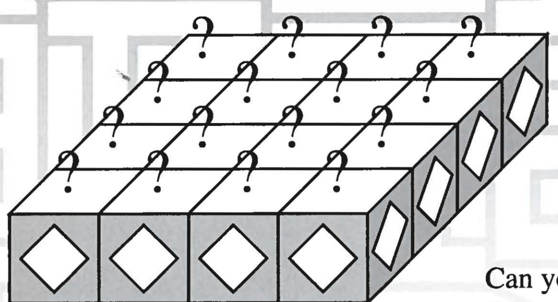

Can you complete the Square of Squares?

Problem 978

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SINGLE RATIONAL ARCTANGENT IDENTITIES FOR π

JACK SAMUEL CALCUT III*

1. Introduction. The most common methods of calculating π to large numbers of decimal places utilize an infinite sum for the arctangent function. All of these infinite sums converge faster when the argument is small. I first began looking at rational arctangent identities for π in 1991. Except for $\arctan(1) = \pi/4$, all rational arctangent identities for π use two distinct angles (e.g., $\arctan(1/2) + \arctan(1/3) = \pi/4$). I wondered why there was no known identity of the form $n * \arctan(x) = \pi$, where n is a large natural number and $|x|$ is a small rational number, and whether such an identity existed. The first major step towards generalizing these identities came in October, 1995, when I independently discovered the pattern preceding Theorem 2. This solved the problem for $\pi/4$ only, however. I was convinced I could solve the problem for all rational multiples of π. The next breakthrough came in September 1998 (at a bus stop no less). The final piece of the puzzle fell into place. This paper contains both of these ideas and all supporting details.

2. Single Rational Arctangent Identities for π. We are interested in identities for π that determine a rational multiple of π with only one evaluation of the arctangent function where the argument is rational.

DEFINITION 1. *A single rational arctangent identity for π is any identity of the form $n\arctan(x) = k\pi$ where n is natural, $x \neq 0$ is rational and k is an integer.*

It follows from the definition that $k \neq 0$ since $n, x \neq 0$. Clearly every identity of the form $\frac{n}{m}\arctan(x) = \frac{a}{b}\pi$, where n, m, b are natural, $a \neq 0$ is an integer, and $x \neq 0$ is rational, reduces to a single rational arctangent identity for π, so we need only generalize the latter. First, we derive a useful expression for $\tan(n\arctan(x))$ where $n = 0, 1, \ldots$ and x is real. Recalling that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

we get:

$$\begin{aligned}
\tan(0 * \arctan(x)) &= \frac{0}{1}\\
\tan(1 * \arctan(x)) &= \frac{x}{1}\\
\tan(2 * \arctan(x)) &= \frac{2x}{1-x^2}\\
\tan(3 * \arctan(x)) &= \frac{x + \frac{2x}{1-x^2}}{1 - x\frac{2x}{1-x^2}} = \frac{3x - x^3}{1-3x^2}\\
\tan(4 * \arctan(x)) &= \frac{x + \frac{3x-x^3}{1-3x^2}}{1 - x\frac{3x-x^3}{1-3x^2}} = \frac{4x - 4x^3}{1 - 6x^2 + x^4}.
\end{aligned}$$

It appears that $\tan(n\arctan(x)) = p_n(x)/q_n(x)$, where $p_n(x)$ is the sum of odd power terms in binomial expansion with alternating signs, and $q_n(x)$ is the sum of even

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power terms in binomial expansion with alternating signs. That is:

$$(1)\qquad p_n(x) = \sum_{i=0}^{\lfloor n/2\rfloor -1} (-1)^i \binom{n}{2i+1} x^{2i+1} \text{ and } q_n(x) = \sum_{i=0}^{\lfloor n/2\rfloor} (-1)^i \binom{n}{2i} x^{2i},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (the binomial coefficients).

Before we prove that equations (1) hold, recall that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, a fundamental ingredient of the Pascal triangle, which can be easily proven by induction [2].

THEOREM 2. $\tan(n\arctan(x))$ *is defined by equations* (1) *for all natural* n *and real* x.

Proof. We will proceed by induction on n. We have seen that equations (1) hold true for $n = 1,2,3,4$. Assume (1) is true for all $n \leq k$. We must show (1) is true for $k+1$. There are two cases to consider: k even and k odd. Suppose k is even. Then $k+1$ is odd. So

$$\begin{aligned}
\tan((1+k)\arctan(x)) &= \frac{x + \frac{p_k(x)}{q_k(x)}}{1 - x\frac{p_k(x)}{q_k(x)}} \frac{xq_k(x) + p_k(x)}{q_k(x) - xp_k(x)} \\
&= \frac{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} x^{2i+1} + \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} x^{2i+1}}{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} x^{2i} - \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} x^{2i+2}} \\
&= \frac{\sum_{i=0}^{\frac{k}{2}-1} (-1)^i \left[\binom{k}{2i} + \binom{k}{2i+1}\right] x^{2i+1} + (-1)^{\frac{k}{2}} x^{k+1}}{1 + \sum_{i=0}^{\frac{k}{2}-1} (-1)^{i+1} \left[\binom{k}{2i+2} + \binom{k}{2i+1}\right] x^{2i+2}} \\
&= \frac{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i+1} x^{2i+1}}{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i} x^{2i}},
\end{aligned}$$

as desired. The proof for k odd is virtually identical, as the reader may wish to verify. □

Note that Theorem 2 applies to all nonzero integral n, since if $n < 0$ then $-n > 0$ and $\tan(n\arctan(x)) = -\tan(-n\arctan(x)) = -p_{-n}(x)/q_{-n}(x)$.

It is transparent from Theorem 2 that $p_n(x)$ and $q_n(x)$ are rational for all integral n and rational x. Clearly then, $\tan(n\arctan(x))$ is rational for rational x provided $q_n(x) \neq 0$.

We now apply Theorem 2 to $\pi/4$.

THEOREM 3. *If we have* $n\arctan(x) = \pi/4$ *for some nonzero integral* n, *then the only possible rational values for* x *are* $x = \pm 1$.

Proof. $n\arctan(x) = \frac{\pi}{4} \Rightarrow \tan(n\arctan(x)) = 1 \Rightarrow p_n(x)/q_n(x) = 1 \Rightarrow p_n(x) = q_n(x) \Rightarrow q_n(x) - p_n(x) = 0 \Rightarrow 1 - \binom{n}{1}x - \binom{n}{2}x^2 + \cdots \pm \binom{n}{n-1}x^{n-1} \pm x^n = 0$.

The only possible rational roots of this polynomial are $x = \pm 1$ by the rational root theorem. □

Theorem 3 characterizes all single rational arctangent identities for $\pi/4$.

Before we characterize all single rational arctangent identities for all rational multiples of π, it is necessary that we make some observations. First, we look at $p_n(1)$ and $q_n(1)$. From (1) we get: $p_1(1) = 1$, $p_2(1) = 2$, $p_3(1) = 2$, $p_4(1) = 0$, $q_1(1) = 1$, $q_2(1) = 0$, $q_3(1) = -2$ and $q_4(1) = -4$.

Further inspection leads one to the following conjecture (where $n = 4d + r$, $r < 4$):

$$(2)\qquad p_n(1) = \begin{cases} 0, & n \equiv 0 (\mathrm{mod}\ 4) \\ (-4)^d, & n \equiv 1 (\mathrm{mod}\ 4) \\ 2(-4)^d, & n \equiv 2 (\mathrm{mod}\ 4) \\ 2(-4)^d, & n \equiv 3 (\mathrm{mod}\ 4) \end{cases} \text{ and } q_n(1) = \begin{cases} (-4)^d, & n \equiv 0 (\mathrm{mod}\ 4) \\ (-4)^d, & n \equiv 1 (\mathrm{mod}\ 4) \\ 0, & n \equiv 2 (\mathrm{mod}\ 4) \\ -2(-4)^d, & n \equiv 3 (\mathrm{mod}\ 4) \end{cases}.$$

Proving equations (2) hold is straightforward after we prove the next proposition.

PROPOSITION 4. $p_{k+1}(1) = q_k(1) + p_k(1)$ *and* $q_{k+1}(1) = q_k(1) - p_k(1)$.

