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Some Solutions of the Generalized Fermat Equation

Herbert E. Salzer

The Diophantine equation

$$pq(p+q) = a^n,\ n \geq 3, \tag{1}$$

is here called the "generalized Fermat equation" because it is equivalent to Fermat's equation

$$x^n + y^n = z^n, \quad n \geq 3, \tag{2}$$

only whenever

a) $(p, q) = 1$, $n \neq 3k$, or b) for any (p, q), $n = 3k$. [1]

For a), in (1) when $(p, q) = 1$, also $(p, p+q) = 1$ and $(q, p+q) = 1$, so that p, q and $p+q$ are n^{th} powers, implying Fermat's equation (2) which, conversely, implies (1) for $p = x^n$, $q = y^n$ and $a = xyz$.

For b), divide (1) by $(p, q)^3 \equiv d^3$ to get $p'q'(p'+q') = a^{3k}/d^3 = (a^k/d)^3 = a'^3$, $(p', q') = 1$, to which we employ the argument in a) applied to $n = 3$.

It was shown in [1] that (1), for $(p, q) \neq 1$, $n \neq 3k$, does have solutions. Following is an additional multiple infinitude of solutions to (1):†

In the simpler case of (1) where $p = q$, $2p^3 = a^n$, it is easily seen, for $n \neq 3k$, that all solutions are given by

$$2(2^{t_0} p_1^{nt_1} \cdots p_k^{nt_k})^3 = (2^s p_1^{3t_1} \cdots p_k^{3t_k})^n, \tag{3}$$

whenever $3t_0 + 1 = sn$, p_i is any odd prime and t_i is any integer, $i = 1, \cdots, k$ for any k.

To satisfy (1) when $p \neq q$, $(p, q) = d > 1$, for $n = 3k + m$, $m = 1$ or 2, and $pq(p+q) = \alpha d \cdot \beta d(\alpha d + \beta d) = \alpha\beta(\alpha+\beta)d^3$, $(\alpha, \beta) = 1$:

When $m = 1$, choose $a = \alpha\beta(\alpha+\beta)$ and $d = [\alpha\beta(\alpha+\beta)]^k$, to get $pq(p+q) = [\alpha\beta(\alpha+\beta)]^{3k+1} = a^{3k+1} = a^n$.

When $m = 2$, choose $a = [\alpha\beta(\alpha+\beta)]^{1/2}$, which is an integer when α, β and $\alpha+\beta$ are squares satisfying the Pythagorean equation, and $d = [\alpha\beta(\alpha+\beta]^{k/2}$, to get $pq(p+q) = \alpha\beta(\alpha+\beta)d^3 = [\alpha\beta(\alpha+\beta)]^{3k/2+1} = a^{3k+2} = a^n$. Another solution is had by choosing $d = [\alpha\beta(\alpha+\beta)]^{2k+1}$ and $a = [\alpha\beta(\alpha+\beta)]^2$, so that $pq(p+q) = \alpha\beta(\alpha+\beta)d^3 = [\alpha\beta(\alpha+\beta)]^{6k+4} = a^{3k+2}$.

The writer's original approach to the solutions for $p \neq q$ did not go beyond

† The present simplification of the original versions of the solutions was suggested by the referee.

taking the easiest one of making the d some suitable power of $\alpha\beta(\alpha + \beta)$. A slight extension is to make $d = \alpha^i\beta^j(\alpha + \beta)^k$, where each of $3i + 1$, $3j + 1$ and $3k + 1$ is congruent to 0 mod n, giving choices of r, s and t for $a = \alpha^r\beta^s(\alpha + \beta)^t$. Going still further, for known prime power factors of α, β or $\alpha + \beta$ say $p_m^{l_m}$, the D is given the factor p_m^x and a has the factor p_m^u, the u obtained from the solution x of the congruence $3x + l_m \equiv 0 \bmod n$. Finding the complete solution to (1) appears to be unattainable, since it requires the knowledge of the complete factorization of $\alpha + \beta$ from that of any α and β when $(\alpha, \beta) = 1$.

This present elementary note has two possible consequences:

I. It might encourage mathematicians to seek a proof of FLT that is easier than the existing one, by trying to prove that every p and q in (1) must have $(p, q) > 1$.

II. It might launch a systematic search for interesting Diophantine equations other than (1), where the solutions must have their g.c.d. greater than 1.

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A Note on the Unity of a Subring

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In abstract algebra courses, occasionally, we encounter examples of a ring with unity, and a subring also with unity, but the two multiplicative identities differ. When does it happen, and how does it occur? The purpose of this short note is to scrutinize this apparent dilemma, and develop a theorem which gives a necessary and sufficient condition. Let us explore an example.

Consider Z_{10}, the residue class ring of integers mod 10 whose unity (multiplicative identity) is the class [1]. Let us consider the subring S = {[0], [2], [4], [6], [8]}. In this subring, the class [6] becomes the unity, since [2] [6] = [2]; [4] [6] = [4]; [6] [6] = [6]; [8] [6] = [8]; and of course, we always have [0][6] = [0]. At the first encounter, this appears a strange result, but it has an easy explanation:

[x] [6] = [x], for every [x] in S

is valid in this subring, simply because [x]([6] - [1]) = [x][5] = [0] for every [x] in S; since x is an even integer. Therefore, [x][6] - [x] = [0] and so [x][6] = [x].

This type of situation can occur only in rings with zero divisors which have more than one solution for the equation $ax = a$ for some a in the ring, because this equation becomes $a(x - e) = 0$; where e is the multiplicative identity (the original unity) of the ring.

Furthermore, if u is the unity of a subring of the ring with original unity e, $u \neq e$, then $u^2 = u$; and thus, u must be an idempotent element of the ring.

Hence, any ring with an idempotent element u other than the unity and zero, will have at least one subring, generated by u, consisting of $\{nu : n \in Z\}$. Obviously, it is a subring, since it is an additive cyclic group; and if nu is an arbitrary element of this subring, then $nu(u) = nu^2 = nu$, which proves that u is the unity of this ring.

Also, only rings with zero divisors can have idempotent elements other than the identity, since it is known that in a ring without zero divisors the only idempotent elements are zero and the multiplicative identity [Refs. 1, 2]. Thus, we have established the following theorem.

Theorem: A necessary and sufficient condition that a subring of a ring $\mathbb{R}$ has a multiplicative identity different from that of the original ring is that $\mathbb{R}$ has an idempotent element other than zero and unity.

Not every ring with zero divisors will have this property. A simple example of such a ring is $\mathbf{Z}_8$, which does not have any idempotent elements besides [0] and [1], and thus cannot have a subring with a unity different from that of the original ring.

I wish to thank Dr. Joseph E. Cicero and the referees for their valuable suggestions.

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A Note On An "Unnatural" Isomorphism

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Let $V_1, V_2, \ldots, V_n$ be subspaces of a finite dimensional vector space. Viar [2] proved the following extension of the second isomorphism theorem for vector spaces:

$$\frac{V_1+\cdots+V_n}{V_1} \cong \frac{V_2}{V_1 \cap V_2} \times \frac{V_3}{(V_1+V_2)\cap V_3} \times \cdots \times \frac{V_n}{(V_1+\cdots+V_{n-1})\cap V_n} \tag{1}$$

Viar's proof does not explicitly give the isomorphism, but instead compares the dimensions of both sides. Viar remarks that it seems unlikely an explicit isomorphism will be found. The purpose of this note is to provide further evidence of the "unnaturalness" of (1) by sketching a different proof that suggests exactly where the trouble lies.

Applying the second isomorphism theorem itself to each of the factors on the right side of (1) gives the equivalent, but more suggestive

$$\frac{V_1+\cdots+V_n}{V_1} \cong \frac{V_1+V_2}{V_1} \times \frac{V_1+V_2+V_3}{V_1+V_2} \times \cdots \times \frac{V_1+\cdots+V_n}{V_1+\cdots+V_{n-1}} \tag{2}$$

which is what we will actually prove. In fact, it is enough to prove (2) for three subspaces V_1, V_2, V_3 For assuming this case proven, we have for $n \geq 3$,

$$\frac{V_1+\cdots+V_n}{V_1} \cong \frac{V_1+\cdots+V_{n-1}}{V_1} \times \frac{(V_1+\cdots+V_{n-1})+V_n}{V_1+\cdots+V_{n-1}} \tag{3}$$

and the induction hypothesis that (2) holds for n-1 subspaces implies that it holds for n subspaces.

The first isomorphism theorem for vector spaces applied in this context gives the isomorphism

$$\frac{V_1+V_2+V_3}{V_1} \Big/ \frac{V_1+V_2}{V_1} \cong \frac{V_1+V_2+V_3}{V_1+V_2} \tag{4}$$

This is equivalent to the exactness of the sequence of homomorphisms

$$0 \to \frac{V_1+V_2}{V_1} \xrightarrow{\iota} \frac{V_1+V_2+V_3}{V_1} \xrightarrow{f} \frac{V_1+V_2+V_3}{V_1+V_2} \to 0 \tag{5}$$

Here ι is the inclusion that sends $x_2 + V_1$ as a coset in $(V_1 + V_2)/V_1$ to $x_2 + V_1$ as a coset in $(V_1 + V_2 + V_3)/V_1$, and f is (well-)defined by $f(x_2 + x_3 + V_1) = x_2 + V_1 + V_2$. *Exactness* means that the image of each homomorphism is the kernel of the next homomorphism in the sequence [1, p.326].

So far, nothing we have said is specific to vector spaces; everything holds for abelian groups. Now comes the "unnatural" part. It is a theorem that every short exact sequence of vector spaces (like our (5)) *splits* [1, p.328]. The meaning of this is quite simple. The exactness of

$$0 \to \frac{V_1+V_2}{V_1} \xrightarrow{\iota} \frac{V_1+V_2+V_3}{V_1} \tag{6}$$

is equivalent to asserting that ι is injective. But every injective linear transformation between vector spaces has a *left inverse*; that is, there exists a linear transformation $\kappa : (V_1 + V_2 + V_3)/V_1 \to (V_1 + V_2)/V_1$ such that $\kappa \circ \iota$ is the identity mapping on $(V_1 + V_2)/V_1$. Similarly, the exactness of

$$\frac{V_1+V_2+V_3}{V_1} \xrightarrow{f} \frac{V_1+V_2+V_3}{V_1+V_2} \to 0 \tag{7}$$

is the same as asserting that f is surjective. But every surjective linear transformation between vector spaces has a *right inverse*; that is, there exists a linear transformation $g: (V_1 + V_2 + V_3)/(V_1 + V_2) \to (V_1 + V_2 + V_3)/V_1$ such that $f \circ g$ is the identity mapping on $(V_1 + V_2 + V_3)/(V_1 + V_2)$.

If a short exact sequence like (5) splits as we have described, then it follows that the middle space in the sequence, which in our case is $(V_1 + V_2 + V_3)/V_1$, is isomorphic to the direct product of the other two spaces, $(V_1 + V_2)/V_1 \times (V_1 + V_2 + V_3)/(V_1 + V_2)$. The desired isomorphism in our case is $\kappa \times f$. As in [1, p.328], we leave the details as an exercise for the reader. This concludes our sketch of the proof of (2).

The "unnaturalness" of this arises in the construction of κ and g. The usual argument is to choose bases for all the spaces and define κ and g in terms of these bases. (The existence of a basis for an arbitrary vector space is guaranteed by Zorn's Lemma (see, for instance, [1, p.231]); this complication is avoided in the finite dimensional case, which is the setting of [2].) In the absence of additional structure such as an inner product, an arbitrary vector space does not have a "canonical" basis. Thus there is no "canonical" choice of κ and g.

We close by noting that a trap awaits the unwary here, but it makes a good exercise in understanding how mappings are defined on coset spaces. It might be tempting to define, say, g by $g(x_3 + V_1 + V_2) = x_3 + V_1$. We leave it to the reader to check that this is actually not well-defined.

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Cubic Connections: An Attack on Antiquity

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The fourth decade of the nineteenth century was to prove a productive period in algebraic advancement. Many of the rapidly unfolding group theory developments of Evariste Galois (1811-1832) from the earlier part of the decade (but published posthumously in 1846) were supplemented by critical findings in the theory of equations by Pierre Laurent Wantzel (1814-1848). Wantzel, of the Ecole Polytechnique of Paris, resolved at long last two of the famous problems of antiquity. The year was 1837, roughly two and a half millennia following the posing of the famous problems. Such a challenge had thus extended from the earliest years of demonstrative Greek mathematics. Resolution rested on a significant insight in the case for algebraic equations and the overall matter of irreducibility.

Wantzel's proof of the impossibility of trisecting the general angle with the Euclidean tools of the unmarked straightedge and compass appeared in *Journal de Mathematiques*, a publication of the mathematician Joseph Liouville (1809-1882). [Coincidentally, this was the same periodical which printed the group theory findings of Galois as mentioned above.] The proof was accompanied in the journal by a demonstration of the impossibility of the Delian construction, that of duplicating a cube. Remarkably, the tools of algebra were utilized to resolve what was perceived historically as a purely geometric problem.

Trisection and the Cubic Criterion

The critical condition employed by Wantzel focused on a cubic equation with integral coefficients and the theorem that if such a cubic equation has no rational roots, then none of its roots are constructible (with the Euclidean instruments). Using a trigonometric identity for cos 3θ, it can be shown that cos $20°$ (and hence, the $20°$ angle) is not constructible. Accordingly, the constructible $60°$ angle cannot be trisected. Interestingly, some attribute this resolution indirectly to Carl Friedrich Gauss (1777-1855). It is regarded as a corollary to his regular polygon constructibility standard as proved in part in the *Disquisitiones Arithmeticae* (1801). Should such a standard express both necessary and sufficient conditions of regular polygon constructibility, then the

regular nine-sided polygon would be shown non-constructible (and similarly for its $40°$ central angle). It would follow that the $120°$ angle could not be trisected. Nevertheless, many prominent historians do not credit Gauss with proving the necessity of the standard (today a well-known theorem), only the sufficiency.

It can be shown that n-secting the general angle is possible if and only if n is a power of 2, a result which follows immediately from Gauss's regular polygon constructibility standard. Moreover, the degree (basic unit of sexagesimal measure) proves non-constructible; otherwise, the $20°$ angle could be constructed by angle repetition.

Duplication and the Cubic Criterion

Wantzel further reduced the problem of cubic duplication to that of constructing the cube root of 2. This is precisely the length by which the edge of a given cube must be multiplied in order to effect the desired duplication. The argument parallels the demonstration above. As $x^3 - 2 = 0$ has no rational roots, none of its roots (including the cube root of 2) are constructible. This famous problem also has a far-reaching generalization. By the same cubic connection, it can be shown that a cube cannot be triplicated (nor multiplied in volume by any non-cubic integer). This is an astronomical advancement over the elementary forerunner problem of duplicating a square by the simple construction of the square root of 2.

The Cubic Criterion Conjecture

The central cubic theorem of Wantzel has thus proved a powerful tool in its attack on antiquity. But, does it admit a generalization to higher degree equations? That is, if an algebraic equation of degree 3 or more has no rational roots, does it follow that none of its roots are constructible? Consider an attempt at generalization by letting the degree of the algebraic equation be any odd integer greater than 3. For example, let the degree of the equation be 5. Also consider the equation

$$(x - \sqrt{2})(x + \sqrt{2})(x^3 - 2) = 0$$

which is

$$(x^2 - 2)(x^3 - 2) = 0.$$

This equation, $x^5 - 2x^3 - 2x^2 + 4 = 0$, clearly has integral coefficients (i.e., is algebraic) but has no rational roots. It is also of degree 5. Yet $\sqrt{2}$ is constructible. It is the hypotenuse of a right triangle with each leg of unit length.

By counterexample, the generalized conjecture above is thus false. Nor can one find select integral exponents n greater than 3 for which the algebraic equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

never has constructible roots in the event none of its roots are rational. Note that

$$(x - \sqrt{2})^n(x + \sqrt{2})^n(x^3 - 2) = 0$$

is an equation of degree 2n + 3, and thus ranges in degree over all odd integers greater than 3. Rejection of the even case follows in a similar manner by use of the equation

$$(x - \sqrt{2})^n(x + \sqrt{2})^n = 0$$

as n ranges over the positive integers.

Clearly, $\sqrt[4]{2}$, $\sqrt[8]{2}$, etc. are all constructible by repeated square root extraction. Note too that $x^4 - 2 = 0$ has no rational roots, yet all of its real roots are constructible. A contradiction would thus appear at such points should one attempt to apply the erroneous generalization above.

The rejected conjecture may otherwise have forced the geometer to look beyond the three dimensions of familiar Euclidean space. The renowned problem of cube duplication, fully resolved, would have been left far behind in a quest for a higher order dimensional variant on the Delian problem and the constructing of the n^{th} root of 2. It is difficult to imagine such a modified problem in a fifth (or higher) dimensional setting. Analysis raises various questions of hyper-space and the counterpart in extended geometric settings to the conventional cube of Euclid's geometry.

Quadrature and the Algebraic Equation Connection

The remarkable year of 1837 witnessed the resolution of two longstanding problems. Disposition of the third famous problem, that of squaring the circle, occurred in 1882. It too was a consequence of advancements in the theory of equations. The cubic connection however was not the sole means to this end.

Essentially, the problem reduced to a demonstration of the non-constructibility of π. Charles Hermite (1822-1901) proved in 1873 that the natural logarithmic base e cannot occur as a root of an algebraic equation (and is thus transcendental). Building on this result, C. Ferdinand Lindemann (1852-1939) just nine years later verified that π is also transcendental. As no transcendental length is constructible, the famous quadrature construction, that of squaring the circle, was shown impossible.

An interesting cubic connection surfaces however. Disposition of the angle trisection problem hinged on showing that cos 20°, though algebraic, is not constructible. Similarly, disposition of the cube duplication problem rested on showing that the cube root of 2, also algebraic, is not constructible. However, the number π falls completely outside the realm of algebraic numbers. It cannot occur as the root of the cubic (algebraic) equation or any other such equation (whatever its degree). This classification of π identifies the impossibility of cubing the sphere, a three-dimensional variation on squaring the circle. [Note: if π is not constructible, neither is $\sqrt[3]{\pi}$. As multiplication of segments is a valid Euclidean construction, it follows that the constructibility of $\sqrt[3]{\pi}$ would imply the constructibility of π.]

The problems of antiquity continue to fascinate the mathematician of today. By means of further abstraction and generalization, implications far transcending the geometry of the ancient world come more clearly into focus. Included among them are such pursuits as asymptotic constructions, famous problems in a non-Euclidean setting (such as the squaring of a circle in spherical geometry), higher dimensions, or the Morley Triangle Theorem and its abundant angle trisectors. The tools of disposition include algebraic approaches as symbolized by cubic connections and the works of Wantzel. Techniques thus extend from the theory of equations and group theory notions to higher forms of analysis. All convey a picture of a remarkably unified field of study.

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Change Ringing: A Connection Between Mathematics and Music

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The object of change ringing is to ring a given set of bells through all possible permutations while moving methodically from one combination to another without repetition of any sequence. There are certain rules that must be followed. Change ringing is connected to mathematics as changes can be written as transpositions of bells in adjacent positions. In addition, changes may be mapped onto a Hamiltonian circuit. This article discusses the breakdown of the permutations into right and left cosets of the nth dihedral group, a subgroup of the symmetric group of permutations on n letters.

