# PI MU EPSILON

# JOURNAL

**VOLUME 10 SPRING 1996 NUMBER 4**

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CONTENTS



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PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF THE

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The PI MU EPSILON JOURNAL is published at DePauw University twice a year—Fall and Spring. One volume consists of five years (ten issues) beginning with the Fall $19n9$ or Fall $19(n+1)4$ issue, $n = 4, 5, \ldots, 8$.

# THE C. C. MACDUFFEE AWARD FOR DISTINGUISHED SERVICE

On August 7, 1995, at the annual meeting of Pi Mu Epsilon at Burlington, Vermont, the C. C. MacDuffee Award for Distinguished Service was presented to Professor Eileen L. Poiani. The text of the citation, by Pi Mu Epsilon President Robert C. Eslinger, is as follows:

"It is with great appreciation and admiration that Pi Mu Epsilon presents Eileen Poiani the C. C. MacDuffee Award for Distinguished Service. Dr. Poiani has served on the Pi Mu Epsilon Council for twenty-one years. After having been elected for an unprecedented four consecutive three-year terms as Councilor, she was elected President-Elect in 1984. While serving as Pi Mu Epsilon's first woman president from 1987 to 1990, Dr. Poiani led the society through the celebration of its 75th anniversary. As Pi Mu Epsilon's ambassador to other organizations she was extraordinarily successful in securing external funding to support the goals of the society. During her tenure on the Council she personally installed over twenty chapters of Pi Mu Epsilon on college and university campuses across the United States.

"Eileen Poiani's service to the mathematical community extends far beyond Pi Mu Epsilon. Having been on the faculty of St. Peter's College since 1967, she currently holds the rank of Professor of Mathematics and serves as Assistant to the

President for Planning. She has been active in the Mathematical Association of America, providing leadership on numerous committees and serving as Governor of the New Jersey Section. In 1994 the Section honored her with its Award for Distinguished Teaching. She has a passionate interest in promoting the status of women and minorities in mathematics.

"Designated in honor of the seventh president of Pi Mu Epsilon, the C. C. MacDuffee Award for Distinguished Service was first awarded to J. Sutherland Frame in 1966. Subsequent recipients were Richard V. Andree, John S. Gold, Francis Regan, J. C. Eaves, Houston Kames, Richard Good, and Milton D. Cox."

Professor Poiani is a graduate of Douglass College, and earned her M. S. and Ph. D. degrees in mathematics at Rutgers University. Besides the items mentioned in the citation, she has been a trustee of St. Peter's Preparatory School (Jersey City, New Jersey) and of Rutgers, the State University of New Jersey. She was a member for eight years, and Chair for five, of the United States Commission on Mathematical Instruction (a commission of the National Research Council of the National Academy of Sciences). She has been a Visitng Lecturer for the Mathematical Association of America and was a Founding and National Director of the Women and Mathematics Program of the Mathematical Association of America. She has been an evaluator for the Middle States Association of Colleges and Schools. She is a member of Phi Beta Kappa, is listed in ***Who's Who in America,*** and has given two commencement addresses. She is an author and speaker on higher education issues, strategic planning, mathematics, and mathematics education.

## THE RICHARD V. ANDREE AWARDS

The Richard V. Andree Awards are given annually to the authors of the three papers written by students that have been judged by the officers and councillors of Pi Mu Epsilon to be the best that have appeared in the ***Pi Mu Epsilon Journal*** in the past year.

Richard V. Andree was, until his death in 1987, Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon long and well in many capacities: as President, as Secretary-Treasurer, and as Editor



of the ***Journal.***

The winner of the first prize for 1994 is **Scott M. Wagner,** for his paper "Group generators and subgroup lattices", this ***Journal*** 10 (1994-99) #2, 106-111.

Since there was a three-way tie for second place, there will be four awards this year. The winners are **Kevin Dennis,** for "Sierpinski *n*-gons" (with Steven Schlicker), this ***Journal*** 10 (1994-99) #2, 81-89, **Lars Serne,** for "Automorphisms of Hasse subgroup diagrams for cyclic groups", this ***Journal*** 10 (1994-99), #3, 215-220, and Julia **Varbalow,** for "An application of partitions to the factorization of polynomials over finite fields" (with David C. Vella), this ***Journal*** 10 (1994-99) #3, 194-206.

At the times the papers were written, Messers. Wagner and Seme were students at Hendrix College, Mr. Dennis at Luther College, and Ms. Varbalow at Skidmore College.

The officers and councillors of the Society congratulate the winners on their achievements and wish them well for their futures.

## Referees

The job of referee is unpaid, anonymous, time-consuming, and can be difficult. Those that take it on do a service to the profession that deserves more thanks than lists such as these provide. The ***Journal*** is grateful for the help the following people have given in the past two years.

Thomas Banchoff (Brown U.), James Becker (Purdue U.). Randall Campbell-Wright (then of the U. of Tampa), Gary Chartrand (Western Michigan U.). John Durbin (U. of Texas—Austin), Joseph Gallian (U. of Minnesota—Duluth), Jennifer Galovich (St. John's U.), Todd Hammond (Northeast Missouri State U.). Richard Johnsonbaugh (DePaul U.), Mark Kannowski (DePauw U.). John Kelingos (Vanderbilt U.), Gail Letzter (Virginia Polytechnic and State U.). Frederick Leysieffer (Florida State U.), Robert Messer (Albion Coll.). Gary Mullen (Penn State U.), Alan Pankratz (DePauw U.). Michael Plummer (Vanderbilt U.). David Stone (Georgia Southern U.). and Jingchen Tong (U. of North Florida).

# NEARNESS OF NORMALS

*Rick Mohr*
*Rose-Hulman Institute of Technology*

What is the distance between a given matrix and the set of normal matrices?

This question, given to me by Dr. Carl Cowen during my Research Experience for Undergraduates at Purdue University during the summer of 1994, is not new. And the underlying general problem—to minimize something subject to a constraint—is much older still. Anyone familiar with calculus has surely seen this idea, for example in Lagrange multipliers. Such problems arise in linear algebra as well.

The question turns up not only as a problem in minimization, but also as part of a real-world problem. Suppose you are a control theorist and want to study the stability of a feedback system. One way to gain stability information is to look at the transfer function matrix. However, analyzing it is not easy unless it is special in some way. For example, you might want the matrix to have perpendicular eigenvectors, making it a normal matrix. (This turns out to be *exactly* the property you want!) Since the transfer function matrix probably doesn't have perpendicular eigenvectors, you might approximate it with a normal matrix. The normal matrix will then give an approximation to the stability of the original system. However, to minimize the error using this estimate, you should try to find the closest normal matrix. For more information on control theory's relation to the problem, see [1].

Finding the closest normal matrix to a given matrix not only solves the question originally posed but also exhibits a solution that achieves the minimum distance. In addition, it solves the associated problem in control theory. The closest normal matrix is the focus of this paper. While it does not contain a general solution for every $n \times n$ matrix, it does contain a solution for any real $2 \times 2$ matrix. Results are also given for the closest Hermitian, skew-Hermitian, and unitary matrix to a given $n \times n$ matrix. Some of these results are in [2].

A matrix N is normal if $N^*N = NN^*$, where * denotes the conjugate transpose. Although this paper deals only with real matrices, the * notation is used because many results carry over directly to complex matrices. Alternatively, a matrix is normal if and only if it has a complete set of orthonormal eigenvectors





(see [4, p. 311]). As noted before, it is this property that makes them so useful in control theory.

To minimize the distance between matrices, we need some notion of what "distance" means. The distance between $n \times n$ matrices $A$ and $B$ will be defined as the norm of $A - B$, $\|A - B\|$, where the norm of a matrix is defined as

$$\|A\| = \max\{\|Av\| : v \in R^n \text{ and } \|v\| = 1\}$$

with the norm of a vector being the usual Euclidean norm. Although there are other definitions for the norm of a matrix, this definition, called the **2-norm**, will be the one used throughout this paper. (Another type of norm is the Frobenius norm, defined as

$$\|A\| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

The problem of finding die closest normal matrix using the Frobenius norm has already been solved [3].)

For example, suppose

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$$

Then the distance between $A$ and $B$ is

$$\|A - B\| = \left\| \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right\| = 4.$$

Why is the norm of the matrix equal to 4? Because the vector $v = (0, 1)$ maximizes $\|(A - B)v\|$, and this value is 4.

Although $B$ is a normal matrix, it is not the *closest* normal matrix. For instance, if $N$ is the normal matrix

$$N = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 2 \end{pmatrix}$$

then the distance between A and $N$ is

$$\|A - N\| = \left\| \begin{pmatrix} 0 & -2.5 \\ 2.5 & 0 \end{pmatrix} \right\| = 2.5.$$

Thus A is closer to N than B. We will see later why N is a normal matrix closest to $A$.

This definition of the norm has several important properties. The fast is unitary **invariance**. This means that if $U_1$ and $U_2$ are unitary matrices ($U$ is unitary if $U^* = U^{-1}$), then

$$\|U_1 A U_2\| = \|U_1 A\| = \|A U_2\| = \|A\|.$$

Other properties include

$$\|A \pm B\| \le \|A\| + \|B\|, \quad \|A^*\| = \|A\|, \text{ and } \|kA\| = |k|\,\|A\|.$$

Another useful tool will be the singular value decomposition of a matrix. This says that any matrix T can be written as $U \Sigma V^*$, where $U$ and V are unitary matrices and $\Sigma$ is a diagonal matrix of the form

$$\begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{pmatrix}$$

where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$ (see [4, p. 442]).

To begin, it is helpful to look at special subclasses of normal matrices. Finding the closest matrix of each special class to our given matrix will not solve the entire problem. However, these matrices can provide initial guesses for the closest normal matrix and in any case will provide bounds on just how far away the closest normal matrix is.

The first special class are Hermitian matrices. Hermitian matrices are characterized by $H = H^*$, and they are clearly normal. To find the closest Hermitian matrix to a given matrix T, note that for any Hermitian H

$$\|T - T^*\| = \|T - H - T^* + H^*\| = \|(T - H) - (T - H)^*\| \le 2\|T - H\|.$$

Thus we see that

$$\|T - H\| \ge \frac{\|T - T^*\|}{2}$$

and that equality is achieved if $H = (T + T^*)/2$. Hence, there is no Hermitian matrix closer to $T$ than $(T + T^*)/2$. Readers might notice that this is analogous to the fact that the closest real number to any complex number $z$ is $\mathrm{Re}(z) = (z + z^*)/2$.

The next subclass of normal martices to be considered are skew-Hermitian matrices. These matrices are characterized by $K = -K^*$. An argument similar to the one above will show that if $K$ is any skew-Hermitian matrix, then for a



given matrix T

$$\|T - K\| \ge \frac{\|T + T^*\|}{2},$$

with equality for $K = (T - T^*)/2$. So no skew-Hermitian matrix is closer to T than $(T - T^*)/2$. The analogous result is that the closest imaginary number to $z$ is $(z - z^*)/2$.

While this result is good, we can do better. Since adding a multiple of the identity matrix to a normal matrix results in another normal matrix, matrices of the form $K + \alpha I$, where $\alpha$ is any real scalar, are also normal and encompass all skew-Hermitian matrices. This allows us to broaden our possibilities for the closest normal matrix. To find the closest matrix of this type, notice that

$$\|T - (K + \alpha I)\| = \|(T - \alpha I) - K\|.$$

To minimize this distance, we need to find the $K$ closest to $T - \alpha I$. For $\alpha$ given value of $\alpha$, we know that K must equal

$$\frac{(T - \alpha I) - (T - \alpha I)^*}{2} = \frac{T - \alpha I - T^* + \alpha I}{2} = \frac{T - T^*}{2}$$

Since this value of $K$ is independent of $\alpha$, our problem is to minimize

$$\|T - (K + \alpha I)\| = \left\|T - \left(\frac{T - T^*}{2} + \alpha I\right)\right\| = \left\|\frac{T + T^*}{2} - \alpha I\right\|.$$

Since $(T + T^*)/2$ is Hermitian, it can be written as $U^* D U$ where $U$ is a unitary matrix and D is a real diagonal matrix. Thus

$$\left\|\frac{T^* + T}{2} - \alpha I\right\| = \left\|U^* D U - \alpha I\right\| = \|D - \alpha I\|.$$

However, the diagonal entries are just the eigenvalues of $(T + T^*)/2$. Therefore, the closest matrix of type $K + \alpha I$ to a given matrix T is

$$\frac{T - T^*}{2} = \frac{\lambda_{\max} + \lambda_{\min}}{2} I$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $(T + T^*)/2$.

Another subclass of normal matrices is the set of unitary matrices. Unitary matrices are normal since

$$U^*U = U^{-1}U = I = UU^{-1} = UU^*.$$

To solve the problem of finding the closest unitary matrix we must find a unitary matrix, R, that minimizes $\|T - R\|$. Taking the singular value decomposition of T to be $U\Sigma V^*$ [4], we have

$$\|T - R\| = \|U\Sigma V^* - R\| = \|U^*(U\Sigma V^* - R)V\| = \|\Sigma - U^*RV\|.$$

Because $U^*RV$ is also unitary, the problem reduces to finding the closest unitary matrix to Σ. As it turns out, this is die identity matrix. (The proof of this is rather long and will be omitted here.) Continuing with our analogies, this corresponds to the fact that the closest point on the unit circle to any positive real number is 1.

If $U^*RV = I$, then $R = UV^*$. Thus, the closest unitary martix to an arbitrary matrix T is $UV^*$.

This result can be extended to find the closest scalar multiple of a unitary matrix, which is also a normal matrix. Let k be a positive real number and R be a unitary matrix. We want to find values for k and R to minimize $\|T - kR\|$. Proceeding as above, we see that

$$\begin{aligned}\|T - kR\| &= \|U\Sigma V^* - kR\| = \|\Sigma - kU^*RV\| = k\|\Sigma/k - U^*RV\| \\ &= k\|\Sigma' - U^*RV\|.\end{aligned}$$

Since $\Sigma'$ has the same properties as 2, die closest unitary matrix to $\Sigma'$ is also the identity matrix. Once again, to minimize the norm, we set $U^*RV$ equal to $I$. Thus, $\|23 - kU^*RV\| = \|\Sigma - kI\|$. We saw before that the closest multiple of I to a diagonal matrix is obtained when k is the average of ther largest and smallest diagonal entries. In this case, the largest and smallest entries are $\sigma_1$ and $\sigma_n$ respectively. As a result, there is no multiple of a unitary matrix closer to T than

$$\frac{\sigma_1 \;\; \sigma_n}{2} UV^*.$$

We now have enough information to find the normal matrix closest to a 2 × 2 real matrix. It is not hard to show that any 2 × 2 real normal matrix must have one of the forms

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Note that the first form is a Hermitian matrix and the second form is a skew-Hermitian matrix plus a multiple of the identity matrix. By determining the



closest matrix of each of these forms, we can find the normal matrix closest to our given 2 × 2 matrix. So, if we are given

$$T = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

then the closest Hennitian matrix will be

$$H = \frac{T + T^*}{2} = \begin{pmatrix} w & \frac{x+y}{2} \\ \frac{x+y}{2} & z \end{pmatrix}$$

and the closest skew-Hermitian plus multiple of the identity will be

$$\begin{aligned}K + \alpha I &= \frac{T - T^*}{2} + \frac{\lambda_{\max} + \lambda_{\min}}{2} I \\ &= \begin{pmatrix} 0 & \frac{x-y}{2} \\ -\frac{x-y}{2} & 0 \end{pmatrix} + \frac{\lambda_{\max} + \lambda_{\min}}{2} I,\end{aligned}$$

where $A$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $H$. But the sum of the eigenvalues of a matrix equals the trace of the matrix, and since H has only two eigenvalues, $\lambda_{\max} + \lambda_{\min} = w + z$. Thus

$$K + \alpha I = \begin{pmatrix} \frac{w+z}{2} & \frac{x-y}{2} \\ -\frac{x-y}{2} & \frac{w+z}{2} \end{pmatrix}$$

So, to find the normal matrix closest to T, we first evaluate

$$\begin{pmatrix} W & \frac{x+y}{2} \\ \frac{x+y}{2} & z \end{pmatrix} \text{ and } \begin{pmatrix} \frac{w+z}{2} & \frac{x-y}{2} \\ -\frac{x-y}{2} & \frac{w+z}{2} \end{pmatrix}$$

Next, we find the distance between T and each of the matrices and choose the closer one. Note that it is possible that the distance will be the same, as in the following example:

$$T = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}, \quad H = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 6 \end{pmatrix}, \quad K + \alpha I = \begin{pmatrix} 4 & -2.5 \\ 2.5 & 4 \end{pmatrix},$$

$$\text{distance to closest normal} = \|T - H\| = \|T - (K + \alpha I)\| = 2.5.$$

It is interesting to note that for $2 \times 2$ matrices the solution to the closest normal problem is the same when using the Frobenius norm, but this is not the case for larger matrices.

At this point we have only candidates for the closest normal approximation to an arbitrary matrix T. On the other hand, the previous results can tell us that some matrices are ***never*** the closest normal to ***any*** matrix. Using the singular value decomposition of $T$, we have

$$\left\|T - \frac{\sigma_1 + \sigma_n}{2} UV^*\right\| = \left\|U\Sigma V^* - \frac{\sigma_1 + \sigma_n}{2} UV^*\right\|$$

$$= \left\|\Sigma - \frac{\sigma_1 + \sigma_n}{2} I\right\| < \|\Sigma - 0I\| = \|T - 0\|.$$

This shows that the zero matrix (which is normal) is never the closest normal matrix to a non-zero matrix T because there is a multiple of a unitary matrix that is closer. This can then be used to show that, if T is not a multiple of I, $\alpha I$ is never the closest normal matrix for any real scalar $\alpha$.

Although these results provide a good stepping-stone for further progess on this problem, it is still far from being solved. The $3 \times 3$ case could be solved if one could find the closest matrix of the form $\alpha U + \beta I$. Unlike the $2 \times 2$ normal martices, not all $3 \times 3$ normal matrices are Hermitian or skew-Hermitian plus a multiple of the identity. The following example shows why this third category is needed:

$$N = \begin{pmatrix} -3 & -2 & 0 \\ 0 & -3 & 2 \\ 2 & 0 & -3 \end{pmatrix} = 2\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + (-3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



It seems as though looking at subclasses of normals may provide the key to solving this problem entirely.

**Acknowledgement**

**I would like to thank Dr. Carl Cowen for all of his help during the REU program. I especially thank Dr. Roger Lautzenhauser for his time and patience in helping me prepare this paper.**

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***Rick Mohr will graduate from Rose-Hulman in May with B. S. degrees in physics and mathematics. He then plans to pursue a Ph. D. degree in theoretical physics. His interestsinclude reading, ultimate frisbee, and martial arts.***

## Watch Those Units!

Oxley banked the twin turboprop-engined Baffin CZ-410 for a better view of Isla Danzante, a steep-sided, 5-square-kilometer (3-square-mile) rock formation that jutted 400 meters (1312 feet) above the sea of Cortez just south of the popular resort town of Loreto.—Clive Cussler, *Inca Gold.* p. 318, Pocket Books, New York, 1995 reprint of the 1994 edition published by Simon & Schuster, New York. Contributed by Mark Kannowski.

# NEW PROOFS OF THE PYTHAGOREAN THEOREM

William **Koerick** and Chris **Soha**
University of North Florida

The Pythagorean Theorem has been known for thousands of years, and many proofs have been given. Here are two more, the first found by the first author and the second by the second, while students in Professor Jingcheng Tong's Modern Geometry course.

Place two congruent right triangles $ABC$ and $BEF$ such that $AB \perp BE$ as in Figure 1. Extend $EF$ and $CA$ to meet at $D$. Draw $GF$. It is easily seen that $BG \perp$ Extend CB to meet EF at

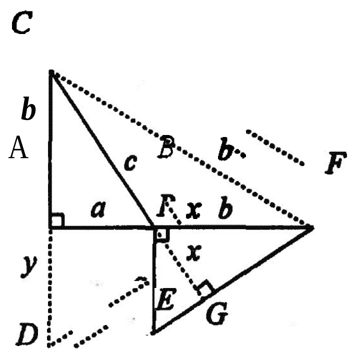


**Figure 1**

$EF$ since $\angle BEG = \angle ABC$ and $\angle ABE = 90°$.

Let $AB = BE = a$, $AC = BF = b$, $BC = EF = c$, $BG = x$, $AD = y$, and $DF = z$. Since triangles $BEG$ and $BEF$ are similar, we have $x/a = b/c$ or $x = ab/c$. Since triangles $ADF$ and $BEF$ are similar, we have $y/(a + b) = a/b$ and $z/(a + b) = c/b$. Hence $y = a(a + b)/b$ and $z = c(a + b)/b$. We can calculate the area of $\triangle CDF$ in two ways:

$$\frac{AF \cdot CD}{2} = \frac{(b + y)(a + b)}{2} \quad \text{and} \quad \frac{CG \cdot DF}{2} = \frac{(c + x)z}{2}$$

Therefore

$$(b + y)(a + b) = (c + x)z,$$

$$\left(b + \frac{a(a + b)}{b}\right)(a + b) = \left(c + \frac{ab}{c}\right)\left(\frac{c(a + b)}{b}\right),$$

$$b^2 + a^2 + ab = c^2 + ab,$$





$$a^2 + b^2 = c^2,$$

completing the proof.

For the second proof, in Figure 2 let $ABC$ and $DEF$ be two congruent right triangles such that $B$ is on $DF$ and $BC$ is perpendicular to $EF$. Draw $FG$ parallel to $BC$ to meet the extension of $AB$ at $G$.

