 = T_1 = 0$, $T_2 = 1$, and in general $T_{n+3} = T_{n+2} + T_{n+1} + T_n$. Specifically, $s_n = T_{n+3}$.

Also solved by Paul S. Bruckman, and Mark Evans. Henceforth we shall print only the names of the also-solvers. The omission of affiliations and locations will save enough space to print an additional article in each issue. We regret the inconvenience and ask your understanding—ed.

825. [Spring 1994, Spring 1995] Proposed by Leon *Bankoff, Los* Angeles, California.

Let O be a point inside the equilateral triangle ABC whose side is of length s. Let OA, OB, OC have lengths a, b, c respectively. Given the lengths a, b, c, find length s.

IV. *Comment* by Henry S. Lieberman, *Waban*, Massachusetts.

In addition to Rex Wu's demonstration of the symmetry in Solution I and the editor's comment, we observe that

$$(2bc)^2 - (b^2 + c^2 - a^2)^2 = (b + c + a)(b + c - a)(c + a - b)(a + b - c),$$

which is clearly symmetric.

836. [Fall 1994] ***Proposed*** by ***the editor.***

Solve this base ten holiday addition alphametic. Since the coming year 1995 is an odd year, you are asked to find that solution such that ***A*** is an odd digit.

MANY
NEW
NEW
YEARS

Solution by Alma College Problem Solving Group, Alma College, Alma Michigan.

Since we can carry at most 1 from the hundreds column, we see that YE = 10 and ***M*** = 9. Since E = 0, we must carry 1 from the units column and R = N + 1. Thus we carry 0 to the hundreds column. From the two ***A's*** in that column, we see that N = 5, so R = 6. Since W > 4, we have W = 7 or 8. Because W = 7 leads to the contradiction ***S*** = N = 5, then W = 8 and ***S*** = 7. We know ***A*** is an odd digit and therefore ***A*** = 3, the only remaining odd digit. Thus our solution is 9351 + 508 + 508 = 10367.

Also solved by Charles Ashbacher, Scott H. Brown, Paul S. Bruckman, Sandra Rená Chandler, William Chau, Mark Evans, Victor G. Feser, Stephen I. Gendler, Sergey Gershtein, Richard I. Hess, Bill Hooper, Carl Libis, Henry S. Lieberman, Raymond Medley, Yoshinobu Murayoshi, Michael R. Pinter, Mike Saparov, Leslie J. Upton, Rex H. Wu, ***and the*** Proposer.

837. [Fall 1994] ***Proposed*** by J. ***Sutherland Frame, Michigan State University, East Lansing, Michigan.***

Evaluate in closed form the integral

$$I = \int_{-a}^{a} \sqrt{a^2 - x^2}\, \ln|z - x|\, dx, \quad |z| < a.$$

Solution by the Proposer.

Note that I is an improper integral because x = z at one point inside the interval of integration. In a small neighborhood of that point the quantity $(a^2 - x^2)^{1/2}$ is essentially a nonzero constant and the integral is equivalent to

$$\lim_{\hat{a},\neg} 2\int_{\epsilon}^{1} n x\, dx = \lim_{\hat{a},\neg} 2[x \ln x - x]_{\epsilon}^{1} = -2 - \lim_{\epsilon \downarrow 0} 2\epsilon \ln \epsilon = -2,$$

so the integral I converges. Now we set x = a cos 6 and $z = a \cos \phi$, then add an equal term to I by replacing 6 by π − 6, getting

$$2I = a^2 \int_0^{\pi} \sin^2\theta [\ln a|\cos\phi - \cos\theta| + \ln a|\cos\theta + \cos\phi|]\, d\theta,$$

$$\frac{2I}{a^2} = \int_0^{\pi} \sin^2\theta \ln[a^2|\sin^2\theta\cos^2\phi - \cos^2\theta\sin^2\phi|]\, d\theta$$

$$= \int_0^{\pi} \sin^2\theta\, [\ln a|\sin(\theta + \phi)| + \ln a|\sin(\theta - \phi)|]\, d\theta.$$

Since both integrand summands have period π, we can replace 6 by 6 − ϕ, or by 6 + 4, and get

$$\frac{2I}{a^2} = \int_0^{\pi} [\sin^2(\theta - \phi) + \sin^2(\theta + \phi)] \ln(a\sin\theta)\, d\theta$$
$$= \int_0^{\pi} [1 - \cos 2\phi \cos 2\theta] \ln(a\sin\theta)\, d\theta,$$

where we used $\sin^2 y = (1 - \cos 2y)/2$ and the formula for cos $(u + v)$. Since

$$(2 - 1)\int_0^{\pi} \ln(a\sin\theta)\, d\theta = \int_0^{\pi} \ln(a^2\sin^2\theta)\, d\theta - \int_0^{\pi/2} \ln(a\sin 2\theta)\, d(2\theta)$$

$$= 2\int_0^{\pi/2} [\ln\frac{a}{2} + \ln\sin\theta - \ln\sin(\pi - \theta)]\, d\theta = \pi \ln\frac{a}{2},$$

so

$$\frac{2I}{a^2} - \pi\ln\frac{a}{2} = -\cos 2\phi \int_0^{\pi} \cos 2\theta \ln(a\sin\theta)\, d\theta$$

$$= \cos 2\phi \left([-\sin\theta\cos\theta \ln(a\sin\theta)]_0^{\pi} + \int_0^{\pi} \sin\theta\cos\theta\cot\theta\, d\theta \right),$$

$$\frac{2I}{a^2} = \pi\ln\frac{a}{2} + (2\cos^2\phi - 1)\left[0 + \frac{\pi}{2}\right] = \pi\left[\ln\frac{a}{2} + \frac{z^2}{a^2} - \frac{1}{2}\right]$$

and finally

$$I = \frac{\pi a^2}{2}\left[\ln\frac{a}{2} + \frac{z^2}{a^2} - \frac{1}{2}\right].$$

Also solved by Paul S. Bruckman.

