PI MU EPSILON JOURNAL

VOLUME 10 FALL 1996 NUMBER 5

CONTENTS

(continued on inside back cover)

PI MU EPSILON JOURNAL

VOLUME 10 FALL 1996 NUMBER 5

CONTENTS

(continued on inside back cover)

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION OF THE NATIONAL HONORARY MATHEMATICS SOCIETY

Editor: Russell Euler
Problems Editor: Clayton W. Dodge
Business **Manager:** Joan Weiss

Editorial correspondence, including manuscripts and news items, should be sent to Russell Euler, Department of Mathematics and Statistics, Northwest Missouri **State** University, **Maryville,** Missouri 64468 (e-mail address, 0100120@acad.nwmissouri.edu). Guidelines for authors **are** found on page 441 of **this issue.**

Problems for solution sad solutions to problems should be sent directly to Clayton W. **Dodge,** Mathematics Department, 5752 Neville Hall, University of Maine, **Orono, Maine** 04469.

Correspondence **regarding** subscriptions, back issues or changes of address should be **addressed** to **Jam Weiss**, Department of Mathematics and Computer Science, Fairfield University, **Fairfield,** CT 06430.

The PI MU EPSILON JOURNAL is published twice a year - Fall and Spring. One volume *consists* of five years(tea issues).

EDITORIAL

Russell Euler

The Pi **Mu** Epsilon **Journal** is a mathematics magazine primarily for students and instructors at a variety of post-secondary levels. Hopefully, the general reader with an interest in mathematics will also **find** interesting content in the **Journal.**

Submissions to the **Journal** can include:

1) research appropriate for the readership
2) undergraduate research projects
3) expository material
4) historical material
5) educational opportunities
6) career opportunities
7) problems and solutions.

Manuscripts from student authors are given top priority.

The transition between editors has been smooth -- primarily because of the excellent guidance from the previous editor Underwood Dudley. The Problems Editor, Clayton Dodge, has also been extremely helpful and cordial. The **Journal** now has a Business Manager -- Joan Weiss. Her help is appreciated very much.

Since this is your journal, suggestions for improving the **Journal** will always be carefully considered.

Red Light, Green Light: A Model of Traffic Signal Systems

Ryan Bennink (student)
Hope College

Introduction

Probably everyone who drives has had the experience of "hitting all the red lights." Especially since moving from a rural area to a town, I have found myself wondering why various lights (red in particular) last as long as they do and how the schedule of traffic signals is determined. What criteria should one use? Can multiple sets of signals on many streets be coordinated to yield optimum traffic flow? And so when a mathematical modeling project was assigned in my senior math seminar, I felt the time was ripe for an investigation of traffic light timing.

At the outset of my project I had a number of objectives in mind. First, I wished to identify the parameters which characterize traffic flow and to determine what restrictions those parameters placed on signal timing. Second, I hoped to find a tuning scheme which would provide for the "best" traffic flow for a model city. And finally, I wanted to compare my proposed timing scheme to real-world traffic signal schedules.

The model I came up with is based on several premises: Drivers always obey the speed limit and all traffic signals; drivers are perfect (they always make correct judgments); and conditions must never force a driver to do something illegal. Furthermore, a city consists of a lattice of intersections with a traffic signal at each intersection. Each block is square, and all blocks have the same side dimension b. Streets are two-way and there are no *a priori* preferred streets or directions to traffic. Finally, traffic signals are uniformly periodic in both time and space. This means that each light cycles from green to yellow to red on a regular, repealing schedule with period T. la addition, there is a number N such that traffic signals which are N intersections apart run on identical schedules.

A Bask Description of Motion

One can describe the motion of a car of length L (say, traveling north) by plotting die position s of the car's front bumper as a function of time t (see Figure 1). Intersections which occupy a width w become bands in the t-s plane. Regions of a particular band can then be color-coded to indicate the status of the light at the intersection for various times. If the speed limit is v_{max} then the maximum slope

of $s(t)$ is v_{max}. We can also characterize starting and stopping by acceleration terms, denoting the maximum braking acceleration by a_{brake}. We can then apply the elementary constant-acceleration motion formulas

$$
\begin{aligned}
s &= s_0 + v_0 t + \frac{1}{2} a t^2 \\
v &= v_0 + at \\
v^2 &= v_0^2 + 2as
\end{aligned}
\tag{1}
$$

For example, the distance d_{stop} required for a car traveling at the speed limit to come to a full stop is

$$d_{stop} = \frac{v_{max}^2}{2|a_{brake}|} \tag{2}$$

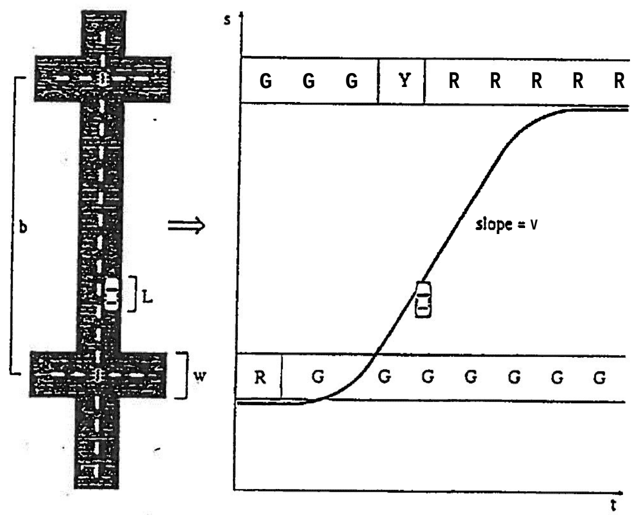


Figure 1.

Implementing Restricting Conditions

Yellow ***Lights***

The amount of time the yellow light must be on is perhaps the simplest feature to determine. The governing rule is this: ***If a car cannot stop at an intersection, it must be able to clear the intersection before the light turns red,*** Consider a car going the speed limit as it approaches an intersection with a green light. Past a certain point, the car is too close to the intersection to stop in time if the light turns yellow. The point of decision, by which time the light must have turned yellow if the driver is to stop, is when the driver is d_{stop} away from the intersection (see Figure 2). Just past this point, the car must keep going and the

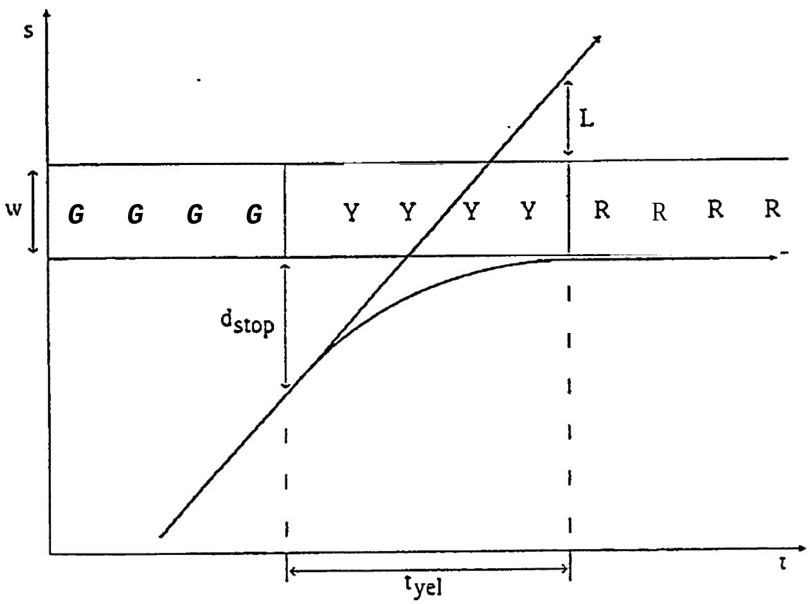


Figure 2.

yellow light must stay on until the car's back bumper leaves the intersection. That is, t_{yel} must be long enough to allow the car to traverse a distance $d_{stop} + w + L$. This yields

$$t_{yel} \geq \frac{v_{\max}}{2|a_{brake}|} + \frac{w+L}{v_{\max}} \quad . \tag{3}$$

The wider the intersection and the longer the car, the longer the yellow light must be. Good braking capability, however, reduces the minimum time. Although the speed limit enters twice, die dominant effect is that die required time lengthens with higher speed limits because stopping distance is increased.

Red and Green Lights: Two-way traffic.

In this model, traffic flow is considered optimal when cars traveling at the speed limit hit the fewest red lights. That is, one would like to time the sequence of traffic lights so that, as much as possible, cars which catch one green light will continue to encounter green lights. The green lights should "follow the speed limit." Perhaps this does not seem too daunting until we remember that since the streets are two-way, the schedule must work for cars traveling in both directions!

Every signal cycles through all three colors with the same period T, and so the times t_{grn}, t_{yel}, and t_{red} must add up to T. If all streets are assumed equally busy, then north-south drivers and east-west drivers should wait for each other equal amounts of time; thus $t_{red} = T/2$. It then follows that $t_{grn} = (T/2) - t_{yel}$. It is important to note that although every signal has period T, each signal has a separate phase delay o (measured in seconds).

The concern here is the relative timing of signals along a single street, which depends primarily on v_{max} and the distance b between intersections. Since w does not affect the time required to go from one intersection to the next, we can represent intersections simply by lines spaced b units apart. And because signals are periodic in both time and space, we can map the entire t-s plane onto a rectangle of width T and length Nb (see Figure 3), where N is the number of intersections after which the schedule of signals repeats. (Notice the wrap-around effect on traffic trajectories.) The period is given by $T = (Nb)/v_{max}$. Hence our task is to situate the green, yellow, and red regions on each intersection timeline so that there exist straight lines with slopes $+v_{max}$ and $-v_{max}$ (representing northbound and southbound cars) which pass through only green regions. In this formulation, the problem is rather difficult since both the signal phases and traffic trajectories are arbitrary. Realizing that northbound and southbound trajectories always make the same " X shape wherever they occur, we can simplify the situation by shifting the origin. We now consider the traffic trajectories fixed and allow the intersections to translate along the s axis.

From this reference frame we can easily measure the time interval Δt between northbound and southbound cars at a particular location s. The left half of Figure 4 shows the "X" formed by northbound and southbound cars over a 3-

block distance. The horizontal separation between the legs of the "X" is the time interval Δt, which is shown as a triangular-looking function of s in the right half of Figure 4. As one can see from the right half of the figure, the time interval between northbound and southbound trajectories is never more than half a period since the space "wraps around". The challenge is to determine if and how green light regions (solid horizontal bars marked with a "G") may be placed so as to intersect both trajectories at each intersection (thin horizontal lines).

If green lights were to last at least half a period, the problem would be trivial; no matter how the intersections were translated, the green light duration would always exceed Δt. The fact that t_{grn} is short of half a period by the amount t_{yel} means there are two forbidden zones (shaded horizontal bars) where the green region is not long enough to catch both trajectories; i.e., where $t_{grn} < \Delta t$. The problem has a solution if we can translate the intersections so that none of them falls in a forbidden zone. Geometrical analysis of Figure 4 reveals that each

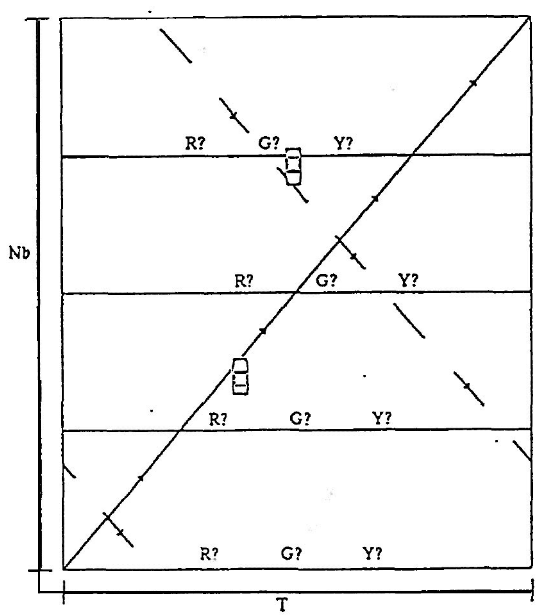


Figure 3.

forbidden zone extends over a distance $v_{max}t_{yel}$ on the s axis. Clearly the intersection spacing b must be at least as large as this. But since two forbidden zones must be avoided, the condition which is sufficient to guarantee a solution becomes $b > 2v_{max}t_{yel}$. Solving this inequality for t_{yel} and combining it with equation (3), we find that a solution exists as long as

$$d_{stop} \leq b - 2(w + L). \tag{4}$$

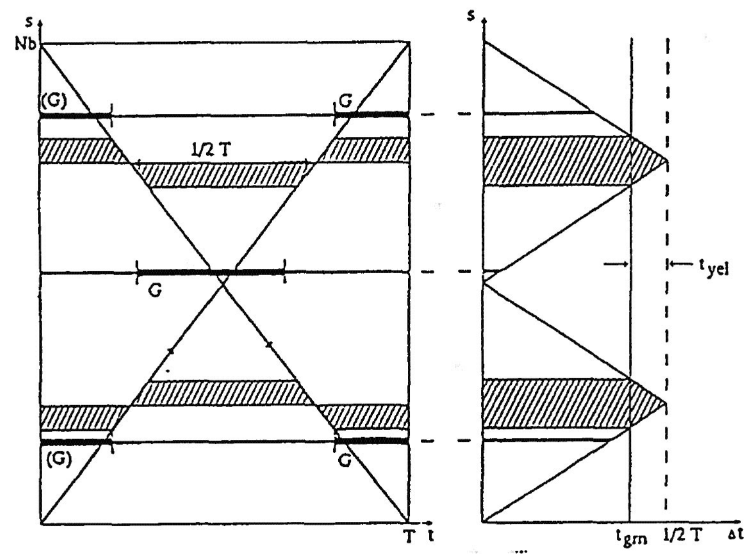


Figure 4.

In practice, this is never a concern. (If you cannot stop your car in a little under a block, either you are driving too fast or you need new brakes!) For a concrete example, let $N = 4$, $b = 330ft$, $w = 30ft$, $L = 15ft$, $v_{max} = 30mi/hr = 44ft/s$, and $a_{brake} = 15ft/s^2$. Then $T = 30s$ and from equation (3) we have $t_{yel} \geq 2.5s$, which means $t_{grn} \leq 12.5s$. Over every 4-block stretch of road there are two sections $v_{max}t_{yel} = 110ft$ long which periodic northbound and southbound cars traverse at times more than $12.5s$ apart. If the lights are scheduled properly, such cars will be able to proceed from intersection to intersection without having to stop for a red light.

As it turns out, many solutions exist. The best ones seem to be obtained with N = 2, 3, 4, 6, or 8. (For typical real-world parameters and N = 1, the light. change too quickly. For $N > 8$, the lights can take several minutes to turn, and most people do not like to wait that long.) Acceptable solutions are optimized by adjusting the placement of intersections so that the sum of the time intervals across all intersections is a minimum. That way, one creates the widest green light intervals through which caravans of many cars can proceed uninhibited. The phase delay for each intersection is then determined by the times at which the trajectories cross the intersection line.

Figures 5 and 6 show typical timing schemes. (The vertical axis is qualitative rather than quantitative, and refers to the particular light within the N-intersection pattern. For each light, a high level represents a green light, a middle level indicates a yellow light, and a low level indicates a red light.) As one can see, traffic signals are generally staggered rather than aligned. Some lights have identical phases, however, because northbound and southbound traffic trajectories are symmetrical and the optimization algorithm favors neither direction. Incidentally, Figure 5 was generated using N = 3, b = $330ft$, and v = 30mi/hr = 44ft/s. Thus $T = Nb/v = 22.5s$, as shown in the graph. Figure 6 was generated with $N = 4$, yielding the period T = 30s.

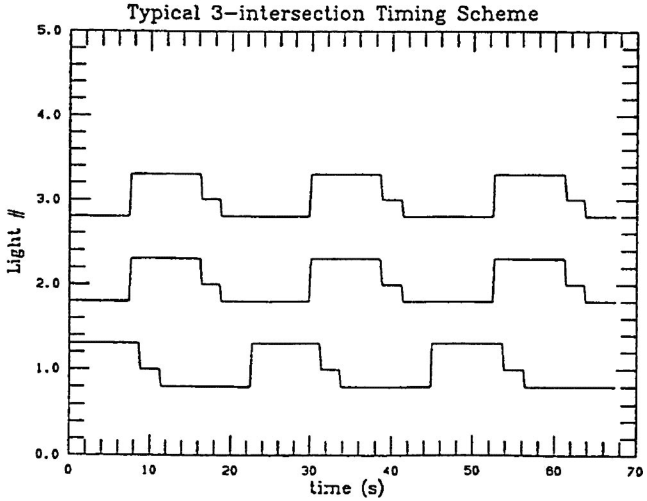


Figure 5

A ***Complete Timing Schedule: Four-way traffic.***

Once we have determined a working time schedule for traffic signals along a single street (1 dimension), we can extend the schedule to accommodate city traffic (2 dimensions). An example will suffice to show how this is accomplished.

Consider the case $N = 4$. Assume that we have come up with appropriate phase delays for traffic signals along a single street, and say that northbound traffic encounters the sequence of signals with phase delays $\phi_0, \phi_1, \phi_2, \phi_3, \phi_0, \ldots$. (Southbound traffic encounters the same sequence in the opposite order.) If we apply this 4-element sequence to adjacent N-S streets but stagger it as shown in equation (5), then the sequence of phases holds for E-W streets as well. Columns of the 4 x 4 lattice represent 4-intersection segments of adjacent N-S streets; rows represent segments of adjacent E-W streets. Northbound and eastbound traffic (i.e., traffic traversing the lattice up or to the right) will encounter the same phase differences between consecutive lights; the same holds true for southbound and westbound traffic.

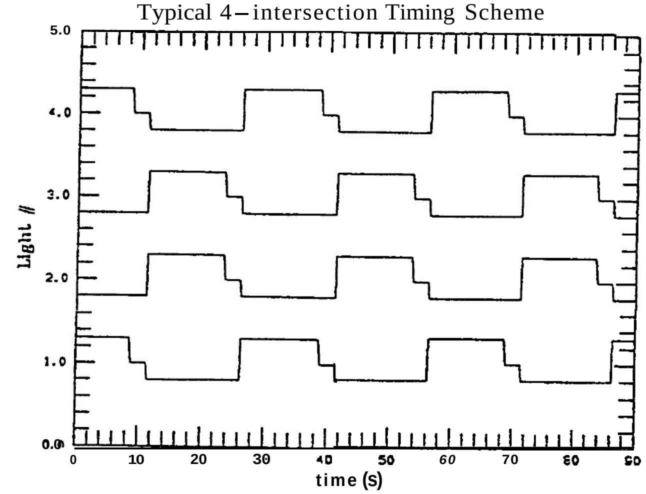


Figure 6.

$$\begin{array}{ll} \emptyset_3 & \emptyset_3\ \emptyset_0\ \emptyset_1\ \emptyset_2 \\ \emptyset_2 & \emptyset_2\ \emptyset_3\ \emptyset_0\ \emptyset_1 \\ \emptyset_1 \rightarrow & \emptyset_1\ \emptyset_2\ \emptyset_3\ \emptyset_0 \\ 0 & \emptyset_0\ \emptyset_1\ \emptyset_2\ \emptyset_3 \end{array} \qquad (5)$$

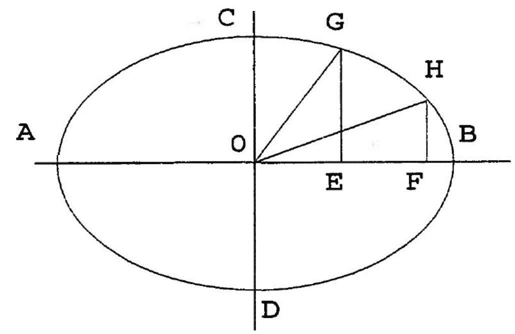
This pattern can be extended indefinitely in all four directions. Thus one can come up with a traffic signal schedule which maximizes unimpeded traffic flow for a lattice of intersections.

Computer Simulation

I thought it would be fun to simulate traffic flow in a city by implementing my lattice of intersections and traffic signals schemes on a computer. Figure 7 shows position-versus-time data extracted from a typical run. In this particular run, four cars were driving north along a street with a 4-intersection signal pattern similar to that shown in Figure 6. One can see smooth curves indicating periods of acceleration or braking, as well as flat regions where cars are waiting at a red light. As one would expect. Figure 7 is reminiscent of Figure 1.

To assess the effectiveness of the 4-intersection pattern suggested by my model, I decided to compare the average speed of cars in a city with the 4-intersection pattern with that of cars in a city with random signal schedules (See Figure 8). The average speed was calculated by taking the total distance traveled by all the cars in the city divided by the length of time the simulation was run (chosen to be 10 periods). As one can see, when there are few cars on the road the average speed in the "planned" city (upper line) is quite close to the given speed limit of 30 *mi/hr* (dashed line). Cars in the "random" city (lower line) average little more than half the speed limit. As the traffic becomes heavier, however, the planned signal scheme seems to be less and less effective, to the point that it produces results hardly better than those of traffic signals given arbitrary phases. This result surprised me; I expected that the 4-intersection pattern would be significantly better up to much heavier traffic loads. Apparently, only a short caravan of vehicles are able to pass uninhibited through green lights at successive intersections.

and H respectively. Show that the areas of sectors OBH and OGC are equal. See the accompanying figure.



I. Solution by Robert Downes, Mountain Lakes High School, *Plainfield*, New Jersey.

Place the ellipse in the Cartesian plane so that we have $B(b, 0)$, $C(0, c)$, $E(e, 0)$, and $F(f, 0)$. The equation of the ellipse then becomes

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1.$$

Since we were given $OE^2 + OF^2 = OB^2$, from which we get $e^2 + f^2 = b^2$, then we find that $H(f, ce/b)$ and $G(e, cf/b)$. The areas of sectors OBH and OCG, $K(OBH)$ and $K(OCG)$, are given by

$$K(OBH) = \int_0^{ce/b} \left(\frac{b}{c}\sqrt{c^2 - y^2} - \frac{bf}{ce}y\right)dy = \frac{bc}{2}\sin^{-1}\frac{e}{b}$$

and

$$K(OGC) = \int_0^{e} \left(\frac{c}{b}\sqrt{b^2 - x^2} - \frac{cf}{be}x\right)dx = \frac{bc}{2}\sin^{-1}\frac{e}{b} = K(OBH).$$

II. Solution by *Skidmore* College Problem Group, Saratoga Springs, New York.

Place the ellipse in the Cartesian plane (as in Solution I above). The linear transformation f, given by $f(x, y) = (x/b, y/c)$, sends the ellipse to the unit circle. We let $f(A) = A'$, etc. The condition that $e^2 + f^2 = b^2$ becomes $e'^2 + f'^2 = 1$. Since applying f multiplies areas by a constant value, we see that the elliptical sectors have the same area if and only if the corresponding circular sectors have the same area. The area of a circular sector of radius r and central angle 9 is $r^2\theta/2$, so we must show the sectors $O'B'H'$ and $O'C'G'$ have equal central angles α and β, respectively. Now $\alpha = \cos^{-1} f'$ and $\beta = \pi/2 - \cos^{-1} e'$. Because $e'^2 + f'^2 = 1$, then $\cos^{-1} f'$ and $\cos^{-1} e'$ are complementary, so $\alpha = \beta$.