Proof. Case 1. (k is even) Then $k+1$ is odd and

$$\begin{aligned}
p_{k+1}(1) &= \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i+1} = \sum_{i=0}^{\frac{k}{2}} (-1)^i \left[\binom{k}{2i+1} + \binom{k}{2i}\right] \\
&= \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} + \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} \\
&= p_k(1) + q_k(1).
\end{aligned}$$

Also,

$$\begin{aligned}
q_{k+1}(1) &= \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i} = \sum_{i=0}^{\frac{k}{2}} (-1)^i \left[\binom{k}{2i} + \binom{k}{2i-1}\right] \\
&= \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} - \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} \\
&= q_k(1) - p_k(1).
\end{aligned}$$

Case 2. (k is odd) The proof is virtually identical to Case 1, as the reader may wish to verify. □

THEOREM 5. *Equations* (2) *hold for all natural* n.

Proof. We will proceed by induction on n. We have seen that equations (2) hold true for $n = 1,2,3,4$. Assume equations (2) hold for all $n \leq k$ and write $k = 4 * d + r$, $r < 4$.

Case 1. $k \equiv 0(\mathrm{mod} 4)$ implies $p_k(1) = 0$ and $q_k(1) = (-4)^d$. Proposition 4 implies $p_{k+1}(1) = (-4)^d$ and $q_{k+1}(1) = (-4)^d$.

Case 2. $k \equiv 1(\mathrm{mod} 4)$ implies $p_k(1) = (-4)^d$ and $q_k(1) = (-4)^d$. Proposition 4 implies $p_{k+1}(1) = 2(-4)^d$ and $q_{k+1}(1) = 0$.

Case 3. $k \equiv 2(\mathrm{mod} 4)$ implies $p_k(1) = 2(-4)^d$ and $q_k(1) = 0$. Proposition 4 implies $p_{k+1}(1) = 2(-4)^d$ and $q_{k+1}(1) = -2(-4)^d$.

Case 4. $k \equiv 3(\text{mod}4)$ implies $p_k(1) = 2(-4)^d$ and $q_k(1) = -2(-4)^d$. Proposition 4 implies $p_{k+1}(1) = 0$ and $q_{k+1}(1) = (-4)^{d+1}$. □

Now, we notice that for a single rational arctangent identity for π $n\arctan(x) = k\pi$ if and only if $0 = \tan(n\arctan(x)) = p_n(x)/q_n(x)$, which occurs if and only if $p_n(x) = 0$ and $q_n(x) \neq 0$. We are inclined to conjecture that if $p_n(x) = 0$ and x is rational then $x = 0$ or ±1, however, it is not at all clear how to prove this straightaway for all natural n. The key is to first prove it is true for all prime $n > 0$.

THEOREM 6. *If we have $n > 0$ prime and x rational such that $p_n(x) = 0$ then $x = 0$ or ±1.*

Proof. We first prove the result for $n = 2$ and then for all $n > 2$. Assume $n = 2$, then $0 = p_2(x) = 2x \Rightarrow x = 0$ as desired. Now, assume $n > 2$ and prime, then n is odd. So,

$$\begin{aligned} p_n(x) = 0 &\Rightarrow \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} x^{2i+1} = 0 \\ &\Rightarrow nx - \binom{n}{3}x^3 + \binom{n}{5}x^5 - \cdots \mp \binom{n}{n-2}x^{n-2} \pm x^n = 0 \\ &\Rightarrow x = 0 \text{ or } n - \binom{n}{3}x^2 + \binom{n}{5}x^4 - \cdots \mp \binom{n}{n-2}x^{n-3} \pm x^{n-1} = 0 \\ &\Rightarrow x = \pm1 \text{ or } \pm n \end{aligned}$$

by the rational root theorem since n is prime. Suppose $x = \pm n$ is a root, then $n \pm \binom{n}{3}n^2 \mp \cdots \pm n^{n-1} = 0 \Rightarrow 1 \pm \binom{n}{3}n^1 \mp \cdots \pm n^{n-2} = 0$, but the only possible rational roots of this polynomial are $\pm1 \Rightarrow n = \pm1$. But this contradicts the assumption that $n > 2$. Hence, $x = 0$ or ±1. □

Next, we show that we can solve the problem for $n\arctan(x) = a\pi$, where n is composite, by "stepping down" by prime factors of n.

THEOREM 7. *If $n > 1$ natural and x is rational such that $\tan(n\arctan(x)) = 0$ then $x = 0$ or ±1.*

Proof. n has a prime factorization, say $n = s_1 s_2 \cdots s_j$, where each $s_i > 1$ and the s_i's are not necessarily distinct. We will proceed by induction on j. First assume $j = 1$. If x is a rational such that $\tan(s_1\arctan(x)) = 0$ then $p_{s_1}(x) = 0$ and $q_{s_1}(x) \neq 0$ and $x = 0$ or ±1 by Theorem 6. Now, assume true for all $j < h$. Let n be some natural such that $n = s_1 s_2 \cdots s_h$, where each $s_i > 1$ and the s_i's are not necessarily distinct. Further, let x be some rational such that $\tan(n\arctan(x)) = 0$ and write $\phi = s_2 \cdots s_h \arctan(x)$. Then we have:

$$0 = \tan(n\arctan(x)) = \tan(s_1 s_2 \cdots s_h \arctan(x)) = \tan(s_1\phi) = \frac{p_{s_1}(\tan\phi)}{q_{s_1}(\tan\phi)},$$

which implies that $p_{s_1}(\tan\phi) = 0$ and $q_{s_1}(\tan\phi) \neq 0$. Now, $(s_2 \cdots s_h)$ is a natural number and x is rational, so $\tan(\phi)$ is defined by equations (1) by Theorem 2. Hence, $\tan(\phi)$ is either rational or undefined. If $\tan(\phi)$ is undefined then $p_{s_1}(\tan\phi) \neq 0$, a contradiction. Therefore, $\tan(\phi)$ is rational, say $y = \tan(\phi)$. Then we have $p_{s_1}(y) = 0$ where y is rational and $s_1 > 1$ is prime, so $y = 0$ or ±1 by Theorem 1.5. Suppose $y = \pm1$, this implies that $\pm1 = y = \tan(\phi) = \tan(s_2 \cdots s_h \arctan(x))$. But then $x = \pm1$ by the proof of Proposition 4. Therefore, suppose $y = 0$, this implies $0 = y = \tan(\phi) = \tan(s_2 \cdots s_h \arctan(x))$, where $s_2 \cdots s_h$ is a natural number composed of $h - 1 < h$ primes. Hence, $x = 0$ or ±1 by the inductive hypothesis. □

Now, we combine our results and generalize all single rational arctangent identities for π.

THEOREM 8. *The following are equivalent:*
(i) *$n\arctan(x) = a\pi$ is a single rational arctangent identity for π.*
(ii) *$x = \pm1$ and $n \equiv 0(\text{mod } 4)$, where n is natural.*

Proof. First, we prove (i) $\Rightarrow$ (ii). Assume (i) is true. Then by definition n is natural, $x \neq 0$ is rational and $a \neq 0$ is an integer. Then $n\arctan(x) = a\pi \Rightarrow \tan(n\arctan(x)) = 0$ and $x = \pm1$ by Theorem7 ($x \neq 0$ by definition). So, we have $\tan(n\arctan(\pm1)) = 0 \Rightarrow \tan(n\arctan(1)) = 0 \Rightarrow p_n(1) = 0$. By equations (2), we have that $n \equiv 0(\text{mod } 4)$.

Now we prove (ii) $\Rightarrow$ (i).

Assume (ii) is true. If $x = -1$, then $p_n(x) = p_n(-1) = -p_n(1)$ and $q_n(x) = q_n(-1) = q_n(1)$ by equations (1). Furthermore, $p_n(1) = 0$ and $q_n(1) \neq 0$ by equations (2), which implies that $\tan(n\arctan(x)) = \dfrac{p_n(x)}{q_n(x)} = \dfrac{p_n(\pm1)}{q_n(\pm1)} = \dfrac{\pm p_n(1)}{q_n(1)} = 0$. Hence, $n\arctan(x) = \arctan(0) = k\pi$ for some integer k. Since $\arctan(x) = \arctan(\pm1) = \pm\pi/4$, we have that $\pm n\pi/4 = k\pi \Rightarrow k = \pm n/4 \neq 0$. □

3. Conclusion. We have seen that there does not exist a single rational arctangent identity for π, hence for any rational multiple of π, that converges faster than $\arctan(1) = \pi/4$. To obtain faster convergence using rational arctangent identities, one must make at least two distinct arctangent evaluations. The logical continuation would be to generalize identities for π that use two arctangent evaluations.