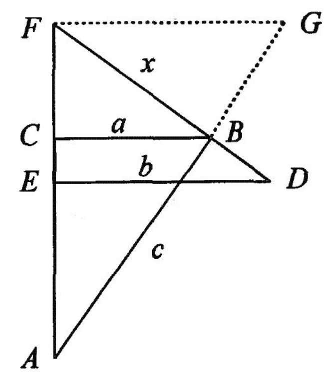


Figure 2

Let $BC = EF = a$, $AC = DE = b$, $AB = DF = c$, and $BF = x$. Because triangles $BCF$ and $DEF$ are similar, $x/a = c/b$ and $CF/x = a/c$. Hence $x = ac/b$ and $CF = a^2/b$. Because triangles $ABF$ and $ABC$ are similar, $AF/x = c/a$, so $AF = c^2/b$. Because triangles $FGB$ and $DEF$ are similar, $FG/x = c/b$ so $FG = cx/b = ac^2/b^2$. Since the area of the triangle $AFG$ is the sum of the areas of the triangle $ABC$ and the trapezoid $BCFG$, we have

$$AF \cdot FG/2 = ab/2 + CF \cdot (a + FG)/2,$$

$$(c^2/b) \cdot (ac^2/b^2) = ab + (a^2/b)(a + ac^2/b^2),$$

$$ac^4 = ab^4 + a^2(ab^2 + ac^2),$$

$$c^4 = b^4 + a^2b^2 + a^2c^2,$$

$$(c^2 + b^2)(c^2 - b^2) = a^2(b^2 + c^2),$$

$$c^2 - b^2 = a^2,$$

so

$$a^2 + b^2 = c^2.$$

---

Both authors graduated in August *1995*. Bill Koerick is enjoying his first year of marriage as well as teaching and coaching. Chris **Soha** is now a teacher at Bishop Kenny High School.

# FIBONACCI PARTIAL SUMS

*Thomas Koshy*
*Framingham State College*

It can sometimes happen that you solve a problem brilliantly when it turns out that your brilliance was not necessary. This note gives an example.

Suppose that we arrange the Fibonacci numbers $(F_1 = F_2 = 1,\ F_{n+1} = F_n + F_{n-1},\ n \geq 2)$ in a triangular array and let $S_n$ denote the sum of the numbers in the nth row, as in Figure *1*. We would like to derive a formula for $S_n$.

| | | | | | $S_n$ |
|---|---|---|---|---|---|
| 1 | | | | | 1 |
| 1 | 2 | | | | 3 |
| 3 | 5 | 8 | | | 16 |
| 13 | 21 | 34 | 55 | | 123 |
| 89 | 144 | 233 | 377 | 810 | 1453 |

Figure *1*

It is not everyone who would observe that the sums are differences of Fibonacci numbers:

$$1 = 2 - 1,\quad 3 = 5 - 2,\quad 16 = 21 - 5,\quad 123 = 144 - 21,\quad 1453 = 1597 - 144$$

That is,

$$S_1 = F_3 - F_2,\quad S_2 = F_5 - F_3,\quad S_3 = F_8 - F_5,\quad S_4 = F_{12} - F_8,$$

and so on. It appears that

$$S_n = F_{b_n} - F_{b_n - n}$$

where $\{b_n\} = \{3, 5, 8, 12, 17, \cdots\}$.

Nor is it everyone who would observe that $b_n = t_n + 2$, where $t_n = n(n+1)/2$, the nth triangular number. Since $b_n - n = t_{n-1} + 2$, we have *a* formula:

$$S_n = F_{t_n + 2} - F_{t_{n-1}+2}.$$





This can be proved by induction.

Fortunately, the observations made above are not necessary. From the formula

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

we get

$$S_n = \sum_{i=1}^{t_n} F_i - \sum_{i=1}^{t_{n-1}} F_i = (F_{t_n+2} - 1) - (F_{t_{n-1}+2} - 1),$$

the same formula as before.

The reader may enjoy, using brilliance or some other method, getting a formula for the sum of the Fibonacci numbers in the nth row of the following array, where the *nth* row has $t_n$ elements:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 2 | 3 | | | | | | | |
| 5 | 8 | 13 | 21 | 34 | 55 | | | | |
| 89 | 144 | 233 | 377 | 610 | 987 | 1597 | 2584 | 4181 | 6765 |

*After graduating from the University of Kerala, India, Thomas Koshy received his Ph. D. degree from Boston University in 1971. Since 1970 he has been on the faculty at Framingham State College, Framingham, Massachusetts.*

Do you know how to determine mathematical talent by looking at someone's scalp? See if the person's hair has square roots.

# HOW ECONOMISTS USE MATHEMATICS TO SHOW WHY SOME PEOPLE WORK SO MUCH FOR SO LITTLE

*John E. Morrill*
*DePauw University*

A standard problem in economic **theory** is to derive supply and demand relationships in various markets for goods and services. When rendered geometrically, they usually result in the familiar supply and demand graph in Figure 1. In the market for a particular consumption good or product, the supplying agents are firms and the demanding agents are individual consumers. These roles are reversed in the market for labor,

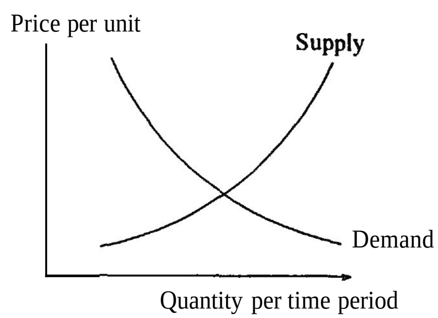


**Figure** 1. Supply and demand.

where the person is the supplier and the firm is the demander or buyer. However, in the determination of all supply and demand relationships the basic method is essentially the **same—begin** with a single person or a single firm and then aggregate the appropriate quantities demanded or supplied, at each price, to produce the market relationships. The appropriate quantities are usually found by solving straightforward optimization problems which, in the elementary theory, are based on the behavioral assumptions that firms wish to maximize economic profits and people want to maximize utility, a numerical measure of happiness or well-being. We will examine one of these **micro-relationships—the** supply of labor provided by one person. More precisely,

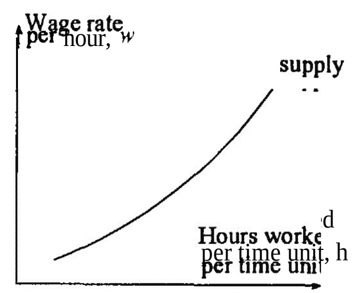


Figure 2. The supply of labor.



**we will seek** to find the **quantity** of labor, h, that a person would provide in a time period as a function of the offered wage rate, $w$. That is, we want to find the analytic description of the geometric story given by the usual economist's graph in Figure 2. (The answer to the question of why the axes in Figure **2** are labeled as they are, **with** $w$, the **independent** variable, on the vertical axis, is that **Alfred Marshall** so labeled than and economists have followed his example. Who **Alfred** Marshall was and why he did it is nicely answered in [2].)

In particular, we will **examine** the **supply** relationship for a class of commonly used utility functions **and** see **that**, though **"correct"** economic and mathematical arguments ***can*** lead to many sensible outcomes, they also lead to one that may be called paradoxical, or nonsensical.

A person's labor supply decision is quite simple. The laborer can choose to work many hours, and thus have a high level of consumption but **little** time for leisure. ***On* the** other hand, the laborer can choose to work less. consequently consuming less but having more leisure tune. **So**, there is a **tradeoff** between labor and leisure or, since we assume that the purpose of labor is to provide for consumption, a **tradeoff** between consumption and leisure.

Following the **usual** modeling assumptions, we will assume that each person has a Utility function, $U(C, L)$, where $U$ measures the utility realized in a time period from a combination of $C$ current consumption **units** and L current leisure units. Let us define variables as follows:

- T the amount of time available for labor per time period
- h the time worked, $0 \le h \le T$
- w the wage rate, $w > 0$
- $M$ the non-labor income available for consumption in a time period
- $k$ the value assigned to a unit of leisure time, $k > 0$
- $L$ the amount of leisure per time period, so $L = T - h$

Then the optimal consumption-leisure allocation follows from maximizing

$$U(C, L) = U(wh + M, k(T - h)) = U(h),$$

a function of the single decision variable $h$.

One well-known book [3, p. 63] says, "a commonly **used** utility function is the Cobb-Douglas utility function" which for this problem would be **written**

$$U(C, L) = C^{\alpha}L^{\beta},$$

where $\alpha$ and $\beta$ arc positive constants. Using this utility function (actually a family of functions), the solution to the labor-leisure problem is the value of h,

$0 \leq h \leq T$, which maximizes

$$U(h) = (wh + M)^{\alpha}(k(T - h))^{\beta}.$$

Elementary calculus shows this value to be

$$= \begin{cases} 0, & \text{if } M \geq \dfrac{\alpha w T}{\beta} \\ \dfrac{\alpha}{\alpha + \beta}T - \dfrac{\beta}{\alpha + \beta}\dfrac{M}{w}, & \text{otherwise.} \end{cases}$$

For example, if $U(h) = (15h + 45)^{2/3}(9(12 - h))^{1/3}$, then the choice that maximizes utility is $h = 7$.

This provides the labor-supply curve of Figure 2 by allowing $w$ to vary with all else held constant. The result also makes some intuitive **sense—as** $M$, the amount of outside income, increases, less work is done, and if $M$ is **sufficiently** large, no work at all will be undertaken.

However, note the implications when $M = 0$. In this case,

$$h = \frac{\alpha}{\alpha + \beta}T,$$

*independent* of $w$. That is, if a person has no outside income, then the number of hours **of work** is the same *no matter what wage is paid.* Also, $h$ is independent of $k$, so no matter what value a person puts on leisure, the number of hours of work to maximize **utility** is the same. This seems to be a paradox.

There are a number of questions that can be asked. For instance, is the paradox real or only apparent? If it is apparent, what is its resolution? If it is real, is the Cobb-Douglas model at fault? Is there a situation that makes economic sense where $M$ can be negative, and if so, is there a mathematical solution to the problem of maximizing utility? What information do the relative sizes of $\alpha$ and $\beta$ tell us about a person's preferences? (The geometry of the Cobb-Douglas family of functions is worth considering.) Who were Cobb and Douglas? **(See** [1], especially page 132.)

Some extensions are also worth **considering**. For one, how does the result change if overtime is possible? That is, suppose there is a two-tiered wage scheme, with

$$w = \begin{cases} w_1, & 0 \leq h \leq T \\ w_2, & T < h \leq S. \end{cases}$$



For another, **in** today's world there are many two-person households. What about the **problem** of maximizing

$$U(h_1, h_2) = C^{\alpha}L_1^{\beta_1}L_2^{\beta_2}$$

where $C = w_1h_1 + w_2h_2 + M$, $0 \leq h_1 \leq T_1$, $0 \leq h_2 \leq T_2$, and so on? What behavioral assumptions are needed? How does the solution relate to the single-person case?

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Here, in case you didn't get them, are the answers to the problems that **occur** later. Don't look at them before trying the problems! 1. vertical asymptote. 2. linear **independence. 3.** repeated roots. 4. radius of convergence. 5. upper and lower bounds.

Now, what is the first line in the following?

**upper bound**

**upper** bound

upper bound

upper bound

upper bound

# THE SQUARE-FREE PROPERTY OF COMPLEX POLYNOMIALS

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In a paper [1] explaining why the real and imaginary parts of a complex polynomial intersect in the complex plane at the polynomial's roots, the authors required that the parts did not display any unusual behavior that might be caused by the presence of a square factor. This note will show that the real and imaginary parts of a complex polynomial are always square-free.

Let $R(x, y)$ and $I(x, y)$ denote the real and imaginary parts of the polynomial $f(z)$:

$$f(z) = R(x, y) + iI(x, y).$$

Replacing $i$ with $-i$, we get

$$2R(x, y) = f(x + iy) + f(x - iy)$$

and

$$2iI(x, y) = f(x + iy) - f(x - iy).$$

Let us denote by $R_H(x, y)$ and $I_H(x, y)$ the terms of the real and imaginary parts with highest total degrees. For example, if $f(z) = z^3 + z + 1$, then

$$R(x, y) = x^3 - 3x^2y + x + 1, \quad I(x, y) = 3x^2y - y^3 + y,$$

and

$$R_H(x, y) = x^3 - 3x^2y, \quad I_H(x, y) = 3x^2y - y^3.$$

If R has a square factor, $R(x, y) = A^2(x,y)B(x,y)$, then $R_H(x, y) = A_H^2(x, y)B_H(x, y)$ does as well. *Thus*, if $R_H$ is **square-free** then R is **square-free.**

Now we are ready for our result.

**THEOREM.** Let $f(z)$ be a **monic** polynomial with real coefficients and degree d. Let $R(x, y)$ and $I(x, y)$ denote the *real* and imaginary parts of $f$. Then R and I are square-free.

*Proof.* It is enough to prove that $R_H$ and $I_H$ *are* square-free. Suppose that $R_H(x, y) = a^2(x, y)b(x, y)$. **Thus**





$$2R_H(x, y) = (x + iy)^d + (x - iy)^d = 2a^2(x, y)b(x, y).$$

Hence, the partial derivatives of $R_H$ give

$$\frac{\partial}{\partial x}R_H(x, y) + i\frac{\partial}{\partial y}R_H(x, y) = d(x - iy)^{d-1}$$
$$= a(x, y)(m(x, y) + n(x, y))$$

for some polynomials $m(x, y)$ *and* $n(x, y)$. Hence, *using* unique factorization of $C(x, y)$, we get $a(x, y) = c(x - iy)^r$ for some complex number $c$ and an integer $r$. Since $a(x, y)$ is real, *this* is possible **only** if $r = 0$. That is, $a(x, y)$ is a constant and $R_H$ does not have a non-trivial square factor. It follows that R is square-free.

To prove that **I** is **square-free,** we start with

$$2iI_H(x, y) = (x + iy)^d - (x - iy)^d = u^2(x, y)v(x, y)$$

and then work as we did for $R_H$ to complete the proof.

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*Steve **McCracken** prepared this paper under the supervision of Professor **Javier** Gomez-Calderon while he was a sophomore at Perm State University.*

# THE ELIMINATION OF A FAMILY OF PERIODIC PARITY VECTORS IN THE $3x + 1$ PROBLEM

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The $3x + 1$ Problem, also known as the Collate Conjecture, is traditionally credited to **Lothar** Collate at the University of Hamburg in the 1930s. **Jeffrey Lagarias** at AT & T Bell Laboratories has written an excellent expository paper on the subject [1] and we will use much of his notation here. Simply put, the $3x + 1$ problem proposes that repeated iteration of the following function $T: Z^+ \to Z^+$ will eventually lead to the value 1 for any $n > 0$:

$$T(n) = \begin{cases} (3n + 1)/2, & \text{if } n \equiv 1 \pmod 2 \\ n/2, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Define the trajectory of n to be the sequence of iterates

$$n,\ T(n),\ T^{(2)}(n),\ T^{(3)}(n),\ \cdots$$

where $T^{(i)}(n)$ represents the $i^{th}$ composition of T with itself We can **classify** these trajectories into three **types** for $n > 0$:

(i) Convergent: $T^{(k)}(n) = 1$ for some k.
(ii) Non-trivial cyclic: The sequence $\{T^{(k)}(n)\}$ eventually becomes periodic and $T^{(k)}(n) \neq 1$ for any $k \geq 1$.
(iii) Divergent: $\lim_{k \to \infty} T^{(k)}(n) = \infty$.

Define the parity vector of n to be the sequence of **0s** and **1s**

$$Q_\infty = s_0(n)s_1(n)s_2(n)\cdots$$

satisfying $s_i(n) \equiv T^{(i)}(n) \pmod 2$ for all $i \geq 0$. The parity vector completely describes the result of k iterations of T, since

$$T^{(k)}(n) = \lambda_k(n)\,n + \rho_k(n)$$

where

$$\lambda_k(n) = \frac{3^{s_0(n) + \cdots + s_{k-1}(n)}}{2^k} \quad \text{and} \quad \rho_k(n) = \sum s_i(n)\frac{3^{s_{i+1}(n) + \cdots + s_{k-1}(n)}}{2^{k-1}}$$





(see [1]).

A non-trivial cyclic trajectory has a periodic parity vector. It has yet to be determined whether or not there are any non-trivial cycles. **Thus,** in order to show that there are no non-trivial cycles, it suffices to show that any periodic **sequence** of **0s** and **1s** is not the parity vector of an integer greater than or equal to 3.

For a (0, 1)-sequence $s$, to eliminate $s$ as a parity vector (or, simply, to eliminate $s$) means to show that $s$ is not the parity vector of a positive integer. Our main result is the elimination of a family of periodic (0, 1)-sequences as parity vectors.

**THEOREM.** Let $s(k) = s_1 s_2 \cdots$, where $s_i = \begin{cases} 1, & \text{if } i \equiv 0 \pmod k \\ 0, & \text{otherwise.} \end{cases}$

There is no positive integer whose parity vector is $s(k)$ for $k \geq 2$.

While this family of (0, 1)-sequences might easily be eliminated by other means, what is of interest in this paper is not only the result but also the method used in the proof. In theory, this method can be used to eliminate any family of (0, 1)-sequences as parity vectors. Also, it gives a very good expository insight into the nature of the problem, especially the relationship with the 2-adic integers.

Any parity vector is a sequence of **0s** and **1s** and thus can be interpreted as **an** element of the 2-adic integers,

$$Z_{(2)} = \{s_0 s_1 s_2 \cdots \mid s_i \in \{0, 1\} \text{ for all } i\}.$$

**One** can define a ring structure on $Z_{(2)}$ by the usual rules for manipulating formal power series where we identify the sequence $s_0 s_1 s_2 \cdots \in Z_{(2)}$ with the formal power series $s_0 + s_1 2 + s_2 2^2 + s_3 2^3 + \cdots$ (see any standard text on p-adic numbers, e. g. **[2],** for details). Note that the integers, Z (and, in fact, the rationals with odd denominators, $Q_{odd}$) can be considered to be **subrings** of $Z_{(2)}$ by associating each positive integer p with its base-two expansion. That is, if $p = \sum_{i=0}^{\infty} b_i 2^i$ is the base-2 representation of $p \in N$, then we associate p with the 2-adic integer $b_0 b_1 b_2 \cdots \in Z_{(2)}$. This inclusion can be extended to **an** embedding of the rings Z and $Q_{odd}$ into $Z_{(2)}$ in a unique way (see **[2]**). In particular, $1/(1 - r) = \sum_{i=0}^{\infty} r^i$ if $r = 2^k$ for some $k \in N$.

Define the set of even 2-adics to be the set of all sequences, $s_0 s_1 s_2 \cdots$, such that $s_0 = 0$ and the set of odd 2-adics to be the complement of this set in $Z_{(2)}$. **Thus,** we can extend T to the **2-adics** in the obvious **manner**, that is $T: Z_{(2)} \to Z_{(2)}$

by

$$T(s) = \begin{cases} (3s+1)/2, & \text{if } s \text{ is odd} \\ s/2, & \text{otherwise.} \end{cases}$$

Similarly, we can define the parity vector $Q_\infty(s)$ for any $s \in Z_{(2)}$ just as was done in the integer case. The map $Q_\infty : Z_{(2)} \to Z_{(2)}$ is a continuous, **measure-preserving, and** onto map on the 2-adic integers $Z_{(2)}$ [1, Theorem L]. Since $Q_\infty$ is onto, every 2-adic is the parity vector of some other parity vector. Therefore we will use the terms parity vector and 2-adic interchangeably to mean any sequence of 0s and 1s.

A natural question to ask when one first encounters the Collate problem is whether or not there is a trajectory whose entries are all odd. In terms of parity vectors this is equivalent to asking if we can eliminate the parity vector $\bar{1}$ consisting of all 1s. (We will sometimes denote the repeating part of a periodic sequence by an over-bar.)

EXAMPLE 1. There is no positive integer n such that $Q_\infty(n) = 111\cdots$.
A straightforward argument that there cannot be such a trajectory might proceed as follows:

First proof d Example 1: We begin by stating some number-theoretic lemmas. The first is a standard result whose proof will be omitted.

LEMMA 1. There is no positive integer n such that $n \equiv -1 \pmod{2^k}$ for all $k \geq 1$.

If $Q_\infty(n) = s_0s_1s_2\cdots$ then define $Q_k(n) = s_0s_1\cdots s_{k-1}$.

LEMMA 2. If $Q_k(n) = 111\cdots1$ (k Is) then $n \equiv -1 \pmod{2^k}$.

Proof. If $k = 1$ then $Q_1(n) = 1 \Leftrightarrow$ n is odd $\Leftrightarrow n \equiv -1 \pmod 2$. Assume the lemma is hue for $k - 1$. Suppose $Q_k(n) \equiv 11-1$ (k 1s). Then $Q_{k-1}(T(n)) = 11\cdots1$ (k − 1 1s) **by definition of** $Q_{k-1}$. Hence, $T(n) \equiv -1 \pmod{2^{k-1}}$. So, by the definition of T, $T(n) = (3n+1)/2$. Therefore, $(3n+1)/2 + 1 = q2^{k+1}$ for some $q \in Z^+$. Therefore $3(n+1) = q2^k$. Since $3(n+1)$ is divisible by 3, $q2^k$ is also divisible by 3. But $2^k$ is not divisible by 3, so q must be divisible by 3. That is, $q = 3x$ for some x. Therefore $3(n+1) = 3x2^k$. Therefore $n + 1 \equiv x2^k$ for some x. Therefore $n \equiv -1 \pmod{2^k}$. QED

To complete the proof of Example 1, assume that there is a positive integer n such that $Q_\infty(n) = 111\cdots$. Then $Q_k(n) = 11\cdots1$ (k 1s) for all $k \geq 1$.



Therefore by Lemma 2, $n \equiv -1 \pmod{2^k}$ for all $k \geq 1$. This contradicts Lemma 1. Therefore there is no positive integer n such that $Q_\infty(n) = 11\cdots$. QED

This elementary method is straightforward, but cannot easily be generalized to eliminate other parity vectors. Let us consider another approach. Since $Q_\infty$ is one-to-one, we can eliminate a parity vector $s$ by showing that $Q_\infty^{-1}(n)$ is not a positive integer.

Second proof d Example 1. Since $T(-1) = -1$, the trajectory of −1 is −1, −1, −1, ⋯. So $Q_\infty = 111\cdots$. **Since $Q_\infty$ is one-to-one, there is no positive** integer n whose parity vector is 111⋯. QED

Thus, in order to generalize this technique to eliminate other sequences, it is necessary to have a method for computing $Q_\infty^{-1}$.