Also solved by Miguel **Amengual Covas,** Paul S. **Bruckman,** George P. Evanovich, **Jayanthi Ganapathy,** Richard I. Hess, Joe Howard, Murray S. Klamkin, Henry S. **Lieberman,** Peter **A. Lindstrom,** David E. Manes, V. S. **Manoranjan,** G. **Mavrigian, Yoshinobu** Murayoshi, H.-J. **Seiffert,** and ***the*** Proposer.

870. [Fall **1995]** ***Proposed by Grattan P. Murphy, University of Maine. Orono, Maine.***

This proposal is based on a problem posed at a recent mathematics meeting and is intended especially for students. Without using machine calculation, that is, without actually finding the digits of the number, show that at least one digit occurs at least 6 times in the decimal representation of the number $(7^7)^7 \cdot 7^7 \cdot 77$.

Solution by Henry S. Lieberman, Waban, Massachusetts.

If $n = (7^7)^7 \cdot 7^7 \cdot 77$, then $\log_{10} n = 57 \log 7 + \log 11 \approx 49.212$ and n has 50 decimal digits. If no digit occurs at least 6 times, then each digit would occur exactly 5 times and the sum of the digits of n would be

$$5(0 + 1 + 2 + \ldots + 9) = 5 \cdot 45 = 225.$$

Since $3 \mid 225$, then $3 \mid n$, but clearly n has no factor of 3. Hence, some digit must appear at least 6 times in the decimal representation of n.

Also solved by Paul S. **Bruckman,** Victor G. Feser, Richard I. Hess, David E. Manes, **Kandasamy Muthuvel,** Michael R. Pinter, Kenneth M. **Wilke,** ***and the*** Proposer.

Rachele Dembowski's Partition Problem

Cecil Rousseau
The University of Memphis

1. The Problem. In her article ***Enumerating Partitions,*** in the Fall, 1995 issue of this **Journal [1],** Rachele Dembowski poses the following problem.

Problem A. Find the **number** of partitions with n parts in which, for $k = 1, 2, \ldots, n$, the kth part is less than or equal to $n - k + 1$ and all parts are odd.

Interpreting the parts of the partition, taken in reverse order, as the values $f(1), f(2), \ldots, f(n)$, we have the following equivalent problem.

Problem B. Find the number of nondecreasing functions

$$f: \{1,2,\ldots,n\} \to \left\{1,3,5,\ldots,2\left[\frac{n-1}{2}\right]+1\right\}$$

satisfying $f(k) \le k$ for $k = 1,2,\ldots,n$.

Another equivalent problem can be phrased in the language of "heads or tails", hi this formulation, an HT sequence (or string) is a finite sequence of symbols, each of which is either H or T.

Problem C. Find the number **of** ***HT*** sequences with $n - 1$ ***H's*** where, from the **begining** up to any point in die sequence, there are at least ***twice*** as many ***H's*** as there **are** ***T's.*** This is the number of outcomes of a "heads or tails" game in which the player, starting with no funds, wins one dollar for each head and owes two dollars for each tail, given (hat the player never goes in debt, and obtains heads a total of $n - 1$ times.

To see that Problem B and Problem C **are** equivalent, associate with each function counted in Problem B a corresponding ***HT*** sequence, namely $HB_1 HB_2 \ldots HB_{n-1}$ where $B_1, B_2, \ldots, B_{n-1}$ are (possibly empty) blocks of T's, with the number of T's in B_k being $(f(k+1) - f(k))/2$. The resulting HI sequence **has** $n - 1$ ***H's*** and is constructed so that at any point in the sequence where there are ***as*** yet k ***H's***, the number of ***T's*** is at most

$$\frac{f(2)-f(1)}{2} + \frac{f(3)-f(2)}{2} + \cdots + \frac{f(k+1)-f(k)}{2} = \frac{f(k+1)-1}{2} \le \frac{k}{2}.$$

Thus the resulting *HT* sequence satisfies the conditions of Problem C. It is clear that the mapping just described is a bijection.

Let a_n denote the common answer to Problems A through C. We shall prove

$$a_n = \begin{cases} \dfrac{1}{3m+1}\dbinom{3m+1}{m} & \text{if } n = 2m, \\[2ex] \dfrac{2}{3m+2}\dbinom{3m+2}{m} & \text{if } n = 2m+1, \end{cases}$$

which agrees with the conjecture made in [1].

2. The **Method.** Note that a function counted in Problem B satisfies $f(n) = n - k + 1$ if and only if the total number of *T's* in the corresponding *HT* sequence is $(f(n) - 1)/2 = (n - k)/2$. Equivalently, $f(n) = n - k + 1$ if and only if the player has a fortune of $(n - 1) - 2((n - k)/2) = k - 1$ dollars after the last coin toss. Let $c(n, k)$ denote the number of *HT* sequences in Problem C so that the player's final fortune is $k - 1$ dollars. Clearly, $c(n, k) = 0$ unless k and n have the same parity, and

$$a_n = \begin{cases} c(n,2) + c(n,4) + \cdots + c(n,n) & \text{if } n \text{ is even,} \\ c(n,1) + c(n,3) + \cdots + c(n,n) & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

If the player's fortune after the last toss is $k - 1$ dollars, then the fortune just before obtaining the last head is $k + 2(j - 1)$ for some $j \geq 0$, from which we have the recurrence relation

$$c(n, k) = c(n-1, k-1) + c(n-1, k+1) + \cdots + c(n-1, n-1). \tag{2}$$

Comparing (1) and (2), we have

$$a_n = \begin{cases} c(n+1, 1) & \text{if } n \text{ is even,} \\ c(n+1, 2) & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

Note that by two applications of (2),

$$c(n, k) = c(n-1, k-1) + c(n-1, k+1) + \cdots + c(n-1, n-1)$$
$$c(n, k+2) = c(n-1, k+1) + c(n-1, k+3) + \cdots + c(n-1, n-1),$$

from which we have

$$c(n, k) = c(n-1, k-1) + c(n, k+2). \tag{4}$$

Thus our plan is mapped out. We want to determine the numbers $c(n, k)$ using (4) and then find a_n using (3). Using (4) and the fact that $c(n, n) = 1$, we can easily compute a table of values for $c(n, k)$ for small values of n a la Pascal's triangle.

n\k	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2		1										
3	1		1									
4		2		1								
5	3		3		1							
6		7		4		1						
7	12		12		5		1					
8		30		18		6		1				
9	55		55		25		7		1			
10		143		88		33		8		1		
11	273		273		130		42		9		1	
12		728		455		182		52		10		1

Table 1 - Values for $c(n, k)$

3. Solution by Generating Functions. We shall find $c(n, k)$ using the method of generating functions. In part, the solution rests on the following result from classical analysis [3, §7.32], [4, p. 138].

Lagrange's Expansion. Let $f(z)$ and $F(z)$ be analytic on and inside a contour C surrounding the origin, and let w be such that $|wf(z)| < |z|$ at all points z on C. Then the equation $w = z/f(z)$ has one root $z = z(w)$ in the interior of C, and $F(z(w))$ has the expansion

$$F(z(w)) = F(0) + \sum_{m=1}^{\infty} \frac{w^m}{m!} \left\{ \frac{d^{m-1}}{dt^{m-1}} F'(t)[f(t)]^m \right\}_{t=0}.$$

Also, the following fact is crucial.

Lemma. Let C_k denote the class of all *HT* sequences for the game in Problem C such that the player is never in debt and has a fortune of $k - 1$ dollars after the last toss. That $C_{k+l} = \{aHb \mid a \in C_k, b \in C_l\}$. Here aHb denotes the sequence obtained

obtained by concatenation of $a \in C_k$, a single H, and $b \in C_l$.

Proof. Given a sequence in C_{k+l}, locate the last point at which the player's fortune is $k - 1$ dollars, and let a denote the *HT* sequence up to that point. Then $a \in C_k$, and the next term of the sequence must be *H*; otherwise the player's fortune would be $k - 1$ dollars at some later point. Letting b denote the remainder of the given sequence, we see that, had the player started with no funds at the beginning of b, he or she would never go in debt and have a final fortune of $l - 1$ dollars. Hence $b \in C_l$. Thus every member of C_{k+l} can be written *aHb* where $a \in C_k$ and $b \in C_l$. We claim that this representation is unique. To see this, suppose for the moment that $a'Hb' = aHb$ where $a, a' \in C_k$ and $b, b' \in C_l$. We may assume that the lengths of a' and a are p and q, respectively, with $p < q$. Then toss $p + 1$ yields *H*, and the player begins at toss $p + 2$ with a fortune of k dollars, The ensuing sequence, b', leads to a fortune of $k - 1$ dollars after $q - p - 1$ tosses, so had the player started this sequence with no funds, he or she would then be in debt. This contradicts $b' \in C_l$ and proves that the representation *aHb* with $a \in C_k$ and $b \in C_l$ is unique.

Note that if there are $r - 1$ *H's* in a and $s - 1$ *H's* in b then *aHb* has $r + s - 1$ *H's*. In view of the uniqueness of the *aHb* decomposition, we have the following important result.

Corollary. For all $n, k, l \geq 1$,

$$c(n, k+l) = \sum_r c(r, k)\, c(n-r, l). \tag{5}$$

Let us introduce the following generating function for column k of the table of $c(n, k)$ values:

$$G_k(x) = \sum_{n=k}^{\infty} c(n,k)\, x^{(3n-k)/2}.$$

Thus, for example,

$$G_1(x) = x + x^4 + 3x^7 + 12x^{10} + 55x^{13} + 273x^{16} + \cdots$$

and

$$G_2(x) = x^2 + 2x^5 + 7x^8 + 30x^{11} + 143x^{14} + 728x^{17} + \cdots .$$

Using (5), we obtain

$$G_{k+l}(x) = G_k(x) G_l(x). \tag{6}$$

For simplicity, let $G_1(x) = G(x)$. Simple induction using (6) yields

$$G_k(x) = [G(x)]^k \quad (k = 1, 2, \ldots). \tag{7}$$

Multiplying both sides of (4) by $x^{(3n-k)/2}$ and summing on n, we find

$$G_k(x) = xG_{k-1}(x) + xG_{k+2}(x). \tag{8}$$

In view of (7) and (8), $G(x)$ satisfies the cubic equation

$$G(x) = x + xG^3(x), \tag{9}$$

so $x = G/f(G)$ where $f(z) = z^3 + 1$. With $F(z) = z^k$ and $f(z) = z^3 + 1$, Lagrange's expansion yields

$$G_k(x) = \sum_{m=k}^{\infty} \frac{x^m}{m!} \left\{ \frac{d^{m-1}}{dt^{m-1}} kt^{k-1}(t^3+1)^m \right\}_{t=0}$$

Let $[x^m]G_k(x)$ denote the cofficient of x^m in the expansion of $G_k(x)$. To simplify the notation, let $r = (n - k)/2$. Then

$$\begin{aligned} c(n, k) &= [x^{n+r}]G_k(x) \\ &= \frac{k}{(n+r)!} \frac{d^{n+r-1}}{dt^{n+r-1}} t^{k-1}(t^3+1)^{n+r} \Bigg|_{t=0} \\ &= \frac{k}{(n+r)!} \binom{n+r}{r} (n+r-1)!. \end{aligned}$$

Thus we have

$$c(n,k) = \frac{k}{(n+r)!} \binom{n+r}{r} \quad \text{where } r = \frac{n-k}{2}. \tag{10}$$

Note: Having found a formula for $c(n, k)$ using generating functions and Lagrange's expansion, it is easy enough to verify that this is indeed the solution of (4) satisfying $c(n, n) = 1$. It is clear that (10) gives $c(n, n) = 1$. Also, assuming that the formula holds for $c(n - 1, k - 1)$ and $c(n, k + 2)$ and setting $r = (n - k)/2$, we have

$$c(n,k) = c(n-1,k-1) + c(n,k+2)$$

$$= \frac{k-1}{n+r-1}\,\frac{(n+r-1)!}{r!(n-1)!} + \frac{k+2}{n+r-1}\,\frac{(n+r-1)!}{(r-1)!n!}$$

$$= \frac{(n+r-2)!}{r!n!}[(k-1)n+(k+2)r]$$

$$= \frac{(n+r-2)!}{r!n!}[k(n+r)-(n-2r)]$$

$$= \frac{(n+r-2)!}{r!n!}[k(n+r-1)]$$

$$= \frac{k}{n+r}\binom{n+r}{r}$$

Finally,

$$a_{2n} = c(2m+1,\ 1) = \frac{1}{3m+1}\binom{3m+1}{m}$$

and

$$a_{2m+1} = c(2m+2,\ 2) = \frac{2}{3m+2}\binom{3m+2}{m},$$

as claimed.

4. **Comments.** The problem dealt with in this note belongs to the subject of ***lattice*** path ***counting,*** and the methods used are a standard part of that subject **[2]**. For more on the subject of generating functions and their many applications in combinatorics, we recommend the excellent book by Herbert Wilf **[4]**.

Acknowledgement. The author **is** indebted to Nicholas **Sekreta,** who obtained an independent proof of the lemma used in this proof.

References

1. **Dembowski,** Rachele, ***Enumerating partitions,*** this journal, ***Fall,*** 1995, pp. 212-214.
2. **Mohanty,** Sri **Gopal,** ***Lattice Path Counting and Applications,*** Academic Press, New York, 1979.
3. Whittaker, E.T. and Watson, G.N., ***A Course in Modem Analysis,*** Cambridge University Press, Cambridge, UK, 1963.
4. Wilf, Herbert S., ***Generatingfunctionology***, Academic Press, Boston, 1990.

Answers to 'What are These?'

1. Sieve of Eratosthenes
2. Brianchon point
3. Markov chain
4. Convex function
5. **Noncollinear** points
6. Dedekind cut

Another Matching Problem with a Matching Probability

Philip J. Byrne
College of St. Benedict

There are several fairly well known probability problems that have a limiting answer of $1 - 1/e$ (see for example [1], [2] p. 107, [3] pp. 104-106, and [4]). This note adds the following, less well known, problem to this family: if n married couples are randomly assigned to 2n chairs lined up in a row, what is the probability, p_n, that at least one couple occupies adjacent chairs? We later vary this problem by arranging the chairs in a circle.

To determine p_n, we first number the chairs 1, 2,...,2n, from left to right. Let A_i denote the event that a couple occupies chairs i and i + 1, i = 1,..., 2n - 1. Then $p_n = P(A_1 \cup A_2 \cup \dots \cup A_{2n-1})$, and by a generalization of the result $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to any finite union of events, we have

$$p_n = \sum_{i_1} P(A_{i_1}) - \sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \dots$$

$$+ \sum_{i_1<\dots<i_{2n-1}} P(A_{i_1} \cap \dots \cap A_{i_{2n-1}}) . \qquad (1)$$

To evaluate (1), we first make the following simple observation. The expression $A_1 \cap A_2$ is the event that there is a couple in chairs 1 and 2, and also a couple in chairs 2 and 3. But the person in chair 2 cannot be the spouse of both of the people in chairs 1 and 3, therefore $A_1 \cap A_2 = \emptyset$. Similarly, if $|i - j| = 1$ then $A_i \cap A_j = \emptyset$. Thus any event $A_{i_1} \cap \dots \cap A_{i_k}$ for which two of the i_j's are consecutive integers has probability zero. One consequence of this is that the last nonzero summation in (1) is

$$(-1)^{n+1} \sum_{i_1<\dots<i_n} P(A_{i_1} \cap \dots \cap A_{i_n}) = (-1)^{n+1}(A_1 \cap A_3 \cap A_5 \cap \dots \cap A_{2n-1}).$$

Now consider any particular collection, $A_{i_1},\dots,A_{i_k}$, in which $i_2 - i_1 > 1$, $i_3 - i_2 > 1$, ..., $i_k - i_{k-1} > 1$. We will determine the probability

of the intersection of these events using an extension of the multiplicative law of conditional probability, $P(A \cap B) = P(A)P(B|A)$, to the intersection of k events: $P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}|A_{i_1})P(A_{i_3}|A_{i_1} \cap A_{i_2}) \cdots P(A_{i_k}|A_{i_1} \cap \dots \cap A_{i_{k-1}})$. First consider $P(A_{i_1})$. No matter who occupies chair i_1, there are 2n - 1 possibilities for the person in chair $i_1 + 1$, only one of whom is the spouse of the person in chair i_1. Since the assignment of persons to chairs is done randomly, $P(A_{i_1}) = 1/(2n - 1)$. Similarly, given that a couple occupies chairs i_1 and $i_1 + 1$, no matter who occupies chair i_2 there are 2n - 3 possibilities for the person in chair $i_2 + 1$, and therefore $P(A_{i_2}|A_{i_1}) = 1/(2n - 3)$. Continuing on in this manner we conclude that

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = \left(\frac{1}{2n-1}\right)\left(\frac{1}{2n-3}\right)\left(\frac{1}{2n-5}\right)\cdots\left(\frac{1}{2n-2k+1}\right). \qquad (2)$$

Since every non-null intersection of k A_{i_j}'s has this same probability, it remains only to count how many of these there are to determine $\sum_{i_1<\dots<i_k} P(A_{i_1} \cap \dots \cap A_{i_k})$, the k[th] summation in (1). Now let $x_1 = i_1 - 1$, $x_j = i_j - i_{j-1} - 2$ for j = 2, 3, ..., k, and $x_{k+1} = 2n - i_k - 1$. Thus x_1 is the number of chairs to the left of chair i_1, x_2 is the number of chairs between chair $i_1 + 1$ and chair i_2, and so on. There is then a one-to-one correspondence between collections of k A_{i_j}'s with a non-null intersection and solutions of $x_1 + x_2 + \dots + x_{k+1} = 2n - 2k$, with $x_1 \ge 0, x_2 \ge 0, \dots, x_{k+1} \ge 0$. The number of such solutions is given by $\binom{2n-k}{k}$ (see [2], p. 38), where the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, k = 0, 1, 2,..., n. We can now express (1) as

$$p_n = \binom{2n-1}{1}\left(\frac{1}{2n-1}\right) - \binom{2n-2}{2}\left(\frac{1}{2n-1}\right)\left(\frac{1}{2n-3}\right) +$$

$$\dots + (-1)^{n+1}\binom{n}{n}\left(\frac{1}{2n-1}\right)\left(\frac{1}{2n-3}\right)\cdots\left(\frac{1}{3}\right)\left(\frac{1}{1}\right). \qquad (3)$$

This can be written more compactly by letting

$a_{k,n} = \prod_{j=1}^{k} \left(\frac{2n - k - j + 1}{2n - 2j + 1} \right)$ for $k = 1, 2, \ldots, n$. Then we have from (3)

$$p_n = \sum_{k=1}^{n} a_{k,n} \frac{(-1)^{k+1}}{k!}. \tag{4}$$

From the Taylor series for e^x about $x = 0$,

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!} \tag{5}$$

converges to $1 - 1/e$ as $n \to \infty$. It should also be noted that $0 \le a_{k,n} \le 1$ for all k and n, and that for each k $\lim_{n\to\infty} a_{k,n} = 1$. These facts taken together would suggest that for n large, $p_n \approx \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$, and therefore that $\lim_{n\to\infty} p_n = 1 - 1/e$. A careful proof of this result is somewhat technical, and is deferred until the end of this note.

If the 2n chairs are arranged in a circle, a couple occupying chairs numbered 2n and 1 would then be in adjacent chairs. The event that at least one couple occupies adjacent chairs in this situation can be thought of as occurring in one of two mutually exclusive ways: (1) at least one couple occupies adjacent chairs other than the two numbered 2n and 1, or (2) a couple occupies chairs 2n and 1 with no other couple occupying adjacent chairs. The first of these events has probability p_n. For the second event we note that the probability of being in the adjacent chairs 2n and 1 is $1/(2n - 1)$, and given that a couple is seated there the probability that no other couples occupy adjacent chairs among the remaining $2(n - 1)$ seats is $1 - p_{n-1}$. Therefore, the probability, $\tilde{p}_n$, of at least one couple occupying adjacent chairs when the chairs are arranged in a circle is given by $\tilde{p}_n = p_n + \left(\frac{1}{2n - 1} \right) (1 - p_{n-1})$, from which it follows that $\tilde{p}_n$ also approaches $1 - 1/e$ as $n \to \infty$.

It is interesting to note that the congruence of both p_n and $\tilde{p}_n$ to $1 - 1/e$ is related to that of $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$, which itself is the answer to a well known probability matching problem: if n married couples are randomly paired up for a dance, what is the probability that at least one pair is a married couple? This problem is sometimes cast in terms of matching hats to men, letters to envelopes, or positions in two decks of cards (see for example [1], [2] p. 107, and [4]). We now present the proof that $\lim_{n\to\infty} p_n = 1 - 1/e$. Let $\epsilon > 0$ be given. Since $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges to e, we can choose N_1 so that

$\left| \sum_{k=0}^{n} \frac{1}{k!} - e \right| = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{\epsilon}{3}$ for all $n \ge N_1$. Since $\sum_{k=1} \frac{(-1)^{k+1}}{k!}$ converges to $1 - 1/e$, we can choose N_2 so that

$\left| \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} - \left(1 - \frac{1}{e}\right) \right| < \frac{\epsilon}{3}$ for all $n \ge N_2$. Let $N_3 = \max(N_1, N_2)$.

