PI MU EPSILON JOURNAL

VOLUME 10 SPRING 1998 NUMBER 8

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PI MU EPSILON JOURNAL

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Approximation Formulas For Primes

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University of Alabama

This note gives two approximation formulas for a prime p_{n+1} when we have knowledge of the primes $p_1, p_2, \cdots, p_n$.

(a) By the definition of the Riemann zeta-function we know

$$\Pi(1-p^{-s})^{-1}=\zeta(s)$$

for $s > 1$, where p goes through all prime numbers. We have

$$\prod_{i=1}^{n}(1-p_i^{-s})^{-1}.(1-p_{n+1}^{-s})^{-1}\approx\zeta(s)$$

since

$$\prod_{i=n+2}^{\infty}(1-p_i^{-s})^{-1}\approx 1$$

for large s. Hence

$$1-p_{n+1}^{-s}\approx\zeta(s)^{-1}\prod_{i=1}^{n}(1-p_i^{-s})^{-1}$$

and we get

$$p_{n+1}\approx\{1-\zeta(s)^{-1}\prod_{i=1}^{n}(1-p_i^{-s})^{-1}\}^{-1/s}. \qquad (1)$$

(b) From

$$\prod(1-p^{-s})^{-1}=\zeta(s)$$

and

$$\prod(1-p^{-2s})^{-1}=\zeta(2s)$$

we get

$$\prod(1+p^{-s})=\zeta(s)/\zeta(2s)$$

where p goes through all prime numbers. Hence we have

$$\prod_{i=1}^{n}(1+p_i^{-s})\cdot(1+p_{n+1}^{-s})\approx\zeta(s)/\zeta(2s)$$

$$\prod_{i=n+2}^{\infty}(1+p_i^{-s})^{-1}\approx 1$$

for large s. Hence,

$$1+p_{n+1}^{-s}\approx\frac{\zeta(s)}{\zeta(2s)}\cdot\prod_{i=1}^{n}(1+p_i^{-s})^{-1}$$

and

$$p_{n+1}\approx\{\frac{\zeta(s)}{\zeta(2s)}\cdot\prod_{i=1}^{n}(1+p_i^{-s})^{-1}-1\}^{-1/s}. \qquad (2)$$

The larger we take s in (1) and (2) the more accurate will be p_{n+1}. Using "MAPLE V" to do the calculation, the approximations are very good for small n. For example, we get p_2=2.999999999999998 by substituting $n = 1$ and $s = 20$ in formula (1)(using the 800 digit calculation). Due to computer limitations, we cannot get accurate results for large n. The first prime 2, corresponding to $n = 0$, can be generated by

$$p_1=\lim_{s\to\infty}\left[1-\zeta(s)^{-1}\right]^{-1/s}$$

and

$$p_1=\lim_{s\to\infty}\left[\frac{\zeta(s)}{\zeta(2s)}-1\right]^{-1/s}.$$

In practice, we just take some reasonable large s, since using larger s needs more calculation. Remark: In [1], Von E. Teuffel gave a similar formula

$$p_n=\left\lceil 1+\{1-\zeta(k)\prod_{i=1}^{n-1}(1-p_i^{-k})^{-1}\}^{1/k}\right\rceil, \qquad (3)$$

where $\lceil x\rceil$ denotes the greatest integer less than or equal to x.

Reference

1. Teuffel, Von E., *Eine Rekursionsformel fur primzahlen*, Jber. Deutsch. math. Verein. 57, Abt. 1(1954), pp.34-36.

Regular Sierpinski Polyhedra[1]

Aimee Kunnen (student)
Steven Schlicker
Grand Valley State University

Introduction: In the paper entitled Sierpinski N-Gons [1], Kevin Dennis and Steven Schlicker constructed Sierpinski n-gons in the plane for each positive integer n. In this paper, we extend these constructions to 3-space to build regular Sierpinski polyhedra.

Background: The process we will use to construct Sierpinski polyhedra is the following: (See [1] and Michael Barnsley's book Fractals Everywhere [2] for more deails). Let $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^3$ with $x_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$ the vertices of a regular polyhedron, A_0. For $r > 0$, we define $w_i\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{r}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{r-1}{r}\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$, for $1 \le i \le n$. Define $A_{m,i}(r) = w_i(A_{m-1}(r))$ and let $A_{m+1}(r) = \bigcup_{i=1}^{n} A_{m,i}(r)$.

For example, let A_0 be the regular tetrahedron, as in figure 1, and let $r = 2$. Then, w_i, when applied to A_0 contracts A_0 by a factor of 2 and then translates the image of A_0 so that the ith vertices of A_0 and the image of A_0 coincide. Then $A_{1,i}$ is the set of all points half way between any point in A_0 and x_i, or $A_{1,i}$ is a tetrahedron half the size of the original translated to the ith vertex of the original. A_0 and A_1 are shown in figures 1 and 2 respectively.

Figure 1

(r = 2)

Figure 2

[1] This project was supported by a grant from the Summer Undergraduate Research Program (SURP) at Grand Valley State University.

We can continue this procedure replacing A_0 with A_1. For $i = 1, 2, 3$, or 4, let $A_{2,i} = w_i(A_1)$ and let $A_2 = \cup A_{2,i}$. A_2 is pictured in figure 3. Again, we can continue this procedure, each time replacing A_i with A_{i+1}. A_3 is shown in figure 4.

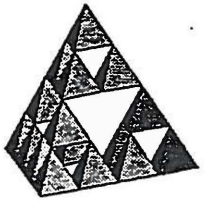


Figure 3
$r = 2$
A_2

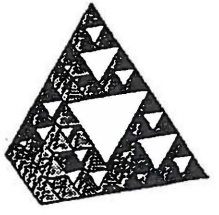


Figure 4
$r = 2$
A_3

Note that, for $r = 2$ the A_i consist of tetrahedra that just touch each other. For smaller values of r, the tetrahedra overlap and for larger r the tetrahedra are disjoint. See figures 5(A_1, $r = 1.5$) and 6(A_1, $r = 3$).

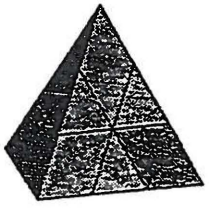


Figure 5
$r = 1.5$
A_1

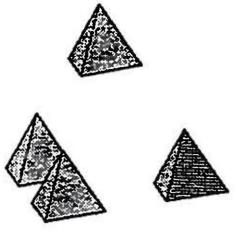


Figure 6
$r = 3$
A_1

If we take the limit as i approaches infinity for the just touching value of r, the resulting figure is the Sierpinski terahedron. This algorithm for building the Sierpinski tetrahedron is called the **deterministic algoithm**.

Sierpinski polyhedra: In the above discussion, there seems to be no reason why we should restrict ourselves to looking at only the regular tetrahedron. Why not consider the other four regular polyhedra (hexahedron, octahedron, dodecahedron, icosahedron)? In this paper, we will determine, for each regular polyhedra, the specific value of r that makes $A_m(r)$ just touching and we will determine the fractal dimension of each of the resulting Sierpinski polyhedra.

Fractal dimension: The figures we will be discussing are all examples of a wide class of objects known as fractals. Every fractal has a number associated to it, the fractal dimension. As a consequence of Theorem 3, p. 184 from [2], the fractal dimension of a Sierpinski polyhedra with n vertices and a scale factor of r is $\ln(n)/\ln(r)$. See [1] for a proof. For example, the fractal dimension of the Sierpinski tetrahedron is $\ln(4)/\ln(2) = 2$ We begin with the regular hexadron, or more commonly, the cube. Figure 7 is a regular hexahedron. By inspection, we can see that the cube has a scale factor of 2. This is illustrated in figure 8. Then, the fractal dimension of the Sierpinski hexahedron is $\ln(8)/\ln(2) = 3$. Note that the Sierpinski hexagon is a cube whose dimension is indeed 3. Next, we look at the regular octahedron. Figure 9 is a regular octahedron. By inspection, we note that each face of the Sierpinski octahedron will be a Sierpinski triangle and thus the regular octahedron has a scale factor of 2. This can be seen in figure 10. The fractal dimension of the regular octahedron is $\ln(6)/\ln(2) \sim 2.585$

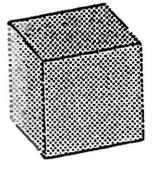


Figure 7

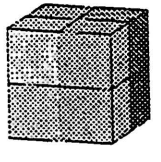


Figure 8

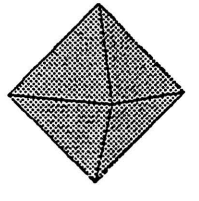


Figure 9

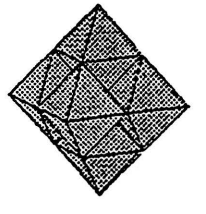


Figure 10

The scale factors for the regular dodecahedron and the regular icosahedron are more difficult to find, and we find those scale factors in the remainder of this paper.

Background Information: The following regular pentagons which are inscribed in a circle of radius r will aid us in determining the scale factor for the dodecahedron and the icosahedron. In figure 11, we let the point a represent the center of the circle. Let d be the length of any side of the face. We let $\overline{ac}$ be the perpendicular bisector of $\overline{eb}$. A little trigonometry shows that

$$d = 2r\sin\left(\frac{\pi}{5}\right), \cos\left(\frac{\pi}{5}\right) = \frac{s}{r}, \text{ and } s = \frac{d\cos\left(\frac{\pi}{5}\right)}{2\sin\left(\frac{\pi}{5}\right)}. \tag{1}$$

In figure 12, we let h represent the length between two vertices which are not adjacent. We see that $h = 2d\cos\left(\frac{\pi}{5}\right)$.

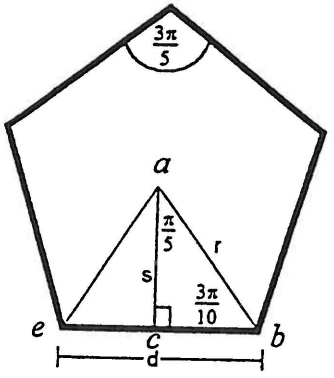


Figure 11

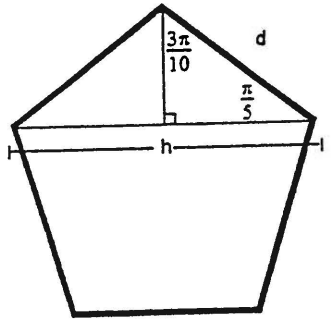
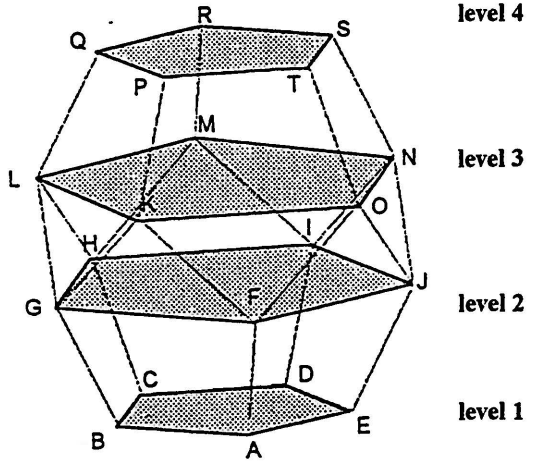


Figure 12

Regular dodecahedron: The regular dodecahedron has 12 faces all of which are regular pentagons. It has 20 vertices: 3 edges meet at each vertex. Figure 13 illustrates a regular dodecahedron.

Figure 13



The vertices of the dodecahedron can be thought of as four sets of five points each of which lie on pentagons parallel to the xy-plane. Each pentagon parallel to the xy-plane is assigned a level as is shown in figure 13. We label the points (x_{ij}, y_{ij}, z_i) where i represents the level and j represents the location of that point on that level. More specifically, on level one, we will start with $A = (x_{11}, y_{11}, z_i)$. We will then move in a clockwise direction when labeling the remaining points. Thus, $B = (x_{12}, y_{12}, z_1)$, $C = (x_{13}, y_{13}, z_1)$, $D = (x_{14}, y_{14}, z_1)$, and $E = (x_{15}, y_{15}, z_1)$. The remaining points on levels 2, 3, and 4 labeled in a similar fashion. We will now show how the scale factor can be found for the dodecahedron. In figure 14 we can see how two small dodecahedra will fit into a large dodecahedron in the just touching case.

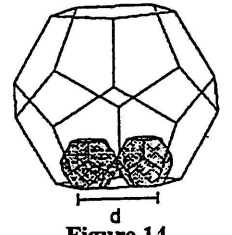


Figure 14

We label the just touching dodecahedron as in figure 15. The scale factor in the just touching case will then be d/d_1. To find d/d_1, we will find the angle between vectors g and k. We will then be able to find the length of f. Since all dodecahedra constructed are similar, it suffices to work with the large dodecahedron. To find g and k, we need to explicitly determine the points A, B, L, and O as labeled in figure 13.

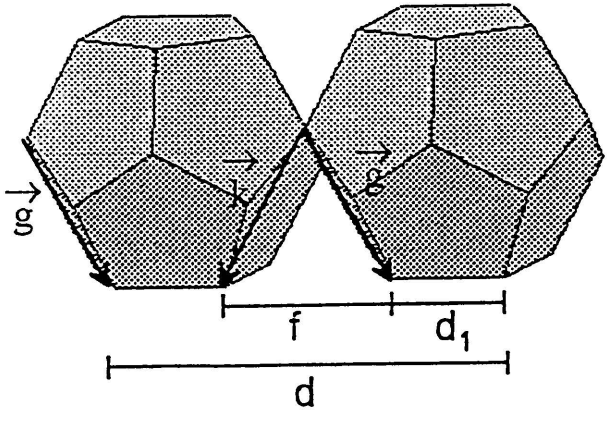


Figure 15

We will assume the base is on the xy-plane with center at (0, 0). We will assume that this base is inscribed in a circle with radius r. The points B and A are illustrated in figure 16. We can see from figure 16 that $A = (r, 0, 0)$ and $B = \left(r\cos\left(\frac{-2\pi}{5}\right), r\sin\left(\frac{-2\pi}{5}\right), 0\right)$. Now we must look at the third level for our other two points. Triangle i, represented in figure 18, will aide us in finding L and O. We let the dihedral angle (the angle between two faces) of the

dodecahedron be represented by θ. Therefore, angle qtv would be $\theta - \frac{\pi}{2}$. We can see from figure 11 that $|\overline{tv}| = s + r$. Then, $z_3 = (s + r)\cos\left(\theta - \frac{\pi}{2}\right)$. We let m be the length of the segment $\overline{qv}$. It follows that $m = (s+r)\sin\left(\theta - \frac{\pi}{2}\right)$. Note that $s + m$ is the distance that the point O, as well as L, is away from the z-axis.

Now we project the dodecahedron from figure 17 into the xy-plane. We see in figure 19 that

$$L = \left((s + m)\cos\left(\frac{-3\pi}{5}\right), (s + m)\sin\left(\frac{-3\pi}{5}\right), (s + r)\cos\left(\theta - \frac{\pi}{2}\right)\right) \quad \text{and}$$

$$O = \left((s + m)\cos\left(\frac{\pi}{5}\right), (s + m)\sin\left(\frac{\pi}{5}\right), (s + r)\cos\left(\theta - \frac{\pi}{2}\right)\right).$$

Now that we know points B, A, L, and O, we can find the vectors $\vec{k}$ and $\vec{g}$ from figure 18.

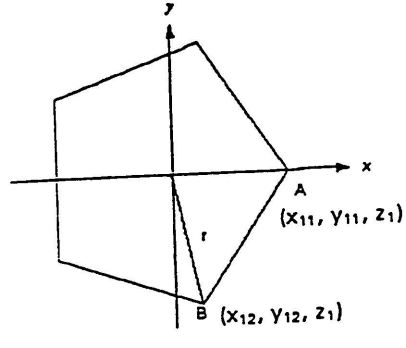


Figure 16

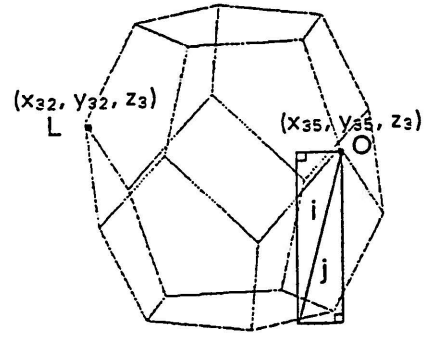


Figure 17

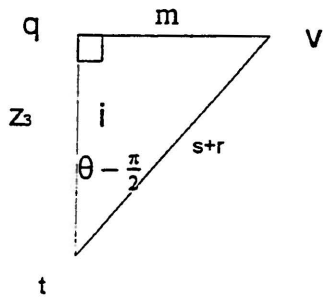


Figure 18

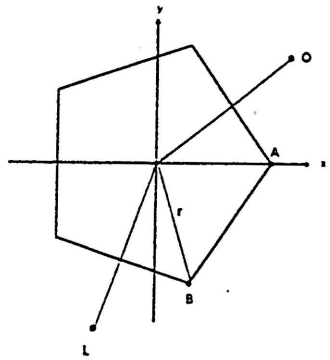


Figure 19

We know that $\vec{k} = A - O$

$$= \left\langle r-(s+m)\cos\left(\frac{\pi}{5}\right), -(s+m)\sin\left(\frac{\pi}{5}\right), -(s+r)\cos\left(\theta-\frac{\pi}{2}\right)\right\rangle$$

Then, $\vec{g} = B - L$,

$$= \left\langle r\cos\left(\frac{-2\pi}{5}\right)-(s+m)\cos\left(\frac{-3\pi}{5}\right), r\sin\left(\frac{-2\pi}{5}\right)-(s+m)\sin\left(\frac{-3\pi}{5}\right), -(s+r)\cos\left(\theta-\frac{\pi}{2}\right)\right\rangle.$$

Next, we used Maple to find the dot product of $\vec{k}$ and $\vec{g}$. We will let β be the angle between $\vec{k}$ and $\vec{g}$. From figure 12, we see that $|\vec{k}| = |\vec{g}| = h$. Then, since $\vec{k} \bullet \vec{g} = |\vec{k}|\ |\vec{g}|\cos\beta$, we know that $\cos\beta = \dfrac{\vec{k}\bullet\vec{g}}{h^2}$. We note from figure 16 that $\sin\dfrac{\beta}{2} = \dfrac{\frac{f}{2}}{h}$ and thus $f = 2h\sin\dfrac{\beta}{2} = 2h\sqrt{\dfrac{1-\cos\beta}{2}}$. So, $d = 2d_1 + 2h\sqrt{\dfrac{1-\cos\beta}{2}}$. Finally, our scale factor is d/d_1. For d/d_1, Maple gives

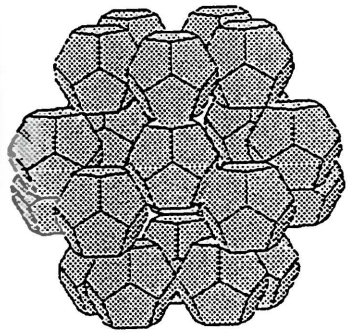
$$2+\frac{1}{8}\left(14-6\sqrt{5}-4\cos\left(\frac{1903}{5400}\pi\right)\sqrt{5}+20\cos\left(\frac{1903}{5400}\pi\right)+30\cos\left(\frac{1903}{5400}\pi\right)^2\right.$$

$$\left.+10\cos\left(\frac{1903}{5400}\pi\right)^2\sqrt{5}\right)^{1/2}\sqrt{5}+\frac{1}{8}\left(14-6\sqrt{5}-4\cos\left(\frac{1903}{5400}\pi\right)\sqrt{5}\right.$$

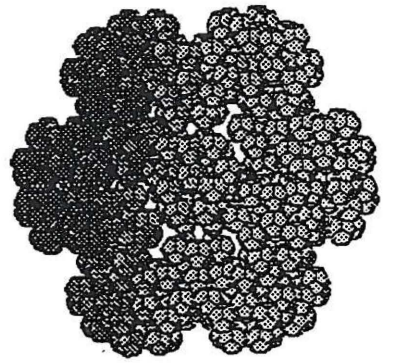
$$\left.+20\cos\left(\frac{1903}{5400}\pi\right)+30\cos\left(\frac{1903}{5400}\pi\right)^2+10\cos\left(\frac{1903}{5400}\pi\right)^2\sqrt{5}\right)^{1/2}.$$

To get an approximation for this scale factor, we substituted $\theta = \dfrac{3497}{5400}\pi$ from [3] for the dihedral angle. Thus, our scale factor for the Sierpinski dodecahedron is approximately 3.618107807.