Let $s = s_0s_1\cdots$ be a periodic parity vector. By [1, Theorem B], $Q_k(n) = Q_k(n + 2^k)$ for all integers n, and there is a non-negative integer $t < 2^k$ such that $Q_k(t) = s_0s_1\cdots s_{k-1}$. Thus if $Q_k(t) = s_0s_1\cdots s_{k-1}$ then either

$$Q_{k+1}(t) = s_0s_1\cdots s_k \text{ or } Q_{k+1}(t+2^k) = s_0s_1\cdots s_k$$

since t and $t + 2^k$ are the only numbers less than $2^{k+1}$ that are congruent to t (mod $2^k$). Thus, we can recursively define a set of integers $t_k$ as follows: let $t_0 = s_0$ and let

$$t_k = \begin{cases} t_{k-1}, & \text{if } T^k(t_{k-1}) \equiv s_k \pmod 2 \\ t_{k-1} + 2^k, & \text{otherwise.} \end{cases}$$

Then $Q_{k+1}(t_k) = s_0s_1\cdots s_k$ for all k. So, the sequence of integers $t_k$ converges to $p = Q_\infty^{-1}(s)$ in $Z_{(2)}$.

Thus, by looking at the binary expansion of $t_k$ for sufficiently large k, one can conjecture what the 2-adic digits of p might be. (For example, if $p$ is rational, its digits will be eventually repeating.) It is then a simple matter of verifying that the conjectured value of p is, in fact, $Q_\infty^{-1}$, by directly computing the parity vector of p. If $p$ is not a positive integer, we have successfully eliminated the vector $s$.

EXAMPLE 2. Let us eliminate the parity vector $s = 001$. By definition, $t_0 = 0$, $t_1 = 0$, $t_2 = 4$, etc. We continue computing t in a similar manner until we reach $t_{14} = 13108$. In (reversed) binary form this number is 00101100-110011, and we see a pattern developing in the binary expansion. We then conjecture that $p = 0010110 = 415$. To verify, we check the parity vector of $4/5$. Since $T(4/5) = 2/5$, $T(2/5) = 1/5$, and $T(1/5) = 415$, die parity vector" of $4/5$ is $\overline{001}$ and we have eliminated this parity vector.

If $\{s(k) \mid k \in N\}$ is a family of parity vectors, one can use this technique to determine $p(k) = Q_\infty^{-1}(s(k))$ for the fast few values of $k$. Using these values we can conjecture what $p(k)$ might be for any $k$. Verification that $p(k) = Q_\infty^{-1}(s(k))$ for all $k$ again can be obtained by direct computation of the parity vector of $p(k)$. This is the method used in the proof of the theorem.

EXAMPLE 3. Let $s(k) = \overline{s_0 s_1 \cdots s_k}$ where $s_i = 1$ if $i = k$ and 0 otherwise. Then a calculation similar to the one used in Example 2 yields the following results.

| k | $s(k)$ | $p(k)$ (2-adic expansion) | $p(k)$ (base 10) |
|---|---|---|---|
| 0 | $\overline{1}$ | $\overline{1}$ | $-1$ |
| 1 | $\overline{01}$ | $\overline{01}$ | 2 |
| 2 | $\overline{001}$ | $001\overline{0110}$ | 4/5 |
| 3 | $\overline{0001}$ | $0001\overline{010001101110}$ | 8/13 |
| 4 | $\overline{00001}$ | $00001\overline{0101100010000110100111011110}$ | 16/29 |

By looking at the values of $p(k)$ in base 10, we are led to the conjecture

$$p(k) = \frac{2^k}{2^{k+1} - 3} \quad \text{for } k \in \mathrm{N}.$$

Having conjectured the values of $p(k)$, we are now ready to prove the theorem.

***Proof.*** Let $p(k) = 2^k/(2^{k+1} - 3)$. Then

$$T^{(k)}(p(k)) = T^{(k)}\left(\frac{2^k}{2^{k+1} - 3}\right) = \frac{1}{2^{k+1} - 3}$$

So,

$$T^{(k+1)}(p(k)) = T(T^{(k)}(p(k)) = T\left(\frac{1}{2^{k+1} - 3}\right)$$

$$= \frac{3\left(\frac{1}{2^{k+1} - 3}\right) + 1}{2} = \frac{2^k}{2^{k+1} - 3} = p(k).$$

Therefore, the trajectory of $p(k)$ is cyclic and the parity vector of $p(k)$ is equal



to $\overline{00 \cdots 01}$ (k 0s) $= s(k)$. It is clear that this will always lead to a fraction for $k > 1$, since the numerator is a power of 2 and the denominator is an odd number greater than 1. Thus $s(k)$ is not the parity vector of a positive integer for $k > 1$. QED

## Acknowledgment

We would like to dedicate this paper to Dr. Kenneth Monks for all of his time, effort, and assistance.

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*The authors wrote this paper while they were senior mathematics majors at the University of Scranton, under the direction of Professor Kenneth Monks. Carolyn Farruggia is currently studying biostatistics at Drexel University, Michael Lawrence is a systems analyst with the SEI Corporation in Wayne, Pennsylvania, and Brian Waterhouse is pursuing a Ph. D. degree at the University of Rochester.*

# Two Ways *ARE* Better Than One

*J. h? Boyd and P. h?Raychowdhury*
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**When** we glance through the problem sections of mathematical journals, we often wonder how the proposers of the problems ever discovered their results in the first place. There is no general rule to guide an explorer to a pretty sight, but a procedure that has given us several pleasant surprises is one which is often employed in **ninth-grade** geometry. It is to compute some quantity correctly in two different ways. If the results **are** $P$ and $Q$, then P = Q.

We will apply this procedure to show that in any right triangle whose legs $a$ and $b$ and hypotenuse $c$ all have integer lengths, $ab/(a+b+c)$ is always an integer. We will then get the same result in another way. Some examples are

| $a$ | $b$ | $c$ | $ab/(a+b+c)$ |
|---|---|---|---|
| 4 | 3 | 5 | 1 |
| 12 | 5 | 13 | 2 |
| 8 | 15 | 17 | 3 |
| 24 | 7 | 25 | 3 |
| 20 | 21 | 29 | 6 |

We will get this by **calculating** the area of the triangle in two different ways. (See Figure 1.) The first way is $ab12$. The second way is as the sum of the areas of the subtriangles $OAB$, $OBC$, and $OCA$ formed with $O$, the center of the inscribed circle. This sum is

$$rc/2 + rail + rb/2.$$

It follows that

$$ab = r(a+b+c)$$

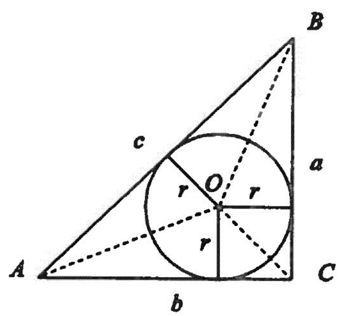


Figure 1. Right triangle and inscribed circle.





or

$$r = \frac{ab}{a+b+c}$$

It remains only to show that r is an integer. It is a standard exercise in elementary plane geometry to show that $r = (a+b-c)/2$. We **know that** $a+b$ **is even if and only if** $(a+b)^2$ **is even.**. Since $(a+b)^2 = a^2 + b^2 + 2ab$, it follows **that** $a^2 + b^2$ is even if and only if $(a+b)^2$ is even. Therefore, $a + b, a^2 + b^2 = c^2$,*andcareallevenoralloddtogether.* Thus $a + b - c$ is even and $r$ is an integer.

Another way to get this is to remember that in a right triangle with integer sides,

$$a = 2kmn, \quad b = k(m^2 - n^2), \quad c = k(m^2 + n^2)$$

for some integers $k$, $m$, and $n$. Then

$$\frac{ab}{a+b+c} = \frac{(2kmn)(k(m^2-n^2))}{2kmn + k(m^2-n^2) + k(m^2+n^2)}$$

$$- \frac{2k^2mn(m+n)(m-n)}{2km(n+m)} = kn(m-n),$$

an integer. Two ways **are** better than one!

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# DETERMINING A DAY OF THE WEEK

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Have you ever needed to know the day of a week that a certain date falls on, but didn't have a calendar handy? This note will show you how to determine it.

If the date is in the current month, just add, modulo seven. If May 5 is a Thursday, to find what day of the week May 28 is, add 23 to Thursday: Thursday + 23 = Thursday + 2 = Saturday.

If the date is in a future month, we need to remember

Thirty days hath September / April, June, and November
All the rest have thirty-one / Except February.

Or, we can remember how many days more than 28 each month has:

| Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0, 1 | 3 | 2 | 3 | 2 | 3 | 3 | 2 | 3 | 2 | 3 |

If February has 28 days, March is identical to February except for the addition of days 29-31. If February has 29 days, then the days in March are shifted to the right on the calendar by one day. If a month has 30 days, the day of the week is two days to the right for the next month, and three days to the right for months with 31 days. So, if May 5 is Thursday, June 5 will be Thursday + 3 = Sunday and July 5 will be Sunday + 2 = Tuesday.

Suppose that January 1 is a Sunday and you want to know what day of the week June 1 is so that you can plan your vacation. If it is not a leap year, then June 1 will **fall on Sunday + (3 + 0 + 3 + 2 + 3) = Sunday + 11 = Sunday + 4 =** Thursday.

The **sum** of all the numbers in the table is 29 (in a non-leap year), so if January 1 is on Sunday, January 1 of the next year will be on Sunday + 29 = Sunday + 1 = Monday.

For leap years, if February 29 comes between your two dates, you need to add one more for the extra day. For example, 1996 is a leap year. (Leap years are those years that can be evenly divided by four and are those years when Americans are supposed to vote for the President of the United States.





Century years are not leap years unless the century is divisible by four. So, 1900 and 2100 are not leap years, but 2000 will be.) Since January 1, 1996 was on a Monday, March 1, 1997 will be on Monday + 2 (for the leap year) + 3 + 0 (for January and February 1997) = Monday + 5 = Saturday. January 1, 1997 is on a Wednesday, so, since 1997 and 1998 are both non-leap years, March 1, 1998 will be on Wednesday + 1 + 3 + 0 = Wednesday + 4 = Sunday.

The idea can be used to determine the day of the week for dates more than one year in the **future**. Since January 1,1994 was on a Saturday, January 1, 1997 will be on Saturday + 1 (for 1994) + 1 (for 1995) + 2 (for 1996) = Saturday + 4 = Wednesday. For another example, we can find the day of the week for New Year's Eve, 1999. Since January 1,1996 was on a Monday, January 1,2000 will be on Monday + 2 + 1 + 1 + 1 =Monday + 5 = Saturday. So, December 31, 1999 will be a Friday.

Working backwards you can determine the days of the week for previous months and years. For example, since January 1, 1994 was a Saturday, you can find what day April 15, 1992 was by going back to January 1,1992 — Saturday - 3 (1992 had a leap year day) = Thursday—and then forward to April 15: Thursday + 3 (January) + 1 (leap year February) + 3 (March) + 14 (April 1 to April 15) = Thursday + 21 = Thursday.

So, the next time you **find** yourself without a calendar and wanting to know what day of the week a certain date falls on, just remember (if the date is in the future)

add one for each year without a leap day and add two if a leap day is involved

add the month numbers

add (or subtract) the difference in days.

Here is a last example to illustrate this: January 6, 1994 was a Tuesday; what day will July 4, 1997 fall on? Tuesday + 2 + 1 (now we are at January 6, 1997) + 3 + 0 + 3 + 2 + 3 + 2 (July 6, 1997) - 2 = Tuesday + 14 = Tuesday.

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# NOTHING IN MODERATION, EVERYTHING IN EXCESS: A NEW WEIGHTED STATISTIC ON PERMUTATIONS

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A statistic on a set of permutations is a function which associates to each permutation some non-negative integer. One of the best known permutation statistics is the major index, which is computed by weighting descents by position. Another statistic on permutations involves excedances—when a number exceeds its position. In this paper, we will consider the **bivariate** distribution that occurs when permutations are grouped according to both the number of excedances and a weight similar to the major index (see Table I). I will call this distribution the P-Distribution.

Before delving into theorems about the excedance-weight distribution, some basic definitions and propositions are needed. Throughout this paper, we will use one-line notation for permutations, writing $w_1 w_2 w_3 \cdots w_n$, where $w_i$ is the number in position i.

We will say that a permutation has an **excedance** where a number is greater than its position. A permutation has a **nonexcedance** where any number is equal to or less than its position.

For example, the permutation 24135 has excedances in positions one and two; positions three, four and five contain nonexcedances.

Permutations can be weighted by summing positions, in this case positions of the excedances. Thus 24135 has a weight of 3. Notice that the number 1 can never exceed, also, any number other than 1 will always exceed in position 1. The number n will exceed any position (except n) while position n can never be exceeded.

Let $S_n$ denote the set of permutations on $\{1, 2, \dots, n\}$ and $S(n, j, k)$ denote the subset of S consisting of permutations which have j excedances and weight k. We will let $N(n, j, k)$ denote the number of elements in the set $S(n, j, k)$. We let $S(n, j)$ indicate the "excedance block" of all permutations in $S_n$ with j





excedances.

The following propositions are easy to establish:

**Proposition 1:** If $x$ is the first position exceeded, positions $1, 2, \cdots, (x - 1)$ are fixed; that is,. $w_i = i$ for $1 \le i \le x - 1$.

**Proposition 2:** The weights k for $S(n, j)$ are exactly the sequence of consecutive integers from $j(j + 1)/2$ to $nj - j(j + 1)/2$.

Proof: The sum of the first j positions is $j(j + 1)/2$, while the sum of the last j positions is $nj - j(j + 1)/2$.

Looking at the distribution data in Table 1, we notice that for each n the sequence of values $N(n, 1, k)$ is $2^m - 1$, $m = n - 1, n - 2, \cdots, 1$, so $N(n, 1, k)$ is apparently $2^{n-k} - 1$. Since every $N(n, 1, k)$ gives the same sequence, with the inclusion of one more term for each successive n, could we establish this result by recursion?

Let us consider an example: the set $S(6, 1, 3)$ contains the permutations 124356, 125346, 125436, 126345, 126354, 126435, and 126453. If we remove the 6 from each permutation where 6 is at the end, we get exactly the elements of $S(5, 1, 3)$: 12435, 12534, 12543 In the other permutations, if we swap position 3 (where the 6 is) and position 6, then remove the 6, we again get 12534, 12435, and 12543, plus the identity 12345. This same process works in general for $j = 1$:

**THEOREM 1.** $N(n, 1, k) = 2N(n - 1, 1, k) + 1$, $1 \le k \le n - 1$.

*Proof:* (i) $N(n, 1, k) \ge 2N(n - 1, 1, k) + 1$. Let $m = N(n - 1, 1, k)$. Consider $u \in S(n - 1, 1, k)$. Place n at the end, to form $un = v \in S_n$. Since $v_n = n$, the only excedance in v is in position k. Make a copy of the new permutation v and switch $v_n$ with $v_k$ to get $v^*$, so that $v_k^* = n$ and $v_n^* = v_k$ (see Example 1.1). Since position n can never be exceeded and n always exceeds anywhere but in position $n$, position k will hold the only excedance. Perform the same procedure on each of the m permutations in $S(n - 1, 1, k)$, obtaining 2m permutations (the v's and the $v^*$'s). In addition, act on the identity permutation, $12 \cdots n - 1$ by placing n at the end and then switching the nth and kth positions as before. Position k will now be exceeded

$u = 12354 \in S(5, 1, 4)$
$v = 123546 \in S(6, 1, 4)$
$v^* = 123645 \in S(6, 1, 4)$

Example 1.1

and we have a total of 2 N(n - 1, 1, k) permutations, each of which is an element of $S(n, 1, k)$.

(ii) $N(n, 1, k) \le 2N(n-1, 1, k) + 1$. Choose any permutation $v \in S(n, 1, k)$. The letter n is either in position n or position k (if it were elsewhere, that position would also have an excedance). If $v_n = n$, remove n to get $u \in S_{n-1}$. The letter which was an excedance in v is still an excedance in $u$ in the same position. There are no other excedances since there was only one to begin with, thus $u \in S(n-1, 1, k)$. If $v_k = n$, swap $v_n$ and $v_k$ to obtain a permutation $v^*$ where $v_n^* = n$ and $v_k^* = v_n = g$ for some $g < n$. Remove n as

$$v = 1253467 \in S(7, 1, 3) \mapsto u = 125346 \in S(6, 1, 3)$$
$$v = 1274563 \in S(7, 1, 3) \mapsto u = 123456 \in S(6, 1, 3)$$
$$v = 1273456 \in S(7, 1, 3) \mapsto u = 126345 \in S(6, 1, 3)$$

Example 1.2

before to get a permutation $u \in S_n$, (see Example 1.2). I claim $g < k$. If $g < k$, some other position in addition to k would have been exceeded in $v \in S_n$. If $v_n = g < k$, then $v_g$ would have to be a number smaller than g since there is only one excedance. However, Proposition 1 states (hat all positions up to k are fixed, therefore $v_g = g$. Thus $g \ge k$. If $g = k$, our new permutation $u \in S_{n-1}$ is the identity permutation (the "+ 1" of the equation). Otherwise, we have a permutation in $S(n-1, 1, k)$. ■

The sequence of numbers $N(n, j, k)$ can now be expressed in closed form by using the preceding theorem and induction on n and k.

Corollary 1.1. $2^{n-k} - 1 = N(n, 1, k)$ for $1 \le k \le n-1$.

Now compare $N(n, j, k)$ with $N(n+1, j, k+j)$ in Table I. The highest few values in block $S(n, j)$ appear in block $S(n+1, j)$. However, the entire excedance block does not carry through from n to n + 1; only the smallest $n - j$ weights from one block appear in the corresponding block for n + 1. What could be so special about the permutations with these weights? First, we note that these permutations always have a 1 in position 1 (hence a non-excedance) while the permutations which do not carry over may or may not have an excedance in position 1. In fact there are no mutations with excedances in position 1 which correspond to the highest $n - j - 1$ weights in any block. These facts follow easily from Proposition 2.

Now let's take a closer look at the role of position 1 for carryover permutations. Consider $13254 \in S(5, 2, 6)$ and $124365 \in S(6, 2, 8)$. Writing



one above the other, $\begin{matrix} 1 & 3 & 2 & 5 & 4 \\ 1 & 2 & 4 & 3 & 6 & 5 \end{matrix}$, we notice that the letters in each position differ by one. So if we add one to each letter in 13254 and then place a 1 at the front, we construct 124365. This same process works for each of the carryover permutations:

THEOREM 2. $N(n, j, k) = N(n+1, j, k+j)$ for all n and j and the top $n - j$ values of k.

Proof: Let us define a map $\phi : S_n \to S_{n+1}$, given by $\phi(v) = w$ where $w_1 = 1$, $w_i = v_{i-1} + 1$, $(i = 2, \cdots, n)$. In other words, $w$ is obtained by adding one to each letter of v and then placing a 1 at the beginning of the new permutation. I claim that if $v \in S(n, j, k)$, then $\phi(v) \in S(n+1, j, k+j)$.

Let $v \in S(n, j, k)$. By Proposition 1, we know that up to the first exceeded position all letters are fixed, $v_1 = 1$, $v_2 = 2$, etc. Apply $\phi$ (see Example 2.1). Every letter increases by 1 and its position is now one greater. Any fixed point in v corresponds to a fixed point in w, a non-excedance corresponds to a non-excedance, and each of the j excedances corresponds to an excedance; everything is just one position higher. Thus, the weight of w increases by j (adding 1 for each of the j excedances of v). Therefore, $w \in S(n+1, j, k+j)$.

$$v = 124563 \in S(6, 3, 12)$$
$$\phi \to 235674$$
$$w = 1235674 \in S(7, 3, 15)$$

Example 2.1

It is easy to see that $\phi$ can be reversed so that we can recover v from w, establishing $\phi$ as a bijection. ■

One of the most striking symmetries in the P-distribution is that the numbers $N(n, j, k)$ are symmetric with respect to excedance blocks. In order to understand why, consider $124365 \in S(6, 2, 8)$ and $143562 \in S(6, 3, 11)$. In the first permutation, excedances occur in positions 3 and 5 while in the second permutation the excedances are in positions 2, 4, and 5. Non-excedances are found in 1, 2 and 4 for the former and in 1 and 3 in the latter (and the inconsequential position 6). Thus, the first five letters in each permutation are of the form nnene and nenee. Look closely and you can see that these are reverse mirror images. That is, one is the other written backwards with $n$'s and $e$'s switched. This same unusual pattern is found throughout all the excedance blocks, and is the basis for the proof of the following theorem:

THEOREM 3. $N(n, j, k) = N(n, n-j-1, k')$ where $k, k'$ range together from the highest to lowest weights for their respective number of excedances.

*Proof.* Let $v \in S(n,j,k)$. Find $v_n$, but reverse the order of the first $n-1$ positions to get $v'$. Next, take the complement of $v'$ with respect to $n+1$, getting $v''$. That is,

v: $132654 \in S(6, 2, 6)$

$v'$: 562314

$v''$: $215463 \in S(6, 3, 9)$

Example 3.1

$$v = v_1 v_2 \cdots v_{n-1} v_n,$$

$$v' = v_{n-1} v_{n-2} \cdots v_2 v_1,$$

$$v'' = (n+1-v_{n-1}) \cdots (n+1-v_{n-i}) \cdots (n+1-v_n)$$

(see Example 3.1).

If $v_m > m$ in σ, I claim $v''_{n-m} \le n-m$. In $v'$, $v_m$ is in position $n-m$. In $v''$, position $n-m$ contains $n+1-v_m$. But if $v_m > m$ then $v_m \ge m+1$, and so

$$v''_{m-n} = n+1-v_m \le n+1-(m+1) = n-m.$$

Similarly, if $v_m \le m$ then

$$v''_{n-m} = n+1-v_m \ge n+1-m > n-m.$$

On the other hand, if $v''_{n-m} \le n-m$, then

$$v''_{n-m} = n+1-v_{n-(n-m)} = n+1-v_m \le n-m.$$

so $v_m > m$. Similarly, if $v''_{n-m} > n-m$, then

$$v''_{n-m} = n+1-v_{n-(n-m)} = n+1-v_m > n-m.$$

so $v_m \le m$.