Since $\lim_{n\to\infty} a_{k,n} = 1$, for each $k = 1, 2, \ldots, N_3$, we can choose n_k so that $1 - a_{k,n} < \epsilon/3e$ for all $n \ge n_k$. Let $N_4 = \max(n_1, n_2, \ldots, n_{N_3})$. Now suppose $n \ge \max(N_3, N_4)$. We have

$$\left| \sum_{k=1}^{n} \left(a_{k,n} \frac{(-1)^{k+1}}{k!} \right) - \left(1 - \frac{1}{e}\right) \right| \le \left| \sum_{k=1}^{n} \left(a_{k,n} \frac{(-1)^{k+1}}{k!} \right) - \sum_{k=1}^{N_3} \left(a_{k,n} \frac{(-1)^{k+1}}{k!} \right) \right|$$

$$+ \left| \sum_{k=1}^{N_3} \left(a_{k,n} \frac{(-1)^{k+1}}{k!} \right) - \sum_{k=1}^{N_3} \frac{(-1)^{k+1}}{k!} \right| + \left| \sum_{k=1}^{N_3} \frac{(-1)^{k+1}}{k!} - \left(1 - \frac{1}{e}\right) \right|. \tag{6}$$

The first term on the right side of (6) is

$$\left| \sum_{k=N_3+1}^{n} a_{k,n} \frac{(-1)^{k+1}}{k!} \right| \le \sum_{k=N_3+1}^{n} \frac{1}{k!} < \sum_{k=N_3+1}^{\infty} \frac{1}{k!} < \frac{\epsilon}{3}$$

since $|a_{n,k}| \le 1$ and $N_3 \ge N_1$. The second term on the right side of (6) is

$$\left|\sum_{k=1}^{N_3}(1-a_{k,n})\frac{(-1)^{k+1}}{k!}\right| < \sum_{k=1}^{N_3}\left(\frac{\epsilon}{3e}\right)\frac{1}{k!} < \frac{\epsilon}{3e}\sum_{k=0}^{\infty}\frac{1}{k!} = \frac{\epsilon}{3}$$

(the first inequality holds because $n \ge N_4$). The last term on the right side of (6) is less than $\epsilon/3$ since $N_3 \ge N_2$. Therefore we have shown that for all $n \ge \max(N_3, N_4)$, $\left|\sum_{k=1}^{n}\left(a_{k,n}\frac{(-1)^{k+1}}{k!}\right) - \left(1-\frac{1}{e}\right)\right| < \epsilon$, and thus $\lim_{n\to\infty} p_n = 1 - 1/e$.

References

1. B. R. Crain, "On the Matching Problem in Probability," Pi Mu *Epsilon Journal,* 9 (Fall 1992), pp. 448-450.
2. W. Feller, *An Introduction* to *Probability Theory and Its Applications,* Volume 1, 3rd Edition, Wiley, 1968.
3. R. Honsberger, ed., *Mathematical Plums,* The Mathematical Association of America, 1979.
4. E. A. Tams, "The Answer is 1-1/e. What is the Question?," Pi Mu *Epsilon Journal,* 8 (Spring 1987), pp. 387-389.

A Generalization of a Dimension Formula and an "Unnatural" Isomorphism

Daniel L. Vim

Let V be a finite dimensional vector space with subspaces V_1 and V_2. Then it is well known that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

where $V_1 + V_2 = \{x + y: x \in V_1 \text{ and } y \in V_2\}$. Anyone who has studied combinatorics will immediately recognize that this is similar to the principle of inclusion/exclusion for two sets (PIE from here on). That is, if S_1 and S_2 are two finite sets and $|S_i|$ denotes the number of elements in S_i, then

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|.$$

Now the PEE generalizes to n sets. Hence a natural question to ask is whether we can find a similar formula for n subspaces. In this paper we present one such possibility and what it translates to in terms of quotient spaces.

First of all, the obvious first guess would be to write down the general formula for the PIE and replace S_i by V_i and $\cup$ by $+$. Unfortunately, this doesn't work. For example, the formula for three subspaces would be

$$\dim(V_1 + V_2 + V_3) = \sum_{i=1}^{3}\dim V_i - \dim(V_1\cap V_2) - \dim(V_1\cap V_3) - \dim(V_2\cap V_3) + \dim(V_1 \cap V_2 \cap V_3)$$

which isn't true in general. For Example, in R^2 let V_1 be the x-axis, V_2 be the y-axis, and V_3 be the line y = x. The left hand side of the above formula is 2 while the right hand side is 3. You might ask yourself why the inductive proof of the PIE doesn't work for subspaces. The reason is that in the inductive step of the PIE you need to know that $\cap$ distributes over $\cup$. Unfortunately, $\cap$ does not distribute over $+$. However, we can prove the following,

Theorem 1. Let $V_1, V_2, \cdots, V_n$ be subspaces of a finite dimensional vector space. Then

$$\dim(V_1 + \cdots + V_n) = \sum_{i=1}^{n}\dim V_i - \dim(V_1\cap V_2) - \dim(V_3\cap(V_1+V_2)) - \dim(V_4\cap(V_1+V_2+V_3)) \cdots - \dim(V_n\cap(V_1+\cdots+V_{n-1})).$$

Proof. We proceed by induction on n, the number of subspaces. If n = 1 then the statement is trivial and if n = 2 it is the well known dimension formula. Suppose that the formula is true up to n - 1. Then viewing $V_1 + \cdots + V_{n-1}$ as a single subspace and applying the case n = 2 we have

$$\dim(V_1+\cdots+V_n)=\dim V_n+\dim(V_1+\cdots+V_{n-1})-\dim(V_n\cap(V_1+\cdots+V_{n-1}))$$

$$=\sum_{i=1}^{n}\dim V_i-\dim(V_1\cap V_2)-\dim(V_3\cap(V_1+V_2))$$

$$-\dim(V_4\cap(V_1+V_2+V_3))$$

$$\cdots$$

$$-\dim(V_{n-1}\cap(V_1+\cdots+V_{n-2}))$$

$$-\dim(V_n\cap(V_1+\cdots+V_{n-1})).$$

Hence the theorem follows by induction. □

The problem with our formula is that it is not very symmetric in $V_1, V_2, \cdots, V_n$. To get a more symmetric formula, write down our formula n! times (once for each permutation of $\{V_1, V_2, \cdots, V_n\}$), and then add each column. For example, in the case n = 3 this leads to

$$3!\dim(V_1+V_2+V_3)=3!\sum_{i=1}^{3}\dim V_i-2\dim(V_1\cap V_2)$$

$$-2\dim(V_1\cap V_3)-2\dim(V_2\cap V_3)$$
$$-2\dim(V_1\cap(V_2+V_3))$$
$$-2\dim(V_2\cap(V_1+V_3))$$
$$-2\dim(V_3\cap(V_1+V_2)).$$

We leave it to the reader to write down a general form for this.

Recall that the second isomorphism theorem for vector spaces says that if V_1 and V_2 are subspaces then

$$\frac{V_1+V_2}{V_1}\cong\frac{V_2}{V_1\cap V_2}$$

Taking dimensions of both sides of this formula yields the well known formula that we started with. One would guess that our dimension formula must also give rise to a quotient isomorphism. In fact, we have the following.

Theorem 2. If $V_1, V_2, \cdots, V_n$ are subspaces of a finite dimensional vector space, then

$$\frac{V_1+\cdots+V_n}{V_1}\cong\frac{V_2}{V_1\cap V_2}\times\frac{V_3}{V_3\cap(V_1+V_2)}\times\cdots\times\frac{V_n}{V_n\cap(V_1+\cdots+V_{n-1})}$$

Proof. This is really a corollary to the first theorem. Take dimensions of both sides. The dimension of the left side is the same as the dimension of the right side by the first theorem. Since the left side and the right side have the same dimension and they are over the same field they must be isomorphic. □

The second theorem isn't very satisfying. It gives us a way of looking at the dimension formula of the first theorem in terms of quotient spaces, but the given isomorphism is "unnatural" in that we haven't given the isomorphism and it seems unlikely that one will be found. Is there a better generalization of the dimension formula, and does it lead to a natural generalization of the second isomorphism theorem? We leave this question the reader.

Self-Similarity and Fractal Dimension of Certain Generalized Arithmetical Triangles

Mark Tomforde (student)
Gustavus Adolphus College

The following two forms of Pascal's (or the arithmetical) triangle are equivalent:

.	.	.	.	.	.	.	.	.	.	.	.
1	4	10	20	.	.	C(0, 3)	C(1, 3)	C(2, 3)	C(3, 3)	.	.
1	3	6	10	.	.	C(0, 2)	C(1, 2)	C(2, 2)	C(3, 2)	.	.
1	2	3	4	.	.	C(0, 1)	C(1, 1)	C(2, 1)	C(3, 1)	.	.
1	1	1	1	.	.	C(0, 0)	C(1, 0)	C(2, 0)	C(3, 0)	.	.

where $C(m, n)$ is the number of combinations of $m + n$ objects taken n at a time. If we let p be a prime and code p different colors to the numbers 0 to $p - 1$, then we can replace each number in the above figure by the color coded to its least positive residue modulo p and thereby "visualize" the relation among the numbers. It is well known that the nonzero residues of Pascal's Triangle modulo a prime p form a fractal image which is **self-similar** and has fractal dimension

$$\frac{\log(p(p+1)/2)}{\log p} \qquad (1)$$

In this paper we investigate a generalized combination $C(m, n)$ which is defined as follows: Given a sequence of integers $C_1, C_2, \ldots, C_n$ we denote the generalized factorial of a number n as $[n]!$, and define it as $[0]! = 1$ and $[n]! = C_1 \cdot C_2 \cdot \ldots \cdot C_n$ for $n \in \mathbb{N}$. We then denote the generalized binomial coefficient by $\left[\begin{matrix} n \\ k \end{matrix}\right]$ and define it as $\left[\begin{matrix} n \\ k \end{matrix}\right] = \dfrac{[n]!}{[k]!\,[n-k]!}$. We can now define $C(m, n) = \left[\begin{matrix} m+n \\ m \end{matrix}\right]$.

For our purposes in this paper we will only discriminate **between** elements which are congruent to 0 modulo a given prime (which will always be colored white) versus those which are not congruent to 0 modulo the prime (and which will be colored **non-white**). We shall also use the following definitions for various types of sequences:

A <u>U-sequence</u> is a sequence $\{U_n\}$ such that $U_0 = 0$, $U_1 = 1$ and $U_{n+2} = aU_{n+1} + bU_n$ for all $n \in \{0, 1, 2 \ldots\}$ and for some fixed integers a and b. In this paper, values of $C(m, n)$

which are determined by a U-Sequence will be denoted $U(m, n)$.

A <u>Gaussian sequence</u> is a sequence $\{Q_n\}$ which is a U-sequence such that $a = 1 + q$ and $b = -q$ for some interger q. Thus $Q_0 = 0$ and $Q_n = 1 + q + q^2 + \ldots + q^{n-1}$. When $q \neq 1$, $Q_n = (1 - q^n)/(1 - q)$; when $q = 1$ this is the sequence 1, 2, 3, 4, ... In this paper, values of $C(m, n)$ which are determined by a Gaussian sequence will be denoted either by $Q(m, n)$ or by $Q_q(m, n)$ to specify the value of q.

A <u>regularly divisible sequence</u> is a sequence $\{C_n\}$ such that $\gcd(C_m, C_n) = C_{\gcd(m,n)}$ for all $m, n > 0$. All U-sequences with $\gcd(a, b) = 1$ are regularly divisible [2, p 132], and thus all Gaussian sequences are also regularly divisible.

In addition, there are some well known facts about U-sequences which will be made use of in this paper. Some of these are listed here:

$$U_{m+n} = U_{m+1}U_n + b\,U_mU_{n-1} \qquad \text{(Fact 1)}$$
$$U(m, n) = U_{m+1}U(m, n-1) + b\,U_{n-1}U(m-1, n) \qquad \text{(Fact 2)}$$
$$U(m, 0) = 1 \qquad \text{(Fact 3)}$$
$$U(m, 1) = U_{m+1} \qquad \text{(Fact 4)}$$
$$U(m, n) = U(n, m) \qquad \text{(Fact 5)}$$

It is now possible to begin proving facts about generalized arithmetical triangles and the sequences used to form them.

Lemma 1: If p is a prime and $p|b$ for a U-sequence, then $U_n \equiv a^{n-1} \pmod p$ for all $n \in \mathbb{N}$ where a and b are as defined before.

Proof: This theorem can *easily* be proven by induction using the basic recurrence in the inductive step and the relationships $U_1 = 1$ and $U_2 = a$ as the base cases.

Theorem 1: If p is a prime, then $p \mid U(m, n)$ for all $m, n \in \mathbb{N}$ **iff** $p|a$ and $p|b$.

Proof: First it will be shown that $p|a$ and $p|b \Rightarrow p \mid U(m, n)$ for all $m, n \in \mathbb{N}$. Since $p|b$ we know from lemma 1 that $U_n \equiv a^{n-1} \pmod p$ which means that for $m, n \in \mathbb{N}$

$$U(m, n) = U_{m+1}U(m, n-1) + b\,U_{n-1}U(m-1, n) \qquad \text{(by fact 2)}$$
$$\equiv a^m U(m, n-1) + b\,a^{n-2}U(m-1, n) \pmod p.$$

Now since $p|a$ and $p|b$ it follows that p divides each of these terms in the sum and hence $p \mid U(m, n)$ for all $m, n \in \mathbb{N}$. Next it will be shown that $p \mid U(m, n)$ for all

$m, n \in \mathbb{N} \Rightarrow p|a$ and $p|b$. Since $p \mid U(m, n)$ for all $m, n \in \mathbb{N}$, then certainly $p|U(m, 1)$ for all $m \in \mathbb{N}$. Therefore $p|U_{m+1}$ (by fact 4) for all $m \in \mathbb{N}$. Thus $p|U_2$, or because $U_2 = a$, $p|a$. Also $p|U_3$, or because $U_3 = a^2 + b$, $p|(a^2 + b)$ and since p also divides a we conclude that $p|b$. Hence $p|a$ and $p|b$.

Theorem 2: If p is a prime for which $p|b$ and $p \nmid a$, then $p \nmid U(m, n)$ for all m, n $\in \{0, 1, 2, \ldots\}$. Furthermore, if $p \nmid U(m, n)$ for all $m, n \in \{0, 1, 2, \ldots\}$, then $p \nmid a$.

Proof: Begin by assuming that $p|b$ and $p \nmid a$. Then because $p|b$, it follows from lemma 1 that $U_n \equiv a^{n-1} \pmod p$ for all $n \in \mathbb{N}$.

Now we will assume that $p \mid U(m, n)$ for some $m, n \in \mathbb{N}$ and arrive at a contradiction. If n = 0 or 1 there is a contradiction since $U(m, 0) = 1$ and $U(m, 1) = U_{m+1} \equiv a^m \pmod p$. If $n \geq 2$, then

$$U(m, n) = U_{m+1}U(m, n - 1) + b\, U_{n-1}U(m - 1, n)$$

$$\equiv a^m U(m, n - 1) + b\, a^{n-2}U(m - 1, n) \pmod p.$$

Now since $p \mid U(m, n)$ and $p \mid b\, a^{n-2}U(m - 1, n)$ (because $p \mid b$) we can conclude that $p \mid a^m U(m, n - 1)$. However, since $p \nmid a$ we can further conclude that $p \mid U(m, n - 1)$. Now if n -1 is equal to 1 there is a contradiction as before. Otherwise we can use the same line of reasoning to see that $p \mid U(m, n - 2)$. By continuing this method we eventually have that $p \mid U(m, 1)$ which is a contradiction. Thus we conclude that $p \nmid U(m, n)$ for all $m, n \in \mathbb{N}$.

For the second part of the proof begin by assuming that $p \nmid U(m, n)$ for all m, n $\in$. Now since $U(1, 1) = U_2$ we know $p \nmid U_2$, and because $U_2 = a$ we can conclude that $p \nmid a$.

Theorem 3: If p is a prime and $q \equiv 0 \pmod p$, then $Q(m, n) \equiv 1 \pmod p$ for all $m, n \in \{0, 1, 2, \ldots\}$.

Proof: Begin by assuming that $q \equiv 0 \pmod p$. Now, recall that a Gaussian sequence is a U-sequence with $a = q + 1$ and $b = q$; it is clear that $a \equiv 1 \pmod p$ and $b \equiv 0 \pmod p$. The theorem will be proven by fixing m in $\{1, 2, 3, \ldots\}$ and using induction on n.

Base case: $Q(m, 0) = 1 \equiv 1 \pmod p$

Inductive step: Assume the lemma to be true for all $Q(m, k)$ such that $k < n$.

$$\begin{aligned} Q(m, n) &= Q_{m+1}Q(m, n - 1) + b\, Q_{n-1}Q(m - 1, n) && \text{(by fact 2)} \\ &\equiv a^m Q(m, n - 1) + b\, Q_{n-1}Q(m - 1, n) \pmod p && \text{(from lemma 1)} \\ &\equiv a^m Q(m, n - 1) \pmod p && (b \equiv 0 \pmod p) \\ &\equiv a^m \pmod p && \text{(since we assumed the theorem true for } k < n) \\ &\equiv 1 \pmod p && (\text{since } a = q + 1 \equiv 1 \pmod p) \end{aligned}$$

Conclusion: By induction the theorem is true for all values of n which are whole numbers. Furthermore, since the value of m can be any arbitrarily chosen whole number, we conclude that the theorem is true for all $m, n \in \{0, 1, 2, \ldots\}$.

Theorem 4: If p is a prime and $r \equiv s \pmod p$, then $Q_r(m, n) \equiv Q_s(m, n) \pmod p$ for all $m, n \in \{0, 1, 2, \ldots\}$.

Proof: This theorem will be proven by double induction on m and n.

We will first show that the theorem holds for m = 0 and $n \in \{0, 1, 2, \ldots\}$. Clearly this is true since $Q_r(0, n) = 1$ and $Q_s(0, n) = 1$ by fact 3, and hence $Q_r(0, n) \equiv Q_s(0, n) \pmod p$.

Next we will show that $Q_r(m, n) \equiv Q_s(m, n) \pmod p$ for all $n \in \{0, 1, 2, \ldots\}$ implies that $Q_r(m + 1, n) \equiv Q_s(m + 1, n) \pmod p$ for all $n \in \{0, 1, 2, \ldots\}$. To do this we will perform induction on n while fixing m.

Base cases: Since

$Q_r(m + 1, 0) = 1$ and $Q_s(m + 1), 0) = 1$ it follows that

$Q_r(m + 1, 0) \equiv Q_s(m + 1, 0) \pmod p$ for any m.

Also,

$$\begin{aligned} Q_r(m + 1, 1) &= 1 + r + r^2 + \ldots + r^{m+1} && \text{(by fact 4 and Q-seq. definition)} \\ & 1 + s + s^2 + \ldots + s^{m+1} \pmod p \\ &\equiv Q_s(m + 1, 1) \pmod p; \end{aligned}$$

it follows that $Q_r(m + 1, 1) \equiv Q_s(m + 1, 1) \pmod p$ for any m.

Inductive step: Assume that $Q_r(m + 1, n) \equiv Q_s(m + 1, n) \pmod p$ for all $k \leq n$ and for some $n \in \{1, 2, 3, \ldots\}$. Then

$$\begin{aligned} & Q_r(m + 1, n + 1) \\ &= Q_r(m + 1, 1)\, Q_r(m + 1, n) - r Q_r(n - 1, 1)\, Q_r(m, n + 1) \pmod p \\ & \qquad \text{(from facts 2 and 4 because } b = -q = -r) \\ &\equiv Q_s(m + 1, 1)\, Q_r(m + 1, n) - r\, Q_s(n - 1, 1)\, Q_r(m, n + 1) \pmod p \\ & \qquad \text{(from the 2nd base case)} \\ &\equiv Q_s(m + 1, 1)\, Q_s(m + 1, n) - r\, Q_s(n - 1, 1)\, Q_s(m, n + 1) \pmod p \\ & \qquad \text{(from the assumptions in the inductive step and previous to the induction on n)} \end{aligned}$$

$\equiv Q(m+1, 1)\, Q_s(m+1, n) - s\, Q_s(n-1, 1)\, Q_s(m, n+1) \pmod p$

(because $r \equiv s \pmod p$)

$\equiv Q_r(m+1, n+1) \pmod p$.

By induction we have that $Q_r(m, n) \equiv Q_s(m, n) \pmod p$ for all $n \in \{0, 1, 2, \ldots\}$ implies that $Q_r(m+1, n) \equiv Q_s(m+1, n) \pmod p$ for all $n \in \{0, 1, 2, \ldots\}$.
From the previous statement and the relationship $Q_r(0, n) \equiv Q_s(0, n) \pmod p$ for all $n \in \{0, 1, 2, \ldots\}$, we use the principle of mathematical induction to conclude that if $r \equiv s \pmod p$, then $Q_r(m, n) \equiv Q_s(m, n)$ for all $m, n \in \{0, 1, 2, \ldots\}$.

Theorem 5: If p is a prime and $r \not\equiv s \pmod p$, then there exist $m, n \in \{0, 1, 2, \ldots\}$ such that $Q_r(m, n) \not\equiv Q_s(m, n)$.
Proof: Look at $m = 1$ and $n = 1$. Then $Q_r(1, 1) = 1 + r$ and $Q(1, 1) = 1 + s$. Since $r \not\equiv s \pmod p$ it follows that $1 + r \not\equiv 1 + s \pmod p$ and therefore $Q_r(1, 1) \not\equiv Q_s(1, 1)$.
Based on these theorems it is now possible to make some statements concerning the generalized arithmetical triangles generated by using a Gaussian sequence. When looking at one of these mangles modulo a prime p one knows that despite the infinite number of possible choices for q, by Theorem 4 we are assured that there will be only p different possible forms for these triangles to have. These correspond to the residues of q modulo p, which have values of $0, 1, 2, \ldots, p-1$. **Furthermore** by Theorem 5 we know that for a given prime p these different forms of the triangles **will** be unique. It is also clear from Theorem 3 that all the triangles for which q is congruent to 0 modulo p will have every element replaced by a black square when looked at modulo p, and because q = 1 corresponds to Pascal's Triangle we know, again from Theorem 4, that all triangles for which q is congruent to 1 modulo p will be identical to Pascal's Triangle modulo p. These results are displayed in the table shown in Figure 1. One should also note that **Malta** Sved has published results on Gaussian coefficient residues modulo a prime [3]. When looking at the pictures generated by these triangles modulo a prime it is natural to wonder what the fractal dimension of these objects is. In order to approach this question it will be necessary to use the following definitions.

Definition: The rank of **apparition** (or sometimes rank) of m in a sequence $\{C_n\}$ is denoted by $r(m)$ and has the following value

$r(m) = \infty$ if m divides no elements in $\{C_n\}$
or $r(m) = \min\{k : m \mid C_k\}$ otherwise.

Definition: A prime p is said to be **ideal** for a sequence $\{C_n\}$ if $\{C_n\}$ is regularly divisible and there is a number $s(p)$ such that the sequence of positive integers $b_1(p) = r(p)$, $b_2(p) = r(p^2)/r(p)$, $b_3(p) = r(p^3)/r(p^2)$, . . . (which either terminates with $b_k(p) = \infty$ for some k or continues indefinitely) is equal to the following:

$$b_k(p) = \begin{cases} 1, & \text{if } 1 < k \le s(p) \\ p, & \text{if } k > s(p) \end{cases}.$$

Definition; In this paper a prime p will be said to be **quasi-ideal** for a sequence $\{C_n\}$ if $\{C_n\}$ is regularly divisible and there is a number $s(p)$ such that $b_k(p) = p$, for all $k > s(p)$.
Note that all primes are **quasi-ideal** for a Gaussian sequence. We can now state a theorem **proven** by Knuth and **Wilt** [4, p. 215].