The following figures illustrate an emerging Sierpinski dodecahedron.



A_1

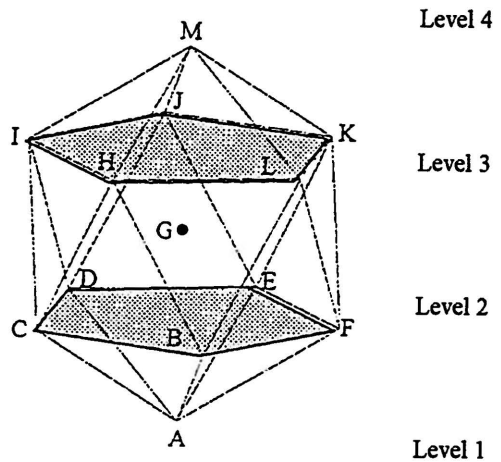


A_2

The fractal dimension of a Sierpinski dodecahedron is $\ln(20)/\ln(d/d_1) \sim 2.329584755$.

Regular icosahedron: The regular icosahedron is another interesting polyhedron. It has 20 faces all of which are equilateral triangles. It has 12 vertices each with 5 edges meeting. Figure 20 is a regular icosahedron. We will assume it is inscribed in the unit sphere.

Figure 20



In figure 20, we see that each vertex of the icosahedron lies on one of four "levels". The first level contains only one point, namely $A = (0, 0, -1)$. As with level 1, level 4 contains only one vertex, namely $M = (0, 0, 1)$. We will label the points (x_i, y_i, z_i) as we did with the dodecahedron. Lastly, $G = (0, 0, 0)$ is the center of the icosahedron. We note here that levels 2 and 3 are regular pentagons and are represented previously in figures 11 and 12. We let r be the radius of these pentagons.

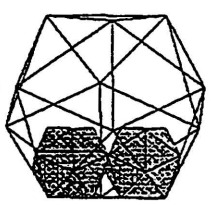


Figure 21

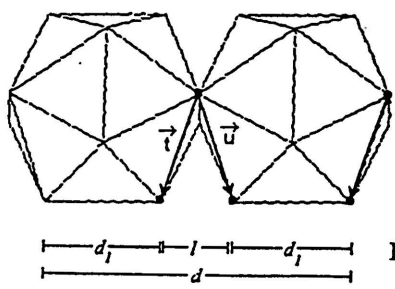


Figure 22

We will now show how the scale factor can be found for the icosahedron. Figure 21 will show us how two icosahedra fit into a larger icosahedron when the smaller ones are just touching. We will view figure 21 as if we are looking down on the top of the icosahedron. From figure 22, our scale factor is d/d_1. We can find d/d_1 by finding the angle between $\vec{u}$ and $\vec{t}$ which will allow us to find l. We know the small icosahedra are similar to the larger one. So, it suffices to find the corresponding points on the large icosahedron.

We need to determine the four points I, H, L, and K as labeled in figure 20. But, in order to do so, we first mus find r. Let (x_3, y_3, z_3) represent any arbitrary vertex on level three. Since our icosahedron is in the unit sphere,

$x_3^2 + y_3^2 + z_3^2 = 1$ or $r^2 + z_3^2 = 1$. It follows directly that

$$z_3 = \sqrt{1 - r^2}. \tag{2}$$

Since d is the distance from $(0, 0, 1)$ to $(x_3, y_3\, z_3)$,

$$d^2 = x_3^2 + y_3^2 + (z_3 - 1)^2. \tag{3}$$

Using (1), (2) and (3), we can use substitution and algebra to determine that

$r = \sqrt{\dfrac{4\sin^2\left(\frac{\pi}{5}\right)}{4\sin^4\left(\frac{\pi}{5}\right)}}$. Using Maple, we get $r = \dfrac{2\sqrt{3-\sqrt{5}}\sqrt{2}}{5-\sqrt{5}}$. Note that Maple expresses r in radical form. To do this one can solve the equation $\sin(5x) = 0$ by expanding $\sin(5x)$ using the standard angle sum formulas (found in any book on trigonometry) for the sine and cosine. One of the solutions to this equation will be $\sin\left(\frac{\pi}{5}\right)$.

Now, we can find I, H, L, and K. We know that each of these points has the same z-value, so we need only look at their x and y-values. Now we will project level 3 into the xy-plane resulting in figure 23.

In figure 23, $I=\left(r\cos\left(\frac{2\pi}{5}\right),r\sin\left(\frac{2\pi}{5}\right)\right)$, $H=\left(r\cos\left(\frac{4\pi}{5}\right),r\sin\left(\frac{4\pi}{5}\right)\right)$,

$L=\left(r\cos\left(\frac{6\pi}{5}\right),r\sin\left(\frac{6\pi}{5}\right)\right)$, and $K=\left(r\cos\left(\frac{8\pi}{5}\right),r\sin\left(\frac{8\pi}{5}\right)\right)$. Then,

$\vec{u}=L-K=\left\langle r\cos\left(\frac{6\pi}{5}\right)-r\cos\left(\frac{8\pi}{5}\right),r\sin\left(\frac{6\pi}{5}\right)-r\sin\left(\frac{8\pi}{5}\right)\right\rangle$. Also,

$\vec{t}=H-I=\left\langle r\cos\left(\frac{4\pi}{5}\right)-r\cos\left(\frac{2\pi}{5}\right),r\sin\left(\frac{4\pi}{5}\right)-r\sin\left(\frac{2\pi}{5}\right)\right\rangle$.

Now we determine the cosine of the angle β between $\vec{u}$ and $\vec{t}$ by using the dot product. We also note that the magnitude of both $\vec{u}$ and $\vec{t}$ is d_1. Thus, we have:

$$\vec{u}\bullet\vec{t} = |\vec{u}||\vec{t}|\cos(\beta) = d_1^2\cos(\beta).$$

Now, $\sin\left(\frac{\beta}{2}\right)=\frac{\frac{1}{2}}{d_1}$ or $2d_1\sin\left(\frac{\beta}{2}\right) = I$. So,

$$d=2d_1+l=2d_1+2d_1\sin\left(\frac{\beta}{2}\right)=2d_1+2d_1\sqrt{\frac{1-\cos(\beta)}{2}}.$$

Since $d_1 = 2r\sin\left(\frac{\pi}{5}\right)$, we can find cos(β). Then, Maple gives $\frac{3}{2}+\frac{\sqrt{5}}{2}\sim 2.618$ for the scale factor for the regular icosahedron.

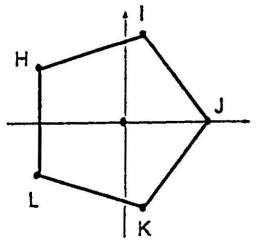
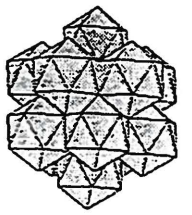
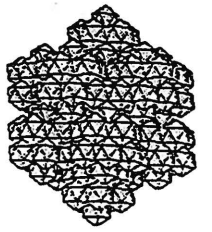


Figure 23

The following figures illustrate an emerging Sierpinski icosahedron.



A_1



A_2

The fractal dimension of a Sierpinski icosahedron is $\ln(12)/\ln(d/d_1) \sim 2.581926003$.

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Mauldin-Williams Graphs With Unique Dimension

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1. Introduction. In the past twenty years, fractals have fascinated people ranging from math dorks to fans of psychedelia. This recent explosion has lead many to believe that fractals are a new invention, which is not the case. Although Benoit Mandelbrot is the modern day beast of fractal geometry, the stylishly goateed Felix Hausdorff and others worked out much of the theory behind fractal geometry around the turn of the century. The actual pictures of fractals could not be easily calculated until the invention of the computer. Combining old theory with new technology, along with new insight, Mandelbrot cashed in on his book "The Fractal Geometry of Nature"[2] in 1982. In that book, Mandelbrot defined a fractal as a metric space whose topological dimension is strictly less than its Hausdorff dimension. We will not define these terms formally in this paper as adequate definitions take more than a few pages. (The interested reader is referred to [1].) We will say, however, that topological dimension is what comes to the layman's mind when someone mentions "dimension": a non-negative integer (or ∞) generalizing the concept of dimension of a vector space. The topological dimension of a single point is 0, of a line is 1, and so on. As one might expect, the topological dimension of the Cantor Set (which we will discuss later) is 0. On the other hand, the Hausdorff dimension of a space can be any non-negative real number (or ∞), and is defined via notions from measure theory. In section 2, we will present theorems which allow for a much easier calculation of the Hausdorff dimension of spaces constructed by using iterated function systems. The reader might be familiar with the notion of self-similarity of such spaces as the Cantor set or the Sierpiński triangle. The focus of the paper, however, will be to generalize this concept to that of graph self-similarity, a notion of several spaces similar to subsets of one another. The relationship between these metric spaces is represented by a Mauldin-Williams graph, named after two mathematicians who have done significant work on the subject ([3]). However, not every Mauldin-Williams graph represents an iterated function system with unique invariant sets. In section 3, we describe the relevant work of Mauldin and Williams, and give some theorems which guarantee the existence of unique invariant sets for a Mauldin-Williams graph. In section 4, we extend this work by presenting further conditions that are

necessary and sufficient for there to be unique invariant sets.

2. Definitions And Simple Cases. If (S, ρ) is a metric space, $f: S \to S$ is a **similarity** if there is a positive number r such that for any $x,y \in S$, $\rho(f(x), f(y)) = r\rho(x,y)$. The number r is called the **ratio** for f. An **iterated function system**, or IFS, is a finite set of similarities on S, $(f_1, f_2, \ldots, f_n)$. The corresponding set of ratios of these similarities, $(r_1, r_2, \ldots, r_n)$, is called the **ratio list** for the IFS. A ratio list is called **contracting** if each $r_i < 1$. An **invariant set** for an IFS is a nonempty compact set, $K \subseteq S$ such that $K = f_1(K) \cup f_2(K) \cup \ldots \cup f_n(K)$. The following theorem and its proof are found in [1, Theorem 4.1.3].

Theorem A. Let S be a complete metric space and $(f_1, f_2, \ldots f_n)$ an *IFS* on S with corresponding contracting ratio list $(r_1, r_2, \ldots, r_n)$. Then there is a unique invariant set for the *IFS*.

A common example of an interated function system is the one that has the Cantor set as its invariant set. The reader may be familiar with the construction of the Cantor set by removing the open middle third of a line segment, and the open middle thirds of the remaining segments, and so on. To construct the Cantor set as the invariant set of an IFS, we let $f_1(x) = \frac{x}{3}$, $f_2(x) = \frac{x+2}{3}$, with corresponding ratio list (1/3, 1/3), where $f_i : [0,1] \to [0,1]$. The reader may check that the Cantor set is indeed the invariant set for this IFS.

Given an IFS with ratio list $(r_1, r_2, \ldots, r_n)$ where $r_i < 1$ for all i, the **similarity dimension** associated with the ratio list is the unique number s such that

$$\sum_{i=1}^{n} r_i^s = 1$$

Thus, the similarity dimension of the Cantor set is $s = (\ln 2)/(\ln 3)$ since this s satisfies $2(1/3)^s = 1$.

An IFS satisfies **Moran's open set condition** if there exists a nonempty open set U such that $f_i(U) \cap f_j(U) = \varnothing$ for $i \neq j$, and $f_i(U) \subseteq U$ for all i. It turns out that if an IFS satisfies Moran's open set condition, then the similarity dimension is equal to the Hausdorff dimension. Therefore, since most of the examples we deal with in this paper satisfy Moran's open set condition, a formal definition of the Hausdorff dimension is not included. The following theorem [1, Theorem

6.3.12] is a formal statement of the relationship between these dimensions.

Theorem B. Given an *IFS* and a corresponding ratio list in which each $r_i < 1$, let s be the similarity dimension. If K is the invariant set for the *IFS*, and Moran's open set condition is satisfied, then the Hausdorff dimension of K is equal to s.

The Cantor set satisfies Moran's open set condition because the open set $U = (0,1)$ satisfies the condition: $f_1(U) = (0, 1/3)$, $f_2(U) = (2/3, 1)$ which are disjoint and are both contained in U. Therefore, by Theorem B the Hausdorff dimension of the Cantor set is $(\ln 2)/(\ln 3)$. Since the topological dimension of the Cantor set is 0, the Cantor set is a fractal by Mandelbrot's definition.

3. Graph Self-Similarity. It is possible to generalize the concept of an *IFS* to apply to two or more nonempty compact sets constructed simultaneously in different complete metric spaces. As an example, we consider the **golden rectangle fractal**. The two metric spaces involved will be called S and R (Square and Rectangle). As implied in Figure 1 below, there are six similarities involved in the construction of the fractal (where φ is the golden ratio):

$$\begin{aligned} f_1, f_2, f_3 &: R \to S \text{ with ratio } \varphi^{-2}, \\ f_4 &: S \to S \text{ with ratio } \varphi^{-3}, \\ f_5 &: S \to R \text{ with ratio } 1, \quad \text{and} \\ f_6 &: R \to R \text{ with ratio } \varphi^{-1}. \end{aligned}$$

These functions are all rotations (not reflections) with orientations indicated by the arrows. For a picture of the invariant set of this system, see [1, Plate 8].

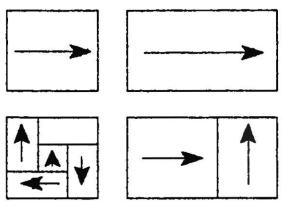

Figure 1

This kind of generalized IFS can be represented by a directed multigraph called a Mauldin-Williams graph, (V, E, i, t, r). In this notation, V is the set of vertices, E is the set of edges, $i : E \to V$ indicates the initial vertex of an edge, $t: E \to V$ indicates the terminal vertex of an edge, and $r : E \to (0, \infty)$ assigns a positive real number to each edge. The corresponding generalized IFS consists of a collection of metric spaces S_v, each corresponding to a vertex v, and functions f_e, each corresponding to an edge $e \in E$, where if $e \in E_{uv}$ (E_{uv} is the set of edges from a vertex u to a vertex v), then $f_e : S_v \to S_u$ represents a similarity with ratio $r(e)$. Note that the direction of a given edge is opposite that of its corresponding similarity. However, this is how Mauldin and Williams originally set things up, so we will continue the notation. A **path** in a Mauldin-Williams graph is a sequence of edges, $e_1, e_2, \ldots, e_m$, such that $t(e_j) = i(e_{j+1})$, for $j = 1, \ldots, m-1$. Figure 2 is the Mauldin-Williams graph associated with the golden rectangle fractal.

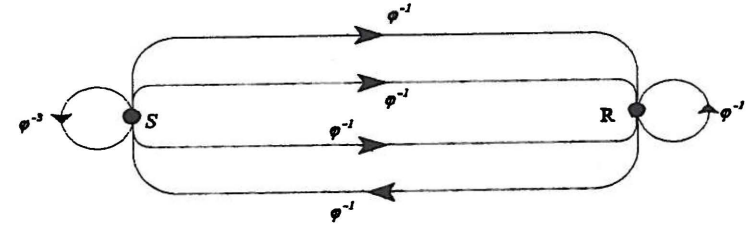

Figure 2

A Mauldin-Williams graph (V, E, i, t, r) is said to be **strictly contracting** if $r(e) < 1$ for all edges $e \in E$. As was the case for an IFS involving a single metric space, certain conditions will guarantee the existence of unique invariant sets for the IFS represented by a Mauldin-Williams graph. (See [1, Theorem 4.3.5]).

Theorem C. Let (V, E, i, t, r) be a strictly contracting Mauldin-Williams graph. Let $(f_e)_{e \in E}$ realize the graph in complete metric spaces S_v. Then there is a unique list $(K_v)_{v \in V}$ of nonempty compact sets ($K_v \subseteq S_v$) such that

$$K_u = \bigcup_{\substack{v \in V \\ e \in E_{uv}}} f_e(K_v)$$

for all $u \in V$.

Recall that in section 2 we presented equations defining invariant sets of an IFS in one metric space. Compare the equations above with those given in

section 2. We define the invariant sets above to be **graph self-similar**.

We can also calculate the Hausdorff dimension of the invariant sets of a Mauldin-Williams graph. We will only consider the case in which the graph is **strongly connected**; that is; for all ordered pairs (u, v) of vertices, there is a path from u to v. When the graph is strongly connected, each of the invariant sets is similar to a subset of each of the others, so they all have the same Hausdorff dimension.

Next we define the dimension of a Mauldin-Williams graph and describe how it is related to the Hausdorff dimension of the invariant sets. For a positive real number s, the s-dimensional **Perron** numbers are positive numbers q_v, one corresponding to each vertex $v \in V$, so that the following equations are satisfied:

$$q_u^s = \sum_{\substack{v \in V \\ e \in E_{uv}}} r(e)^s \cdot q_r^s$$

for all $u \in V$. The following theorem is found in [1, Theorem 6.6.6].

Theorem D. *Let (V, E, i, t, r)* be a strongly connected, strictly contracting Mauldin-Williams graph. There is a unique number $s \geq 0$ such that the s-dimensional Perron numbers exist.

This unique s is called the **dimension** of the Mauldin-Williams graph. Similar to Moran's open set condition in Section 2, there is an analogous open set condition for graph self-similar sets. An *IFS* with corresponding Mauldin-Williams graph (V, E, i, t, r) satisfies the open set condition provided there exist nonempty open sets U_v for each $v \in V$ such that if $u, v \in V$ and $e \in E_{uv}$, then $f_e(U_v) \subseteq U_u$, and for all $u, v, v' \in V$, $e \in E_{uv}$, $e' \in E_{uv'}$, and $e \neq e'$, $f_e(U_v) \cap f_{e'}(U_{v'}) = \varnothing$. Once again, we have a theorem that makes calculating the Hausdorff dimension easy. For a proof, see [1, Theorem 6.4.8].

Theorem E. Suppose we have an *IFS* with a strongly connected, strictly contracting Mauldin-Williams graph *(V, E, i, t, r)*, with dimension s, and satisfying the open set condition. Then the Hausdorff dimension of the unique invariant sets of the *IFS* is s.

4. Contracting and Cycle-Contracting Graphs. Note that the Mauldin-Williams graph for the golden rectangle fractal is not strictly contracting. However, its *IFS* still has a unique invariant set. This is because the metrics in the two metric spaces, S and R, may be rescaled so that the resulting graph is

strictly contracting. Suppose we redefine the metric in S to be $\rho'_s = \varphi \cdot \rho_s$, where ρ_s is the original metric of S. Now, if $e \in E_{SR}$, we have the following:

$$\rho'_s(f_e(x), f_e(y)) = \varphi\rho_s(f_e(x), f_e(y)) = \varphi r(e)\rho_R(x,y).$$

This has the effect of multiplying $r(e)$ by φ for all $e \in E_{SR}$ and dividing $r(e)$ by φ for all $e \in E_{RS}$. Note that $r(e)$ is unchanged if e is a loop (an edge with the same initial and terminal vertices). Figure 3 illustrates the modified Mauldin-Williams graph, in which all values of $r(e)$ are less than 1.