OVERVIEW

Bell ringing may be seen as an art by most people, but to change ringers it is also a science with a basis in mathematics. Change ringing involves a set of bells that are rung through all possible permutations (Tufts, 1961). The main objectives are to:

(1) ring all possible sequences of a given set of bells;
(2) move methodically from one combination to the next;
(3) avoid repetition of any sequence. (DeSimone, 1992)

Fabian Stedman, a Cambridge printer, investigated the possibilities of change ringing (Camp, 1974) and published its formal rules in 1668 in his Tintinnalogia (The Art of Change Ringing) (White, 1987). These rules include:

(1) the first and last change are both rounds (ringing the bells in order from the highest to the lowest);
(2) no other change is repeated;
(3) no bell moves more than one position from one change to the next;
(4) no bell remains in the same position for more than two consecutive changes;
(5) each bell does the same amount of work;
(6) the method used is palindromic. (White, 1983)

Change ringing societies began to form in the 1400's and competitions used to be quite popular (Hatch, 1964). Churches had peals of bells, thus allowing

the bells to be used for both religious purposes and change ringing (Coleman, 1971).

"The church bell is usually hung in a frame on an axle. The bell is rung by pulling on a rope that goes around a wheel fastened securely to the axle" (Hatch, 1964, p. 32). Such a setup allows control over when the bell is rung in relation to others and thus is suitable for change ringing. There are change ringing teams that practice and perform, with generally one person for each bell and at times an additional person to conduct (P. Price, 1983). The ringers are allowed no memory aids to assist in their ringing (White, 1983). The purpose of the conductor, therefore, is to instruct modification of the ringing pattern at the appropriate time (White, 1987).

Change ringing is almost exclusively an English sport. Because of this England has been called the "Ringing Isle." There are a few churches that change ring in Australia, Africa, Canada, and the United States. All the bells used, however, were imported from England (Camp, 1974).

In the Eighteenth Century, change ringing in England ranked as a popular sport with both hunting and football. It is used as both a form of entertainment and exercise. Change ringing uses the same number of muscles as rowing. While it is tiring, it does not overtax the muscles (Coleman, 1971).

Since change ringing is also a sport that can cause thirst, churches used to provide jugs of beer in the bell tower to sustain the ringers (P. Price, 1983). The presence of refreshment is often necessary as peals can take some time to perform. In common, seven bells are used and the resulting 5040 changes take slightly more than three hours to ring. In 1767, a peal on eight bells lasted twenty-seven hours (Tufts, 1961). This record was broken in 1923 when a peal was rung at the Church of St. Chad in Chesire, England by the Chester Diocesan Guild on eight bells. The peal of Kent Treble Bob Major rang 17,280 changes and took ten hours to complete. A ten bell peal (3,556,800 changes) is estimated at 105 days of continuous ringing. A peal on twelve bells (479,001,600 changes) would take 137 years to ring (Hatch, 1964).

GENERAL INFORMATION

A full peal on n bells produces n! changes. These changes can be produced in many different ways. The easiest way to begin creating changes is to use a plain hunt. In the plain hunt, the first or lead bell moves through each position from one change to the next until it reaches the back position or is behind where it remains for another change. It then returns to the lead by reversing its path. For n bells, the plain hunt produces 2n changes with each bell ringing twice in every position (Wilson, 1965).

The speed (or frequency of repetition) with which a bell is rung varies depending on where it is within the hunt. When it is hunting up from the lead, there are n other bells rung between each strike. While going down to the lead, n-2 bells are struck between rings of the same bell. Thus, a bell hunting down moves "faster" than a bell hunting up from the lead. When rounds are rung over and over (round ringing), there are n-1 bells ringing between each strike so a bell in round ringing moves with intermediate speed (Wilson, 1965).

The plain hunt on three bells produces all the possible changes (2n = n! for n = 3 only). The resulting changes with the first bell hunting are as follows:

123
213
231
321
312
132

A return to rounds (123) at this point would complete the peal (DeSimone, 1992).

For groups of more than three bells it is necessary to vary the plain hunt in order to avoid a premature return to rounds and produce more of the possible changes. An example of the types of variations is easily illustrated by the four bell case. With four bells, the variations used are called making second place (a specific bell rings twice in the second position) and dodging (two bells interchange places in ringing order from one change to the next). Beginning as always with rounds, the four bells are taken through a plain hunt by the lead bell. When the lead bell returns to the first position, it remains there for the next change while the second bell makes second place. The third and fourth bells dodge at this point. The resulting change is called a lead end and is produced in order to avoid returning to rounds before all the possible changes have been rung. Then, the first bell plain hunts once again. Second place is made and the last two bells dodge at the hunt's conclusion to make the second lead end. This process is repeated a third time to return to rounds and complete the full peal of 4! or 24 changes (Wilson, 1965).

This example shows that there are n-1 lead ends produced by this type of method on n bells. For four bells the method is called Plain Bob Minimus and on more than four (such as six) bells it is Plain Bob Minor. With six bells, five leads are produced with the fifth and sixth bells dodging at the same time as the third and fourth. In the Plain Bob method, n-1 of n bells are working bells (bells that dodge and make place in addition to plain hunting) which corresponds to the number of lead ends produced (Wilson, 1965).

The Plain Bob Minor method on six bells produces five leads of twelve changes each or only sixty of the full 720 changes associated with a full peal on six bells. Therefore, other variations are necessary to obtain the other 660 changes. Bobs and singles are two other variations which can be used to procure more changes.

The bob alters the position of three bells. The lead bell remains in the lead while bells two and three continue in their respective hunting courses. The bell in position four remains there, making fourth place. The remaining bells (five and six for the six bell case) dodge in pairs (Wilson, 1965).

The use of bobs in Plain Bob Minor on six bells, however, produces a maximum of 360 out of the possible 720 chances. The use of singles increases this number. The single affects only two bells and has the second, third, and fourth bells making place. The lead bell remains at the front and the remaining bells (five and six in this example) dodge in pairs. As the peal continues, the third place bell will hunt to the back without going to lead (Wilson, 1965).

THE MATHEMATICS OF CHANGE RINGING: GROUP THEORY

A permutation is a rearrangement of elements in a set. It must be both one-to-one and onto. A function mapping elements from one set to another is called one-to-one if distinct elements in the first set are assigned to distinct elements in the second set. A function is onto if for all elements in the second set, a corresponding element in the first set can be found.

The set of all permutations on a given set with n elements is called the symmetric group on n letters and generally denoted by S_n. This group has n! elements. The symmetric group has many subgroups, one of which is the dihedral group denoted by D_n. D_n is the nth dihedral group and contains as elements the symmetries of a regular n-gon. The order of dihedral groups is 2n whereas the order of the symmetric group is n!. These two orders will be equal only for n = 3 (Fraleigh, 1994).

If H is a subgroup of a given group G, then the left and right cosets of H are subsets of G defined as follows: for left cosets, a given element of G is multiplied on the left of every element in H to generate the new subset ($aH = \{ah|h\in H\}$). Likewise, right cosets are generated by multiplying every element in H on the right with a given element from G ($Ha = \{ha|h\in H\}$).

Change ringing can be used to generate dihedral groups out of a set of permutations. D_n is called the hunting group and is generated by a simple plain hunt. It gives a block of 2n changes. A move such as making second place and dodging remaining bells is used to get out of the hunting group and link various hunting groups together (White, 1983).

Table 1: The set of changes on four bells

1	2	3	4	1	3	4	2	1	4	2	3
2	1	4	3	3	1	2	4	4	1	3	2
2	4	1	3	3	2	1	4	4	3	1	2
4	2	3	1	2	3	4	1	3	4	2	1
4	3	2	1	2	4	3	1	3	2	4	1
3	4	1	2	4	2	1	3	2	3	1	4
3	1	4	2	4	1	2	3	2	1	3	4
1	3	2	4	1	4	3	2	1	2	4	3
								1	2	3	4

Transposition Rule: $[(AB)^3(AC)]^3$
with A = (12)(34) B = (23) C = (34)

Table 1 shows the full set of changes on four bells. The first column of eight changes represents the dihedral subgroup D_4, also called the octic group, which holds the symmetries of a square (Fraleigh, 1994). We use transposition C to get out of that group into two more subgroups before returning to rounds and completing the sequence. If the first eight changes are denoted as the set H, a subset of S_4, then the second column of eight changes forms the right coset of H using (243) as the element from S_4. Likewise, the final eight changes form the right coset of H using the element (234) from S_4 (Budden, 1972). If the entire group of changes on four bells is denoted as S_4, then we can represent the changes on four bells as a decomposition into the following disjoint right cosets:

$S_4 = D_4 \cup D_4(243) \cup D_4(234)$. In fact, the methods used by change ringers decompose symmetric groups into LaGrange cosets (Fletcher, 1956). Change ringers were doing this type of coset decomposition into symmetric groups a full century before LaGrange (White, 1983). Similarly, change ringing on five or more bells can be represented as right cosets of the initial dihedral subgroup formed using the plain hunt.

From Table 1, we see that $(AB)^3AC = (243)$ corresponds to the change 1342 at the top of the second column. If we denote 1342 by w and recall that the first eight changes on four bells (Table 1, first column) represent D_4, then the two subsequent sets of eight changes can be denoted by the left cosets wD_4 and w^2D_4. The interested reader may find a discussion of left cosets in Budden's The Fascination of Groups.

The transpositions denoted by A = (12)(34), B = (23), and C = (34) for the four bell case are also used as the generators for the symmetric group on four letters. The word denoted by $w = (AB)^3AC$ is a plain lead end as it takes the bell in the treble position from the lead and returns it there after a number of changes. It can be used successively to form the plain course of the extent. Such a course is denoted by w^3 = rounds for the plain bell case. In general w^{n-1} will give n - 1 plain leads on n bells (White, 1987).

CONCLUSION

The evolution of bells from hand held rattles to the modern day inverted cup shape mounted on a wheel led to many interesting discoveries not only in bell making, but also in bell ringing. Mounting the bell on a full wheel instead of a half wheel allowed the ringer to use a rope and have better control over both the swinging of the bell and the speed of such a swing. Because of this, it was found that a bell could be kept upright and therefore allowed different ringers to change the place in which their bell sounded. This led directly to the art and science of change ringing.

As bells are rung through their various changes according to the rules of change ringing, various mathematical concepts take shape. Perhaps the most interesting mathematical connection to change ringing is in the area of group theory. Changes can be used effectively to find dihedral subgroups of the symmetric group on n letters as well as produce left and right cosets of various groups within the set of all permutations on n letters.

There are other mathematical connections not discussed in this article.

Since bells may move only one place in ringing order, changes may be represented easily as a string of transpositions which correspond to pairs of bells which swap places. In addition, change ringing also provides an algorithm by which a Hamiltonian circuit can be found on a Cayley graph. It is one of the few ways in which such a circuit can be found quickly and simply.

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The Periods of the Digits of the Fibonacci Numbers

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The present paper is a record of the work in the Junior Seminar of the Department of Mathematics and Computer Science at Seton Hall University in the Spring Semester of 1996. The pariticipants were Jennifer Aigner, Patricia Hegarty, Thin Duc Nguyen, Deanna L. Rusnak, Keith Sadlowski, Daniel Stewart, Amy Troy, and Sherwood Washburn.

The topic of the Seminar was the paper by Dov Jarden, "On the Periodicity of the Last Digits of the Fibonacci Numbers", in the Fibonacci Quarterly [4]. In this paper Jarden stated the Theorem below without proof, and the Seminar was devoted to working out the proof.

The paper of Wall [6] is an excellent reference for the sort of questions we shall discuss. One could also consult [1], [2], [3], and [5].

This paper was much improved by the comments of the referee, who deserves my warmest thanks, and the thanks of all the participants in our Seminar.

The *Fibonacci sequence* is defined as follows: the initial values are

$$F_0 = 0 \text{ and } F_1 = 1$$

with the recursion

$$F_n = F_{n-1} + F_{n-2}$$

if $n \geq 2$.

Here is the statement of Jarden's Theorem:

Theorem 1. The period of the last digit of the Fibonacci numbers is 60; the period of the last two digits is 300; and if $d \geq 3$, the period of the last d digits is $15 \cdot 10^{d-1}$.

The following Theorem was proved by Lagrange.

Theorem 2. The Fibonacci sequence in Z_n is periodic, for every positive integer n.

The Theorem is easily proved by applying the Pigeon-Hole Principle to consecutive pairs of congruence classes in the Fibonacci sequence to show that some pair of consecutive classes in Z_n must recur, and then running the recursion forward and backwards.

Wall [[6], Theorem 1, 525-526] gives the same proof of this result. Let us compute the Fibonacci sequence in Z_2, Z_3, and in Z_4.

The Fibonacci sequence in Z_2: 0, 1, 1, 0, 1, 1, ... The period in Z_2 is 3.

The Fibonacci sequence in Z_3: 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, ... The period in Z_3 is 8.

The Fibonacci sequence in Z_4: 0, 1, 1, 2, 3, 1, 0, 1, 1, ... The period in Z_4 is 6.

In general we may ask

What is the period of the Fibonacci sequence in Z_n?

Our question is

What is the period of the last d digits of the Fibonacci numbers?

Or equivalently,

What is the period of the Fibonacci numbers in Z_{10^d}?

Recall the

Chinese Remainder Theorem 3. If m and n are positive integers and if g.c.d.(m, n) = 1, then the mapping $Z_{mn} \to Z_m \times Z_n$ which sends [k] to ([k], [k]) is a ring isomorphism.

Lemma 4. If A(n) is the period of the Fibonacci sequence in Z_n, and if

$$n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$$

is the prime factorization of n, then

$$A(n) = \text{l.c.m.}\left\{A\left(p_1^{s_1}\right), A\left(p_2^{s_2}\right), \ldots, A\left(p_r^{s_r}\right)\right\}$$

Wall [[6], Theorem 2, p. 526] gives a proof of this result.

In the problem of finding the periods of the digits we are concerned with finding the periods of the digits modulo powers of ten, so our problem can be reduced to the following:

What is the period of the Fibonacci sequence in Z_{2^d} and in Z_{5^d}?

We can solve this problem in two steps if we make a simple observation about the period of the Fibonacci sequence in Z_n for any positive integer n: namely that the elements of Z_n in a period can be arranged in the following way:

$$\begin{array}{cccccc}
0 & 1 & 1 & 2 & \cdots & m \\
0 & m & m & 2m & \cdots & m^2 \\
0 & m^2 & m^2 & 2m^2 & \cdots & m^3 \\
\cdots & & & & & \\
0 & m^{r-1} & m^{r-1} & 2m^{r-1} & \cdots & m^r = 1 \\
0 & 1 & & & &
\end{array}$$

We shall call the integer m the *multiplier* of the Fibonacci sequence in Z_n. Notice that m must be a unit in Z_n, and if r is the order of m in Z_n^*, and if a(n) is the length of any row in the array above, then the period of A(n) satisfies the equation $A(n) = r \cdot a(n)$.

The concept of the multiplier seems to be due to Carmichael [[2], pp. 354-355], who also proved the formula which we have just stated.

We shall use this equation to compute the period. First consider two representative examples.

The Fibonacci sequence in Z_8:

0	1	1	2	3	5
0	5	5	2	7	1
0	1				

The Fibonacci sequence in Z_5:

0	1	1	2	3
0	3	3	1	4
0	4	4	3	2
0	2	2	4	1
0	1			

Notice that each of these arrays has the general form which we have described, and also that the multiplier in Z_8 has order 2 and that the multiplier in Z_5 has order 4. The period in Z_8 is 12 and the period in Z_5 is 20.

Using the isomorphism

$$Z_{10} \to Z_2 \times Z_5$$

and the facts above we see that the period of the Fibonacci sequence in Z_{10} is

$$\text{l.c.m.}(3,20) = 60$$

In general, we can find the period of the Fibonacci sequence in Z_n for any n if we can solve two problems:

<u>The Divisibility Problem</u>:

Given a positive integer n, what is the smallest positive integer a(n) such that n divides $F_{a(n)}$?

<u>The Multiplier Problem:</u>

If m is the multiplier for the Fibonacci sequence in Z_n, what is the order r of m in Z_n^*?

Before we solve these two problems in the case where n is a power of 2 or a power of 5, and hence solve the problem of the digits, let us find the period of the first two digits of the Fibonacci numbers. We will use the isomorphism

$$Z_{100} \to Z_4 \times Z_{25}$$

and since we know that the period in Z_4 is 6 we need the period in Z_{25}. In Z_{25} we have

0	1	1	2	3	5	8	13	21	9
5	14	19	8	2	10	12	22	9	6
15	21	11	7	18	0	18	18	11	4
15	19	9	3	12	15	2	17	19	19
5	16	21	12	8	20	3	23	1	24
0	24	24	23	22	20	17	12	4	16
20	11	6	17	23	15	13	3	16	19
10	4	14	18	7	0	7	7	14	21
10	6	16	22	13	10	23	8	6	14
20	9	4	13	17	5	22	2	24	1
0	1								

We see that the period in Z_{25} is 100. Therefore the period in Z_{100} is

$$\text{l.c.m.}(6,100) = 300$$

Solutions to the Divisibility Problem and the Multiplier Problem can be found in the papers of J.H. Halton [3] and J. Vinson [5]. We shall give our own solutions to these problems, first in the case of a power of 2. We have given solutions for n = 2 and n = 4 above, so assume that

$$n = 2^d \text{ where } d \geq 3$$

The solution is given by the following Theorem.

Theorem 5. If $d \geq 3$ then

(i). 2^d divides $F_{3 \cdot 2^{d-2}}$, 2^{d+1} does not divide this Fibonacci number, and 2^d does not divide any smaller Fibonacci number.

(ii). The multiplier for the Fibonacci sequence in Z_{2^d} is congruent to $2^{d-1} + 1 \pmod{2^d}$.

In order to prove this Theorem we need to recall a few facts about Fibonacci numbers and several identities.

First recall that the *Lucas numbers* are a companion sequence to the Fibonacci numbers. The Lucas numbers are defined by the initial values

$$L_0 = 2 \text{ and } L_1 = 1$$

and the recursion

$$L_n = L_{n-1} + L_{n-2}$$

if $n \geq 2$.

The Fibonacci numbers and the Lucas numbers are expressed by the following.

Binet's Formula 6. If $\phi = \frac{1+\sqrt{5}}{2}$ and $\phi' = \frac{1-\sqrt{5}}{2}$, then

$$F_n = \frac{(\phi)^n - (\phi')^n}{\sqrt{5}}$$

$$L_n = (\phi)^n + (\phi')^n.$$

The following two formulas follow easy from Binet's Formula.

$$F_{2n} = F_n L_n$$

$$L_n^2 - 5F_n^2 = (-1)^n \cdot 4$$

We will need one other formula:

$$F_{g.c.d.(m,n)} = g.c.d.(F_m, F_n)$$

Carmichael [[1], Theorem VI, p. 38] gave a proof of this.

Now assume that $d \geq 3$. First we must prove that 2^d divides $F_{3\cdot 2^{d-2}}$, and that the same Fibonacci number is not divisible by 2^{d+1}, and we shall prove this by induction on d. Notice that $F_6 = 8$: this starts the induction.
Now

$$F_{3\cdot 2^{d-1}} = F_{3\cdot 2^{d-2}} \cdot L_{3\cdot 2^{d-2}}$$

By induction 2^d divides the Fibonacci number on the right hand side, and 2^{d+1} does not. The identity

$$L_n^2 - 5F_n^2 = (-1)^n \cdot 4$$

with $n = 3 \cdot 2^{d-2}$ shows that L_n is divisible by 2 but not by 4, and this completes the proof by induction.

We must also prove that 2^d does not divide F_n for any index smaller than $n = 3 \cdot 2^{d-2}$. To prove this, apply the identity

$$F_{g.c.d.(m,n)} = g.c.d.(F_m, F_n)$$

with $m < 3 \cdot 2^{d-2}$ and $n = 3 \cdot 2^{d-2}$. Then the conclusion is obvious by induction if 3 divides g.c.d(m, n). If not, a separate inductive argument, using the identities above, proves that F_{2^d} is always odd and this finishes the solution of the Divisbility Problem for powers of two.