Theorem (Knuth and Wilf): Let p be an ideal prime for a sequence $\{C_n\}$. Then **the** exponent of the highest power of p that divides $C(m, n)$ is equal to the number of carries that occur to the left of the radix point when the numbers $m/r(p)$ and $n/r(p)$ are added in **p-ary** notation, plus an **extra** $s(p)$ if a carry occurs across the radix point itself.
According to Knuth and **Wilt**, a similar result holds when p is not an ideal prime, but we must use a mixed-radix number system in the addition.

Corollary: Given a regularly divisible sequence $\{C_n\}$ and a prime p, it follows that $p \mid C(m, n)$ iff there are carries when m and n are added in the mixed-radix system

$$m = x_n B_n + x_{n-1} B_{n-1} + \cdots + x_1 B_1 + x_0$$

$$n = y_n B_n + y_{n-1} B_{n-1} + \cdots + y_1 B_1 + y_0 \qquad 0 \le x_k, y_k \le b_{k+1}$$

where $B_i = b_i \cdot b_{i-1} \cdot \ldots \cdot b_1$ and $b_1(p) = r(p)$, $b_2(p) = r(p^2)/r(p)$, $b_3(p) = r(p^3)/r(p^2)$ $\cdots$. Using this corollary the following theorem can now be proven.

Theorem 6: Let D_0 be the $B_1 \times B_1$ matrix

$$\left|\begin{array}{cccc} \chi_p(0, B_1 - 1) & \cdot & \cdot & \chi_p(B_1 - 1, B_1 - 1) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \chi_p(0, 1) & & & \cdot \\ \chi_p(0, 0) & \chi_p(1, 0) & \cdot\ \cdot & \chi_p(0, B_1 - 1) \end{array}\right| \qquad \chi_p = \begin{cases} 0 \text{ if } p \mid C(m, n) \\ 1 \text{ if } p \nmid C(m, n) \end{cases}$$

Note that this matrix does <u>not</u> follow the usual matrix indexing with (0, 0) at the upper left. Instead it follows the "Cartesian" pattern with (0, 0) at the bottom left. Also let D_n be generated by D_{n-1} in the following way:

$$D_n = \left|\begin{array}{cccccc} D_{n-1} & 0 & 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ D_{n-1} & D_{n-1} & 0 & 0 & & \cdot \\ D_{n-1} & D_{n-1} & D_{n-1} & 0 & & \cdot \\ \cdot & & \cdot & & & \cdot \\ & & \cdot & D_{n-1} & D_{n-1} & 0 \\ D_{n-1} & & \cdot\ \cdot\ \cdot\ \cdot & D_{n-1} & D_{n-1} & D_{n-1} \end{array}\right|$$

where D_n is a square matrix comprised of $b_{n+1} \times b_{n+1}$ **submatrices**.

Then for any whole number **n**, when the entries of a generalized arithmetical triangle generated by a regularly divisible sequence are written as

$$\begin{array}{ccccc} C(0, B_n) & \cdot & \cdot & \cdot\ \cdot & C(B_n, B_n) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ C(0, 2) & & & & \cdot \\ C(0, 1) & C(1, 1) & & & \cdot \\ C(0, 0) & C(1, 0) & C(2, 0) & \cdot\ \cdot\ \cdot & C(B_n, B_n) \end{array}$$

D_n has a 1 corresponding to those elements which are not congruent to 0 modulo p and a 0 corresponding to those elements which are congruent to 0 modulo p, for any prime p.

Proof: This theorem can be proven by induction. It will be similar in structure to that of a less general theorem proven by A. Jaeger and K. **Saldanha** in an unpublished paper **[5]**.

Base case: The proof for D_0 is trivial since by the definition of D_0 the theorem is true.

Inductive step: Assume the theorem true for D_{n-1}. Recall the definition of D_n and

consider the (i, **j)th** element from the bottom left (i.e., starting at the bottom left coiner, count i elements (not submatrices) over and j elements up). We can write i and j as

$$i = x_nB_n + x_{n-1}B_{n-1} + \ldots + x_1B_1 + x_0$$
$$j = y_nB_n + y_{n-1}B_{n-1} + \ldots + y_1B_1 + y_0 \qquad 0 \le x_k, y_k \le b_{k+1}$$

so that the x_k**'s** are the digits of **i** and the y_k**'s** are the digits of j in a mixed radix system. Now the values of these digits tell which submatrices the **(i, j)th** element from the bottom belongs to. For instance, this element belongs to the (x_n, y_n)**th** submatrix of D_n, the (x_{n-1}, y_{n-1})**th submatrix** of D_{n-1}, etc.

It is also clear that this element corresponds to C(j, i) = C(i, j) in the generalized arithmetical triangle. We shall look at the following **two** cases:

Case 1: If $x_n + y_n \ge b_n$, then we know that the elements is in one of the **submatrices** above the diagonal in the D_n matrix. We also know that i + j must yield at least one cany in mixed radix addition (since $x_n + y_n \ge b_n$). Therefore, by the corollary to **Knuth** and **Wilf's** theorem, **C(i,** j) is divisible by p. Hence **C(i,** j) corresponds to a 0 in the matrix D_n and we can conclude that every element above the diagonal corresponds to 0.

Case 2: If $x_n + y_n < b_n$, then the (i, **j)th** element from the bottom is on or below the diagonal **submatrices** of D_n. Now consider i′ and j′ where

$$i' = x_{n-1}B_{n-1} + y_{n-2}B_{n-2} + \ldots + x_1B_1 + x_0$$
$$j' = y_{n-1}B_{n-1} + y_{n-2}B_{n-2} + \ldots + y_1B_1 + y_0.$$

Notice that $i = x_nB_n + i'$ and $j = y_n B_n + j'$. As **mentionsed** earlier (x_n, y_n) determines the location of a D_{n-1} submatrix and (i′, j′) determines the location of an element within this submatrix.

Now suppose that i′ + j′ produces no cany. Since $x_n + y_n < b_n$ there is no *cany* out of the B_1 position so we know i + j yields no cany. On the other hand, if i′ + j′ produces a cany, then i + j yields at least one cany. Therefore i + j and i′ + j′ when added in this mixed radix system either both yield no carries or both yield at least one cany. Thus $\chi_p(i, j) = \chi_p(i', j')$ and the elements below die diagonal in the matrix D_n (and which are constructed by the submatrices D_{n-1}) correspond to the elements of the generalized arithmetical triangle.

Conclusion: By induction the matrix D_n does in fact correspond **0's** to those elements in the generalized arithmetical triangle congruent to **0(mod** p) and 1's to all other elements.

Theorem 7: Let N_n be the number of nonzero elements in the matrix D_n and let

D_n correspond to a regularly divisible sequence with a quasi-ideal prime p. Then, for $n > s(p)$, $N_n = (p(p+1)/2)^{n-s(p)+1} \cdot \beta$ where $\beta = (b_{s(p)}(b_{s(p)}+1)/2) \cdot (b_{s(p)-1}(b_{s(p)-1}+1)/2) \cdot \ldots \cdot (b_1(b_1+1)/2)$ and $s(p)$ is the same as in the definition of a quasi-ideal prime.

Proof: From the definition of D_0 and Knuth and Wilf's theorem we know that only the elements on or below the diagonal of D_0 are nonzero, so

$$N_0 = (1 + 2 + \ldots + (b_1 - 1) + b_1) = b_1(b_1 + 1)/2.$$

Now from Theorem 6 we know that D_n contains $(1 + 2 + \ldots + b_{n+1})$ copies of D_{n-1}. Therefore

$$N_n = (1 + 2 + \ldots + b_{n+1}) \cdot N_{n-1} = (b_{n+1}(b_{n+1} + 1)/2) \cdot N_{n-1} \text{ and}$$
$$N_1 = (b_2(b_2 + 1)/2) \cdot N_0 = (b_2(b_2 + 1)/2) \cdot (b_1(b_1 + 1)/2),$$
$$N_2 = (b_3(b_3 + 1)/2) \cdot N_1 = (b_3(b_3 + 1)/2) \cdot (b_2(b_2 + 1)/2) \cdot (b_1(b_1 + 1)/2),$$

and by induction

$$N_n = (b_{n+1}(b_{n+1} + 1)/2) \ldots\ldots (b_2(b_2 + 1)/2) \cdot (b_1(b_1 + 1)/2),$$

and for $n > s(p)$

$$N_n = (p(p+1)/2) \cdot \ldots\ldots (p(p+1)/2) \cdot (b_{s(p)}(b_{s(p)} + 1)/2 \ldots\ldots (b_2(b_2 + 1)/2) \cdot (b_1(b_1 + 1)/2)$$

so $N_n = (p(p+1)/2)^{n-s(p)+1} \cdot \beta.$

Now fractal dimension, as calculated by Mandelbrot, requires a geometric construction to be carried out ad infinitum while scaling it to fit, say, the unit square. The fractal, or self-similarity, dimension is then given by dim = log N/log(1/R) where N equals the number of self-similar pieces in the limit structure and R equals the linear scaling ratio required from one stage to the next in order to fit the structure to the unit square. For our purposes we will use an equivalent measure of fractal dimension called the Entropy Index [6, p. 184] or Mass/Cluster Dimension [7, p.32] which is defined as

$$\text{Dim} = \lim_{n\to\infty} \frac{\log N_n}{\log S}$$

where N_n is as described in Theorem 7, $S = 1/r$, and r is the linear scaling ratio needed to shrink the entire figure to the unit square.

Theorem 8: The Entropy Index or Mass/Cluster Dimension for a generalized arithmetical triangle which is looked at modulo p, where p is a quasi-ideal prime, is given by $\text{Dim} = \dfrac{\log(p(p+1)/2)}{\log p}$.

Proof: The value for N_n is given by Theorem 7, and from Theorem 6 we know that D_n is a $B_{n+1} \times B_{n+1}$ matrix so $r = 1/B_{n+1}$ and $S = B_{n+1}$. Therefore

$$\text{Dim} = \lim_{n\to\infty} \frac{\quad}{\log S}$$

$$= \lim_{n\to\infty} \frac{\log(p(p+1)/2)^{n-s(p)+1}\beta}{\log B_{n+1}} \qquad \text{(from Theorem 7)}$$

$$= \lim_{n\to\infty} \frac{\log((p(p+1)/2)^{n-s(p)+1}\beta}{\log(p^{n-s(p)+1}B_{s(p)})} \qquad \text{(by the definition of } B_{n+1})$$

$$= \lim_{n\to\infty} \frac{(n-s(p)+1)\log(p(p+1)/2) + \log\beta}{(n-s(p)+1)\log p + \log B_{s(p)}}$$

$$= \frac{\log((p(p+1)/2)}{\log p}$$

It is interesting to note that the dimension found in Theorem 8 is the same as that obtained for Pascal's triangle modulo a prime [1]. The formula also gives the fractal dimension of the triangles generated by a Gaussian Sequence.

References

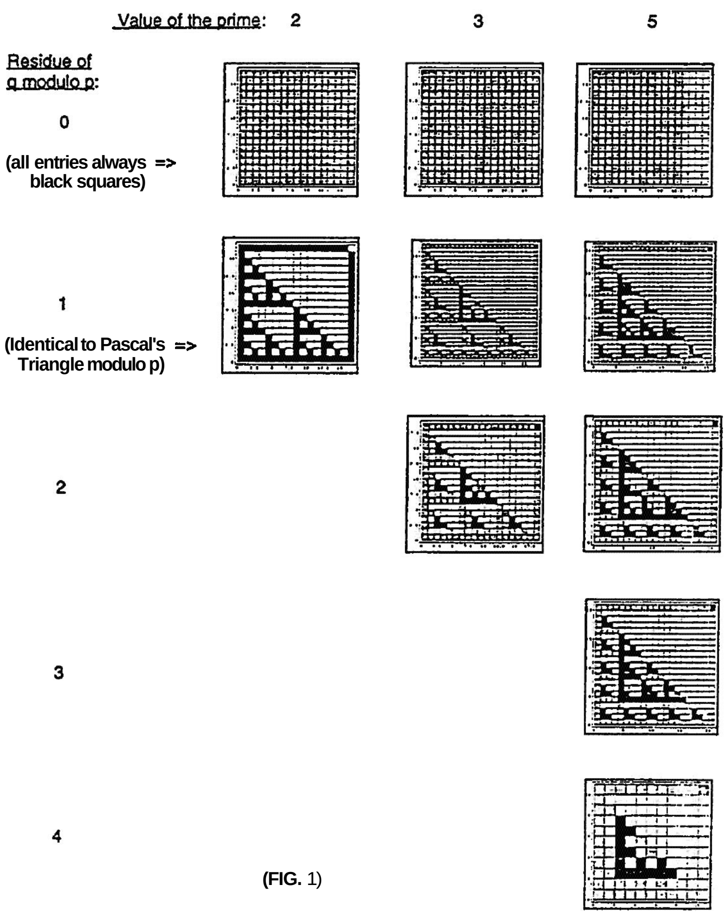
1. S. Wolfram, "Geometry of Binomial Coefficients," American Mathematical Monthly, 1984, p. 566-571.
2. P. Horak and L. Skula, "A Characterization of the Second-Order Strong Divisibility Sequences," The Fibonacci Quarterly, May 1985, p. 126-132.
3. M.Sved, "Divisibility-With Visibility," Mathematical Intelligencer, Vol. 10, No. 2 (1988), p. 54-56.
4. D. Knuth and H. Wilf, "The Power of a Prime that Divides a Generalized Binomial Coefficient," *Journal fur die reine und angewande Mathematik,* 1989, p. 212-219.
5. A. Jaeger and K. Saldanha, "Self-Similarity and Fractal Dimension of the Fibonomial Triangle, "Pi Mu Epsilon conference, St. John's University, Collegeville, MN, 4 April 1992.

6. G. Edgar, ***Measure, Topology. and Fractal Geometry,*** Springer-Verlag, New York, 1990.

7. J. Feder, ***Fractals,*** Plenum Press, New York, 1988.

This paper was prepared under the supervision and guidance of Dr. John Holte while Mark Tomforde was a sophomore at Gustavus Adolphus College.

Generalized Arithmetical Triangles Generated by Gaussian Sequences



(FIG. 1)

Approximating $e^n/2$ with nearly $n + 1/3$ Terms

J.S. Frame and C.L. Frenzen
Michigan State University and Naval Postgraduate School.

Let $S_n = \sum_{k=0}^{n-1} n^k/k!$ denote the sum of the first n terms of the power series for e^n, whose largest terms are the n^{th} and $(n+1)^{th}$, both equal to $n^n/n!$ and denoted by M. It was shown by Frenzen [1] that S_n is less than but near to $e^n/2$. Question! How much of the next term M needs to be added to S_n to get $e^n/2$? With two different methods we obtain:

$e^n/2 = S_n + M(1/3 + 4/(135n + f(n)))$, with $f(n)$ near $90/(7 - 16/(3n + 13581495))$

To show this by successive curve fittings we require successive limits as $n \to \infty$. We first evaluate for $n = 2^m$, $0 \le m \le 9$, to 20 or 25 decimals the values of

$$p_n = e^n/2M = (e/n)^n n!/2, \quad s_n = S_n/M + 1/3 = 4/3 + \sum_{k=1}^{n-1} \prod_{j=1}^{k} (1-j/n).$$

Stirling's formula with remainder approximates p_n by

$$(\pi n/2)^{1/2} \exp(1/12n - 1/360n^3 + \ldots)$$

We notice that $d_n = p_n - s_n$ decreases and that nd,,appears to have a limit for large n. For example,

$d_{128} = 0.00023\ 13083\ 53860\ 07169\ 8269$, $128d_{128} = 0.02960\ 747 \ldots$

$d_{256} = 0.00011\ 56975\ 70464\ 71564\ 4961$, $256d_{256} = 0.02961\ 858 \ldots$

$d_{512} = 0.00005\ 78595\ 91786\ 55076\ 2801$, $512d_{512} = 0.02962\ 411 \ldots$

Since these differences of nd_n values decrease by about ½ we add the last difference 0.00000 553 to $512d_{512}$ to get 0.02962964, which is near 41135 = 0.0296296296... This suggests evaluating the function $4/nd_n - 135 = f(n)/n$ for 10 values of $n = 2^m$ as follows.

n	$4/nd_n$ - 135	n	$4/nd_n$ - 135	weights
1	19.99321 75390 61752	32	0.41130 36841 90604	1/315
2	8.56789 21217 29290	64	0.20327 85644 92804	-2121
4	3.79347 85832 66614	128	0.10104 36004 59236	8/9
8	1.75661 91688 24471	256	0.05037 25983 74131	-64121
16	0.84142 86281 27822	512	0.02514 89644 27800	1024/315

We fit a quartic polynomial in $1/n$, namely $a(n) = a_{\infty} + a_1/n + a_2/n^2 + a_3/n^3 + a_4/n^4$ to the values above for arguments n, $n/2$, $n/4$, $n/8$, $n/16$ with n = 512, and sum these values multiplied by the weights in the last column above to obtain a value a_{∞} which estimates the limit for $n \to \infty$ with an error of order n^{-5}. The weights shown were found in the first row of the inverse of the 5x5 Vandermonde matrix with i, j-entries $2^{(i-1)(j-1)}$. We get $a_{\infty} = 4.1 \times 10^{-11}$, indicating that the limit of $4/nd_n$ is 135 to at least 12 significant digits.

Next we use the same technique to obtain a limit for $f(n) = 4/d_n - 135n$. Since this limit is close to 12.857 which resembles 90/7, we examine $7f(n) - 90$ instead, with values 2.132025259695, 1.068796892776, 0.535066011476, 0.267696286445, 0.133888509235. Applying the same weights we get 5.0×10^{-10} for our limit estimate a_{∞}, near 512^4. The limit of $f(n)$ appears to be 90/7. We compute five values of $n(f(n) - 90/7)$ and apply the same weights to get an estimated limit $a_{\infty} = 9.79591836678 = 9 + 1/(1 + 1/3.899999986)$, close to $9 + 1/(4.9/3.9) = 480149 = (90/7)(16/21)$. Next assume $90n/f(n) = 7n - 16/(3 + r(n))$.

n	$n(f(n) - 90/7)$	$3 + r(n)$	$nr(n)$
32	9.74640 11871 77	3.08667 02527 07	2.77344 77666
64	9.77185 73053 81	3.04310 11297 55	2.75847 23043
128	9.78406 42099 47	3.02149 18770 27	2.75096 02594
256	9.79003 56185 60	3.01073 12438 06	2.74719 84147
512	9.79298 81040 46	3.00536 19473 69	2.74531 70528
a_0	9.79591 83667 82	3.00000 00033	2.74343 59807

This estimate a_{∞} indicates that $r(\infty) = 0$ and $nr(n)$ approaches a limit near 2.4 + .34343... which is about 12/5 + 34/99 = 1358/495. This gives the 4 digit approximation 2.718 for n = 1 which is not noteworthy but could be changed to e = 2.7182816 by replacing 495 by 495 - 96/11. However, for $n \geq 64$ the approximation for $f(n)$ in the fast paragraph yields 12 significant digits for p_n and for $n \geq 512$ yields 18 significant digits,

A second way to obtain $f(n)$ without some good guessing is to express factorials as integrals, and to make suitable changes of variable such as expressing z as a power series [2] in w if $z - \ln(1 + z) = w^2/2$. We shall express d_n as a power series in $1/n$ which yields the same continued fraction obtained above. First we sum the first n + 1 terms of the $e^n n!/n^n$ series getting

$$s_n + 2/3 = \sum_{k=0}^{n} n^k n!/n^n k! = \sum_{k=0}^{n} \binom{n}{n-k} (n-k)!/n^{n-k}$$

$$= \int_0^{\infty} (1 + t/n)^n e^{-t} dt = n \int_0^{\infty} e^{-n(z - \ln(1+z))} dz$$

where t = nz. Then set $w^2/2 = z - \ln(1 + z) = (z^2/2)(1 - 2z/3 + 2z^2/4 - 2z^3/5 + \ldots)$ and get $wdw = (1 - 1/(1 + z))dz = zdz/(1 + z)$ or $1 + z = (z/w)(dz/dw)$. We assume a series expansion

$1 + z = 1 + w + a_1w^2 + a_2w^3 + \ldots = (1 + a_1w + a_2w^2 + \ldots)(1 + 2a_1w + 3a_2w^2 + \ldots)$ so that $a_1 = 1/3$ and $a_{n-1} = (n+2)(a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1 + a_n)/2$. To simplify the fractions in the recurrence relations for a_n we set $b_n = 6^n a_n$ and get

$$b_1 = 2,\ b_2 = 1,\ b_3 = -4/5,\ b_n = (2 - 2n)b_{n-1}/(n + 2) - \sum_{k=2}^{n-2} b_k b_{n-k}/2,\quad n \geq 4.$$

Hence, $b_4 = 3/10$, $b_5 = 16/35$, $b_6 = -3(139)/350$, $b_7 = 48/35$, $b_8 = -57111400$, $b_9 = -64(281)/9625$. Thus we can compute $s_n + 2/3$ as an asymptotic series [3]

$$s_n + 2/3 = \int_0^{\infty} \exp(-nw^2/2)(dz/dw)n\,dw$$
$$= \int_0^{\infty} \exp(-nw^2/2)\left(1 + \sum_{k=1}^{\infty} (k+1)b_k(w/6)^k\right)n\,dw$$

Next we evaluate $2p_n = e^n n!/n^n = \int_0^{\infty} (t/n)^n e^{n-t} dt = \int_{-1}^{\infty} (1+z)^n e^{-nz} n\,dz$ where $t = n(1 + z)$. Again we set $w^2/2 = z - \ln(1 + z)$, but now the lower limit is $-\infty$.

$$2p_n = \int_{-\infty}^{\infty} \exp(-nw^2/2)(dz/dw)n\,dw$$
$$= 2\int_0^{\infty} \exp(-nw^2/2)\left(1 + \sum_{k=1}^{\infty} (2k+1)b_{2k}(w/6)^{2k}\right)n\,dw$$

since the odd powers of w drop out and the even powers double up on integration. Thus $s_n + 2/3 - p_n = \int_0^{\infty} \exp(-nw^2/2)\left(\sum_{k=1}^{\infty} 2kb_{2k-1}(w/6)^{2k-1}\right) n\,dw$.