Note that the resulting graph would be strictly contracting if we had chosen any number between 1 and φ^2 as the rescaling number for the metric on S. A Mauldin-Williams graph is called **contracting** if it may be rescaled in this way so that it is strictly contracting. In the general case, for a Mauldin-Williams graph (V, E, i, t, r), with vertices $v_1, v_2, \ldots, v_n \in V$, if the corresponding metric spaces are rescaled by positive numbers

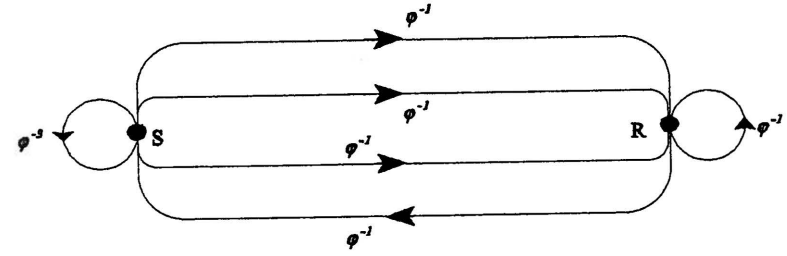

Figure 3

$a_1, a_2, \ldots, a_n$, then for $e \in E_{v_i v_j}$, $r'(e) = a_i r(e)/a_j$, where $r'(e)$ is the rescaled ratio.

Given a path p consisting of edges $e_1, e_2, \ldots, e_m$, define $r(p)$ to be

$$\prod_{i=1}^{m} r(e_i).$$

Also, define $i(p) = i(e_1)$ and $t(p) = t(e_m)$. A cycle is a path p such that $i(p) = t(p)$.

Lemma 1. For any cycle c, $r(c)$ is invariant under rescaling.

Proof: Given a Mauldin-Williams graph with vertices $v_1, v_2, \ldots, v_n$, let the graph be rescaled by factors $a_1, a_2, \ldots, a_n$. Let c be a cycle consisting of edges $e_1, e_2, \ldots, e_m$ where $e_1 \in E_{v_{\alpha(1)}v_{\alpha(2)}}, e_2 \in E_{v_{\alpha(2)}v_{\alpha(3)}}, \ldots, e_m \in E_{v_{\alpha(m)}v_{\alpha(1)}}$, with ratios $r(e_1), r(e_2), \ldots, r(e_m)$. Now, $r(c) = \prod_{i=1}^{m} r(e_i)$. After rescaling,

$$r'(c) = \prod_{i=1}^{m} \frac{a_{\alpha(i)}}{a_{\alpha(i+1(\mathrm{mod}\, m))}} r(e_i)$$

it is easy to see that the rescaling factors all cancel out in this product, and we are left with $r(c) = r'(c)$. ■

We say a Mauldin-Williams graph is called **cycle-contracting** if $r(c) < 1$ for any cycle c.

The next theorem, which is the solution to exercise 4.3.9 in [1], is the key to the main results of the paper. It will lead to a generalization of Theorem D by extending the uniqueness of the dimension of Mauldin-Williams graphs to those which are cycle-contracting. The result is implicit in [3], but our method is significantly different.

Theorem 1. A Mauldin-Williams graph is contracting if and only if it is cycle-contracting.

Proof. First, if a Mauldin-Williams graph is contracting, then it may be rescaled so that it is strictly contracting, and thus cycle contracting. Since the ratios of cycles are invariant under rescaling by Lemma 1, the original graph is cycle contracting.

Fix a cycle-contracting Mauldin-Williams graph with vertices $v_1, v_2, \ldots, v_n$. The proof will be by induction on the number of vertices being rescaled. Assume we can rescale the vertices in the set $U_{k-1} = \{v_1, v_2, \ldots, v_{k-1}\}$ with positive numbers $a_1, a_2, \ldots, a_{k-1}$, such that all paths p beginning and ending at one of these vertices satisfies $r(p) < 1$. This is obvious for $k = 1$. Now let

$$M_k = \max\{r(\alpha) : \alpha \text{ is a path with } i(\alpha) \in U_{k-1} \text{ and } t(\alpha) = v_k\}$$

and

$$m_k = \min\{\frac{1}{r(\beta)} : \beta \text{ is a path with } i(\beta) = v_k \text{ and } t(\beta) \in U_{k-1}\}.$$

Since there are only a finite number of paths in the graph which do not contain cycles, and since adding a cycle to a path results in a new path whose ratio is strictly less than the ratio of the original path, M_k and m_k are well-defined.

Now let α and β be paths such that $r(\alpha) = M_k$ and $\frac{1}{r(\beta)} = m_k$. Since $t(\alpha) = i(\beta)$, we may consider the path $\alpha\beta$. Since $i(\alpha\beta), t(\alpha\beta) \in U_{k-1}$, we have $r(\alpha\beta) = r(\alpha)r(\beta) < 1$. Thus $\frac{M_k}{m_k} < 1$, so $M_k < m_k$. Choose a_k so that $M_k < a_k < m_k$, and rescale vertex v_k by a_k.

Consider any path ϖ such that $i(\varpi) \in U_{k-1}$ and $t(\varpi) = v_k$. Let $r(\varpi)$ be the ratio value before rescaling v_k by a_k, and $r'(\varpi)$ the value afterwards. So,

$$r'(\varpi) = \frac{r(\varpi)}{a_k} \le \frac{M_k}{a_k} < \frac{M_k}{M_k} = 1$$

Similarly, for any path ϖ with $i(\varpi) = v_k$ and $t(\varpi) \in U_{k-1}$, we have

$$r'(\varpi) = a_k \cdot r(\varpi) \le \frac{a_k}{m_k} < \frac{m_k}{m_k} = 1.$$

Finally, if ϖ is either a cycle or has initial and terminal vertices in U_{k-1}, then $r'(\varpi) = r(\varpi) < 1$. Thus all paths p with initial and terminal vertices in $U_k = \{v_1, v_2, \ldots, v_k\}$ satisfy $r(p) < 1$. By the Principle of Mathematical Induction, we may rescale all of the vertices so that the resulting graph is strictly contracting. ■

The following two lemmas generalize Theorem D, and bring us to our conclusion.

Lemma 2. A strongly connected, contracting Mauldin-Williams graph has unique dimension.

Proof. We will show that the dimension s is invariant under rescaling of a single vertex, and thus under any rescaling. Recall that if the vertex v is rescaled by a factor of a_v, then $r'(e) = a_v \cdot r(e)$ when $i(e) = v$, $r'(e) = \frac{r(e)}{a_v}$ for $t(e) = v$, and $r'(e) = r(e)$ otherwise. We define a new set of Perron numbers by letting $q_v' = a_v \cdot q_v$ and $q_u' = q_u$ for $u \neq v$. Then

$$(q_v')^s = \sum_{\substack{u\in V\\ e\in E_{vu}}} r'(e)^s \cdot q_u^s = \sum_{\substack{u\in V\\ e\in E_{vu}}} a_v^s \cdot r(e)^s \cdot q_u^s = a_v^s \cdot q_v^s$$

Therefore, the number s still satisfies the equations for the Perron numbers, and so the dimension s is invariant. ■

Lemma 3. A strongly connected Mauldin-Williams graph that has unique dimension is contracting.

Proof. Fix a strongly connected Mauldin-Williams graph (V, E, i, t, r) that is not contracting. Then there is some cycle ϱ such that $r(\varrho) \geq 1$ by Theorem 1. Let the vertices of ϱ be $v_1, v_2, \ldots, v_k$ and the edges be $e_{1,2}, e_{2,3}, \ldots, e_{k,1}$ in the obvious manner. Suppose there exist s-dimensional Perron numbers $q_1, q_2, \ldots, q_k$ corresponding to these vertices. Then we have $q_i^s \geq r(e_{i,i+1})^s \cdot q_{i+1}^s$, where $i + 1$ is taken modulo k, and by repeated substitutions, we have $q_1^s \geq r(\varrho)^s \cdot q_1^s$. Since $r(\varrho) \geq 1$, we have $q_1^s \geq q_1^s$. Obviously, $q_1^s = q_1^s$, which implies $r(\varrho) = 1$, and that there are no edges $e \notin \varrho$, such that $i(e) \in \varrho$, because otherwise there would be extra terms in the right side of the inequality $q_1^s \geq r(\varrho)^s \cdot q_1^s$, which would give strict inequality. Thus if there were any vertices not in ϱ, the graph would not be strongly connected. If follows that the entire graph is just the cycle ϱ, in which case Perron numbers exist for any value of s, so the dimension is not unique. ■

Theorem 2. A strongly connected Mauldin-Williams graph has unique dimension if and only if it is cycle-contracting. This is equal to the Hausdorff dimension of the invariant sets provided that the open set condition is satisfied.

Proof. This result follows directly from Theorem E, Theorem 1 and Lemmas 2 and 3. ■

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Where Do My Sequences Lead?

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One of the things I find most exciting about mathematics is that seemingly simple ideas often lead to interesting and surprisingly complex results. Here is one example. Consider the sequence

$$\frac{1}{3}, \frac{1+3}{5+7}, \frac{1+3+5}{7+9+11}, \frac{1+3+5+7}{9+11+13+15}, \ldots .$$

Notice that the first four fractions in this sequence reduce to 1/3. It would seem that these fractions formed by partial sums of consecutive, odd integers with an equal number of terms in the numerator and denominator all reduce to 1/3. This is in fact the case. To see this, let n be the number of terms in the numerator (and in the denominator). Then the terms in the numerator are

$$1+3+5+7+\cdots+(2n-1)=n^2 \qquad \text{(see [1]).}$$

The denominator is

$$(2n+1)+(2n+3)+\cdots+(2(2n)-1).$$

Since $1 + 3 + 5 + \cdots + (2(2n) - 1) = (2n)^2$, the denominator can be found by subtracting from this sum the sum of the terms in the numerator: $(2n)^2 - n^2 = 3n^2$. Thus, any term in the sequence has the form

$$\frac{1+3+5+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(4n-1)} = \frac{n^2}{3n^2} = \frac{1}{3}. \qquad ■$$

Other Arithmetic Sequences With Fractions That All Reduce To 1/3

A logical next step is to consider if this property is special to the odd integers. That is, are there other arithmetic sequences that create fractions which all reduce to 1/3? To explore this, we will consider arithmetic sequences with arbitrary terms. First, consider the terms in the numerator: a, $a + b$, $a + 2b$, $a + (n - 1)b$, where a is the starting value and adding consecutive multiples of b generates successive terms. Notice that for the odd integers, $a = 1$ and $b = 2$. We know that these values of a and b result in fractions that reduce to 1/3. Which other values of a and b yield a constant 1/3? To answer this

question, we will evaluate the terms of the sequence. We can begin by summing the terms in the numerator.

$$\sum_{i=0}^{n-1}(a+bi)=an+b\sum_{i=1}^{n-1}i=an+\frac{n(n-1)b}{2}=n\left(a+\frac{n-1}{2}b\right).$$

Similarly, each term in the sequence of ratios is

$$\frac{\sum_{i=0}^{n-1}(a+ib)}{\sum_{i=n}^{2n-1}(a+ib)}=\frac{n\left(a+\frac{n-1}{2}b\right)}{2n\left(a+\frac{2n-1}{2}b\right)-n\left(a+\frac{n-1}{2}b\right)}$$

$$=\frac{a+\frac{n-1}{2}b}{a+\frac{3}{2}nb-\frac{b}{2}}$$

$$=\frac{2a+nb-b}{2a+3nb-b}.$$

Notice that when $b = 2a$, this expression reduces to 1/3, so every term in the sequence will reduce to 1/3. For example, let $a = 2$ and $b = 2a = 4$. Then

$$\frac{2}{6}=\frac{2+6}{10+14}=\frac{2+6+10}{14+18+22}=\dots=\frac{1}{3}.$$

(Note that this is a constant multiple of the sequence by consecutive odd integers.)

Limits For Any *a* And *b*

Obviously if every term in a sequence reduces to 1/3 then the limit of that sequence is 1/3. Not so obvious, however, is the limit when this is not the case. Consider limits of sequences for any a and b. We already saw that the nth term in the sequence reduces to:

$$\frac{2a+bn-b}{2a+3bn-b}.$$

When we calculate the limit, we get

$$\lim_{n\to\infty}\frac{2a+bn-b}{2a+3bn-b}=\lim_{n\to\infty}\frac{\frac{2a}{n}+b-\frac{b}{n}}{\frac{2a}{n}+3b-\frac{b}{n}}=\frac{1}{3}.$$

Thus, for any values of a and $b \neq 0$, the limit of our sequence is 1/3.

Think Polynomials And Increase The Degree

So far we have considered sequences of the form

$$a,\ a+b,\ a+2b,\dots,\ a+(n-1)b.$$

We can also think of generating this sequence by evaluating a function at the non-negative integers. That is, the ith term is

$$f(i)=a+bi \quad \text{where } i=0,1,2,\dots.$$

Noticing that this is a first degree polynomial, we might then consider second degree polynomials:

$$f(i)=ci^2+bi+a \quad \text{where } i=0,1,2,\dots.$$

We can ask the same questions as before. Begin this time by finding the limit of this sequence:

$$\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}f(i)}{\sum_{i=0}^{2n-1}f(i)-\sum_{i=0}^{n-1}f(i)}=\lim_{n\to\infty}\left(\frac{\sum_{i=0}^{n-1}(ci^2+bi+a)}{\sum_{i=0}^{2n-1}(ci^2+bi+a)-\sum_{i=0}^{n-1}(ci^2+bi+a)}\right)$$

$$=\lim_{n\to\infty}\frac{\frac{c(n-1)(n)(2n-1)}{6}+\frac{b(n-1)(n)}{2}+an}{\frac{c(2n-1)(2n)(4n-1)}{6}+\frac{b(2n-1)(2n)}{2}+2an-\text{numerator}}.$$

Since this is a limit, we are only interested in the coefficients of the term of largest degree. For both the numerator and denominator, this is n^3. Thus, this limit is the ratio of the coefficients of n^3:

$$\frac{\frac{c}{3}}{\frac{8c}{3}-\frac{c}{3}}=\frac{1}{8-1}=\frac{1}{7}.$$

Now we know that the limit for any values of a, b, and c is 1/7. For what special values of a, b, and c will the terms all reduce to 1/7? Let's start with the last simplifications we obtained before we started ignoring terms.

$$\frac{\dfrac{c(n-1)(n)(2n-1)}{6} + \dfrac{b(n-1)(n)}{2} + an}{\dfrac{c(2n-1)(2n)(4n-1)}{6} + \dfrac{b(2n-1)(2n)}{2} + 2an - [\text{numerator}]}$$

$$= \frac{\dfrac{c(n-1)(n)(2n-1)}{6} + \dfrac{b(n-1)(n)}{2} + an}{\dfrac{c(2n-1)(2n)(4n-1) - c(n-1)(n)(2n-1)}{6} + \dfrac{b(2n-1)(2n) - b(n-1)(n)}{2} + 2an - an}$$

$$= \frac{c\left(\dfrac{2n^3 - 3n^2 + n}{6}\right) + b\left(\dfrac{n^2 - n}{2}\right) + an}{c\left(\dfrac{14n^3 - 9n^2 + n}{6}\right) + b\left(\dfrac{3n^2 - n}{2}\right) + an}.$$

This fraction reduces to 1/7 if

$$7\left(c\left(\frac{2n^3-3n^2+n}{6}\right) + b\left(\frac{n^2-n}{2}\right) + an\right) = c\left(\frac{14n^3-9n^2+n}{6}\right) + b\left(\frac{3n^2-n}{2}\right) + an.$$

Through simplification we find the identity $36a - 18b + 6c = n(12c - 12b)$ for all values of n. This implies that $b = c$ and $3a = b$. Thus, the function $f(n) = cn^2 + bn + a$ yields a constant 1/7 when $3a = b = c$. The constant sequence corresponding to $f(n) = 3n^2 + 3n + 1$ looks like this:

$$\frac{1}{7}, \frac{1+7}{19+37}, \frac{1+7+19}{37+61+91}, \ldots.$$

Summary

- The limit is 1 if the sequence of fractions is generated by the constant polynomial $f(n) = a \quad \forall a \neq 0$.

- The limit is 1/3 if the sequence of fractions is generated by the first degree polynomial $f(n) = bn + a$.
- The limit is 1/7 if the sequence of fractions is generated by the second degree polynomial $f(n) = cn^2 + bn + a$.
- Every term of the sequence of fractions reduces to 1 when it is generated by the constant polynomial $f(n) = a \quad \forall a \neq 0$.
- Every term of the sequence of fractions reduces to 1/3 when it is generated by nonzero multiples of the first degree polynomial $f(n) = 2n + 1$.
- Every term of the sequence of fractions reduces to 1/7 when it is generated by nonzero multiples of the second degree polynomial $f(n) = 3n^2 + 3n + 1$.

You might be detecting a pattern. (See if you can convince yourself of the following results.)

Generalization

- The limit is 1/15 if the sequence of fractions is generated by the third degree polynomial $f(n) = dn^3 + cn^2 + bn + a$.
- The limit is $1/(2^{k+1} - 1)$ if the sequence of fractions is generated by the kth degree polynomial $f(n) = a_k n^k + \cdots + a_1 n + a_0$.
- Every term of the sequence of fractions reduces to 1/15 when it is generated by nonzero multiples of the third degree polynomial $f(n) = 4n^3 + 6n^2 + 4n + 1$.
- Every term of the sequence of fractions reduces to $1/(2^{k+1} - 1)$ when it is generated by the kth degree polynomial $f(n) = (n + 1)^{k+1} - n^{k+1}$.

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Equivalent Conditions for Fibonacci and Lucas Pseudoprimes To Contain A Square Factor

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Introduction. The *Fibonacci Pseudoprimes* (or FPP's) are those composite positive integers n that are relatively prime to 10 and satisfy the relation: $F_{n'} \equiv 0 (\text{mod } n)$; here, $\{F_n\}$ is the Fibonacci sequence defined by the recurrence relation: $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$, $F_1 = 1$, $n' = n - \epsilon_n$, and ϵ_n represents the Jacobi symbol $(5/n)$. We note that $\epsilon_n = +1$ if $n = \pm 1 (\text{mod } 10)$, and $\epsilon_n = -1$ if $n = \pm 3 (\text{mod } 10)$.

The *Lucas Pseudoprimes* (or LPP's) are those composite positive integers n such that $L_n = 1$ (mod n), where $\{L_n\}$ is the Lucas sequence defined by the same recurrence relation as the Fibonacci sequence, but with the initial values $L_0 = 2$, $L_1 = 1$.

Both of these sets of pseudoprimes have been studied extensively (see [1] - [6]); each set is known to be infinite and intersects with the other set. Tables of LPP's have been published [7]; tables of FPP's are being produced as of this writing by the author and his collaborator Dr. Peter G. Anderson of the Rochester Institute of Technology, and will be published at a future date. The first few FPP's are: $323 = 17 \cdot 19$, $377 = 13 \cdot 29$, $1891 = 31 \cdot 61$, etc. The first few LPP's are: $705 = 3 \cdot 5 \cdot 47$, $2465 = 5 \cdot 17 \cdot 29$, $2937 = 7 \cdot 17 \cdot 23$, etc.

From a study of such tables, it would appear that all of these numbers are quadratfrei, that is, contain no square factor (other than one). The author commented on this observation in [1], in connection with the LPP's. In that paper, it was shown that if a LPP does contain a square factor p^2, say (where p is prime), such p must satisfy the condition $Z(p^2) = Z(p)$ (where $Z(n)$ is the entry-point of n in the Fibonacci sequence, i.e., the smallest positive index m such that $n \mid F_m$). Indeed, it was demonstrated in [1] that this condition is necessary and sufficient for a LPP to contain a square factor. It is easily shown that the same condition holds for FPP's. If this condition holds for a given prime p, one may easily show that p^2 itself is both a FPP and a LPP.

The aim of this expository paper is to derive certain conditions that are equivalent to the condition $Z(p^2) = Z(p)$. It should be remarked that the latter condition holds *for no known p*. As a result indicated on pp. 85 - 86 of [13],

H.C. Williams determined that the *negative* of this condition, namely $Z(p^2) = pZ(p)$, which we term the "Williams condition", is satisfied for all $p < 10^9$. Montgomery [9] has extended this result to all $p < 2^{32}$. More recently, an unpublished result by McIntosh extends this bound further to $1.8 \cdot 10^{12}$. On the basis of this evidence, it is tempting to conjecture that the Williams condition holds for all p. However, the statistical evidence, thus far at least, is not sufficient to settle this obvious conjecture one way or the other.