838. . [Fall 1994] Proposed by Florentin Smarandache, Phoenix, Arizona.

Let $d_n = p_{n+1} - p_n$, n = 1, 2, 3, ..., where p_n is the nth prime number. Find the nature of the series

$$\sum_{n=1}^{\infty} \frac{1}{d_n}.$$

I. Solution by Richard *I. Hess,* Rancho Palos Verdes, California.

For large x the probability of x being a prime is approximately $1/\ln x$. Thus there is on average one prime between x and $x + \ln x$. Hence

$$\sum \frac{1}{p_n} \approx \int \frac{dx}{x \ln x}.$$

Since

$$\int_2^N \frac{dx}{x \ln x} = \ln(\ln N) - \ln(\ln 2),$$

the integral diverges when N becomes infinite, so the sum of the reciprocals of the primes diverges. Now

$$\frac{1}{d_n} = \frac{1}{p_{n+1} - p_n} > \frac{1}{p_{n+1}},$$

so $\Sigma\, 1/d_n$ diverges by the comparison test.

II. Comment by Paul S. Bruckman, *Edmonds,* Washington.

The same problem by the same author appeared as Problem B-726 in The Fibonacci Quarterly, Vol. 30, No. 4 (Nov. 1992). The published solution, ibid, Vol. 32, No. 1 (Nov. 1994) showed that the indicated series is divergent.

Also solved by Joe Howard, David E. Manes, Rex H. Wu, and the Proposer.

839. [Fall 1994] Proposed by James Chew, *North* Carolina Agricultural and Technical Stare *University,* Greensboro, North Carolina.

a) A ticket buyer chooses a number from 10 through 99 inclusive. A number is randomly picked as winner. If, for example, 63 is the winner, then each ticket number 63 that has been sold is awarded \$A. The reversal ticket number 36 is awarded \$*B*. That is, the second prize goes to any ticket with both digits correct, but in the wrong order. The third prize of \$*C* is paid to any ticket that contains at least one of the correct digits, e.g. 33, 43, 34, 65, 76, etc. A ticket can win only one prize and prizes are not shared. If you have bought 5 tickets numbered 63, you win \$5*A*. Find the fair price for a ticket.

*h) Find the fair price for the game of part (a) if prizes are shared. That is, the ticket seller pays out a total of at most \$(A + B + *C*) in winnings for any one game, \$A is shared among all winning tickets (number 63), if any. Then \$*B* is shared among all holders of second prize tickets (number 36). Finally, all third prize winners share the one amount \$C.

I. Solution to part (a) *by* Mark Evans, *Louisville,* Kentucky.

The ticket price should actually he a function of the number the player chooses. There are three cases.

Case 1. To illustrate the nine numhers that end in zero, suppose you pick the number 10. You win \$A with probability 1/90, you cannot win \$B, and you win \$*C* if any of 11, 12, ..., 19, 20, 21, 30, 31, ... 90, 91 is chosen, with probability 25/90. The fair price is $P_1 = \$(A + 25C)/90$.

Case 2. Suppose the buyer picks one of the nine numbers with two like digits. For example, suppose you pick 11. You win \$A if 11 is chosen, with probability 1/90, you cannot win \$B, and you win \$*C* if any of 10, 12, 13, ..., 19, 21, 31, ..., 91 is chosen, with probability 17/90. Hence the fair price for your ticket is $P_2 = \$(A + 17C)/90$.

Case 3. There are 72 remaining numbers, such as 12, having two distinct digits, neither of which is 0. Then you win \$A with probability 1/90, now \$*B* (for the number 21) with probability 1/00, and \$*C* for any of the numbers 10, 11, 13, 14, ... 19, 20, 22, 23. .... 29, 31, 32. 41. 42. ....

91, 92, a total of 32 numbers. Here the fair price is $P_3 = \$(A + B + 32C)/90$.

II. ***Solution to part (a)*** by ***the Proposer.***

Continuing Solution I, suppose the player selects a number completely at random without regard to the three cases considered. (Perhaps the number is assigned by a drawing.) Then the probabilities of picking case 1, 2, or 3 are 9/90 = 1/10, 9/90 = 1/10, and 72/90 = 8/10, so the fair price should be

$$P = \frac{1}{10}P_1 + \frac{1}{10}P_2 + \frac{8}{10}P_3 = \$\frac{10A + 8B + 298C}{900}$$

III. ***Solution to part (b)*** by ***Richard*** I. ***Hess, Rancho Palos Verdes, California.***

Assume you follow a mixed strategy with probabilities p, q, and $r = 1 - p - q$ of choosing numbers from cases 1, 2, and 3 (from Solution 1 above) respectively. We shall solve for p and q so that your expectation is indifferent to whatever strategy the remaining population chooses. Let the population consist of n people including yourself. We suppose the others all pick pure strategies from cases 1, 2, and 3.

Suppose the remaining population all pick from case 1, say they pick the number 10. The table shows your expectation for each choice you make.