Now set $w^2 = 2t/n$ and $p_n - s_n = d_n$ to obtain

$$2/3 - d_n = \int_0^\infty e^{-t} \sum_{k=1}^{\infty} 2k\, b_{2k-1} (t/18n)^{k-1} dt/6$$

$$= \sum_{k=1}^{\infty} b_{2k-1} k!/3(18n)^{k-1}$$

$= 213 - 4/(135n) + (2/3)^3/(105\, n^2) + (2/3)^4/(105n^3) - (2/3)^5(281/(51975n^4) + \ldots$
and so
$d_n = (4/135n)(1 - (2/3n)/7 - (2/3n)^2/7 + (2/3n)^3 281/3465 + \ldots)$.
We expand this in a continued fraction to simplify the coefficients. Set
$x = 2/(21\, n)$.
$d_n = (4/135n)/(1 + x + 8x^2 - 6344x^3 1495 + \ldots)$
$= 4/(135n + (90/7)/(1 - 8x + 56(679)x^2/495 + \ldots))$
$= 4/(135n + 90/(7 - 16/3n + 16(1358)/495(9n^2) + \ldots))$
$= 4/(135n + 90/(7 - 16/(3n + 13581495 + \ldots)))$.

This continued fraction agrees with the one obtained by the first method, and shows that the previous limit guess 13581495 was correct. Actually, this guess helped to correct an error made in computing b_9 by the second method. The second method *makes* dear why the denominator of the k^{th} of the successive limits 113, 413'5,9017, 1613, 1358/5(9)11 has prime factors only from the first k odd integers greater than 1. It can yield more terms if desired.

References

1. **Frenzen**, C.L., ***A** New **Elementary Proof of Stirling's Formula,*** Mathematics Magazine, Vol. 86, No. 1, February 1995, pp. 55-58.
2. Frame, J.S., ***Power Series for Inverse Functions,*** American Mathematical Monthly, Vol. 64, No. 4, April 1957, pp. 236-240.
3. Olver, **F.W.J.**, ***Asymptotic*** and ***Special Functions,*** Academic Press, 1974, **pp**. 85-88.

A Particular Polarity

John A. Frohliger
Richard Tuggle (student)
Jennifer Van Sistine (student)

INTRODUCTION

Start with the Euclidean plane, represented by the Cartesian coordinate system. There is a one-to-one correspondence between points **(a,b)** in the Euclidean plane and **nonvertical** lines $y = ax - b$. (See Figure 1.) Let us refer to l_p as the line corresponding to point P and to X_l as the point corresponding to line l. This correspondence leads to some interesting results, pairing curves with curves, inflection points with cusps, and conic sections with conic sections.

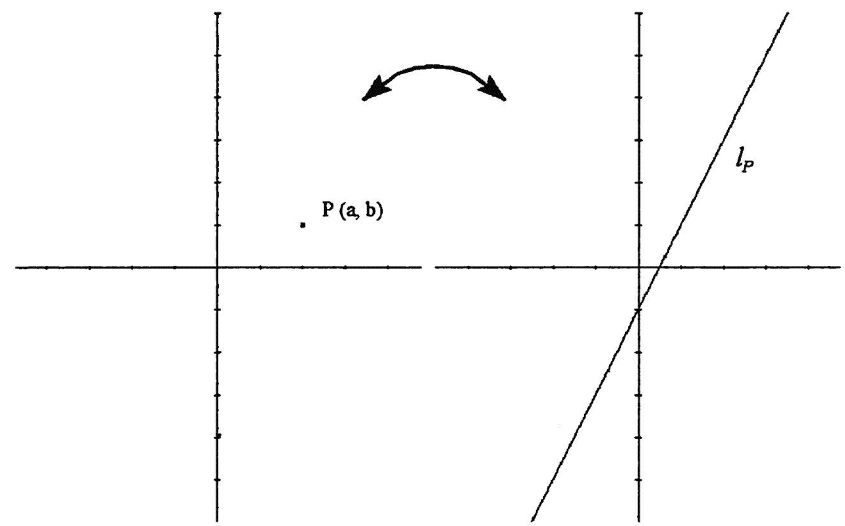


Figure 1

Property 1: If P and Q are points with different first coordinates then l_P and l_Q intersect at a point whose first coordinate is the slope of $\overleftrightarrow{PQ}$ and whose second coordinate is the negative of the y-intercept of $\overleftrightarrow{PQ}$.

Proof: Let P and Q have coordinates (a,b) and **(c,d)**, respectively. The corresponding lines l_P and l_Q are described by the equations y = ax - b and y = cx - d. (See Figure 2.) It can be easily shown that the intersection of these lines is the point $\left(\frac{d - b}{c - a}, \frac{ad - bc}{c - a}\right)$. A (wick check reveals that $\overleftrightarrow{PQ}$ has slope $m = \frac{d - b}{c - a}$ and y-intercept $(0, e) = \left(0, \frac{bc - ad}{c - a}\right)$.

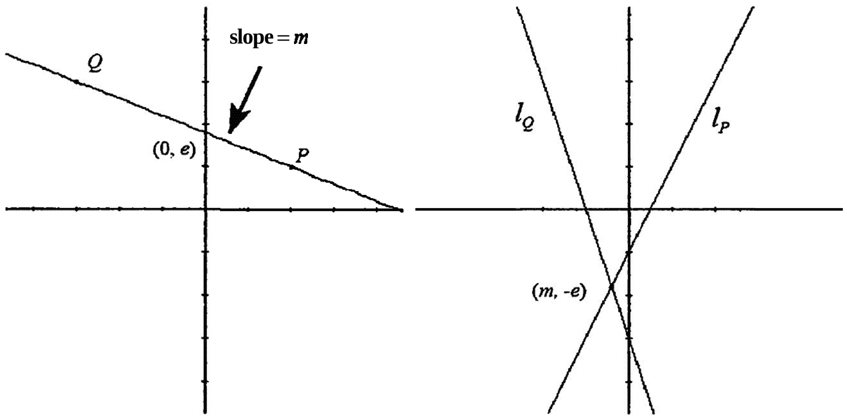


Figure 2

If points P and Q have coordinates (a,b) and **(c,d)**, respectively, then the line $\overleftrightarrow{PQ}$ can be parameterized as:

$$\begin{cases} x = at + c(1 - t) \\ y = bt + d(1 - t). \end{cases}$$

Corresponding to the point with coordinates (at + c(1 - t), bt + d(1 - t)) is the line y = (at + c(1 - t))x - (bt + d(1 - t)). In particular, if *M* is the midpoint of $\overline{PQ}$, then the line l_M, given by the equation $y = \frac{a + c}{2}x - \frac{b + d}{2}$, is the "midline" of l_P and l_Q. That is, l_M has the property that, for every value of x, the point on l_M with first coordinate x is midway between the corresponding points on l_P and l_Q. Furthermore, l_M is the only line with this property. This fact and Property 1 can be used to give a quick, though somewhat **unusual**, proof of a

common geometric property.

Proposition: The diagonals of a parallelogram bisect each other.

Proof: **Let** A, B, C, and D be vertices of a parallelogram, as shown in Figure 3. **La** M be the midpoint of $\overline{AC}$. **Since** $\overleftrightarrow{AB} \parallel$ fie, the intersection of l_A and l_B has the same first coordinate x_1 **as** the intersection of lines l_D and l_C. Similarly, the intersection of l_A and l_D has the same first coordinate x_2 **as** the intersection of lines l_B and l_C. Now l_M is midway between l_A and l_C **in** the sense described above. At x_1 and x_2 then, l_M is midway between l_B and l_D. Consequently, l_M is midway ***between*** l_B and l_D at every x; hence, we may conclude that M is the midpoint of $\overline{BD}$, and the property follows.

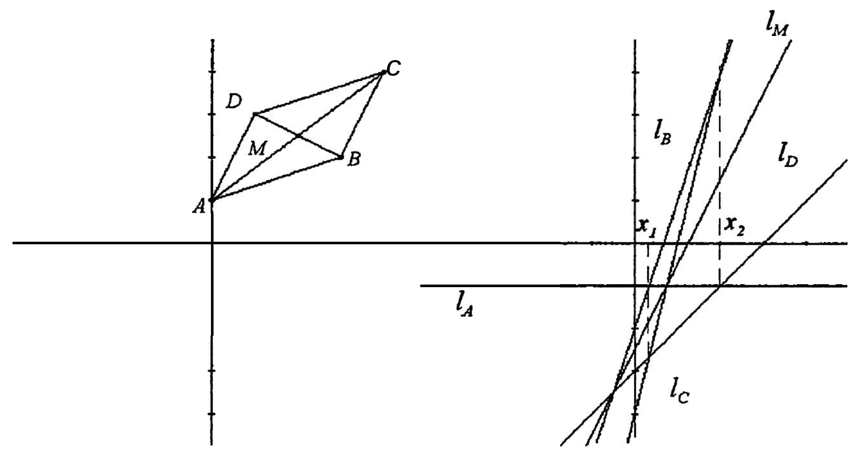


Figure 3

This correspondence between points and **nonvertical** lines has another important **property**.

Property 2: Let P be a point and l_P be its corresponding line. Then the set of lines corresponding for the points on l_P is the pencil of **nonvertical** lines through P.

Proof: (See Figure 4.) Let P be the point (a, b). **Then** l_P has equation

$y = ax - b$. Now the line corresponding to a point $R(t, at - b)$ on l_P has the equation $y = tx - (at - b)$; consequently, this line passes through P. Similarly, it can be shown that every nonvertical line through P corresponds to a point on l_P.

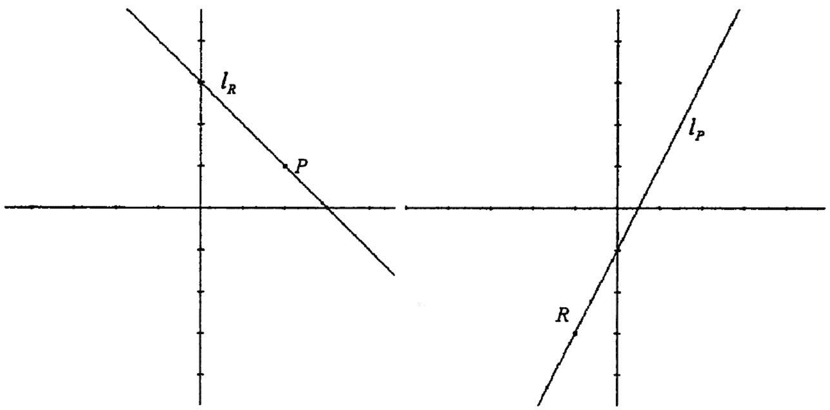


Figure 4

In projective geometry, a transformation such as ours, which takes points to lines and lines to points, is called a correlation if it preserves incidence. That is, if point P is on line I, then under a correlation the point corresponding to l would be on the line corresponding to P. We can extend our correlation to the extended Euclidean plane, which contains ideal as well as ordinary points and lines, by having the vertical line $x = a$ correspond to the ideal point associated with the bundle of parallel lines with slope a and having the ideal line correspond to the ideal point associated with the bundle of vertical lines. With this new, expanded definition, the next property shows that our correlation is a special type of transformation called a polarity. Readers unfamiliar with projective geometry may wish to skip the property and its proof.

Property 3: This correlation is projective with period 2; hence, it is a polarity.

Proof: To prove this, we need only show that the points on one line are

transformed into one pencil of lines in a projective fashion. It follows then that the points on every line are mapped to the corresponding pencil in a projective fashion. (See [1], page 57.) In our case, we will use the x-axis as the line and the lines through the origin as the corresponding pencil. Let l_1 and l_2 be the y-axis and the line $y = 1$, respectively, and let O_1 and O_2 be the points $(1,1)$ and $(1,0)$. (See Figure 5.) One can easily see that, for $a \neq 0, 1$, a point $(a, 0)$ is projected through O_1 to the point $(0, a/(a - 1))$ on l_1 and this point is projected through O_2 to the point $(1/a, 1)$ on l_2. Finally, the line through this point and the origin is the line $y = ax - 0$, the image of $(a, 0)$ under the correlation.

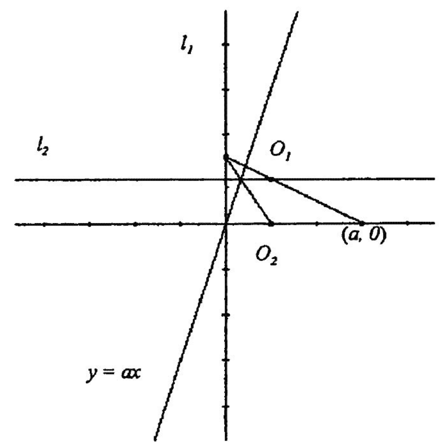


Figure 5

Using ideal points, one can easily see that this projectivity takes the point $(0,0)$ to the x-axis and the point $(1,0)$ to the line $y = x$.

It is easy to see that the correlation has period 2. It follows from Property 3 that, if point P is transformed into line l_P then l_P is transformed into P.

<u>DUAL **CURVES** INDUCED BY THE **POLARITY**</u>

Let I be an open interval on the real line and $f(x)$ be a function on I whose second derivative is continuous with isolated zeros. Let C be the graph of f.

(See Figure 6.) At the point P on C with coordinates $(t, f(t))$ the line l with equation $y = f'(t)x - (tf'(t) - f(t))$ is tangent to the graph. Under our correlation, l is related to a point X_l whose coordinates are $(f'(t),\ tf'(t) - f(t))$. Let C' be the set of all of these points X_l. That is, C' is the curve parameterized by the equations

$$\begin{cases} x = f'(t) \\ y = tf'(t) - f(t). \end{cases}$$

We will call C' the dual of C under this polarity. Notice that, at the point $(x, y) = (f'(t),\ tf'(t) - f(t))$, a tangent line l' has slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{tf''(t) + f'(t) - f'(t)}{f''(t)} = t$$ and equation $y = tx - f(t)$. (This also holds at those points where $f''(t) = 0$ since f'' is continuous with isolated zoos.) Under the correlation this $X_{l'}$ is the original point $P(t, f(t))$ on C. Using the chain rule again, we see that if $f''(t) \neq 0$ then, on C',

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{dt/dt}{dx/dt} = \frac{1}{f''(t)} \text{ at } X_l.$$

We could have started by assuming that C was a curve parameterized by a function $g \times h: I \to \mathbb{R} \times \mathbb{R}$ for some open interval I. We can compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of the parameter. As long as we have die condition that, on C, $\frac{dy}{dx}$ is continuous and $\frac{d^2y}{dx^2}$ has isolated zeros and discontinuities, we will achieve the same relationship between C and C' as the one described above. Therefore, it follows that C is also the dual of C' under the polarity.

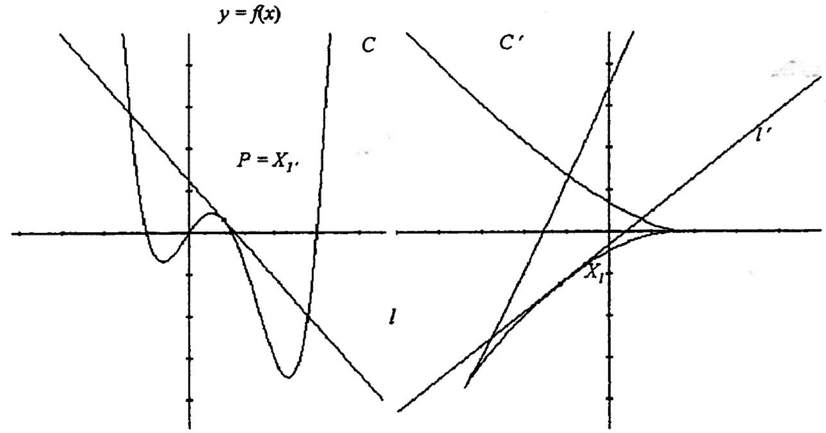


Figure 6

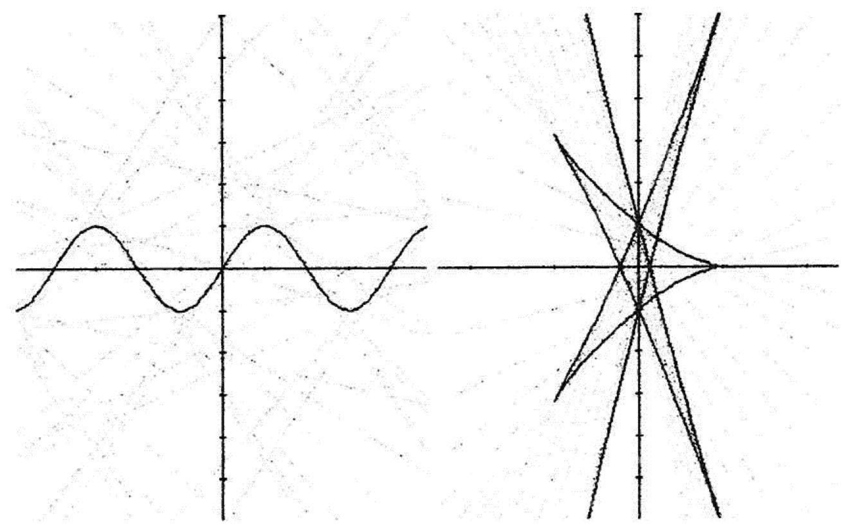


Figure 7

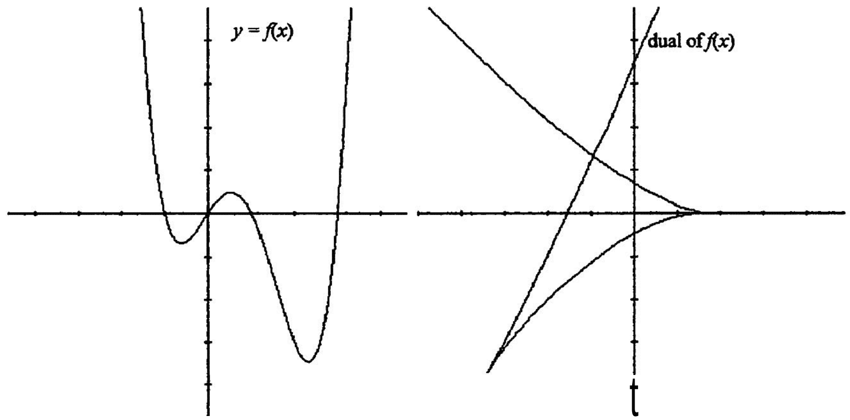


Figure 8

PROPERTIES OF THE DUAL CURVE

The dual of a curve can become very complicated. Figure 7, for example, shows the graphs of the sine function and its dual with some tangent lines. But what information does the dual of a curve C provide? For example, compare the graph of $f(x) = 0.5(x+1)x(x-1)(x-3)$ and its dual (Figure 8). First of all, the y-intercepts of C' correspond to the points where C has a horizontal tangent. The x-intercepts of C', on the other hand, correspond to points on C whose tangent lines pass through the origin. Furthermore, where $f''(x) \neq 0$, the concavity of C at a point has the same sign (positive or negative) as the concavity of C' at the corresponding point.

One of the striking features of the dual is the cusp. At this point the first coordinate $f'(t)$ was changing from decreasing to increasing and the concavity $\frac{1}{f''(t)}$ changed signs. This means that cusps on the dual curve correspond to inflection points on the original. It is curious that an inflection point, which is often difficult to pinpoint with the naked eye, would correspond to something as pronounced as a cusp.

How are translations and reflections of C manifested in the dual? It is easy to

prove the following:

1. If C is translated vertically a units, then C' is translated -a units vertically.
2. If C is translated horizontally b units, then C' is replaced by $C' + l$ where l is given by $y = bx$. That is, every point (x, y) on C' is replaced by (x, y + bx). Conversely, if C is replaced by C + 1, then C' is translated horizontally b units.
3. If C is reflected about the x-axis, C' is rotated π units about the origin.
4. If C is reflected about the y-axis, so is C'.

Figure 9 shows the effects of such translations and reflections on the example in Figure 8.

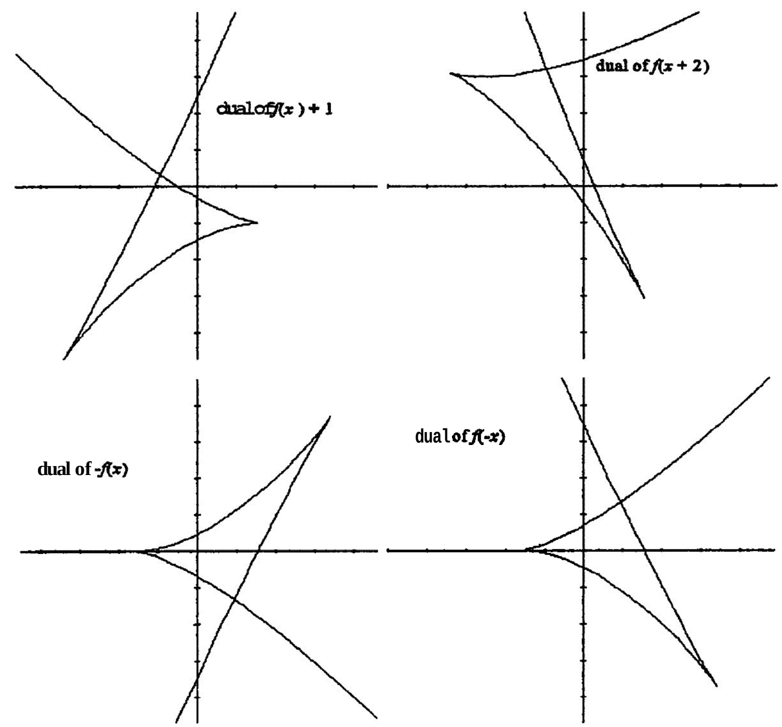


Figure 9

COMICS AND SELF-DUALITY

Since our correlation is a polarity, the dual of a conic is a conic [1]. The images of some basic conies are given in the following table. For the sake of completeness, ideal points and their images are included.

Equation for C	Equation for corresponding C'
$y = ax^2$	$y = \frac{x^2}{4a}$
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$a^2x^2 - y^2 = b^2$
$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$	$a^2x^2 + y^2 = b^2$
$y = \frac{c}{x}$	$x = -\frac{y^2}{4c}$

Figure 10 shows four graphs, each one showing one of these basic conics and its dual. Every other conic in the plane results from one of these by a combination of reflections, translations, and "adding" lines. This is true since the equation of a conic section has the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, which can be transformed into one of the forms in the table by the transformations $x \to x + d$, $y \to y + e$, and $y \to y + mx$. Therefore, the correspondences listed in the table are **sufficient** to characterize the images of all conics.