Thus the **non-excedances** in v'' come from excedances in v and the excedances in v'' come from the non-excedances in $v$. We conclude that v has $j$ excedances if and only if $v''$ has $n-j-1$ excedances.

Now, all that is left to show is that these reversals actually land us in the proper places for excedances and so give the correct value of $k'$. In other words, we want to show that if a permutation in $S(n, j)$ has an excedance in position i, then the corresponding permutation in $S(n, n-j-1)$ has a non-excedance in the "swapped position.

First, from Proposition 2 we know that the least possible k is $1+2+\cdots+j$ and the least $k'$ is $1+2+\cdots+(n-1-j)$. Also from Proposition 2, it is obvious that $k' > k$ for $j < (n-1)/2$. Since the weights form a consecutive sequence, each pair $k, k'$ **differs** by a constant. We have



$$k' - k = (1+2+\cdots+n-1-j) - (1+2+\cdots+j)$$
$$= (j+1) + (j+2) + \cdots + (n-1-j) = (n^2-n-2nj)/2.$$

So $k' = (n^2-n-2nj)/2 + k$.

Let $e_1, e_2, \cdots, e_j$ be the excedance places for $v$ and let $f_1, f_2, \cdots, f_{n-j}$ be the non-excedance places for v. Recall that $f_{n-j} = n$, so $n-f_1$, $n-f_2$, $\cdots$, $n-f_{n-j-1}$ are the excedance places for $v''$. The sum of all excedance places in v is $\sum_{i=1}^{n} e_i = k$, and the sum of all positions is $\sum_{i=1}^{n} i = n(n+1)/2$, so the sum of all nonexcedance places in v is

$$\sum_{i=1}^{n-j} f_i = (n(n+1)/2) - k$$

and

$$\sum_{1=1}^{n-j-1} r_i = (n(n+1)/2) - k - n.$$

Therefore the sum of all excedance places in $v''$ is

$$\sum_{i=1}^{n-j-1} (n-f_i) = n(n-j-1) - \sum_{i=1}^{n-j-1} f_i$$
$$= n(n-j-1) - ((n(n+1)/2) - k - n)$$
$$((n^2-n-2nj)/2) + k = k'. \blacksquare$$

We have shown that the P-Distribution on permutations of $\{1, 2, \cdots, n\}$ has several interesting symmetry properties. One might also consider whether or not similar **theorems may** hold for multiset **permutations**. In a preliminary analysis we noted some symmetries in special cases, but more significant results await farther research.

A table of P-distributions for n = 4, 5, 6, and 7 follows. In the table, $N_i$ stands for N(i, j, k).

**Those** interested in further reading should see Stanley, Richard P., Enumerative Combinatorics, volume 1, **Wadsworth** and Brooks. Belmont. California, 1986.

| $j$ | $k$ | $N_4$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 2 |
| 1 | 2 | 3 |
| 1 | 3 | 1 |
| 2 | 3 | 2 |
| 2 | 4 | 3 |
| 2 | 5 | 1 |
| 3 | 6 | 1 |

| $j$ | $k$ | $N_5$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 15 |
| 1 | 2 | 7 |
| 1 | 3 | 3 |
| 1 | 4 | 1 |
| 2 | 3 | 31 |
| 2 | 4 | 17 |
| 2 | 5 | 14 |
| 2 | 6 | 3 |
| 2 | 7 | 1 |
| 3 | 6 | 15 |
| 3 | 7 | 7 |
| 3 | 8 | 3 |
| 3 | 9 | 1 |
| 4 | 10 | 1 |

| $j$ | $k$ | $N_6$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 31 |
| 1 | 2 | 15 |
| 1 | 3 | 7 |
| 1 | 4 | 3 |
| 1 | 5 | 1 |
| 2 | 3 | 115 |
| 2 | 4 | 69 |
| 2 | 5 | 68 |
| 2 | 6 | 32 |
| 2 | 7 | 14 |
| 2 | 8 | 3 |
| 2 | 9 | 1 |
| 3 | 6 | 115 |
| 3 | 7 | 69 |
| 3 | 8 | 68 |
| 3 | 9 | 32 |
| 3 | 10 | 14 |
| 3 | 11 | 3 |
| 3 | 12 | 1 |
| 4 | 10 | 31 |
| 4 | 11 | 15 |
| 4 | 12 | 7 |
| 4 | 13 | 3 |
| 4 | 14 | 1 |
| 5 | 15 | 1 |

| $j$ | $k$ | $N_7$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 63 |
| 1 | 2 | 31 |
| 1 | 3 | 15 |
| 1 | 4 | 7 |
| 1 | 5 | 3 |
| 1 | 6 | 1 |
| 2 | 3 | 391 |
| 2 | 4 | 245 |
| 2 | 5 | 260 |
| 2 | 6 | 146 |
| 2 | 7 | 99 |
| 2 | 8 | 32 |
| 2 | 9 | 14 |
| 2 | 10 | 3 |
| 2 | 11 | 1 |
| 3 | 6 | 675 |
| 3 | 7 | 445 |
| 3 | 8 | 522 |
| 3 | 9 | 385 |
| 3 | 10 | 219 |
| 3 | 11 | 105 |
| 3 | 12 | 47 |
| 3 | 13 | 14 |
| 3 | 14 | 3 |
| 3 | 15 | 1 |

| $j$ | $k$ | $N_7$ |
|---|---|---|
| 4 | 10 | 391 |
| 4 | 11 | 245 |
| 4 | 12 | 260 |
| 4 | 13 | 146 |
| 4 | 14 | 99 |
| 4 | 15 | 32 |
| 4 | 16 | 14 |
| 4 | 17 | 3 |
| 4 | 18 | 1 |
| 5 | 15 | 63 |
| 5 | 16 | 31 |
| 5 | 17 | 15 |
| 5 | 18 | 7 |
| 5 | 19 | 3 |
| 5 | 20 | 1 |
| 6 | 21 | 1 |

*This paper is based on the senior thesis of Ann Marie Paulukonis, written in 1993-94 under the direction of Professor Jennifer Galovich.*

# A SOLUTION STRATEGY FOR DIFFERENTIAL EQUATIONS

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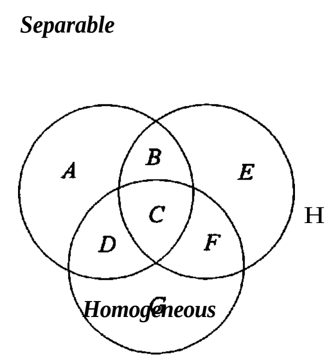
A student in an introductory differential equations class, after being exposed to the various kinds of first-order equations, asked in what order the solution methods should be tried. For example, if separating variables was unsuccessful, what next? This was a reasonable question since the standard treatment of first-order ordinary differential equations has been described as "a collection of special 'methods,' 'devices,' 'tricks,' or recipes,' in descending order of kindness!" *[4,* p. 25]. In particular, the relationships that exist, or fail to exist, among separable, exact, homogeneous, and linear equations are not always made clear.

The Venn diagram on the right divides the space of first-order equations into eight parts, for which examples are given, when possible both linear (L) and nonlinear (NL). They may be found useful even in differential equations courses that no longer emphasize special solution methods, and may be found of interest in any event.




For consistency, all equations will be written in the differential form $M(x, y)\,dx + N(x, y)\,dy = 0$.

**A.** ***Separable, but neither exact nor homogeneous:***

(L) $(y - 1)\,dx + dy = 0$

(NL) $(xy^3 - 4x)\,dx + (x^2y + y)\,dy = 0$

**B.** ***Separable and exact, but not homogeneous:***

(L) $x\,dx + (y + 1)\,dy = 0$

(NL) $(xy^2 - 4x)\,dx + (x^2y + y)\,dy = 0$

*C. Separable, exact, and homogeneous:*

(L) $2xy\,dx + x^2\,dy = 0$

(NL) $x^2\,dx - y^2 dy = 0$

*D. Separable and homogeneous, but not exact:*

(L) $y\,dx - x\,dy = 0$

(NL) $x^2 y\,dx - y^2 x\,dy = 0$

*E. Exact, but neither separable nor homogeneous:*

(L) None: if exact, then separable

(NL) $(x^2 + 2y^2)\,dx + (4xy - y^2 + 1)\,dy = 0$

*F. Exact and homogeneous, but not separable:*

(L) None: if exact, then separable

(NL) $(x^2 + 2y^2)\,\&\; + (4xy - y^2)\,dy = 0$

*G. Homogeneous, but neither exact nor separable:*

(L) $(y + x)\,dx - x\,dy = 0$

(NL) $y^2\,dx + (3xy - 1)\,dy = 0$

*H. Neither separable, exact, nor homogeneous:*

(L) $(xy - x^3)\,\& + dy = 0$

(NL) $y^2\,dx + (3xy - 1)\,dy = 0$

For those differential equation courses that still treat these types of equations in some detail (and such an approach is pedagogically defensible), here is a solution strategy that can be offered to a student facing a first-order ordinary differential equation:

First, *try* to ***separate variables.*** (Some equations, *as*

$$y' = 1 + x + y^2 + xy^2$$

are not instantaneously recognizable as separable. Scott [5] has a simple test for separability. A less useful criterion is provided by Plaat [2, ex. 8, p. **38]**. Plaat [3] has some interesting comments on the algorithm for solving an equation by separation of variables.)

Next, see if the equation is ***homogeneous—that*** is, see if it can be written in the form $dy/dx = f(y/x)$. This is *as* easy (or **as difficult)** as recognizing separability. The change of variable $y = ux$ gives a separable equation for $u$.

Then see if the equation is ***linear,*** in either variable. If it is, then multiplication by the proper integrating factor leads to the solution.

There is a test involving partial derivatives to see if an equation is ***exact.*** If



it is not **exact,** there **may** be an integrating factor that will make it exact. There is a **useful** table of integrating factors in **[1,** p. **28].**

If the equation has *still* not yielded, it **may** have a special form that a change-of variables will change to a solvable equation. The Bernoulli, **Riccati,** and **Clairaut** equations are examples.

If the equation is a textbook exercise, then it ***must*** be solvable by one of the above methods. If the equation is a **real** one, then it is possible that none will work, and something **else—numerical** solution, solution in series, inspection of integral curves generated by a calculator or computer—will have to be tried.

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# A GENERALIZATION OF LINEAR MAPS


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In linear algebra, when we define linear maps between vector spaces we always assume that the vector spaces are over the same field. Of course this is done for a good reason. If $T: V \to W$ is a linear map between the vector spaces V and W, $v \in V$, and $c$ is a scalar, then $T(cv) = cT(v)$ makes sense only if V and W are over the same field. However, if our goal is to be able to compare different vector spaces, then it seems natural that we would want to be able to define linear maps between vector spaces over different fields. The purpose of this note is to investigate that possibility.

Let V and W be vector spaces over the fields F and $F'$ respectively. Let us suppose that if $v_1, v_2 \in V$ then $T(v_1 + v_2) = T(v_1) + T(v_2)$. Let $\phi$ be a map from $F$ to $F'$ and let us suppose that $\phi$ and T satisfy $T(cv) = \phi(c)T(v)$ for all $v \in V$ and $c \in F$. We then have

THEOREM 1. Let V and W be vector spaces of the fields $F$ and $F'$ respectively. Let $T: V \to W$, $T \neq 0$ and $\phi : F \to F'$ be maps such that

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$
$$T(cv) = \phi(c)T(v)$$

for all $v_1, v_2 \in V$ and $c \in F$. Then $\phi$ is a homomorphism of fields.

***Proof.*** Let $a, b \in F$ and $0 \neq v \in V$. Then

$$\phi(a+b)T(v) = T((a+b)v) = T(av+bv) = T(av) + T(bv)$$
$$= \phi(a)T(v) + \phi(b)T(v) = (\phi(a) + \phi(b))T(v).$$

Since $T$ is not the zero map and $v \neq 0$ we may assume that $T(v) \neq 0$. Hence it follows that $\phi(a+b) = \phi(a) + \phi(b)$.

So far we have shown that $\phi$ is an additive homomorphism. Exactly the same argument as the one above, replacing + with $\cdot$, shows that $\phi(ab) = \phi(a)\phi(b)$ so that $\phi$ is a multiplicative homomorphism.

For $\phi$ to be a nontrivial homomorphism of fields, it remains only to show that $\phi$ takes 1 to $1'$ where $1 \in F$ and $1' \in F'$ are the multiplicative identities. We have

$$1' \cdot T(v) = T(v) = T(1 \cdot v) = \phi(1) \cdot T(v),$$





which is what we wanted.

What else can we say about $\phi$? It would be nice if $\phi$ turned out to be an isomorphism. Then this whole discussion would be moot. However, everyone knows that homomorphisms are not necessarily isomorphisms. For example, if we let R and $C$ denote the real and complex numbers, then $\phi: R \to C$ given by $\phi(a) = a + 0i$ is a field homomorphism but not an isomorphism. However, if we put a restriction on $T$ we can prove

THEOREM 2. **If** $T$ is injective than $\phi$ is injective.

***Proof.*** Let $a, b \in F$ and $0 \neq v \in V$. Suppose that $\phi(a) = \phi(b)$. We want to show that $a = b$. Note that $\phi(a) = \phi(b)$ implies that $\phi(a)T(v) = \phi(b)T(v)$. We have

$$T(av) = \phi(a)T(v) = \phi(b)T(v) = T(bv).$$

However, $T$ is injective *so* that $T(av) = T(bv)$ implies $av = bv$ so that, since $v \neq 0$, $a = b$.

Now it would be natural to ask, "What happens if T is surjective?" **Unfortunately,** if $T$ is surjective, it does not follow that $\phi$ is surjective. For example, let the real and complex numbers be denoted as above. Let V be the complex numbers as a vector space over the reals (i. e., F = R) and the $W$ be the complex numbers as a vector space over itself (i. e., $F' = C$). Let $T: C \to C$ be the identity map and let $\phi: R \to C$ be the embedding map $\phi(a) = a + 0i$. Then $T$ and $\phi$ satisfy all the conditions of the two theorems and T is surjective, but $\phi$ is not.

We have seen how one might begin to define linear transformations between vector spaces over different fields. However, it is not clear (to the author) what the appropriate notion of isomorphism might be. Certainly, if $\phi$ is a bijection then we have shown that F and $F'$ are isomorphic. Must this be the case to have a notion of isomorphism? For instance, how does the structure of the reals over the rationals compare to the complex numbers over the rational complex numbers (i. e., all complex numbers of the form $r + qi$ where $r$ and q are rational)? What about generalizations to modules over a ring? We leave such questions to the interested reader. The author would be interested in any solutions.

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# A Triangle of Coefficients and Its Uses

*Joe Howard*
*New Mexico Highlands University*

Consider the following identities:

$$x = x$$
$$x^2 = x + x(x-1)$$
$$x^3 = x + 3x(x-1) + x(x-1)(x-2)$$
$$x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3).$$

The **coefficients** are

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 7 & & 6 & & 1 \end{array}$$

and the next row would be

$$1 \quad 15 \quad 25 \quad 10 \quad 1.$$

The method for getting row $n+1$ from row $n$ can be seen in the example

$$15 = 1 + 2\cdot 7,\quad 25 = 7 + 3\cdot 6,\quad 10 = 6 + 4\cdot 1,$$

and the row after, $\{1, 31, 90, 65, 15, 1\}$, would come from

$$31 = 1 + 2\cdot 15,\quad 90 = 15 + 3\cdot 25,\quad 65 = 25 + 4\cdot 10,\quad 15 = 10 + 5\cdot 1.$$

These numbers, the **Stirling** numbers of the second kind, have been known for a long time, but do not often appear in the undergraduate mathematics curriculum. Let $S_n(k)$ denote the number in row n and column k. Then $S_n(k)$ is the number of ways of partitioning a set of k elements into n non-empty subsets. For example, $S_4(2) = 7$, counting the seven partitions

$$(\{a,b,c\},\{d\}),\ (\{a,b,d\},\{c\}),\ (\{a,c,d\},\{b\}),(\{b,c,d,\},\{a\}),$$
$$(\{a,b\},\{c,d\}),(\{a,c\},\{b,d\}),(\{a,d\},\{b,c\}).$$

The numbers can be calculated directly from





$$S_n(k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n.$$

As an application of their use, let us derive a formula for

$$\sum_{k=0}^{n}\binom{n}{k}k^4.$$

We use the binomial formula and differentiate repeatedly:

$$(1+x)^n = \sum\binom{n}{k}x^k$$

(1) $n(1+x)^{n-1} = \sum k\binom{n}{k}x^{k-1}$

(2) $n(n-1)(1+x)^{n-2} = \sum k(k-1)\binom{n}{k}x^{k-2}$

(3) $n(n-1)(n-2)(1+x)^{n-3} = \sum k(k-1)(k-2)\binom{n}{k}x^{k-3}$

(4) $n(n-1)(n-2)(n-3)(1+x)^{n-4}$

$$= \sum k(k-1)(k-2)(k-3)\binom{n}{k}x^{k-4}.$$

Set $x = 1$ in (1)-(4) and form the sum $(1) + 7\cdot(2) + 6\cdot(3) + (4)$. Using the identity for $k^4$, the right-hand side of the sum is just $\sum_{k=0}^{n}\binom{n}{k}k^4$ while the left-hand side is

$$n2^{n-1} + 7n(n-1)2^{n-2} + 6n(n-1)(n-2)2^{n-3}$$
$$+\ n(n-1)(n-2)(n-3)2^{n-4}.$$

Simplifying, we get

$$\sum_{k=0}^{n}\binom{n}{k}k^4 = (n+1)n(n^2+5n-2)2^{n-4}.$$

For another example, we can evaluate $\sum_{n=0}^{\infty}\frac{n^k x^n}{n!}$.

We have

$$\sum_{n=0}^{\infty} \frac{n^k x^n}{n!} =$$

$$\sum_{n=0}^{\infty} \frac{(S_k(1)n + S_k(2)n(n-1) + \cdots + S_k(k)n(n-1)\cdots(n-k+1))x^n}{n!}$$

$$= S_k(1)x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} + S_k(2)x^2 \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} + \cdots$$

$$+ S_k(k)x^k \sum_{n=k}^{\infty} \frac{x^{n-k}}{(n-k)!}$$

$$= (S_k(1)x + S_k(2)x^2 + \cdots + S_k(k)x^k)e^x.$$

In particular, if we put $x = 1$ and $k = 4$ we get

$$\sum_{n=0}^{\infty} \frac{n^4}{n!} = (1 + 7 + 6 + 1)e = 15e.$$

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*Joe Howard received his education from Eastern New Mexico University and New Mexico State University. He has taught for several years at New Mexico Highlands University.*

# A STATISTICAL ANALYSIS OF BASEBALL'S 1987 HOME RUN PHENOMENON

*Terry McMahan and Mike Surrey*
*Centre College*

The most dramatic hit in baseball is the home run. A record 4458 home runs were hit in 1987, more than were hit even in the days of Babe Ruth, Lou Gehrig, Joe DiMaggio, and Roger Maris. Why so many? We conclude, because of statistical data, that the home run phenomenon was caused by a production of lively baseballs.

What are the possible causes of such an offensive explosion? One explanation is league expansion, which has historically produced offensive booms. For example, when the Seattle Mariners and Toronto Blue Jays entered the American League in 1977, the other twelve teams in the league hit 658 more home runs than the year before. But expansion can be excluded from our discussion because no new teams were added to either league in 1987.

Another possible reason for the phenomenon is that there was simply more offensive power and talent in the league in 1987. But the numbers do not support this theory. Home runs per game decreased by over 0.5 per game between the 1987 and 1988 seasons in both leagues. Perhaps the umpires are the cause of the phenomenon—did they tighten their strike zones? The numbers also contradict this theory: record numbers of strikeouts were recorded in both leagues in 1987.

Perhaps the phenomenon was caused by poor pitching. A multiple regression model with home runs per game as the dependent variable and strikeouts per game and bases on balls per game as the independent variables measures the relationship between home runs and pitching. The model is based on data gathered for twenty-one seasons, 1973-1993, and results in the following regression lines (see the appendix for a list of data):

**AL:** HR = −0.59145 + 0.12295 SO + 0.15337 BB

**NL:** HR = −2.00911 + 0.17440 SO + 0.23633 BB.

This model yields interesting results. It shows, unexpectedly, a positive correlation between strikeouts and home runs (e. g., $r = .55$ in the American League). Although this contradicts our initial intuition, this relationship has a reasonable explanation: as more home runs are hit during a season, more players begin to swing for the fences. Thus they begin to swing at pitches out of the strike zone, resulting in higher numbers of strikeouts.

The phenomenon was not merely coincidental. The American League model produces results consistent with the observed data for every year except **1987,** where it tells us to expect **1.92** home runs per game. The actual rate for **1987** was **2.32** home runs per game. Similarly, the predicted and observed values for the National League are **1.68** and **1.88.** These results lead us to believe that an uncharacteristically large number of home runs was recorded for each league in **1987,** despite record numbers of strikeouts and near record numbers of bases on balls.

An expected response hypothesis test (see [1, p. 526]) supports our claim. For the American League, a test of the null hypothesis of 2.32 HR/game against the alternative of less than 2.32 HR/game produces a p-value of 0 (more than four standard deviations from the mean), and the p-value for the corresponding test for the National League is .0015.

Inspection of the data in the appendix shows that the increases and decreases in home runs per game from year to year seem to be close to the same for both leagues. Not only does the direction of change tend to be the same, even the magnitudes of the changes are close. This pattern consistently repeats itself over the entire **1973-1993** time span.

**A** paired difference test **confirms** this claim. Let ALC denote die change in the number of home runs per game in the American League from one year to the next and NLC the corresponding number for the National League. If D = ALC - NLC, then a test of the hypothesis that the mean of D is zero against the alternative that it is not zero produces a p-value of .8474.

The regression analysis shows that the home run phenomenon existed for both leagues in **1987.** The paired difference test supports the hypothesis that the phenomenon was caused by a production of lively baseballs. The two leagues have different umpires, players, coaches, and managers. The only common factor between the two leagues is the equipment used, the baseballs, bats, helmets, and so on. Only the bats and balls can affect the number of home runs, but the bats are manufactured by many different companies. Thus we have excluded all possible explanations, except the official major league baseballs, manufactured by only one company and under contract with the Major League Baseball Association.