You pick (one of)	With probability	Your expectation
10	$\frac{p}{9}$	$\frac{1}{90}\left(\frac{A}{n} + \frac{25C}{n}\right)$
20, 30, ..., 90	$\frac{8p}{9}$	$\frac{1}{90}\left(A + \frac{16C}{n} + 9C\right)$
11	$\frac{q}{9}$	$\frac{1}{90}\left(A + C + \frac{16C}{n}\right)$
22, 33, ..., 99	$\frac{8q}{9}$	$\frac{1}{90}\left(A + 14C + \frac{3C}{n}\right)$
12-19, 21, 31, ..., 91	$\frac{16r}{72}$	$\frac{1}{90}\left(A + B + \frac{16C}{n} + 16C\right)$
23, 24, ..., 98 (no 0, 1, or repeat)	$\frac{56r}{72}$	$\frac{1}{90}\left(A + B + \frac{6C}{n} + 26C\right)$

Hence the expectation E(1) is given by

$$E(1) = \frac{A}{90} - \frac{pA}{810}\left(1 - \frac{1}{n}\right) + \frac{rB}{90} + \frac{pC}{90}\left(\frac{17}{n} + 8\right) + \frac{qC}{810}\left(113 + \frac{40}{n}\right) + \frac{rC}{810}\left(214 + \frac{74}{n}\right).$$

Similar counting for the cases where the remaining population chooses purely from sets 2 and 3 gives

$$E(2) = \frac{A}{90} - \frac{qA}{810}\left(1 - \frac{1}{n}\right) + \frac{rB}{90} + \frac{pC}{810}\left(\frac{40}{n} + 185\right) + \frac{qC}{810}\left(120 + \frac{33}{n}\right) + \frac{rC}{855}\left(230 + \frac{58}{n}\right)$$

and

$$E(3) = \frac{A}{90} - \frac{rA}{6480}\left(1 - \frac{1}{n}\right) + \frac{rB}{90} - \frac{rB}{6480}\left(1 - \frac{1}{n}\right) + \frac{pC}{810}\left(\frac{74}{n} + 151\right) + \frac{qC}{810}\left(95 + \frac{58}{n}\right) + \frac{rC}{540}\left(119 + \frac{73}{n}\right).$$

To make the expectations equal we set

$$0 = E(3) - E(2) = \left(-p \cdot \frac{34C}{810} + q \cdot \frac{A - 25C}{810} - r \cdot \frac{A + B + 412C}{6480}\right)\left(1 - \frac{1}{n}\right)$$

and

$$0 = E(2) - E(1) = \left(p \cdot \frac{A + 113C}{810} + q \cdot \frac{-A + 7C}{810} + r \cdot \frac{16C}{810}\right)\left(1 - \frac{1}{n}\right),$$

which, since $r = 1 - p - q$, reduce to

$$p(A + B + 140C) + q(9A + B + 212C) = A + B + 412C$$

and

$$-p(A + 97C) + (A + 9C) = 16C.$$

After much algebra, during which we find that $A/C \geq \sqrt{27^2 + 2587} - 27 \approx 30.58472$ in order for r to be nonnegative, we find the expectation E to be

$$\frac{91A^3+362860C^3+91A^2B+13280A^2C+368449AC^2-173999BC^2+5020ABC}{810(10A^2 + 21824C^2 + 2AB + 1234AC + 106BC)}.$$

If $A/C < \sqrt{27^2 + 2587} - 27$, the problem simplifies and you choose only from sets 1 and 2, taking $r = 0$, as the reader may wish to verify.

Also solved by Paul S. Bruckman, William Chau, Mark Evans, Richard I. Hess, *and the* Proposer.

Editorial note—It is interesting to see a problem where five solvers submit solutions with five different answers. Some of these differences were due to different assumptions about strategies and about the conditions of the problem. Combinatorial and probability problems attract different interpretations, clearly illustrated in the recent furor over the so-called Monty Hall problem.

840. [Fall 1994] *Proposed by Seung-Jin Bang. Seoul, Republic of Korea.*

Prove that, for $n \geq 2$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \ln n + \frac{n + 1}{2n}.$$

Solution by George P. Evanovich, Saint Peter's college. Jersey City, New Jersey.

We have that

$$\ln n = \int_1^n \frac{1}{x}\,dx,$$

Draw the graph of $f(x) = 1/x$ and approximate the area under the curve on the interval $[1, n]$ by the trapezoidal rule. Since the curve is concave upward, the area is less than the approximation, that is,

$$\ln n < \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + \cdots + 2\left(\frac{1}{n-1}\right) + \frac{1}{n}\right]$$

Now add $112 + 1/(2n)$ to each side to obtain the desired result.

Also solved by Alma College Problem Solving Group, Paul S. Bruckman, Mark Evans, **Jayanthi** Ganapathy, Edward Hamilton, Richard I. Hess, Joe Howard, Murray S. Klamkin, Henry S. Lieberman, Peter A. Lindstrom, David E. Manes, Can. A. **Minh**, Yoshinobu Murayoshi, Bob **Prielipp**, St. Olaf Problem Solving Group, Selvaratnam Sridharma, Sammy Yu *and* Jimmy Yu, *and the* Proposer.

841. [Fall 1994] *Proposed by Seung-Jin Bang, Seoul. Republic of Korea.*

For given real constants a, b, and c, let $\{a\}$ be the sequence satisfying the recursion equation $na_n = aa_{n-1} + ba_{n-2}$ for $n > 1$, $a_0 = 0$, $a_1 = c$. Find the sum of the series

$$\sum_{n=0}^{\infty} a_n.$$

Solution by Paul S. Bruckman, Edmonds, Washington.

Initially we ignore questions of convergence. Let $S = \sum_{n=0}^{\infty} a_n$. We first deal with degenerate cases. If $a = b = 0$, then clearly $S = c$. If $a \neq 0$ and $b = 0$, we see that $a_n = ca^{n-1}/n!$ for $n \geq 1$, and $S = (c/a)(e^a - 1)$. By letting $a \to 0$ we see that the former case is a limiting instance of the latter.