Under a polarity a point is called **self-conjugate** if it lies on its corresponding line. Furthermore, the set of self-conjugate points forms a conic and their corresponding lines are the tangents to the conic. Under our correlation there is such a **self-dual** curve which the reader can easily verify is the graph of $y = \frac{1}{2}x^2$. This is the only curve in which every point corresponds to itself in its dual. However, there **are other** curves which are identical to their duals, though individual points do not map to themselves. These include the graphs of $y = -\frac{x^2}{2}$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ when $b = a^2$

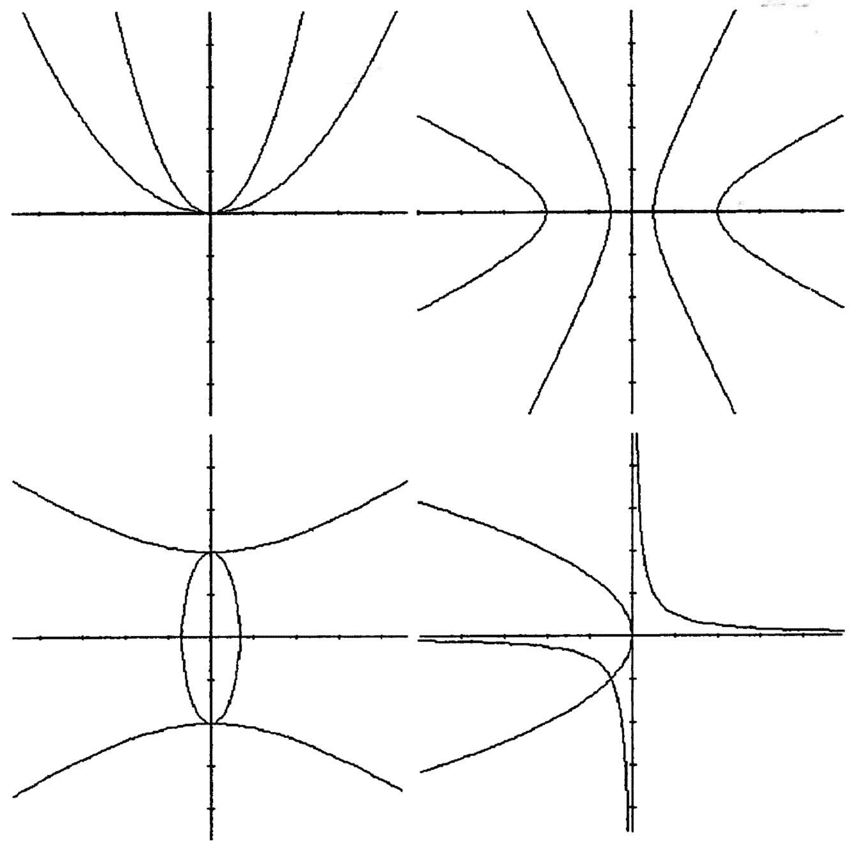


Figure 10

REFERENCE

1. H. S. Coxeter, ***Projective Geometry*, Blasidell** Publishing Company, New York, **1964.**

The Power Means Theorem via the Weighted AM-GM Inequality

N. Schaumberger and M. Steiner
Hofstra University

For any real number $r \neq 0$, the power mean of order r, M_r, is defined as

$$M_r = \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{\frac{1}{r}}$$

where $a_i > 0$. One of the fundamental theorems involving means states that if $\alpha > \beta$, then

$$(1) \qquad M_\alpha = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n} \right)^{\frac{1}{\alpha}} \ge M_\beta = \left(\frac{a_1^\beta + a_2^\beta + \cdots + a_n^\beta}{n} \right)^{\frac{1}{\beta}}$$

with equality iff $a_1 = a_2 = \cdots = a_n$.

In particular, it follows from (1) that the harmonic mean M_{-1} does not exceed the arithmetic mean M_1 which in turn does not exceed the quadratic mean M_2. Other applications of (1) to problems in elementary mathematics can be found in [2].

The usual proof of (1) uses both the ordinary AM-GM inequality and the Bernoulli inequality and is not particularly simple. (See, for example, [1]).

The weighted AM-GM inequality states that if $x_1, x_2, \cdots, x_n$ are positive real numbers with $\sum_1^n x_i = 1$, then for nonnegative real numbers $q_1, q_2, \cdots,$ a, we have

$$(2) \qquad x_1q_1 + x_2q_2 + \cdots + x_nq_n \ge q_1 \; q_2$$

with equality iff $q_1 = q_2 = \cdots =$ a.

In this note we show that for α and β either both positive or both negative, (1) is an almost immediate consequence of (2). Start out with $\gamma > \delta > 0$, and $c_1, c_2, \cdots, c_n > 0$.

Since

$$(3) \qquad \frac{c_1^\delta}{\sum_1^n c_i^\delta} + \frac{c_2^\delta}{\sum_1^n c_i^\delta} + \cdots + \frac{c_n^\delta}{\sum_1^n c_i^\delta} = 1$$

it follows from (2) that

$$\frac{\sum_1^n c_i^\gamma}{\sum_1^n c_i^\delta} = \frac{c_1^\delta c_1^{\gamma-\delta} + c_2^\delta c_2^{\gamma-\delta} + \cdots + c_n^\delta c_n^{\gamma-\delta}}{\sum_1^n c_i^\delta}$$

$$\ge (c_1^{\gamma-\delta})^{\frac{c_1^\delta}{\sum_1^n c_i^\delta}} (c_2^{\gamma-\delta})^{\frac{c_2^\delta}{\sum_1^n c_i^\delta}} \cdots (c_n^{\gamma-\delta})^{\frac{c_n^\delta}{\sum_1^n c_i^\delta}} .$$

Hence

$$(4) \qquad \left(\frac{\sum_1^n c_i^\gamma}{\sum_1^n c_i^\delta} \right)^{\frac{1}{\gamma-\delta}} \ge c_1^{\frac{c_1^\delta}{\sum_1^n c_i^\delta}} c_2^{\frac{c_2^\delta}{\sum_1^n c_i^\delta}} \cdots c_n^{\frac{c_n^\delta}{\sum_1^n c_i^\delta}}$$

with equality iff $c_1 = c_2 = \cdots = c_n$.

Again using (2) with (3) gives

$$\frac{n}{\sum_1^n c_i^\delta} = \frac{c_1^\delta \cdot \left(\frac{1}{c_1^\delta} \right) + c_2^\delta \cdot \left(\frac{1}{c_2^\delta} \right) + \cdots + c_n^\delta \cdot \left(\frac{1}{c_n^\delta} \right)}{\sum_1^n c_i^\delta}$$

$$\ge \left(\frac{1}{c_1^\delta} \right)^{\frac{c_1^\delta}{\sum_1^n c_i^\delta}} \left(\frac{1}{c_2^\delta} \right)^{\frac{c_2^\delta}{\sum_1^n c_i^\delta}} \cdots \left(\frac{1}{c_n^\delta} \right)^{\frac{c_n^\delta}{\sum_1^n c_i^\delta}}$$

It follows that

$$(5) \qquad c_1^{\frac{c_1^\delta}{\sum_1^n c_i^\delta}} c_2^{\frac{c_2^\delta}{\sum_1^n c_i^\delta}} \cdots c_n^{\frac{c_n^\delta}{\sum_1^n c_i^\delta}} \ge \left(\frac{\sum_1^n c_i^\delta}{n} \right)^{\frac{1}{\delta}} .$$

Equality holds iff $c_1 = c_2 = \cdots = c_n$.

Equations (4) and (5) give

$$(6) \qquad \left(\frac{\sum_1^n c_i^\gamma}{\sum_1^n c_i^\delta} \right)^{\frac{1}{\gamma-\delta}} \ge \left(\frac{\sum_1^n c_i^\delta}{n} \right)^{\frac{1}{\delta}} .$$

Write (6) as

$$\frac{\sum_1^n c_i^\gamma}{n} \ge \left(\frac{\sum_1^n c_i^\delta}{n}\right)^{\frac{\gamma-\delta}{\delta}} \left(\frac{\sum_1^n c_i^\delta}{n}\right) = \left(\frac{\sum_1^n c_i^\delta}{n}\right)^{\frac{\gamma}{\delta}}$$

In other words

$$(7) \qquad \left(\frac{\sum_1^n c_i^\gamma}{n}\right)^{\frac{1}{\gamma}} \ge \left(\frac{\sum_1^n c_i^\delta}{n}\right)^{\frac{1}{\delta}}$$

with equality iff $c_1 = c_2 = \cdots = c_n$.

To get (1) for $\alpha > \beta > 0$, use (7) with $\gamma = \alpha$, $\delta = \beta$, and $c_i = a_i$, $1 \le i \le n$. Similarly, (1) can be obtained for $0 > \alpha > \beta$ by using (7) with $\gamma = -\beta$, $\delta = -\alpha$, $c_i = \frac{1}{a_i}$, $1 \le i \le n$, and inverting both sides of the inequality. Finally, the ordinary AM-GM inequality can be used to complete the argument for $\alpha > 0 > \beta$. **Thus**

$$\left(\frac{\sum_1^n a_i^\alpha}{n}\right)^{\frac{1}{\alpha}} \ge \left(\prod_1^n a_i^\alpha\right)^{\frac{1}{\alpha n}} = \left(\prod_1^n a_i\right)^{\frac{1}{n}} = \left(\prod_1^n a_i^\beta\right)^{\frac{1}{\beta n}} \ge \left(\frac{\sum_1^n a_i^\beta}{n}\right)^{\frac{1}{\beta}}.$$

References

1. N.D. **Kazarinoff,** Analytic Inequalities, Holt, Rinehart, and Winston, N.Y., 1961, **pp.** 62-64.
2. P.P. Korovkin, Inequalities, **Blaisdell** Publishing Co., N.Y., **1961, Ch. 2, 3**.

MISCELLANY

Chapter Reports

Professor Joanne Snow reports that the INDIANA EPSILON **Chapter** (Saint Mary's College) was addressed by Dr. Maura Mast (University of Northern Iowa) at the department's annual Open House. The chapter performed various service activities during the year.

Professor Chris **Leary** reports that sixteen talks were presented to the NEW YORK OMEGA Chapter (Saint **Bonaventure** University) during the **1995-96** academic year. Students P. **J.** Darcy and David Tascione were members of St **Bonaventure's** team in the Mathematical Contest in Modeling. **SIAM** selected the team's solution to Problem B as the outstanding solution for that problem.

Professor Joan Weiss reports that the CONNECTICUT GAMMA Chapter **(Fairfield** University) was addressed by A. Michael White **from** the Defense Research Agency. The chapter was involved with various mathematical contests during the year.

Errata

Sandra Chandler found die following errors in her paper "Determining a Day of the **Week"** (Volume **10(1994-99),** Number 4,283-284). **In** paragraph two on page 284, every occurrence of Thursday' should be replaced with 'Wednesday'. Also, the last example should read: "Here is a last example to illustrate this: January 6, 1994 was a Thursday; what day will July 4, 1997 fall on? Thursday + 1 + 1 + 2 (now we are at January 6, 1997) + 3 + 0 + 3 + 2 + 3 + 2 (July 6, 1997) - 2 = Thursday + 15 = Friday."

Paul S. Bruckman and **Robyn M. Carley** should have been listed as solvers to Mathacrostic 41.

The 1995 Game

In die Fall 1995 issue of the **Journal, Paul** S. **Bruckman** challenged readers to represent die *integers* **starting** from 1 **using** the digits **1, 9, 9,** and 5 in that order. The challenge was accepted by **Victor** G. **Feser** from the University of ***Mary.*** He represented the integers from 0 to 154 in the prescribed manner. Some of his representations are listed below. Contrary to **Bruckman's** expectations, **Feser** felt that the expressions for 20 and 25 were easy to find. He was more challenged by the representations for **63, 78,** and 79.

$$0 = 1 - \sqrt{9} + \sqrt{9-5}$$

$$5 = 1^9 + 9 - 5$$

$$10 = 1 + \sqrt{9}^{\sqrt{9-5}}$$

$$20 = (1+9)\sqrt{9-5}$$

$$25 = (-1 + \sqrt{9} + \sqrt{9}) \cdot 5$$

$$28 = (1 + \sqrt{9})! + 9 - 5$$

$$63 = (1 + (\sqrt{9})!) \cdot 9 \cdot \lceil \sqrt{\sqrt{5}} \rceil$$

$$78 = -1 + 9 \cdot 9 - \lceil \sqrt{5} \rceil$$

$$79 = 1 \cdot 9 \cdot 9 - \lceil \sqrt{5} \rceil$$

$$106 = (-1 + (\sqrt{9})!)! - 9 - 5$$

$$154 = 1 + 9 + ((\sqrt{9})!)!/5$$

Victor claims that every integer can be represented as directed.

Cryptogram Solution

Paul S. Bruckman provided the following solution to the **cryptogram** which appeared on page 185 of the Fall 1995 issue of the **Journal:**

> Should the Pi Mu **Epsilon Journal** publish mathematical cryptograms? Or should it have no puzzles at all?

What are These?

The following items were created by **Florentin Smarandache** and submitted by **Charles Ashbacher.**

1. **E A O T E E**
r t s h n s

2. **Brianchon**

3. **R V R V R V**
M K M K M K
A O A O A O

4. **F N**
U O
N I
C T

5. **OI**
P N S P N S
OI T T

6. **DEDE/KIND**

Answers, should you need them, are on page 370.

MATHACROSTICS

Solution to **Mathacrostics** 42, by **Jeanette Bickley** (Spring **1996).**

Words:

A.	sieve of Eratosthenes	P.	opine
B.	Windows	Q.	first law of hydrostatics
C.	even	R.	august
D.	icosahedron	S.	foolish
E.	**Noah**	T.	injure
F.	bowlers	U.	null set
G.	Elements	V.	absolute
H.	relativity **theory**	W.	logarithms
I.	**grown**	X.	thirteen
J.	Dipper	Y.	heighten
K.	rack	Z.	Euclidean geometry
L.	**Euler**	a.	owning
M.	**amuck**	b.	race
N.	**moa**	c.	yesterday
O.	Stephen Hawking		

Author and title: S. **Weinberg,** ***Dreams of a Final Theory***

Quotation: Although we do not yet have a sure **sense** of where in our work we should rely on our sense of beauty, still in elementary particle physics, aesthetic judgments seem to be **working** increasingly **well.** I take this as evidence that we are moving in the right direction and perhaps not *so* far from our goal.

Solvers: Avraham and **Chara** G. Adler (jointly); Thomas **Banchoff;** Frank P. Battles; Paul S. **Bruckman;** Keith G. **Calkins;** Charles R. Diminnie; Clayton W. Dodge; **Thomas Drucker;** Robert C. **Gebhardt;** Jennifer Hake; Geoff **Inman**, Brooke **Bentley** and Curt Evans (jointly); Henry S. and Elizabeth C. **Lieberman** (jointly); Rachael Lott; **Thom** Mitchell; Matt **Reason;** Laurie **Schlenkermann;** Naomi **Shapiro;** Stephanie **Sloyan;** and the proposer.

Mathacrostic 43 by **Gerald M. Leibowitz appears** on the following pages. The directions for solving acrostics are also given. To be listed as a solver, send your solution to the editor.

A	Coral structures	132	27	139	34	67
B.	Probability space, usually	35	28	19	116	88
C.	He wrote Operations **Linéaires**	105	29	111	55	3
				107		
D.	Member of a set	10	135	73	23	58
				115	84	
E.	**Risqué**	49	137	18	106	
F.	Dentifrice	130	101	124	25	79
		64	83	20	71	113
G.	________ **boundary**	33	9	48	14	61
				98		
H.	Algebraic systems	22	68	7	121	96
				43		
I.	Steel tool	80	59	69		
J.	23rd pan of a treatise, abbr.	91	4	17	122	8
				128		
K.	American newt	41	74	30		
L.	Word associated with C. and R.	95	125	42	138	119
M.	Sworn statement	39	120	51	131	
N.	Name with series and integrals	47	21	108	36	12
				70	118	

O. Co-author of convexity theorem — 109 90 126 63 75 56

P. Covered with water — 134 38 66 6 85

Q. Parallel to canal — 37 44 54 102 94 77 26

R. Inventor of ℓ^2 — 72 129 100 112 15 40 60

S. Lambs' moms — 127 1 133 50

T. Paired — 87 24 78 32 65 97 104

U. Attribute of $f(x) = ax + b$ — 110 136 16 114 45 99

V. Bird sound — 81 93 2 103 11

W. Member of a list — 31 13 62 82

X. Translates of a subgroup — 76 5 46 53 89 117

Y. British snacks — 140 52 86 123 57 92

The mathacrostic is a keyed anagram. The 140 letters to be entered in the diagram in the numbered spaces will be identical with those in the 25 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution.

When completed, the initial letters of the words will give the name of an author and the title of a book; the completed diagram will be a quotation from that book.

1 S	2 V		3 C	4 J	5 X	6 P	7 H		8 J	9 G	10 D		11 V	12 N	13 W	14 G
15 R		16 U	17 J	18 E	19 B	20 F		21 N	22 H		23 D	24 T	25 F	26 Q	27 A	28 B
29 C	30 K	31 W	32 T	33 G		34 A	35 B	36 N		37 Q	38 P	39 M		40 R	41 K	42 L
43 H	44 Q	45 U	46 X		47 N	48 G	49 E	50 S	51 M		52 Y	53 X		54 Q	55 C	56 O
57 Y	58 D	59 I		60 R	61 G		62 W	63 O	64 F	65 T	66 P	67 A	68 H	69 I	70 N	
71 F	72 R	73 D		74 K	75 O	76 X	77 Q		78 T	79 F	80 I	81 V		82 W	83 F	84 D
85 P	86 Y	87 T	88 B	89 X	90 O	91 J	92 Y		93 V	94 Q	95 L		96 H	97 T	98 G	99 U
100 R	101 F	102 Q	103 V	104 T		105 C	106 E		107 C	108 N	109 O	110 U	111 C		112 R	113 F
114 U	115 D	116 B	117 X		118 N	119 L	120 M	121 H		122 J	123 Y	124 F	125 L	126 O	127 S	
128 J	129 R	130 F	131 M		132 A	133 S	134 P	135 D		136 U	137 E	138 L	139 A	140 Y		

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: **dodge@ganss.nmemat.maine.edu** *Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed to arrive by July 1, 1997.*

Problems for Solution

888. *Proposed by the Editor.*

In 1953 Howard Eves' book *An Introduction to the History of Mathematics* was first published. It quickly became the definitive undergraduate text in mathematics history. It still is today. To honor this outstanding text and its equally outstanding author, solve this base nine **alphametic,** finding the unique value of *HEVES*:

$$MATH + HIST = HEVES.$$

889. *Proposed by M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Prove that

$$\frac{a^x - 1}{x} - \frac{a^y - 1}{y} > \frac{x - y}{2} \cdot \left(\frac{a - 1}{a}\right)^2$$

where $a > 1$ and $x > y > 0$.

890. *Proposed by Peter A. Lindstrom, Irving, Texas.*

Express the following sum in closed form, where real number $a \neq 1$:

$$\sum_{i=1}^{n} ia^{n-i}.$$

891. *Proposed by John Wahl, Mt. Pocono, Pennsylvania, and Andrew Cusumano, Great Neck, New York.*

Solve for d the equation

$$\frac{bcd + cda + dab + abc}{a + b + c + d} = \sqrt{abcd}.$$

892. *Proposed by Bill Correll, Jr., student, Denison University, Granville, Ohio.*

Prove that the average of the eigenvalues of a real, symmetric, idempotent matrix is at most one.

893. *Proposed by Peter A. Lindstrom, Irving, Texas.*

Show that the sequence $\{x_n\}$ converges and find its limit, where $x_1 = 2$ and, for $n \geq 1$,

$$x_{n+1} = \frac{2x_n \sin x_n + \sin x_n + \cos x_n}{2 \sin x_n}$$

894. *Proposed by Andrew Cusumano, Great Neck, New York.*

Let us lake $P_2 = 4\sqrt{2 - \sqrt{2}}$, $P_3 = 8\sqrt{2 - \sqrt{2 + \sqrt{2}}}$, $P_4 = 16\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$, and so forth. Find the value of $\lim_{n\to\infty} n(P_n - P_{n-1})$.

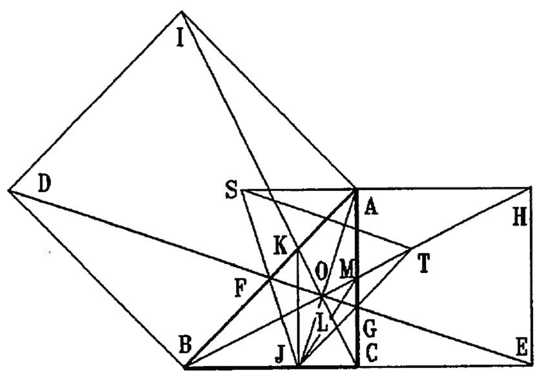
895. *Proposed by Andrew Cusumano, Great Neck, New York.*

Let *ABC* be an isosceles right triangle with right angle at *C*. Erect squares *ACEH* and *ABDI* outwardly on side *AC* and hypotenuse *AB*. Let *CI* meet *BH* at *O* and *AB* at *K*, and let *AO* meet *BC* at *J*. Let *DE* cut *AB* at F and *AC* at G. It is known (Problem 817, Fall 1994, page 72) that *DE* passes through *O*. Let *JF* meet *AH* at *S* and let *JG* meet *BH* at T. Finally, let *BH* and *AC* meet at *M* and let *JM* and *CI* meet at *L*. See the figure.

a) Prove that

i) *ST* is parallel to *DOE*,

ii) *JK* is parallel to AC,
iii) *JG* is parallel to *AB*,
iv) *AI* passes through T,
v) JF passes through **I**,
vi) *EK* passes through *M*, and
vii) BL passes through G.
*b) Which of these results generalize to an arbitrary triangle?



896. ***Proposed by Peter A. Lindstrom, Irving, Texas.***

For arbitrary positive integers k and n, find each summation:

a) $\displaystyle\sum_{i=1}^{n} (i)(i+1)(i+2)\cdots(i+k).$

b) $\displaystyle\sum_{i=1}^{n} (i)(i-1)(i-2)\cdots(i-k),$ where $n \geq k+1$.

c) $\displaystyle\sum_{i=1}^{n} (i-k)(i-k+1)\cdots(i-1)(i)(i+1)(i+2)\cdots(i+k),$

where $n \geq k+1$.

897. ***Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.***

Show that all non-negative integral solutions of the Diophantine equation

$$3(x_i - x_{i-1})^2 = 2(x_i + x_{i-1})^2 + 1$$

are given by consecutive terms of an **infinite** sequence of integers x_i with $x_0 = 0$, $x_1 = 1$, and $x_{i+1} = ax_i + bx_{i-1}$. Find a and b and the first seven terms of the sequence x_i. Generalize this procedure and determine the solution x_i for the equation

$$(2c+1)(x_i - x_{i-1})^2 = 2c(x_i + x_{i-1})^2 + 1.$$

898. ***Proposed by Paul S. Bruckman, Edmonds, Washington.***

An n-digit number N is defined to be a ***base*** 10 ***Armstrong number of order n*** if

$$N = \sum_{k=0}^{n-1} d_k 10^k = \sum_{k=0}^{n-1} d_k^n,$$

where the d_k are decimal digits, with $d_{n-1} > 0$. (See Miller and **Whalen**, "**Armstrong** Numbers: $153 = 1^3 + 5^3 + 3^3$," *Fibonacci Quarterly* **30.3, (1992)**, pp. 221-224.) Prove that there are no base ten Armstrong numbers of order 2; that is, prove the impossibility of the equation

$$10y + x = x^2 + y^2,$$

where x and y are integers with $0 \leq x \leq 9$ and $1 \leq y \leq 9$.