After the completion of the work presented here, it was found that Gauss had done just that, but only for $\pi/4$, $2\pi/4$, and $3\pi/4$. Gauss' method is outlined in Wrench [4], the key relation being:

$$\arctan(x) = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right) \tag{3}$$

However, using equation (3) to prove the result of this paper only leads one to the polynomials in (1). To see this, suppose $n\arctan(x) = a\pi$ is a single rational arctangent identity for π. Equation (3) implies that $\frac{1}{2i}\ln\left(\left(\frac{1+ix}{1-ix}\right)^n\right) = a\pi$. Now, we use the fact that:

$$\tan(\phi) = \frac{1}{i}\frac{e^{2i\phi}-1}{e^{2i\phi}+1} = z \Rightarrow e^{2i\phi} = \frac{1+iz}{1-iz},$$

to see that:

$$e^{\ln\left(\left(\frac{1+ix}{1-ix}\right)^n\right)} = \frac{1+i(\tan(a\pi))}{1-i(\tan(a\pi))} = 1.$$

Simplifying gives us $\left(\frac{1+ix}{1-ix}\right)^n = 1$, which implies that $(1+ix)^n - (1-ix)^n = 0$. The Binomial Theorem and some simplification shows that $(1+ix)^n - (1-ix)^n = 0$ is exactly equivalent to $p_n(x) = 0$. It should be mentioned that even the best rational arctangent identities for π have their limitations. Using logarithms it is easy to see that each iteration of the Gregory series for $\arctan(x)$, where $|x| < 1$ and $x \neq 0$, yields approximately $|2\log_{10}(x)|$ more digits accuracy, which is linear in $\log_{10}(x)$. There are recursive formulas for π based on elliptic integrals that have quadratic, cubic, quadruple, and septet convergence rates. The interested reader may refer to Kanada [1].

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In Memoriam.

A. D. Stewart. Pi Mu Epsilon notes with sadness the passing of A. D. Stewart, Councilor from 1978-1984 from Prairie View A & M University in Prairie View, Texas. A. D. Stewart was a gentle man who could always be counted upon as a valuable presence at Pi Mu Epsilon meetings. His loosely knotted tie and warm smile were always recognizable characteristics. Prof. Stewart encouraged his students from Prairie View A&M to be active mathematically on campus and to take part in the national PME meetings. He was a loyal ΠME supporter and will be very much missed.

Robert G. Kane. Professor Robert G. Kane, Associate Professor of Mathematics and Computer Science at the University of Detroit Mercy (Michigan Beta chapter), died July 12 at age 65. Professor Kane was the moderator of Michigan Beta chapter, Pi Mu Epsilon, for many years. He taught mathematics at the University of Detroit (pre-merger) since 1957.

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LEFT IDENTITIES AND ASSOCIATED SUBNEARRINGS OF THE NEARRING OF POLYNOMIALS IN TWO VARIABLES*

G. ALAN CANNON AND TINA STEPHENS †

1. Introduction. Polynomials have been studied quite extensively, both as functions and as algebraic structures. With the operations of addition and multiplication, the set of all polynomials in one variable with real coefficients forms a ring. A less-studied approach is to examine the set of polynomials with the operations of addition and composition. With these operations we obtain a different algebraic structure called a nearring.

Let N be a nonempty set on which two binary operations called addition, $+$, and multiplication, $\cdot$, are defined. Then $(N, +, \cdot)$ is a (right) *nearring* with respect to the given addition and multiplication provided the following properties hold for all a, b, and c in N:

i) $(a+b)+c = a+(b+c)$;
ii) There exists a zero element 0 in N such that for each a in N, $a+0 = 0+a = a$;
iii) For each element $a \in N$, there exists an element $-a$ in N such that $a+(-a) = (-a)+a = 0$;
iv) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
v) $(a+b) \cdot c = a \cdot c + b \cdot c$.

Every ring is a nearring since rings satisfy all of the above axioms, have commutative addition, and satisfy the left distributive law. An example of a nearring that is not a ring can be found by taking the set of all functions mapping the real numbers to the real numbers using function addition and composition as the operations.

If we consider the set of all polynomials in two variables x and y with coefficients from the real numbers, we obtain a nearring under polynomial addition and composition, denoted $(\mathbb{R}[x,y], +, \circ)$ where composition, suggested by Clay ([1], p. 12), is defined by $f(x,y) \circ g(x,y) = f(g(x,y), g(x,y))$ for all $f(x,y)$ and $g(x,y)$ in $\mathbb{R}[x,y]$. Clay notes that the nearring $\mathbb{R}[x,y]$ has been virtually ignored in the literature. This might be explained by the lack of a two-sided identity and that there are many ways in which one could define the composition of these polynomials.

We are following the suggestion of an Exploratory Problem in Clay ([1], 2.12) and determine all left identities of $\mathbb{R}[x,y]$, and for each left identity, $i(x,y)$, we find a subnearring of $\mathbb{R}[x,y]$ for which $i(x,y)$ is also a right identity. Then we determine the invertible elements in each of these subnearrings.

2. Main Results. In this section we look at some of the basic properties of $\mathbb{R}[x,y]$. First we show that $\mathbb{R}[x,y]$ is not a ring. Although addition in $\mathbb{R}[x,y]$ is commutative, the left distributive law does not hold. To illustrate, let $f(x,y) = 2x+2$, $g(x,y) = 2x^2 - y$, and $h(x,y) = x+y$. So $f(x,y)$, $g(x,y)$, and $h(x,y)$ are elements of $\mathbb{R}[x,y]$. It follows that $[f \circ (g+h)](x,y) = f((g+h))(x,y) = f((g+h)(x,y), (g+h)(x,y)) = f(2x^2+x, 2x^2+x) = 2(2x^2+x)+2 = 4x^2+2x+2$ and $[(f \circ g) + (f \circ h)](x,y) = (f \circ g)(x,y) + (f \circ h)(x,y) = f(g(x,y), g(x,y)) + f(h(x,y), h(x,y)) = f(2x^2-y,\, 2x^2-y) + f(x+y,\, x+y) = 2(2x^2-y)+2+2(x+y)+2 = 4x^2+2x+4$. Hence $f \circ (g+h) \neq f \circ g + f \circ h$, and $\mathbb{R}[x,y]$ is not a ring.

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†Southeastern Louisiana University

We also note that $\mathbb{R}[x,y]$ is not a commutative nearring; in other words, composition of polynomials is not commutative. As an example, consider $f(x,y) = x^2$ and $g(x,y) = x+y$. Thus $f(x,y), g(x,y) \in \mathbb{R}[x,y]$. Then $(f \circ g)(x,y) = f(g(x,y), g(x,y)) = f(x+y, x+y) = (x+y)^2$ and $(g \circ f)(x,y) = g(f(x,y), f(x,y)) = g(x^2, x^2) = x^2 + x^2 = 2x^2$. Clearly $f \circ g \neq g \circ f$ and $\mathbb{R}[x,y]$ is not commutative.

In addition to lacking commutativity of composition, $\mathbb{R}[x,y]$ does not have a right identity. Therefore, there is no two-sided identity either. We verify this in our first lemma.

LEMMA 1. *The nearring* $(\mathbb{R}[x,y], +, \circ)$ *does not contain a right identity.*

Proof. Suppose that $i(x,y) \in \mathbb{R}[x,y]$ is a right identity. Then $f \circ i = f$ for all $f \in \mathbb{R}[x,y]$. In particular, let $f(x,y) = x$ and $g(x,y) = y$. Then $f(x,y)$ and $g(x,y)$ are elements of $\mathbb{R}[x,y]$. So $x = f(x,y) = (f \circ i)(x,y) = f(i(x,y), i(x,y)) = i(x,y)$ and $y = g(x,y) = (g \circ i)(x,y) = g(i(x,y), i(x,y)) = i(x,y)$. Hence $i(x,y) = x = y$, a contradiction. So $\mathbb{R}[x,y]$ does not have a right identity. $\square$

Left identities do exist as shown in the next theorem. We use the notation $f(x,y) = \sum_{0 \le j+k \le n} (a_{j,k})\, x^j y^k$ to represent an arbitrary polynomial of degree n in $\mathbb{R}[x,y]$, where $a_{j,k} \in \mathbb{R}$ for all j and k.