Henceforth we suppose that $b \neq 0$. To develop a differential equation, suppose

$$y = y(x) = \sum_{n \geq 0} a_n x^n,$$

whose coefficients a satisfy the given recursion equation. Since

$$xy' = \sum_{n \geq 1} n a_n x^n, \quad axy = a\sum_{n \geq 2} a_{n-1} x^n, \quad \text{and} \quad bx^2y = b\sum_{n \geq 2} a_{n-2} x^n,$$

then $xy' - a_1x = axy + bx^2y$, or

$$y' = c + ay + bxy,$$

subject to the conditions $y(0) = 0$ and $y'(0) = c$. To solve this system we make the substitution $y = u \exp(ax + bx^2/2)$ and we get

$$u' = c \exp(-ax - bx^2/2) \text{ with } u(0) = 0 \text{ and } u'(0) = c.$$

From this equation we obtain the solution $u = c \int_0^x \exp(-at - bt^2/2)\,dt$ and

$$y = c \exp\left[\frac{(a + bx)^2}{2b}\right] \int_0^x \exp\left[-\frac{(a + bt)^2}{2b}\right] dt.$$

We observe that the integral exists for all x and hence that the given series converges. Since $S = y(1)$, we obtain

$$S = c \exp\left[\frac{(a + b)^2}{2b}\right] \int_0^1 \exp\left[-\frac{(a + bt)^2}{2b}\right] dt$$

or equivalently,

$$S = c \exp\left[a + \frac{b}{2}\right] \int_0^1 \exp\left[-at - \frac{bt^2}{2}\right] dt.$$

If $b > 0$, then by making the substitution $a + bt = u\sqrt{b}$, we can obtain

$$S = c\sqrt{\frac{2\pi}{b}} \exp\left[\frac{(a + b)^2}{2b}\right] \cdot \left[\Phi\left(\frac{a + b}{\sqrt{b}}\right) - \Phi\left(\frac{a}{\sqrt{b}}\right)\right],$$

where $\Phi(x)$ is the cumulative function of the normal probability distribution, defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}u^2} du.$$

Also solved by Murray S. Klamkin, and the Proposer.

842. [Fall 19941 Proposed by Russell *Euler*, Northwest Missouri State University, Maryville, Missouri.

Let x_i be a positive real number for i = 1, 2, ..., n. Prove that

$$\left[\sum_{i=1}^n \frac{1}{x_i}\right] \left[\sum_{i=1}^n (x_i)^2\right]^{1/2} \geq n\sqrt{n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Solution by Joe Howard, New Mexico Highlands University, *Las Vegas*, New Mexico.

By the Cauchy-Schwartz inequality we have that

$$(1) \qquad \left(\sum \frac{1}{x_i}\right)\left(\sum x_i\right) \geq \left(\sum \sqrt{\frac{1}{x_i} \cdot x_i}\right)^2 = n^2$$

and

$$(2) \quad \left(\sum x_i^2\right)\left(\sum 1\right) \geq \left(\sum x_i\right)^2 \text{ or } \left(\sum x_i^2\right)^{1/2} \cdot \sqrt{n} \geq \sum x_i.$$

Combining (1) and (2) we get that

$$\left(\sum \frac{1}{x_i}\right)\left(\sum x_i^2\right)^{1/2} \cdot \sqrt{n} \geq \left(\sum \frac{1}{x_i}\right)\left(\sum x_i\right) \geq n^2,$$

and the theorem follows. It is easy to see that we have equality if and only if $x_1 = x_2 = \ldots = x_n$.

Also solved by **Miguel Amengual** Covas, Seung-Jin Bang, Scott H. Brown, Paul S. Bruckman, Philip **A.** D. Castoro, William **Chau**, Richard I. Hess, Murray S. Klamkin, Henry S. Lieberman, David E. Manes, Can. **A. Minh**, Yoshinobu Murayoshi, Bob Prielipp, St. **Olaf** Problem Solving Group, Selvaratnam Sridharma, Sammy Yu and Jimmy **Yu**, and the Proposer.

Klarnkin showed more generally that

$$\left[\sum_{i=1}^n \frac{1}{x_i^r}\right]^p \left[\sum_{i=1}^n x_i^s\right]^q \geq n^{p+q},$$

with equality if and only if the x_i are constant. Here p, q, r, s are positive numbers such that $pr = qs$.

843. [Fall 1994] *Proposed by Bill Correll, Jr., student, Denison University. Granville, Ohio.*

Let $s(n)$ denote the sum of the binary digits of the positive integer n. Find a value for c so that

$$\sum_{n=1}^{c} \frac{1}{s(n)} = \frac{2342173}{5544}$$

Solution by David E. Manes. SUNY College at Oneonta, Oneonta, New York.

The value of c is *2050*. Let m be a nonnegative integer and n any integer such that $2^m \le n < 2^{m+1}$. Then the number of digits for n in base *2* is m + *1*. Since the leading digit for these numbers in base *2* must be *1*, it follows that the number of these integers with k ones in the base *2* representation is $\binom{m}{k-1}$, $1 \le k \le m + 1$. Therefore,

$$\sum_{n=2^m}^{2^{m+1}-1} \frac{1}{s(n)} = \sum_{k=1}^{m+1} \frac{1}{k}\binom{m}{k-1}.$$

Since

$$\frac{1}{k}\binom{m}{k-1} = \frac{1}{m+1}\binom{m+1}{k},$$

this sum can be written in closed form as

$$\sum_{n=2^m}^{2^{m+1}-1} \frac{1}{s(n)} = \frac{1}{m+1}\sum_{k=1}^{m+1}\binom{m+1}{k} = \frac{1}{m+1}(2^{m+1}-1).$$

Consequently,

$$T_m = \sum_{n=1}^{2^{m+1}-1} \frac{1}{s(n)} = \sum_{r=0}^{m}\sum_{n=2^r}^{2^{r+1}-1} \frac{1}{s(n)} = \sum_{r=0}^{m} \frac{1}{r+1}(2^{r+1}-1).$$

Then we find that

$$T_{10} = \frac{2331085}{5544} \text{ and } T_{11} = \frac{4222975}{5544}.$$

Thus $2047 = 2^{11} - 1 < c < 2^{12} - 1 = 4095$. Fortunately $s(2048) = 1$ and $s(2049) = s(2050) = 2$, so that

$$\sum_{n=1}^{2050} \frac{1}{s(n)} = \frac{2331085}{5544} + 2 = \frac{2342173}{5544},$$

and the solution is complete.