Finally we must prove that if $d \geq 3$, then the multiplier in Z_{2^d} is congruent to $2^{d-1} + 1 (\mathrm{mod}\ 2^d)$. Notice that this implies that the multiplier has order in 2 in Z_{2^d}.

Once again we start the induction with $d = 3$, and since we have seen that the multiplier is $5 = 1 + 2^2$, this starts the induction. Now assume the statement by induction: the binary expansion of the multiplier is

$$m = 1 + 2^{d-1} + \dots$$

Now remember that in $Z_{2^{d+1}}$

$$m = F_{3\cdot 2^{d-1}-1} = F_{3\cdot 2^{d-1}+1}$$

Using the identity

$$F_{2k+1} = F_k^2 + F_{k+1}^2$$

We have

$$F_{3\cdot 2^{d-1}+1} = F^2_{3\cdot 2^{d-2}} + F^2_{3\cdot 2^{d-2}+1} \equiv (2^d)^2 + (2^{d-1}+1)^2 \equiv 0 + 2^d + 1 (\mathrm{mod}\ 2^{d+1})$$

which completes the proof.

In paticular we have proved that for $d \geq 3$,

if $n = 2^d$, then $A(n) = 2a(n) = 3 \cdot 2^{d-1}$.

It remains to solve the Divisibiliy Problem and the Multiplier Problem for powers of 5.

Theorem 7. For all positive integers d

(i). The Fibonacci number F_{5^d} is divisible by 5^d but not by 5^{d+1}, and 5^d does not divide any smaller Fibonacci number.

(ii). The multiplier m for the Fibonacci sequence in Z_{5^d} has order 4.

To prove the first part of this Theorem we will use the Second Form of Binet's Formula, which follows easily by applying the Binomial Theorem to Binet's Formula as stated above.

Binet's Formula, Second Form 8. We have

$$F_n = 2^{-n+1}\left(\binom{n}{1} + \binom{n}{3} \cdot 5 + \binom{n}{3} \cdot 5^2 + \dots \right).$$

We shall use this formula with $n = 5^d$: clearly the first term of the sum is equal to 5^d. A formula of Lucas implies that if $0 < k < 5^d$ then the binomial coefficient $\binom{5^d}{k}$ is divisible by 5^{d-e}, where $e = \lfloor k/5 \rfloor$, and this implies that all the terms of the above sum except for the first are divisible by 5^{d+1}. It follows that F_{5^d} is divisible by 5^d but not by 5^{d+1}.

Using the formula

$$F_{g.c.d.(m,n)} = g.c.d.(F_m, F_n)$$

with $n = 5^d$ and $1 \leq m < 5^d$ we see that F_m cannot be divisible by 5^d.

To prove that the order of the multiplier m in Z_{5^d} is 4, consider the identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

We have proved that $5^d \mid F_{5^d}$, and it follows that $5^d \mid F_{2\cdot 5^d}$. Now use the identity, with $n = 2 \cdot 5^d$, and notice that $m^2 = F_{2\cdot 5^d - 1} = F_{2\cdot 5^d + 1}$. It follows that

$$m^4 = F^2_{2\cdot 5^d - 1} \equiv 1 (\bmod 5^{2d})$$

so

$$m^2 \equiv \pm 1 (\bmod 5^d)$$

To show that the order of m is 4, we must prove that the sign is -1. Remember that the order of m is Z_5 is 4, and consider the exact sequence

$$0 \to (5)Z_{5^d} \to Z_{5^d} \to Z_5 \to 0$$

The definition of the multiplier involves only additions and multiplications, and so the multiplier is preserved by the maps in this sequence. Since the order of the multiplier in Z_5 is 4, it must be 4 in Z_{5^d}. This completes the proof.

Now suppose that $d \geq 3$. Then

$$A(10^d) = \text{l.c.m.}(A(2^d), A(5^d)) = \text{l.c.m.}(3 \cdot 2^{d-1}, 4 \cdot 5^d) = 15 \cdot 10^{d-1}$$

Finally let us remark that checking this formula for small values of d makes an excellent computer exercise.

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Rearrangement of Series

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I. Introduction.

Commutativity of addition for a finite number of summands is a fundamental property of real numbers. It was therefore naturally assumed by mathematicians that commutativity of addition also held for infinitely many summands, i.e., infinite series. In other words, it was commonly believed that no matter how the terms of an infinite series were rearranged, the sum would not be altered. As G. Aurello noted in her article "On the Rearrangement of Infinite Series" [1, p. 641], this assumption was shown to be false by A. Cauchy in 1833. Since that time, the ideas and generalizations concerning rearrangements of the terms of an infinite series have piqued the interest of mathematicians.

B. Riemann proved what some consider to be the ultimate result on rearranging series of real numbers. In a paper published in 1866 on representing functions by trigonometric series, Riemann demonstrated that every conditionally convergent infinite series of real numbers can be rearranged to converge to any real number [8, p. 97]. For a concrete example of this, consider rearranging the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

to converge to an arbitrary number, say 2. One simply adds positive terms of the series until the sum is greater than 2, then adds on negative terms until the sum is less than 2, etc. The reader can verify that the following rearrangement of the alternating harmonic series is converging to 2:

$$\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{62}\right) + (-1) + \left(\frac{1}{64} + \cdots + \frac{1}{454}\right) + \left(\frac{-1}{3}\right) + \cdots .$$

These results are summarized in the following theorem (Rearrangement theorem for real numbers).

Theorem 1. Let $\sum_{k=1}^{\infty} x_k$ be a series of real numbers. The set of all sums of rearrangements of $\sum_{k=1}^{\infty} x_k$ is either Ø (the empty set), a single real number, or all real numbers.

It took 39 years for a similar result to be obtained for infinite series of complex numbers. P. Levy proved the result in a paper published in 1905. E. Steinitz proved the result for rearranging infinite series of vectors in finite dimensional spaces in a paper published in 1913.

While Theorem 1 is generally known to mathematicians, the post-1866 results on rearranging infinite series are not as widely known. P. Rosenthal published an article in 1987 on the results of Levy and Steinitz with the intention of making them more well-known in mathematical circles [9, p. 342]. He presents a very polished and refined version of their results as stated in the following theorem (Rearrangement theorem for finite-dimensional spaces), [9, p. 342].

Theorem 2. Let X be a finite-dimensional vector space and $\sum_{k=1}^{\infty} x_k$ an infinite series of vectors from X. Then the set of all sums of rearrangements of $\sum_{k=1}^{\infty} x_k$ is either Ø, a linear manifold, or the whole space X.

However, this result is somewhat unmotivated, and Rosenthal's work leaves the following questions unanswered: How did Levy and Steinitz approach the problem of rearranging infinite series? Is there any connection between their works? The purpose of this paper is to answer these questions by examining the original works of Levy and Steinitz to determine how they arrived at their results and to explore the connection between them.

II. Levy's Results

Levy's results on conditionally convergent series were published in 1905. In the first paragraph of his paper, he states that it is known that a conditionally convergent series of complex numbers may take on an infinite number of values when the terms are rearranged. Levy then raises the question of whether a series of complex numbers may be rearranged to have any complex number as its sum [6, p.506] and essentially proves the following result.

Theorem 3. Let $\sum_{k=1}^{\infty} x_k$ be an infinite series of complex numbers. Then the set of all sums of rearrangements of $\sum_{k=1}^{\infty} x_k$ is either Ø, a single complex number, a line in the complex plane, or the entire complex plane.

Levy alludes to the algorithm in Riemann's theorem for rearranging series by distinguishing the terms with positive real parts from those with negative real parts. Noting this technique is not applicable to complex numbers, he carries the idea one step further. He decomposes the original series into three partial series and sets up a procedure based on a special case [6, p. 507].

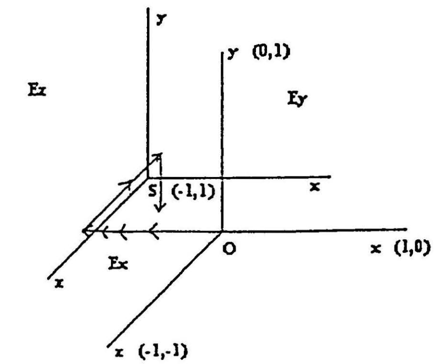
The following is an example applying Levy's procedure modified for $\mathbb{R}^2$, i.e., representing complex numbers as vectors in $\mathbb{R}^2$.

Let i = (1, 0) and j = (0, 1) and consider the following series:

$$\sum_{n=1}^{\infty} \frac{-i}{n} + \sum_{n=1}^{\infty} \frac{i+j}{n} + \sum_{n=1}^{\infty} \frac{-j}{n}$$

Let S = -i + j = (-1, 1) be the desired sum of the series. Using Levy's approach, a rearrangement of the given series which converges to S is obtained in the following manner.

Beginning with the vectors *Ox* = (1, 0), *Oy* = (0, 1), and *Oz* = (-1, -1), three half lines are drawn originating from S, parallel to these vectors. This partitions the plane into three regions *Ex, Ey,* and *Ez* as indicated on the diagram below.



Diagram

Levy's procedure is to start with a vector from the series, say the first one parallel to *Ox* in region *Ex*, and to add on vectors parallel to *Ox* until the line of vectors formed enters another region, say *Ez*. Once in *Ez*, start adding on vectors parallel to *Oz* until another region is reached (see Diagram). Continuing in this fashion will yield a rearrangement of the original series whose sum is S = (-1, 1).

By proceeding in the manner described in the preceding paragraph, the following rearrangement of the given series is obtained:

$$-1(1,0)-\frac{1}{2}(1,0)-\frac{1}{3}(1,0)-\frac{1}{4}(1,0)-1(-1,-1)-\frac{1}{2}(-1,-1)-1(0,1)$$
$$-\frac{1}{5}(1,0)-\frac{1}{6}(1,0)-\frac{1}{7}(1,0)-\frac{1}{8}(1,0)-\frac{1}{9}(1,0)-\frac{1}{10}(1,0)-\frac{1}{11}(1,0)$$
$$-\frac{1}{3}(-1,-1)-\frac{1}{4}(-1,-1)-\frac{1}{2}(0,1)-\frac{1}{12}(1,0)-\frac{1}{13}(1,0)-\frac{1}{14}(1,0)$$
$$-\frac{1}{15}(1,0)-\frac{1}{16}(1,0)-\frac{1}{17}(1,0)-\frac{1}{18}(1,0)-\frac{1}{19}(1,0)-\cdots,$$

which appears to be converging to S = (-1, 1). Levy then uses a complicated geometric approach to show that the rearranged series actually converges to S = (-1, 1).

In the last paragraph of his paper, Levy claims that his method may be generalized to consider rearrangements of infinite series of vectors in n-dimensional space [6, p. 511]. His main idea is to decompose the series under consideration into n + 1 partial series. This corresponds to the idea of decomposing a series of complex numbers, or equivalently a series in $\mathbb{R}^2$, into three partial series. Levy does not, however, elaborate on exactly how this procedure would work.

Levy's article is very concise. It offers neither proofs nor examples, and is somewhat poorly written. Knopp [5, p. 398] and Kadets [4, p. 1] state that Levy proved the result on the description of the set of all rearrangements of a series of complex numbers but they make no mention of the fact that he at least conjectured the result for n-dimensional space. Rosenthal claims [9, p. 342] that it was Levy who first proved the result for n-dimensional space. The fact is that Levy did not prove the result, but he was at least aware of it.

III. Steinitz' Results.

Steinitz begins the introduction to his paper on conditionally convergent series in convex systems by briefly tracing the history of the study of the rearrangement of series of real numbers. He introduces the term "summation range" of a series.

Definition 1. Let X be a finite dimensional vector space. The summation range of the series $\sum_{k=1}^{\infty} x_k$, where $x_k \in X$, is the set of all vectors $x \in X$ such that $\sum_{k=1}^{\infty} x_{\pi(k)}$ converges to x where $\sum_{k=1}^{\infty} x_{\pi(k)}$ is a rearrangement of $\sum_{k=1}^{\infty} x_k$.

Steinitz then restates Riemann's result as follows: The summation range of a conditionally convergent series of real numbers is the set of all real numbers [10, p. 128]. Next he considers the summation range of a complex series and gives two brief examples in the complex plane [10, p. 128-129]. The following example, Steinitz first, demonstrates that the summation range of a complex series may be a line in the complex plane.

Let g be any line in the plane, α a point on g, and $w \neq (0, 0)$ any point on the line through the origin parallel to g. Let $\sum_{n=1}^{\infty} a_n$ be any conditionally convergent series of real numbers. Using standard vector addition it is easy to see that any number of the form $\alpha + rw$, where r is a real number, will be a point on the line g. By Riemann's theorem, the series $\sum_{n=1}^{\infty} a_n$ can take on any real value r. Hence the complex series $\alpha + a_1w + a_2w + \cdots$ can take on every complex value of the form $\alpha + rw$. Thus the line g is the summation range of the complex series.

Steinitz second example demonstrates that a series of complex numbers can be rearranged to converge to any point in the complex plane. He begins with two conditionally convergent series of real numbers $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. Again using Riemann's theorem, $\sum_{n=1}^{\infty} a_n$ can be rearranged to converge to any real number a, and $\sum_{n=1}^{\infty} b_n$ can be rearranged to converge to any real number b. Thus the series formed from the numbers $a_1, b_1i, a_2, b_2i, \cdots$ can be rearranged to converge to any point in the complex plane.

Steinitz makes two assertions in his introduction: first, that the summation range of a conditionally convergent series of complex numbers will always be either a line in the complex plane or the entire plane; second, that the summation range of a conditionally convergent series in n-dimensional space will always be a linear manifold or the whole space [10, p. 129].

Steinitz initial work was read by E. Landau who informed him of Levy's work and expressed some doubt about the correctness of Levy's results. In the introduction to his paper, Steinitz states that he arrived at his results in 1906 and was totally unaware of Levy's work [10, p.130]. This is obviously the case, as the reader will see, for his approach to determining the set of all sums of a conditionally convergent series is somewhat different from and more modern than Levy's.

Levy's work, according to Steinitz, is brief, disconnected, and unclear. He says the work is incomplete, especially in the higher dimensional cases. He acknowledges that Levy must be given credit for proving the result for conditionally convergent series of complex numbers, but says that, although Levy stated the result for series in n-dimensional space, an actual proof requires more than just generalizing his (Levy's) procedure [10, p. 130]. In section V of his paper Steinitz gives his original proof for n-dimensional space and specifically indicates which points can be deduced from Levy's work and which cannot.

In section VI of his paper Steinitz offers a proof using an entirely different approach, one which is shorter and more direct. In order to do this Steinitz develops an extensive and cumbersome amount of material on convex systems of rays and convex bodies. However, one gains some insight into his general approach when he discusses a series rearrangement theorem for a vector space of complex numbers with n units [10, p. 173]. Steinitz discusses in detail how certain conditionally convergent series of complex numbers may be rearranged to converge to any given complex number in the vector space. A slight modification of his approach leads to the following ideas for $\mathbb{R}^n$.

Let $\sum_{k=1} x_k$ be a conditionally convergent series of vectors in $\mathbb{R}^n$ with the following property. If $\Gamma = \{x_k\}_{k=1}^{\infty}$ and Γ^* is the set of all possible partial sums of elements from Γ, then Γ^* is totally unbounded, i.e., unbounded in all directions. Now let σ be any vector in $\mathbb{R}^n$. To obtain a rearrangement of $\sum_{k=1}^{\infty} x_k$ which converges to σ, Steinitz proceeds in the following manner. Define

$$g_k = \mathrm{lub}_{l \ge k}\{\|x_l\|\}.$$

Using the dyadic number system, Steinitz defines an ordering on the partial sums as follows. Given any partial sum Π, replace each summand x_k by 2^k. This determines a natural number for each partial sum, and thus also an ordering on the set of partial sums.

Using results proven earlier in his paper, Steinitz shows that based on this ordering one can obtain a partial sum Π which has x_1 as a summand and which satisfies the condition

$$\|\Pi - \sigma\| \le ng_1$$

where n is the dimension of the space. He chooses the first partial sum Π (using the ordering above) that satisfies this condition and calls it Π_1.

Now let Γ_1 denote the set of vectors which remain when the vectors occurring in Π_1 are removed from Γ. Then one can choose a partial sum Π from Γ_1^* which contains the first element from Γ_1 as a summand and satisfies the condition

$$\|\Pi_1 + \Pi - \sigma\| \le ng_2.$$

Steinitz chooses the first such partial sum (using the ordering above) and defines $\Pi_2 = \Pi_1 + \Pi$.

Continuing in this fashion yields a sequence of partial sums $\Pi_k \to \sigma$ and thus a rearrangement of $\sum_{k=1}^{\infty} x_k$ which has sum σ.

The following example demonstrates Steinitz procedure. In $\mathbb{R}^2$, let Γ be the sequence

$$(1,0),(0,1)\left(\frac{-1}{2},0\right),\left(0,\frac{-1}{2}\right),\left(\frac{1}{3},0\right)\left(0,\frac{1}{3}\right),\left(\frac{-1}{4},0\right),\left(0,\frac{-1}{4}\right),\ldots,$$

and let $\sigma = (-1, 1)$. In this case, $g_1 = 1$, so $ng_1 = 2$. We want to find the first partial sum Π such that $\|\Pi - \sigma\| \le 2$. Taking the first vector from the sequence yields $\|(1, 0) - (-1, 1)\| = \|(2, -1)\|$ which is greater than 2, so another partial sum must be found. Consider the partial sum consisting of the first two vectors in Γ, i.e., (1, 0) and (0, 1). Then $\|(1, 0) + (0, 1) - (-1, 1)\| = \|(2, 0)\| \le 2$. Since this is the first partial sum (using Steinitz dyadic ordering scheme) that satisfies the condition $\|\Pi - \sigma\| \le 2$, $\Pi_1 = (1, 0) + (0, 1)$.

Now Γ_1 is the set

$$\left\{\left(\frac{-1}{2},0\right),\left(0,\frac{-1}{2}\right),\left(\frac{1}{3},0\right)\left(0,\frac{1}{3}\right),\left(\frac{-1}{4},0\right),\left(0,\frac{-1}{4}\right),\ldots\right\}.$$

In this case, $ng_2 = 2$. If we begin by taking the first vector from Γ_1 we have

$$\left\| (1,0)+(0,1)+\left(\frac{-1}{2},0\right)-(-1,1)\right\| = \left\|\left(\frac{3}{2},0\right)\right\|$$

which is less than 2. Thus $\Pi_2 = (1,0)+(0,1)+\left(-\frac{1}{2},0\right) = \left(\frac{1}{2},1\right)$.

Continuing in this fashion, we find that

$$\Pi_3 = (1,0)+(0,1)+\left(\frac{-1}{2},0\right)+\left(0,\frac{-1}{2}\right)+\left(0,\frac{1}{3}\right)+\left(\frac{-1}{4},0\right)+\left(0,\frac{1}{5}\right)$$

$$+\left(\frac{-1}{6},0\right)+\left(\frac{-1}{8},0\right)=\left(\frac{-1}{24},\frac{31}{30}\right),$$

and Π_4 is

$$(1,0)+(0,1)+\left(\frac{-1}{2},0\right)+\left(0,\frac{-1}{2}\right)+\left(0,\frac{1}{3}\right)+\left(\frac{-1}{4},0\right)+\left(0,\frac{1}{5}\right)+\left(\frac{-1}{6},0\right)$$

$$+\left(\frac{-1}{8},0\right)+\left(\frac{1}{3},0\right)+\left(0,\frac{-1}{4}\right)+\left(\frac{-1}{10},0\right)+\left(\frac{-1}{12},0\right)+\left(\frac{-1}{14},0\right)+$$

$$+\left(-\frac{1}{16},0\right)=\left(\frac{-43}{1680},\frac{47}{60}\right).$$

IV. Conclusion.

There is a marked contrast between the works of Levy and Steinitz. Although the two arrived at the same conclusions for rearranging infinite series of complex numbers and infinite series of vectors in finite dimensional vector spaces, their approaches are quite different. Levy's approach is very geometric, using the geometry of the plane. Steinitz' approach, while also geometric, is more analytic in nature, using norms, convergence, etc. Levy proves the result for complex numbers while Steinitz fully investigates the case for finite dimensional spaces.