THEOREM 2. *The following are equivalent:*

(*i*) $i(x,y) = \sum_{0 \le j+k \le n} (a_{j,k}) x^j y^k$ *is a left identity of* $\mathbb{R}[x,y]$;

(*ii*) $i(x,x) = x$;

(*iii*) $a_{0,0} = 0$, $a_{1,0} + a_{0,1} = 1$, *and* $\sum_{j+k=m} a_{j,k} = 0$ *for all* $2 \le m \le n$.

Proof. Assume condition (i) holds. We show that condition (ii) is true. Since $i(x,y)$ is a left identity of $\mathbb{R}[x,y]$, then $(i \circ f)(x,y) = i(f(x,y), f(x,y)) = f(x,y)$ for all $f(x,y) \in \mathbb{R}[x,y]$. In particular, let $f(x,y) = x$. Then $x = f(x,y) = (i \circ f)(x,y) = i(f(x,y), f(x,y)) = i(x,x)$. So condition (i) implies condition (ii).

Now assume that condition (ii) is true. We show that condition (iii) holds. Then $x = i(x,x) = \sum_{0 \le j+k \le n} (a_{j,k}) x^j x^k = \sum_{0 \le j+k \le n} (a_{j,k}) x^{j+k}$. But $x = \sum_{0 \le j+k \le n} (a_{j,k}) x^{j+k}$ implies that $a_{0,0} = 0$, $a_{1,0} + a_{0,1} = 1$, and $\sum_{j+k=m} a_{j,k} = 0$ for all $2 \le m \le n$. This gives that condition (ii) implies condition (iii).

It is straightforward to verify that if $i(x,y) = \sum_{0 \le j+k \le n} (a_{j,k}) x^j y^k$ such that $a_{0,0} = 0$, $a_{1,0} + a_{0,1} = 1$, and $\sum_{j+k=m} a_{j,k} = 0$ for all $2 \le m \le n$, then $i(x,x) = x$. This gives (iii) implies (ii).

Finally, assume condition (ii) is true and let $f(x,y) \in \mathbb{R}[x,y]$. Then $(i \circ f)(x,y) = i(f(x,y), f(x,y)) = f(x,y)$. So $i(x,y)$ is a left identity for $\mathbb{R}[x,y]$. $\square$

By Theorem 2, the polynomial $i_1(x,y) = x^2 - 3y^2 + 2xy + 3x - 2y \in \mathbb{R}[x,y]$ is a left identity since the coefficients of the quadratic terms sum to zero, the coefficients of the linear terms sum to one, and the constant term is zero. Similarly, $i_2(x,y) = 14x - 13y$ and $i_3(x,y) = 4x^3 - 3x^2y - y^3 - 11x + 12y$ are also left identities in $\mathbb{R}[x,y]$.

While we have shown that there are no right identities in $\mathbb{R}[x,y]$, there are some subnearrings of $\mathbb{R}[x,y]$ for which a left identity is also a right identity. For each left identity $i(x,y) \in \mathbb{R}[x,y]$, define the set $S_i = \{f(x,y) \in \mathbb{R}[x,y] \mid f \circ i = f\}$.

THEOREM 3. *For each left identity* $i(x,y) \in \mathbb{R}[x,y]$, S_i *is a subnearring of* $\mathbb{R}[x,y]$.

Proof. To show that S_i is a subnearring of $\mathbb{R}[x,y]$, we need to show that $(S_i, +)$ is a subgroup of $(\mathbb{R}[x,y], +)$ and that S_i is closed under composition ([2], p. 5).

Let $s_1, s_2 \in S_i$. So $s_1 \circ i = s_1$ and $s_2 \circ i = s_2$. Then $(s_1 - s_2) \circ i = (s_1 \circ i) - (s_2 \circ i) = s_1 - s_2$. Hence $s_1 - s_2 \in S_i$ and $(S_i, +)$ is a subgroup of $(\mathbb{R}[x,y], +)$.

Also, $(s_1 \circ s_2) \circ i = s_1 \circ (s_2 \circ i) = s_1 \circ s_2$. Hence, $s_1 \circ s_2 \in S_i$, and S_i is closed under composition. Therefore, S_i is a subnearring of $\mathbb{R}[x,y]$. $\square$

Even though we are not using the operation of standard polynomial multiplication in our nearring, it is useful to note that the subnearring S_i is also closed under this multiplication.

LEMMA 4. *For each left identity* $i(x,y) \in \mathbb{R}[x,y]$, S_i *is closed under standard polynomial multiplication.*

Proof. Let $s_1, s_2 \in S_i$. So $s_1 \circ i = s_1$ and $s_2 \circ i = s_2$. Therefore $(s_1(x,y) \cdot s_2(x,y)) \circ i(x,y) = ((s_1 \cdot s_2)(x,y)) \circ i(x,y) = (s_1 \cdot s_2)(i(x,y), i(x,y)) = s_1(i(x,y), i(x,y)) \cdot s_2(i(x,y), i(x,y)) = [s_1(x,y) \circ i(x,y)] \cdot [s_2(x,y) \circ i(x,y)] = s_1(x,y) \cdot s_2(x,y)$ and $s_1 \cdot s_2 \in S_i$. $\square$

We now completely determine the specific elements in S_i. To begin, we first notice if $f(x,y) = c$ where c is any real number, then $f(x,y) \in S_i$ since $(f \circ i)(x,y) = f(i(x,y), i(x,y)) = c = f(x,y)$. Also, $i(x,y) \in S_i$ since $i(x,y)$ is a left identity for $\mathbb{R}[x,y]$ and $(i \circ i)(x,y) = i(x,y)$.

Since S_i is closed under addition, then $f(x,y) = i(x,y) + c$ is an element of S_i for any real number c. Furthermore, since S_i is closed under standard polynomial multiplication, then $g(x,y) = c[i(x,y)]^n$ is an element of S_i for any constant c and any integer $n \ge 1$. It follows that any polynomial of the form $h(x,y) = \sum_{j=0}^{n} (a_j)[i(x,y)]^j$ is an element of S_i. In fact, these are all of the elements of S_i since if $f(x,y) = \sum_{0 \le j+k \le n} (a_{j,k}) x^j y^k \in S_i$, then $f = f \circ i$ implies that $f(x,y) = (f \circ i)(x,y) = f(i(x,y), i(x,y)) = \sum_{0 \le j+k \le n} (a_{j,k})[i(x,y)]^j [i(x,y)]^k = \sum_{m=0}^{n} (a_m)[i(x,y)]^m$ for appropriately chosen values of a_m. We have just proven the following characterization theorem.

THEOREM 5. *For each left identity* $i(x,y) \in \mathbb{R}[x,y]$,

$$S_i = \left\{ f(x,y) \in \mathbb{R}[x,y] \;\middle|\; f(x,y) = \sum_{j=0}^{n} (a_j)[i(x,y)]^j \text{ for some } 0 \le n \in \mathbb{Z} \right\}.$$

Now we focus on finding invertible elements in S_i. To facilitate this, we first note that S_i is more familiar to us than it might appear.

THEOREM 6. *For each left identity* $i(x,y) \in \mathbb{R}[x,y]$, $(S_i, +, \circ)$ *is nearring isomorphic to* $(\mathbb{R}[x], +, \circ)$, *the nearring of polynomials in one variable.*

Proof. The mapping $\Psi : S_i \to \mathbb{R}[x]$ given by $\Psi\left(\sum_{j=0}^{n} (a_j)[i(x,y)]^j\right) = \sum_{j=0}^{n} (a_j) x^j$ is a nearring isomorphism. The details of the proof are left to the reader. $\square$

Thus, to find invertible elements in S_i, we only need to look for invertible elements in $\mathbb{R}[x]$. The following known result answers this question.

THEOREM 7. *The invertible elements in $\mathbb{R}[x]$ are exactly the non-constant linear polynomials in x.*

Proof. Let $f(x) \in \mathbb{R}[x]$ such that $f(x)$ is invertible in $\mathbb{R}[x]$. Since constant polynomials are not invertible, we can assume that $f(x)$ is of degree n with $n \geq 1$. So $f(x)$ is of the form $f(x) = \sum_{j=0}^{n}(a_j)x^j$ where $a_n \neq 0$. Since $f(x)$ is invertible, then $f^{-1}(x)$ exists and is of the form $f^{-1}(x) = \sum_{j=0}^{m}(b_j)x^j$ where $b_m \neq 0$ and $m \geq 1$. Therefore

$$\begin{aligned} x = (f \circ f^{-1})(x) = f(f^{-1}(x)) &= f\Big(\sum_{j=0}^{m}(b_j)x^j\Big) \\ &= a_n\Big(\sum_{j=0}^{m}(b_j)x^j\Big)^n + a_{n-1}\Big(\sum_{j=0}^{m}(b_j)x^j\Big)^{n-1} + \cdots + a_1\Big(\sum_{j=0}^{m}(b_j)x^j\Big) + a_0. \end{aligned}$$

Expanding the right-hand side, we see that the term of highest power is $a_n b_m^n x^{nm}$. Since the right-hand side is equal to x, we conclude that $a_n b_m^n = 0$ or $mn = 1$. Thus $mn = 1$ since we assumed that $a_n \neq 0$ and $b_m \neq 0$. Hence $m = n = 1$ and $f(x)$ and $f^{-1}(x)$ are non-constant linear polynomials.