899. ***Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.***

Find the average number of times an ordinary six-sided die must be tossed in order that each of its six faces comes up at least once.

900. ***Proposed by Howard Eves, Lubec, Maine.***

Given the lengths of two sides of a triangle and that the medians to those two sides are perpendicular to each other, construct the triangle with Euclidean tools.

Solutions

862. [Fall **1995**] Proposed by Philip Tate, student, University of Maine, Orono, Maine.

"Solve this base ten addition alphametic."

"But it doesn't have a unique solution."

"It does if I give you the value of T."

"Never mind, I found it. Furthermore, it has a unique solution in base eight. Let me show it to you."

$$\begin{array}{r} DODGE \\ +\ \underline{THE} \\ GREAT \end{array}$$

Solution by William *H.* Peirce, Delray Beach, Florida.

Let $b \geq 8$ be the base. Since $G \neq D$ and $R \neq O$, there must be a carry from the hundreds column to the thousands column and from the thousands to the ten thousands. Then $D + 1 = G$, $O = b - 1$, and $R = 0$. From the units column we have either $T = 2E$ or $T = 2E - b$. In the latter case the hundreds column requires that $c + D + 2E - b = E + b$, where c is the carry **from** the tens column and hence is 0 or 1. Then $D + E = 2b - c$, which is impossible, so $T = 2E < b$ and there is no carry from the units column. The alphametic now reads

$$\begin{array}{ccccc} 1 & 1 & c & 0 & \\ D & b-1 & D & D+1 & E \\ & & 2E & H & E \\ \hline D+1 & 0 & E & A & 2E \end{array}$$

where c is the carry into the hundreds column.

Suppose $c = 0$. Because then $D + 2E = E + b$, we have $D = b - E$. Also $A = D + 1 + H = b - E + 1 + H \leq b - 2$ implies that $E \geq 3 + H \geq 4$. For base 8, $E \geq 4$ contradicts $2E < b$. In base ten we must have $E = 4$, so $T = 8$ and $D = 10 - 4 = 6$. Now $A = D + 1 + H \geq 7 + H \geq 8$, which is impossible.

Thus $c = 1$. We have $1 + D + 2E = E + b$, so $D = b - 1 - E$ and $G = D + 1 = b - E$. Then $G + H = A + b$ implies that $A = H - E$. The alphametic now reads

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & \\ b-1-E & b-1 & b-1-E & b-E & E \\ & & 2E & H & E \\ \hline b-E & 0 & E & H-E & 2E \end{array}$$

in which we have $E \geq 2$ because $G = b - E < b - 1$.

In base ten, $E = 2$, 3, or 4. If $E = 3$, then $G = 10 - 3 = 7$, $D = G - 1 = 6$, and $T = 2E = 6$, a contradiction. If $E = 2$, then $G = 10 - 2 = 8$, $D = G - 1 = 7$, and $T = 2E = 4$, leaving 1, 3, 5, and 6. Then $A = 1$ and $H = 3$, or $A = 3$ and $H = 5$ provide solutions $79782 + 432 = 80214$ and $79782 + 452 = 80234$. If $E = 4$, then $G = 10 - 4 = 6$, $D = G - 1 = 5$, and $T = 2E = 8$, leaving 1, 2, 3, and 7. Then $A = 3$ and $H = 7$ yields the solution $59564 + 874 = 60438$. Therefore, T cannot be specified randomly—T *can* only be 4 or 8—and the solution is unique **only** when $T = 8$.

In base 8, $E = 2$ or 3. If $E = 3$, then $G = 8 - 3 = 5$, $D = G - 1 = 4$, and $T = 2E = 6$, leaving only 1 and 2 remaining. Then A and H cannot be chosen to satisfy $A = H - E$. If $E = 2$, then $G = 8 - 2 = 6$, $D = G - 1 = 5$, and $T = 2E = 4$, leaving 1 and 3. Then $A = 1$ and $H = 3$ provide the unique base 8 solution $57562 + 432 = 60214$.

These solutions are listed below:

Base 10			Base 8
$T = 4$:	$T = 4$:	$T = 8$:	$T = 4$:
79782	79782	59564	57562
432	452	874	432
80214	80234	60438	60214

For general base b, the number of solutions is a quadratic function of b having the form $(3b^2 - pb + q)/6$, and there are twelve such expressions in a congruence class of b modulo 12. These expressions **are**, for $b \equiv 8, 9, \ldots, 19 \pmod{12}$, $(p, q) = (41, 142)$, $(48, 201)$, $(43, 148)$, $(44, 163)$, $(45, 174)$, $(46, 181)$, $(41, 136)$, $(48, 195)$, $(43, 154)$, $(44, 169)$, $(45, 168)$, $(46, 175)$. For bases 8 and 10, these give 1 and 3 respectively.

Also solved *by* Charles Ashbacher, Paul S. **Bruckman,** James Campbell, Mark Evans, Victor G. **Feser,** S. **Gendler,** Richard I. Hess, Carl **Libis,** Henry S. Lieberman, David E. Manes, Brandon Marsee, Greg Mitts, Yoshinobu Murayoshi, Jeffrey Pierce, H.-J. Seiffert, Kelly Straughen, Kenneth M. Wilke, Rex H. Wu, and the Proposer.

*863. [Fall 1995] Proposed *by* James Chew, North Carolina Agricultural *and* Technical *State University,* Greensboro, North Carolina.

Here is a problem especially for undergraduates. Everyone is familiar with the story of the absent-minded professor who wears different colored socks on his feet. Suppose a month's supply of socks are in the clothes drier; specifically, let there be n pairs of socks in a drier containing only these socks.

a) Assume the socks are of n different colors. The professor draws socks one at a time from the drier without replacement, noting the color as he draws each sock. To get a pair of matching socks, at least 2 and at most $n+1$ socks must be drawn. On average, how many socks would have to be drawn to get a matching pair?

b) Repeat part (a), assuming k different colors of socks: n_1 pairs of red socks, n_2 pairs of blue socks, etc., where $n_1 + n_2 + \ldots + n_k = n$.

Solution *by* Paul *S. Bruckman, Salmiya,* Kuwait.

We fust solve the more general problem in Part (b). Some preliminary definitions are in order. Let $x_i = 2n_i$, $i = 1, 2, \ldots, k$. Let U_i denote the ith elementary symmetric function of the numbers x_i. Note that $U_1 = 2n$; also $U_{k+1} = 0$. We define $U_0 = 1$ for convenience. Let V_i denote the sum of all the permutations of terms such as $x_1^2x_2x_3\ldots x_i$. It is easily verified that

$$V_i = U_1U_i - (i+1)U_{i+1}. \tag{1}$$

Let θ_i denote the probability of requiring exactly i draws to obtain a pair of socks. Note that $2 \le i \le k+1$. The event defining θ_i involves fust drawing $i-1$ different-colored socks, which may be done in $(i-1)!U_{i-1}$ ways, and then drawing another sock of a previously drawn color. Having previously drawn color c, there are $x_c - 1$ ways to draw a matching sock. Thus, the total number of ways to draw a pair of socks in exactly i draws is given by $(i-1)![V_{i-1} - (i-1)U_{i-1}]$. The total number of ways to draw i socks, irrespective of results and again counting permutations, is $(2n)^{(i)} =$

$2n(2n-1)\cdots(2n-i+1)$. Therefore,

$$\theta_i = \frac{(i-1)![V_{i-1} - (i-1)U_{i-1}]}{(2n)^{(i)}} = \frac{(i-1)![(2n+1-i)U_{i-1} - iU_i]}{(2n)^{(i)}}$$

for i = 2, 3, ..., k + 1, obtained by using Equation (1). It is instructive to verify that $\theta_2 + \theta_3 + \ldots + \theta_{k+1} = 1$, as required. We omit the proof, but note that it is easily demonstrated by the use of telescoping series. Let $\mu(\mathbf{x})$ denote the required mean of the distribution, where $\mathbf{x}$ is the k-tuple $(x_1, x_2, \ldots, x_k)$. Then

$$\mu(\mathbf{x}) = \sum_{i=2}^{k+1} i\theta_i = \sum_{i=2}^{k+1} \frac{i![(2n+1-i)U_{i-1} - iU_i]}{(2n)^{(i)}}$$

$$= \sum_{i=1}^{k} \frac{(i+1)!\,U_i}{(2n)^{(i)}} - \sum_{i=2}^{k} \frac{i\cdot i!\,U_i}{(2n)^{(i)}} = 2 + \sum_{i=2}^{k} U_i \binom{2n}{i}^{-1},$$

or

$$\mu(\mathbf{x}) = \sum_{i=0}^{k} U_i \binom{2n}{i}^{-1}. \tag{3}$$

The result in (3) is the most general and depends on the values of the n_i for its evaluation in closed form. For Part (a) we have $k = n$, $n_i = 1$, $1 \le i \le n$, so that $U_i = 2^i\binom{n}{i}$. We denote the mean of the distribution by μ_n in this case. Then we obtain

$$\mu_n = \sum_{i=0}^{n} 2^i \binom{n}{i}\binom{2n}{i}^{-1}. \tag{4}$$

We may evaluate this sum in closed form. Identity 1.9 in [1] states

$$\sum_{i=0}^{n} \binom{x}{i} y^i = \sum_{i=0}^{n} \binom{n-x}{i} (1+y)^{n-i}(-y)^i. \tag{5}$$

Setting $x = -n-1$ and $y = -1/2$ in (5), we obtain

$$\sum_{i=0}^{n} \binom{-n-1}{i}\left(\frac{-1}{2}\right)^i = \sum_{i=0}^{n} \binom{2n+1}{i}\left(\frac{1}{2}\right)^n. \tag{6}$$

On the other hand, since $\binom{n}{i}\binom{2n}{n} = \binom{2n}{i}\binom{2n-i}{n-i}$, we see from (4) that

$$\mu_n = \binom{2n}{n}^{-1}\sum_{i=0}^{n} 2^i\binom{2n-1}{n-1} = \binom{2n}{n}^{-1}\sum_{i=0}^{n} 2^{n-i}\binom{n+i}{i}$$

$$= 2^n\binom{2n}{n}^{-1}\sum_{i=0}^{n}\left(\frac{-1}{2}\right)^i\binom{-n-1}{i}.$$

From (6) we obtain the desired result for Part (a),

$$(7)\qquad \mu_n = \binom{2n}{n}^{-1}\sum_{i=0}^{n}\binom{2n+1}{i} = 4^n\binom{2n}{n}^{-1}$$

since $\sum_{i=0}^{n}\binom{2n+1}{i} = 2^{2n}$. Note that, from Stirling's formula, $\binom{2n}{n} \approx 4^n(n\pi)^{-1/2}$ as $n \to \infty$. Consequently, we see from (7) that $\mu_n \approx (n\pi)^{1/2}$ as $n \to \infty$.

Reference

1. H. W. Gould, Combinatorial Identities, Morgantown, W. Va., University, 1972.

Also solved by William Chau, and Mark Evans.

Editorial note: So much for a simple, little problem intended for undergraduates!

864. [Fall 1995] Proposed by Charles *Ashbacher,* Geographic Decisions Systems, Cedar Rapids, Iowa.

On page 11 of the booklet Only *Problems,* Not Solutions! by Florentine Smarandache, there is the following problem.

Let $a_1, a_2, \ldots, a_m$ be digits. Are there primes, on a base b, which contain the group of digits $\overline{a_1 \cdots a_m}$ into its writing? But n!? But n°?

Prove that for any such sequence of digits $a_1, a_2, \ldots, a_m$, no matter how generated, there exists a prime such that the sequence is found in that prime.

I. Solution by H.-J. *Seiffert,* Berlin, *Germany.*

Let $b > 1$ be a natural number and $a_1, \ldots, a_m$ be base b digits. By Dirichlet's theorem, the arithmetic progression, where the quantity in parentheses is the base b numeral $a_1 \ldots a_m c$ and c is a base b digit chosen so that b and $a_1 \ldots a_m c$ have no common factors greater than 1,

$$kb^{m+1} + \left(\sum_{i=1}^{m} a_i b^{m-i+1} + c\right),\ k = 1, 2, \ldots,$$

contains infinitely many primes. Clearly, each such prime has a base b representation of the form $\ldots a_1 \ldots a_m c$. In base 10, we may take $c = 1$.

11. Comment by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

On pages 154-155 of [3], Sierpinski established the following result. For an arbitrary finite sequence $c_1, c_2, \ldots, c_m$ of digits there exists a prime number whose first m digits are $c_1, c_2, \ldots, c_m$. George Barany [1] proved there are infinitely many such primes. Sierpinski [4] also proved given two arbitrary finite sequences of digits (of the decimal system) $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$, where $b_n = 1, 3, 7$, or 9, there exist arbitrarily many prime numbers whose first m digits are the string $a_1, a_2, \ldots, a_m$ and whose last n digits are $b_1, b_2, \ldots, b_n$. Borucki and Diaz [2] proved both of Sierpinski's results for a Dirichlet arithmetic progression.

References

1. American Mathematical Monthly, Advanced Problem 5738, 78(1971)683.
2. L. J. Borucki and J. B. Diaz, "A Note on Primes, with Arbitrary Initial or Terminal Decimal Ciphers, in Dirichlet Arithmetic Progression," American Mathematical Monthly, 81(1974)1001-1002.
3. Waclaw Sierpinski, Elementary *Theory* of Numbers, Hafner Publishing Company, New York, 1964.
4. Waclaw Sierpinski, A Selection of Problems in the *Theory* of Numbers, The Macmillan Company, New York, 1964, page 40.

Also solved by Paul S. Bruckman, Pat Costello, Thomas C. Leong, David E. Manes, and the Proposer.

865. [Fall 1995] Proposed by Miguel *Amengual Covas,* Mallorca, Spain.

Let ABC be a triangle with sides of lengths a, b, and c, semiperimeter s, and area K. Show that, if $\Sigma a(s - a) = 4K$, then the three circles centered at the vertices A, B, and C and of radii $s - a$, $s - b$, and $s - c$, respectively, are all tangent to the same straight line.

Solution by *William* H. Peirce, *Delray* Beach, Florida.

Let $r_1 = s - a$, $r_2 = s - b$, and $r_3 = s - c$ be the radii of the three circles centered respectively at A, B, and C. From these definitions we get $a = r_2 + r_3$, $b = r_3 + r_1$, $c = r_1 + r_2$, and $s = r_1 + r_2 + r_3$. Then

$$K^2 = s(s-a)(s-b)(s-c) = r_1r_2r_3(r_1 + r_2 + r_3).$$

Since we have

$$4K = \Sigma a(s-a) = 2(r_1r_2 + r_2r_3 + r_3r_1),$$

we may eliminate K between these two expressions to obtain

$$F \equiv r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_1^2 - 2r_1r_2r_3(r_1 + r_2 + r_3) = 0.$$

That is, $F = 0$ is equivalent to the statement $\Sigma a(s - a) = 4K$.

Consider any two extemally tangent circles of radii r_1 and r_2. By simple geometry the distance between the two points of tangency along an external tangent line is $2\sqrt{r_1r_2}$. Therefore, when the three externally tangent circles of the problem are tangent to the same line, one and only one of the following three situations will occur:

$$\sqrt{r_1r_2} + \sqrt{r_2r_3} - \sqrt{r_3r_1} = 0, \quad \sqrt{r_1r_2} - \sqrt{r_2r_3} + \sqrt{r_3r_1} = 0,$$

$$\sqrt{r_1r_2} - \sqrt{r_2r_3} - \sqrt{r_3r_1} = 0.$$

These three expressions, along with the nonzero factor

$$\sqrt{r_1r_2} + \sqrt{r_2r_3} + \sqrt{r_3r_1},$$

are the four factors of F, so that $F = 0$ is equivalent to the three circles being tangent to the same straight line. The theorem follows.

Also solved by Paul S. Bruckman, David Iny, *and* the Proposer.

866. [Fall 1995] Proposed by J. Rodriguez, Sonora, *Mexico.*

For any nonzero integer n, the *Smarandache function* is the smallest integer $S(n)$ such that $(S(n))!$ is divisible by n. Thus $S(12) = 4$ since 12 divides 4! but not 3!.

a) Find a strictly increasing infinite sequence of integers such that for any consecutive three of them the Smarandache function is neither increasing nor decreasing.

*b) Find the longest increasing sequence of integers on which the Smarandache function is strictly decreasing.

I. Solution by David *Iny,* Baltimore, Maryland.

a) Obviously, if p is prime, then $S(p) = p$. Also, if p is an odd prime greater than or equal to 5, then $p + 1$ is divisible by distinct integers 2 and $(p + 1)/2$, so that $S(p + 1) \le (p + 1)/2$. Thus, if $p_1, p_2, \ldots$ is any increasing sequence of primes each greater than or equal to 5, then the sequence of integers $p_1, p_1 + 1, p_2, p_2 + 1, \ldots$ is an increasing sequence whose Smarandache function values alternately increase and decrease.

b) We extend the observation of Part (a) to note that, if p is prime and $k < p$ is a positive integer, then $S(p^k) = kp$. Now we find primes $p_1, p_2, \ldots, p_n$, all greater than n, so that $S(p_k^k) = kp_k$. If $p_1, p_2^2, p_3^3, \ldots, p_n^n$ is increasing with $p_1, 2p_2, \ldots, np_n$ decreasing, we are done. Recall that asymptotically there are $m/\ln m$ primes less than or equal to m.

The construction now follows. Fix $n >> 1$ and pick prime $p_n >> n^n$. Suppose we have already picked $p_n, p_{n-1}, \ldots, p_{k+1}$ for $k \ge 1$. Then we pick a prime p_k so that $p_k^k < p_{k+1}^{k+1}$ and $kp_k > (k+1)p_{k+1}$. That is,

$$\frac{k+1}{k}p_{k+1} < p_k < p_{k+1}^{(k+1)/k}.$$

Since $n^n << p_n < p_{n-1} < \ldots < p_{k+1}$, then we have that

$$p_{k+1}^{(k+1)/k} \ge (n^n)^{1/k}p_{k+1} \ge np_{k+1} \text{ since } k \le n-1.$$

Thus it is sufficient to pick p_k prime so that

$$\frac{k+1}{k}p_{k+1} < p_k < np_{k+1}.$$

We estimate the number of possible primes as

$$\frac{np_{k+1}}{\ln(np_{k+1})} - \frac{\frac{k+1}{k}p_{k+1}}{\ln\left(\frac{k+1}{k}p_{k+1}\right)} >> 1$$

since n >> 1. This completes the construction.

11. Comment by Paul S. *Bruckman, Salmiya,* Kuwait.

This problem appeared almost verbatim in *The* Fibonacci *Quarterly,* **Vol. 32,** No. 1, February 1994, by the same proposer, as Problem H-484. My solution appeared in the same journal, **Vol.** 33, No, 2, May 1995, pp. **189-192.**

Also solved *by* Charles Ashbacher, James Campbell, William Chau, Thomas C. **Leong, H.-J. Seiffert,** Rex H. Wu, and the Proposer.

867. [Fall 1995] Proposed by *Seung-Jin* Bang, AJOU *University,* Suwon, Korea.

Find the number of solutions (x, y, z, w) to the system

$$\begin{aligned} x + y + z + w &= 7 \\ x^2 + y^2 + z^2 + w^2 &= 15 \\ x^3 + y^3 + z^3 + w^3 &= 37 \\ xyzw &= 6. \end{aligned}$$

Solution *by* Henry S. *Lieberman, Waban,* Massachusetts.

The solutions to the system are **all** of the permutations of (1, **1, 2, 3),** of which there are 12. The left sides of the equations are rational integral symmetric functions of x, y, z, and w. We note the following **elementary** symmetric **functions** of x, y, z, and w:

$$\begin{aligned} \sigma_1 &= x + y + z + w, \\ \sigma_2 &= xy + yz + zw + wx + xz + yw, \\ \sigma_3 &= xyz + yzw + zwx + wxy, \text{ and} \\ \sigma_4 &= xyzw. \end{aligned}$$

Let $s_i = x^i + y^i + z^i + w^i$. Then the system can be written as

$$s_1 = \sigma_1 = 7,\ s_2 = 15,\ s_3 = 37, \text{ and } \sigma_4 = 6.$$

We claim that (x, y, z, w) solves the system if and only if these values are the zeros of the polynomial in t, $t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4$. From formulas in [1] we see that

$$(1) \qquad s_2 - s_1\sigma_1 + 2\sigma_2 = 0 \text{ and } s_3 - s_2\sigma_1 + s_1\sigma_2 - 3\sigma_3 = 0.$$

We have $\sigma_1 = 7$ and $\sigma_4 = 6$, and from (1),

$$15 - 7\cdot 7 + 2\sigma_2 = 0 \text{ and } 37 - 15\cdot 7 + 7\sigma_2 - 3\sigma_3 = 0,$$

which yield $\sigma_2 = \sigma_3 = 17$. It is easy to discover that the zeros of

$$t^4 - 7t^3 + 17t^2 - 17t + 6$$

are 1, **1, 2,** and 3. Hence, (1, **1, 2, 3)** and its permutations solve the system.

Reference

1. B. L. Van der Waerden, Modem Algebra, vol. 1, Frederick Ungar Publishing Company, New York, 1953, page 81.

Also solved by **Miguel Amengual Covas, Paul** S. **Bruckman,** William Chau, Russell Euler, Mark Evans, Robert **C.** Gebhardt, S. **Gendler,** Richard I. **Hess,** David Iny, Carl **Libis,** David E. Manes, **Yoshinobu** Murayoshi, **Kenneth** M. **Wilke,** *and the* Proposer.