The works of Levy and Steinitz essentially expanded the results of Cauchy and Riemann on rearranging series. As noted previously the results on rearranging series of real numbers, complex numbers, and vectors in a finite dimensional space are completely known. The natural extension to a consideration of rearranging series of vectors in infinite dimensional spaces was begun by Orlicz and continued by Dvoretsky, Rogers and others. A perusal of the Mathematical Reviews indicates that research is continuing in this area.

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Partitions of n into Prime Parts

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The number of partitions of a number n is the number of ways that n can be divided into unordered parts that are positive integers. For example, $5 = 3 + 1 + 1$ is the same as $5 = 1 + 3 + 1$, and other partitions of 5 include 5, $4 + 1$, $3 + 2$, $2 + 1 + 1 + 1$, $2 + 2 + 1$, and $1 + 1 + 1 + 1 + 1$. To make things more interesting, one can look at specific types of partitions, such as those where all parts are odd, or all parts are distinct. One of the main things mathematicians investigate about partitions is the attempt to define for which types of partitions the number of partitions of n are equal for both types of partitions for all n. That is, they ask whether there exists a one-to-one correspondence between partitions of n from two different types of partitions (for example, the famous 18th century mathematician Euler proved a classical theorem that the number of partitions into odd parts is equal to the number of partitions into distinct parts for all n).

Generating functions are a way of representing the number of partitions. Consider the infinite series $\Sigma a_i x^i$. When each a_i is the number of partitions of i, then the series is the generating function for the number of partitions of i.

This paper investigates the consequences of restricting the partitions to prime parts, such as $5 = 5$, and $5 = 3 + 2$, but disregarding $5 = 2 + 2 + 1$, because 1 is not a prime. Four theorems are proven that demonstrate several properties of partitions of n into prime parts. At the end of the paper is a listing of all the prime partitions of the integers 2 through 11.

Theorem 1: The sequence of the number of partitions of n into prime parts is non-decreasing for all n.

Proof: For every partition of n, we will find a unique partition of $n + 1$. Write the parts in non-increasing order. Each partition of n has 2 as a part or it does not. If it does not have a 2, then add 1 to the last number, divide that number by 2, and replace the last number by that many 2's. For example, $22 = 7 + 7 + 5 + 3$ ------ $23 = 7 + 7 + 5 + 2 + 2$.

Note that the new partition has at least two 2's and if the new partition has a 3, then it has exactly two 2's. If the partition has one or two 2's, then change the last 2 into a 3. Since there is at most one 2, this is a unique partition. If the

partition has exactly three 2's, then change two of the 2's into a 5, and keep the last 2. Again, as it has only one 2, thus it is a unique partition. Finally, if there are four or more 2's, then change a 2 into a 3. There are at least three 2's, hence the partition is again unique.

A problem arises, however, when you consider the case where your partitions of n include $\ldots 5 + 3 + \ldots + 3 + 2 + 2$ and $\ldots 3 + \ldots + 3 + 2 + 2 + 2$, as these will map to the same thing. However, this is solved by the following trick. In the case where you have $\ldots 3 + \ldots + 3 + 2 + 2 + 2$, let the number of times 3 appears be called y. Either y is even or it is odd. If y is odd, then $3 + \ldots + 3 + 2 + 2 + 2$ is odd and divisible by three, thus not a prime number. Add one to this number, and then divide it into a bunch of 2's. For example, $\ldots + 3 + 3 + 3 + 3 + 3 + 2 + 2 + 2$ would map to $\ldots + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$. Since if there is anything in the ... (where ... denotes the list of the numbers preceding those written explicitly) of this partition of $n + 1$ it is at least 5, this is guaranteed not to be generated by one of our previous cases. In the case where the ... is empty, we still have a unique partition because the only numbers that will map to a partition of all 2's are prime, and $3 + 3 + 3 + 3 + 3 + 2 + 2 + 2$ is not prime. If y is even, then $y < 5$ or $y > 5$. If $y > 5$, then map $\ldots + 3 + \ldots + 3 + 2 + 2 + 2$ to $\ldots + 7 + 5 + 2 + .. + 2$. This is guaranteed to generate a unique partition of $n + 1$ because 5 will only appear followed by one or two 2's in the absence of 3's. If $y < 6$, then $y = 2$ or $y = 4$. If $y = 2$, then map $\ldots + 3 + 3 + 2 + 2 + 2$ to $\ldots + 11 + 2$, which is not generated in any of the above cases. Likewise, when $y = 4$, map $\ldots + 3 + 3 + 3 + 3 + 2 + 2 + 2$ to $\ldots + 17 + 2$, which is not generated by any of the above cases. Note that every new partition has at least one 2 or one 3.

Theorem 2: For $n > 8$, the sequence of the number of partitions of n into prime parts is strictly increasing.

Proof: The partitions of 8 are $5 + 3$, $3 + 3 + 2$, and $2 + 2 + 2 + 2$. The partitions of 9 are $5 + 2 + 2$, $3 + 3 + 3$, $3 + 2 + 2 + 2$, and $7 + 2$. Hence, there are more partitions of 9 then of 8. For $n > 9$, we will find a partition of n that we have not observed from $n - 1$ by Theorem 1. To do this, we will find partitions without 2's or 3's. For $10 \leq n \leq 14$, $10 = 5 + 5$, $11 = 11$, $12 = 7 + 5$, $13 = 13$, and $14 = 7 + 7$.

For $n > 14$, add an appropriate number of 5's to one of the above partitions.

We can do this for any number, as the above numbers are congruent to 0, 1, 2, 3, and 4 modulo 5. As none of these partitions has a 2, or a 3, it is a partition we did not form from $n - 1$ by Theorem 1. Therefore, for $n > 8$, the sequence of the number of partitions of n is strictly increasing.

Theorem 3: The sequence of the number of partitions of n into prime parts increases by arbitrarily large amounts.

Proof: We will derive an algorithm for finding a number n such that the number of partitions of n into prime parts is at least k greater than the number of partitions of $n - 1$ into prime parts.

Consider multiples of 35. There are several partitions without any 2's and 3's: 5 + 5 + 5 + 5 + 5 + 5 + 5, 7 + 7 + 7 + 7 + 7, 19 + 11 + 5, 17 + 13 + 5, 13 + 7 + 5 + 5 + 5, 11 + 7 + 7 + 5 + 5 ... . If we let $h = [n/35]$, where $[x]$ is the greatest integer less than or equal to x, there are at least $6h$ partitions that did not come from the techniques of Theorem 1. Hence if you want a jump of at least k in the sequence of partitions into prime parts, consider $n = 35([k - 1)/6] + 1)$. Note that this means there is a minimum difference in the number of partitions of n and $n - 1$. In other words, there is a minimum number m where, or all numbers greater than or equal to m, the difference is arbitrarily large.

Theorem 4: The number of partitions of n into prime parts is equal to the number of partitions of n into parts that are powers of primes, p^k, repeated no more than $p - 1$ times. (For example, 2, 4, 8, 16, ... are powers of 2 repeated no more than 1 time; 3, 9, 27, 81, ... are powers of 3 repeated no more than 2 times; etc.)

Proof: Examine partitions of the form n into parts that are powers of primes, p^k, repeated no more than $p - 1$ times. To find the corresponding partition into purely prime parts, take every power p^k, and express it in its prime factorization. Then, write this as a sum of the prime. For example, 25 = (5)(5) = 5 + 5 + 5 + 5 + 5. This gives you a unique partition of n into purely prime parts. This is unique because prime factorizations are unique. The same process works backwards. For example, 2 + 2 + 2 + 2 + 2 + 2 = (2)(2)(2) + (2)(2) = 8 + 4 since 12 can be written in the binary system as a sum of powers of 2 in only one way.

Alternate Proof: The generating function for the number of partitions of n into parts that are powers of primes, p^k, but no part repeated more than $p - 1$ times is given by

$$(1+x^2)(1+x^3+x^6)(1+x^4)(1+x^5+x^{10}+x^{15}+x^{20})\cdots = (1+x^2)(1+x^4)(1+x^8)\cdots(1+x^3+x^6)(1+x^9+x^{18})(1+x^{27}+x^{54})\cdots(1+x^5+x^{10}+x^{15}+x^{20})$$

$$= \frac{1-x^4}{1-x^2}\cdot\frac{1-x^8}{1-x^4}\cdot\frac{1-x^{16}}{1-x^8}\cdots\frac{1-x^9}{1-x^3}\cdot\frac{1-x^{27}}{1-x^9}$$

$$\frac{1-x^{81}}{1-x^{27}}\cdots\frac{1-x^{25}}{1-x^5}\cdot\frac{1-x^{125}}{1-x^{25}}\cdots\frac{1-x^{625}}{1-x^{125}}$$

$$= \frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots,$$

which is simply the generating function for the number of partitions of n into primes.

In summary, we have deduced and proven four theorems with regards to partitioning an integer n into prime parts. While we have only looked into partitions of n into prime parts thus far, further lines of inquiry could include partitions of n into parts that satisfy some polynomial expression, or any other class of natural numbers.

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Prime Partitions of the integers 2 through 11

2:	2				
3:	3				
4:	2 + 2				
5:	2 + 3,	5			
6:	2 + 2 + 2,	3 + 3			
7:	2 + 2 + 3,	5 + 2,	7		
8:	2 + 2 + 2 + 2,	3 + 3 + 2,	5 + 3		
9:	3 + 2 + 2 + 2,	3 + 3 + 3,	5 + 2 + 2,	7 + 2	
10:	2 + 2 + 2 + 2 + 2,	3 + 3 + 2 + 2,	5 + 3 + 2,	5 + 5,	7 + 3
11:	3 + 2 + 2 + 2 + 2,	3 + 3 + 3 + 2,	5 + 2 + 2 + 2,	5 + 3 + 3,	7 + 2 + 2, 11.

Power Series and Inversion of an Integral Transform

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Introduction. An *integral transform* is a transformation of the form

$$(Tf)(s) = \int_D f(t)K(s,t)dt$$

which transforms a given function f on a domain D into a function $Tf = T(f)$ on some other domain. The function $K(s,t)$ is called the kernel of the transformation. For example, if D is the real line and $K(s,t) = (1/\sqrt{2\pi})e^{ist}$ then $T(f) = \hat{f}$ is the Fourier transform of f, and if D is the positive real line and $K(s,t) = e^{-st}$ then $T(f) = \tilde{f}$ is the Laplace transform of f. Integral transforms have been studied for over 200 years and are of interest both for their intrinsic beauty and for their numerous applications to the study of solutions of differential equations and other topics throughout science and engineering (see [2]). An important problem in connection with the study of integral transforms is the inversion problem, that is, the problem of finding the function $f(t)$ when its image $(Tf)(s)$ under the transform is known.

A slightly different type of transform, called the Radon transform (see [6]), begins with a function f on $\mathbb{R}^n$ and gives a new function on the set of affine subsets of $\mathbb{R}^n$ of some dimension $k < n$. If $n = 3$ and $k = 1$, for example, we start with a function f on $\mathbb{R}^3$ and then form a new function Rf on the space A of all affine lines in $\mathbb{R}^3$ by saying that, for a line ξ, the value $Rf(\xi)$ should be the integral of the original function f over ξ,

$$(Rf)(\xi) = \int_\xi f = \int_{\mathbb{R}} f|_\xi,$$

as long as this integral converges for all $\xi \in A$. The Radon transform has applications in the field of medical x-ray tomography, since the intensity of the spot of light on an x-ray is actually an integral of the density of the body being x-rayed along the line followed by the x-ray's beam. Thus the problem of figuring out the three-dimensional shape of organs inside the body from a two-dimensional x-ray is the problem of inverting a Radon transform. A modern generalization of the Radon transform, called the Penrose transform (see [12]), applies to functions of complex, instead of real, variables and involves integration over complex subspaces of the original space. The Penrose transform

is used to study the differential equations of mathematical physics such as Maxwell's equations and the wave equation.

In this paper we consider a special case of an integral transform similar to a Penrose or Radon transform [10]. The problem we wish to solve is to determine the image of the transform, that is, to somehow characterize which functions will occur in the image, and to find an inversion formula for the transform. The special case involves functions $f(z_1, z_2)$ of two complex variables. The beauty of complex analysis is that a differentiable function automatically is expressible in terms of a convergent power series, enabling us to use power series as a major tool.

Our paper begins with the presentation of background material on complex numbers and complex functions which will be used later. In the second section we develop the spaces of functions on which the transform will act. Then in Section 3 we study the geometry of the one-dimensional subspaces of $\mathbb{C}^2$ to which our functions will be restricted. Section 4 contains the definition of our integral transform. Section 5 contains the inversion formula for the transform and a characterization of the functions in the image of the transform.

1. Preliminaries.

We begin with some facts and useful lemmas about complex numbers. Recall that a complex number $z \in \mathbb{C}$ is a number $z = x + iy$, where x and y are real numbers. The complex conjugate of z is the number $\bar{z} = x - iy$, and the norm, or absolute value, of z satisfies $|z|^2 = x^2 + y^2 = z\bar{z}$. By $z \in \mathbb{C}^2$ we mean an ordered pair $z = (z_1, z_2)$ of complex numbers with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. The norm of $z \in \mathbb{C}^2$ is then given by

$$|z|^2 = |z_1|^2 + |z_2|^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

In integrating functions of a complex variable, we use the identification of $\mathbb{C}$ with $\mathbb{R}^2$ and a scaling factor. Thus, for $z = x + iy \in \mathbb{C}$ we let $dm(z) = \frac{1}{\pi}\, dx\, dy$ and for $z = (z_1, z_2) \in \mathbb{C}^2$, as above, we let $dm(z) = dm(z_1)dm(z_2) = \frac{1}{\pi^2}\, dx_1 dy_1 dx_2 dy_2$. The next two lemmas explain the scaling factor and give integration formulas that will be very useful in later sections.

Lemma 1. For $n = 1, 2$,
$$\int_{\mathbb{C}^n} e^{-|z|^2}\, dm(z) = 1.$$

Proof. In evaluating the $n = 2$ case, the integral breaks into a product of two integrals $\int_{\mathbb{C}^2} e^{-|z|^2}\, dm(z) = \int_{\mathbb{C}} e^{-|z_1|^2}\, dm(z_1) \int_{\mathbb{C}} e^{-|z_2|^2}\, dm(z_2)$, and so we need just prove the case $n = 1$. We use polar coordinates, letting $z = r\, e^{i\theta}$, so that $|z|^2 = r^2$ and $dm(z) = \frac{1}{\pi}\, r\, dr\, d\theta$. The integral again breaks up into a product, as follows:
$$\int_{\mathbb{C}} e^{-|z|^2}\, dm(z) = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}\, r\, dr = \int_0^{\infty} e^{-u}\, du = 1. \blacksquare$$

Lemma 2. If j, k are nonnegative integers, and if $\alpha \in \mathbb{C}$ has positive real part, then
$$\int_{\mathbb{C}} z^k \bar{z}^j e^{-\alpha|z|^2}\, dm(z) = \begin{cases} 0, & j \neq k, \\ k!\,\alpha^{-(k+1)}, & j = k. \end{cases}$$

Proof. We use polar coordinates, as above. The case for $j \neq k$ is obtained immediately by noticing that $\int_0^{2\pi} e^{i(k-j)\theta}\, d\theta = 0$. However, for $j = k$, we have $\int_0^{2\pi} e^{i(k-j)\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi$. For this case we must proceed with the evaluation of the r-integral
$$\int_0^{\infty} r^{j+k} e^{-\alpha r^2}\, r\, dr = \int_0^{\infty} r^{2k} e^{-\alpha r^2}\, r\, dr = \frac{1}{2}\, k!\, \alpha^{-(k+1)},$$
by induction and integration by parts. $\blacksquare$

Before discussing differentiation of functions of a complex variable, we note that, since $z = x + iy$, a function $f(z)$ may be expressed in terms of the two independent real variables x and y or in terms of the two variables z and $\bar{z}$, since $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. We then use the operators $\partial/\partial z$

and $\partial/\partial\bar{z}$ instead of $\partial/\partial x$ and $\partial/\partial y$. We say that the function $f(z)$ is *holomorphic* in z if $\partial f/\partial\bar{z} = 0$ and *antiholomorphic* in z if $\partial f/\partial z = 0$. We know that a holomorphic function f may be expressed as the sum of a convergent power series in z and that an antiholomorphic function f is the sum of a convergent power series in $\bar{z}$ (see [11]).

Now we discuss the inner product structure on $\mathbb{C}^2$. The standard inner product $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ on $\mathbb{C}^2$ is both hermitian, since $\langle z, w\rangle = \overline{\langle w, z\rangle}$ and positive definite, since $\langle z, z\rangle \geq 0$ for all $z \in \mathbb{C}^2$ with equality only when $z = 0$. Other hermitian forms may be negative definite, satisfying $\langle z, z\rangle \leq 0$ for all z with equality only for $z = 0$, or may satisfy that $\langle z, z\rangle$ is positive for some vectors and negative for others. Hermitian forms with mixed signs arise naturally in physics and are also interesting mathematically (see [10], [7]). In our case we define the hermitian form h on $\mathbb{C}^2$ by $h(z, w) = z_1\bar{w}_1 - z_2\bar{w}_2$. The geometry resulting from the use of this form will be discussed further in Section 3 below.

2. The Function Space.

Our goal in this section is to define the spaces $\mathcal{H}_\alpha$ of complex-valued functions on $\mathbb{C}^2$ on which our integral transform will act. First we need an inner product on functions $f(z) = f(z_1, z_2)$. For functions f and g on $\mathbb{C}^2$, we define
$$\langle\langle f, g\rangle\rangle = \int_{\mathbb{C}^2} f(z)\overline{g(z)}\, e^{-|z|^2}\, dm(z).$$
This formula differs from the standard inner product on functions by the addition of the exponential weighting factor. We wish to compute this inner product in some specific examples. Let f_{ij} denote the monomial $f_{ij}(z) = z_1^i\, \bar{z}_2^j$, where i and j are non-negative integers.

Lemma 3. For non-negative integers i, j, k, and l, with f_{ij} and f_{kl} as defined above, then
$$\langle\langle f_{ij}, f_{kl}\rangle\rangle = \begin{cases} 0, & i \neq k \ or \ j \neq l, \\ i!\,j!, & i = k \ and \ j = l. \end{cases}$$

Proof. To calculate
$$\langle\langle f_{ij}, f_{kl}\rangle\rangle = \int_{\mathbb{C}^2} z_1^i\, \bar{z}_2^j\, \bar{z}_1^k\, z_2^l\, e^{-|z|^2}\, dm(z),$$
we apply Lemma 2 to the resulting integrals in z_1 and z_2 separately. $\blacksquare$

We may now define the space of functions $\mathcal{H}$ to be

$$\mathcal{H} = \left\{ f(z) = f(z_1, z_2) \mid \frac{\partial f}{\partial \bar{z}_1} = 0, \ \frac{\partial f}{\partial z_2} = 0, \ \langle\langle f, f \rangle\rangle < \infty \right\}.$$

The space $\mathcal{H}$ is sometimes called the Bargmann-Segal-Fock space, first studied in [3]. The occurrence of the mixed differential operators $\partial/\partial\bar{z}_1$ and $\partial/\partial z_2$ corresponds to our choice of hermitian form h of mixed signature (1, 1) (see [4] for more details). Note that $\mathcal{H}$ consists of functions f which are holomorphic in z_1, antiholomorphic in z_2, and for which the integral $\langle\langle f, f \rangle\rangle$ converges. From Lemma 3, we note that any monomial or any polynomial in z_1 and $\bar{z}_2$ will be in $\mathcal{H}$. In fact, these polynomials in z_1 and $\bar{z}_2$ are dense in $\mathcal{H}$ [3]. The other functions f in $\mathcal{H}$ are expressible as infinite power series in z_1 and $\bar{z}_2$, $f(z_1, z_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z_1^j \bar{z}_2^k$, where the condition that $\langle\langle f, f \rangle\rangle < \infty$ controls the growth of the coefficients a_{jk} [3].