For the purpose of discussion, we define *exceptional primes* as those primes p (if any) that do not satisfy the Williams condition, i.e. such that $Z(p^2) = Z(p)$. Whether vacuous or not, the equivalent conditions that we will obtain will be necessary and sufficient for primes p to be exceptional (and therefore for FPP's and LPP's to contain square factors). We will express these conditions in terms of certain functions of p that are defined below.

Congruence Results Given an *odd* prime p, we let R_p denote any complete set of residues (mod p), e.g. $R_p = \{0,1,2,\ldots, p-1\}$. For any $x \in R_p$, define the polynomial $S_p(x)$ as follows :

$$S_p(x) = \sum_{k=1}^{p-1} x^k/k. \tag{1}$$

We are only interested in determining $S_p(x)$ (mod p); in such determination, it is of course understood that the terms $1/k$ refer to the *inverses* of k (k^{-1}(mod p)). Consequently, we may regard $S_p(x)$ as a polynomial in x with integral coefficients .

We may also extend the domain of $S_p(x)$ to include algebraic integers in certain fields, e.g. $F(\sqrt{5})$ or $F(\sqrt{-3})$. Such extensions would, of course, encompass the original domain R_p . In some cases, we consider x to be an element of $F(\sqrt{c})$, where c is a quadratic non-residue (mod p); that is, $x = a + b\sqrt{c}$, where a and b are in R_p. We may also allow c to be a quadratic residue (mod p); in this case, $x \in R_p$. Our main result, from which the others follow, is given below.

Theorem 1 . For all $x \in F(\sqrt{c})$, $S_p(x) \equiv \{1 - x^p - (1-x)^p\}/p$ (mod p).

Proof: $(1-x)^p = \sum_{n=0}^{p} {}_pC_n(-x)^n$, and so $1 - x^p - (1-x)^p = -\sum_{n=1}^{p-1} {}_pC_n(-x)^n$

$= -p \sum_{n=1}^{p-1} {}_{p-1}C_{n-1}(-x)^n/n$. Then $\{1 - x^p - (1-x)^p\}/p \equiv -\sum_{n=1}^{p-1} {}_{(-1)}C_{n-1}(-x)^n/n \pmod p$

$$= -\sum_{n=1}^{p-1}(-1)^{n-1}(-1)^n x^n/n = \sum_{n=1}^{p-1} x^n/n = S_p(x). \quad \blacksquare$$

COROLLARY 1: $S_p(\alpha) \equiv (1 - L_p)/p \pmod p$

Proof: This follows from Theorem 1 by setting $x = \alpha \equiv (1+\sqrt{5})/2$, and from the Binét expression: $L_n = \alpha^n + \beta^n$, where $\beta = 1 - \alpha = (1 - \sqrt{5})/2$.

Corollary 2: $S_p(x) \equiv S_p(1-x) \pmod p$.

Proof: This is immediately obvious from Theorem 1.

Corollary 3: $S_p(\alpha) \equiv S_p(\beta) \pmod p$.

Proof: This follows from Corollary 2, by setting $x = \alpha$.

Corollary 4: $$\sum_{n=1}^{p-1} F_n/n \equiv 0 \pmod p.$$

Proof: The Binét expression for F_n is given by $(\alpha^n - \beta^n)/(\alpha - \beta) = (\alpha^n - \beta^n)/\sqrt{5}$. This result, along with that of Corollary 3, implies the stated result, provided $p \neq 5$. If $p = 5$, the sum is easily verified to equal $35/12 \equiv 0 \pmod 5$.

Corollary 5: $$S_p(0) \equiv S_p(1) \equiv 0 \pmod p.$$

Proof: This follows from Corollary 2 and the definition of $S_p(x)$. Note that we may express the relation $S_p(1) \equiv 0 \pmod p$ as follows: $H_{p-1} \equiv 0 \pmod p$, where $H_n = \sum_{k=1}^{n} 1/k$; this is a well-known number-theoretic result.

As a consequence of Corollary 2, we may make the substitution $q = x(1 - x)$ and regard $S_p(x)$ as a polynomial in q, say $U_p(q)$. Since the degree of $S_p(x)$ is $p - 1$, that of $U_p(q)$ is $(p - 1)/2$. Note that $dS_p(x)/dx = \sum_{k=1}^{p-1} x^{k-1} = (x^{p-1} - 1)/(x-1)$; as a consequence of Fermat's "Little" Theorem, we see that $S_p'(x)$ has the zeros $2,3,4,\ldots,p-1$; that is to say, $(x^{p-1} - 1)/(x-1) \equiv (x-2)(x-3) \cdots (x-p+1) \pmod p$. On the other hand, $S_p'(x) = dU_p(q)/dq \cdot (1-2x)$, by the chain rule for differentiation. Since $x^2 - x + q = 0$, we may solve for x in terms of q. This yields two roots x_1 and x_2, say; however, $x_2 = 1 - x_1$, and in view of Corollary 2, there is nothing to choose between these two roots. We choose to define x as $(1 - \theta)/2$, where $\theta = \sqrt{1-4q}$. Then $1 - 2x = \theta$, and we find that $U_p'(q) = (x - x^p)/q\theta$, where $x = (1-\theta)/2$. Then $x^p = \{(1-\theta)/2\}^p \equiv (1 - \theta^p)/2 \pmod p$; hence, $U_p'(q) \equiv (\theta^{p-1} - 1)/2q \pmod p$, or equivalently:

$$U_p'(q) \equiv \{(1-4q)^{(p-1)/2} - 1\}/2q \pmod p. \tag{2}$$

Since $U_p(0) \equiv 0 \pmod p$, it follows that $U_p(q) = \int_0^q \{(1-4t)^{(p-1)/2} - 1\}/2t \, dt$; by the substitution $1 - 4t = u$, then

$$U_p(q) \equiv \tfrac{1}{2} \int_1^{1-4q} (u^{(p-1)/2} - 1)/(u-1)\,du \pmod p. \tag{3}$$

Now the integrand in (3) is equal to $G_p'(u)$, say, where

$$G_p(x) = \sum_{k=1}^{(p-1)/2} x^k/k. \tag{4}$$

Therefore, we have the alternative expression:

$$U_p(q) \equiv (G_p(1 - 4q) - G_p(1))/2 \pmod p. \tag{5}$$

In particular, if $x = \alpha$, then $q = \alpha\beta = -1$, and so

$$S_p(\alpha) \equiv (G_p(5) - G_p(1))/2 \equiv U_p(-1) \pmod p. \tag{6}$$

We now derive an alternative expression for $U_p(q)$ from the integral expression in (3). From (2) above,

$$U_p'(q) \equiv 1/2 \sum_{n=1}^{(p-1)/2} {}_{1/2(p-1)}C_n(-4)^n q^{(n-1)} \equiv 1/2 \sum_{n=1}^{(p-1)/2} {}_{-1/2}C_n(-4)^n q^{n-1} \pmod p$$

$$\equiv 1/2 \sum_{n=1}^{(p-1)/2} {}_{2n}C_{nq}^{n-1} \pmod p.$$ Therefore,

$$U_p(q) \equiv 1/2 \sum_{n=1}^{(p-1)/2} {}_{2n}C_n\, q^n/n \pmod p. \tag{7}$$

In particular,

$$U_p(-1) \equiv \tfrac{1}{2} \sum_{n=1}^{(p-1)/2} {}_{2n}C_n\,(-1)^n/n \pmod p \tag{8}$$

Note also from (3) that $U_p'(q) = -2\{(u^{(p-1)/2} - 1)/(u-1)\}\,|_{u=1-4q}$;

since $u^{(p-1)/2} \equiv (u/p) \pmod p$ for all $u \in R_p$ (Legendre symbol), then $U_p'(q) \equiv -2\prod_{\rho}(1-4q-\rho)$, the product taken over all quadratic residues $\rho \pmod p \neq 0$ or 1 (there being $(p-3)/2$ such residues). Equivalently:

$$U_p'(q) \equiv -2 \prod_{k=2}^{(p-1)/2} (1 - 4q - k^2) \pmod p \tag{9}$$

All of the above derivations are interesting, but it is not yet clear how they relate to the original problem of determining exceptional primes. Such relationship is clarified in the following section.

3. <u>Equivalent Conditions for Exceptional Primes</u>. It was indicated in [1] that if n is a LPP, it contains a square factor p^2 iff p is an exceptional prime. The same equivalent conditions may easily be shown to hold for a FPP. As previously mentioned, it has apparently been verified that there are no exceptional primes p with $p < 1.8 \cdot 10^{12}$. Accepting this statistic as valid, we can say with certainty that if n is a LPP (FPP) with $n < 3.24 \cdot 10^{24}$, it must be quadratfrei ; for if $p^2|n$, then $Z(p^2) = Z(p)$ and also $p^2 < 3.24 \cdot 10^{24}$, or $p < 1.8 \cdot 10^{12}$, which contradicts the assumed statistic. Incidentally, it is of interest to note (and relatively easy to demonstrate) that if p is an exceptional prime, then p^2 itself is both a FPP and a LPP.

We will now demonstrate how these conditions are related to the congruence relations given in the preceding section.

It is well-known that for *all* primes p, $L_p \equiv 1 \pmod p$. Thus, the ratio $(1 - L_p)/p$ given in Corollary 1 is an integer. Now p^2 is a LPP as well as a FPP

iff $Z(p^2) = Z(p)$, in which case $p^2 \mid (L_p - 1)$ and $p^2 \mid F_{p'}$, where $p' = p - (5/p)$ (we will accept the result attributed to McIntosh , so that p is a prime greater than $1.8 \cdot 10^{12}$). However, by Corollary 1, we see that this condition is equivalent to $S_p(\alpha) \equiv 0 \pmod p$. In light of the preceding derivations, we have the following equivalent conditions:

(a) p is an exceptional prime;

(b) p^2 is a FPP and a LPP;

(c) $S_p(\alpha) \equiv S_p(\beta) \equiv 0 \pmod p$;

(d) $\sum_{k=1}^{p-1} F_{k-1}/k \equiv 0 \pmod p$;

(e) $\sum_{k=1}^{p-1} F_{k+n}/k \equiv 0 \pmod p$ for all n;

(f) $\sum_{k=1}^{(p-1)/2} (5^k - 1)/k \equiv 0 \pmod p$;

(g) $\sum_{k=1}^{(p-1)/2} {}_{2k}C_k\,(-1)^k/k \equiv 0 \pmod p$.

Clearly, (e) implies (d). On the other hand, Corollary 4, (d) and the recurrence relation for the F_n's imply (e), by induction. Thus (d) and (e) are equivalent. Based on our previous comments, the preceding conditions apply for any p such that p^2 is a divisor of n, where n is either a FPP or LPP.

We should also mention a congruence relation that was proposed as a problem [12] by Morgan Ward. Using our notation, Ward's problem was as follows :

(h) $Z(p^2) = Z(p)$ iff $G_p(5/9) \equiv S_p'(3/2)/p \pmod p$, i.e.

$$\sum_{k=1}^{(p-1)/2} (5/9)^k/k \equiv 2\{(3/2)^{p-1} - 1\}/p \pmod p .$$

Yet another condition that is easily shown to be equivalent to the preceding ones in (*a*)-(*h*) is the following:

(i) $\alpha^{p'} \equiv (5/p) \pmod{p^2}$.

Also, as indicated in [6], p^2 is both a FPP and a LPP iff it satisfies one of the following congruences :

(j) $p^2 \mid F_{p'/2}$ if $p \equiv 1 \pmod 4$, or $p^2 \mid L_{p'/2}$ if $p \equiv 3 \pmod 4$.

We would be remiss if we omitted another equivalent condition for exceptional primes, also due to H. C. Williams [13]:

(k) $\sum_{k=1}^{p-1-[p/5]} (-1)^k/k \equiv 0 \pmod{p}$, if $p \neq 5$.

There are also the two equivalent congruences mentioned previously:

(l) $p^2 \mid (L_p - 1)$;

(m) $p^2 \mid F_{p'}$.

The relation in (i) is seen to be akin to the condition for Wieferich primes (a Wieferich prime is a prime p such that $2^{p-1} \equiv 1 \pmod{p^2}$; the only Wieferich primes less than $4 \cdot 10^{12}$ are 1,093 and 3,511) . There are also some scattered known solutions of the more general congruence $m^{p-1} \equiv 1 \pmod{p^2}$, where m is a given positive integer. See [10] for a list of such solutions, and [8] for a related discussion.

Less is apparently known about such congruences when m is an algebraic integer, as is the case when $m = \alpha = (1+\sqrt{5})/2$; note that the exponent $p-1$ is replaced by p' in that instance .

Finally, we should include another pair of conditions equivalent to those for exceptional primes (and to each other), due to Z.H Sun and Z.W. Sun [11]:

(n) $\sum_{\substack{k=1, \\ 5 \mid (2p-k)}}^{p-1} 1/k \pmod{p}$;

(o) $\sum_{\substack{k=1, \\ 5 \mid (p+k)}}^{p-1} 1/k \pmod{p}$.

It is expected that any exceptional primes should be exceedingly rare, if they exist at all. We cannot infer from the referenced searches that they are non-existent. It is hoped that this paper will encourage computer scientists and/or students to carry on the search for any such exceptional primes, so that the basic question of whether there exist FPP's and LPP's containing square factors might be settled. Various algorithms have been given in this paper, to determine the existence of exceptional primes. Some of these algorithms are more computationally efficient than others; it is left to computer scientists and other interested researchers to determine the relative merits of the various algorithms and to use this information to extend our search for exceptional primes. Of course, if there are *no* exceptional primes, any such search would be fruitless (although it would strengthen our faith in the non-existence of exceptional primes); in this event, additional theoretical considerations would need to be advanced.

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Symmetric Chromatic Functions

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A graph G with d vertices has an associate symmetric function of degree d in variables $x_1, x_2, \cdots, x_n$. This function is related to the chromatic polynomial of G in that if we set $x_1 = \cdots = x_n = 1$ in the symmetric function, we obtain the chromatic polynomial of G evaluated at n. The purpose of this paper is to investigate the symmetric chromatic polynomials of graphs as expressed in terms of the standard bases for the symmetric functions. In particular, expressions in terms of the elementary symmetric functions are examined. New expressions are proven for previously characterized graphs, and conjectures are made regarding uninvestigated classes of graphs.

1. Symmetric Chromatic Functions

Graphs. A *graph* is a set of vertices $V = V(G) = \{v_1, \cdots, v_n\}$ and a set of edges $E = E(G) = \{e_1, \cdots, e_m\}$ where each edge is an unordered pair of vertices. If two vertices are connected by an edge, they are said to be *adjacent*. In this paper we will not consider graphs with self adjacent vertices.

A *coloring* of a graph G is a map κ: $V \to \{1, 2, \cdots\}$, where we denote a set of colors by $\{1, 2, \cdots\}$. A proper coloring is a coloring in which no two adjacent vertices are assigned the same color. Note that this is why we do not want to consider self adjacent vertices, as there would be no proper coloring for a graph that included one.

The *chromatic polynomial* $\chi_{G(n)}$ of a graph G is the number of proper colorings of G using n colors. It is not immediately obvious that $\chi_{G(n)}$ is a polynomial. For a proof of this and other results on chromatic polynomials see, for example, Brualdi [Bru]. The chromatic polynomials do not yield any actual colorings, so we will describe another way to think about them using symmetric functions.

Symmetric Functions. Denote the set of all polynomials in variables $x_1, \cdots, x_n$ with coefficients in the integers, **Z**, by $\mathbf{Z}[x_1, \cdots, x_n]$. Under polynomial addition

and multiplication $\mathbf{Z}[x_1, \ldots, x_n]$ forms a ring. If $f(x_{i_1},\ldots,x_{i_n}) = f(x_1,\ldots,x_n)$ for every possible permutation $x_{i_1},\cdots,x_{i_n}$ of the variables $x_1,\ldots,x_n$, the polynomial $f \in \mathbf{Z}[x_1, \ldots, x_n]$ is said to be *symmetric*. For example, the polynomial $x + y$ is symmetric in $\mathbf{Z}[x, y]$ but $2y + x$ is not. The set of all symmetric polynomials forms a subring of $\mathbf{Z}[x_1,\ldots, x_n]$ denoted $\Lambda[x_1,\ldots, x_n]$. The set of all symmetric polynomials of degree d is denoted $\Lambda^d = \Lambda^d[x_1,\ldots,x_n]$. The set Λ^d has a finite integer basis which we define below. It is important to note that $\Lambda^d \cap \Lambda^e = \{0\}$ for all $d \neq e$. Also, $\Lambda = \oplus_{d\geq 0}\Lambda^d$, meaning that every symmetric polynomial p can be uniquely written as a finite sum $p = \sum P_d$ where $P_d \in \Lambda^d$. Therefore, a basis for the infinite dimensional ring Λ is the union of the bases for Λ^d for all d.

Before we define a basis for Λ^d, we introduce the notion of integer partitions. A *partition of n*, denoted $\lambda \vdash n$, is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2,\ldots,\lambda_l)$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $\sum_{i=1}^{l} \lambda_i = n$. The length of λ is $\ell(\lambda) = l$. An alternative notation is to write $\lambda = (1^{a_1}, 2^{a_2},\cdots,n^{a_n})$, where a_i is the number of times i appears in the partition. For example, $\lambda = (4, 4,3, 1, 1, 1, 1, 1)$ is a partition of 16 which is also expressed as $\lambda = (1^5, 3^1, 4^2)$. Partitions can be ordered under lexicographic ordering: i.e., if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ then the partition λ is greater than the partition γ if and only if the first non-zero entry of $(\lambda_1 - \gamma_1, \lambda_2 - \gamma_2, \cdots)$ is greater than zero.

Given variables $x_1, \cdots, x_n$, we define the elementary symmetric functions $e_1,\cdots,e_n \in \mathbf{Z}[x_1,\cdots,x_n]$ the formulas

$$e_1 = x_1 + \cdots + x_n,$$
$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n$$
$$\vdots$$
$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r},$$
$$\vdots$$
$$e_n = x_1x_2\cdots x_n,$$

and for each $\lambda \vdash n$ define $e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}$. We have the following well known property of elementary symmetric functions, see Sagan [Sag] or MacDonald [Mac] for a proof.

Theorem 1.1. *The set* $\{e_\lambda | \lambda \vdash d, \lambda_1 \leq n\}$ *is a basis for* Λ^d.

For a finite graph G with d vertices, Stanley [Stan] defines a homogeneous symmetric function X_G of degree d in the variables $x_1, x_2, \cdots, x_n$ as follows

$$X_G = X_G(x_1, x_2, \cdots, x_n) = \sum_{\kappa} x_{\kappa(v_1)}x_{\kappa(v_2)}\cdots x_{\kappa(v_n)},$$

where the sum ranges over all proper colorings κ: $V \to \{1, 2, \ldots, n\}$. This symmetric function is called the *symmetric chromatic polynomial* of G. If we set $x_1 = \cdots = x_n = 1$, then we obtain $\chi_{G(n)}$ the usual chromatic polynomial of G evaluated at n. Each monomial of this polynomial is denoted x^κ, where $x^\kappa = x_{\kappa(v_1)}\cdots x_{\kappa(v_n)}$. This is a monomial in which the degree of x_i is the number of times color i appears in the coloring κ.

Since the set of elementary symmetric functions $\{e_\lambda | \lambda \vdash d, \lambda_1 \leq n\}$ forms a $\mathbf{Z}$-basis of Λ^d, we can expand X_G in terms of this basis so that

$$X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda \tag{1.2}$$

where $c_\lambda \in \mathbf{Z}$. The coefficients in this expansion are the main focus of this paper.