Also solved by Charles Ashbacher, Paul S. Bmckman, Mark Evans, Richard I. Hess, Michael R. Pinter, Rex H. Wu, *and the* Proposer.

845. [Fall 1994] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Let A, B, and C be subsets of $U = \{1, 2, 3, \ldots, m\}$. An ambitious student wants to prove that if $A \subseteq B$, then $A \cup (B \cap C) = (A \cup C) \cap B$ for all A, B, and C. Express in closed form the number of specific cases the student must consider.

Solution by William Chau, New York, New York.

We must consider only sets A and B such that $A \subseteq B$. There are $\binom{m}{k}$ sets B of k elements each, and each has 2^k subsets A. Therefore, the total number of choices for A and B is

$$\sum_{k=0}^{m}\binom{m}{k}2^k = \sum_{k=0}^{m}\binom{m}{k}2^k 1^{m-k} = (2+1)^m = 3^m.$$

Considering the 2^m different subsets C, one obtains a total of $2^m 3^m = 6^m$ possibilities.

Also solved by Paul S. Bmckman, Mark Evans, Stephen I. Gendler *and* Daniel Schall, David E. Manes, Rex H. Wu, *and the* Proposer.

846. [Fall 1994] *Proposed by M. A. Khan, Lucknow, India.*

Let N, L, M be points on sides AB, BC, CA of a given triangle ABC such that

$$0 < \frac{AN}{AB} = \frac{BL}{BC} = \frac{CM}{CA} = k < 1.$$

Let AL meet CN at P and BM at Q, and let BM and CN meet at R. Draw lines parallel to CN through A, parallel to AL through B, and parallel to BM through C. Let XYZ be the triangle formed by these three new lines. Prove

that:

a) Triangles *ABC*, PQR, and *XYZ* have a common centroid, and

b) If the areas of triangles PQR, ABC, and *XYZ* are in geometric progression, then k = $\sqrt{3} - 1$.

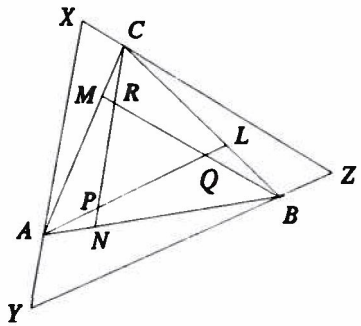


Figure 3. Problem 846

Solution by William H. Peirce, Defray Beach, Florida.

Place the figure in the complex plane and let the complex affix of each point be denoted by the corresponding lower case letter. Then

$$l = (1 - k)b + kc,\ m = (1 - k)c + ka, \text{ and } n = (1 - k)a + kb.$$

Next, P lies on line *AL*, so for some real number λ, we have

$$p = \lambda a + (1 - \lambda)l = \lambda a + (1 - \lambda)(1 - k)b + (1 - \lambda)kc.$$

Since P lies also on line CN, there is a real constant μ such that

$$p = \mu c + (1 - \mu)n = (1 - \mu)(1 - k)a + (1 - \mu)kb + \mu c.$$

Since the representation for p must be unique, we may equate the coefficients of a, those of b, and those of c in the two expressions for p, obtaining

$$\lambda = (1 - \mu)(1 - k),\ (1 - \lambda)(1 - k) = (1 - \mu)k, \text{ and } (1 - \lambda)k = \mu,$$

which we solve simultaneously to get

$$\lambda = \frac{(1 - k)^2}{1 - k + k^2} \text{ and } \mu = \frac{k^2}{1 - k + k^2}.$$

Note that the denominator $1 - k + k^2$ is positive for all real k. Substituting for λ and μ in either expression above gives the expression for p in terms of a, b, and c. Similarly, q and *r* are found, yielding

$$p = \frac{(1 - k)^2 a + k(1 - k)b + k^2 c}{1 - k + k^2},$$

$$q = \frac{(1 - k)^2 b + k(1 - k)c + k^2 a}{1 - k + k^2},$$

and

$$r = \frac{(1 - k)^2 c + k(1 - k)a + k^2 b}{1 - k + k^2}$$

We develop similar expressions for x, y, and z. Since *XZ* is parallel to ***BM*** and passes through C, there is a real constant λ such that

$$x = c + \lambda(m - b) = k\lambda a - \lambda b + (1 + \lambda - k\lambda)c.$$

Also, *XY* is parallel to CN and passes through A, so for some real μ, we have

$$x = a + \mu(n - c) = (1 + \mu - k\mu)a + k\mu b - \mu c.$$

Again we equate the coefficients of a, b, and of c in these two expressions for x to solve for λ and μ. Then we substitute back into either equation to find an expression for x. Similarly, we find y and *z*. We get

$$x = \frac{k^2 a - kb + c}{1 - k + k^2},\ y = \frac{k^2 b - kc + a}{1 - k + k^2}, \text{ and } z = \frac{k^2 c - ka + b}{1 - k + k^2}.$$

a) The centroid of a triangle is the intersection of the medians and is equal to the average of its vertices. Thus the centroid G of triangle ABC is given by

$$g = \frac{a + b + c}{3}.$$

Similarly we average the affixes for triangles PQR and *XYZ*. Easy algebra shows each centroid coincides with point G. Furthermore, the centroid of triangle *LMN*, too, is at G.