We are interested in the subsets of $\mathcal{H}$ given for each integer $m \in \mathbb{Z}$ by

$$\mathcal{H}_m = \{f \in \mathcal{H} \mid f(e^{-i\theta}z) = e^{im\theta} f(z) \text{ for all } z \in \mathbb{C}^2, \theta \in \mathbb{R}\}. \tag{1}$$

Lemma 4. The power series expansion of any $f \in \mathcal{H}_m$ has the form

$$f(z_1, z_2) = \begin{cases} \sum_{j=0}^{\infty} c_j z_1^j \bar{z}_2^{j+m}, & m \geq 0, \\ \sum_{j=0}^{\infty} c_j z_1^{j+|m|} \bar{z}_2^j, & m < 0, \end{cases} \tag{2}$$

where the coefficients c_j are uniquely determined and satisfy

$$\langle\langle f, f \rangle\rangle = \sum_{j=0}^{\infty} j!(j + |m|)!|c_j|^2 < \infty. \tag{3}$$

Proof. Any monomial f_{jk} satisfies

$$f_{jk}(e^{-i\theta}z) = (e^{-i\theta}z_1)^j \overline{(e^{-i\theta}z_2)^k} = e^{-ij\theta} z_1^j e^{ik\theta} \bar{z}_2^k = e^{i\theta(k-j)} f_{jk}(z).$$

Thus a monomial $f_{jk} \in \mathcal{H}_m$ if and only if $k - j = m$, or $k = j + m$. If $m \geq 0$, then the degree of $\bar{z}_2$ is greater than the degree of z_1, so the monomial of least degree is $z_1^0 \bar{z}_2^m$. If $m < 0$, then we must write $j = k - m = k + |m|$, and the monomial of least degree is $z_1^{|m|} \bar{z}_2^0$. Now condition (1) implies that any $f \in \mathcal{H}_m$ will satisfy $\langle\langle f, f_{j,k} \rangle\rangle = 0$ if $k - j \neq m$, and so the only monomials $f_{j,k}$ that can appear in the power series expansion of $f \in \mathcal{H}_m$ are those satisfying $k - j = m$. The power series expressions in (2) and the uniqueness of the c_j now follow from the theory of complex power series.

We know from (1) that any $f \in \mathcal{H}_m$ satisfies $\|f\|^2 = \langle\langle f, f \rangle\rangle < \infty$. In the case $m \geq 0$ we rewrite the power series for f as $f(z) = \sum_{j=0}^{\infty} c_j f_{j,j+m}(z)$. Applying Lemma 3 to the expression for $\langle\langle f, f \rangle\rangle$ gives equation (3). The case $m < 0$ is analogous. ∎

Any $f \in \mathcal{H}$ may be expressed uniquely as a sum of functions in $\mathcal{H}_m$, by grouping the terms in the power series for f according to their "homogeneity degree" $k - j$. This says that $\mathcal{H}$ is the direct sum of the subspaces $\mathcal{H}_m$, written as $\mathcal{H} = \oplus_{m \in \mathbb{Z}} \mathcal{H}_m$.

3. Subspaces of $\mathbb{C}^2$. In this section we discuss a family of subspaces of $\mathbb{C}^2$. These will be used in the construction of our integral transform. Of course, since the complex dimension of $\mathbb{C}^2$ is two, the only non-trivial complex subspaces of $\mathbb{C}^2$ are one-dimensional, or complex lines. The set of all one-dimensional subspaces of $\mathbb{C}^2$ is called complex projective space, written $\mathbb{CP}^1$. Any line in $\mathbb{C}^2$ has a basis consisting of a single non-zero vector $z = (z_1, z_2)$. As long as we can find such a vector with $z_2 \neq 0$, we may multiply by the complex scalar z_2^{-1} to obtain a basis vector of the form $(\zeta, 1)$, where $\zeta \in \mathbb{C}$. Two different choices of ζ now clearly determine different lines. The only line this parameterization misses is the line $L \in \mathbb{CP}^1$ consisting of all vectors $(z_1, 0)$, with $z_1 \in \mathbb{C}$. Notice that we have parameterized $\mathbb{CP}^1$ by the set of all $\zeta \in \mathbb{C}$ together with the single line L, which may be thought of as a "point at infinity." This gives the correspondence of $\mathbb{CP}^1$ with a sphere, usually called the Riemann sphere.

We now investigate the restriction of our hermitian form h to each of the lines in $\mathbb{CP}^1$. There are three possibilities for the restrictriction of h to a one-dimensional subspace of $\mathbb{C}^2$: it will be either positive definite, negative definite, or identically zero. Such lines are called positive, negative, or null, respectively. First, for the line L, note that $h((z, 0), (z, 0)) = |z|^2 \geq 0$, with equality only if $z = 0$. Thus h restricted to L is positive definite, so L is positive.

We now calculate the restriction of h to the complex line $V(\zeta)$ spanned by the vector $(\zeta, 1)$, that is,

$$V(\zeta) = \{(w\zeta, w) \mid w \in \mathbb{C}\}. \tag{4}$$

Lemma 5. Let $V(\zeta)$ be described above. Then $V(\zeta)$ is positive, null, or negative when ζ satisfies, respectively, $|\zeta| > 1$, $|\zeta| = 1$, or $|\zeta| < 1$.

Proof. Fix $\zeta \in \mathbb{C}$, and consider the point $(w\zeta, w) \in V(\zeta)$. Then

$$h((w\zeta, w), (w\zeta, w)) = |w\zeta|^2 - |w|^2 = |w|^2 (|\zeta|^2 - 1).$$

Thus the restriction of h to $V(\zeta)$ is now positive definite exactly when $|\zeta|^2 - 1 > 0$, identically zero exactly when $|\zeta|^2 - 1 = 0$, and negative definite exactly when $|\zeta|^2 - 1 < 0$. ∎

We are interested particularly in the space $M\_$ of negative lines in $\mathbb{C}P^1$. Lemma 5 implies that the space $M\_$ is parameterized by the set $D = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$. This is exactly the unit disk in the complex plane.

4. Construction of the Integral Transform.

In this section we construct the integral transform Φ_m on $\mathcal{H}_m$. First we choose $f \in \mathcal{H}_m$ and restrict f to a line $V(\zeta) \in M\_$, where $\zeta \in D$. From (4) we may think of $f|_{V(\zeta)}$ as a function of $w \in \mathbb{C}$ where $f|_{V(\zeta)}(w) = f(\zeta w, w)$. We begin with a lemma concerning these restricted functions.

Lemma 6. Consider the monomial $f_{jk}(z) = z_1^j \bar{z}_2^k$, as before, and let $n \in \mathbf{Z}$, $n \geq 0$. Then

$$\int_{\mathbb{C}} f_{jk}(\zeta w, w)\, w^n e^{-|w|^2}\, dm(w) = \begin{cases} 0, & n \neq k - j, \\ k!\ \zeta^j, & n = k - j. \end{cases}$$

Proof. The result follows by applying Lemma 2 to the integral

$$\int_{\mathbb{C}} f_{jk}(\zeta w, w)\, w^n e^{-|w|^2}\, dm(w) = \zeta^j \int_{\mathbb{C}} w^{j+n} \bar{w}^k e^{-|w|^2}\, dm(w). \ ∎$$

We now define a family of integral transforms Φ_m on $\mathcal{H}_m$, for each $m \geq 0$. For any monomial $f_{jk} \in \mathcal{H}_m$, it follows from the proof of Lemma 4 that $k - j$ is constant, with $k - j = m$. It follows that the integral in Lemma 6 is now nonzero exactly when $n = m$. Thus for any $f \in \mathcal{H}_m$ we define $\Phi_m f$ by

$$(\Phi_m f)(\zeta) = \int_{\mathbb{C}} f(\zeta w, w)\, w^m e^{-|w|^2} dm(w). \tag{5}$$

Notice that we are first restricting our function f to the one-dimensional subspace $V(\zeta)$ before integrating. The resulting function $\Phi_m f$ may now be considered to be a function on the space $M\_$ of negative lines in $\mathbb{C}^2$, or it may be thought of as a function on the parameter space D for $M\_$, the unit disk in the complex plane. It is this latter interpretation which is most helpful.

Lemma 6 allows us to use power series to compute the effect of the transform Φ_m on any function $f \in \mathcal{H}_m$.

Lemma 7. Let $m \geq 0$ As in equation (2), let $f \in \mathcal{H}_m$ have power series expansion

$$f(z_1, z_2) = \sum_{j=0}^{\infty} c_j\, z_1^j\, \bar{z}_2^{j+m} = \sum_{j=0}^{\infty} c_j\, f_{j,j+m}(z).$$

Then the image $\Phi_m(f)$ may be expressed as a power series in $\zeta \in D$ by

$$(\Phi_m f)(\zeta) = \sum_{j=0}^{\infty} c_j (j + m)!\ \zeta^j. \tag{6}$$

This series converges for $|\zeta| < 1$ and so determines a holomorphic function of $\zeta \in$ D.

Proof. Lemma 6 says that $(\Phi_m f_{j,\, j+m})(\zeta) = (j + m)! \zeta^j$. Therefore the power series expression in (6) would follow from Lemma 6 and the linearity of Φ_m if we knew that we could interchange the order of summation and integration. Thus we must show that the series (6) converges. We use the root test (see [5]).

We showed in (3) that the series $\sum_{j=0}^{\infty} j!(j + m)!|c_j|^2$ converges. The root test applied to series (3) implies that

$$\rho = \limsup_{j \to \infty} (j!(j+m)!\,|c_j|^2)^{1/j} \leq 1.$$

The radius of convergence of the series (6) is $R = 1/r$, where

$$r = \limsup_{j \to \infty} (|c_j|(j+m)!)^{1/j}.$$

Now,

$$\begin{aligned} r^2 &= \limsup_{j \to \infty} (|c_j|^2((j+m)!)^2)^{1/j} \\ &= \limsup_{j \to \infty} (j!(j+m)!\,|c_j|^2)^{1/j} \lim_{j \to \infty} ((j+1)\cdots(j+m))^{1/j} \\ &= \rho \cdot 1 \leq 1, \end{aligned}$$

since $\lim_{j\to\infty}((j+1)\cdots(j+m))^{1/j} = 1$. Thus $r \le 1$, so $R = 1/r \ge 1$, so the series (6) converges for all $|\zeta| < 1$. That the resulting function of ζ is holomorphic on D follows from complex analysis. ∎

As is customary, we let $O(D)$ denote the space of all holomorphic functions on the unit disk D. The main result concerning the transform Φ_m is now the following.

Theorem 1.. The transform $\Phi_m: \mathcal{H}_m \to O(D)$ as defined in (5) is well-defined and injective for $m \ge 0$.

Proof. That Φ_m is well-defined was shown in Lemma 7. The injectivity of Φ_m follows from the power series expression (6). We know from complex analysis that two power series are equal if and only if their coefficients are equal. Thus if two power series of the form (6) are equal, their coefficients are equal. Hence, by the uniqueness of the coefficients c_j in the series (2), the original functions must be equal, so Φ_m is injective. ∎

5. Inversion of the Transform and Description of the Image.

Now we turn our attention to the development of an inversion process for this integral transform. We develop this process prior to describing the image of the transform. The reason for this is that, with the inversion process already established, we may apply it to an arbitrary function in $\Phi_m(\mathcal{H}_m)$. To the result, we apply the restrictions that come from the definition of the domain $\mathcal{H}_m$. These restrictions are then translated into information about the image.

Theorem 2. Let $m \ge 0$. For $f \in \mathcal{H}_m$, if $(\Phi_m f)(\zeta) = g(\zeta)$ where g has power series expansion $g(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$, then we may recover the function f as

$$f(z) = \sum_{j=0}^{\infty} \frac{a_j}{(j+m)!} z_1^j \bar{z}_2^{j+m}.$$

Proof. This result follows immediately from Lemma 7. ∎

We have demonstrated a process for inverting the operator if we already know that a given function is in the image. This creates a need for a criterion

that will determine whether or not a given function $g \in O(D)$ satisfies $g \in \Phi_m(\mathcal{H}_m)$ for any $m \ge 0$. This criterion is given in the following theorem.

Theorem 3. Let $g \in O(D)$ have power series expansion $g(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$, and let $m \ge 0$. Then $g \in \Phi_m(\mathcal{H}_m)$ if and only if the coefficients a_j satisfy

$$\sum_{j=0}^{\infty} \frac{j!}{(j+m)!} |a_j|^2 < \infty. \tag{7}$$

Proof. A function lies in the image of Φ_m if and only if we may apply the inversion process given in Theorem 2 and obtain a function in the domain $\mathcal{H}_m$. Thus, for an arbitrary $g \in O(D)$ we will apply the inversion process, then check for three conditions:

1. The resulting power series must converge for all $z \in \mathbb{C}^2$.
2. This power series must represent a function $f(z_1, z_2)$ which is holomorphic in z_1 and antiholomorphic in z_2.
3. The function $f(z_1, z_2)$ must satisfy $\langle\langle f, f\rangle\rangle < \infty$.

We will prove that criteria 1 and 2 hold for any g and that criterion 3 is equivalent to (7). So we assume that $g(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$ is holomorphic on the unit disk D and that $(\Phi_m^{-1} g)(z) = f(z_1, z_2) = \sum_{j=0}^{\infty} c_j z_1^j \bar{z}_2^{j+m}$, where $c_j = a_j/(j+m)!$.

Since g is holomorphic on D, we know that we may apply the root test to its power series, concluding that $\limsup_{j\to\infty} |a_j|^{1/j} = 1/R$ where $R \ge 1$ is the radius of convergence (if the series is convergent everywhere, then the lim sup is zero). Applying the root test to the coefficients of f yields

$$\limsup_{j\to\infty} |c_j|^{1/j} = \limsup_{j\to\infty} |a_j|^{1/j} \lim_{j\to\infty}\left(\frac{1}{(j+m)!}\right)^{1/j}$$

$$= \frac{1}{R} \lim_{j\to\infty}\left(\left(\frac{1}{(j+m)!}\right)^{1/(j+m)}\right)^{(j+m)/j} = 0$$

if R is finite (if R is not finite, then both limits in the product above tend to zero, and thus so does their product). Hence if we were to create a power series $\sum_{j=0}^{\infty} c_j \xi^j$, it would converge for all $\xi \in \mathbb{C}$. Now consider $f(z_1, z_2) = \sum_{j=0}^{\infty} c_j z_1^j \bar{z}_2^{j+m}$. For any $(z_1, z_2) \in \mathbb{C}^2$, we may rewrite the series above as

$$f(z_1, z_2) = \bar{z}_2^m \sum_{j=0}^{\infty} c_j (z_1 \bar{z}_2)^j$$

where the final summation may now be considered as one in a single variable $\xi = z_1 \bar{z}_2$, and thus it converges for all z_1, z_2. It is clearly holomorphic in z_1 and antiholomorphic in z_2. Thus criteria 1 and 2 are satisfied. For criterion 3, we computed in Lemma 4 that

$$\langle\langle f, f \rangle\rangle = \sum_{j=0}^{\infty} j!(j+m)!|c_j|^2 = \sum_{j=0}^{\infty} \frac{j!}{(j+m)!}|a_j|^2.$$

Thus the finiteness of this norm is exactly equivalent to (7). $\blacksquare$

Theorem 3 gives a characterization of the image of Φ_m via power series, but we also would like a simpler and more easily computable criterion for when a function $g \in O(D)$ satisfies $g \in \Phi_m(\mathcal{H}_m)$. The next theorem gives a preliminary result along these lines.

Theorem 4. Let $g \in O(\bar{D})$ be holomorphic on the closed unit disk. Then $g \in \Phi_m(\mathcal{H}_m)$ for every $m \geq 0$.

Proof. Let g have power series expansion $g(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$. Since g is holomorphic on the closed disk $\bar{D}$, we know that g is actually holomorphic on a larger disk D' with radius $R > 1$. We apply criterion (7) to this series, using the root test. We compute that

$$\limsup_{j\to\infty}\left(|a_j|^2 \frac{j!}{(j+m)!}\right)^{1/j} = \limsup_{j\to\infty}|a_j|^{2/j} \lim_{j\to\infty}((j+1)\cdots(j+m))^{-(1/j)} = \frac{1}{R^2}$$

if R is finite (the limit is zero if R is not finite). Since $R > 1$, then $1/R^2 < 1$, so the series converges by the root test. Theorem 3 thus implies that $g \in \Phi_m(\mathcal{H}_m)$ for

any $m \geq 0$. $\blacksquare$

Notice that Theorem 3 implies that if $g \in \Phi_m(\mathcal{H}_m)$ for some m, then $g \in \Phi_k(\mathcal{H}_k)$ for any $k \geq m$. This follows from (7) and the comparison test, since $1/(j+k)! \leq 1/(j+m)!$. Theorems 3 and 4 thus show that

$$O(\bar{D}) \subseteq \Phi_0(\mathcal{H}_0) \subseteq \Phi_1(\mathcal{H}_1) \subseteq \Phi_2(\mathcal{H}_2) \subseteq \cdots \subseteq O(D).$$

The question now arises as to which of these containments are proper. We investigate this question in the following examples.

Example 1. Consider the holomorphic function $g \in O(D)$ given by $g(\zeta) = -\ln(1-\zeta) = \sum_{j=1}^{\infty} \frac{1}{j}\zeta^j$. By (7) the norm of the pre-image $\Phi_m^{-1}(g)$ satisfies

$$\|\Phi_m^{-1} g\|^2 = \sum_{j=1}^{\infty} \frac{j!}{j^2(j+m)!} = \sum_{j=1}^{\infty} \frac{1}{j^2(j+m)\cdots(j+1)}.$$

This converges for every $m \geq 0$, so $g \in \Phi_m(\mathcal{H}_m)$ for all $m \geq 0$. However, the function g is undefined at $\zeta = 1$ and so does not extend to the boundary of D. Thus $g \notin O(\bar{D})$.

Example 2. For any $n \geq 1$ we consider the function

$$g_n(\zeta) = \sum_{j=0}^{\infty}\left(\frac{(j+n-1)!}{(j+1)!}\right)^{1/2} \zeta^j.$$

By the ratio test, the series converges for $|\zeta| < 1$, so $g_n \in O(D)$. However, the series diverges for $\zeta = 1$ and so $g_n \notin O(\bar{D})$. By (7), the pre-image $\Phi_m^{-1}(g_n)$, if it exists, has norm given by

$$\|\Phi_m^{-1} g_n\|^2 = \sum_{j=0}^{\infty} \frac{(j+n-1)!j!}{(j+1)!(j+m)!} = \sum_{j=0}^{\infty} \frac{(j+n-1)!}{(j+1)\ (j+m)!}.$$

By comparing the degrees of numerator and denominator we see that this series converges if and only if $m > n-1$, and diverges if $m \leq n-1$. Thus, if $m \geq 1$ is fixed, we choose $n = m$. Then $\|\Phi_m^{-1} g_m\|$ is finite, but $\|\Phi_{m-1}^{-1} g_m\|$ is not. This implies that $g_m \in \Phi_m(\mathcal{H}_m)$, but $g_m \notin \Phi_{m-1}(\mathcal{H}_{m-1})$.