Richard Levin, a spokesman for the Association, denied that the phenomenon is explained by the baseball. "The ball is the same as it always has been," he said **[2,** p. **72].** Ex-big league manager Whitey **Herzog** performed his own test by unraveling and bouncing two baseballs, one from **1986** and one from **1987.** The **1987** baseball bounced higher. We agree with his conclusion **[2,** p. **72]**: "You didn't have to be no scientist to figure that one out."

## References

1. **Mendenhall, William, Dennis** Wackerly, and Richard **Schaefer,** ***Mathematical Statistics with Applications,*** PWS-Kent, Boston, **1990.**
2. **Starr, Mark,** Kiss that baby goodbye, ***Newsweek,*** May **10, 1993.**

## Appendix

American League

| Year | HR | H | SO | ERA | BB |
|---|---|---|---|---|---|
| 1973 | 1.60 | 17.89 | 10.13 | 3.82 | 6.84 |
| 1974 | 1.41 | 16.18 | 9.80 | 3.62 | 6.31 |
| 1975 | 1.51 | 17.35 | 9.76 | 3.78 | 6.86 |
| 1976 | 1.17 | 17.30 | 9.41 | 3.52 | 6.30 |
| 1977 | 1.78 | 18.14 | 9.91 | 4.06 | 6.41 |
| 1978 | 1.48 | 17.59 | 8.96 | 3.77 | 6.43 |
| 1979 | 1.77 | 18.24 | 8.92 | 4.22 | 6.54 |
| 1980 | 1.63 | 18.48 | 9.14 | 4.04 | 6.37 |
| 1981 | 1.40 | 17.38 | 9.27 | 3.66 | 6.36 |
| 1982 | 1.83 | 18.14 | 9.63 | 4.07 | 6.47 |
| 1983 | 1.68 | 18.22 | 9.67 | 4.08 | 6.26 |
| 1984 | 1.75 | 18.11 | 10.20 | 3.99 | 6.32 |
| 1985 | 1.92 | 17.80 | 10.39 | 4.15 | 6.58 |
| 1986 | 2.02 | 17.85 | 11.51 | 4.18 | 6.76 |
| 1987 | 2.32 | 18.18 | 11.85 | 4.46 | 6.89 |
| 1988 | 1.68 | 17.61 | 10.87 | 3.97 | 6.34 |

| Year | HR | H | SO | ERA | BB |
|---|---|---|---|---|---|
| 1989 | 1.51 | 17.71 | 10.84 | 3.89 | 6.42 |
| 1990 | 1.58 | 17.55 | 11.19 | 3.91 | 6.73 |
| 1991 | 1.72 | 17.81 | 11.41 | 4.09 | 6.82 |
| 1992 | 1.57 | 17.64 | 10.75 | 3.94 | 6.79 |
| 1993 | 1.83 | 18.22 | 11.42 | 4.32 | 7.06 |

National League

| Year | HR | H | SO | ERA | BB |
|---|---|---|---|---|---|
| 1973 | 1.60 | 17.30 | 10.81 | 3.86 | 6.64 |
| 1974 | 1.32 | 17.39 | 10.23 | 3.62 | 7.02 |
| 1975 | 1.27 | 17.49 | 10.08 | 3.62 | 6.92 |
| 1976 | 1.15 | 17.26 | 9.88 | 3.50 | 6.44 |
| 1977 | 1.68 | 17.97 | 10.79 | 3.91 | 6.67 |
| 1978 | 1.31 | 17.03 | 10.19 | 3.57 | 6.46 |
| 1979 | 1.47 | 17.73 | 10.20 | 3.73 | 6.37 |
| 1980 | 1.28 | 17.68 | 10.13 | 3.60 | 6.14 |
| 1981 | 1.13 | 17.41 | 9.89 | 3.50 | 6.42 |
| 1982 | 1.34 | 17.58 | 10.60 | 3.60 | 6.14 |
| 1983 | 1.44 | 17.26 | 11.06 | 3.63 | 6.61 |
| 1984 | 1.31 | 17.33 | 11.24 | 3.69 | 6.33 |
| 1985 | 1.47 | 17.07 | 10.98 | 3.59 | 6.56 |
| 1986 | 1.57 | 17.12 | 11.98 | 3.72 | 6.75 |
| 1987 | 1.88 | 17.77 | 11.99 | 4.05 | 6.77 |
| 1988 | 1.32 | 16.75 | 11.31 | 3.45 | 5.96 |
| 1989 | 1.40 | 16.68 | 11.68 | 3.50 | 6.43 |
| 1990 | 1.56 | 17.40 | 11.49 | 3.80 | 6.40 |
| 1991 | 1.47 | 16.83 | 11.78 | 3.68 | 6.43 |
| 1992 | 1.30 | 17.01 | 11.67 | 3.50 | 6.15 |
| 1993 | 1.72 | 18.01 | 11.78 | 4.04 | 6.25 |



*Terry McMahan, a 1994 graduate from Centre College, was a mathematics major and had a batting average of .529 as a sophomore. Mike Surrey, who also was graduated from Centre with a mathematics major and who also played baseball, is now attending law school. Their faculty advisor was Professor Bill Johnston.*

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## What are these?

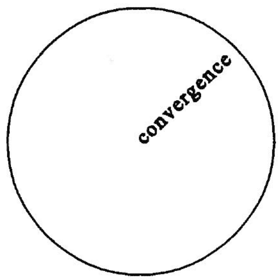
(From Professor K. R. Johnson of North Dakota State University.)

1. A<br>S<br>Y<br>M<br>P<br>T<br>O<br>T<br>E

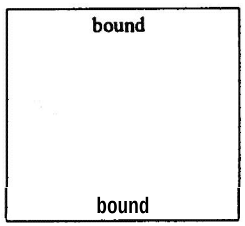
2. DEPENLINEARENCE

3. ROOTS ROOTS ROOTS

4.




5.




Answers, shoud you need them, are on page 272.

# LAPLACE TRANSFORMS AND TAYLOR SERIES

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In [2], basic properties of Laplace transforms are discussed. In [1], a formal power series was used to prove a result involving Laplace transforms. The purpose of this paper is to show how Taylor series expansions can be used to find the Laplace transforms of certain functions.

We will assume that $f(t)$ can be expanded in a Taylor series

$$(1) \qquad f(t) = \sum_{n=0}^{\infty} a_n t^n$$

on $|t| < R$ for some $R > 0$, where $a_n = f^{(n)}(0)/n!$. Since $\mathcal{L}[t^n] = n!/s^{n+1}$ for $s > 0$, one can use (1) to find $\mathcal{L}[f(t)]$ provided that the Laplace transform of a power series can be computed termwise. Since power series are uniformly convergent on compact subsets of the interval of convergence, power series can be integrated termwise. Also, since Laplace transforms are integral operators, it is reasonable to assume

$$(2) \qquad \mathcal{L}[f(t)] = \sum_{n=0}^{\infty} a_n \mathcal{L}[t^n] = \sum_{n=0}^{\infty} a_n \frac{n!}{s^{n+1}}$$

for $s > 0$. In many cases, it is possible to express the right-hand side of (2) in closed form.

As an example, since $e^{at} = \sum_{n=0}^{\infty} (at)^n/n!$ for $|t| < \infty$ and any nonzero constant $a$,

$$\mathcal{L}[e^{at}] = \sum_{n=0}^{\infty} \frac{a^n \mathcal{L}[t^n]}{n!} = \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}}$$

for $s > 0$. But this is a geometric series with first term $1/s$ and ratio $a/s$, so

$$\mathcal{L}[e^{at}] = \frac{1/s}{1 - a/s} = \frac{1}{s - a}$$

for $|a/s| < 1$ (i. e., $s > |a|$). If the restriction that $a \neq 0$ is removed and we





let $a = 0$, we get that $\mathcal{L}[1] = 1/s$ for $s > 0$.

As another example, it is well known that

$$\sin at = \sum_{n=0} \frac{(-1)^n (at)^{2n+1}}{(2n+1)!}$$

for $|t| < \infty$. So, for $s > 0$,

$$\mathcal{L}[\sin at] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} \mathcal{L}[t^{2n+1}]}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{s^{2n+2}},$$

another geometric series, from which $\mathcal{L}[\sin at] = a/(s^2 + a^2)$ follows.

In [2] it was shown using integration by parts that, with certain growth restrictions on $f(t)$, $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$. This can be obtained using series as follows. From (1),

$$f'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}.$$

So,

$$\mathcal{L}[f'(t)] = \sum_{n=1}^{\infty} n a_n \mathcal{L}[t^{n-1}] = \sum_{n=1}^{\infty} n a_n \frac{(n-1)!}{s^n}$$

$$= \sum_{n=1}^{\infty} n! \frac{a_n}{s^n} = \sum_{n=0}^{\infty} n! \frac{a_n}{s^n} - a_0 = s\mathcal{L}[f(t)] - f(0).$$

The last equality follows from (2) and the fact that $a_0 = f^{(0)}(0)/0! = f(0)$.

The above result is easy to generalize using power series. If $k$ is a positive integer, then

$$f^{(k)}(t) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n t^{n-k}$$

and so

$$\mathcal{L}[f^{(k)}(t)] = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n \frac{(n-k)!}{s^{n-k+1}}$$

$$= \sum_{n=k}^{\infty} n! \frac{a_n}{s^{n-k+1}} = \sum_{n=0}^{\infty} n! \frac{a_n}{s^{n-k+1}} \quad \sum_{n=0}^{k-1} n! \frac{a_n}{s^{n-k+1}}$$

$$= s^k \mathcal{L}[f(t)] - \sum_{n=0}^{k-1} f^{(n)}(0) s^{k-n+1}.$$

Using the definition of Laplace transform, it is easy to show that

$$\mathcal{L}[e^{bt}t^n] = \mathcal{L}[t^n]\big|_{s\to s-b} \text{ and so } \mathcal{L}[e^{bt}t^n] = \frac{n!}{(s-b)^{n+1}}.$$

Hence,

$$\mathcal{L}[e^{bt}f(t)] = \sum_{n=0}^{\infty} a_n \mathcal{L}[e^{bt}t^n] = \sum_{n=0}^{\infty} n!\frac{a_n}{(s-b)^{n+1}} = \mathcal{L}[f(t)]\big|_{s\to s-b}.$$

This result was given in Table *1* of **[2]**.

Although the computations in this paper were done formally, perhaps the main justification of the techniques is that the results agree with those obtained using standard techniques.

## References


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2. **Denton,** B. H., **Laplace** transforms in theory and practice, ***Mathematical Spectrum*** 27 ***(199415)***#1, ***8-10.***


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***Russell Euler is a member of the faculty at Northwest Missouri State University and is a frequent contribtor of papers and problems to this and other journals.***

# WHEN BLESSED EVENTS COINCIDE

*Amanda Beck and A. J. Mitchell*
*Carthage College*

What is the probability of a couple having their first child on their first wedding anniversary? To answer this question mathematically in the real world would be too **difficult**, so we **will** create a "perfect world" by making the following assumptions.

- All women will get pregnant within the **first** year of trying.
- No birth control methods or fertility drugs are used.
- All couples start **trying** to get pregnant immediately after their wedding.
- The time it takes to get pregnant and the term of pregnancy are independent and normally distributed.

Let $\mu_1$ be the mean length of time that it takes a couple to conceive their first child from their wedding day and let $\sigma_1$ be the standard deviation. Let $\mu_2$ and $\sigma_2$ be the mean duration and standard deviation of a pregnancy term. The distribution of the total time from wedding until birth will, because of the independence assumption, be normal with mean $\mu_1 + \mu_2$ and standard deviation $\sqrt{\sigma_1^2 + \sigma_2^2}$.

According to doctors the average time it takes a couple to conceive is six months, or ***180*** days. The mean length of the term of pregnancy is ***40*** weeks, or ***280*** days. Doctors say that ***10%*** of babies are born on their due date. If we consider the due **date** to be the exact middle of the actual day, we can say that ***10%*** of babies are born within ± .5 days of their due date. Using these facts, we can assign values to the parameters.

Since ***100%*** of women get pregnant within the first year after their wedding in this perfect world, we assume that all times to conception lie within ± 3 standard deviations from $\mu_1$. That is, $3\sigma_1$ = **6 months,** so $\sigma_1 = 2$ months.

Using a table of the normal distribution we find that ***10%*** of data falls within **± .125** standard deviations of $\mu_2$. Since ***10%*** of babies are born within ± **.5** days of their due date, we see that ***0*** 125 ***a,*** = .5 days, so $\sigma_2 = 4$ days.

So, $\mu_1 + \mu_2 =$ ***460*** days and $\sqrt{\sigma_1^2 + \sigma_2^2} = 60.1$ days are the mean and standard deviation of the time to birth. The probability that the time will be between 365 and 366 days is **.0019286** or **.19%**.

This probability is less than the random probability of 11365 or **.27%.** The reason for this is that some couples do not conceive soon enough after marriage to have a baby on their first anniversary. It might be interesting to see what is the probability that a second child is born on a wedding anniversary.

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Amanda Beck is a senior mathematics major at Carthage College with a strong interest in computers. A. *J. **Mitchell*** graduated from Carthage in 1995 with a major in business and a minor in mathematics

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A medical doctor in Japan has a question, as follows:

Here is a perfect die. When I throw it once, one of the six numbers **1, 2, 3,** 4, 5, 6 must come up. The probability of each is exactly 116. Suppose that the outcomes of the n tosses are a,, a,, ··· , a where each $a_i$ is one of the integers from 1 to 6.

We can make a rational number from these n integers, 0.a,a, ··· a . Then we can make the following finite sequence, $s_1, s_2, \cdots, s_n$,

$$s_1 = 0.a_1,\ s_2 = 0.a_1a_2,\ \cdots\ s_n = 0.a_1a_2\cdots a_n.$$

The larger n becomes, the **larger** $s_n$ becomes.

If I continue to throw the die infinitely often, this finite sequence will become an infinite sequence. The **infinite** sequence is bounded from above (by 213 = **.66666··· )** and is monotone increasing. Does it converge?

For example, suppose that the first five tosses were 1, **5, 3, 2,** and 4. If the sequence converges, then it has a limit. Let the limit be $s$. Then

$$0.15324111\cdots \le s \le 0.15324666\cdots.$$

Then we ***can*** write $s = 0.15324a_6a_7a_8\cdots$. Each a , n = **6, 7,** 8, ···, is one of the integers from 1 to 6. The limit, $s$, is a fixed real number. That means that the number a, has already been decided before the sixth toss.

This is a contradiction, since the probability that a, will **turn** up is 116, not 1. The same is true for a,, $a_8, \cdots$. Therefore the sequence does not converge.

But there is a theorem that a sequence that is monotone increasing and bounded above must converge. How can this contradiction be resolved?

## MATHACROSTICS

Solution to Mathacrostic 41, by **Corine** Bickley (Fall 1995).

Words:

| | | | |
|---|---|---|---|
| A. | moonstone | K. | **off and** on |
| B. | order | L. | whole |
| C. | rabble | M. | equate |
| D. | raft | N. | right hand |
| E. | itch | **O.** | speed |
| F. | shell sort | P. | ocean |
| G. | on high | Q. | foggiest |
| H. | Newton | R. | tete a tete |
| I. | shear stress | S. | **eigensystem** |
| J. | parameters | T. | night |

Author and title: **Morrisons,** Powers of Ten

Quotation: The step from one scene to its neighbor is always made a tenfold change. The edge of each square represents a length ten times longer or shorter than that of its two neighbors.

Solvers: Thomas Banchoff, Jeanette Bickley, Barbara Buckley, Charles R. **Diminnie,** Thomas L. **Drucker,** Victor G. **Feser,** Richard C. Gebhardt, Henry S. Lieberman, **Naomi Shapiro,** and the proposer.

Mathacrostic 42, by Jeanette Bickley appears on the next four pages. Directions for solving **acrostocs** appear at the end of the clues. To be listed as a solver, send your **solation** to the editor.

**A** Method of finding primes (3 wds)

___ ___ ___ ___ ___ ___ ___ ___ ___
110 144 27 172 10 239 72 67 147

___ ___ ___ ___ ___ ___ ___
1 225 46 150 77 197 207

___ ___ ___
85 29 24

**B.** A graphical computer-user interface.

___ ___ ___ ___ ___ ___ ___
183 100 212 174 5 137 31

**C.** Exact

___ ___ ___ ___
39 190 32 91

**D.** A polyhedron of twenty faces

___ ___ ___ ___ ___ ___ ___ ___ ___
118 177 138 28 93 219 90 122 186

___ ___
12 223

**E.** He experienced much rain

___ ___ ___ ___
142 71 111 180

**F.** Those who hope to knock down pins

___ ___ ___ ___ ___ ___ ___
73 227 9 87 70 65 222

**G.** Euclid's __________

___ ___ ___ ___ ___ ___ ___ ___
37 2 130 89 136 62 99 226

**H.** Einstein's achievement (2 wds)

___ ___ ___ ___ ___ ___ ___ ___ ___
237 112 59 240 161 81 21 151 117

___ ___ ___ ___ ___ ___ ___
155 18 36 74 134 26 16

**I.** Bigger than it was

___ ___ ___ ___ ___
238 98 14 45 126

**J.** A little one and a big one might be visible at night

___ ___ ___ ___ ___ ___
11 141 216 104 58 230



| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A 1 | G 2 | Q 3 | S 4 | B 5 | Z 6 | R 7 | W 8 | | F 9 | A 10 | | J 11 | D 12 | |
| U 13 | I 14 | X 15 | | H 16 | X 17 | H 18 | | X 19 | K 20 | H 21 | T 22 | | M 23 | |
| A 24 | R 25 | H 26 | A 27 | | D 28 | A 29 | X 30 | B 31 | C 32 | | V 33 | Q 34 | | a 35 |
| H 36 | G 37 | K 38 | C 39 | | Q 40 | T 41 | | Q 42 | R 43 | T 44 | | I 45 | A 46 | Q 47 |
| O 48 | | O 49 | L 50 | | W 51 | Y 52 | S 53 | V 54 | Q 55 | Z 56 | | X 57 | J 58 | H 59 |
| Z 60 | | N 61 | G 62 | | P 63 | T 64 | F 65 | | c 66 | A 67 | Y 68 | S 69 | F 70 | |
| E 71 | A 72 | | F 73 | H 74 | V 75 | M 76 | A 77 | Q 78 | | Q 79 | R 80 | H 81 | L 82 | U 83 |
| | Y 84 | A 85 | | Y 86 | F 87 | P 88 | G 89 | D 90 | C 91 | U 92 | D 93 | Z 94 | c 95 | |
| P 96 | O 97 | I 98 | G 99 | B 100 | Z 101 | S 102 | U 103 | | J 104 | Y 105 | c 106 | U 107 | Q 108 | K 109 |
| A 110 | | E 111 | H 112 | R 113 | V 114 | O 115 | O 116 | H 117 | D 118 | M 119 | | T 120 | L 121 | D 122 |
| O 123 | W 124 | Z 125 | I 126 | Z 127 | Q 128 | | V 129 | G 130 | Z 131 | N 132 | | Q 133 | H 134 | |
| V 135 | G 136 | | B 137 | D 138 | Q 139 | M 140 | J 141 | E 142 | Z 143 | | A 144 | Z 145 | Q 146 | A 147 |
| V 148 | Q 149 | A 150 | H 151 | O 152 | Y 153 | W 154 | H 155 | | Q 156 | Z 157 | U 158 | V 159 | | S 160 |
| | H 161 | b 162 | K 163 | L 164 | | X 165 | Q 166 | P 167 | Q 168 | | R 169 | O 170 | | c 171 |
| A 172 | O 173 | B 174 | Z 175 | P 176 | D 177 | Y 178 | | O 179 | E 180 | Z 181 | Y 182 | | B 183 | c 184 |
| | W 185 | D 186 | b 187 | | M 188 | S 189 | C 190 | T 191 | a 192 | W 193 | | X 194 | O 195 | |
| W 196 | A 197 | X 198 | | b 199 | W 200 | a 201 | O 202 | Q 203 | | Q 204 | a 205 | W 206 | A 207 | b 208 |
| c 209 | Z 210 | W 211 | B 212 | | N 213 | a 214 | c 215 | | J 216 | O 217 | L 218 | D 219 | Q 220 | O 221 |
| F 222 | | D 223 | a 224 | A 225 | | G 226 | F 227 | | S 228 | c 229 | J 230 | | Q 231 | c 232 |
| Q 233 | Z 234 | | Z 235 | U 236 | H 237 | | I 238 | A 239 | H 240 | Z 241 | | | | |

| | | |
|---|---|---|
| K. | To subject to extreme physical cruelty | 38 20 109 163 |
| L. | He discovered that $e^{ix} = \cos x + i \sin x$ | 50 121 82 164 218 |
| M. | In a frenzied manner | 23 188 76 119 140 |
| N. | An extinct flightless bird | 132 61 213 |
| O. | He was born on the $300^{th}$ anniversary of **Galileo's** death **(2 wds)** | 170 179 217 221 202 116 195 115 97 / 49 48 173 152 123 |
| P. | Believe | 63 96 167 176 88 |
| Q. | A body immersed in liquid is buoyed up by a force equal to the weight of the displaced liquid **(4** wds) | 34 40 47 168 3 55 149 156 233 / 231 166 78 204 139 42 79 / 133 220 203 108 146 128 |
| R. | Stately | 169 43 7 25 113 80 |
| S. | Not wise | 228 53 189 102 160 69 4 |
| T. | Damage | 191 41 120 64 44 22 |
| U. | There's nothing in it **(2 wds)** | 13 236 83 158 107 103 92 |



| | | |
|---|---|---|
| V. | ________ value | 75 135 129 33 159 54 114 148 |
| W. | John Napier's remarkable invention | 154 211 193 185 206 200 196 8 124 / 51 |
| X. | A baker's dozen | 15 19 194 57 165 198 17 30 |
| Y. | Elevate | 52 86 84 153 105 182 178 68 |
| Z. | Frequently studied by teens **(2 wds)** | 125 6 101 241 210 56 175 181 145 / 143 157 235 234 131 127 94 / 60 |
| a. | Possessing | 224 35 192 205 214 201 |
| b. | Presidential ______ | 199 162 208 187 |
| c. | It's past now | 106 171 66 209 184 232 215 229 95 |

The **mathacrostic** is a keyed **anagram.** The 241 letters to be entered in the diagram in the numbered spaces will be identical with those **in** the 29 keyed words at the **matching** numbers. The **key** numbers have been entered in the diagram to assist **in** constructing the solution.