b) The area of a triangle ABC, denoted by $K(ABC)$, in the complex

plane is given by

$$K(ABC) = \pm\frac{i}{4}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}.$$

If $P = \lambda a + \mu b + \nu c$, $Q = \lambda c + pa + vb$, and $R = \lambda b + \mu c + va$, where λ, μ, and ν are real numbers such that $\lambda + \mu + \nu = 1$, then

$$K(PQR) = \pm\frac{i}{4}\begin{vmatrix} \lambda a+\mu b+\nu c & \lambda\bar{a}+\mu\bar{b}+\nu\bar{c} & 1 \\ \lambda c+\mu a+\nu b & \lambda\bar{c}+\mu\bar{a}+\nu\bar{b} & 1 \\ \lambda b+\mu c+\nu a & \lambda\bar{b}+\mu\bar{c}+\nu\bar{a} & 1 \end{vmatrix} = \pm\begin{vmatrix} \lambda & \mu & \nu \\ \mu & \nu & \lambda \\ \nu & \lambda & \mu \end{vmatrix} \cdot K(ABC)$$

Now the multiplier determinant $D(PQR)$ is given by

$$D = \begin{vmatrix} \lambda & \mu & \nu \\ \mu & \nu & \lambda \\ \nu & \lambda & \mu \end{vmatrix} = 3\lambda\mu\nu - \lambda^3 - \mu^3 - \nu^3 = 3(\mu\nu + \nu\lambda + \lambda\mu) - 1.$$

For the given triangles PQR and XYZ we have

$$D(PQR) = \frac{(1-2k)^2}{1-k+k^2} \text{ and } D(XYZ) = \frac{(1+k)^2}{1-k+k^2}$$

For $K(PQR)$, $K(ABC)$, and $K(XYZ)$ to be in geometric progression, then $D(PQR)$ and $D(XYZ)$ must be reciprocals of one another, so their product must be 1. We have

$$\frac{(1-2k)^2}{1-k+k^2} \cdot \frac{(1+k)^2}{1-k+k^2} = 1.$$

This equation simplifies to $3k^2(k^2 + 2k - 2) = 0$, whose only root in the allowable range for k is $\sqrt{3} - 1$.

Also solved by Miguel Amengual Covas, Paul S. Bruckman, Murray S. Klamkin, Henry S. Lieberman, and the Proposer.

***847.** [Fall 1994] Proposed by *Dmitry* P. Mavlo, Moscow, Russia.

From the SYMP-86 Entrance Examination: The midline of an isosceles triangle has length L and its acute angle is a. Determine the trapezoid's area, if it is known that a circle can be inscribed in the trapezoid.

Solution by George W. Rainey, Los Angeles, California.

Let the inscribed circle have radius R. Then $R = (L/2)\sin a$ and the trapezoid's altitude $h = 2R$, as seen in the figure. The trapezoid's area A is given by

$$A = \frac{(b_1 + b_2)h}{2} = L(2R) = L^2 \sin\alpha.$$

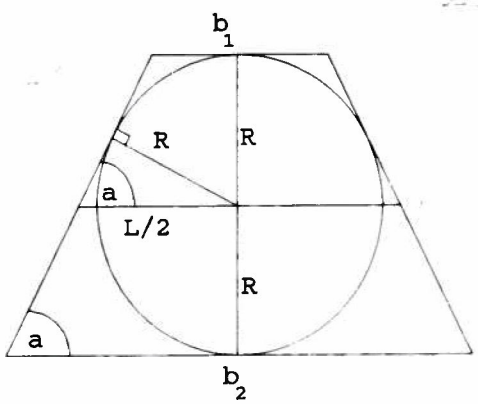


Figure 4. Problem 847.

Also solved by Alma College Problem Solving Team, Miguel Amengual Covas, Paul S. Bmckman, William Chau, Richard I. Hess, Henry S. Lieberman, Can. A. Minh, Kandasamy Muthuvel, Selvaratnam Sridharma, Rex H. Wu, and Sammy Yu and Jimmy Yu (two solutions).

848. [Fall 1994] Proposed by Rex H. *Wu*, *SUNY* Health Science Center, Brooklyn, New York.

a) Given a non-trivial group (a group having more than one element) such that, if x, y are any members, then (i) $x \neq y$ implies $x^2 \neq y^2$ and (ii) $xy = y^2x^2$, prove the group is abelian (commutative).

b) Prove part (a) if the term group is replaced by semigroup.

1. Solution by Kandasamy Muthuvel*. University of *Wisconsin-Oshkosh*, Oshkosh, *Wisconsin*.

a) Let e be the identity and x any element of the group. Then

$$x = xe = e^2x^2 = x^2 = x^2e,$$

so that $x = e$. Thus there is no non-trivial group satisfying condition (ii) and the theorem is vacuously true.

b) For any two elements x and y of the semigroup we have

$$(xy)^2 = (xy)(xy) = x(yx)y = x(x^2y^2)y = x^2(xy)y^2$$

$$= x^2(y^2x^2)y^2 = (x^2y^2)(x^2y^2) = (x^2y^2)^2 = (yx)^2.$$

By the contrapositive of (i) we have that $xy = yx$.

More generally, if $xy = y^nx^n$ for some ***fixed*** positive integer n and all members x and y of a semigroup, then it is commutative. See Problem 1400, ***Mathematics Magazine, 66*** (1993) p. 198.