Now assume that $f(x)$ is a non-constant linear polynomial in $\mathbb{R}[x]$. Then $f(x) = a_1x + a_0$ for some $a_1, a_0 \in \mathbb{R}$ with $a_1 \neq 0$. One can easily check that $f^{-1}(x) = a_1^{-1}x - a_0a_1^{-1} \in \mathbb{R}[x]$. Hence $f(x)$ is invertible. This completes the proof. □

Note that while the polynomial function $f(x) = x^3$ is an invertible real-valued function, it is not an invertible element of $\mathbb{R}[x]$ since $f^{-1}(x) = \sqrt[3]{x} \notin \mathbb{R}[x]$. From Theorems 6 and 7 we immediately get:

THEOREM 8. *For each left identity $i(x,y) \in \mathbb{R}[x,y]$, the invertible elements in S_i are the non-constant linear polynomials in i.*

In conclusion, we have found all left identities in $\mathbb{R}[x,y]$ and corresponding subnearrings of $\mathbb{R}[x,y]$ for which a left identity is also a right identity. In addition, we have also found the invertible elements of these subnearrings. These results might prove useful in other investigations of this nearring.

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WHICH GRAPHS HAVE PLANAR SHADOW GRAPHS?

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Abstract. The shadow graph $S(G)$ of a graph G is that graph obtained from G by adding to G a new vertex z' for each vertex z of G and joining z' to the neighbors of z in G. All those graphs G for which $S(G)$ is planar are determined.

1. Introduction. One of the most studied numbers associated with graphs is the chromatic number and one of most studied properties that a graph can possess is that of being planar. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors that can be assigned to the vertices of G (one color to each vertex) so that adjacent vertices are colored differently. A graph G is *planar* if G can be embedded (or drawn) in the plane so that no edges cross. These two concepts come together in the famous Four Color Theorem where it was shown by K. Appel and W. Haken (with J. Koch) [1] that $\chi(G) \leq 4$ for every planar graph G.

If a graph G contains a complete subgraph K_k order k (that is, k mutually adjacent vertices), then $\chi(G) \geq k$, since these k vertices alone require k distinct colors. Therefore, if G contains a triangle K_3, then $\chi(G) \geq 3$. However, a graph G may have chromatic number 3 or more without containing a triangle, that is, G is *triangle-free*. For example, the cycle C_5 of order 5 shown in Figure 1 has chromatic number 3 but is triangle-free. (A coloring with the three colors 1, 2, 3 is shown in Figure 1 as well.)

C_5 :

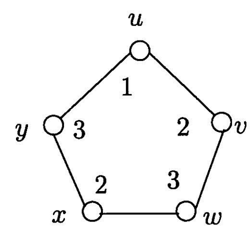

FIG. 1. *The cycle of order 5*

Figure 2 shows a triangle-free graph with chromatic number 4. This graph can be constructed from the cycle C_5 of Figure 1 by adding a vertex z' for each vertex z of C_5 and joining z' to the neighbors of z (the vertex z' is called the *shadow vertex* of z) and adding a vertex s^* (called the *star vertex*) that is then joined to all shadow vertices. In fact the graph that is constructed in this manner is called the *star shadow graph* $S^*(C_5)$ of C_5.

The star shadow graph $S^*(C_5)$ is also known as the *Grötzsch graph.* In 1955, J. Mycielski [4] showed that if G is a triangle-free graph having chromatic number k, then $S^*(G)$ is a triangle-free graph having chromatic number $k+1$. Hence Mycielski's result shows that for every integer $k \geq 3$, there exists a triangle-free graph having chromatic number k. Other researchers have established this fact by different methods (see [2] for example).

A graph closely related to the star shadow graph is the shadow graph. The *shadow graph* $S(G)$ of G is obtained from G by adding a shadow vertex z' for each vertex z

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$G = S^*(C_5)$:

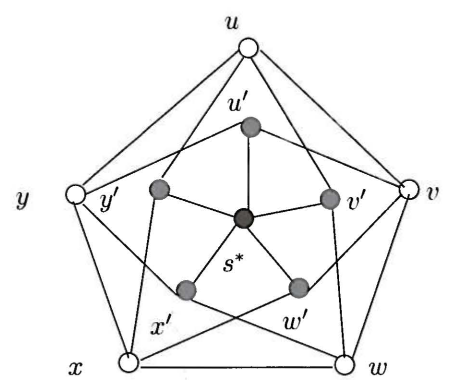

FIG. 2. *The star shadow graph* $S^*(C_5)$

of G and joining z' to the neighbors of z in G. In other words, $S(G)$ is obtained from $S^*(G)$ by deleting the star vertex (and all incident edges). The shadow graph $S(C_5)$ is shown in Figure 3.

$S(C_5)$:

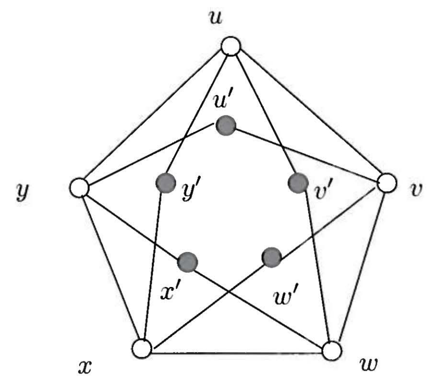

FIG. 3. *The shadow graph* $S(C_5)$

It is not difficult to see that $\chi(S(G)) = \chi(G)$ for every graph G. Indeed, since G is a subgraph of $S(G)$, it follows that $\chi(S(G)) \geq \chi(G)$. By assigning the same color to z' as that assigned to z for every vertex z of G, we obtain a proper coloring of $S(G)$. In this paper, however, we are concerned with the planarity of shadow graphs, not with their chromatic numbers. Indeed, our goal is to determine all graphs G for which $S(G)$ is planar.

If $S(G)$ is planar, then G too is planar since G is a subgraph of $S(G)$. A well-known theorem that will be of great use to us is K. Kuratowski's characterization [3] of planar graphs. A *subdivision* of a graph G is a graph obtained from G by inserting vertices of degree 2 into one or more edges of G. Figure 4 shows the complete graph K_5, the complete bipartite graph $K_{3,3}$, and a subdivision of each of these.

THEOREM 1. (Kuratowski) *A graph is planar if and only if it contains no subgraph isomorphic to* K_5 *or* $K_{3,3}$ *or a subdivision of one of these graphs.*

Since the complete graph K_5 is nonplanar, $S(K_5)$ is nonplanar as well. However both K_4 and $S(K_4)$ are planar (see Figure 5).

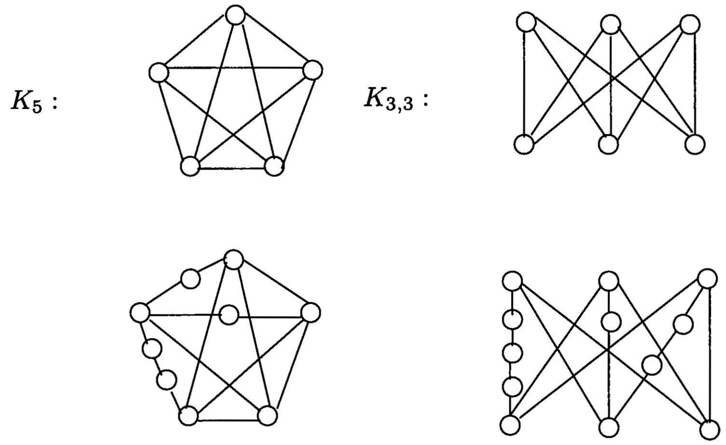

FIG. 4. *Some nonplanar graphs*

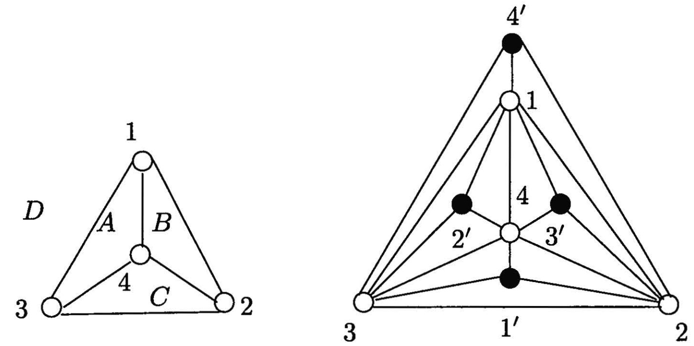

FIG. 5. K_4 *and its shadow graph*

The embedding (drawing) of K_4 in the plane shown in Figure 5 has four regions, namely three interior regions denoted by A, B, C, and the exterior region D. The *boundary* of a region R in a graph is the subgraph of G consisting of the vertices and edges of G incident with R. The boundary of the region D in Figure 5 is the subgraph consisting of the vertices $1, 2, 3$ and edges 12, 13, 23. It is a fact that every graph embedded in the plane can be re-embedded so that any region becomes the exterior region.