When completed, the **initial** letters of the words will give the name of **an** author **and** the title of a book; the completed diagram will be a quotation from that book.

# PROBLEM DEPARTMENT

*Edited by Clayton W. Dodge*
*University of Maine*

*This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

*All communications should be addressed to C. W. Dodge,* 5752 ***Neville/Math,*** *University of Maine,* ***Orono,*** *ME* 04469-5752. *E-mail:* ***dodge@gauss.umemat.maine.edu.*** *Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly* ***identified*** *with name and address. Solutions to problems in this issue should be mailed by December* 1, *19%.*

## Problems for Solution

***875.*** *Proposed by Howe Ward Johnson, Iceboro, Maine.*

**A** certain restaurant chain used to advertise "**28** flavors" of ice cream. In remembrance of many pleasant stops there, this problem is proposed. Replace each letter by a digit to reconstruct this base ten equation:

$$(ICE)^2 + 28 = ICONE.$$

***876.*** *Proposed by Peter A. Lindstrom, Irving, Texas.*

Consider the portion of a typical calculator keyboard shown on the next page:

a) Define a *small square number* to be a four-digit number formed by pressing in cyclic order four keys that form a small square, **e.g. 1254** or **8569.** Show that each small square number is divisible by 11.

b) Define a *large square number* to be a four-digit number formed by pressing in cyclic order the four keys that form the vertices of the large





square, **e.g. 9713** or **3179.** Show that each large square number is divisible by **11.**

| | | |
|---|---|---|
| 7 | 8 | 9 |
| 4 | 5 | 6 |
| 1 | 2 | 3 |

c) Define a *diamond number* to be a four-digit number formed by pressing in cyclic order the four keys that form a diamond, **e.g. 6842** or **2486.** Show that each diamond number is divisible by **22.**

d) Define a *big square number* to be an eight-digit number formed by pressing in cyclic order the eight keys that form the vertices and sides of the large square, **e.g. 98741236** or **14789632.** Show that each big square number is divisible by **11** and is divisible by neither **3** nor **5.**

e) Define a *rectangular number* to be *a* six-digit number formed by pressing in cyclic order six keys that form the vertices and sides of a rectangle, **e.g. 987456** or **478521.** Show that each rectangular number is divisible by **111.**

**f)** Define a *double triangle number* to be a six-digit number formed by pressing in any order the six keys that form the vertices of two right triangles with a common hypotenuse, **e.g. 958956** or **421245.** Show that each double triangle number is divisible by **3.**

**877.** *Proposed by the late John M. Howell, Littlerock, California.*

For given constants $a, b, c, d$, let $a_0 = a$, $a_1 = b$, and, for $n > 1$, let $a_n = ca_{n-1} + da_{n-2}$.

a) Find $a_n$ in terms of $a$, $b$, $c$, and $d$.

b) Find $\lim_{n\to\infty} (a_n / a_{n-1})$.

c) Find integers $a, b, c, d$ so that the limit of part (b) is **3.**

**878.** *Proposed by Andrew* ***Cusumano,*** *Great Neck, New York.*

If $x$ is a solution to the equation $x^2 - ax + 1 = 0$, where $a$ is an integer greater than **2,** then show that $x^3$ can be written in the form $p +$

$q\sqrt{r}$, where p, q, and r are integers.

**879.** Proposed by Barton L. *Willis,* University of Nebraska at *Kearney,* Kearney, Nebraska.

A Mystery Space. Let S be a set of ordered pairs of elements. Define binary operations $+$, $*$, and $\div$ on S by

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, b)*(c, d) = (ac, ad + bc),$$

and

$$(a, b) \div (c, d) = (a \tilde{\Lambda} c, b \tilde{\Lambda} c - ad \div c^2).$$

Although it might be fun to deduce properties of space S (commutativity, associativity, etc.), the problem is to find an application for S.

**880.** Proposed by Rex H. Wu, Brooklyn, New York.

Evaluate, where $i = \sqrt{-1}$,

$$\lim_{n\to\infty} \frac{n}{4i}\left(e^{2ai/n} - e^{-2ai/n}\right).$$

881. Proposed by Andrew *Cusumano,* Great Neck, New York.

Let ABC be an equilateral triangle with center $D$. Let a be an arbitrary positive angle less than $30°$. Let $BD$ meet CA at F. Let $G$ be that point on segment $CD$ such that angle CBG = a, and let E be that point on FG such that angle FCE = a. Prove that DE is parallel to BC.

**882.** Proposed by Rex Wu, Brooklyn, New York.

Define, for any nonnegative integer m and any real number n,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}. \text{ Otherwise } \begin{bmatrix} n \\ m \end{bmatrix} = 0.$$

Find the values of

$$\text{(a)} \sum_{i=n}^{k} \begin{bmatrix} i \\ n \end{bmatrix} \quad \text{and} \quad \text{(b)} \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix}.$$



**883.** Proposed by Sammy Yu (student), University of South Dakota, *Vermillion,* South Dakota.

M. N. Khatri [*Scripta Mathematica,* 1955, vol. 21, p. 94] found that from the identity T(4) + 719) = $T(10)$, where $T(n) = n(n+1)/2$ is the nth triangular number, Pythagorean triples (5, 12, 13) and (8, 15, 17) produce the more general formulas $T(4 + 5k) + T(9 + 12k) = T(10 + 13k)$ and $T(4 + 8k) + T(9 + 15k) = T(10 + 17k)$, where k is a positive integer. Given p, q, r, so that $T(p) + T(q) = T(r)$, find Pythagorean triples (a, b, c) so that $a^2 + b^2 = c^2$ and $T(p + ak) + T(q + bk) = T(r + ck)$ for any positive integer k.

**884.** Proposed by *Seema Chauhan, Lucknow,* India.

a) Held every day is a tutorial class in which 2m students are enrolled. Exactly $m$ of these students, selected at random, attend class on any given day. If the class meets for exactly 2r days, find the probability that in the end each student has attended exactly r classes.

*b) The class of part (a) contains $m$ boys and m girls. For each p, $0 \le p \le r$, find the probability that each girl attends exactly r + p classes and each boy attends just r − p classes.

**885.** Proposed by Arthur Marshall, Madison, Wisconsin.

Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 6}{3^{(2n-1)/2}(2n-1)}.$$

**886.** Proposed by $R$. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Find the general solution in integers to the equation $x^2 - 8y + 7 = 0$.

**887.** Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

The Fibonacci numbers $F_n$ are defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for k > 1. Compute the following sums involving Fibonacci numbers:

$$S_{1,n} = \sum_{k=1}^{n} \frac{1}{F_{2k-1}F_{2k+1}} \quad \text{and} \quad S_{2,n} = \sum_{k=1}^{n} \frac{1}{F_{2k}F_{2k+2}}.$$

Also find their limits $S$, and $S_2$ as $n \to \infty$. Express the finite sums as rational numbers in lowest terms. Finally, simplify each of the following expressions:

$$a = \frac{S_1^2}{S_2}, \quad b = \frac{1}{S_{2,n}} - \frac{1}{S_{1,n}}, \quad c = S_1 - S_2,$$

$$d = \frac{S_{1,n}}{S_{2,n}} - S_{1,n}, \text{ and } e = \frac{1}{S_{1,n}^2} - \frac{1}{S_{2,n}}.$$

## Solutions

844. [Fall *1994*, Spring **1995**] *Proposed* by *Bill* ***Correll,*** *Jr., student,* ***Denison*** *University, Granville, Ohio.*

If $F_n$ denotes the *nth* Fibonacci number ($F_1 = F_2 = 1$ and $F_{k+2} = F_k + F_{k+1}$, for $k$ a positive integer), evaluate

$$\sum_{n=1}^{\infty} \binom{n}{k} F_n / 2^{n+k}.$$

*I. Solution by the Proposer.*

For $0 < |x| < 1$, recall that $\sum_{n=1}^{\infty} x^n = -1 + 1/(1-x)$. Differentiation k times yields

$$\sum_{n=1}^{\infty} n(n-1)(n-2)\cdots(n-k+1)x^{n-k} = k!\,(1-x)^{-k-1},$$

from which we get that

$$\sum_{n=1}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} = \frac{1}{x}\left[\frac{x}{1-x}\right]^{k+1}.$$

Let $4 = (1 + \sqrt{5})/2$. Then

$$\sum_{n=1}^{\infty} \binom{n}{k} F_n / 2^{n+k} = \frac{1}{2^k\sqrt{5}} \sum_{n=1}^{\infty} \binom{n}{k} \left[\frac{\phi}{2}\right]^n - \frac{1}{2^k\sqrt{5}} \sum_{n=1}^{\infty} \binom{n}{k} \left[-\frac{1}{2\phi}\right]^n$$

$$= \frac{1}{2^k\sqrt{5}} \left[\frac{2}{\phi}\left[\frac{\phi}{2-\phi}\right]^{k+1} + 2\phi\left[-\frac{1}{2\phi+1}\right]^{k+1}\right]$$



$$= \frac{1}{2^k\sqrt{5}}\left[(\sqrt{5}-1)(2+\sqrt{5})^{k+1} + (\sqrt{5}+1)(2-\sqrt{5})^{k+1}\right]$$

since we have $-1 < -1/24 < 0 < \phi/2 < 1$.

II. *Solution by Paul S.* ***Bruckman, Edmonds,*** *Washington.*

Recall the well-known "Binet formula" for the Fibonacci numbers,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \text{ where } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } ß = \frac{1-\sqrt{5}}{2}.$$

It is **easy** to verify that $\alpha ß = -1$, $2 - a = ß^2$, and $2 - ß = \alpha^2$. *Recall* also that $\binom{n}{k}$ is zero if k $< 0$ or if $n$ is a positive integer and k $> n$, that $\binom{n+k}{k} =$ [illegible] and that $\binom{n-m-1}{n} = (-1)^n\binom{m}{n}$. Now, fork $= 0, 1, 2, 3, \ldots$, take

$$S_k = \sum_{n=1}^{\infty} \binom{n}{k} \frac{F_n}{2^{n+k}} = \sum_{n=k}^{\infty} \binom{n}{k} \frac{F_n}{2^{n+k}} = \sum_{n=0}^{\infty} \binom{n+k}{n} \frac{F_{n+k}}{2^{n+2k}}$$

$$= 4^{-k} \sum_{n=0}^{\infty} \binom{-k-1}{n} \left[\frac{-1}{2}\right]^n F_{n+k}$$

$$= \frac{4^{-k}}{\sqrt{5}} \sum_{n=0}^{\infty} \binom{-k-1}{n} \left[\frac{-1}{2}\right]^n (\alpha^{n+k} - \beta^{n+k}).$$

Since $|ß/2| < |all| < 1$, we obtain

$$S_k = \frac{1}{4^k\sqrt{5}}\left[\alpha^k\left[1-\frac{\alpha}{2}\right]^{-k-1} - \beta^k\left[1-\frac{\beta}{2}\right]^{-k-1}\right]$$

$$= \frac{1}{2^{k-1}\sqrt{5}}\left[\alpha^k\beta^{-2k-2} - \beta^k\alpha^{-2k-2}\right] = \frac{1}{2^{k-1}\sqrt{5}}\left[\alpha^{3k+2} - \beta^{3k+2}\right] = \frac{F_{3k+2}}{2^{k-1}}.$$

*Editor's* ***note—The*** *misprint in the original statement of the problem prompted the following two submissions.*

**III.** *Comment by Bob* ***Prielipp,*** *University of Wisconsin, Oshkosh, Wisconsin.*

We assume the proposer intended the following summation:

$$\sum_{k=1}^{\infty}\binom{n}{k}\frac{F_n}{2^{n+k}} = \sum_{k=1}^{n}\binom{n}{k}\frac{F_n}{2^{n+k}} = \frac{F_n}{2^n}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k$$

$$= \frac{F_n}{2^n}\left[\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k - 1\right] = \frac{F_n}{2^n}\sum_{k=0}^{n}\binom{n}{k}1^{n-k}\left(\frac{1}{2}\right)^k - \frac{F_n}{2^n}$$

$$= \frac{F_n}{2^n}\left(1+\frac{1}{2}\right)^n - \frac{F_n}{2^n} = \frac{F_n}{2^n}\left[\left(\frac{3}{2}\right)^n - 1\right].$$

***IV. Comment by Paul S. Bruckman, Edmonds, Washington.***

As it was stated originally, the statement made no sense. This solution is based on the assumption that the proposer intended the following summation:

$$S(n) = \sum_{k=0}^{n}\binom{n}{k}\frac{F_{n-k}}{2^{n+k}}.$$

Letting $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, we have that

$$S(n) = \frac{1}{2^n\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k(\alpha^{n-k}-\beta^{n-k})$$

$$= \frac{1}{2^n\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k\alpha^{n-k} - \frac{1}{2^n\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k\beta^{n-k}$$

$$= \frac{1}{2^n\sqrt{5}}\left[\left(\alpha+\frac{1}{2}\right)^n - \left(\beta+\frac{1}{2}\right)^n\right] =$$

$$\frac{1}{4^n\sqrt{5}}[(2\alpha+1)^n - (2\beta+1)^n].$$

Since $2\alpha + 1 = \alpha^3$ and $2\beta + 1 = \beta^3$, then

$$S(n) = \frac{1}{4^n\sqrt{5}}(\alpha^{3n}-\beta^{3n}) = \frac{F_{3n}}{4^n}.$$



***Also solved by*** Henry S. Lieberman and Bob **Prielipp.**

849. [Spring **1995**] ***Proposed*** by ***L. A. Bohr, Great Works. Maine.***

Solve this base 8 addition alphametic: ***THIS*** + IS = ***WORK.***

***Solution by Victor G. Feser, University of Mary, Bismarck, North Dakota.***

The unique solution is 1756 + 56 = 2034.

Immediately we have H = 7, ***O*** = 0, ***W*** = T + 1, and I $\geq$ 4. There are six possible values for S; from each we get a value for K. This process eliminates S = 4, *K* = 0. For each of the remaining five pairs we choose all available values for **I**, and thus get ***R.*** Eliminating duplications of previous values, we are left with just four cases: ***(S,*** K, **I**, ***R)*** = (1, ***2***, 6, **4**), (3, 6, 5, 2), (5, 2, 4, **1**), and (6, 4, 5, 3). The remaining two values are T and ***W.*** In only the one case (6, 4, 5, 3) can we have T + 1 = W, the case given above.

***Also solved by*** Charles **Ashbacher,** Adelicia **Beckhama,** Aaron Beeler, Laura **Bolton,** Scott H. Brown, Paul S. **Bruckman,** James Campbell, Sandra Rena Chandler, William Chau, Shaw **Cunningham,** Martin Davis, Mark **Denton,** Jack T. **Dunn,** Mark **Eckstein,** Mark Evans, Robert C. Gebhardt, **Brandi** Hamilton, Michael Hamilton, **Lynette** Harvey, Richard I. Hess, Jamie **Kiner, Kee-Wai Lau,** Carl **Libis,** Henry S. **Lieberman,** Peter **A.** Lindstrom, Yoshinobu Murayoshi, Chuck Pierce, Mike **Pinter,** Medley Raymond, H.-J. **Seiffert,** Carla **Strassle,** Kenneth M. **Wilke,** Rex H. Wu, ***and the*** Proposer.

***850.** [Spring **1995**] ***Proposed*** by ***Bill Correll, Jr., student, Denison University, Granville. Ohio.***

Clearly the following integral evaluation is incorrect. Find the flaw. By letting $u = -x$ we get that

$$I = \int \ln(e^x+1)\,dx = -\int \ln(e^{-u}+1)\,du = -\int \ln\frac{e^u+1}{e^u}\,du$$

$$= -\int \ln(e^u+1)\,du + \int \ln(e^u)\,du = -I + u^2/2 + C,$$

so that $I = x^2/4 + C'$. (See Problem 828.).

Solution by David *Tascione* and Christopher W. Murphy, students, St. Bonaventure University, St. Bonaventure, New *York*.

Each step of the solution proves valid except for the final substitution

$$I = \int \ln(e^u + 1)\,du.$$

These are not equivalent expressions for the indefinite integral. The initial integral should be described as

$$I(x) = \int \ln(e^x + 1)\,dx,$$

whereas the latter integral is actually

$$I(-x) = \int \ln(e^{-x} + 1)\,d(-x) = -\int \ln(e^{-x} + 1)\,dx.$$

The correct final statement would then become

$$I(x) = I(-x) + \frac{x^2}{2} + C.$$

Also solved by Charles Ashbacher, Paul S. **Bruckman,** James Campbell, Russell Euler, Mark Evans, Victor G. **Feser,** Robert C. Gebhardt, Richard I. **Hess,** Henry S. **Lieberman,** Peter **A.** Lindstrom, V. S. **Manoranjan,** Kandasamy Muthuvel, Mike Pinter, John F. Putz, H.-J. Seiffert, Selvaratnam **Sridharma,** and Rex H. Wu.

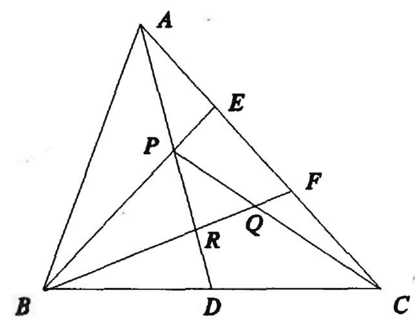
851. [Spring **1995]** Proposed by Bill *Correll,* Jr., student, *Denison* University, Granville, Ohio.

In triangle ABC let Cevian AD bisect side BC and let Cevians BE and BF trisect side *CA*. Let AD intersect BE at P and BF at R, and let CP meet BF at Q. See the figure. If the area of triangle ABC is 1, find the area of triangle PQR.

I. Solution by William *H.* Peirce, *Rangeley,* Maine.

This problem **and** Problem 846 in the Fall, 1994, issue are special cases






of a more general problem: If *ABC* and *PQR* are two coplanar triangles with a known linear relationship between the vertices P, Q, R and the vertices A, B, C, find the ratio of the triangle areas.

The solution to the general problem makes use of three lemmas, the first two of which are stated without proof. They apply equally well to rectangular coordinates in the Cartesian plane or to affixes in the complex plane.

Lemma 1. Any point on the line through two distinct points can be expressed uniquely as a linear combination of the two points in which the coefficients add to 1.

Thus, if $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$, are Cartesian points with $C$ **lying** on line *AB*, then there are unique real constants m and $n$ such that $m + n = 1$, $c_1 = ma_1 + nb_1$, and $c_2 = ma_2 + nb_2$. If a, b, $c$ are the affixes of A, B, $C$ in the complex plane, then $c = ma + nb$ for the same m and n. In either case we will write $C = mA + nB$.

Lemma 2. Any point in the plane of three **non-collinear** (and therefore distinct) points can be expressed uniquely as a linear combination of the three points, in which the three coefficients add to one.

Lemma **3.** If points **P,** Q, R are related to points A, B, C by

$$P = u_1A + u_2B + u_3C, \quad Q = v_1A + v_2B + v_3C, \quad R = w_1A + w_2B + w_3C,$$

where

$$u_1 + u_2 + u_3 = v_1 + v_2 + v_3 = w_1 + w_2 + w_3 = 1,$$

then the areas $K(PQR) = \pm H \cdot K(ABC)$, where H is the $3 \times 3$ determinant of the coefficients

$$H = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The plus sign is used if the two triangles have the same clockwise or counterclockwise orientation, the minus sign if they are opposite.

To establish Lemma 3, recall that the area of triangle *ABC* is given by either of these two determinant formulas, the former for rectangular coordinates, the latter for complex coordinates:

$$K(ABC) = \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = \pm\frac{i}{4}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix},$$

using whichever sign makes the value nonnegative. It is easy to check that the product of determinant H by either of these two determinants yields the corresponding area determinant for triangle PQR. Since the determinant of a product equals the product of the determinants, Lemma 3 is established.

Problems such as 851 and 846 are solved by finding the determinant whose rows are the coefficients of P, Q, and R when each is written as a linear combination of A, B, and C. Problem 851 is now solved by using the defining intersections to determine these coefficients.

Since D is the midpoint of BC, then $D = B/2 + C/2$. Likewise $E = 2A/3 + C/3$ and $F = A/3 + 2C/3$. For some $\lambda$,

$$P = \lambda A + (1 - \lambda)D = \lambda A + \frac{1-\lambda}{2}B + \frac{1-\lambda}{2}C.$$

Also, for some $\mu$,

$$P = \mu B + (1 - \mu)E = \frac{2(1-\mu)}{3}A + \mu B + \frac{1-\mu}{3}C.$$

The coefficients of A, B, and C in each expression for P add to 1. Now the uniqueness property of Lemma 2 allows us to equate like coefficients to get three equations in $\lambda$ and $\mu$:

$$3\lambda + 2\mu = 2,\ \lambda + 2\mu = 1, \text{ and } 3\lambda - 2\mu = 1,$$

which are consistent and have the unique solution $\lambda = 112$ and $\mu = 114$. Using these values in either expression gives

$$P = \frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C.$$



Point Q lies on BF and CP and point R is on BF and AD. By the method of the **preceding** paragraph we obtain

$$Q = \frac{2}{7}A + \frac{1}{7}B + \frac{4}{7}C \text{ and } R = \frac{1}{5}A + \frac{2}{5}B + \frac{2}{5}C.$$

The determinant of Lemma 3 is now readily calculated. We have

$$H = \begin{matrix} 1/2 & 1/4 & 1/4 \\ 2/7 & 1/7 & 4/7 \\ 1/5 & 2/5 & 2/5 \end{matrix} = \frac{9}{140}.$$

Since $K(ABC) = 1$, then $K(PQR) = 91140$.