2. Stanley's Generating Functions. One way to find the symmetric chromatic polynomial for a graph is to use the generating function for that class of graph. A generating function is defined in the following way. Given a sequence $a_0, a_1, a_2, \cdots$, the corresponding *generating function* is the power series

$$f(t) = \sum_{n\geq 0} a_n t^n.$$

Define a chain to be a graph P_d with vertex set $V = \{v_1, \cdots, v_d\}$ and with edges v_iv_{i+1}, $1 \leq i \leq d - 1$. Define a cycle to be a graph C_d with vertex set $V = \{v_1, \cdots, v_d\}$, $d \geq 3$ and with edges v_iv_{i+1}, $1 \geq i \geq d - 1$ and v_1v_d. Richard Stanley [Stan] has derived generating functions for the symmetric-chromatic polynomials of chains and cycles (see 2.2 and 2.3). In these generating functions, the symmetric chromatic polynomial for a graph with n vertices is the coefficient of t^n. The main purpose of this paper is to use these generating functions to find a formula for the coefficient c_λ defined in (1.2).

Stanley determines the generating functions for the symmetric chromatic polynomials of two special kinds of graphs, chains and cycles.

Theorem 2.1. [Stan, Proposition 5.3] *The qenerating function for the symmetric chromatic polynomials of chains length d is*

$$\sum_{d\geq 0} X_{P_d}\cdot t^d = \frac{\sum_{i\geq 0} e_i t^i}{1-\sum_{i\geq 1}(i-1)e_i t^i} \tag{2.2}$$

One consequence of this theorem is derived from the geometric series, $\frac{1}{1-r} = \sum_{k\geq 0} r^k$. The generating function is of this form, and when we express it as a geometric sum it is easy to see that our generating function is e-positive. That is, the coefficients are positive for an expansion of the symmetric chromatic polynomials for chains in terms of the elementary symmetric functions.

Theorem 2.3. [Stan, Proposition 5.4] *The generating function for the symmetric chromatic polynomials of cycles of length d is*

$$\sum_{d\geq 0} X_{C_d}\cdot t^d = \frac{\sum_{i\geq 0} i(i-1)e_i t^i}{1-\sum_{j\geq 1}(j-1)e_j t^j}. \tag{2.4}$$

Again, it is easy to see from this generating function that the cycles are e-positive. Stanley conjectures e-positivity for many other classes of graphs, but fails to find generating functions for them.

3. Coefficients in the Basis of Elementary Symmetric Functions.

Chains. The elementary symmetric functions $\{e_\lambda | \lambda \vdash d\}$ form a **Z**-basis of the symmetric functions of degree d. Let $\lambda = (1^{a_1}, 2^{a_2}, \cdots, d^{a_d})$, and let c_λ denote the coefficient of $e_\lambda = e_1^{a_1} e_2^{a_2} \cdots e_d^{a_d}$ in the expansion of X_{P_d} in terms of the elementary symmetric functions. Thus:

$$X_{P_d} = \sum_{\lambda\vdash d} c_\lambda e_\lambda \tag{3.1}$$

It follows from the generating function that $c_\lambda \geq 0$ for all $\lambda \vdash d$. We give an

explicit formula for c_λ in the following theorem.

Theorem 3.2. Let $\lambda = (1^{a_1}, 2^{a_2}, \cdots, d^{a_d})$, be a partition of d. The coefficient c_λ of $e_\lambda = e_1^{a_1} e_2^{a_2} \cdots e_d^{a_d}$ in the expansion of X_{P_d} is given by

$$c_\lambda = \binom{a_1 + \cdots + a_d}{a_1, \cdots, a_d} \prod_{j=1}^{d} (j-1)^{a_j} +$$

$$\sum_{i\geq 1} \left(\binom{(a_1 \cdots + a_d) - 1}{a_1, \cdots, a_i - 1, \cdots, a_d} \left(\prod_{\substack{j=1 \\ j\neq i}}^{d} (j-1)^{a_j} \right) (i-1)^{a_i - 1} \right)$$

Proof. Apply the geometric series, $\frac{1}{1-r} = \sum_{k\geq 0} r^k$, to Stanley's generating function (2.2). This gives

$$\sum_{d\geq 0} X_{P_d}\cdot t^d = \frac{\sum_{i\geq 0} e_i t^i}{1-\sum_{j\geq 1}(j-1)e_j t^i} = \sum_{i\geq 0} e_i t^i \cdot \left(\sum_{k\geq 0} \left(\sum_{j\geq 1} (j-1)e_j t^j \right)^k \right). \tag{3.3}$$

The right side of (3.3) is the product of two sums, one sum over i and one over j. We consider two cases in our coefficient formula, i = 0 and $i > 0$. If $i = 0$ then X_{P_d} is generated entirely by $\sum_{k\geq 0}(\sum_{j\geq 1}(j-1)e_j t^j)^k$. Note that in this case, if e_n appears in the expansion, it is multiplied by $(n-1)$. Thus, $e_n^{a_n}$ is multiplied by $(n-1)^{a_n}$. If e_n is not in the expansion, then $a_n = 0$ and it follows that $(n-1)^{a_n} = 1$. Furthermore, the multinomial coefficient $\binom{a_1 + \cdots + a_d}{a_1, \cdots, a_d}$ keeps track of the number of different ways to obtain $e_1^{a_1} e_2^{a_2} \cdots e_d^{a_d}$ in the expansion of $\sum_{k\geq 0}(\sum_{j\geq 1} e_j t^j)^k$ [Bru]. It follows that the coefficient of $e_1^{a_1} e_2^{a_2} \cdots e_d^{a_d}$ in $\sum_{k\geq 0}(\sum_{j\geq 0}(j-1)e_j t^j)^k$ is $\binom{a_1 + \cdots + a_d}{a_1, \cdots, a_d} \cdot \prod_{j\geq 1}^{d} (j-1)^{a_j}$.

In the case where $i \neq 0$, we must generate the coefficient X_{P_d} in the same way, but as if the partition had one less part of size i. Looking at the right half

of the generating function, $\sum_{i\geq 0} e_i t^i \cdot \left(\sum_{k\geq 0}\left(\sum_{j\geq 1}(j-1)e_j t^j\right)^k\right)$ it is evident that parts

of size j have a weight of $(j - 1)$ because e_j is multiplied by $(j - 1)$. However, parts of size i have a weight of 1. Therefore if we have a partition, we must consider the case in which no part is generated by the sum over i, and each case in which exactly one part is generated by the sum over i. We have done the former, but in the latter we find the coefficient as if one part of size i has a weight of 1 instead of $(i - 1)$. We must do this for each possible i, so we will sum over i. First we find the product of all the weights of all of the parts with a size not equal to i. This is very simply $\prod_{j\geq 1, j\neq i}(j-1)^{a_j}$. Then we multiply by the combined weight of the parts of size i, less one part. This neglected part is the part generated by the sum over i (the first part of our product), which has a weight of one (not affecting the product) and looks like: $(i-1)^{a_i - 1}$.

Now we are ready to put the whole formula together. We start with the product of the weights of the non-i parts $\prod_{j\geq 1, j\neq i}(j-1)^{a_j}$, and we multiply by the weight of the parts of size i to obtain:

$$\left(\prod_{j\geq 1, j\neq i}(j-1)^{a_j}\right)(i-1)^{a_i - 1}.$$

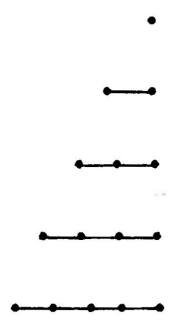
Next we throw in the multinomial coefficient for the parts generated by the sum over j. As mentioned above, we must do this for each possible i:

$$\sum_{i\geq 1}\left(\binom{(a_1 \cdots + a_2 + \cdots + a_d) - 1}{a_1, \cdots, a_i - 1, \cdots, a_d}\left(\prod_{j=1, j\neq i}^{d}(j-1)^{a_j}\right)(i-1)^{a_1 - 1}\right).$$

If $a_i - 1 < 0$, then $\binom{(a_1 + a_2 + \cdots + a_d) - 1}{a_1, \cdots, a_1 - 1, \cdots, a_d}$ is zero. Thus, this is just a sum over all

possible ways you could substitute a part generated by the i sum in the generating function for one generated by the j sum. Finally, we add the case in which $i = 0$ and obtain our formula.

Figure 3.4 gives the expansion of X_{P_d} for $1 \leq d \leq 5$.



$1e_1$

$2e_2$

$3e_3 + e_2e_1$

$4e_4 + 2e_3e_1 + 2e_2^2$

$5e_5 + 3e_4e_1 + 7e_3e_2 + e_2^2e_1$

Figure 3.4. X_{P_d} expanded in the basis of elementary symmetric functions.

Remarks. (1) If $a_1 \geq 2$, the coefficient of $e_1^{a_1}e_2^{a_2}\cdots e_d^{a_d}$ is zero. This is clear from two perspectives. In the generating function, the only place an e_1 could come from is from the sum over i. A part of size 1 generated by the sum over j has a coefficient of zero because $j = 1$ and there is a factor of $j - 1$ in that sum. Thus, the coefficient of $e_1^{a_1}e_2^{a_2}\cdots e_d^{a_d}$ is at most zero. Also, if you look at Theorem (3.2) for a partition in which $a_1 > 1$, you will see that the left product and the right product each lead to a circumstance in which you get 0^n, $n > 1$, which is just zero. This is avoided if you have only one e_1 because $(i-1)^{a_1 - 1} = (1 - 1)^{1-1} = 0^0$. Though 0^0 is undefined, we can assign it the value 1 from the generating function. Thus, if $a_1 \geq 2$, the coefficient is zero.

(2) If $a_1 = 1$ then the coefficient is just $\binom{(a_1 + \cdots + a_d)}{a_2, \cdots, a_d}\prod_{j=2}^{d}(j-1)^{a_j}$.

This was hinted at in remark (1). The first product in our formula is just zero, because of its $(j-1)^{a_j}$ term. The $(i-1)^{a_i - 1}$ in the second product is simply 1. Finally since $a_i = a_1 = 1$, the top half of the multinomial coefficient is $(a_1 + a_2 + \cdots + a_d) - 1 = (1 + a_2 + \cdots + a_d) - 1 = (a_2 + \cdots + a_d)$ and the bottom half is $(a_1 \cdots, a_i - 1, \cdots, a_d) = (a_1 - 1, a_2, \cdots, a_d) = (a_2, \cdots, a_d)$.

(3) There are a number of patterns that arise in the coefficients of the partitions if you modify the partitions in a systematic way. Any partition for which $a_n = 1, n \neq 2$ will have the coefficient 2. Any partition for which $a_1 = 1$, $a_{n>2} = 0$ will have a coefficient of 1. In the former case, you can express the coefficient

as $\binom{a_2}{a_2}(2-1)^{a_2}+\binom{a_2-1}{a_2-1}\cdot 1\cdot(2-1)^{a_2}-1$, which is just 2. In the latter case, you may invoke the result from (2) and the formula for the coefficient reduces to $\binom{(a_2+\cdots+a_d)}{a_2,\cdots,a_d}\prod_{j=2}^{d}(j-1)^{a_j}=\binom{a_2}{a_2}(2-1)^{a_2}=1$.

Example. Consider the polynomial generated when $d=5$. The partitions of 5 are (5), (1,4), (2,3),(1^2, 3), (1,2^2), (1^3, 2) and (1^5). From remark 1 we know that the coefficients of $e_3e_1^2, e_2e_1^3$, and e_1^5 are zero. Remark 2 tells us that the coefficient of e_4e_1 is just 3. Finally, from remark 3 we know that $e_2^2e_1$ has a coefficient of 1. This leaves us two to work out. The coefficient of e_5 is

$\binom{1}{1}\cdot(5-1)^1+\binom{0}{0}\cdot 1\cdot(1-1)^{(1-1)}=5$. The coefficient of e_3e_2 is

$\binom{2}{1,1}\cdot(3-1)^1\cdot(2-1)^1+\binom{1}{1}\cdot(2-1)^1\cdot(3-1)^{(1-1)}+\binom{1}{1}\cdot(3-1)^1\cdot(2-1)^{(1-1)}$

$=2\cdot 2+1\cdot 1+2\cdot 1=7$. Thus, the symmetric chromatic polynomial for a chain of length 5 is $X_{P_5}=5e_5+3e_4e_1+7e_3e_2+e_2^2e_1$.

Cycles. The generating function for cycles is like the generating function for chains, in that we get a symmetric-chromatic polynomial whose monomial terms consist of elementary symmetric functions multiplied by an integer. For a partition $\lambda=(1^{a_1},2^{a_2},\cdots,d^{a_d})$ of d, let r_λ denote the coefficient of $e_\lambda=e_1^{a_1}e_2^{a_2}\cdots e_d^{a_d}$ in the expansion of x_{C_d} in terms of the elementary symmetric functions. We give a formula for r_λ:

Theorem 3.5. Let $\lambda=(1^{a_1},2^{a_2},\ldots,d^{a_d})$ be a partition of d. The coefficient r_λ of $e_\lambda=e_1^{a_1}e_2^{a_2}\cdots e_d^{a_d}$ in the expansion of X_{C_d} is given by

$$r_\lambda=\sum_{i\geq 1}\binom{(a_1+a_2+a_d)-1)}{a_1,\cdots,a_i-1,\ldots,a_d}i\cdot\prod_{j=1}^{d}(j-1)^{a_j}.$$

Proof: The proof is quire simililar to the proof for our formula for chains. Once again, we start off by manipulating the original generating function (2.4) slightly, using our identity for infinite geometric sums:

$$\sum_{d\geq 0}X_{C_d}\cdot t^d=\sum_{i\geq 0}i(i-1)e_it^i\cdot\sum_{k\geq 0}\left(\sum_{j\geq 1}(j-1)e_jt^j\right)^k. \tag{3.6}$$

As in the case for the chains, the coefficient is made up of the multinomial expansion of the right half. The appropriate terms of this expansion for the coefficient X_{C_d} are those such that $i+(j_1+\cdots+j_k)=d$ because $t^d=t^i\cdot t^{j_1}\cdot\ldots\cdot t^{j_k}=t^{i+j_1+\cdots+j_k}$. The sum over j is raised to the kth power again, so we include the multinomial coefficient, $\binom{(a_1+a_2+\cdots+a_d)-1}{a_1,\cdots,a_i-1,\cdots,a_d}$. We remove one part of size i to take into account the fact that the sum over i is not raised to the kth power.

The portion of the coefficient generated by the weights of the sums is exactly the same as it was for the chain case, except that the weight for the sum over i is no longer 1, it is $i\cdot(i-1)$. Note that this means that there is always a contribution from the sum over i. In the case for the chains, if $i=0$ the term in the product that involved i was 1, and the coefficient was due entirely to the sum over j. In the case of the cycles, if $i=0$ then $i\cdot(i-1)=0$ and the whole product is 0. Thus, we must first find the product of the weights for all $a_j, j\neq i$, which is just $\prod_{j\geq 1,j\neq i}(j-1)^{a_j}$. Then we multiply in those parts generated by the sum over j for $j=i$, which are just $(i-1)^{a_i-1}$. Finally, we multiply in the contribution from the sum over i: $i\cdot(i-1)$. Putting it all together:

$$\left(\prod_{j\geq 1,j\neq i}(j-1)^{a_j}\right)(i-1)^{a_i-1}(i-1)\cdot i=\left(\prod_{j\geq 1,j\neq i}(j-1)^{a_j}\right)(i-1)^{a_i}\cdot i=\prod_{j\geq 1}(j-1)^{a_j}\cdot i$$

Now we multiply by the multinational coefficient and sum over all possible i.

$$\sum_{i\geq 1}\binom{(a_1+a_2+a_d)-1)}{a_1,\cdots,a_i-1,\ldots,a_d}\cdot i\cdot\prod_{j=1}^{d}(j-1)^{a_j}.$$

(Once again note that if, as we sum over all i, i is a value that is not found in the given partition then the multinomial coefficient is zero.) We have the

formula we set out to prove. ■

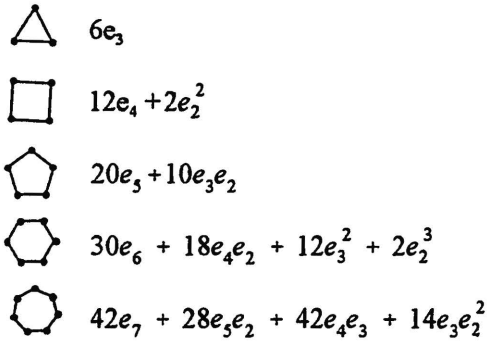


Figure 3.7. The coefficients for the polynomials of cycles.

Remarks. (1) The coefficient of e_n is just $n \cdot (n - 1)$. If $i \neq n$, then

$\binom{(a_1+a_2+\cdots+a_d)-1}{a_1,\cdots,a_i-1,\cdots,a_d} = 0$ and the whole product is zero. If $i = n$, then

$\binom{(a_1+a_2+\cdots+a_d)-1}{a_1,\cdots,a_1-1,\cdots,a_d} = 1$. But if $j \neq n$, then $a_j = 0$, and that part of the product is 1. Finally, if $j = i = n$ then we get $(j - 1)^1 \cdot i = (n - 1) \cdot n$.

(2) The coefficient for a term with $a_1 \geq 1$ is zero. If $j = 1$, then $a_j = a_1 \geq 1$. But one term of our product is $(j-1)^{a_j}$. If $j \geq 1$ then $a_j \geq 0$ and the term of the product mentioned is $(1-1)^{a_j} = 0^{a_j} = 0$ making the whole product is zero.

Example. Consider the polynomial generated for the cycle when $d = 5$. The partitions of 5 are (5), (1, 4), (2, 3), (1^2, 3), (1, 2^2), (1^3, 2), and (1^5). From remark 2 we know that the coefficients of $e_4e_1, e_3e_1^2, e_2^2e_1, e_2e_1^3$, and e_1^5 are zero. From remark 1 we know that the coefficient of e_5 is $5 \cdot (5-1) = 20$. This just leaves e_3e_2. The coefficient of e_3e_2 is

$$\binom{1}{1} \cdot (2-1)^1 \cdot (3-1)^1 \cdot 2 + \binom{1}{1} \cdot (2-1)^1 \cdot (3-1)^1 \cdot 3 = 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 3 = 10.$$

Therefore, the symmetric chromatic polynomial for a cycle of length 5 is $X_{C_5} = 20e_5 + 10e_3e_2$.

4. A Graph Basis for the Symmetric Functions. Theorem (3.2) leads to an interesting consequence. We can define a basis for the symmetric functions in terms of graphs that are single chains and graphs that are disjoint collections of chains. The following property is important in showing this.

Property 4.1. Let G be a graph made up of disjoint subgraphs E and F. Then the symmetric chromatic polynomial of G is $X_G = X_E \cdot X_F$ where X_E is the symmetric chromatic polynomial for E and X_F is the symmetric chromatic polynomial for F.

Proof. The colorings of E are completely independent of the colorings of F. Thus χ_E is completely independent of χ_F, so if you fix a coloring of E, then you have χ_F ways to color the whole disjoint graph G. There are χ_E ways to fix a coloring of E, therefore $\chi_G = \chi_E \cdot \chi_F$ and $\chi_G = \chi_E \cdot \chi_F$. ■

The basis of the symmetric functions is as follows.

Theorem 4.2. Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be a partition of n and let $G_\lambda = P_{\lambda_1} \cdot P_{\lambda_2} \cdots P_{\lambda_n}$ be the graph consisting of disjoint chains whose lengths form the partition λ of n. Then $\{X_{G_\lambda} | \lambda \vdash n\}$ is a basis for Λ^n.

Proof. Using Theorem 3.2, if we expand X_{P_d}, (the symmetric chromatic polynomial for a single chain) in terms of the elementary functions, the result is a polynomial in which the coefficient of e_d is d. The orders of the elementary functions form a partition of d, so the polynomial for a single chain of n vertices is $ne_n + M$, where M is some sum of terms in which each term is a partition of n. Thus, the partition of n that corresponds to a term of M is less than $\lambda = (n, 0, \ldots, 0)$ in the lexicographic ordering. The symmetric chromatic polynomial of the graph of disjoint subchains whose polynomials have leading terms $e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}$ will have a leading term $e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}$.