II. ***Solution by Henry S. Lieberman, Waban, Massachusetts.***

b) By (ii) we have

$$x^2 = xx = x^2x^2 = (x^2)^2,$$

so then

$$xy = y^2x^2 = (y^2)^2(x^2)^2 = x^2y^2 = yx.$$

Also solved by Douglas L. Bedsaul, Paul S. **Bruckman,** William Chau, David Del Sesto, Victor G. Feser*, **Jayanthi** Ganapathy, Linda **Gellings** ***part (a) only,*** Stephen I. Gendler*, Peter **A.** Lindstrom, David E. Manes, Can. **A.** Minh, John F. Putz, Selvaratnam Sridharma, David C. Vella, Sammy Yu ***and*** Jimmy Yu, ***and the*** Proposer.

Solvers whose names are marked with an asterisk (*) showed the non-existence of the group of part (a). Many solvers proved part (b) only, stating that that was sufficient to prove part (a) also.

MISCELLANEOUS

Chapter Report

The INDIANA EPSILON Chapter (St. Mary's College) organized the mathematics department's Open House. which featured an address by Professor Thomas Bachoff on "Flatland, linkages. and interactive computer **graphics.**" The Chapter also (reports Professor Joanne Snow) prepared two mathematics activities for the kindergarten class at the Early Childhood Development Center and prepared displays for Matliematics Awareness Week.

Comment

We're lucky, you and I. Not just for being alive in this time and place (though that has a lot to be said for it), nor for being able to spend time with the Pi Mu ***Epsilon Journal*** (though that does as well), but for having mathematics.

I will explain. We have minds and the question is, what are we going to do with them? After we get through with the **dailyness** of dealing with the details of life that must be dealt with every day, that is. **The same** qucstion arises for us as a **species**: after doing what is necessary to **see** to it that **we** survive for another generation, what do we do then to **keep** our minds occupied?

There are, of course, many answers, and the variety of human mental pursuits is as amazing as the variety in human beings. Did you **know**, for example, that **there** are numerological literary critics? I **never** did until I picked up a copy of ***Triumphal Forms,*** by **Alistair** Fowler (Cambridge University Press, **1970**), who explained that Shakcspcarc's sonnets numbered 99. 126. and 145 are irregular because 153 is a triangular number. and who said (p. 200) that "Some critics regard numerology as **the** key to all literary knowledge." That is amazing.

There arc **many** answers to **the** qucstion of what **we** should occupy our minds with, but **none** is **better** than mathematics. There arc **many** that arc as good, or nearly as good, but **none** better. I do not say that because I think that mathcniatics is **the most** glorious creation of **the** human mind. It is. but **other people—poets** and philosophers perhaps—might argue **strenuously** that it is not. and **they** must **be** granted **the** right to disagree.

Whatever the degree of its glory, mathematics has matter. Mathematics has problems to be solved, problems with substance. Crossword puzzles, and the mathacrostic that appears in this issue, are problems to be solved as well, but once they are solved, what of them? They are insubstantial, and they are discarded and forgotten. Their solution does not get us anywhere.

Solving problems in mathematics does get us somewhere. First, we solve quadratic equations. Then. a couple of thousand years later, we solve cubics and, almost immediately, quartics. After another two hundred years, we show that we can't solve quintics. Not all of them, that is, but only some—which ones? Galois finds out, and starts group theory. Just a little while ago we (and it took a lot of us. working together) found all the finite simple groups. We are getting somewhere. Progress is being made. What do we do next—are we done? Certainly not: there are plenty of problems left—infinitely many, in fact—and we will never be done.

This is not so in all lines of intellectual endeavor. Take philosophy for example, whose date of birth was approximately the same as mathematics'. There are many problems and many questions that can be asked, but there are no answers. Someone said that the history of philosophy consists of attempts to answer questions that Plato asked. Philosophy does not seem to get anywhere. Progress can be said to be made—the questions become more clear—but it is not the same as progress in mathematics. A philosopher will write a paper and some other philosopher will write another paper saying that the first philosopher is an idiot. Well, not quite that, but the second philosopher will point out things that the first philosopher, who was not as acute as the first philosopher, failed to notice, or interpreted wrongly. The first philosopher (or a third. a friend of the first) can write a third paper explaining why the second philosopher is all wet. Well, not all wet, but damper than he or she should be. The cycle can go on and on, bringing us not much closer to the answer to. for example. the question of what is knowledge and how do we know it.

Theology's problem of evil will never be solved. In history, all we can have are reinterpretations. And pity the poor classicists! It is possible that, lying in an attic somewhere, there is someone's dairy for the years 1862-65 that will shed a whole new light on the Civil War. Historians thus have a hope. however slim, of getting new material, but the chance of finding Sophocles's diary is nil: classicists have a fixed amount of material and all they can do is rearrange it in different ways.

In mathematics we havc it better. The amount of material is infinite, and

we make progress. Of course, progress is made in the sciences, but it is not the same. Once the physicists construct their grand theory of everything and verify it sufficiently by experiment, that's it. They're finished. They can all be given gold watches and put out to pasture, or be put into classrooms explaining $F = ma$ to the next generation. It will take longer to wind up chemistry, but there are only finitely many elements and I think only finitely many compounds that could exist and be stable. Once everything is known, there is nothing more to find out. Biology is harder yet and we have a long, long way to go, but the end of biology is also conceivable. But mathematics will have no end, ever. The race may get tired of the subject and stop pursuing it, but that will be because the race is exhausted, not mathematics.

Another huge advantage of mathematics is that it has matter at all levels to be worked on. Very few of us have the ability and the courage to attack the Riemann Hypothesis, but more of us can do things like finding equivalent statements of it (they might be easier), or of verifying that the next few million zeros of $\zeta(s)$ lie on the critical strip (a counterexample might turn up). Contributions on lower levels can be made. May I mention the best theorem that I ever proved? It was known that the fractional parts of $\{n \cos n\}$ are uniformly distributed on $[0, 1]$ and those of $\{\cos n\}$ are not: where does the switch occur? The answer is that for any $f(n)$ that goes to infinity, no matter how slowly, the fractional parts of $\{f(n) \cos n\}$ are uniformly distributed. Not an important result, but satisfying.