**II.** Solution by ***Jianming*** Wu, student. ***Denison*** University, Granville, ***Ohio.***

Draw segment $DF$ to intersect PC at J. Since EF = FC and $BD$ = DC, then $FD$ is parallel to EB and $FD = EB/2$. Also AP = PD because $AE$ = EF, $PE = DF/2$, and $JF = PE/2$. Let $JF = x$. Then $PE = 2x$, $DF = 4x$, $BE = 8x$, and $BP = 6x$. Since triangles BPQ and $FJQ$ are similar as are triangles BPR and $FDR$, we have

$$\frac{BR}{RF} = \frac{PR}{RD} = \frac{BP}{FD} = \frac{6x}{4x} = \frac{3}{2},\quad \frac{BQ}{QF} = \frac{BP}{FJ} = \frac{6x}{x} = 6,$$

$$\frac{BR}{RQ + QF} = \frac{3}{2} \text{ and } \frac{BR \bullet RQ}{QF} = 6,$$

$$2BR = 3RQ + 3QF,\quad BR + RQ = 6QF,$$

$$BR + RQ = 4BR - 6RQ, \text{ and } \frac{BR}{RQ} - \frac{7}{3}.$$

Since BD = DC, then $K(ABD) = 112$. Since AP = PD, then $K(BPD) = K(ABD)/2 = 114$. From $PR/RD = 312$ we get that

$$K(BPR) = \frac{3}{5}K(BPD) = \frac{3}{20}.$$

Finally, $BR/RQ = 713$ gives us

$$K(PQR) = \frac{3}{7}\cdot\frac{3}{20} = \frac{9}{140}.$$

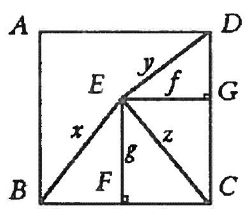
***Also solved by*** **Miguel Amengual** Covas, Paul **S. Bruckman,** William Chau, Mark Evans, Richard I. Hess, Henry S. Lieberman, Yoshinobu **Murayoshi,** William **H.** Peirce ***(second solution),*** Rex **H.** ***Wu,*** and ***the*** Proposer.

852. [Spring **1995]** ***Proposed by Rex H. Wu, Brooklyn, New York.***

Let $E$ ***be*** a point inside square $ABCD$ with $BE = x$, $DE = y$, and $CE = z$. If $x^2 + y^2 = 2z^2$, find the area of $ABCD$ in **terms** of $x$, $y$, and $z$.

I. ***Solution*** by ***Victor G. Feser, University of Mary, Bismarck, North Dakota.***

Let the square have sides of length 1. Drop perpendiculars from point E to $F$ on $BC$ and to $G$ on $CD$, of lengths $g$ **and** $f$, respectively, as shown in the accompanying figure.




By the Pythagorean theorem we have

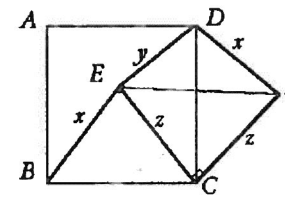
$$y^2 = f^2 + (1-g)^2,\quad x^2 = g^2 + (1-f)^2,\quad \text{and}\quad z^2 = f^2 + g^2.$$

Substitute these values into the equation $x^2 + y^2 = 2z^2$ and simplify to get $f + g = 1$. It follows that BFE and $DGE$ are both isosceles right triangles and thus D, $E$, and $B$ ***are*** **collinear,** forming a diagonal of the square. Then by familiar formulas, the area of the square $ABCD$ is $(x + y)^2/2$.

II. ***Solution*** by ***the Proposer.***

Rotate triangle $BCE$ $90°$ about point $C$ so that $BC$ coincides with $DC$ and let $E$ map to $E'$. Then $\angle ECE' = 90°$ and $EE' = z\sqrt{2}$. In triangle $EDE'$ we have $x^2 + y^2 = 2z^2$, which implies that $\angle EDE' = 90°$, so then






quadrilateral $CEDE'$ can be inscribed in a circle (with center at the midpoint of $EE'$). Now we apply **Ptolemy's** theorem to get

$$DC\cdot EE' = DE\cdot CE' + CE\cdot DE',\quad DC\cdot z\sqrt{2} = yz + xz,$$

so that

$$K(ABCD) = DC^2 = \frac{(x+y)^2}{2} = z^2 + xy.$$

**III.** ***Comment by William H. Peirce, Rangeley, Maine.***

The locus of $E$ is the diagonal $BD$ of the square, so that $x + y = BD$. The theorem can be generalized to allow E to lie outside or on the square, with the understanding that the numerically smaller of $x$ and $y$ will be replaced by its negative. Then $E$ still lies on the diagonal (extended) and the area of the square is still $(x + y)^2/2 = z^2 + xy$. Of course, in this case, $xy \le 0$.

***Also solved by*** Scott H. Brown, Paul S. **Bruckman,** William Chau, Mark Evans, Robert **C.** Gebhardt, Richard I. Hess, **Jamshid Kholdi,** Henry S **Lieberman,** David E. Manes, V. S. **Manoranjan,** Can A. Minh, **Kandasamy** Muthuvel, William **H.** Peirce ***(two solutions),*** **H.-J. Seiffert,** George **Tsapakidis,** Kenneth **M. Wilke** ***and*** Sammy and Jimmy Wu.

853. [Spring **1995]** ***Proposed*** by ***Charles Ashbacher, Cedar Rapids, Iowa.***

This problem was submitted by Vietnam for the 1990 International Mathematical Olympiad and has appeared in booklets overseas. If real numbers $x \ge y \ge z > 0$, then prove that

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \ge x^2 + y^2 + z^2$$

I. Solution by Henry ***S. Lieberman, Waban,*** Massachusetts.

By using the A.M.-G.M. inequality, we obtain that

$$\frac{y}{z} + \frac{z}{x} + \frac{x}{y} \geq 3,$$

from which we get

$$P = z^2\left[\frac{y}{z} - 1\right] + z^2\left[\frac{z}{x} - 1\right] + z^2\left[\frac{x}{y} - 1\right] \geq 0.$$

Denote by Q the **left** side minus the right side of the desired inequality. Then

$$Q = x^2\left[\frac{y}{z} - 1\right] + y^2\left[\frac{z}{x} - 1\right] + z^2\left[\frac{x}{y} - 1\right]$$

and $Q \geq 0$ if $Q - P \geq 0$. We establish this latter inequality thus:

$$Q - P = x^2\left[\frac{y}{z} - 1\right] + y^2\left[\frac{z}{x} - 1\right] - z^2\left[\frac{y}{z} - 1\right] - z^2\left[\frac{z}{x} - 1\right]$$

$$= (x^2 - z^2)\frac{y - z}{z} + (y^2 - z^2)\frac{z - x}{x}$$

$$= (x - z)(y - z)\left[\frac{x + z}{z} + (y + z)\left(-\frac{1}{x}\right)\right]$$

$$= \frac{(x - z)(y - z)}{xz}(x^2 + xz - yz - z^2)$$

$$= \frac{(x - z)(y - z)}{xz}[(x^2 - z^2) + z(x - y)] \geq 0.$$

**II.** Solution and generalization by Murray S. ***Klamkin,*** University of Alberta, Edmonton, Alberta, ***Canada.***

Let $(x, y, z) = (1/c, 1/b, 1/a)$, so that we want to prove equivalently that

$$a^3b + b^3c + c^3a \geq b^2c^2 + c^2a^2 + a^2b^2 \tag{1}$$



where $a \geq b \geq c > 0$. More generally, we show that if $a_1 \geq a_2 \geq \cdots \geq a_r > 0$ and $m \geq n \geq 0$, then

$$S(m, n) \geq S(m, m - 1) \tag{2}$$

where

$$S(m, n) = a_1^{m+n}a_2^{m-n} + a_2^{m+n}a_3^{m-n} + \cdots + a_r^{m+n}a_1^{m-n}.$$

Since it is known [2] that $S(m, n) \geq S(m, -n)$, we have by **Cauchy's** inequality that

$$S^2(m, n) \geq S(m, n)\,S(m, -n) \geq S^2(m, 0).$$

For the special case $r = 3$, $m = 2$, $n = 1$, we get inequality (1).

By Holder's inequality we get

$$S(m, n) \geq [S(m, n)]^{(n-1)/n}[S(m, -n)]^{1/n} \geq S(m, n-1).$$

This latter inequality allows us to interpolate the inequality $S(m, n) \geq S(m, 0)$, i.e., in terms of other exponents:

$$\Sigma a_1^{2m-1}a_2 \geq \Sigma a_1^{2m-2}a_2^2 \geq \cdots \geq \Sigma a_1^m a_2^m,$$

where the sums are cyclic over the indices 1, 2, ..., $r$.

Also solved by **Miguel Amengual** Covas, Paul S. **Bruckman,** J. S. Frame, Richard I. Hess, Joe Howard, Kee-Wai **Lau,** David E. Manes, Yoshinobu **Murayoshi, Kandasamy** Muthuvel, **H.-J.** Seiffert, George **Tsapakidis,** J. Ernest **Wilkins,** Jr., **Rex** H. Wu, and the Proposer. Amengual ***Covas*** found the problem in references 1 ***and 3,*** and Howard also supplied reference 1.

## References


1. *Crux Mathematicorum*, **20(1994)43-44,** Problem 6.
2. M. S. **Klamkin,** Crux ***Mathematicorum*****, 6(1980)**107.
3. The Vietnamese National Olympiad in Mathematics for Secondary Schools, Hanoi, February, 1991.

854. [Spring **1995]** Proposed by ***Jayanthi*** Ganapathy, University of Wisconsin at Oshkosh, Oshkosh, Wisconsin.

Let a and b be two nonzero real numbers such that

$$a^3(3a^2 - 5ab + 3b^2) = b^3(5a^2 - 3ab + 5b^2).$$

Find the values of the expressions $(a^2 + b^2)/a^2$ and $(a^2 - b^2)/ab$.

Solution by Can Anh Minh, student, University of California, Berkeley, California.

Substitute $b = ta$, so that $t = b/a$. The given equation reduces to

$$3 - 5t + 3t^2 = t^3(5 - 3t + 5t^2)$$

and

$$5t^5 - 3t^4 + 5t^3 - 33 + 5t - 3 = 0,$$

which factors easily to yield

$$(5t - 3)(t^4 + t^2 + 1) = 0.$$

Since the latter factor has no real roots, we must have t = 315. Hence

$$\frac{a^2 + b^2}{b^2} = 1 + t^2 = \frac{34}{25} \text{ and } \frac{a^2 - b^2}{ab} = \frac{1}{t} - t = \frac{16}{15}.$$

Also solved by **Anurag Agarwal,** Miguel **Amengual Covas, Seung-Jin** Bang, Scott H. Brown, Paul S. Bruckman, James Campbell, S Ira Rena Chandler, William Chau, Russell Euler, George P. **Evanovich,** Mark Evans, Victor G. Feser, Robert C. Gebhardt, Richard I. Hess, Joe Howard, Jamshid **Kholdi,** Murray S. **Klamkin, Kee-Wai Lau,** Carl **Libis,** Henry S. **Lieberman,** Peter A. **Lindstrom,** David E. Manes, **Kandasamy** Muthuvel, Yoshinobu Murayoshi, William H. Peirce, Bob Prielipp, H.-J. Seiffert, Selvaratnam **Sridharma,** Kenneth M. Wilke, Rex H. Wu, and the Proposer.

**855.** [Spring **1995]** Proposed by the late Florentin Smarandache, Phoenix, Arizona.

Prove that a square matrix of integers, having in each row and in each column a unique element not divisible **by** a given prime $p$, is nonsingular.



Solution by H.-J. ***Seiffert,*** Berlin, Germany.

Let $A = (a_{ij})$, $i, j = 1, 2, \ldots, n$, be a square matrix having the described properties. Then there exists one and only one permutation $\pi \in S_n$ such that $p \nmid a_{i\pi(i)}$ for all $i \in \{1, 2, \ldots, n\}$. Since p is a prime, then $p \nmid \prod_{i=1}^{n} a_{i\pi(i)}$. For all other permutations $\sigma \in S_n$, $\sigma \neq \pi$, we have that $p \mid \prod_{i=1}^{n} a_{i\sigma(i)}$. Since

$$\det(A) = \operatorname{sgn}(\pi)\prod_{i=1}^{n} a_{i\pi(i)} + \sum_{\substack{\sigma\in S_n \\ \sigma\neq\pi}} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i\sigma(i)},$$

we see that $\det(A)$ is an integer of the form $\det(A) = r + ps$, where $r, s \in Z$ and $p \nmid r$. Hence $r + ps$ cannot be 0, so $\det(A) \neq 0$ and A is nonsingular.

Also solved by Paul S. Bruckman, James Campbell, Richard I. Hess, Murray S. **Klamkin,** Henry S. **Lieberman,** Can A. **Minh, Skidmore** College Problem Group, and the Proposer.

856. [Spring **1995]** Proposed by Paul ***S.*** Bruckman, ***Edmonds,*** Washington.

Starting with a regular n-gon whose side is of unit length, snip off congruent isosceles triangles from each of its vertices, resulting in a regular 2n-gon. Repeat the process indefinitely. Find the ratio of the area of the limiting circle to that of the original n-gon.

Solution by H.-J. ***Seiffert,*** Berlin, Germany.

It is easily seen that all the regular polygons obtained by the described process have the same **inradius** $r$ **as** the original n-gon. The area of the original n-gon is S = ***mil*** and of the **incircle** is $C = \pi r^2$, where we have $\tan(\pi/n) = 1/(2r)$. Since the **incircle** is the limiting circle, we have

$$\frac{C}{S} = \frac{2\pi r^2}{rn} = \frac{\pi/n}{\tan(\pi/n)} = \frac{\pi}{n}\cot\left(\frac{\pi}{n}\right).$$

Also solved by James Campbell, Mark Evans, Richard I. Hess, Murray S. **Klamkin,** Henry S. **Lieberman,** William H. Peirce, **Skidmore** College Problem Group, Rex H. Wu, and the Proposer.

***857.*** [Spring **1995**] Proposed by Andrew ***Cusumano,*** Great Neck, New *York.*

Find all prime numbers whose reciprocals have **repetends** of exactly seven decimal places.

I. Solution by J. Ernest ***Wilkins,*** Jr., Clark Atlanta University, Atlanta, Georgia.

If p is such a prime number, then $1/p$ can be written as a fraction whose numerator is the seven-digit **repetend** and whose denominator is 9999999. Hence p is a factor of $9999999 = 3^2 \cdot 239 \cdot 4649$, so p is 3, 239, or 4649. Clearly, $p = 3$ does not satisfy the conditions of the problem, but $p = 239$ and $p = 4649$ do; the **repetends** for $1/239$ and for $1/4649$ are **0048141** and **0002151**, respectively.

II. Solution by Bob ***Prielipp,*** University of ***Wisconsin-Oshkosh, Oshkosh,*** Wisconsin.

Theorem 4 on pages 123-124 of [1] states that if $\gcd(n, 10) = 1$, then the period of $1/n$ is $r$, where r is the smallest positive integer such that $10^r \equiv 1 \pmod{n}$. Now $10^7 \equiv 1 \pmod{p}$ if and only if p divides $10^7 - 1 = 9999999 = 3^2 \cdot 239 \cdot 4649$. Now 7 is the smallest positive integral exponent r such that $10^r \equiv 1 \pmod{p}$ for $p = 239$ and for $p = 4649$, but $10^r \equiv 1 \pmod{3}$. Thus 239 and 4649 are the desired primes.

Also solved by Paul S. Bruckman, Russell Euler, Robert C. Gebhardt, Richard I. Hess, Jamshid **Kholdi,** Henry S. Lieberman, Peter **A.** Lindstrom, David E. Manes, Thomas E. Moore, H.-J. Seiffert, Kenneth M. **Wilke**, Rex H. Wu, Sammy and Jimmy Wu, and the Proposer.

## Reference


1. U. Dudley, Elementary Number Theory, 2nd ed, W. H. Freeman and Company, San Francisco, **1978.**


***858.*** [Spring **1995**] Proposed by David ***Iny,*** Baltimore, Maryland.

It is known that the rational numbers in the interval $[0, 1]$ can be enumerated. Let $\{r_k\}_{k=1}^{\infty}$ be such an enumeration and pick $\epsilon$ such that



$0 < \epsilon < 1$. Take an interval $I_k$ of length $\epsilon 2^{-k}$ centered on each $r_k$. Then the sum of all these interval lengths $\Sigma_{k=1}^{\infty} r_k = \epsilon < 1$. Show how **to find** a real number in $[0, 1]$ and not contained in any of the intervals $I_k$.

Solution by Henry S. Lieberman, Waban, Massachusetts.

The Cantor diagonal method works here. Consider a denumerable listing of the **nonterminating** decimal expansions of the rationals $\{r_k\}_{k=1}^{\infty}$ in $[0,1]$. Without loss of generality we assume that the tenths digit in each of $r_1$ through $r_4$ is zero. Construct a real number $s$ as follows. Let the tenths digit of $s$ be $7$. Then $|s - r_1| \geq 0.6 > 2^{-2} > 2^{-2}\epsilon$. For each $r_k$, $k = 1, 2, 3, 4$, we have $|s - r_k| \geq 0.6 > 2^{-k-1}\epsilon$, so $s$ lies outside intervals $I_1, I_2, I_3$, and $I_4$. For each $k \in N$, $k > 1$, let the *k*th entry of $s$ be the smallest of 3, **4, 5, 6,** and 7 that is not equal to the *k*th entry of $r_m$ for $m = 4k + 1$, $4k + 2$, $4k + 3$, and $4k + 4$. Then, assuming the worst possible case for the $(k + 1)$st decimal place, $|s - r_{4k+1}| \geq 0.3 \cdot 10^{-k} > 0.3 \cdot 2^{-4k} > 2^{-4k-2} > 2^{-4k-2}\epsilon$, so $s$ lies outside the interval $I_{4k+1}$. Similarly, it lies outside $I_{4k+2}$, $I_{4k+3}$, **and** $I_{4k+4}$. Hence $s$ is a decimal in $(0,1)$ and lies outside all the $I_k$.

Also solved by Paul S. Bruckman, Selvaratnam **Sridharma,** Rex H. Wu, and the Proposer.

***859.*** [Spring **1995**] Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

Sum in closed form the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{-1/2}{n}^2, \quad \text{where} \quad \binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.$$

I. Solution by Paul S. Bruckman, *Edmonds,* Washington.

We first show that $S$ is a well defined constant, **i.e.** the series converges (absolutely). **From** Stirling's formula,

$$\binom{2n}{n}^2 \approx \frac{4^{2n}}{n\pi} \quad \text{as } n \to \infty.$$

Note that

$$S = \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2$$

Then, for some constant C we have

$$0 < S < C\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = C,$$

which proves the series converges.

Using Pochhammer's symbol $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$, we next express $S$ in the form

$$S = \sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n\, n!}.$$

Therefore, $S$ may be expressed in terms of the hypergeometric function F as

$$S = -1 + F(½, ½; 2; 1).$$

It is well known, where $\Gamma$ is the gamma function, that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

provided that $\mathrm{Re}(c - a - b) > 0$ and c is not a nonnegative integer. Thus

$$S = -1 + F(½, ½; 2; 1) = -1 + \frac{\Gamma(2)\Gamma(1)}{\Gamma^2(3/2)}$$

$$= -1 + \frac{4}{\pi} \approx 0.2732395,$$

since $\Gamma(1) = \Gamma(2) = 1$ and $\Gamma(3/2) = \sqrt{\pi}/2$.

II. Solution by the Proposer.

We have that

$$S = \sum_{n=1}^{\infty}\frac{1}{n+1}\binom{-1/2}{n}^2$$

$$= \sum_{n=1}^{\infty}\binom{-1/2}{n}\frac{2}{\pi}\int_0^{\pi/2}\sin^{2n}\theta\, d\theta\int_0^1(-x)^n\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\int_0^1\left[(1 - x\sin^2\theta)^{-1/2} - 1\right]dx\, d\theta.$$



We set $x = (\sin^2\phi)/(\sin^2\theta)$, so $dx = 2\sin\phi\cos 4\, d\phi/\sin^2\theta$, and we get

$$S = \frac{2}{\pi}\int_0^{\pi/2}\sin^{-2}\theta\int_0^{\theta}(\sec\phi - 1)2\sin\phi\cos\phi\, d\phi\, d\theta$$

$$= \frac{}{\pi}\int_0^{\pi/2}\sin^{-2}\theta\left[1 - \text{cost} - \frac{\sin^2\theta}{2}\right]d\theta$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\sec^{-2}\left[\frac{\theta}{2}\right]d\left[\frac{\theta}{2}\right] - 1 = \frac{4}{\pi} - 1.$$

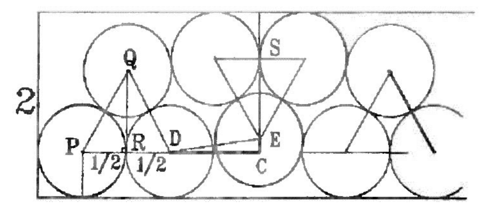
Also solved by Murray S. Klamkin, Carl Libis, and H.-J. Seiffert.

**860.** [Spring 1995] Proposed by Richard I. *Hess*, Rancho *Palos* Verdes, *California*.

This problem originally appeared in a column by the Japanese problems columnist Nob Yoshigahara. Find the minimal positive integer n so that $2n + 1$ circles of unit diameter can be packed inside a 2 by n rectangle.

Solution by the Proposer *and* the Problems Editor.

The "usual" packing of pairs of circles side-by-side will allow only $2n$ circles in a 2 by n rectangle, so we must use a different packing. Let us "glue" equilateral triangles of 3 circles each, and then pack them into the 2 by n rectangle, as shown in the figure.