Although $S(K_4)$ is planar, no connected graph containing K_4 as a proper subgraph has a planar shadow graph. To see this, we add a pendant edge e at vertex 4 in the graph K_4 denoting the resulting graph by K_4+e (see Figure 6). The shadow graph $S(K_4 + e)$ contains a subdivision of $K_{3,3}$. Thus by Kuratowski's theorem, $S(K_4 + e)$ is nonplanar.

We now turn to cycles. Every even cycle has a planar shadow graph. This fact is illustrated in Figure 7 for C_6 and C_8.

For odd cycles, the situation is quite different. Since K_3 is a subgraph of K_4 and $S(K_4)$ is planar, certainly $S(K_3)$ is planar. From Figure 3, we can see that $S(C_5)$ is itself a subdivision of K_5 and is therefore nonplanar. We now consider $S(C_{2k+1})$ for $k \geq 3$. Figure 8 shows C_{2k+1} and a subgraph of $S(C_{2k+1})$. Paths from vertex 1 to

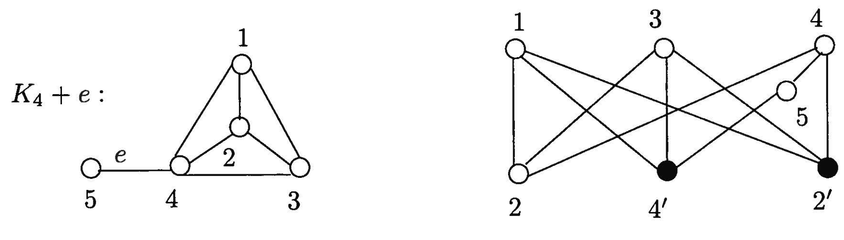

FIG. 6. *The nonplanarity of* $S(K_4 + e)$

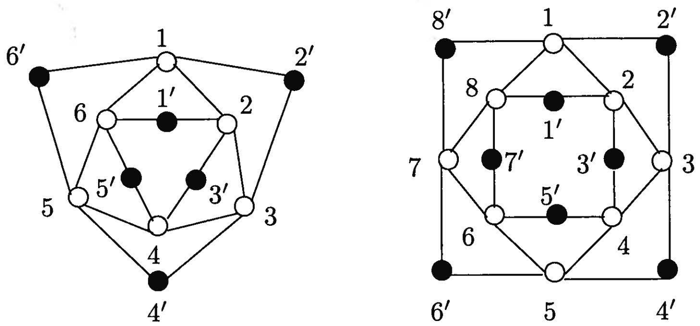

FIG. 7. *The planarity of* $S(C_{2k})$, $k \geq 2$

vertex 6 and from vertex 5 to vertex $2k+1$ are shown in Figure 8 as well. Placing these paths into the subgraph of $S(C_{2k+1})$ shows that $S(C_{2k+1})$ contains another subgraph, this one a subdivision of $K_{3,3}$. Hence $S(C_{2k+1})$ is nonplanar if and only if $k \geq 2$.

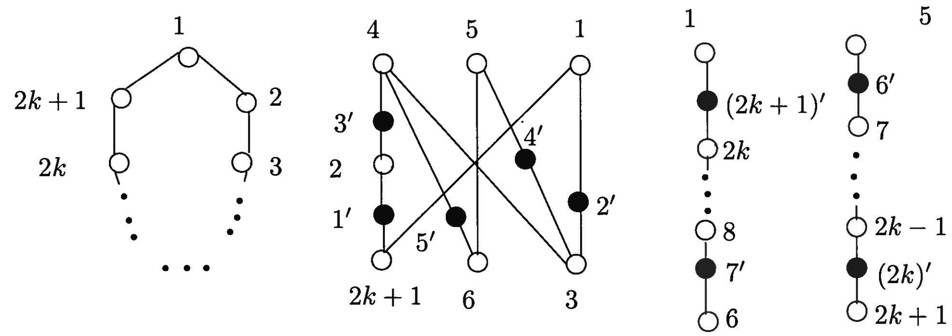

FIG. 8. *The cycle* C_{2k+1} *and a subgraph of* $S(C_{2k+1})$

Before presenting our main result, we have a few remarks to make concerning the structure of graphs. If G is a graph whose components are $G_1, G_2, \cdots, G_k$, then $S(G)$ is planar if and only if $S(G_i)$ is planar for all i $(1 \leq i \leq k)$. Hence it suffices to consider connected graphs only. A vertex v in a connected graph G is a *cut-vertex* of G if the graph obtained by removing v and all incident edges is disconnected. A subgraph B of a connected graph G is called a *block* of G if B is a maximal connected subgraph of G such that B itself contains no cut-vertices. A connected graph G containing the five blocks $B_1, B_2, \cdots, B_5$ is shown in Figure 9. The graph G has three cut-vertices, namely u, v, and w. Necessarily, each block B_i $(1 \leq i \leq 5)$ contains at least one of u, v and w, but, of course, no vertex of B_i is a cut-vertex of B_i itself.

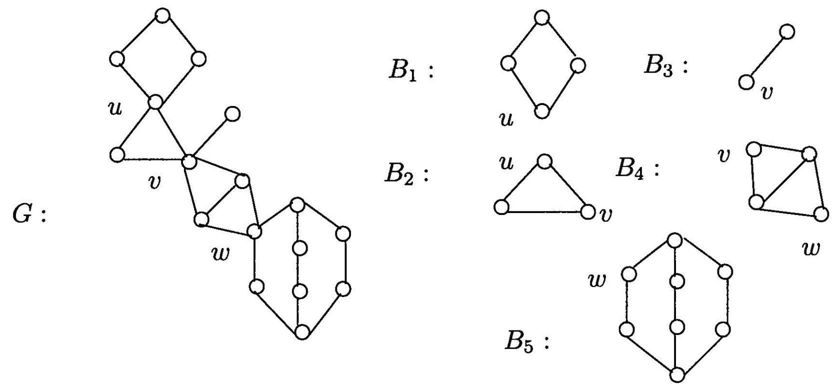

FIG. 9. *A connected graph with five blocks*

If G is a connected graph with a cut-vertex, then G has two or more blocks, each of which contains at least one cut-vertex of G. Every two distinct blocks either have no vertices in common or exactly one vertex in common. If they have one vertex in common, then this vertex must be a cut-vertex of G. An *end-block* of G is a block containing exactly one cut-vertex of G. Every connected graph containing a cut-vertex has at least two end-blocks. If B is an end-block of a connected graph G containing the cut-vertex v of G, then we write $G - B$ to mean the graph obtained by deleting all vertices of B from G except v. If G has $k + 1$ blocks, then $G - B$ is a connected graph with k blocks.

The simplest type of block contains only a single edge and is therefore K_2. The block B_3 in Figure 9 is such a block. A block may consist of a single cycle, as B_1 and B_2 in Figure 9. Otherwise, a block contains two vertices x and y connected by at least three paths that have no vertices in common other than x and y.

We are now prepared to present our main result.