Example 3. Consider the function g defined by

$$g(\zeta) = \sum_{j=0}^{\infty} e^{\sqrt{j}} \zeta^j.$$

We compute that

$$\limsup_{j\to\infty} (e^{\sqrt{j}})^{1/j} = \limsup_{j\to\infty} e^{1/\sqrt{j}} = 1.$$

Thus, by the root test, g has radius of convergence 1, so $g \in O(D)$. However, for a fixed non-negative integer m, applying (7) yields that

$$\|\Phi_m^{-1} g\|^2 = \sum_{j=0}^{\infty} \frac{j!}{(j+m)!} |e^{\sqrt{j}}|^2 = \sum_{j=0}^{\infty} \frac{e^{2\sqrt{j}}}{(j+1)\cdots(j+m)}.$$

(If $m = 0$, the last denominator is 1.) Notice that the numerator is an exponential in $\sqrt{j}$, while the denominator is a polynomial of degree $2m$ in $\sqrt{j}$. By L'Hopital's rule we know that

$$\lim_{j\to\infty} \frac{e^{2\sqrt{j}}}{(j+1)\cdots(j+m)} = +\infty.$$

Thus the terms do not approach zero, so the series diverges for every $m \geq 0$.

Example 1 shows that the containment $O(\overline{D}) \subset \Phi_m(\mathcal{H}_m)$ is proper for all $m \geq 0$. Example 2 shows that the containment $\Phi_{m-1}(\mathcal{H}_{m-1}) \subset \Phi_m(\mathcal{H}_m)$ is proper for each $m \geq 1$, and it follows that the containment $\Phi_m(\mathcal{H}_m) \subset O(D)$ is proper for all $m \geq 0$. Example 3 shows that there exists a function in $O(D)$ which is not in $\Phi_m(\mathcal{H}_m)$ for any m. Thus the union of all the sets $\Phi_m(\mathcal{H}_m)$ is properly contained in $O(D)$. Therefore, we may write

$$O(\overline{D}) \subset \Phi_0(\mathcal{H}_0) \subset \Phi_1(\mathcal{H}_1) \subset \Phi_2(\mathcal{H}_2) \subset \cdots \subset O(D).$$

6. Further Study. There are three main areas where further study could be concentrated. First, the image space of the transform should be characterized in a more "global" way, without using power series. A precise formula such as an integral norm would be much easier to apply in practice. The second area concerns developing a closed form expression such as an integral operator for the inverse of the transform. Both of these questions are answered in [1]. Finally, this paper has considered a special case of a more general problem. The transform may be considered for higher dimensions, that is, for functions of $p + q$ complex variables, where $p \geq 1$ and $q \geq 1$. This, of course, introduces many interesting complexities but may also facilitate the discovery of important patterns. The inversion problem for the generalized transform is addressed in [8] and [9].

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BIOGRAPHICAL SKETCHES OF THE AUTHORS

Herbert E. Salzer received his Ph.D. in applied mathematics from Columbia University in 1953. He has 54 years of experience and research in applied mathematics, computation and numerical analysis. Dr. Salzer has been a member of **Pi Mu Epsilon** for 60 years and he has 190 papers published.

Subhash C. Saxena received his Ph.D. in differential geometry from the University of Delhi, India. Over the last 35 years, he has taught at several universities in the United States. Dr. Saxena has published in the areas of geometry, analysis, algebra and discrete mathematics. He is also the author of a book entitled 'Introduction to Real Variable Theory.'

Michael Kinyon received his B.S., M.S., and Ph.D. degrees in mathematics from the University of Utah in 1986, 1988, and 1991, respectively. Since 1992, he has been at Indiana University South Bend. His research interests focus on using algebraic methods to understand differential equations, hyperbolic geometry, and special relativity. His teaching interests include analysis, differential equations, calculus, and students, not necessarily in that order.

Richard L. Francis is a professor of mathematics at Southeast Missouri State University. He received a B.S. degree from Southeast Missouri State University and master's and doctoral degrees from the University of Missouri (Columbia). Major scholarly interests include number theory and the history of mathematics. Previous articles in these areas of interest have appeared in the publications of the National Council of Teachers of Mathematics, the American Mathematical Society, Consortium for Mathematics and Its Applications, and the Mathematical Association of America (from which he received the George Polya Award). He, his wife Joyce, and three children, Richard Lee, Marie, and Michael, reside in Cape Girardeau, Missouri.

Danica Nowosielski received her B.S. in chemistry and mathematics from Russell Sage College, Troy, New York, in May 1996. Her change ringing work was done as an honors project in mathematics during her senior year. Ms. Nowosielski is now working towards a Ph.D. in physical chemistry at Rensselaer Polytechnic Institute, Troy, New York. She is active in the community as a musician on both piano and violin.

Sherwood Washburn is Chair of the Department of Mathematics and Computer Science at Seton Hall University. He has a B.A. from Berkeley and a Ph.D. from Columbia. His thesis advisor was H. Hironaka. He is interested in several topics in algebra and combinatorics. His book 'Discrete Mathematics' is under contract with Addison-Wesley. He lives in New York with his wife Katharine, a writer, and they have three sons.

Kerry Smith McNeill received her Bachelor's degree in Mathematics from Seton Hall University in May 1995, and her Master's in Mathematics from Stevens Institute of Technology, where she is a Teaching Assistant, in May 1997. She is currently pursuing her doctorate at Stevens. The research for this paper was funded by a Clare Boothe Luce Grant for Women in the Sciences, and was supervised by Professor **John J. Saccoman**, Department of Mathematics, Seton Hall University.

Michael Last spent several years attending mathematics classes at the University of California, Davis, while in high school. He is currently a first year undergraduate at the University of California, San Diego, majoring in mathematics with minors in Physics and Japanese Studies. Presently, his primary mathematical interests lie in Analysis and Topology. Outside of class he is active in both the school's Ballroom Dance Team and Aerial Robotics Team, as well as several clubs. He plans to graduate in June of 2000.

Mike Oehrtman is a graduate student in mathematics at the University of Texas at Austin. This paper is based on his Senior Honors Thesis written in 1992 at Oklahoma State University under the direction of **Lisa Mantini.** Professor Mantini received her Ph. D. in mathematics at Harvard University in 1983. She has been at OSU since 1985.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed to arrive by July 1, 1998. Solutions by students are given preference.

Problems for Solution

914. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Solve this base ten addition alphametic, dedicated to the memory of the late Leon Bankoff:

$$FRIEND + INDEED = BANKOFF.$$

***915.** *Proposed by the late John Howell, Littlerock, California.*

Prove or disprove that, if $n \geq 0$, $k \geq 0$, and $n + k \geq 1$, then

$$n! = (n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots + (-1)^n k^n.$$

916. *Proposed by Morris Katz, Macwahoc, Maine.*

Prove these two formulas:

$$1^2(2n-1)^2 + 2^2(2n-2)^2 + \cdots + n^2n^2 = \frac{1}{30}n(n+1)(16n^3 - n^2 + n - 1)$$

and

$$1^2(2n-1)^2 - 2^2(2n-2)^2 + \cdots + (-1)^{n+1}n^2n^2 = \frac{1}{2}n[1-(-1)^n n^3].$$

917. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine necessary and sufficient conditions on the real numbers w_1, w_2, ..., w_n so that for all vectors $\mathbf{v}_i$ in E^m,

$$|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n|^2 \le w_1|\mathbf{v}_1|^2 + w_2|\mathbf{v}_2|^2 + \cdots + w_n|\mathbf{v}_n|^2.$$

918. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Evaluate the integral

$$I = \int_0^{\pi/3} \ln(1 + \sqrt{3}\tan x)\,dx.$$

919. *Proposed by the Editor.*

Erect directly similar nondegenerate triangles *DBC, ECA, FAB* on sides *BC, CA, AB* of triangle *ABC*. At *D, E, F* center circles of radii *k•BC, k•CA, k•AB* respectively for fixed positive *k*. Let *P* be the radical center of the three circles. If *P* lies on the Euler line of the triangle, show that it always falls on the same special point.

***920.** *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

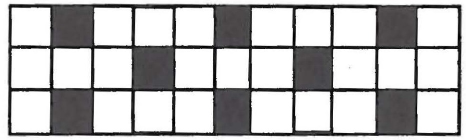
The *sorted Fibonacci sequence* is produced by starting with the first two terms 1 and 1 and defining each succeeding term as the sum of the prior two terms with the digits sorted into ascending order. Thus we have 1, 1, 2, 3, 5, 8, 13, 12, 25, 37, 26, ... This sequence eventually falls into a repeating cycle.

a) Are there any two initial terms that produce a diverging sequence?

b) How many different repeating cycles can you find?

***921.** *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Place 13 three-digit square numbers in the spaces in the accompanying grid. (The solution is unique.)



922. *Proposed by David Iny, Baltimore, Maryland.*

Suppose that $f(f(x)) = 0$ for all real x. Show that a necessary and sufficient condition that ensures that $f(x) = 0$ for all x is that f be infinitely differentiable on the real line.

923. *Proposed by A. Stuparu, Vâlcea, Romania.*

Let $S(n)$ denote the *Smarandache function*: if n is a positive integer, then $S(n) = k$ if k is the smallest nonnegative integer such that $k!$ is divisible by n. Thus $S(1) = 0$, $S(2) = 2$, $S(3) = 3$, and $S(6) = 3$, for example. Prove that the equation $S(x) = p$, where p is a given prime number, has just 2^{p-1} solutions between p and $p!$.

***924.** *Proposed by George Tsapakidis, Agrino, Greece.*

Find an interior point of a triangle so that its projections on the sides of the triangle are the vertices of an equilateral triangle.

925. *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Given positive integers s and c and any integer k such that $0 \le k \le s$, prove that

$$(-1)^k \sum_{j=k}^{s} \binom{c+j-1}{c-1}\binom{j}{k} = \sum_{j=0}^{k} (-1)^j \binom{s+c}{s+j-k}\binom{s+j-k}{j}.$$

926. *Proposed by Tom Moore, Bridgewater State College, Bridgewater, Massachusetts.*

Students were asked the question, "How many times is $x := x + 1$ executed in the following nested loop?

```
For i = 2 to n
    For j = 1 to ⌊i/2⌋
        x:= x + 1
    Next j
Next i
```

Discover which of the following ten actual student answers are correct, where $\lfloor\ \rfloor$ is the floor function and $\lceil\ \rceil$ is the ceiling function (so that $\lfloor\pi\rfloor = 3$ and $\lceil\pi\rceil = 4$):

$$a(n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \frac{n^2-1}{4}, & n \text{ odd.} \end{cases} \qquad b(n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, & n \text{ odd.} \end{cases}$$

$$c(n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \left(\frac{n-1}{2}\right)\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right), & n \text{ odd.} \end{cases} \qquad d(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor.$$

$$e(n) = \left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor. \qquad f(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.$$

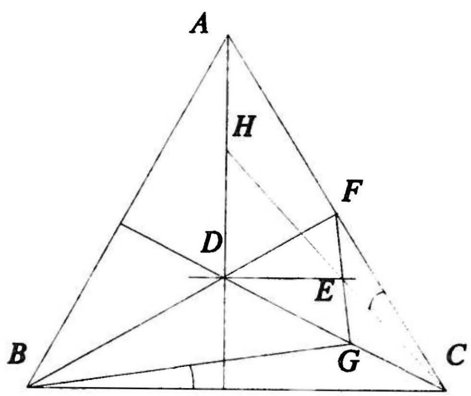
$$g(n) = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + [n \pmod 2] \right). \qquad h(n) = \left\lfloor \frac{n^2+2}{4} \right\rfloor.$$

$$i(n) = \left\lfloor \frac{n^2}{4} \right\rfloor. \qquad j(n) = \sum_{k=2}^{n} \left\lfloor \frac{k}{2} \right\rfloor.$$

Solutions

881. [Spring 1996, Spring 1997] *Proposed by Andrew Cusumano, Great Neck, New York.*

Let ABC be an equilateral triangle with center D. Let α be an arbitrary positive angle less than 30°. Let BD meet CA at F. Let G be that point on segment CD such that angle $CBG = \alpha$, and let E be that point on FG such that angle $FCE = \alpha$. Prove that DE is parallel to BC.



II. *Solution by Henry S. Lieberman, Waban, Massachusetts.*

Since triangle ABC is equilateral and D is its center, then $\angle BDC = 120°$, so $\angle FDC = 60°$. Since $\angle FBC = 30°$, it will suffice to prove that DE bisects $\angle FDC$. To that end we have, where ΔABC denotes the area of triangle ABC,

$$\frac{FE}{EG} = \frac{\Delta FCE}{\Delta ECG} = \frac{(\frac{1}{2})FC\cdot CE\cdot\sin\alpha}{(\frac{1}{2})GC\cdot CE\cdot\sin(30° - \alpha)} = \frac{FC}{GC}\cdot\frac{\sin\alpha}{\sin(30° - \alpha)}.$$

Similarly,

$$\frac{DG}{GC} = \frac{BD}{BC}\cdot\frac{\sin(30° - \alpha)}{\sin\alpha} = \frac{\sqrt{3}}{3}\cdot\frac{\sin(30° - \alpha)}{\sin\alpha}$$

because $BD = (\sqrt{3}/3)BC$. Since $FD/FC = \tan 30° = \sqrt{3}/3$, then

$$\frac{FE}{EG} = \frac{FC}{CG}\cdot\frac{GC}{DG}\cdot\frac{\sqrt{3}}{3} = \frac{FC}{DG}\cdot\frac{FD}{FC} = \frac{FD}{DG}.$$

Thus DE is the bisector of $\angle FDG$ in triangle FDG since it divides the opposite side in segments proportional to the adjacent sides. This completes the proof.

888. [Fall 1996] *Proposed by the Editor.*

In 1953 Howard Eves' book *An Introduction to the History of Mathematics* was first published. It quickly became the definitive undergraduate text in mathematics history. It still is today. To honor this outstanding text and its equally outstanding author, solve this base nine alphametic, finding the unique value of *HEVES*:

$$MATH + HIST = HEVES.$$

Amalgam of essentially similar solutions submitted independently by Heidi Barek, Alma College, Alma, Michigan, Karen Ellison, Oxford, Ohio, and Corie Kreps, Oxford, Ohio.

Clearly $H = 1$ and, from the 9^3 column, $M + 1 + (\text{carry}) \geq 10_9$, so $M = 7$ or 8 and $HE = 10_9$ since $HE = 11_9$ is not possible. We now have

```
  MAT1
+ 1IST
 10V0S.
```

In the ones column we have $1 + T = S$ because $T = 8$ would make $S = 0$ and $S \neq E$. From the 9^1 column we have $T + S = 10_9$, so $T = 4$ and $S = 5$. Thus we have

```
  1 1
   MA41
+  1I54
  10V05.
```

Now 2, 3, 6, and one of 7 and 8 are left for A, I, V, and M. If $M = 7$, then $1 + A + I = 10_9 + V \geq 12_9$, which requires $1 + 8 + 6 = 16_9$, $1 + 8 + 3 = 13_9$, or $1 + 8 + 2 = 12_9$, all of which are not possible since then V would equal either A or I.

We are left with $M = 8$ and $1 + A + I = V$ using only 2, 3, 6, and 7. Then A and I are 2 and 3 and $V = 6$. There are two solutions, in each of which $HEVES = 10605$, specifically

```
   8241          8341
+  1354       +  1254
  10605         10605.
```

Also solved by Avraham Adler, *New York, NY,* Charles Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* Laura Batt, *Reno, NV,* Scott H. Brown, *Auburn University, AL,* Paul S. Bruckman, *Highwood, IL,* William Chau, *New York, NY,* Lynette J. Daig, *Alma College, MI,* Kenneth B. Davenport, *Pittsburgh, PA,* Josh Delbacker, *Alma College, MI,* Jen Ditchik, *SUNY College at Fredonia, NY,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* Victor G. Feser, *University of Mary, Bismarck, ND,* Robert C. Gebhardt, *Hopatcong, NJ,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Carl Libis, *University of Alabama, Tuscaloosa,* Henry S. Lieberman, *Waban, MA,* Peter A. Lindstrom, *Batavia, NY,* Glen R. Marr, *University of Florida, Longwood,* Jill McEachin, *Alma College, MI,* Yoshinobu Murayoshi, *Okinawa, Japan,* Shannon Nielsen, *Alma College, MI,* William H. Peirce, *Delray Beach, FL,* Michael R. Pinter, *Belmont University, Nashville, TN,* H.-J. Seiffert, *Berlin, Germany,* University of Central Florida Problems Group, *Orlando,* Kenneth M. Wilke, *Topeka, KS,* Rex H. Wu, *Brooklyn, NY, and the* Proposer.

889. [Fall 1996] *Proposed by M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Prove that

$$\frac{a^x - 1}{x} - \frac{a^y - 1}{y} > \frac{x - y}{2} \cdot \left(\frac{a - 1}{a}\right)^2$$

where $a > 1$ and $x > y > 0$.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.

By the mean value theorem with $f(x) = (a^x - 1)/x$ we have

$$\frac{\dfrac{a^x - 1}{x} - \dfrac{a^y - 1}{y}}{x - y} = \frac{ca^c \ln a - a^c + 1}{c^2} \text{ for some } c \text{ between } x \text{ and } y.$$

Now let $g(x) = xa^x \ln a - a^x + 1 - (x^2/2)(\ln a)^2$ for $x \geq 0$. Then $g(0) = 0$ and

$$g'(x) = xa^x(\ln a)^2 - x(\ln a)^2 = x(\ln a)^2(a^x - 1) > 0,$$

so it follows that

$$\frac{ca^c \ln a - c^c + 1}{c^2} > \frac{(\ln a)^2}{2}.$$

Finally we show that $\ln a > (a - 1)/a$ for $a > 1$. For an easy proof of this fact, define $h(x) = x \ln x - x + 1$. Then $h(1) = 0$ and $h'(x) = \ln x > 0$ for $x > 1$. Hence $h(x) > 0$ when $x > 1$. The result follows by putting these results together.

Also solved by Avraham Adler, *New York, NY,* Dip Bhattacharya *and* S. Gendler, *Clarion University, PA,* Paul S. Bruckman, *Highwood, IL,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Richard I. Hess, *Rancho Palos Verdes, CA,* David Iny, *Baltimore, MD,* Henry S. Lieberman, *Waban, MA,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

890. [Fall 1996] *Proposed by Peter A. Lindstrom, Irving, Texas.*

Express the following sum in closed form, where real number $a \neq 1$:

$$\sum_{i=1}^{n} ia^{n-i}.$$

Solution by SUNY Fredonia Problem Group, SUNY at Fredonia, Fredonia, New York.