Clearly we lose at the start, since a 2 by 2 rectangle then holds just 3 circles, circles ($P$), (D), and ($Q$). Since the height $QP$ is only $\sqrt{3}/2$, there is a slight gain in space when the next triangle of circles is [illegible]ed in. Since $DE = 1$ and $CE = 1 - \sqrt{3}/2$, then $CD = \sqrt{\sqrt{3} - 3/4} \approx 0.9909847666$ by the Pythagorean theorem. Hence, although 3 circles fit in a rectangle of length 2, we have 4 circles fit in one of length $1.5 + CD \approx 2.49$, 5 circles fit in length $2 + CD \approx 2.99$, 6 in length $2.5 + CD \approx 3.49$. In general, $3n - 2$ circles fit in length $(n - 1)CD + (n + 1)/2$, $3n - 1$ fit in $(n - 1)CD + (n + 2)/2$, and $3n$ circles fit in length $(n - 1)CD + (n + 3)/2$. Since we want the number of circles to be twice the length plus one, we examine the equation

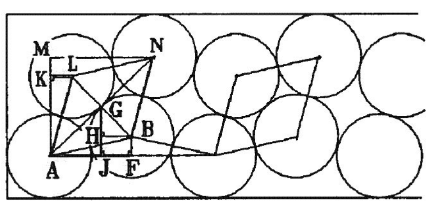
$$3n - 2 = 2[(n - 1)CD + (n + 1)/2] + 1$$

and solve it for $n$ to get

$$n = \frac{2 - CD}{1 - CD} \approx 111.92.$$

Using the second general case with $n = 112$, we find that $3n - 1 = 335$ circles fit in length $(n - 1)CD + (n + 2)/2 = 166.999 < 167$. Furthermore, $n = 111$ produces 331 circles in length $165.008 > 165$, which does not save enough length. In the figure above notice that if we cut off the left 1 unit, we remove space for exactly the first two circles. Doing so, we find that the smallest solution to the problem is 166 units of length enclosing 333 circles.

Are there other configurations that might produce smaller solutions? One might try packing "rhombi" of four circles "glued" together, as shown in the figure below.




A derivation similar to that above shows the distance

$$AF = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{6} \approx 0.995781916.$$

Here the smallest solution is 238 units of length for 477 circles. Thus the first solution is more efficient.

Also solved by Rex H. Wu. One incorrect *solution was* received.



861. [Spring **1995**] Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

Evaluate in closed form the sum

$$S(n,k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{2j}{k}.$$

Solution by Paul S. *Bruckman*, Edmonds, Washington.

Let

(1) $$f_n(x) = (-x^2 - 2x)^n = [1 - (1 + x)^2]^n,\ n = 0, 1, 2, \ldots.$$

Then

$$f_n(x) = \sum_{j=0}^{n}\binom{n}{j}(-1)^j(1 + x)^{2j} = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\sum_{k=0}^{2j}\binom{2j}{k}x^k$$

$$= \sum_{k=0}^{2n}x^k\sum_{k \le 2j \le 2n}(-1)^j\binom{n}{j}\binom{2j}{k} = \sum_{k=0}^{2n}x^k\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2j}{k}$$

$$= \sum_{k=0}^{2n}S(n,k)\,x^k$$

since $\binom{2j}{k} = 0$ if $0 \le 2j \le k$.

From (1) we see that the expansion of $f_n(x)$ contains no terms for powers of $x$ that are less than n. Therefore, $S(n, k) = 0$ if $0 \le k < n$. Thus

(2) $$f_n(x) = \sum_{k=n}^{2n}S(n,k)\,x^k.$$

On the other hand,

$$f_n(x) = (-x)^n(x + 2)^n = (-x)^n\sum_{k=0}^{n}\binom{n}{k}x^k 2^{n-k} =$$

$$(-1)^n\sum_{k=n}^{2n}\binom{n}{k-n}2^{2n-k}x^k.$$

By comparison of the coefficients of this last expression and (2), we obtain

$$S(n,\ k) = \begin{cases} (-1)^n \left[ {n \atop k-n} \right] 2^{2n-k} & \text{if } n \le k \le 2n \\ 0 & \text{otherwise.} \end{cases}$$

***Also solved by*** Murray **S. Klamkin,** H.-J. Seiffert, ***and the*** Proposer. ***Klamkin found the inequality in H. W. Gould, Combinatorial Identities, Morgantown Printing and Binding Co., Morgantown, 1972, p. 29, #364.***

---

**Norman** Schaumberger shows that $\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ by a new method:

$$S_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{k=1}^{2n} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{1}{2k} = \sum_{k=n+1}^{2n} \frac{1}{k}.$$

$$\ln 2 = \ln \prod_{k=n+1}^{2n} \frac{k}{k-1} = \sum_{k=n+1}^{2n} \frac{1}{k} \ln \left[ \frac{k}{k-1} \right]^k > \sum_{k=n+1}^{2n} \frac{1}{k} \ln e$$

$$= S_n > \sum_{k=n+1}^{2n} \frac{1}{k} \ln \left[ \frac{k+1}{k} \right]^k = \ln \prod_{k=n+1}^{2n} \frac{k+1}{k} = \ln \frac{2n+1}{n+1}.$$

Now let $n$ go to infinity.



## Trisecting the Paper Better

Professor **MONTE** ZERGER of Adams **State** College (Alamosa, Colorado), referring to the method of dividing a sheet of paper into vertical thirds using neither ruler, straightedge, nor compass in the spring 1995 issue (10 (1994-99) #2, pages 128-130) says that he has found a method that is just as simple and

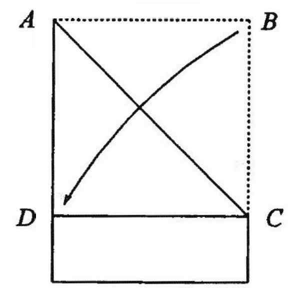



Figure 1

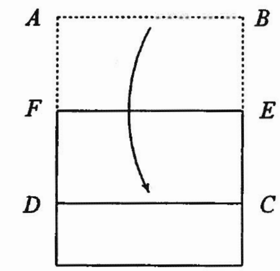



Figure 2

does not depend on the length of the paper. Here is how to do it.

"(I) Create a square on the upper portion of the sheet by bringing the upper and left edges of the sheet into coincidence. Label the vertices of the square *ABCD* as shown in Figure 1, and unfold..

"(2) Locate E, the midpoint of $\overline{BC}$, by **folding** $\overline{AB}$ to coincide with $\overline{DC}$. Unfold. See Figure 2.

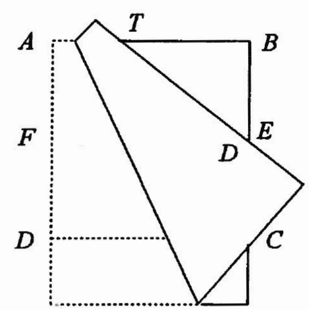



Figure 3

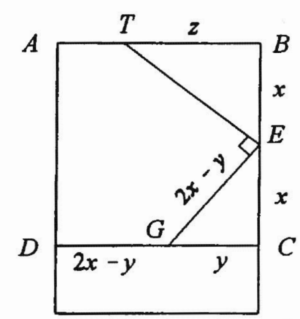



Figure 4

'(3) Fold to bring D into coincidence with E. Then T, the intersection point of the left and top edges of the paper, is the desired point. See Figure 3.

"To prove this, let BE = EC = x, $GC$ = y, and TB = $z$. Then DG = 2 x - y. Since the folding brought $\overline{DG}$ into coincidence with $\overline{GE}$, and $\angle ADC$ into coincidence with $\angle$TEG, we know GE = $2x$ - y and $\angle$TEG is a right angle. See Figure 4.

"From AGEC, $x^2 + y^2 = (2x - y)^2$ which leads to y = 3x14.

"Since AGEC ~ AETB, we have $x/y = z/x$ or $x^2 = yz = 3xz/4$. Thus $z$ = 4x13 = **(2/3)*AB*** so that AT is (1/3)***AB***.

"When can '*n*-secting' a paper be accomplished? For $n = 2^k$ it is trivial, and since we can 3-sect we can certainly $3 \cdot 2^k$-sect, by repeatedly bisecting our 3-sected result. What about 5-secting? There is a simple way.

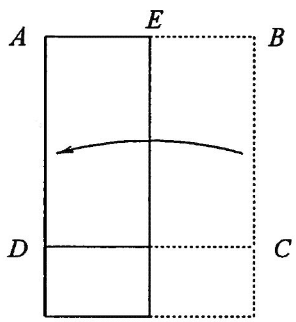



Figure 5

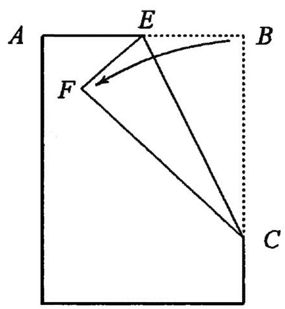



Figure 6

"(1) Again create a square on the upper portion of the sheet by bringing the upper and left edges of die sheet into coincidence. Label the vertices of the square ***ABCD*** as shown before in Figure 1.

"(2) This time fold the sheet in half vertically, bringing $\overline{BC}$ into coincidence with $\overline{AD}$. This will locate the midpoint E of $\overline{AB}$. Unfold. See Figure 5.

"(3) Now fold B down in such a **way** that the resulting crease passes through both ***C*** and E. Let the point where B touches the paper be F. Then die distance from F to the left edge of

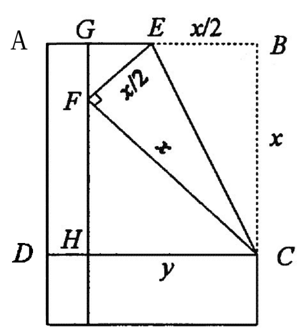



Figure 7



the paper is 115 of the width. See Figure 6.

"To prove this, label as in Figure 7. Since $\Delta EFG$ is similar to $\Delta FCH$, we can write

$$\frac{x/2}{x} = \frac{y - x/2}{\sqrt{x^2 - y^2}}, \quad 2y - x = \sqrt{x^2 - y^2}, \quad 4y^2 - 4xy = -y^2,$$

so y = ***4x15***, which means that the specified distance is ***x/5***.

**"This leaves 7-secting a paper as the** first pesky case. At least, I don't know a **relatively simple way. Do any readers?"**

**After** that, the problem of **11-secting** arises and, after that is solved, the problem of what use would be an 8 **1/2**-by-11 sheet of paper with eleven vertical **colums**, each **.77272727...** inches in width.

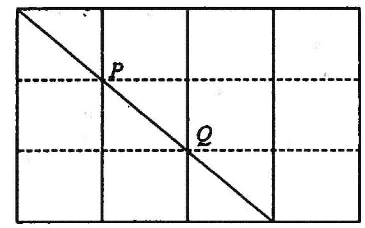



Figure 8

Here is a **different** way of **trisecting the** paper, from Professor Emil **Slowinski** of the chemistry department of **Macalester** College. Fold the paper in half and in half again **so as** to make four strips. **Take the rectangle** formed by three of them and fold so as to get die diagonal, as in Figure 8. It intersects the original folds at points P and $Q$. These points do the job.

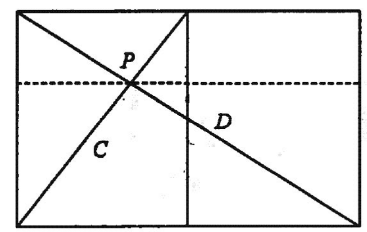



Figure 9

There is a similar **mathod** for accomplishing the same task (see Figure 9). Fold the paper in half and fold to get the main diagonal D. Construct the diagonal of one of the half-sheets, $C$. The intersection of D and C at P does the **trisection.**

**The** alert reader might have **asked** the question, "But how do you fold those diagonals?" Stan Wagon, of **Macalester**

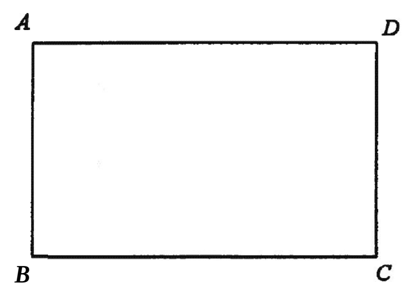



Figure 10

College, shows how. Given a rectangle $ABCD$ as in Figure 10, fold so that C touches A. Then, without unfolding, fold so that $B$ touches $D$. Unfold, and there you have the diagonal from $A$ to $C$.

When **Augustus** De Morgan asked how to fold paper into thirds in 1872 he got, as far as I know, no satisfactory answer. Look at the progress since then! Readers of the ***Journal*** are capable of feats that were beyond the capacity of people in De Morgan's time.

## An Application of War to Mathematics

The applications usually go the other way, but not this time.

**Fermat** (1601-1665) asked, probably out of nothing more than curiosity, for the location of the point $P$ in a triangle $ABC$ so that the sum of the distances from $P$ to the verices is as small as possible. The answer is that it is where the three angles around $P$ are all equal to $120°$.

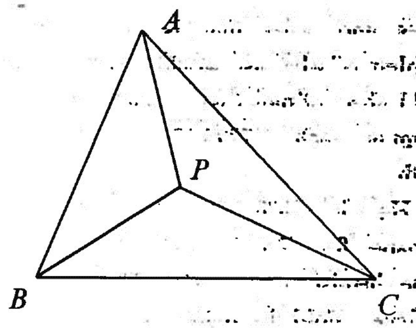



Figure 1. Minimize $AP + BP + CP$.

Here is a very clever proof of that, due to J. E. **Hoffmann**, that can be found on pages 21 and 22 of H. S. M. Coxeter's ***Introduction to Geometry*** (**Wiley**, New York, second edition 1969).

In Figure 2, rotate the triangle $APB$ through $60°$ around $B$ to get triangle $C'P'B$. Then triangles $ABC'$ and $PBP'$ are equilateral triangles. (The **figure** is not very accurate, but pictures are for illustration

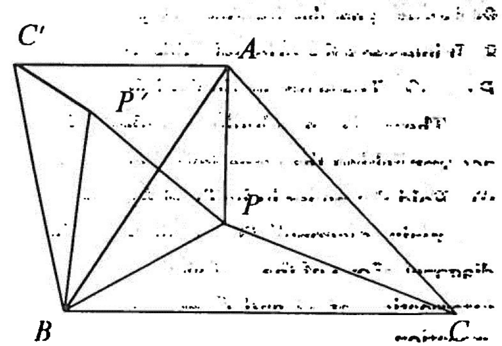



Figure 2. Finding the Fermat point.



only.) Thus $AP + BP + CP = C'P' + P'P + PC$. The right-hand sum will be minimal when the three segments form a straight line. When that is the case,

$$\angle BPC = 180° - \angle BPP' = 120°$$

and

$$\angle APB = \angle C'P'B = 180° - \angle PP'B = 120°.$$

Thus $\angle CPA$ is $120°$ too.

Mr. **Woodson** W. **Baldwin**, Jr., **of Torrance**, California, independently rediscovered **Fermat's result, and proved it as well**. The rediscovery, a good **illustration** of how theorems **can come** into being, came about **because** of the **Persian** Gulf war. During the war, Mr. **Baldwin** was employed by a corporation that provided the U. S. Air Force with information and advice on satellites and missiles. Mr. **Baldwin** writes

"During the Gulf war there were many Scud missiles being launched, which **launches** were observed by Air Force **geosynchronous** satellites. For every Scud missile launch, three different satellite/ground-station combinations produced three different **estimates** of the geographic location of the Scud launch point. I **weighed** the three points equally, **determined** their **center** of **gravity**, measured the sum of die radii from the center **of gravity to the three points, calculated** the mean radius, **and multiplied it** by a constant to yield an estimate of the **standard deviation** of a circular-normal **distribution**.

"**On a few** Such **Scud-launch occasions** we **were** supplied **also a** *fourth* estimate of the Scud launch point, which was provided by some undisclosed intelligence sources, **which point** I **generally ignored**. **However, out of curiosity, or boredom,** I did occasionally measure the sum of the radii **from the** intelligence point to the three **satellite-based points**, with which the former had no logical **connection**, of course, and this measurement was, as I expected, **usually greater than** the radial sum from the satellite-based center of gravity. However, **on** one **earth-shaking** day, the intelligence-based radial sum was *smaller* than **the regular radial sum**! **How could this** ***be***, **I asked** myself. The center **of gravity is** an **unbiased** estimate **of the true center** of the **distribution**, **I reminded myself**. Measuring from die center (or its estimate) ***ought*** to **minimize such measurements**, **I thought**. **I checked the figures**, **and re-checked**. **The figures were correct**. **This fact** raised the more fundamental question: given three points, if the **center of gravity is not the point which minimizes the radial sum to the three basic points, where IS the point which does so?**

"I set up a hypothetical triangle, **and wrote a computer program to compute a** fast radial sum from any point Then I combed the area of the triangle, using finer and finer spacings, until I found the minimum of all minima, good to about eight significant figures. The final measuring point I knew, the three final radii I knew. I calculated the three central angles. They were *exactly* 120° each! *I* was astounded. ...

"The above exercises provided the *experimental* proof of the location of the point which minimizes the radial sum to the vertices of a triangle. The *mathematical* proof is furnished in the attached document"

Mr. Woodson's proof is longer than the proof given above, but no less correct.

## The 1995 National Pi Mu Epsilon Meeting

The meeting took place in conjunction with the summer meeting of the Mathematical Association of America and the American Mathematical Society in Burlington, Vermont, August 5-7,1995.

There were twenty-two student papers delivered in four sessions:

Charles Sanders Peirce, or the consequences of a hypothesis, by Ivana Metodieva Alexandrova (Furman University)

Iteration of the greatest integer function, by Jason Calmes (Southeastern Louisiana University)

Applications of the Polya-Burnside theorem to teaching, toys and jewelry, by Ashley Carter (University of Wisconsin—Parkside)

Mastermind: breaking the codes, by Shawn Chiappetta and Steven Gannaway (Carthage College)

Check digits and license numbers, by Alayne Clare (Youngstown State University)

The triangle peg game, by Scott E. Clark (Youngstown State University)

Pursuit curves: the mathematics of coyotes, roadrunners, and ants, by Philip J. Darcy (St. Bonaventure University)

Hamiltonian properties of Petersenlike graphs, by Dan Diminnie (Allegheny College)

Is there (ever) an end?, by Jacqueline Goss (St. Norbert College)

Pertubation expansion for hermitian gaussian randon matrices, by Nancy Heinschel (University of California—Davis)



Groebner bases, by Dennis Keeler (Miami University)

Commutativity of matrices, by Gee Yoke Lan (Wichita State University)

The secret behind the Keebler cards, by Jason Martin (Youngstown State University)

Subspaces of the Sorgenfrey line, by Justin Moore (Miami University)

Derivative rings, by Dan Nordman (St John's University)

The seven guests: no longer a guess!, by Dennis Schmidt (St. Norbert College)

Indiana Jones and the quest for anticonnected digraphs, by Nick Sousanis (Western Michigan University)

Images and inverse *images* of iterates of the line graph operator, by Donna R. Svvoid (Hendrix College)

Solving general nonlinear multivariate polynomial systems using algebraic geometry, by Wayne Tarrant (Wake Forest University)

Special relativity: the Lorentz transformation and the hyperbolic geometry of spacetime, by Michael Theriot, Jr. (Louisana State University)

A function and its "dual", by Richard Tuggle (St. Norbert College)

Dirichlet's theorem and an improved lower bound for an L-function, by Sonny Vu (University of Illinois—Urbana-Champaign).

Four prizes for papers of unusualk merit were awarded to Aron Atkins, Ashley Carter, Alayne Clare, and Scott Clark.

The National Security Agency again awarded the Society a grant of $5000 for the support and encouragement of student speakers.

The J. Sutherland Frame Lecture was delivered by Marjorie Senechal of Smith College, whose subject was "Tilings as diffraction gratings."

## Unparalleled Opportunity

At the business meeting of the Society, the decision was made to raise the subscription price of the *Journal.* The price *has* been unchanged since 1980, when the cost of living (which includes reading the *Journal*) was less *than* half of what it is now. The new rates are $20 for two years and $40 for five years.

However, present subscribers have the opportunity to extend their present subscriptions at the old rates through the end of the millennium. There will be no similar opportunity for at least the next one thousand years.

To take advantage of this offer, calculate the number of copies of the Journal that will be issued between the time of the expiration of your subscription (indicated on your address label) and the fall 1999 issue. For example, if your address label contains an "F 96", the number of issues would be six (S 97, F 97, S 98, F 98, S 99, F 99). Then multiply that integer by two and send a check, marked "extension" (so as to avoid confusion), for that amount to the Journal's business manager,

Robert S. Smith
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Subscribers outside the United States should multiply the integer by three and proceed similarly.

# PI MU EPSILON T-SHIRTS

**The shirts are white, Hanes® BEEFY-T®, pre-shrunk, 1 % cotton. The front of the shirt has a large Pi Mu Epsilon shield (in black), with the line "1914 - ∞" below it. The back of the shirt has a "Π M E" tiling in the PME colors of gold, lavender, and violet. This tiling of the plane was designed by Doris Schattschneider, on the occasion of PME's 75th anniversary in 1989. The shirts are available in sizes large and X-large. The price is only $10 per shirt, which includes postage and handling. To obtain your shirt, send your check or money order, payable to Pi Mu Epsilon, to:**

**Rick Poss**
**Mathematics - Pi Mu Epsilon**
**St. Norbert College**
**100 Grant Street**
**De Pere, WI 54115**

# SPEECHLESS IN SEATTLE?

## Don't be!

Present a paper at the national Pi Mu Epsilon meeting at the University of Washington, in Seattle, WA, August 10-12, 1996. This meeting is being held in conjunction with the annual MAA **MathFest.** Pi Mu Epsilon student speakers are eligible for **free travel** to the meeting! (See below for details.) Any student member of Pi Mu Epsilon not having received a master's degree by May, 1996, is eligible to speak at the national meeting.

Pi Mu Epsilon will provide travel support for student speakers at the national meeting. If a chapter is not represented by a student speaker, Pi Mu Epsilon will provide one-half support for a student delegate. Full support is defined to be full round-trip air fare (including ground transportation) from the student's school or home to Seattle, WA, up to $600. (Delegates will receive up to $300.) A student who chooses to drive will receive 25 cents per mile for the round trip from school or home to Seattle, up to $600. (Delegates will receive 12% cents per mile, up to $300.)

If there is more than one speaker from a chapter, each of the additional speakers (up to four) will be eligible for 20% of what the first speaker receives. For example, if the distance traveled (by car or van) is over 2400 miles (round trip distance), a single speaker would receive $600, two student speakers would receive $720 (to share in any way they wish), three speakers would share $840, four speakers would share $960, and five or more speakers from this single chapter would share $1080.

For information on how to apply to speak and to receive travel funds, see your Pi Mu Epsilon Advisor

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