THEOREM 2. *Let G be a nontrivial connected graph. Then $S(G)$ is planar if and only if every block of G is K_2, K_3, $K_4 - e$, K_4, or an even cycle and G has the following properties:* (1) *every cut-vertex of G has degree at most 2 in every block containing it, and* (2) *if K_3 is a block of G, then not all three vertices of the block are cut-vertices of G.*

Proof. First, let G be a nontrivial connected graph such that $S(G)$ is planar. As we have seen, G contains no odd cycle of length 5 or more and G does not contain K_4 as a proper subgraph. If G contains a block B that is different from $K_2, K_3, K_4 - e, K_4$, or an even cycle, then B contains two vertices x and y connected by three paths, all of even length or all of odd length, such that no two of these paths have vertices in common other than x and y. Assume first that these paths have lengths $r, s, t \geq 2$. This situation is illustrated in Figure 10. Also Figure 10 contains a subgraph of $S(G)$ that is a subdivision of $K_{3,3}$ and so $S(G)$ is nonplanar, a contradiction.

If one of these paths has length 1, that is, if $xy \in E(G)$, then the other two paths have odd length. This situation is shown in Figure 11. Here too $S(G)$ contains a subdivision of $K_{3,3}$ and again a contradiction is produced.

Consequently, the only possible blocks of G are K_2, K_3, $K_4 - e$, K_4, or even cycles. To show that every cut-vertex of G has degree at most 2 in every block containing it, it suffices to show that the shadow graph of the graph obtained by adding a pendant

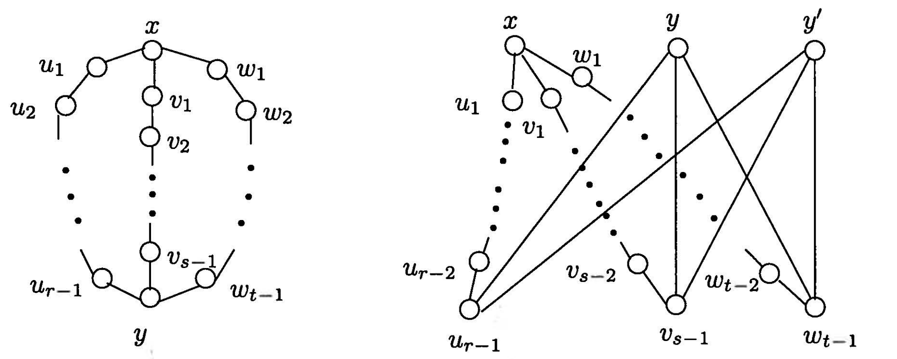

FIG. 10. *A nonplanar subgraph*

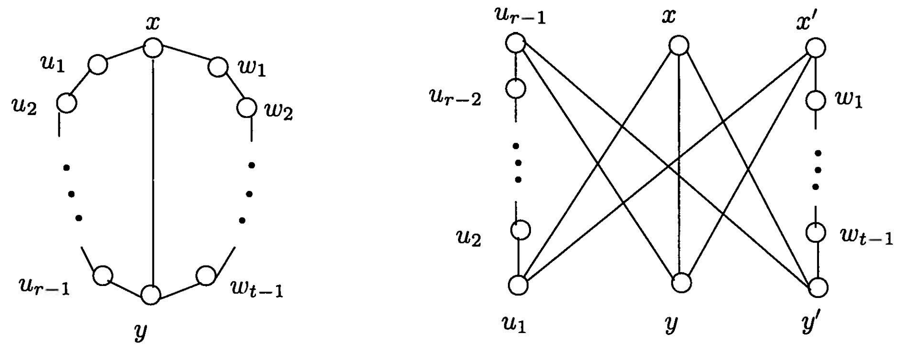

FIG. 11. *Another nonplanar subgraph*

edge to a vertex of degree 3 in $K_4 - e$ is nonplanar. This follows immediately, however, since the subdivision of $K_{3,3}$ shown in Figure 6 exists in the shadow graph of the graph $K_4 + e$ in which the edge joining vertices 1 and 3 is deleted. Thus the shadow graph of the graph obtained by adding a pendant edge at a vertex of degree 3 in $K_4 - e$ is nonplanar. It is straightforward to show that the shadow graph of the graph obtained by adding pendant edges at the two vertices of degree 2 in $K_4 - e$ is planar however.

It remains only to show that the shadow graph of the graph obtained by adding a pendant edge to each vertex of K_3 is nonplanar. This fact, however, is verified in Figure 12. If, on the other hand, only two pendant edges are added to K_3, then it can be shown that the shadow graph of the resulting graph is planar.

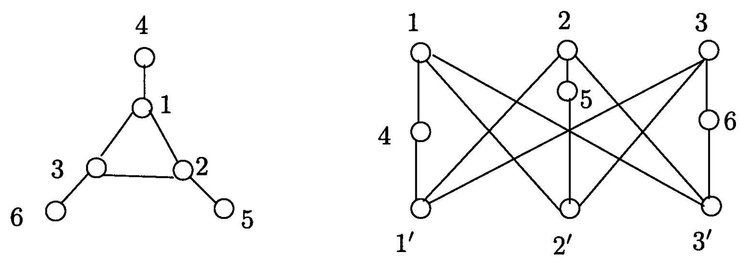

FIG. 12. *The nonplanarity of the shadow graph of K_3 with three pendant edges*

Before considering the converse, we present a definition. Let G be a connected graph every block of which is $K_2, K_3, K_4 - e, K_4$, or an even cycle such that G satisfies (1) and (2). A set U of vertices of G is called an *E-set* if U consists of

(a) all cut-vertices of G,

(b) all vertices in a block that is an even cycle,

(c) both vertices of degree 2 in a block that is a $K_4 - e$,

(d) any two vertices of every block that is a K_3, where the excluded vertex of K_3 is not a cut-vertex of G, and

(e) at most one end-vertex of G.

Then G has a nonempty E-set unless G has exactly one block and $G = K_4$ or possibly $G = K_2$. In the graph G of Figure 13, $U = \{a, d, p, r, s, t, u, x, z\}$ is an E-set. If we replace x in U by y and/or replace a in U by b or c or delete a altogether, then the resulting set is also an E-set.

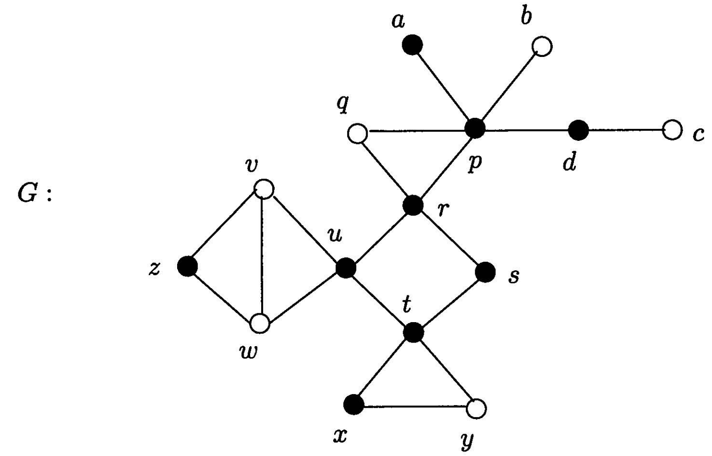

FIG. 13. *An E-set in a graph*

We now establish the converse. Actually we verify the somewhat stronger result that if G is a connected graph every block of which is K_2, $K_3, K_4 - e$, K_4, or an even cycle such that G satisfies (1) and (2), then for every E-set U of G, there exists a planar embedding of $S(G)$ such that for every vertex $v \in U$, there is a $v - v'$ path of length 2 (where v' is the shadow vertex of v) that lies on the boundary of some region R_v and if $u, v \in U$ with $u \neq v$, then $R_u \neq R_v$. A planar embedding of $S(G)$ with this property is said to have property P. The proof proceeds by induction on the number of blocks of G.

Assume first that G consists of a single block. We have already seen that $S(K_4)$ is planar and that K_4 has an empty E-set. Then the fact that $S(K_4)$ has a planar embedding with property P is satisfied vacuously. Therefore, we are left only with the cases that G is $K_2, K_3, K_4 - e$, or an even cycle. If $G = C_{2k}, k \geq 2$, then $V(G)$ is the unique E-set of G and there is a planar embedding of $S(G)$ having property P. (See Figure 7 for C_6 and C_8.) The graphs K_2, K_3, and $K_4 - e$ are shown in Figure 14 together with planar embeddings of their shadow graphs. The graph K_2 has two possible nonempty E-sets, namely $\{a\}$ and $\{b\}$, and this embedding of $S(K_2)$ has property P. The graph K_3 has three possible E-sets, namely $\{a, b\}, \{a, c\}$, and $\{b, c\}$, and the embedding of $S(K_3)$ shown in Figure 14 has property P for the first two E-sets. (Another planar embedding with property P can easily be given for the E-set $\{b, c\}$.) The unique E-set of $K_4 - e$ is $\{b, d\}$ and the embedding of $S(K_4 - e)$ shown in Figure 14 has property P.