Every term of M is some product of two or more elementary symmetric

functions and a constant coefficient. Again, property (4.1) says that given any product of elementaries we can find a graph of disjoint subchains whose symmetric chromatic polynomial will have that product of elementaries as its leading term. Furthermore, that graph will be the graph of disjoint chains that have each of those elementaries as the leading terms in their respective polynomials. Thus we can always find a graph whose polynomial has the same leading term as X (up to a constant multiple), subtract the two polynomials, and end up with a new polynomial that has a smaller leading term. Since the lexicographic ordering is a well ordering, we will eventually have a remainder of zero. This, the set $\{X_{G_\lambda} \mid \vdash n\}$ spans all of the elementary functions e_λ of a given degree d, where $\lambda \vdash d$.

The polynomials for single chains are linearly independent, because the polynomials for chains have terms that correspond to partitions of the chain length. Thus, no two chains of different lengths will have like terms. The symmetric chromatic polynomials of graphs comprised of disjoint chains are also linearly independent, because the symmetric chromatic polynomials of any two different graphs of disjoint chains will have different leading terms. This is because the leading terms are the products of the leading terms of the subchains, and if the graphs are different, they have different subchains.

Therefore, since the set of the symmetric chromatic polynomials of the chains and the disjoint chains spans the elementary symmetric functions of degree n and is linearly independent, it is a basis for the elementary symmetric functions of degree n. In section 1 we showed that $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for Λ^n thus the set of symmetric chromatic polynomials for graphs of disjoint chains is also a basis for Λ^n. ■

5. Conjectures about Other Types of Graphs. A cycle with a tail is a cycle with chain connected to one of the vertices in the cycles. Only cycles with a single tail are considered. The natural first case to consider is the three cycle with an arbitrarily long tail. We may attack the problem of finding a generating function for this symmetric-chromatic polynomial using the method that Stanley used on chains. Define L_d, a 3-cycle with a tail, to have vertex set $V = \{v_1, \cdots, v_d\}$ with edges $v_i v_{i+1}, 1 \geq i \geq d-1$ and $v_1 v_3$. Let $X_d^i = \sum_\kappa x^\kappa$ summed over all

proper colorings $\kappa: V \to [n]$ of L_d such that $\kappa(v_d) = i$. Then let us define

$$F^i(t) = \sum_{d \geq 3} X_{L_d}^i \cdot t^d.$$

We index from $d \geq 3$ because it does not make sense to talk about a two vertex three cycle with a tail. We make the substitution $X_d^i = \sum_\kappa x^\kappa$ and we sum over the index $j, j \neq i$. This gives us:

$$F^i(t) = \left(\sum_{\substack{j \geq 1 \\ j \neq i}}^{n} \sum_{d \geq 3} \sum_{\substack{\kappa' \\ \kappa'(d-1) \neq j}} x^{\kappa'} t^d \right) x_i.$$

Now we need to take the first term out of the sum over d. This term is the symmetric chromatic polynomial for a 3-cycle in which one of the vertices has a fixed color, i. One of the other vertices can be any color except i, and the third vertex can be any color except i or the color of the second vertex. Thus, the equation with this term pulled out of the sum is

$$F^i(t) = \sum_{\substack{y=1 \\ y \neq i}}^{n} \sum_{\substack{z=1 \\ z \neq y,i}}^{n} x_y x_z x_i t^3 + \left(\sum_{\substack{j \geq 1 \\ j \neq i}}^{n} \sum_{d \geq 4} \sum_{\substack{\kappa \\ \kappa'(d-1) \neq j}} x^{\kappa'} t^d \right) x_i.$$

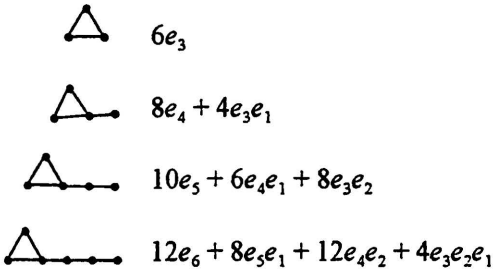
However, $\sum_{y=1, y \neq i}^{n} \sum_{z=1, z \neq y,i}^{n} x_y x_z = 2e_2(\hat{x}_i)$. Also, we can advance the d index by one by substituting $d = d + 1$. Then, $\sum_{d \geq 3} \sum_{\substack{\kappa \\ \kappa'(d) \neq j}} x^{\kappa'} t^{d+1} = F^i(t) \cdot t$, so,

$$F^i(t) = 2e_2(\hat{x}_i) x_i t^3 + \left(\sum_{\substack{j=1 \\ j \neq i}} F^j(t) \right) x_i t \tag{5.1}$$

The solution to this equation yields the generating function for 3-cycles with tails. It seems evident that generating similar equations would be easy for any cycle with one tail, or even for a simple tree that had one fork. If this function can be solved, generating functions for several more classes of graph would be easy to find.

The following table was computed with help from Timothy Chow's C-

program [Chow].



$6e_3$

$8e_4 + 4e_3e_1$

$10e_5 + 6e_4e_1 + 8e_3e_2$

$12e_6 + 8e_5e_1 + 12e_4e_2 + 4e_3e_2e_1$

Figure 5.2. The syrmmetric chromatic polynomials of lariats

From observations of symmetric chromatic polynomials generated by a C program written by Timothy Chow [Chow], we made some conjectures about all of the polynomials generated by this function. First of all, there is a very close relationship between this group of polynomials and those generated for chains. The products of elementary symmetric functions that appear in these polynomials are a subset of the products of elementary functions that appear in the polynomials for chains of the same number of vertices (that is, if a term with e_4e_2 appears in the symmetric chromatic polynomial for a 3-cycle with a 3-chain, then a term with e_4e_2 appears in the symmetric polynomial for 6-chain). It appears that the only terms that are missing are those which contain no elementary symmetric function of order greater than two. Furthermore, the coefficients of products of elementary symmetric functions in the cycle-tail polynomial where no elementary functions of degree two are present seem to be exactly twice their counterparts in the polynomial for a chain with the same number of vertices. Those terms that do contain e_2 have coefficients somewhat less than twice those of the same term in the function for chains. That terms with e_2 are somehow different is supported by (5.1). This also seems to support a conjecture that the other coefficients are very closely related to their counterparts in the chains, as (5.1) is otherwise very similar to the equation that Stanley arrived at in his proof of Theorem 2.1. Other conjectures:

1. The coefficient of $e_{n-2}e_2$ is $4(n - 3)$
2. The coefficient of $e_{n-3}e_2e_1$ is $2(n - 4)$
3. The coefficient of $e_{n-4}e^2$ is $4(n - 5)$

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[Chow] Timothy Chow, *C-Program for Computing Symmetric Polynomials.*

BIOGRAPHICAL SKETCHES OF THE AUTHORS

Xuming Chen is a recipient of the Henry C. Miller, Jr. Endowed Mathematics Fellowship Award and an honor student at the University of Alabama, Tuscaloosa.

Aimee Kunnen and **Steve Schlicker** wrote their paper based on work conducted as part of the Summer Undergraduate Research Program at Grand Valley State University. Aimee received her Bachelor's degree in mathematics from GVSU in 1995 and is currently teaching as an adjunct at Grand Valley. Steve is an Associate Professor of Mathematics at GVSU.

Johanna Miller is a junior at Duke University majoring in mathematics and dabbling in chemistry, music, and physics. She spends much too much of her time writing problems for Duke's high school math contest, playing Scarlatti sonatas on the harpsichord, baking cookies, and ruining grade distributions in chemistry classes. **C. Ryan Vinroot** is a senior undergraduate mathematics major at North Carolina State University. He spent the 1996-97 academic year studying mathematics in Budapest, Hungary and plans to attend graduate school in mathematics in the fall. Other than mathematics, Ryan also enjoys mountain biking, traveling, and playing percussion.

Tricia Stone (Hovorka) graduated from John Carroll University in May 1997 with a B.S. degree, majoring in mathematics. She married Joseph Hovorka in the summer of 1997 and is currently living in St. louis where she is beginning a career as an actuary.

Paul S. Bruckman was born in Florence, Italy, and received an M.S. degree in mathematics from the University of Illinois at Chicago in 1974. From 1960 through 1990, he was employed with private actuarial firms, most recently as a pension plan actuary. Mr. Bruckman has been and continues to be a frequent contributor to The Fibonacci Quarterly and the Pi Mu Epsilon Journal .

Michael Wolfe wrote this paper as a senior honors project at Macalester College. It was written under the direction of Tom Halverson and David Bressoud, whom Michael thanks profusely.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 1, 1998.

Problems for Solution

927. *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

In the following base ten alphametic,

a) find the maximum value for *FINAL*,

b) find the minimum value for *FINAL*, and

c) can you find a solution yielding any other value for *FINAL*?

$$PASS + THE = FINAL.$$

928. *Proposed by the late J. L. Brenner, Palo Alto, California.*

Is it true that, as n increases through the integers, the number of primes in the open interval $(n, 2n)$ can stay the same, increase by one, or decrease by one, but never change by two or more? Student solutions are especially invited.

929. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

On the ground floor of a building there are on the wall three light switches of the usual kind that show whether they are on or off. One of them controls a lamp with an ordinary incandescent 100-watt bulb located on the third floor. The other two switches are not connected to anything, although you have no way of telling which switch is the live one. You are allowed to toggle the switches at will before climbing to the third floor, which you can do just once. From the ground floor you cannot tell whether the lamp is on or off, but you have full access to the lamp when you are on the third floor.

a) Tell how to determine which switch controls the lamp.

b) Solve the problem if the switches are not marked with on and off positions, but you know that the lamp is initially off.

c) Solve part (b) if you do not know if the lamp is initially on or off.

930. *Proposed by the late J. L. Brenner, Palo Alto, California.*

By direct calculation, that is, without using published theorems, show that the permutation group generated by (127) and (135)(246) contains all of the following types (*shapes*): 3^1, 2^2, $4^1 2^1$, 3^2, $3^1 2^2$, 7^1, and 5^1, where (127) is of type 3^1 and (135)(246) is of type 3^2.

931. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine the maximum value of

$$\frac{1}{S_u^r}\{S_{nur_1} + S_{nur_2} + \cdots + S_{nur_m}\},$$

where the x_i, r_i, and u all are positive and

$$S_t = x_1^t + x_2^t + \cdots + x_n^t \quad \text{and} \quad r = r_1 + r_2 + \cdots + r_m \geq 1.$$

932. *Proposed by David Iny, Baltimore, Maryland.*

a) For $0 < \mu < \varepsilon \leq 1$, define a sequence recursively by $x_0 = \varepsilon$, and $x_{n+1} = \sin x_n$ for $n \geq 0$. Thus $\{x_n\}$ is a monotone decreasing sequence of positive numbers. Estimate the smallest value of m such that $x_m \leq \mu$.

b) Repeat Part (a) using the recursion formula $x_{n+1} = \ln(1 + x_n)$.

933. *Proposed by David Iny, Baltimore, Maryland.*

Define for nonnegative integers k and n the sums

$$J_{kn} = \frac{1}{1+k}\binom{n}{0} + \frac{1}{2+k}\binom{n}{1} + \cdots + \frac{1}{n+1+k}\binom{n}{n}.$$

a) Find closed form expressions for J_{kn} for $k = 0, 1, 2, \ldots$

b) Let p be any nonnegative real number and $[x]$ the greatest integer less than or equal to x. Evaluate

$$\lim_{n\to\infty} n2^{-n}(J_{[pn]n}).$$

934. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Evaluate the integral

$$I = \int_{-\infty}^{\infty} \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx.$$

***935.** *Proposed by M. A. Khan, Lucknow, India.*

From a deck of n cards numbered 1, 2, ..., n, select m cards ($3 \leq m \leq n$) at random. Show that the probability p that the numbers on the selected cards are in arithmetic progression is given by

$$p = \frac{(q+1)(R+n+1-m)}{2\binom{n}{m}},$$

where q is the integral quotient and R the remainder when $n-m+1$ is divided by $m-1$.

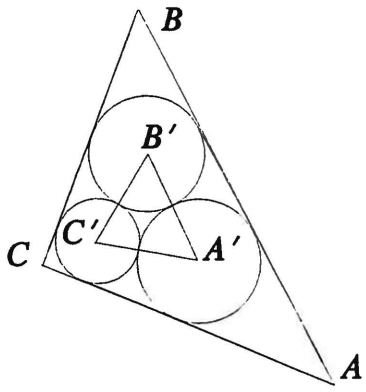
936. *Proposed by the late Jack Garfunkel, Flushing, New York.*

Given the Malfatti configuration, where three mutually external, mutually tangent circles with centers A', B', C' are inscribed in a triangle ABC so that circle (A') is tangent to the two sides of angle A, circle (B') is tangent to the sides of angle B, and (C') to the sides of C. See the figure. If $\angle A \leq \angle B \leq \angle C$ and $\angle A < \angle C$, then prove that we have $\angle C' - \angle A' < \angle C - \angle A$.

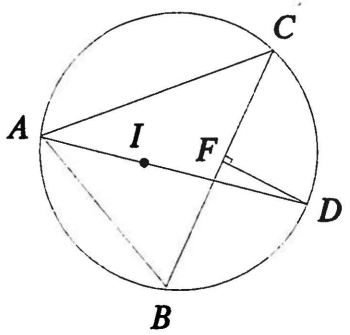
937. *Proposed by R. S. Luthar, Janesville, Wisconsin.*

Let I be the incenter of triangle ABC, let AI cut the triangle's

circumcircle (again) at point D, and let F be the foot of the perpendicular dropped from D to side BC, as shown in the figure. Prove that $DI^2 = 2R \cdot DF$, where R is the circumradius of triangle ABC.



Problem 936



Problem 937

938. *Proposed by R. S. Luthar, Janesville, Wisconsin.*

Find the locus of the midpoints M of the line segments in the first quadrant lying between the two axes and tangent to the unit circle centered at the origin. See the figure.

939. *Proposed by Khiem Viet Ngo, Virginia Polytechnic Institute, Blacksburg, Virginia.*

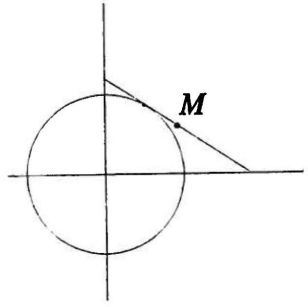
In the accompanying figure both quadrilaterals $ABCD$ and $MNPQ$ are squares, each side of square $ABCD$ has length 1, and the five inscribed circles are all congruent to one another. Find their common radius.

Solutions

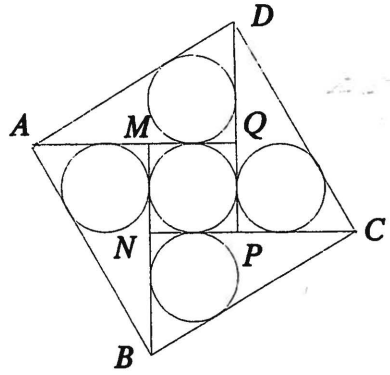
901. [Spring 1997] *Proposed by Elizabeth Andy, Limerick, Maine.*

Solve this base twelve multiplication alphametic

$$PROF \times EVES = GEOMETRY.$$



Problem 938



Problem 939

Answer by the Proposer.

Using e for eleven, the unique solution is $e136 \times 9592 = 89379410$.

Alphametics will stimulate int'rest
When clever solutions are expressed.
When all institutions
Send computer solutions,
Excitement is certainly repressed.

Also solved by Charles Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* Mark Evans, *Louisville, KY,* Richard I. Hess, *Rancho Palos Verdes, CA, and* Rex H. Wu, *Brooklyn, NY.*

902. [Spring 1997] *Proposed by Bill Correll, Jr., Student, Denison University, Granville, Ohio.*

For all positive integers n, prove that

$$n + 1 \geq \sqrt{2^{1/n}(n^{4/3} + n^{5/3})}.$$

I. *Solution by* Mayumi Dubree, *Graduate Student, Angelo State University, San Angelo, Texas.*

We have that $(1 + 1/n)^n$ is a well-known monotone increasing sequence (converging to $e = 2.718...$). Thus, if n is a positive integer, then

$$\left(1 + \frac{1}{n}\right)^n \geq \left(1 + \frac{1}{1}\right)^1 = 2, \quad \text{so that} \quad n + 1 \geq 2^{1/n}n. \tag{1}$$

Furthermore, since $(1 - n^{1/3})^2 \geq 0$, then we have $1 - n^{1/3} + n^{2/3} \geq n^{1/3}$ and

$$n + 1 = (1 + n^{1/3})(1 - n^{1/3} + n^{2/3}) \geq n^{1/3}(1 + n^{1/3}). \tag{2}$$

Finally, multiply (1) and (2) side for side and then take square roots of each side to get the desired inequality.

II. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*
We must show that

$$f(x) = (x + 1)^2 - 2^{1/x}(x^{4/3} + x^{5/3})$$

is nonnegative for natural numbers x. Let $u = x^{1/3}$ and define

$$g(u) = (u^3 + 1)^2 - 2^{1/u^3}(u^4 + u^5)$$

and

$$h(u) = (u^3 + 1)^2 - \frac{3}{2}(u^4 + u^5).$$

When $x > (\ln 2)/\ln(3/2) \approx 1.7095$, then $u > [(\ln 2)/\ln(3/2)]^{1/3}$ and we have $g(u) > h(u)$. To find when $h(u) \geq 0$, we calculate that

$$\frac{dh}{du} = 6u^2(u^3 + 1) - 6u^3 - \tfrac{15}{2}u^4 = \tfrac{3}{2}u^2(4u^3 - 5u^2 - 4u + 4),$$

which is zero at $u_0 = 0$, $u_1 = 0.724$, and $u_2 = 1.467$. We find that $h'(u) > 0$ for $u > u_2$ and $h(u_2) = 0.143 > 0$ is the minimum on the interval (u_1, ∞), so $g(u) > 0$ and $f(x) > 0$ for $x > 1.7095$. Since $f(1) = 0$, the desired inequality is established.

Also solved by Paul S. Bruckman, *Highwood, IL,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans *and* Matthew Evans, *Louisville, KY, and University of Nebraska, Lincoln,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Murray S. Klamkin, *University of Alberta, Canada,* Yoshinobu Murayoshi, *Okinawa, Japan,* Cecil Rousseau, *The University of Memphis, TN,* H.-J. Seiffert, *Berlin, Germany,* Rex H. Wu, *Brooklyn, NY, and the* Proposer.