868. [Fall **1995]** ***Proposed*** by ***William H. Peirce, Delray Beach, Florida.***

1. Enter total amount of all **social security** benefits . . . . , ...	**1.**___S___
2. Enter one-half of line 1	**2.**________
7. Enter your provisional income	**7.**___*P*___
8. Enter $32,000 if married filing jointly	**8.__32,000_**
9. Subtract line 8 from line 7. If zero or **less,** enter 0	9.________
Is line 9 zero? If **yes,** enter 0 on line 18. If no, continue to line 10.	
10. Enter $12,000 if married filing jointly	**10._12,000_**
11. Subtract line 10 from line 9. If zero or **less,** enter 0	11.________
12. Enter the smaller of line 9 or line 10	12.________
13. Enter one-half of **line** 12	13.________
14. Enter the **smaller** of line 2 or line 13	14.________
15. Multiply line 11 by 0.85	15.________
16. Add lines 14 and 15	16.________
17. Multiply line 1 by 0.85	17.________
18. Taxable social security benefits. Enter the smaller of line 16 or **line** 17	18.___T___

Social Security **Benefits** Worksheet (somewhat simplified)

Computation of the taxable portion of social **security benefits** in 1994 is considerably more complicated than in past years, and the IRS **has designed** the 1994 accompanying worksheet to determine **these** taxable **benefits.** Let S be the total social **security benefits** on line 1, ***P*** the provisional income on line 7, and T the taxable **benefits** on line 18. For married couples filing jointly, find T as a function of S and ***P.*** Exhibit the solution graphically by showing the function T for each pertinent region of the SP-plane, and give the boundary equations for each region. **Assume** S $>$ 0 and ***P*** $>$ 32,000 and ignore their practical upper **limits.**

Solution by Paul S. ***Bruckman, Salmiya, Kuwait.***

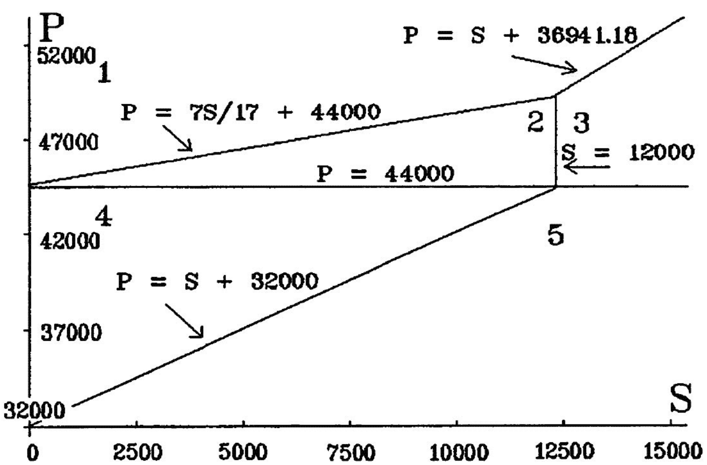
We **consider** the **possibilities** in the table below.

Line	Value			
1.	S			
2.	$S/2$			
7.	P			
8.	32000			
9.	P - 32000			
10.	12000			
11.	$P > 44000$: P - 44000		$P \le 44000$: 0	
12.	12000		P - 32000	
13.	6000		$P/2$ - 16000	
14.	$S > 12000$: 6000	$S \le 12000$: $S/2$	$S > P$-32000: $P/2$ - 16000	$S \le P$-32000: $S/2$
15.	.85P-37400	.85P-37400	0	0
16.	.85P-31400	.5S + .85P - 37400	$P/2$ - 16000	$S/2$
17.	.85S	.85S	.85S	.85S
18.	$S >$ P-36941.18: .85P-31400	$S >$ 17P/7 -106857.14: .5S+.85P - 37400	$S >$ 10P/17 - 18823.53: .5P-16000	.5S
	$S \le$ P-36941.18: .85S	$S \le$ 17P/7 -106857.14: .85S	$S \le$ 10P/17 - 18823.53: .85S	

We may display these results in the following five regions:

1. $T = .85S$ if $P > 7S/17 + 44000$ and $S \le 12000$, or if $P > S + 36941.18$ and $S > 12000$;
2. $T = .5S + .85P - 37400$ if $44000 < P \le 75/17 + 44000$ and $S \le 12000$;
3. $T = .85P - 31400$ if $44000 < P \le S + 36941.18$ and $S > 12000$;
4. $T = .5S$ if $S + 32000 < P \le 44000$ and $S \le 12000$; and
5. $T = .5P - 16000$ if $P \le S + 32000$ and $S \le 12000$.

We see that T is a positive, piecewise continuous, linear function of the two variables P and S, subject to the conditions $S > 0$ and $P > 32000$. We graph the boundaries of these five regions into which the SP-plane is divided. (We note there is a void region for $S > 12000$ and $P > 44000$.)



Also solved by Mark Evans *and the* Proposer.

869. [Fall **1995**] ***Proposed*** by ***Rasoul Behboudi, University of North Carolina, Charlotte, North Carolina.***

Consider an ellipse with center at O and with major and minor axes AS and CD respectively. Let E and F be points on segment OB so that we have $OE^2 + OF^2 = OB^2$. At E and F erect perpendiculars to cut arc BC at G

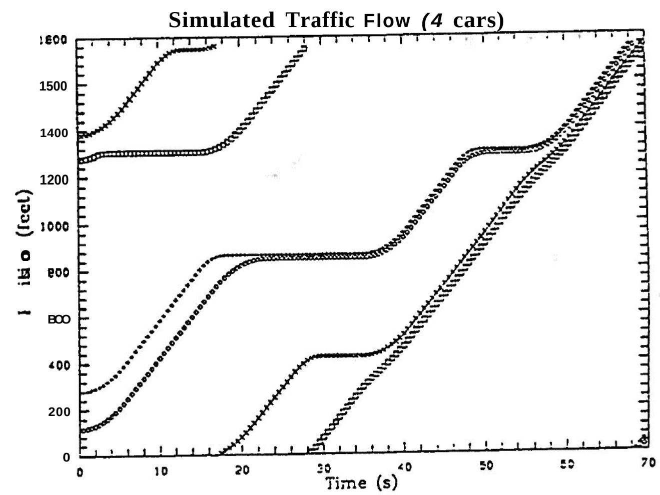


Figure 7.

Real-World Comparison

The table below shows yellow light durations at several intersections in the Hope College **area**. As one can see, the model (which predicts **minimum** values for t_{yel}) for die most part agrees with actual traffic signals. Agreements and discrepancies should not be **taken** too seriously, since the predictions were based *on* very rough estimates of braking accelerations and intersection widths (it's hard to measure the width of an intersection when pesky cars keep getting in the way).

Comparisons with Actual Traffic Lights

Location	Speed Limit	Predicted t^*_{yel}	Actual t_{yel}
8 @ River	25 mi/hr	3.6s	3.7s
8^{th} @ Central	25 mi/hr	3.6s	4.0s
8^{th} @ College	25 mi/hr	3.6s	3.5s
8^{th} @ US 31	35 mi/hr	5.3s	6.2s
US 31 @ 8^{th}	55 mi/hr	5.6s	4.5s
9^{th} @ Central	30 mi/hr	3.7s	4.0s
9 @ College	30 mi/hr	3.8s	4.0s

Comparison of Random and Planned Timing Schedules

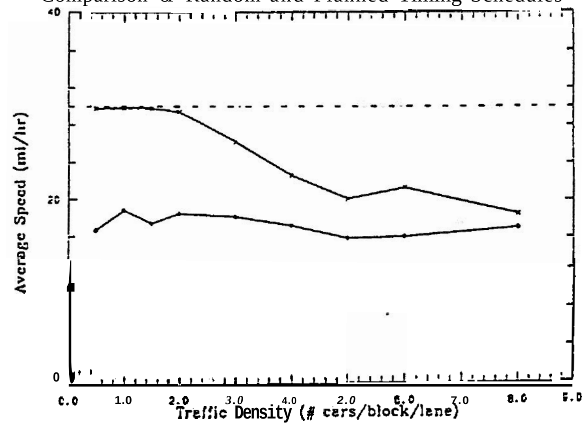


Figure 8.

Summary

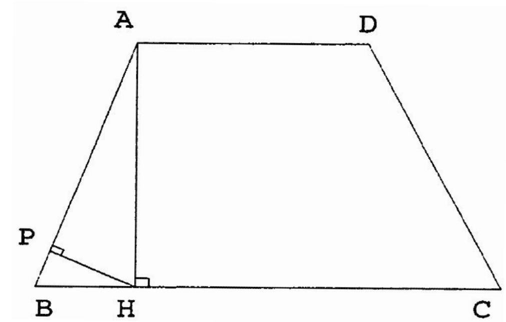
In conclusion, one can develop a suitable traffic signal timing schedule for a simple city based on a few parameters and a few formulas of motion. Yellow light timing depends primarily on speed limit, braking acceleration, and the width of intersections. Red and green light timing depends on the spacing of intersections, the speed limit, and the chosen pattern length N. The model shows that some degree of unimpeded traffic flow is almost always possible for regularly spaced intersections. Furthermore, timing schemes for two-way traffic can generally be extended to four-way traffic systems. And finally, the model provides results which seem to be in general agreement with actual traffic signals.

The model leaves many questions unanswered, however. How well do the proposed timing schedules prevent traffic jams? What if the city contains heavy-load and light-load streets? What if the lattice is irregular (e.g., the spacing varies)? I believe the model developed in this paper could be extended to handle these situations, and that the computer simulation would especially help in answering the first question. In any case, it was a fun project which I think gives a reasonable description of traffic systems. And who knows? Perhaps someone will pick up where I have left off I would be interested in hearing about the results.

Editorial note: *F.Y.*I., n = **16,292,246,262,786,755,156,105,160,857, 651,258,074,334,634,907,277.**

871. [Fall **1995]** ***Proposed*** by ***Miguel Amengual Covas, Mallorca, Spain.***

Let ABCD be an isosceles trapezoid with major base ***BC.*** If the altitude AH is the mean proportional between the bases, then show that each side is the arithmetic mean of the bases, and show that the projection AP of the altitude on side ***AB*** is the harmonic mean of the bases. See the figure.



Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let $BC = b$, $AD = a$, and $AH = h$. Since $h^2 = ab$, then

$$AB^2 = h^2 + \left(\frac{b-a}{2}\right)^2 = \left(\frac{b+a}{2}\right)^2,$$

so $AB = (a+b)/2$, the average of the bases. Let 6 = angle BAH. Then

$$\cos\theta = \frac{AH}{AB} - \frac{2\sqrt{ab}}{a+b}, \text{ so } AP = AH\cos\theta = \frac{2ab}{a+b},$$

whence AP is the harmonic mean of a and b.

Also solved by Paul S. **Bruckman,** William **Chau,** Russell Euler and **Jawad** Sadek, George P. Evanovich, Joe Howard, Tommy **Jarrett** ***and*** **David** Lindsey ***and*** Laura Ramdarass ***and*** Robyn Carley, Henry S. Lieberman, **Kandasamy** Muthuvel, H.-J. **Seiffert, Skidmore** Problem Group, Kenneth M. **Wilke,** Rex H. Wu, ***and the*** Proposer.

872. [Fall 19951 ***Proposed*** by ***Paul*** S. ***Bruckman, Edmonds, Washington.***

Given A,, A_2, and A, are the angles of a triangle and $4 < k < 12$, let

$$S_k = S_k(A_1, A_2, A_3) = \Sigma_{i=1}^{3} (k \cos A_i + \cos 2A_i),$$

defined on the triangular plane region R: $0 < A_1 < \pi$, $0 < A_2 < \pi$, $0 < A_1 + A_2 < u$. Find the maximum value of S_k for all triangles.

Solution by David Iny, Baltimore, Maryland.

In order for S_k to take on its **maximum** value for all choices of k, we will **take** the closure of the indicated region allowing the degenerate case. We show that $\max(S_k) = k + 3$ for $k \le 9$, achieved with the degenerate triangle $(0, 0, \pi)$, and $\max(S_k) = 3(k - 1)/2$ for $k \ge 9$, achieved with the equilateral triangle. To do this, we write

$$S_k = (k - 9)(\Sigma \cos A_i) + S_9$$

and we show that S_9 has (up to permutations) two distinct global maximizers, the degenerate triangle $(0, 0, \pi)$ and the equilateral triangle. We then combine this result with the inequality $1 \le \Sigma \cos A_i \le 312$ with lower and upper bounds achieved respectively by the degenerate and equilateral triangles.

Consider first the **function** $f(x) = 9\cos x + \cos 2x$. Since we **have** $f''(x) = -8\cos^2 x - 9\cos x + 4$, **then** f'' has a zero at $a = \cos^{-1}[(-9 + \sqrt{209})/16]$. We see that f is convex (concave downward) for $0 < x < a$ and concave (upward) for $a < x < \pi$. Thus, if $\max(A_i) \le a$, then

$$S_9 = \Sigma f(A_i) \le 3f(\tfrac{1}{3}\textstyle\sum A_i) = 12 \text{ with equality for } A_1 = A_2 = A_3 = \frac{\pi}{3}.$$

Because f is concave on (a, u), a maximizer of $\Sigma f(A_i)$ can have at most one angle greater than a. Thus, without loss of generality, we assume A, and A_2 do not exceed a. Since

$$f(A_1) + f(A_2) \le 2f\left(\frac{A_1 + A_2}{2}\right)$$

by convexity, we must have A, = A, and we let $y = A_1 = A_2$. Then

$$S_9 = 2(9\cos y + \cos 2y) + 9\cos(r - 2y) + \cos 2(\pi - 2y)$$

$$= 8\cos^4 y - 22\cos^2 y + 18\cos y + 8,$$

which on the interval $0 \le \cos y \le 1$ has maxima at $\cos y = 112$ and $\cos y = 1$, which occur at $y = \pi/3$ and $y = 0$. This establishes that

$$S_9 = \Sigma(9\cos A_i + \cos 2A_i) \le 12$$

with equality for the degerate and equilateral triangles.

It remains to show that $\Sigma \cos A_i \le 312$. If $\max(A_i) \le \pi/2$, then $\cos x$ is convex, whence

$$\Sigma \cos A_i \le 3\cos\left(\frac{1}{3}\Sigma A_i\right) = 3\cos\frac{\pi}{3} = \frac{3}{2}$$

with equality for $A_1 = A_2 = A_3 = \pi/3$. If $A_3 \ge \pi/2$, then

$$\Sigma \cos A_i \le 2\cos\left(\frac{A_1 + A_2}{2}\right) + \cos[\pi - (A_1 + A_2)].$$

But $2\cos z - \cos 2z = -2\cos^2 z + 2\cos z + 1$ has a maximum when $\cos z = 1/2$. Hence $\Sigma \cos A \le 312$. To get the lower bound, we take $A_1 \le A_2 \le \pi/2$ and $A_2 \le A_3$ without loss of generality. By convexity, $A_1 = 0$ whenever $\Sigma \cos A_i$ is a **minimum**. Then

$$\Sigma \cos A_i = 1 + \cos A_2 + \cos(\pi - A_2) = 1.$$

Thus $\Sigma \cos A_i \ge 1$ and the result is established. Furthermore, we see that the result holds for all real k.

Also solved by William H. **Peirce *and the*** Proposer.

873. [Fall **1995] *Proposed*** by ***Mohammad K. Azarian, University of Evansville. Evansville. Indiana.***

For p and q positive real numbers and any positive integer m let

$$f(x) = \left[1 + x + \frac{x^m}{m!} + \frac{x^{m+1}}{m!}\right]^{pq} \left[1 + \frac{x}{p}\right]^{p^3} \exp\left[\frac{q^3 x}{q + x}\right],$$

where $x \ge 0$. Prove that

$$0 < \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \int_0^{\infty} f(x) \exp(-[(p + q)^2 + k^n]x)\,dx \le 1.$$

Solution *by* H.-J. Seiffert, Berlin, Germany.

Since $1 + x + \ldots + x^n/n! < e^x$ when $x > 0$ and n is a positive integer, we have that

$$0 < f(x)\exp[-(p+q)^2 x]$$

$$= \left[(1+x)\left[1+\frac{x^m}{m!}\right]\right]^{pq}\left[1+\frac{x}{p}\right]^{p^3}\exp\left[\frac{q^3x}{q+x}-(p+q)^2x\right]$$

$$< \exp\left(2pqx + p^2x + \frac{q^3x}{q+x} - (p+q)^2x\right) = \exp\left(-\frac{q^2x^2}{q+x}\right) < 1.$$

Hence

$$0 < \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\int_0^{\infty} f(x)\exp(-[(p+q)^2 + k^n]x)\,dx$$

$$< \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\int_0^{\infty}\exp(-k^n x)\,dx = \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{k^n}\int_0^{\infty}\exp(-t)\,dt$$

$$= \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{k^n} = \sum_{k=2}^{\infty}\frac{1}{k^2}\sum_{n=2}^{\infty}\frac{1}{k^{n-2}} = \sum_{k=2}^{\infty}\frac{1}{k^2}\frac{k}{k-1}$$

$$= \sum_{k=2}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right) = 1.$$

Also solved by Paul S. Bruckman, Russell **Euler,** David Iny, and the Proposer.

874. [Fall **1995]** Proposed **by** David Iny, Westinghouse Electric Corporation, Baltimore, Maryland.

a) Given real numbers x_i and z_i for $1 \le i \le n$, prove that

$$n[\sum x_i^2 \sum z_i^2 - (\sum x_i z_i)^2] \ge$$

$$(\sum x_i)^2(\sum z_i^2) + (\sum x_i^2)(\sum z_i)^2 - 2(\sum x_i)(\sum z_i)(\sum x_i z_i).$$

b) Determine a necessary and sufficient condition for equality.

I. Solution *by* Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

It will be shown that the given inequality is equivalent geometrically to the square of the volume of a tetrahedron being nonnegative.

Let X, Z, and I denote, respectively, the n-dimensional vectors $(x_1, x_2, \ldots, x_n)$, $(z_1, z_2, \ldots, z_n)$, and $(1, 1, \ldots, 1)$ and let $|\mathbf{X}| = x$, $|\mathbf{Z}| = z$, and here $|\mathbf{1}| = \sqrt{n}$. Also let α, β, and γ denote the angles between X and I, Z and I, and X and Z, respectively. The given inequality can now be rewritten in the form

$$n[\mathbf{X}^2\mathbf{Z}^2 - (\mathbf{X}\cdot\mathbf{Z})^2] \ge (\mathbf{X}\cdot\mathbf{I})^2\mathbf{Z}^2 + (\mathbf{Z}\cdot\mathbf{I})^2\mathbf{X}^2 - 2(\mathbf{X}\cdot\mathbf{I})(\mathbf{Z}\cdot\mathbf{I})(\mathbf{X}\cdot\mathbf{Z})$$

or equivalently

$$nx^2z^2(1-\cos^2\gamma) \ge nx^2z^2\cos^2\alpha + nx^2z^2\cos^2\beta - 2nx^2z^2\cos\alpha\cos\beta\cos\gamma$$

and finally

$$nx^2z^2(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma) \ge 0.$$

The left side of the latter inequality is 36 times the square of the volume of the tetrahedron having X, Z, and I as three **coterminal** edges. There is equality if and only if the volume is 0 or equivalently if X, Z, and I are linearly dependent. That is, if either X or Z equal 0 or k**I** or there are constants a and b such that $z_i = ax_i + b$ for all i.

II. Comment *by* H.-J. Seiffert, Berlin, Germany.

This inequality is known. See **[1],** page 227, Equation 20.1.

Reference

1. D. S. **Mitrinović,** J. E. **Pečarić,** and A. M. Fink, Classical *and* New Inequalities in Analysis, **Kluwer, 1993.**

Also solved *by* Paul S. **Bruckman,** Joe Howard, **Yoshinobu Murayoshi,** and the Proposer.

Subscription and Change of Address

If your address label contains the symbols "F96" then this is the last issue in your current subscription. We hope that you agree that the ***Journal*** provides good value and that you will renew your subscription. The rates are:

United States:	\$20 for 2 years \$40 for 5 years
Foreign:	\$25 for 2 years
Back issues:	\$ 5 each
Complete volumes:	\$50 (5 years, 10 issues)
All issues:	\$400 (9 back volumes and volume 10)

If you have moved, please let us know. The ***Journal*** is not forwarded so it is important that we have a current mailing address for you.

To subscribe or change your address, complete the form below (or a copy thereof) and send it, with a check payable to the Pi Mu ***Epsilon Journal*** for subscriptions, to

Joan **Weiss**
Department of Mathematics and Computer Science
Fairfield University
Fairfield CT 06430

Name: ______________________ Chapter: ______________

Address: ____

____

____

Address change? ________________ Subscription: __________

REFEREE **APPLICATION FORM**

The contributions of a referee can enhance the Quality of any journal. If you would like to volunteer to serve as a referee for the Pi Mu Epsilon Journal, please provide the information requested below so that the appropriate manuscripts can be sent to you for your consideration. Since manuscripts are not to exceed ten pages in length, the editor believes that a referee's report should be **sent** back to the editor in at most two months from the date of receipt. Please keep this in mind before volunteering your valuable time and expertise. Your support of the Journal is appreciated. Please send the completed form to:

Russell **Euler**
Department of Mathematics and Statistics
Northwest Missouri State University
Maryville, MO 64468.
(816)562-**1229**

(Please type or print neatly)

Name ____

Affiliation ____

Mailing Address ____

Phone Number ____

E-mail Address ____

"lease list the mathematical areas in which you feel competent to referee.

Please **list** any **specific** times of the year that you would prefer **NOT** to receive a manuscript to review.

Editor's Note

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. All submissions to the Journal should be directed toward this group.

Undergraduate and beginning graduate students are urged to submit papers to the Journal for consideration for publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are also welcome. Some priority areas are listed in the Editorial (p. 352) in the Fall 1996 issue of the Journal. Some guidelines include:

1. Manuscripts must be original work, not have been previously published nor should they be under consideration for publication elsewhere.
2. Manuscripts must be typewritten, double spaced with wide margins, on only one ride of 8.5" x 11" white paper. Whenever possible, manuscripts should be typewritten in WordPerfect.
3. Including figures, manuscripts should not exceed ten pages in length.
4. Figures should be placed on separate sheets and photo-ready. Camera-ready figures should measure a maximum of 4.5" wide by 6" long.
5. In order to insure anonymity in the review process, the author's name, affiliation (academic or otherwise), and address should appear only on a separate title page.
6. The tide must also appear on the first page of the exposition.
7. Each page should be numbered.
8. Students submitting manuscripts should give their school, date of graduation, and the name of their faculty advisor for their submission.
9. Two copies of each manuscript and a copy of the disk (whenever the manuscript is typed in WordPerfect) should be submitted to the editor at the address listed on the inside of the front cover.

Eleventh Annual Moravian College

Student Mathematics Conference

Saturday, February 22, 1997

Invited Lecture

Newton's Original Method

--or--

Though this be Method, yet there is madness in't

Professor William Dunham

Truman Koehler Professor of Mathematics

Muhlenberg College

The one-day conference, sponsored by Moravian College's chapter of Pi Mu Epsilon, is a unique opportunity for undergraduate students in the Tri-State area to meet and discuss mathematics. Students are invited to give talks in the fields of mathematics, statistics, operations research, and computing. In October, mathematics departments in the region will receive detailed information on the conference, registration forms for students talks, and registration forms for attendance at the conference. Mark your calendars and encourage your students to come, to give a talk, or just to listen to some interesting talks by others and to socialize with students and faculty from many other institutions.

For further information, contact:
Doris Schattschneider
Moravian College, 1200 Main St., Bethlehem, PA 18018-6650
email: schattdo@moravian.edu
tel (610) 861 1373 fax (610) 861 1462