Differentiate the known formula

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}, \text{ where } x \neq 1,$$

to obtain

$$\sum_{i=1}^{n} ix^{i-1} = \frac{(1 - x)[-(n + 1)]x^n + 1 - x^{n+1}}{(1 - x)^2} = \frac{nx^{n+1} - (n + 1)x^n + 1}{(1 - x)^2}.$$

Thus we have

$$\sum_{i=1}^{n} ia^{n-i} = a^{n-1}\sum_{i=1}^{n} ia^{1-i} = a^{n-1}\sum_{i=1}^{n} i\left(\frac{1}{a}\right)^{i-1}$$

$$= a^{n-1}\left(\frac{n\left(\frac{1}{a}\right)^{n+1} - (n + 1)\left(\frac{1}{a}\right)^n + 1}{\left(1 - \frac{1}{a}\right)^2}\right) = \frac{n - (n + 1)a + a^{n+1}}{(a - 1)^2}.$$

Also solved by Avraham Adler, *New York, NY,* Ayoub B. Ayoub, *Pennsylvania State University, Ogantz Campus, Abington,* Paul S. Bruckman, *Highwood, IL,* William Chau, *New York, NY,* Kenneth B. Davenport, *Pittsburgh, PA,* Charles R. Diminnie, *San Angelo, TX,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* George P. Evanovich, *Saint Peter's College, Jersey City, NJ,* Mark Evans, *Louisville, KY,* Amanda J. Gambino, *Alma College, MI,* Jayanthi Ganapathy, *University of Wisconsin-Oshkosh,* Robert C. Gebhardt, *Hopatcong, NJ,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Michael K. Kinyon, *Indiana University South Bend,* Carl Libis, *University of Alabama, Tuscaloosa,* Henry S. Lieberman, *Waban, MA,* David E. Manes, *SUNY College at Oneonta,* Kandasamy Muthuvel, *University of Wisconsin-Oshkosh,* William H. Peirce, *Delray Beach, FL,* Bob Prielipp, *University of Wisconsin-Oshkosh,* John F. Putz, *Alma College, MI,* Henry J. Ricardo, *Medgar Evers College, Brooklyn, NY,* Shiva K. Saksena, *University of North Carolina at Wilmington,* H.-J. Seiffert, *Berlin, Germany,* George Tsapakidis *(two solutions), Agrinio, Greece,* University of Central Florida Problems Group, *Orlando,* Stan Wagon, *Macalester College, St. Paul, MN,* Lamarr Widmer, *Messiah College, Grantham, PA,* Kenneth M. Wilke, *Topeka, KS,* Rex H. Wu, *Brooklyn, NY,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer.

891. [Fall 1996] *Proposed by John Wahl, Mt. Pocono, Pennsylvania, and Andrew Cusumano, Great Neck, New York.*

Solve for d the equation

$$\frac{bcd + cda + dab + abc}{a + b + c + d} = \sqrt{abcd}.$$

Solution by Edward J. Koslawska, graduate student, Angelo State University, San Angelo, Texas.

By squaring both sides of the equation and collecting like terms, we obtain

$$(abc)d^3 - (a^2b^2 + b^2c^2 + c^2a^2)d^2 + (abc)(a^2 + b^2 + c^2)d - (abc)^2 = 0.$$

We factor to get $(da - bc)(db - ac)(dc - ab) = 0$, which yields the solutions

$$d = \frac{bc}{a},\ d = \frac{ca}{b},\ \text{and } d = \frac{ab}{c},$$

provided $bc > 0$ and $a \neq 0$, $ca > 0$ and $b \neq 0$, or $ab > 0$ and $c \neq 0$, respectively. These conditions are necessary since substitution of these solutions into the original equation yields $bc = |bc|$, $ca = |ca|$, and $ab = |ab|$, respectively.

If $a = 0$, then $bcd/(b + c + d) = 0$, which implies $b + c + d \neq 0$ and $bcd = 0$. There are two possibilities: first, $d = 0$ and $b + c \neq 0$. Otherwise, d is arbitrary, $b = 0$ or $c = 0$ or both, and $b + c + d \neq 0$.

A similar set of conditions holds if either $b = 0$ or $c = 0$.

Also solved by Avraham Adler, *New York, NY,* Dip Bhattacharya *and* S. Gendler, *Clarion University, PA,* Paul S. Bruckman, *Highwood, IL,* Russell Euler, *Northwest Missouri State University, Maryville,* George P. Evanovich, *Saint Peter's College, Jersey City, NJ,* Mark Evans, *Louisville, KY,* Todd Fischer, *Oxford, OH,* Robert C. Gebhardt, *Hopatcong, NJ,* Richard I. Hess, *Rancho Palos Verdes, CA,* Murray S. Klamkin, *University of Alberta, Canada,* Peter A. Lindstrom, *Batavia, NY,* William H. Peirce, *Delray Beach, FL,* University of Central Florida Problems Group, *Orlando,* Stan Wagon, *Macalester College, St. Paul, MN,* Rex H. Wu, *Brooklyn, NY,* David W. Yunghans *and* Eric M. Smith, *SUNY College at Fredonia, NY,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposers.

892. [Fall 1996] *Proposed by Bill Correll, Jr., student, Denison University, Granville, Ohio.*

Prove that the average of the eigenvalues of a real, symmetric, idempotent matrix is at most one.

I. *Solution by Michael K. Kinyon, Indiana University South Bend, South Bend, Indiana.*

If A is an idempotent matrix, λ is an eigenvalue, and $\mathbf{x}$ is an eigenvector corresponding to λ, then $\lambda\mathbf{x} = A\mathbf{x} = A^2\mathbf{x} = \lambda^2\mathbf{x}$. Thus $\lambda^2 = \lambda$, so $\lambda = 0$ or λ

= 1. Therefore, the average of the eigenvectors is no less than 0 nor more than 1; the upper bound being achieved by the identity matrix. The hypotheses that A is real and symmetric are not needed.

II. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

By hypothesis A is an $n \times n$ idempotent matrix. Therefore, the trace of A equals the rank r of A. (See Exercise 4(ii) on p. 239 with solution of p. 426 of Daniel T. Finkbeiner II, *Introduction to Matrices and Linear Transformations, 3rd ed.,* W. H. Freeman, 1978.) Since the sum of the eigenvalues of A is equal to the trace of A and $r \le n$, then the average of the eigenvalues equals $r/n \le 1$.

Also solved by Paul S. Bruckman, *Highwood, IL,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Joe Howard *and* John Jeffries, *New Mexico Highlands University, Las Vegas,* Murray S. Klamkin, *University of Alberta, Canada,* Henry S. Lieberman, *Waban, MA,* David E. Manes, *SUNY College at Oneonta,* Kandasamy Muthuvel, *University of Wisconsin-Oshkosh,* William H. Peirce, *Delray Beach, FL,* Henry J. Ricardo, *Medgar Evers College, Brooklyn, NY,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

893. [Fall 1996] *Proposed by Peter A. Lindstrom, Irving, Texas.*

Show that the sequence $\{x_n\}$ converges and find its limit, where $x_1 = 2$ and, for $n \ge 1$,

$$x_{n+1} = \frac{2x_n \sin x_n + \sin x_n + \cos x_n}{2\sin x_n}.$$

I. *Solution by Michael K. Kinyon, Indiana University South Bend, South Bend, Indiana.*

If $f(x) = e^{-x}(\sin x + \cos x)$, then observe that

$$x_{n+1} = x_n - \frac{e^{-x_n}(\sin x_n + \cos x_n)}{-2e^{-x_n}\sin x_n} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Thus the recurrence is just the Newton-Raphson method applied to the function f. If it converges at all, it must converge to one of the zeroes of f, which are $(k + 3/4)\pi$, k an integer. In this case the method starts at 2 and converges to $3\pi/4$. Note that $f(x)f''(x) > 0$ on the open interval joining the initial point 2 to the appropriate zero of f, a sufficient condition for convergence. Here $f(x)f''(x) = -2e^{-2x}\cos 2x$ and is positive on the interval $(\pi/4, 3\pi/4)$, which contains 2.

It is perhaps somewhat intellectually dishonest to pretend to cleverly pull f out of thin air. Observing the recursion formula in the form

$$x_{n+1} = x_n - \frac{\sin x_n + \cos x_n}{2\sin x_n},$$

I guessed the problem had been constructed by application of the Newton-Raphson method to a function of the form $f(x) = g(x)(\sin x + \cos x)$. Then differentiate $f(x)$ and substitute into the equation

$$\frac{f(x)}{f'(x)} = \frac{\sin x + \cos x}{2\sin x}.$$

Equating coefficients yields the differential equations

$$\frac{g'(x) - g(x)}{g(x)} = -2 \quad \text{and} \quad \frac{g'(x) + g(x)}{g(x)} = 0.$$

The solution to the second equation is $g(x) = ce^{-x}$ for some constant c, which also satisfies the first equation. I took $c = 1$.

II. *Solution by Lisa M. Croft, Messiah College, Grantham, Pennsylvania.*

The recursion equation can be written in the form

$$x_{n+1} = x_n + \frac{1}{2} + \frac{1}{2}\cot x_n,$$

so define

$$g(x) = x + \frac{1}{2} + \frac{1}{2}\cot x.$$

Now $g(x)$ and $g'(x)$ are continuous and $0 \le g'(x) \le 1/2$ on the interval $[a,b] = [\pi/4, 3\pi/4]$. Furthermore $a \le g(x) \le b$ on $[a, b]$. Then the contraction mapping theorem (*Elementary Numerical Analysis* by Kendall Atkinson, p. 84) guarantees a unique point α such that $\alpha = g(\alpha)$ in the interval $[a, b]$. In addition, for any initial x_0 in that interval, the iterates x_n will converge to α. We find that fixed point by solving the equation $x = g(x)$, which reduces to $\cot x = -1$, so $x = 3\pi/4$. Hence the sequence $\{x_n\}$ converges to $3\pi/4$.

III. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*
Since $x_{n+1} = x_n + (1 + \cot x_n)/2$, we take $x_n = 3\pi/4 - \varepsilon_n$. Then

$$\cot x_n = \frac{1 + \tan\frac{3}{4}\pi \tan\epsilon_n}{\tan\frac{3}{4}\pi - \tan\epsilon_n} = -1 + \frac{2\tan\epsilon_n}{1 + \tan\epsilon_n},$$

so that we have

$$x_{n+1} = \frac{3}{4}\pi - \epsilon_n + \frac{\tan\epsilon_n}{1 + \tan\epsilon_n} \quad \text{and} \quad \epsilon_{n+1} = \epsilon_n - \frac{\tan\epsilon_n}{1 + \tan\epsilon_n}.$$

We find that, if $x_1 = 2$, then $\varepsilon_1 = 0.356$, $\varepsilon_2 = 0.0850$, $\varepsilon_3 = 0.00649$, and in general, for small ε_n, $\varepsilon_{n+1} \approx \varepsilon_n^2$. Hence $\lim_{n\to\infty} \varepsilon_n = 0$ and the sequence converges to $3\pi/4$.

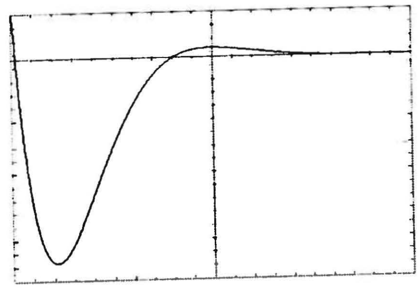
IV. *Solution and comment by Robert C. Gebhardt, Hopatcong, New Jersey.*

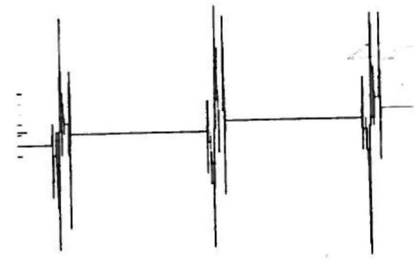
If the sequence converges, then as $n \to \infty$, $x_{n+1} - x_n \to 0$. If indeed $x = \lim_{n\to\infty} x_n$, then we must have $\cot x = -1$, so $x = (k + 3/4)\pi$ for some integer k. A pocket calculator shows that when $x_1 = 2$, then the sequence converges to $3\pi/4$.

There are some values of x_1 (e.g., 0) for which the sequence does not converge. The sequence has curious properties. A plot of the limit x, when that limit exists, vs. x_1 shows a sort of stair-step pattern, with a scattering of many different values in the vicinity of certain values of x_1. All such limits x satisfy the equation $x = (k + 3/4)\pi$ for some integer k.

Also solved by Paul S. Bruckman, *Highwood, IL,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* Joe Howard and John Jeffries, *New Mexico Highlands University, Las Vegas,* David Iny, *Baltimore MD,* H.-J. Seiffert, *Berlin, Germany,* Lamarr Widmer, *Messiah College, Grantham, PA, and the* Proposer.

Comment by the editor. The accompanying figures show the graph of the damped sine curve function $f(x) = e^{-x}(\sin x + \cos x)$ of Solution I and the limit x vs. the initial value x_1 of Solution IV. The domain of each graph is [-4, 4]. the ranges are [-25, 5] for the first graph and [-40, 40] for the second one. The long tic mark on the second graph shows $y = 0$ and the other tic marks show $y = (k + 3/4)\pi$ for $k = -3, -2, ..., 3$. Notice that the erratic behavior in the second graph occurs (not surprisingly) at the horizontal tangents of the first graph.





894. [Fall 1996] *Proposed by Andrew Cusumano, Great Neck, New York.*

Let us take $P_2 = 4\sqrt{2 - \sqrt{2}}$, $P_3 = 8\sqrt{2 - \sqrt{2 + \sqrt{2}}}$, $P_4 = 16\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$, and so forth. Find the value of $\lim_{n\to\infty} n(P_n - P_{n-1})$.

Solution by H.-J. Seiffert, Berlin, Germany.

We will use the well-known trigonometric relations

$$2\cos\frac{x}{2} = \sqrt{2 + 2\cos x}, \quad 0 \le x \le \pi, \tag{1}$$

$$2\sin\frac{x}{2} = \sqrt{2 - 2\cos x}, \quad 0 \le x \le \pi, \tag{2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}, \quad x \in \mathbb{R}, \tag{3}$$

and

$$\lim_{x\to 0} \frac{\sin \pi x}{x} = \pi. \tag{4}$$

Using (1), it is easy to prove by mathematical induction that

$$2\cos\frac{\pi}{2^n} = \sqrt{2 + \sqrt{2 + \sqrt{\cdots + \sqrt{2}}}},$$

where there are $n - 1$ twos on the right side. Hence, by (2),

$$P_n = 2^{n+1}\sin\frac{\pi}{2^{n+1}}, \quad n \ge 2.$$

Now, from (2) and (3) we obtain

$$4^n(P_n - P_{n-1}) = 4^n\left(2^{n+1}\sin\frac{\pi}{2^{n+1}} - 2^n\sin\frac{\pi}{2^n}\right) = 2^{3n+1}\sin\frac{\pi}{2^{n+1}}\left(1 - \cos\frac{\pi}{2^{n+1}}\right)$$

$$= 2^{3n+2}\sin\frac{\pi}{2^{n+1}}\sin^2\frac{\pi}{2^{n+2}} = \frac{1}{8}\left(2^{n+1}\sin\frac{\pi}{2^{n+1}}\right)\left(2^{n+2}\sin\frac{\pi}{2^{n+2}}\right)^2,$$

so that by (4),

$$\lim_{n\to\infty} 4^n(P_n - P_{n-1}) = \frac{\pi^3}{8}.$$

It follows that the required limit is zero.

Also solved by Paul S. Bruckman, *Highwood, IL*, Kenneth B. Davenport, *Pittsburgh, PA*, Charles R. Diminnie, *San Angelo, TX*, Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville*, Richard I. Hess, *Rancho Palos Verdes, CA*, David Iny, *Baltimore MD*, Kandasamy Muthuvel, *University of Wisconsin-Oshkosh*, Khiem V. Ngo, *Virginia Polytechnic Institute and State University, Blacksburg, and the* Proposer.

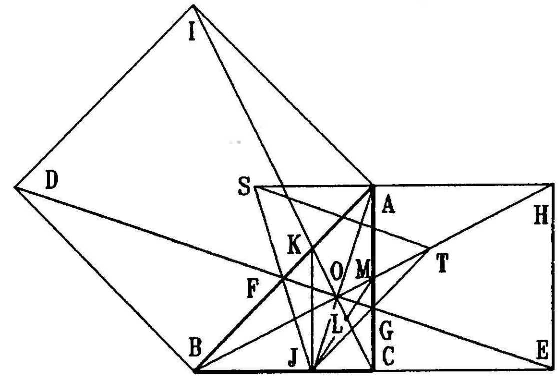
895. [Fall 1996] *Proposed by Andrew Cusumano, Great Neck, New York.*

Let ABC be an isosceles right triangle with right angle at C. Erect squares $ACEH$ and $ABDI$ outwardly on side AC and hypotenuse AB. Let CI meet BH at O and AB at K, and let AO meet BC at J. Let DE cut AB at F and AC at G. It is known (Problem 817, Fall 1994, page 72) that DE passes through O. Let JF meet AH at S and let JG meet BH at T. Finally, let BH and AC meet at M and let JM and CI meet at L. See the figure.

a) Prove that

i) ST is parallel to DOE,
ii) JK is parallel to AC,
iii) JG is parallel to AB,
iv) AI passes through T,
v) JF passes through I,
vi) EK passes through M, and
vii) BL passes through G.

*b) Which of these results generalize to an arbitrary triangle?



Solution by William H. Peirce, Delray Beach, Florida.

With no loss of generality, let triangle ABC be given by $A(a,b)$, $B(-1,0)$, and $C(1,0)$. The outward squares on AC and AB have vertices $D(-1-b, 1+a)$, $E(1+b, 1-a)$, $H(a+b, 1-a+b)$, and $I(a-b, 1+a+b)$ (which are inward squares if $b < 0$).

a) Since $\angle ACB = 90°$ and $AC = BC$, we have $A(1,2)$, $D(-3,2)$, $E(3,0)$, $H(3,2)$, and $I(-1,4)$. From the defining intersections we get $O(3/5, 4/5)$, $K(1/3, 4/3)$, $J(1/3, 0)$, $F(0,1)$, $G(1, 2/3)$, $S(-1/3, 2)$, $T(5/3, 4/3)$, $M(1,1)$, and $L(5/7, 4/7)$.

Letting m_{AB} denote the slope of line AB, it is easy now to see that

$$m_{ST} = m_{DOE} = -\tfrac{1}{3}, \quad \text{neither } m_{JK} \text{ nor } m_{AC} \text{ exists}, \quad m_{JG} = m_{AB} = 1,$$

so we see that (i), (ii), and (iii) are satisfied.

Three points (x_1,y_1), (x_2,y_2), (x_3,y_3), in the plane are collinear if and only if the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

For each of the following four sets of points, it is readily checked that the associated determinant does indeed vanish:

$$AIT: \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 1 & \frac{5}{3} & \frac{4}{3} \end{vmatrix}, \quad JFI: \begin{vmatrix} 1 & \frac{1}{3} & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 4 \end{vmatrix}, \quad EKM: \begin{vmatrix} 1 & 3 & 0 \\ 1 & \frac{1}{3} & \frac{4}{3} \\ 1 & 1 & 1 \end{vmatrix}, \text{ and } BLG: \begin{vmatrix} 1 & -1 & 0 \\ 1 & \frac{5}{7} & \frac{4}{7} \\ 1 & 1 & \frac{2}{3} \end{vmatrix}$$

This completes the proof of part (a).

b) It is necessary here to use the general triangle $A(a, b)$, $b > 0$, $B(-1, 0)$, $C(1, 0)$, in which there are many degenerate cases where points are either coincident or not defined. (For example, if A lies on the circle whose diameter is side BC, then points A, O, K, F, G, S, T, M, and L all coincide.) We shall ignore these special cases.

Following part (a), develop the coordinates for the thirteen points D, E, H, I, O, K, J, F, G, S, T, M, and L. Then, as in Part (a), equate the three pairs of slopes and set each of the four determinants to zero. We omit the rather tedious details.

i) We have ST parallel to DOE if and only if

$$(1 + a + b - a^2 + ab - a^3 - ab^2)(-a + b + a^2 + b^2) = 0.$$

This equation is the product of two factors, the first of which we shall call Z and return to in part (iv), and the second factor, which when set to zero, represents a circle of radius $1/\sqrt{2}$ and center $(1/2, -1/2)$ in the ab-plane.

ii) Line JK is parallel to AC if and only if $-3 + 2a - 2b + a^2 + b^2 = 0$, a circle of radius $\sqrt{5}$ and centered at $(-1, 1)$ in the ab-plane.

iii) Lines JG and AB are parallel if and only if $-2 - a - b + a^2 + b^2 = 0$, a circle of radius $\sqrt{5/2}$ and center $(1/2, 1/2)$ in the ab-plane.

iv and v) Points A, I, T are collinear and J, F, I are collinear if and only if the expression Z of part (i) is zero.

vi) Points E, K, M are collinear if and only if $-1 + a = 0$, that is, when triangle ABC has a right angle at C.

vii) Finally, B, L, G are collinear for all triangles ABC.