903. [Spring 1997] *Proposed by Peter A. Lindstrom, Batavia, New York.*
Evaluate the indefinite integral

$$\int \frac{x\ln[x(x-1)] - \ln(x-1)}{x(x-1)}\,dx.$$

I. *Solution by Barry Brunson, Western Kentucky University, Bowling Green, Kentucky.*

As Bo Diddley's song says, "You can't judge a book by lookin' at the cover." Rewriting the integrand, using the properties of logarithms, is all that is necessary to find the integral. Thus use $\ln[x(x-1)] = \ln x + \ln(x-1)$ to obtain

$$\int \frac{x\ln[x(x-1)] - \ln(x-1)}{x(x-1)}\,dx = \int\left(\frac{1}{(x-1)}\ln x + \frac{1}{x}\ln(x-1)\right)dx.$$

At this point we say "AHA!", recognizing the *Hi-de-Ho and Ho-de-Hi* footprint of the product rule. Hence the antiderivative is

$$\ln x \ln(x-1) + C.$$

II. *Solution by Joseph C. Fursch, student, Angelo State University, San Angelo, Texas.*

Simplify the integral

$$\int \frac{x\ln[x(x-1)] - \ln(x-1)}{x(x-1)}\,dx = \int \frac{1}{(x-1)}\ln x\,dx + \int \frac{1}{x}\ln(x-1)\,dx.$$

Now use integration by parts with $u = \ln(x-1)$ and $dv = dx/x$ to get

$$\int \frac{\ln(x-1)}{x}\,dx = (\ln x)\cdot\ln(x-1) - \int \frac{\ln x}{x-1}\,dx + C.$$

Therefore,

$$\int \frac{x\ln[x(x-1)] - \ln(x-1)}{x(x-1)}\,dx = (\ln x)\cdot\ln(x-1) + C.$$

Also solved by Prem N. Bajaj, *Wichita State University, KS,* Frank P. Battles, *Massachusetts Maritime Academy, Buzzards Bay,* James D. Brasher, *Colsa Corp., Huntsville, AL,* Jim Brown,

Michigan State University, Laingsburg, Paul S. Bruckman, *Highwood, IL,* Eleanore Chadderdon, *Kalamazoo, MI,* Kenneth B. Davenport, *Pittsburgh, PA,* Mayumi Dubree, *Angelo State University, San Angelo, Texas,* Russell Euler and Jawad Sadek, *Northwest Missouri State University, Maryville,* George P. Evanovich, *Saint Peter's College, Jersey City, NJ,* Mark Evans *and* Matthew Evans, *Louisville, KY, and University of Nebraska, Lincoln,* Robert C. Gebhardt, *Hopatcong, NJ,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Murray S. Klamkin, *University of Alberta, Canada,* Michael W. Lanstrum, *Kent State University, OH,* Henry S. Lieberman, *Waban, MA,* David E. Manes, *SUNY College at Oneonta,* Yoshinobu Murayoshi, *Okinawa, Japan,* Kandasamy Muthuvel, *University of Wisconsin-Oshkosh,* Bob Prielipp *(2 solutions), University of Wisconsin-Oshkosh,* Cecil Rousseau, *The University of Memphis, TN,* Shiva K. Saksena, *University of North Carolina at Wilmington,* Harry Sedinger, *St. Bonaventure University, NY,* H.-J. Seiffert, *Berlin, Germany,* Skidmore College Problems Group, *Saratoga Springs, NY,* Leon Varjian, *Midland Park High School, NJ,* Stan Wagon, *Macalester College, St. Paul, MN,* Rex H. Wu, *Brooklyn, NY, and the* Proposer.

Comment by the Editor. Other forms for the answer included

$$\ln[x(x-1)]\bullet\ln(x-1) - \ln^2(x-1),$$

$$[\ln^2 x + \ln^2(x-1) - \ln^2\{x/(x-1)\}]/2,$$

and

$$[\ln^2\{x(x-1)\} - \ln^2 x - \ln^2(x-1)]/2,$$

each of which reduces to the answer given above by application of the formula $\ln ab = \ln a + \ln b$ or $\ln a/b = \ln a - \ln b$.

904. [Spring 1997] *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Let n be a positive integer and let $\sigma(n)$ denote the sum of the positive integer divisors of n. If $A = \{n/\sigma(n);\ n \text{ is a positive integer}\}$, prove that A is dense in the interval $(0, 1)$.

Solution by Cecil Rousseau, The University of Memphis, Memphis, Tennessee.

Given any number a in the interval $(0, 1)$, we shall show a sequence $\{n_k\}$ such that $\{n_k/\sigma(n_k)\}$ is a monotone nonincreasing sequence converging to a. To that end, let p_i, $i = 1, 2, \ldots$, be the primes in increasing order. If, for

some finite subset S of the natural numbers, we have

$$n = \prod_{i\in S} p_i, \text{ then } \frac{n}{\sigma(n)} = \prod_{i\in S}\frac{p_i}{p_i+1} = \prod_{i\in S}\left(1 - \frac{1}{p_i+1}\right).$$

Now choose the n_k by the following rule. Let $n_1 = p_i$, where p_i is the smallest prime such that $n_1/\sigma(n_1) \geq a$. At each stage n_k, if $n_k/\sigma(n_k) = a$, then $n_{k+1} = n_k$. Otherwise, $n_{k+1} = n_k p_j$, where $j > i$ and p_j is the smallest prime such that $n_{k+1}/\sigma(n_{k+1}) \geq a$. Then $\{n_k/\sigma(n_k)\}$ is a monotone nonincreasing sequence bounded below to a. Since

$$\sum_{p \text{ prime}} \frac{1}{p} \text{ diverges, then } \prod_{p \text{ prime}}\left(1 - \frac{1}{p+1}\right)$$

diverges to zero. If $\{n_k/\sigma(n_k)\}$ converges to $a + \varepsilon$, then every $p_j > a/\varepsilon$ would be chosen and $\{n_k/\sigma(n_k)\}$ would converge to zero. This contradiction establishes a as the limit of the sequence.

Also solved by Paul S. Bruckman, *Highwood, IL, and the* Proposer.

905. [Spring 1997] *Proposed by the late Charles W. Trigg, San Diego, California.*

A permutation of the digits of the four-digit integer 1030 in the decimal system converts it to its equivalent 3001 in the septenary system. Find all four-digit base-ten numerals that can be converted to their base seven equivalents by permuting their digits.

I. *Solution by Paul S. Bruckman, Highwood, Illinois.*

Note that the digits must be between 0 and 6 inclusive with the initial digit ≥ 1. Let $N = abcd = (abcd)_{10}$ denote the decimal representation. Let the base 7 numeral be $N_7 = (efgh)_7$, where the digits e, f, g, and h are a permutation of the digits a, b, c, and d. Then $1000 < N < 2400 = (6666)_7$, so we see that $10 \leq ab \leq 24$.

If $N = 10cd$, then $1000 \leq N \leq 1066$ and $(2626)_7 \leq N_7 \leq (3052)_7$. Hence $cd = 26$ or 62 or that c or d is 3. Testing each possibility, we find that $1026 = (2664)_7$, $103d = (3001)_7 + d$, $1062 = (3045)_7$, and $10c3 = (2632)_7 + 10c$, where the $10c$ must be converted to base 7. We try all possible digits 0 to 6 for c and for d, finding only the given solution $N = 1030 = (3001)_7$.

Similarly, we test $ab = 11, 12, 13, 14, 15, 16, 20, 21, 22$, and 23. The most troublesome case is $ab = 14$, which requires trying 49 different possibilities. In this manner we find all solutions, namely $1030 = (3001)_7$, $1234 = (3412)_7$, $1366 = (3661)_7$, $1431 = (4113)_7$, $1454 = (4145)_7$, $2060 = (6002)_7$, and $2116 = (6112)_7$, a total of seven solutions.

II. *Comment by Robert C. Gebhardt, Hopatcong, New Jersey.*

If we allow initial zeros, then obviously 0000, 0001, ..., 0006 are the same in both bases and are solutions. Other solutions with initial zeros are $0023 = (0032)_7$, $0046 = (0064)_7$, $0265 = (0526)_7$, $0316 = (0631)_7$, and $0641 = (1604)_7$.

Also solved by Charles Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* Mark Evans, *Louisville, KY,* Robert C. Gebhardt, *Hopatcong, NJ,* Richard I. Hess, *Rancho Palos Verdes, CA,* Yoshinobu Murayoshi, *Okinawa, Japan,* Rex H. Wu, *Brooklyn, NY,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer.

906. [Spring 1997] *Proposed by Norman Schaumberger, Douglaston, New York.*

If a, b, and c are positive real numbers, prove that

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

I. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

From four applications of the arithmetic mean-geometric mean (AM-GM) inequality, we have that

$$\frac{1}{3}\left[\left(\frac{b}{c}\right)^3 + \left(\frac{c}{a}\right)^3 + 1^3\right] \geq \frac{b}{c}\cdot\frac{c}{a}\cdot 1 = \frac{b}{a},$$

$$\frac{1}{3}\left[\left(\frac{c}{a}\right)^3 + \left(\frac{a}{b}\right)^3 + 1^3\right] \geq \frac{c}{a}\cdot\frac{a}{b}\cdot 1 = \frac{c}{b},$$

$$\frac{1}{3}\left[\left(\frac{a}{b}\right)^3 + \left(\frac{b}{c}\right)^3 + 1^3\right] \geq \frac{a}{b}\cdot\frac{b}{c}\cdot 1 = \frac{a}{c},$$

and

$$\frac{1}{3}\left[\left(\frac{a}{b}\right)^3 + \left(\frac{b}{c}\right)^3 + \left(\frac{c}{a}\right)^3\right] \geq \frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a} = 1.$$

Now add the four inequalities and subtract 1 from each side to obtain the required result. The solution generalizes.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

More generally, it will be shown that

$$S = \sum x_i^m \geq \max\left\{\sum x_i, \sum \frac{1}{x_i}\right\},$$

where the sums here and subsequently are over $i = 0$ to n, $m \geq n \geq 1$, $x_i > 0$, and $x_0x_1\cdots x_n = 1$. Firstly (here it suffices for $m \geq 1$),

$$\frac{\sum x_i^m}{n+1} \geq \left[\frac{\sum x_i}{n+1}\right]^m \geq \frac{\sum x_i}{n+1}. \tag{1}$$

The left inequality follows from the power mean inequality

$$\left(\frac{\sum x_i^p}{n+1}\right)^{1/p} \geq \left(\frac{\sum x_i^q}{n+1}\right)^{1/q},$$

where $p > q$, the x_i and all positive, the sums are from $i = 0$ to n, and equality holds only when the x_i are all equal. See Theorem 22, page 64, in *Analytic Inequalities* by N. D. Kazarinov, Holt, Rinehart and Winston, 1961. The right inequality in (1) follows from the AM-GM inequality.

We now rewrite S in the form

$$S = \frac{1}{n}[(S - x_0^m) + (S - x_1^m) + \cdots + (S - x_n^m)].$$

By the AM-GM inequality for $i = 0, 1, \ldots, n$, and then (1),

$$\frac{S - x_i^m}{n} \geq \left(\frac{1}{x_i}\right)^{m/n},$$

so that

$$S \geq \sum\left(\frac{1}{x_i}\right)^{m/n} \geq \sum \frac{1}{x_i}.$$

The somewhat less general case for $m = n$ was proved in my Olympiad Corner 64, *Crux Mathematicorum* 11(1985)118.

Also solved by Paul S. Bruckman, *Highwood, IL*, William Chau, *A T & T Laboratories, Middletown, NJ*, Russell Euler *and* Jawad Sadek *(2 solutions), Northwest Missouri State University, Maryville*, Richard I. Hess, *Rancho Palos Verdes, CA*, Joe Howard, *New Mexico Highlands University, Las Vegas*, Henry S. Lieberman, *Waban, MA*, Cecil Rousseau, *The University of Memphis, TN*, H.-J. Seiffert, *Berlin, Germany, and the* Proposer. *One incorrect solution was received.*

Comment by the Editor. The problem is not new, but the featured solutions are interesting and well worth publishing. In addition to the *Crux* reference given in Solution II, also mentioned by Euler and Sadek, Chau gave Problem 7.2.4, page 251, of *Problem Solving Through Problems* by L. C. Larson.

907. [Spring 1997] *Proposed by the late J. S. Frame, Michigan State University, East Lansing, Michigan.*

For $k \geq 0$, evaluate the determinant of the $n \times n$ matrix $A_{n,k}$ whose (i,j)-entry is $(i + j + k - 2)!$. Denote by $n!!$ the product $\Pi_{m=1}^{n}\, m!$.

Solution by Carl Libis, University of Alabama, Tuscaloosa, Alabama.

We have

$$|A_{n,k}| = \begin{vmatrix} k! & (k+1)! & \cdots & (k+n-1)! \\ (k+1)! & (k+2)! & \cdots & (k+n)! \\ & & \cdots & \\ (k+n-1)! & (k+n)! & \cdots & (k+2n-2)! \end{vmatrix}$$

$$= k!(k+1)!\cdots(k+n-1)!\begin{vmatrix} 1 & k+1 & \cdots & (k+1)(k+2)\cdots(k+n-1) \\ 1 & k+2 & \cdots & (k+2)(k+3)\cdots(k+n) \\ & & \cdots & \\ 1 & k+n & \cdots & (k+n)(k+n+1)\cdots(k+2n-2) \end{vmatrix}$$

by factoring the first element from each row. Denoting this last determinant by $D_{n,k}$, we subtract each row but the last from the succeeding row to get

$$D_{n,k} = \begin{vmatrix} 1 & k+1 & (k+1)(k+2) & \cdots & (k+1)(k+2)\cdots(k+n-1) \\ 0 & 1 & (k+2)(2) & \cdots & (k+2)(k+3)\cdots(k+n-1)\cdot(n-1) \\ & & & \cdots & \\ 0 & 1 & (k+n)(2) & \cdots & (k+n)(k+n+1)\cdots(k+2n-3)\cdot(n-1) \end{vmatrix}$$

$$= (n-1)!\begin{vmatrix} 1 & k+2 & \cdots & (k+2)(k+3)\cdots(k+n-1) \\ 1 & k+3 & \cdots & (k+3)(k+4)\cdots(k+n) \\ & & \cdots & \\ 1 & k+n & \cdots & (k+n)(k+n+1)\cdots(k+2n-3) \end{vmatrix} = (n-1)!\,D_{n-1,k+1}.$$

Since we have

$$D_{2,k} = \begin{vmatrix} 1 & k+1 \\ 1 & k+2 \end{vmatrix} = 1 = 1!$$

for all k, it follows by mathematical induction that

$$D_{n,k} = (n-1)!(n-2)!\cdots 1! = (n-1)!!$$

and hence that

$$|A_{n,k}| = k!(k+1)!\cdots(k+n-1)!\,D_{n,k} = \frac{(k+n-1)!!\,(n-1)!!}{(k-1)!!}.$$

Also solved by Paul S. Bruckman, *Highwood, IL*, Cecil Rousseau, *The University of Memphis, TN*, H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

908. [Spring 1997] *Proposed by Andrew Cusumano, Great Neck, New York.*

Evaluate

$$\lim_{n\to\infty}\left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} - \frac{(n+1)^{n+1}}{n^n}\right].$$

Solution by Paul S. Bruckman, Highwood, Illinois.

Let $u_n = (1 + 1/n)^n$. Then we have

$$\ln\left(1 + \frac{1}{n}\right)^n = n\ln\left(1 + \frac{1}{n}\right) = n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$= 1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots = 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right).$$

Thus

$$\left(1 + \frac{1}{n}\right)^n = e^{1-1/2n+O(1/n^2)} = e\left[1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right],$$

so that

$$(n+1)u_{n+1} - nu_n = e\left[n + 1 - \frac{1}{2} + O\left(\frac{1}{n}\right) - n + \frac{1}{2} + O\left(\frac{1}{n}\right)\right] = e + O\left(\frac{1}{n}\right).$$

Therefore, the desired limit becomes

$$\lim_{n\to\infty}[(n+2)u_{n+1} - (n+1)u_n] = \lim_{n\to\infty}[e + O(1/n) + u_{n+1} - u_n]$$

$$= e + 0 + e - e = e.$$

Also solved by Paul Bateman, *Urbana, IL,* Charles R. Diminnie, *Angelo State University, San Angelo, TX,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Murray S. Klamkin, *University of Alberta, Canada,* Cecil Rousseau, *The University of Memphis, TN,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer. *Eight incorrect solutions were received.*

Comment by the Editor. Most incorrect solvers wrote to the effect that

$$\lim_{n\to\infty} n(u_{n+1} - u_n) = \left(\lim_{n\to\infty} n\right)\lim_{n\to\infty}(u_{n+1} - u_n) = \left(\lim_{n\to\infty} n\right)\cdot(e - e) = \lim_{n\to\infty} 0 = 0,$$

trying to use the theorem that the limit of a product is the product of the limits. Of course, the rest of the theorem states, "provided the limits of both factors exist." The limit of n does not exist. Hence a discussion similar to

that in the featured solution must be given to prove that the displayed limit is indeed equal to zero. To see this fact, consider applying the faulty reasoning as follows:

$$\lim_{n\to\infty} n\left(\frac{2}{n} - \frac{1}{n}\right) = \left(\lim_{n\to\infty} n\right)\lim_{n\to\infty}\left(\frac{2}{n} - \frac{1}{n}\right) = \left(\lim_{n\to\infty} n\right)\cdot 0 = \lim_{n\to\infty} 0 = 0.$$

The true value of this limit is 1, not 0.

909. [Spring 1997] *Proposed by the late John M. Howell, Littlerock, California.*

For nonnegative integers n and k, let $a(n, 0) = n$, $a(0, k) = k$, and $a(n+1, k+1) = a(n+1, k) + a(n, k+1)$. Find a closed formula for $a(n, k)$.

Solution by David Vella, Skidmore College, Saratoga Springs, New York.

We recognize the recursion to be the same as that for Pascal's triangle, with slightly different boundary conditions (when $n = 0$ or $k = 0$). With values of n and k being represented by diagonals reading down, the first few rows of our triangle look like this:

```
      0
     1 1
    2 2 2
   3 4 4 3
  4 7 8 7 4
```

We observe that the row sums are 2 less than twice the row sums for the Pascal triangle, the index sum $n + k$ is constant in any one row, that the boundary diagonal columns appear in the Pascal triangle just inside the outermost column of ones, and that $a(n, k) = a(k, n)$. Thus reading the 1 column down to the right and the 3 column down to the left, we observe that $a(1, 3) = 7$ (the fourth element in row 4). These observations suggest superimposing two copies of the Pascal triangle, offset horizontally by two elements and adding the superimposed pairs of elements, and then deleting the outside columns of ones. We obtain:

```
      1                1                1   1
     1 1              1 1              1 1 1 1
    1 2 1   plus     1 2 1   equals   1 2 2 2 1
   1 3 3 1          1 3 3 1          1 3 4 4 3 1
  1 4 6 4 1        1 4 6 4 1        1 4 7 8 7 4 1
```

and finally

```
    1 1
   2 2 2
  3 4 4 3
 4 7 8 7 4
```

If we put a zero on top, we see that the result is our triangle, suggesting that

$$a(n,k) = \binom{n+k}{n-1} + \binom{n+k}{n+1} = \binom{n+k}{k-1} + \binom{n+k}{k+1}.$$

For $k = 0$ we have

$$a(n,0) = \binom{n}{n-1} + \binom{n}{n+1} = n + 0 = n$$

and similarly $a(0,k) = 0 + k = k$. Thus, in particular, the formula is true for $n = 0$ and for all k. We proceed by induction on n. Assuming the formula is true for some fixed n, then we have

$$a(n+1, k+1) = a(n+1, k) + a(n, k+1)$$

$$= \binom{n+k+1}{n} + \binom{n+k+1}{n+2} + \binom{n+k+1}{n-1} + \binom{n+k+1}{n+1}$$

$$= \left[\binom{n+k+1}{n} + \binom{n+k+1}{n-1}\right] + \left[\binom{n+k+1}{n+2} + \binom{n+k+1}{n+1}\right]$$

$$= \binom{n+k+2}{n} + \binom{n+k+2}{n+2},$$

and the proof is complete.

Also solved by Paul S. Bruckman, *Highwood, IL*, Robert C. Gebhardt, *Hopatcong, NJ*, Richard I. Hess, *Rancho Palos Verdes, CA*, Murray S. Klamkin, *University of Alberta, Canada*, Cecil Rousseau, *The University of Memphis, TN*, H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

910. [Spring 1997] *Proposed by William Chau, New York, New York.*

A triangle whose sides have lengths a, b, and c has area 1. Find the line segment of minimum length that joins two sides and separates the interior of the triangle into two parts of area α and $1 - \alpha$, where α is a given number between 0 and 1.

Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.