The capacity for satisfaction exists at all levels. All journals of mathematics aimed at general audiences, including this one. have problem sections. They do not have them because it is the right thing to do, they have them because their readers like them. They are sometimes the most popular parts of the journals. The reason is obvious: they provide matter, matter for readers to work with and sometimes triumph over. And any reader can grapple with it.

Lucky us! All of us have matter that we can deal with. Do journals of philosophy have problem sections? I have not made a survey, but I doubt it. The vast majority of people with training in philosophy do not have any matter to occupy them. They can read and appreciate the works of the masters of the field, just as we can, but they do not have anything to do. The same holds for students of history, classics, and almost anything else.

Mathematics is wonderful. Not only does it have matter that can engage. any of us, it also gives us the satisfaction of knowing that we have mastered the matter. When you know calculus, you know it, once and for all, and for

certain. Anyone who has been teaching calculus for a while could, if locked in a room, supplied with nothing but food, water, and reams of paper and told that the door would not be unlocked until a calculus textbook had been written, do the job. It might be excruciatingly tedious, but we could produce an acceptable calculus text because we know calculus.

Not everyone is as lucky. In some fields, not only do its practitioners not have the satisfaction of knowing that they have mastered a body of knowledge, there is a question of whether there is a body of knowledge to be mastered.

Mathematicians have the assurance that comes with mastery. This has been widely observed, as by Rebecca Goldstein, a philosopher and novelist, in The Mind-Body Problem (Random House. 1983, Penguin reprint, 1993, pp. 201-202):

> Observers of the academic scene may be aware that there are distinct personality types associated with different disciplines. The types can be ordered along the line of a single parameter: the degree of concern demonstrated over the presentation of self, or "outward focus." ...
>
> Thus at the end of the spectrum occupied by the sociologists and professors of literature, where there is uncertainty as how to discover the facts, the nature of the facts to be discovered, and whether indeed there are any facts at all, all attention is focused on one's peers, whose regard is the sole criterion for professional success. Great pains are taken in the development of the impressive persona, with excessive attention given to distinguished and faultless sentence structure.
>
> At the other end, where, as the mathematicians themselves are fond of pointing out, "a proof is a proof," no concern need be given to making oneself acceptable to others; and as a rule none whatsoever is given.

To sum up, mathematics is the best of all possible places to be, intellectually. There is matter, plenty of it, that can be mastered. We cannot master all of mathematics, but in the part we have mastered there are problems to be worked on and solved, no matter what our level of brilliance is. Solving the problems can give satisfaction, and can also advance the subject. Further, the supply of problems will never dry up. We are finite but mathematics is infinite. Who could ask for anything more? We are lucky.

Play the 1995 Game!

Though the year is no longer young. there is still time to play the 1995 game. Paul S. Bruckman challenges you to represent the integers from 1 on up using the digits 1. 9. 9. and 5 in that order. For example.

$1 = -1 \cdot \sqrt{9} + 9 - 5$ $\quad 2 = 1 + 9 - \sqrt{9} - 5$

$3 = 1 - 9 + (\sqrt{9})! + 5$ $\quad 4 = -1 + 9 - 9 + 5$

$5 = 1 \cdot 9 - 9 + 5$ $\quad 6 = 1 + 9 - 9 + 5$.

You may have some difficulty with 20 and 25 Using various subterfuges, Mr. Bruckman got all the way to 139 before quitting. and closed with a four de *force* by noting that

$$1995 = (19 \cdot \sqrt{9} \cdot 5) \cdot (1 \cdot \sqrt{9} + 9 - 5).$$

After the 1995 game there is always the 1996 game to look forward to. though 1 think it may be a bit harder. We should enjoy these games while we can. since the 2000 game will not be very rewarding.

Errata

Whatever his other virtues, your editor is not very good at proofreading. The Journal always has errors. too many of them. For example, in the last issue (p. 103) the sequence was misprinted: as several readers pointed out, it should have been

1, 10, 3, 9, 5, 8, 7, 7, 9, 6, 11

which makes the next term. 5. obvious if you see the pattern.

Rex Wu found three errors in his "A note on an exponential equation" (10 (1994-99) #1, 22-25): on page 23, line 5. $(A_n, A_1, \ldots, k_n)$ should have the subscripts running from 1 to n; on page 23, line 24 in $\gcd(k_0, M)$ s the term to the right of the divides sign should be $(ck_0 - s)$; and on page 24, line 11 the four-tuple of powers of 2 should be $(2^{13}, 2^{20}, 2^{36}, 2^{30})$.

He also notes that $3^6 + 18^3 = 3^8$ provides an answer to his last question in the paper's last paragraph.

Corrections generally have a hard time catching up with the original errors, but the *Journal* will continue to print errata as space permits.

PI MU EPSILON

T-SHIRTS

The shirts are white, Hanes' BEEFY-T', pre-shrunk, 100% cotton. The front of the shirt has a large Pi Mu Epsilon shield (in black), with the line "1914 - ∞" below it. The back of the shirt has a "Π M E" tiling in the PME colors of gold, lavender, and violet. This tiling of the plane was designed by Doris **Schattschneider**, on the occasion of PME's 75th anniversary in 1989. The shirts are available in sizes large and X-large. The price is only $10 per shirt, which includes postage and handling. To obtain your shirt, send your check or money order, payable to **Pi** Mu Epsilon, to:

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