Thus the answer to part (b) is that property (vii) is the only property that generalizes to all triangles.

Also solved by Paul S. Bruckman, *Highwood, IL*, Mark Evans, *Louisville, KY*, Victor G. Feser, *University of Mary, Bismarck, ND, and partially by the* Proposer.

896. [Fall 1996] *Proposed by Peter A. Lindstrom, Irving, Texas.*

For arbitrary positive integers k and n, find each summation:

a) $\displaystyle\sum_{i=1}^{n} (i)(i+1)(i+2)\cdots(i+k)$.

b) $\displaystyle\sum_{i=1}^{n} (i)(i-1)(i-2)\cdots(i-k)$, where $n \ge k + 1$.

c) $\displaystyle\sum_{i=1}^{n} (i-k)(i-k+1)\cdots(i-1)(i)(i+1)(i+2)\cdots(i+k)$,

where $n \ge k + 1$.

Amalgam of essentially identical solutions by SUNY Fredonia Student Group, Fredonia, New York, and University of Central Florida Problems Group, Orlando, Florida.

a) For all positive integers n and nonnegative integers k, we claim that

$$\sum_{i=1}^{n} (i)(i + 1)(i + 2)\cdots(i + k) = \frac{(n + k + 1)!}{(n - 1)!(k + 2)},$$

which we prove by mathematical induction. The case for $n = 1$ is clear, so assume the statement is true for some positive integer n. Then, for $n + 1$ we have

$$\sum_{i=1}^{n+1} (i)(i + 1)(i + 2)\cdots(i + k) = \frac{(n + k + 1)!}{(n - 1)!(k + 2)} + \frac{(n + k + 1)!}{n!}$$

$$= \frac{(n + k + 1)!\,n}{n!(k + 2)} + \frac{(n + k + 1)!(k + 2)}{n!(k + 2)} = \frac{(n + k + 2)!}{n!(k + 2)},$$

and the proof is complete.

b) We use part (a) and note that the first k terms of this series are zero, yielding

$$\sum_{i=1}^{n} (i)(i - 1)(i - 2)\cdots(i - k) = \sum_{i=k+1}^{n} (i)(i - 1)(i - 2)\cdots(i - k)$$

$$= \sum_{i=1}^{n-k} (i)(i + 1)(i + 2)\cdots(i + k) = \frac{(n + 1)!}{(n - k - 1)!(k + 2)}.$$

c) Using the same technique as in Part (b), we see that the first k terms vanish and we have

$$\sum_{i=1}^{n}(i-k)(i-k+1)\cdots(i)(i+1)(i+2)\cdots(i+k)$$

$$=\sum_{i=1}^{n-k}(i)(i+1)(i+2)\cdots(i+2k)=\frac{(n+k+1)!}{(n-k-1)!(2k+2)}.$$

Also solved by Ayoub B. Ayoub, *Pennsylvania State University, Ogantz Campus, Abington,* Paul S. Bruckman, *Highwood, IL,* William Chau, *New York, NY,* Kenneth B. Davenport, *Pittsburgh, PA,* Charles R. Diminnie, *San Angelo, TX,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* George P. Evanovich, *Saint Peter's College, Jersey City, NJ,* Mark Evans, *Louisville, KY,* Murray S. Klamkin, *University of Alberta, Canada,* Carl Libis, *University of Alabama, Tuscaloosa,* David E. Manes, *SUNY College at Oneonta,* William H. Peirce, *Delray Beach, FL,* H.-J. Seiffert, *Berlin, Germany,* Kenneth M. Wilke, *Topeka, KS,* Rex H. Wu, *Brooklyn, NY, and the* Proposer.

897. [Fall 1996] *Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.*

Show that all non-negative integral solutions of the Diophantine equation

$$3(x_i - x_{i-1})^2 = 2(x_i + x_{i-1})^2 + 1$$

are given by consecutive terms of an infinite sequence of integers x_i with $x_0 = 0$, $x_1 = 1$, and $x_{i+1} = ax_i + bx_{i-1}$. Find a and b and the first seven terms of the sequence x_i. Generalize this procedure and determine the solution x_i for the equation

$$(2c+1)(x_i - x_{i-1})^2 = 2c(x_i + x_{i-1})^2 + 1.$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

Let $p_i = x_i - x_{i-1}$ and $q_i = x_i + x_{i-1}$, so that $x_i = (p_i + q_i)/2$. Then p_i and q_i are solutions of the Fermat-Pell equation

$$3P^2 - 2Q^2 = 1. \qquad (1)$$

All solutions in positive integers are given by the equation

$$p_i\sqrt{3} + q_i\sqrt{2} = (\sqrt{3}+\sqrt{2})(5+2\sqrt{6})^i,$$

where $p_1 = q_1 = 1$ is the solution of (1) in smallest nonnegative integers and $5 + 2\sqrt{6}$, that is, $x = 5$ and $y = 2$, is the fundamental solution of the Fermat-Pell equation $x^2 - 6y^2 = 1$. The first few solutions and corresponding x_i are given in the Table 1.

Table 1. Solutions to the Original Problem.

i	p_i	q_i	x_i
1	1	1	1
2	9	11	10
3	89	109	99
4	881	1079	980
5	8721	10681	9701
6	86329	105731	96030
7	854569	1046629	950599

Substituting $x_0 = 0$, $x_1 = 1$, $x_2 = 10$, and $x_3 = 99$ into the equation $x_{i+1} = ax_i + bx_{i-1}$, we obtain $10 = a + 0$ and $99 = 10a + b$, so $a = 10$ and $b = -1$.

More generally, the equation $(2c+1)(x_i - x_{i-1})^2 = 2c(x_i + x_{i-1})^2 + 1$ is equivalent to the Fermat-Pell equation

$$(2c+1)P^2 - (2c)Q^2 = 1, \qquad (2)$$

where p_i and q_i are defined as above. The solutions to (2) are given by

$$p_i\sqrt{2c+1} + q_i\sqrt{2c} = (\sqrt{2c+1}+\sqrt{2c})[(4c+1)\sqrt{2c+1} + (4c+3)\sqrt{2c}]^i,$$

where $(4c+1) + 2\sqrt{2c(2c+1)}$ is the fundamental solution of the equation

$$x^2 - [2c(2c+1)]y^2 = 1.$$

The first few solutions and corresponding x_i are given in Table 2. As in the first case, we solve for a and b, obtaining $a = 8c + 2$ and $b = -1$, so that $x_{n+1} = (8c+2)x_n - x_{n-1}$ in the generalized case.

Table 2. Solutions to the Generalized Problem.

i	p_i	q_i	x_i
1	1	1	1
2	$8c + 1$	$8c + 3$	$8c + 2$
3	$64c^2 + 24c + 1$	$64c^2 + 40c + 5$	$64c^2 + 32c + 3$
4	$512c^3 + 320c^2 + 48c + 1$	$512c^3 + 448c^2 + 112c + 7$	$512c^3 + 384c^2 + 80c + 4$

Also solved by Paul S. Bruckman, *Highwood, IL,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* Richard I. Hess, *Rancho Palos Verdes, CA,* Murray S. Klamkin, *University of Alberta, Canada,* William H. Peirce, *Delray Beach, FL,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

Klamkin referred to his article "Perfect Squares of the form $(m^2 - 1)a_n^2 + t$," *Math. Mag.*, 40(1969)111-113.

898. [Fall 1996] *Proposed by Paul S. Bruckman, Edmonds, Washington.*

An n-digit number N is defined to be a *base 10 Armstrong number of order n* if

$$N = \sum_{k=0}^{n-1} d_k 10^k = \sum_{k=0}^{n-1} d_k^n,$$

where the d_k are decimal digits, with $d_{n-1} > 0$. (See Miller and Whalen, "Armstrong Numbers: $153 = 1^3 + 5^3 + 3^3$," *Fibonacci Quarterly* 30.3, (1992), pp. 221-224.) Prove that there are no base ten Armstrong numbers of order 2; that is, prove the impossibility of the equation

$$10y + x = x^2 + y^2,$$

where x and y are integers with $0 \le x \le 9$ and $1 \le y \le 9$.

I. *Solution by Charles R. Diminnie, San Angelo, Texas.*

By completing the squares, we reduce the equation to

$$(2x - 1)^2 + 4(y - 5)^2 = 101.$$

Since 101 can be written as a sum of two squares in exactly one way, $101 = 100 + 1$, and since $2|y - 5|$ is even, then we must have $2|y - 5| = 10$ and hence $y = 0$ or $y = 10$. Therefore, there are no integer solutions with $1 \le y \le 9$.

II. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Since $10y - x(x - 1) = y^2$, then y^2 is even, so y is even. Thus $y = 2, 4, 6$, or 8. If $y = 2$ or 8, then the given equation becomes

$$x^2 = x + 16, \text{ that is, } (2x - 1)^2 = 65.$$

When $x = 4$ or 6, the equation becomes

$$x^2 = x + 24, \text{ that is, } (2x - 1)^2 = 97.$$

Since neither 65 nor 97 is a square number, there is no solution with the given constraints.

III. *Solution by James Campbell, University of Missouri, Columbia, Missouri.*

We must have $(10 - y)y = x(x - 1)$. Now $9 \le (10 - y)y \le 25$ whenever $1 \le y \le 9$ and $x(x - 1)$ is an increasing function, so if there is a solution, $x = 4$ or $x = 5$. Substituting these values in for x, we get the equations

$$y^2 - 10y + 12 = 0 \text{ and } y^2 - 10y + 20 = 0.$$

In neither case is the discriminant a perfect square, so no integer solution exists.

IV. *Solution by Robert C. Gebhardt, Hopatcong, New Jersey.*

By the quadratic formula,

$$y = 5 \pm \sqrt{25 - x^2 + x}.$$

The radicand, $25 - x^2 + x$, is a perfect square only if $x = 0$ or $x = 1$. Then $y = 0$ or $y = 10$, neither of which is permitted. Thus there is no solution.

V. *Solution by Rex H. Wu, Brooklyn, New York.*

The given equation can be written in the form

$$\frac{x}{y} = \frac{10 - y}{x - 1},$$

so therefore, $x = k(10 - y)$ and $y = k(x - 1)$, where $k = 1, 2, ..., 9$, or perhaps $k = 1/m$, where m is such a digit. Eliminate y between these two equations and solve for x to get

$$x = \frac{11k}{1 + k^2} \quad \text{or} \quad x = \frac{11m}{m^2 + 1}.$$

For x to be an integer, either $(1 + k^2)$ or $(m^2 + 1)$ must divide 11, which is impossible since this occurs only when $k = 10$ or $m = 10$.

Also solved by Avraham Adler, *New York, NY,* Charles Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* Victor G. Feser, *University of Mary, Bismarck, ND,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Carl Libis, *University of Alabama, Tuscaloosa,* Henry S. Lieberman, *Waban, MA,* Peter A. Lindstrom, *Batavia, NY,* David E. Manes, *SUNY College at Oneonta,* William H. Peirce, *Delray Beach, FL,* H.-J. Seiffert, *Berlin, Germany,* Kenneth M. Wilke, *Topeka, KS, and the* Proposer.

Editor's comment. Although it may seem wasteful to publish five solutions to a rather simple problem, it is the very variety and cleverness of the solutions that make the problem interesting.

899. [Fall 1996] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find the average number of times an ordinary six-sided die must be tossed in order that each of its six faces comes up at least once.

I. *Solution by Paul S. Bruckman, Highwood, Illinois.*

Let n denote the number of times a normal die is cast in order for each face to turn up at least once. Thus $n \geq 6$. Let p_n denote the probability that exactly n casts of the die are required to fulfill this requirement. Clearly, $p_6 = 6!/6^6 = 5/324$. If $n \geq 6$, we see that p_n is the probability that exactly five different faces have turned up in $n - 1$ casts and that the sixth face has come up in the nth cast. There are ${}_5C_6 = 6$ ways to select the first five faces that turn up, and with such a choice, there are 5 possible outcomes for the first cast, 5 for the next cast, and so on, for a total of 5^{n-1} outcomes. We must,

however, subtract from this the number of ways in which only four faces turn up in the $n - 1$ casts, and this can happen in $5 \cdot 4^{n-1}$ ways, except for the cases where only 3 faces turn up, which can happen in $10 \cdot 3^{n-1}$ ways, and so on. Thus the net number of ways to have exactly 5 faces turn up in $n - 1$ casts is equal to

$$6(5^{n-1} - 5 \cdot 4^{n-1} + 10 \cdot 3^{n-1} - 10 \cdot 2^{n-1} + 5) \equiv A_{n-1},$$

say. Therefore, $p_n = A_{n-1}/6^n$, $n = 6, 7, ...$ It is easily verified that $\Sigma_{n\geq 6}\, p_n = 1$ and also that $A_5 = 720 = 6!$.

We seek the expectation E of the distribution, namely $E = \Sigma_{n\geq 6}\, np_n$. We find, using $\Sigma_{n\geq 0}\, r^n = 1/(1 - r)$ and $\Sigma_{n\geq 0}\, nr^n = r/(1 - r)^2$ when $|r| < 1$, that

$$E = \sum_{n\geq 5} \frac{(n + 1)(5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5)}{6^n}$$

$$= \sum_{n=0}^{\infty} (n + 6)\left[\left(\frac{5}{6}\right)^{n+5} - 5\left(\frac{4}{6}\right)^{n+5} + 10\left(\frac{3}{6}\right)^{n+5} - 10\left(\frac{2}{6}\right)^{n+5} + 5\left(\frac{1}{6}\right)^{n+5}\right]$$

$$= \left(\frac{5}{6}\right)^5 (6^2 + 5 \cdot 6) - 5\left(\frac{4}{6}\right)^5 (3^2 + 5 \cdot 3) + 10\left(\frac{3}{6}\right)^5 (2^2 + 5 \cdot 2)$$

$$- 10\left(\frac{2}{6}\right)^5 \left[\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right)\right] + 5\left(\frac{1}{6}\right)^5 \left[\left(\frac{6}{5}\right)^2 + 5\left(\frac{6}{5}\right)\right] = 14.7,$$

exactly.

II. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

With k (= 1 to 6) faces yet to come up, the expected number of rolls to get one of the k to appear is

$$E_k = \frac{k}{6} + 2 \cdot \frac{k}{6} \cdot \frac{6 - k}{6} + 3 \cdot \frac{k}{6} \cdot \left(\frac{6 - k}{6}\right)^2 + \cdots$$

$$= \frac{k}{6}\left[1 + 2 \cdot \frac{6 - k}{6} + 3 \cdot \left(\frac{6 - k}{6}\right)^2 + \cdots\right]$$

$$= \frac{k}{6}\cdot\frac{1}{\left(1-\frac{6-k}{6}\right)^2} = \frac{k}{6}\cdot\frac{36}{k^2} = \frac{6}{k}.$$

Now the expected number of turns to get all 6 faces to appear at least once is

$$E = E_6 + E_5 + E_4 + \cdots + E_1 = \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7.$$

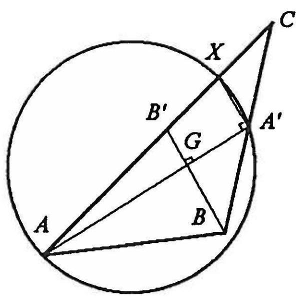
Also solved by Avraham Adler, *New York, NY,* Mark Evans, *Louisville, KY,* David Iny, *Baltimore MD,* Corie Kreps *and* Karen Ellison, *Oxford, OH,* Shiva K. Saksena, *University of North Carolina at Wilmington,* University of Central Florida Problems Group, *Orlando,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer. One incorrect solution was received.

Zerger pointed out that this problem is an example of the famous "coupon problem" which is mentioned in many probability books, as well as in *Fifty Challenging Problems in Probability* by Mosteller, Dover, 1965.

900. [Fall 1996] *Proposed by Howard Eves, Lubec, Maine.*

Given the lengths of two sides of a triangle and that the medians to those two sides are perpendicular to each other, construct the triangle with Euclidean tools.

I. *Solution by Ben Smith, Miami University, Oxford, Ohio.*



Let the given lengths be a and b. Construct a segment AC of length b. Construct its midpoint B' and the midpoint X of segment $B'C$. Draw the circle on AX as diameter and swing an arc of length $a/2$ from C to cut the circle at A'. See the accompanying figure. Extend CA' its own length to point B. Then $CB = a$. Now ABC is the desired triangle.

Proof: Clearly $\angle AA'X = 90°$ since it is inscribed in a semicircle. Since X and A' are the midpoints of two sides of triangle $CB'B$, then $A'X$ is parallel to $B'B$, so the medians AA' and BB' are perpendicular. Since $AC = b$ and $BC = a$, the proof is complete.

The construction can be carried out if and only if $a > b/2$ and $b > a/2$.

II. *Solution by the Problems Editor.*

Referring to the figure in Solution I, where G is the intersection of the medians, recall that G trisects each median. Let m_a and m_b denote the lengths of medians AA' and BB' respectively. From right triangles AGB' and BGA' we get the equations

$$\left(\frac{2}{3}m_a\right)^2 + \left(\frac{1}{3}m_b\right)^2 = \left(\frac{b}{2}\right)^2 \quad\text{and}\quad \left(\frac{2}{3}m_b\right)^2 + \left(\frac{1}{3}m_a\right)^2 = \left(\frac{a}{2}\right)^2.$$

These equations can be solved simultaneously to find that

$$m_a^2 = \frac{3}{5}\left[b^2 - \left(\frac{a}{2}\right)^2\right] \quad\text{and}\quad m_b^2 = \frac{3}{5}\left[a^2 - \left(\frac{b}{2}\right)^2\right].$$

The lengths of the medians are constructible, so triangle AGB' is readily constructed and its sides extended to locate points A', C, and B. Also

$$c^2 = \left(\frac{2}{3}m_a\right)^2 + \left(\frac{2}{3}m_b\right)^2 = \frac{1}{5}(a^2 + b^2),$$

which also is readily constructed with Euclidean tools.

Also solved by Paul S. Bruckman, *Highwood, IL,* William Chau, *New York, NY,* George Tsapakidis, *Agrinio, Greece,* University of Central Florida Problems Group, *Orlando,* Rex H. Wu, *Brooklyn, NY, and the* Proposer.

Corrections

Henry S. Lieberman, Waban, MA, was inadvertently overlooked as an also-solver for problems 875, 878, and 880.

MISCELLANY

Chapter Reports

Professor Joanne Snow reported that the **INDIANA EPSILON Chapter** (Saint Mary's College) was addressed by **John Emert** (Ball State University) at the department's annual Open House. The chapter performed various service activities during the year.

It was reported that three talks were presented to the **NEW YORK OMEGA Chapter** (Saint Bonaventure University) during the 1995-96 academic year. **David Tascione**, chapter Secretary-Treasurer, was awarded the Mathematics Medal, and the Myra J. Reed Award (Pi Mu Epsilon Award) went to graduating President **Nicole Giovanniello**.

Professor Cathy Talley reported that the **TEXAS ZETA Chapter** (Angelo State University) was addressed by **Mr. Trey Smith** at the fall induction ceremony and by **Dr. Charles Diminnie** at the spring induction ceremony. The chapter co-sponsored three mathematics forms during the year.

Dr. David Sutherland reported that the **ARKANSAS BETA Chapter** (Hendrix College) was addressed by five speakers during the year. Their Undergraduate Research Program was very active. The McHenry-Lane Freshman Math Award was given to **Zachary Manis**. The Hogan Senior Math Award was given to **Diana Hua** and **Jennifer Powell**. The Phillip Parker Undergraduate Research Award was given to **Jac Cole.** Graduating with Distinction was **Jac Cole**, **Diana Hua**, and **Jennifer Powell**.

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