Let ABC be the given triangle and let MN be the line segment that joins sides AB and AC so that triangle AMN has area α. Let x, y, and z be the lengths of AM, AN, and MN respectively. By the law of cosines we have

$$z^2 = x^2 + y^2 - 2xy \cos A.$$

Since twice the area of triangle AMN is given by $xy \sin A = 2\alpha$, then

$$f(x) = z^2 = x^2 + \frac{4\alpha^2}{x^2 \sin^2 A} - \frac{4\alpha \cos A}{\sin A}.$$

To minimize $f(x)$, we calculate that

$$f'(x) = 2x - \frac{8\alpha^2}{x^3 \sin^2 A}$$

and set $f'(x) = 0$ and solve to find $x = \sqrt{2\alpha/\sin A}$. It follows that $x = y$ and $z = 2\sqrt{\alpha \tan(A/2)}$. So, the minimum length of the line segment MN is

$$\min\{2\sqrt{\alpha \tan(A/2)},\ 2\sqrt{\alpha \tan(B/2)},\ 2\sqrt{\alpha \tan(C/2)}\}.$$

The minimum length corresponds to the smallest angle of triangle ABC.

Also solved by Paul S. Bruckman, *Highwood, IL*, Cecil Rousseau, *The University of Memphis, TN, and the* Proposer.

911. [Spring 1997] *Proposed by Norman Schaumberger, Douglaston, New York.*

If a, b, and c are the lengths of the sides of a triangle with semiperimeter s and area K, show that

$$\left(\frac{s}{s-a}\right)^{\frac{a}{s-a}} + \left(\frac{s}{s-b}\right)^{\frac{b}{s-b}} + \left(\frac{s}{s-c}\right)^{\frac{c}{s-c}} \geq \frac{s^4}{K^2}.$$

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.

Note that

$$\frac{a}{s-a} + 1 = \frac{s}{s-a} \quad \text{and} \quad \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = 1.$$

We use the weighted arithmetic mean-geometric mean inequality,

$$\sum m_i x_i \geq \prod x_i^{m_i},$$

where $\Sigma m_i = 1$, $m_i > 0$, and $x_i > 0$ for all $i = 1$ to n. We let

$$m_1 = \frac{s-a}{s} \quad \text{and} \quad x_1 = \left(\frac{s}{s-a}\right)^{\frac{s}{s-a}}, \quad \text{etc.}$$

to get that

$$\left(\frac{s}{s-a}\right)^{\frac{a}{s-a}} + \left(\frac{s}{s-b}\right)^{\frac{b}{s-b}} + \left(\frac{s}{s-c}\right)^{\frac{c}{s-c}}$$

$$= \frac{s-a}{s}\left(\frac{s}{s-a}\right)^{\frac{s}{s-a}} + \frac{s-b}{s}\left(\frac{s}{s-b}\right)^{\frac{s}{s-b}} + \frac{s-c}{c}\left(\frac{s}{s-c}\right)^{\frac{s}{s-c}}$$

$$\geq \left[\left(\frac{s}{s-a}\right)^{\frac{s}{s-a}}\right]^{\frac{s-a}{s}} \cdot \left[\left(\frac{s}{s-b}\right)^{\frac{s}{s-b}}\right]^{\frac{s-b}{s}} \cdot \left[\left(\frac{s}{s-c}\right)^{\frac{s}{s-c}}\right]^{\frac{s-c}{s}}$$

$$= \frac{s}{s-a} \cdot \frac{s}{s-b} \cdot \frac{s}{s-c} = \frac{s^4}{K^2}$$

by Heron's formula.

Also solved by Paul S. Bruckman, *Highwood, IL,* William Chau, *A T & T Laboratories, Middletown, NJ,* Murray S. Klamkin, *University of Alberta, Canada,* Henry S. Lieberman, *Waban, MA,* Yoshinobu Murayoshi, *Okinawa, Japan,* Cecil Rousseau, *The University of Memphis, TN,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

912. [Spring 1997] *Proposed by Paul S. Bruckman, Highwood, Illinois.*

Let p be a prime such that $p \equiv 1 \pmod{60}$. Show that there are positive integers r and s with $p = r^2 + s^2$ and 3 divides r or s and 5 divides r or s.

Solution by Jim Brown, student, Michigan State University, Laingsburg, Michigan.

There is a theorem due to Fermat that states that a prime p is the sum of two positive squares if and only if $p \equiv 1 \pmod 4$. Since $p \equiv 1 \pmod{60}$, then we have $p \equiv 1 \pmod 3$, $p \equiv 1 \pmod 4$, and $p \equiv 1 \pmod 5$. Hence p is the sum of two squares, $p = r^2 + s^2$.

If 3 divides r, we are done. Otherwise, $r^2 \equiv 1 \pmod 3$. Since $r^2 + s^2 = p$, then $1 + s^2 \equiv 1 \pmod 3$, so $s \equiv 0 \pmod 3$. That is, 3 divides s and we are done.

Similarly, if 5 does not divide r, then $r^2 \equiv 1 \pmod 5$ or $r^2 \equiv 4 \pmod 5$. Then either $1 + s^2 \equiv 1$ or $4 + s^2 \equiv 1 \pmod 5$. The first congruence requires that 5 divides s and the second is impossible, so we are done.

Also solved by Charles R. Diminnie, *Angelo State University, San Angelo, TX,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Henry S. Lieberman, *Waban, MA,* Kandasamy Muthuvel, *University of Wisconsin-Oshkosh,* Bob Prielipp, *University of Wisconsin-Oshkosh,* Cecil Rousseau, *The University of Memphis, TN,* H.-J. Seiffert, *Berlin, Germany,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer.

913. [Spring 1997] *Proposed by Kenneth B. Davenport, Pittsburgh, Pennsylvania.*

Find a closed formula for the sum

$$1^3 - 2^3 + 3^3 - 4^3 + \cdots + (-1)^{n+1} n^3.$$

I. *Solution by Joseph C. Furtsch, student, Angelo State University, San Angelo, Texas.*

The given series, S_n, can be thought of as a sum of odd minus even cubes. There are two cases, depending upon whether the final term is odd or

even. Thus, if $n = 2m + 1$, we have

$$S_n = \sum_{k=0}^{m} [(2k + 1)^3 - (2k)^3] = \sum_{k=0}^{m} (12k^2 + 6k + 1)$$

$$= \tfrac{12}{6}m(m + 1)(2m + 1) + \tfrac{6}{2}m(m + 1) + m + 1 = \tfrac{1}{4}n^2(2n + 3) - \tfrac{1}{4}.$$

If $n = 2m$, then we have

$$S_n = \sum_{k=1}^{m} [(2k - 1)^3 - (2k)^3] = \sum_{k=1}^{m} (-12k^2 + 6k + 1)$$

$$= -\tfrac{12}{6}m(m + 1)(2m + 1) + \tfrac{6}{2}m(m + 1) - m = -\tfrac{1}{4}n^2(2n + 3).$$

So the general sum is

$$S_n = \frac{n^2(2n + 3)(-1)^{n+1}}{4} + \frac{(-1)^n - 1}{8}.$$

II. *Solution by Jim Brown, student, Michigan State University, Laingsburg, Michigan.*

From the equation

$$S_n = \sum_{k=0}^{n} k^3(-1)^{k+1},$$

it is easy to see that the desired sum can be found by applying elementary calculus to the power series

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Differentiate this equation and then multiply both sides of the result by x to get

$$\sum_{k=0}^{n} kx^k = \frac{x(1 - x^n - nx^n + nx^{n+1})}{(x - 1)^2}.$$

Differentiate and multiply by x again to obtain

$$\sum_{k=0}^{n} k^2x^k = \frac{x(-1 - x + x^n + 2nx^n + n^2x^n + x^{n+1} - 2nx^{n+1} - 2n^2x^{n+1} + n^2x^{n+2})}{(x - 1)^3}.$$

Now differentiate and multiply by x^2, which yields

$$\sum_{k=0}^{n} k^3x^{k+1} = \frac{x^2[1 + 4x + x^2 + x^n(-1 - 3n - 3n^2 - n^3)]}{(x - 1)^4} +$$

$$+ \frac{x^2[x^{n+1}(-4 + 6n^2 + 3n^3) + x^{n+2}(-1 + 3n - 3n^2 - 3n^3) + n^3x^{n+3}]}{(x - 1)^4}.$$

Finally, replacing x by -1, we get

$$\sum_{k=0}^{n} k^3(-1)^{k+1} = \frac{-1 + (-1)^n(1 - 6n^2 - 4n^3)}{8}.$$

Also solved by Avraham Adler, *New York, NY,* Doru Popescu Anastasiu, *Slatina, Romania,* Frank P. Battles, *Massachusetts Maritime Academy, Buzzards Bay,* Paul S. Bruckman, *Highwood, IL,* William Chau, *A T & T Laboratories, Middletown, NJ,* Rob Downes, *Plainfield, NJ,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* George P. Evanovich, *Saint Peter's College, Jersey City, NJ,* Mark Evans, *Louisville, KY,* Robert C. Gebhardt, *Hopatcong, NJ,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Carl Libis *(2 solutions), University of Alabama, Tuscaloosa,* Henry S. Lieberman, *Waban, MA,* Peter A. Lindstrom, *Batavia, NY,* David E. Manes, *SUNY College at Oneonta,* Kandasamy Muthuvel, *University of Wisconsin-Oshkosh,* David E. Penney, *The University of Georgia, Athens,* Bob Prielipp, *University of Wisconsin-Oshkosh,* Cecil Rousseau, *The University of Memphis, TN,* Shiva K. Saksena, *University of North Carolina at Wilmington,* Harry Sedinger, *St. Bonaventure University, NY,* H.-J. Seiffert, *Berlin, Germany,* Michelle J. Sharpe *and* Mike Baysdell, *Alma College, MI,* SUNY Fredonia Student Group, *NY,* David C. Vella, *Skidmore College, Saratoga Springs, NY,* Stan Wagon, *Macalester College, St. Paul, MN,* Al White, *St. Bonaventure University, NY,* Rex H. Wu, *Brooklyn, NY,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer.

THE RICHARD V. ANDREE AWARDS

The Richard V. Andree Awards are given annually to the authors of the three papers written by students that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the **Pi Mu Epsilon Journal** in the past year.

Until his death in 1987, Richard V. Andree was Professor Emeritus of Mathematics at the University of Oklahoma. He had served **Pi Mu Epsilon** for many years and in a variety of capacities: as President, as Secretary-Treasurer, and as Editor of the **Journal**.

Listed alphabetically, the three winners for 1997 are:

1. **Marc Fusaro** for his paper "A Visual Representation of the Sequence Space", this **Journal** 10(1994-99)#6, 466-481.
2. **Kerry Smith McNeill** for her paper "Rearrangement of Series", this **Journal** 10(1994-99)#7, 547-555.
3. **Danica Nowosielski** for her paper "Change Ringing: A Connection between Mathematics and Music", this **Journal** 10(1994-99)#7, 532-539.

At the time the papers were written **Mr. Fusaro** was at the University of Scranton; **Ms. McNeill** was at Seton Hall University; and **Ms. Nowosielski** was at Russell Sage College.

The officers and councilors of the Society congratulate the winners on their achievements and wish them well for their futures.

1997 NATIONAL PI MU EPSILON MEETING

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held at Georgia Tech and the Renaissance Hotel in Atlanta from August 2 through August 3. As in the past, the meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections.

The J. Sutherland Frame Lecturer was **Philip J. Straffin**, Beloit College. His presentation was on "Excursions in the Geometry of Voting."

The following thirty-one student papers were presented at the meeting. An asterisk (*) before the name of the presenter indicates that the speaker received a best paper award.

Program-Student Paper Pi Mu Epsilon Sessions

An Introduction to the Fourier Series	**Michael G. Baker** Mount Union College Ohio Omicron
Continuing Work on a Mathematical Model of Blood Flow in the Abdominal Aorta of a Rabbit	**Michael Bice** University of California, Davis California Lambda
P-Adic Integrals	**Christine Carracino** University of Virginia Virginia Kappa
Is Cola-Cola an Underachiever?	***Jeff Clouse** Youngstown State University Ohio Xi
A Possible Win for Wile E. Coyote	**Christy Conn** Youngstown State University Ohio Xi
The Rolling Ball Problem	**Michael DeCoster** St. Norbert College Wisconsin Delta

Walks on Triangulated Surfaces
Thomas Dorsey
University of Wisconsin-Madison
Wisconsin Beta

How to Keep the Golden Years Golden
Jodi Faloba
Youngstown State University
Ohio Xi

Clusters of Soap Bubbles and Immiscible Fluids
David Futer
University of Pennsylvania
Pennsylvania Alpha

DNA Computing and Graph Theory Problems
Nathan Gibson
Worcester Polytechnic Institute
Massachusetts Alpha

Stable Matchings in the Couples Problem
Stephen Hartke
University of Dayton
Ohio Zeta

Three Dimensional Graphics for Calculators
Donald Hixon
University of South Dakota
South Dakota Alpha

Meetings, Bloody Meetings
***Joshua Horstman**
***Jayme Moore**
Rose-Hulman Institute
of Technology
Indiana Gamma

Sex, Drugs, and Alcohol: How do they relate to Upper Level Mathematics?
Kimberly Johnson
Siena College
New York Epsilon

The Case of the Missing Case: The Completion of a Proof by Ron Graham
Julie Jones
Randolph-Macon College
Virginia Iota

Violent Activity in Northeast Ohio High Schools: An Analysis of Survey Results
Lori Kaserman
University of Akron
Ohio Nu

Improving Your Golf Game with Mathematics
Ben Keck
Youngstown State University
Ohio Xi

Why is 9 Prime?
***Vincent Lucarelli**
Youngstown State University
Ohio Xi

Aesthetically Pleasing, Acoustically Atrocious
Adam Messner
Youngstown State University
Ohio Xi

Vertex Defining Sets of Graphs
Thayer Morrill
Miami University
Ohio Delta

Chaos and Nonlinear Dynamics
Michael Van Opstall
Hope College
Michigan Delta

On the Bank of Bankruptcy: A Mathematical Model Describing the Social Security System in the United States
*** Michael Perry**
University of Akron
Ohio Nu

Symmetry Structure Analysis of Finite Designs and Infinite Patterns in Decorative Art Work: Amish Quilt Patterns and Other Rural Designs
***Sheryle Proper**
Allegheny College
Pennsylvania Sigma

The Dynamics of the Family of Tent Maps on the Interval [0, 1]
Brian Raines
Hendrix College
Arkansas Beta

And the Winner is: The Cycloid! Fastest Path From Point A to Point B	**Robert Reed** University of Arkansas Arkansas Alpha
Game Theory	**Wendy Rigterink** Miami University Ohio Delta
Some History and Special Cases of Fermat's Last Theorem	**John Slanina** Youngstown State Univeristy Ohio Xi
Extensions on the Question " Can You Hear the Shape of a Drum?"	**Greg Sloan** Western Kentucky University Kentucky Beta
Two Applications of the Theory of Diophantine Functions	**Harry Smith** St. Joseph's University Pennsylvania Xi
Do You Always Have to Put in What You Get Out?	**Jessica Thelen** St. Norbert College Wisconsin Delta
3D Optical Illustions From a Mathematical Perspective	**Naomi Yaekel** Carthage College Wisconsin Epsilon

The **American Mathematical Society** and the **National Security Agency** have given **Pi Mu Epsilon** grants to be used as a monetary awards for excellent student presentations. The six speakers indicated above each received $100.

List of Referees

The editor wishes to acknowledge the substantial contributions made by the following mathematicians who reviewed manuscripts for the **Pi Mu Epsilon Journal** during the past year.

Brian Bradie, Christopher Newport University
Marc Brodie, The College of St. Benedict
Paul S. Bruckner
Underwood Dudley, DePauw University
Paul Fishback, Grand Valley State University
John A. Frohliger, St. Norbert College
Michael Kinyon, Indiana University South Bend
Ken McDonald, Northwest Missouri State University
Cecil Rousseau, The University of Memphis
Radhakrishnan Sivakumar, Intel Corporation
Lawrence Somer, The Catholic University of America
John Stoughton, Hope College
Osman Yürekli, Ithaca College

Miscellany

Andrew Cusumano made the following observation regarding the harmonic series

$$S = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Let N be a fixed positive integer. For any positive integer M,

$$S \geq \frac{1}{N} + \left[\frac{1}{MN - (M-1)} + \frac{1}{MN-(M-2)} + \cdots + \frac{1}{MN}\right]$$

$$+ \left[\frac{1}{M^2N - (M^2-1)} + \frac{1}{M^2N-(M^2-2)} + \cdots + \frac{1}{M^2N}\right]$$

$$+ \left[\frac{1}{M^3N - (M^3-1)} + \frac{1}{M^3N-(M^3-2)} + \cdots + \frac{1}{M^3N}\right] + \cdots$$

$$\geq \frac{1}{N} + \frac{M}{MN} + \frac{M^2}{M^2N} + \frac{M^3N}{M^3} + \cdots$$

$$= \frac{1}{N} + \frac{1}{N} + \frac{1}{N} + \frac{1}{N} + \cdots$$

and S diverges.

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PI MU EPSILON

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The shirts are white, Hanes® BEEFY-T®, pre-shrunk, 100% cotton. The front has a large Pi Mu Epsilon shield (in black), with the line "1914 - ∞" below it. The back of the shirt has a "Π M E" tiling, designed by **Doris Schattschneider**,in the PME colors of gold, lavender, and violet. The shirts are available in sizes **large** and **X-large**. The price is only $10 per shirt, which includes postage and handling. To obtain a shirt, send your check or money order, payable to **Pi Mu Epsilon**, to:

Rick Poss
Mathematics - Pi Mu Epsilon
St. Norbert College
10 Grant Street
De Pere, WI 54115

FREE*

INTERNATIONAL

TRAVEL!!!

The 1998 Meeting of the Pi Mu Epsilon National Honorary Mathematics Society will be held in Toronto, Ontario, Canada, from July 15-17. The meeting will be held in conjunction with the MAA Mathfest, which will run from July 15-18. Pi Mu Epsilon will again coordinate its national meeting with that of the MAA student chapters.

The Pi Mu Epsilon meeting will begin with a reception on the evening of Wednesday, July 15. On Thursday morning, July 16, the Pi Mu Epsilon Council will have its annual summer meeting. The student presentations will begin later that same day. The presentations will continue on Friday, July 17. The Pi Mu Epsilon Banquet will take place that evening, followed by the J. Sutherland Frame Lecture. This year's Frame lecture will be given by Joe Gallian, of the University of Minnesota - Duluth. Pi Mu Epsilon members are encouraged to participate in the MAA Student Chapter Workshop and Student Lecture, both of which will take place on Saturday, July 18.

***TRAVEL SUPPORT FOR STUDENT SPEAKERS**

Pi Mu Epsilon will provide travel support for student speakers at the national meeting. The first speaker is eligible for 25 cents per mile, up to a maximum of $600. If a student chooses to use public transportation, PME will reimburse for the actual cost of transportation, up to a maximum of $600. In case this request exceeds 25 cents per mile, receipts should be presented. The first four additional speakers from a given chapter are eligible for 20% of whatever amount the first speaker receives. In the case of more than one speaker from one chapter, the speakers may share the allowance in any way that they see fit. If a chapter is not represented by a student speaker, Pi Mu Epsilon will provide one-half support for a student delegate. **Every Pi Mu Epsilon student member is encouraged to give a presentation at this summer meeting!** For further information about attending the meeting, preparing a talk to present, and receiving travel support:

SEE YOUR PI MU EPSILON ADVISOR

Π M E

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Thirteenth Annual

PI MU EPSILON

Regional Undergraduate Math Conference

November 6-7, 1998

Featured Speaker: **Marty Isaacs**

University of Wisconsin - Madison

Sponsored by: St. Norbert College Chapter of ΠME

and

St. Norbert College ΣΝΔ Math Club

The conference will begin on Friday evening and continue through Saturday noon. Highlights of the conference will include sessions for student papers and two presentations by Professor Isaacs, one on Friday evening and one on Saturday morning. Anyone interested in undergraduate mathematics is welcome to attend. All students (who have not yet received a master's degree) are encouraged to present papers. The conference is free and open to the public.

For information, contact:

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(920) 403-3198
FAX: (920) 403-4098 .
e-mail: possrl@sncac.snc.edu

Σ N Δ