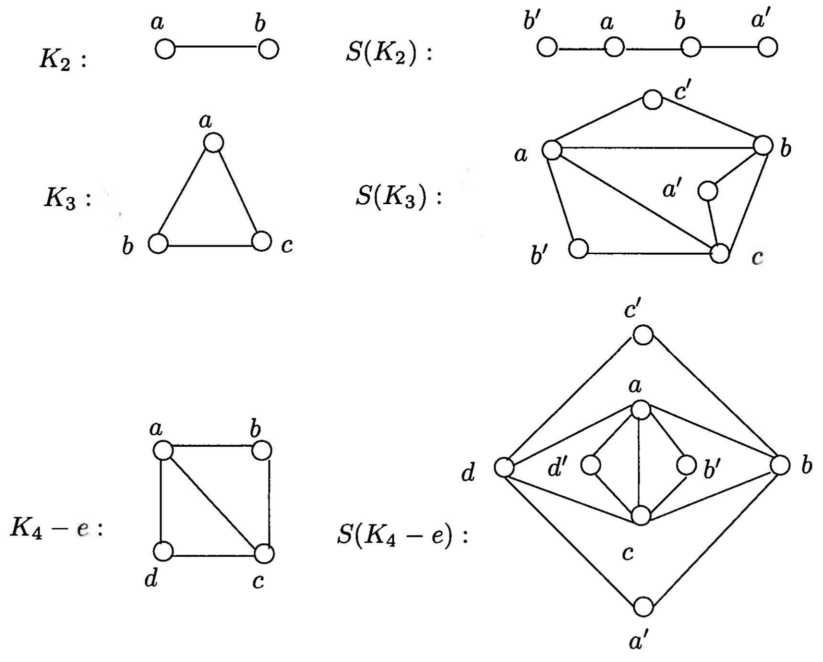

FIG. 14. *Planar embeddings with property P*

Assume that if G is a connected graph with exactly k blocks, $k \geq 1$, each of which is a $K_2, K_3, K_4 - e, K_4$ or an even cycle such that G satisfies (1) and (2), then for every E-set W of G, there exists a planar embedding of $S(G)$ with property P. Of courses, K_4 can only be a block of G if $k = 1$. Let H be a connected graph with exactly $k + 1$ blocks, each of which is a $K_2, K_3, K_4 - e$, or an even cycle such that H satisfies (1) and (2). We show that for every E-set W of H, there exists a planar embedding of $S(H)$ with property P. Let U be an E-set of H. If there exists an end-vertex $w \in U$, let B denote the K_2 block containing w; otherwise, let B denote any end-block of H. Let x be the cut-vertex of H in B.

The connected graph $H - B$ has exactly k blocks, each of which is a $K_2, K_3, K_4 - e$, or an even cycle. Furthermore, $H - B$ satisfies (1) and (2). The set $U' = U \cap V(H - B)$ is necessarily an E-set of $H - B$. By the induction hypothesis, there exists a planar embedding of $S(H - B)$ with property P (with respect to the E-set U'). Let P_x : x, y, x' be the $x - x'$ path of length 2 associated with this embedding, where P_x lies on the boundary of some region R_x. (Thus for $v \in U'$, where $v \neq x$, there is a $v - v'$ path P_v of length 2 lying on the boundary of a region R_v distinct from R_x.) We may assume that R_x is the exterior region.

Assume first that there is an end-vertex $w \in U$. Thus B is a K_2 block and we can obtain the planar embedding of $S(H)$ shown in Figure 15. If $y \in U$, then there is a $y - y'$ path P_y of length 2 on the boundary of a region R_y that is not the exterior region. In the embedding of $S(H)$ shown in Figure 15, the path P_x is on the boundary of the region bounded by the 4-cycle x, y, x', w, x; while the path w, x, w' is on the

boundary of the exterior region. Hence this embedding of $S(H)$ has property P. The argument is similar if U contains no end-vertex and B is a K_2 block.

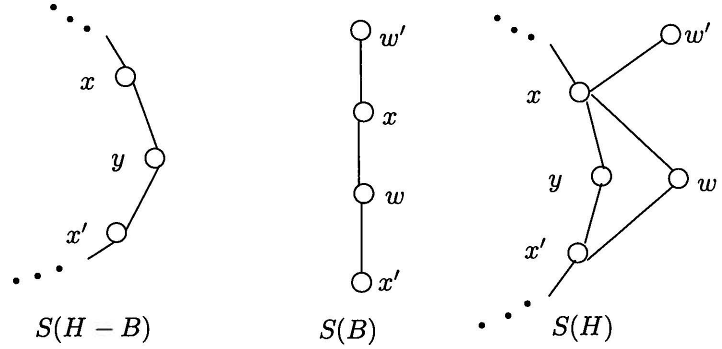

FIG. 15. *Producing a planar embedding of $S(H)$ with property P when $B = K_2$*

Thus we may assume that B is an end-block which is a $K_3, K_4 - e$, or an even cycle. The set $U'' = U \cap V(B)$ is an E-set in B and hence there exists a planar embedding of $S(B)$ with property P (with respect to U''). Let P'_x : x, z, x' be the $x - x'$ path of length 2 associated with this embedding, where P'_x lies on the boundary of a region R'_x. Again, we may assume that R'_x is the exterior region. We can now combine the embeddings of $S(H - B)$ and of $S(B)$ to produce the embedding of $S(H)$ shown in Figure 16.

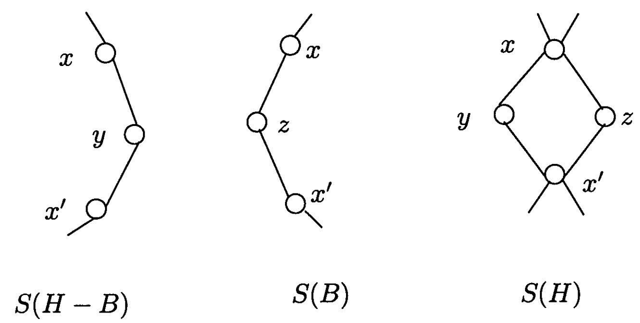

FIG. 16. *Producing a planar embedding of $S(H)$ with property P when $B \neq K_2$*

If either y or z belongs to U, then the associated paths P_y and P_z lie on the boundary of a non-exterior region. In the embedding of $S(H)$ shown in Figure 16, both P_x and P'_x lie on the boundary of the region bounded by the 4-cycle x, y, x', z, x. Thus either P_x or P'_x completes the required condition for this embedding of $S(H)$ to have property P, and the proof is complete. $\square$

Acknowledgement. The research advisor for this project was Prof. Gary Chartrand.

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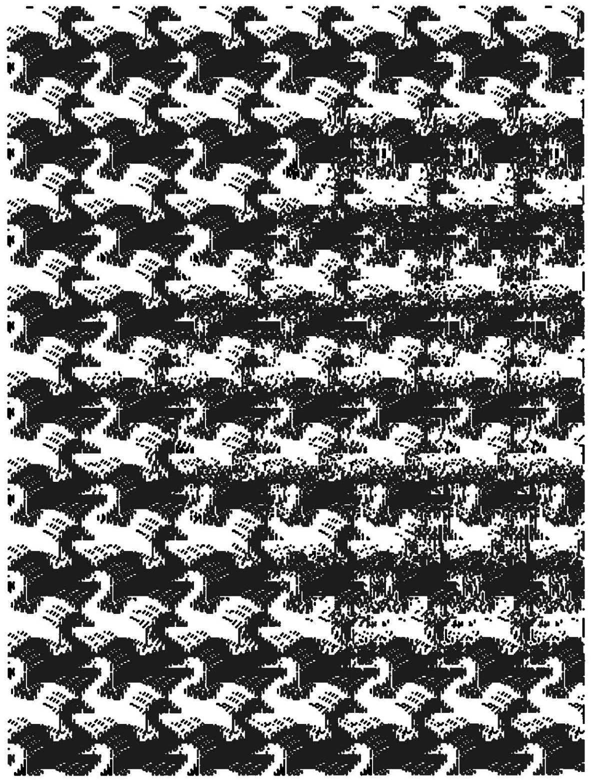
From the Right Side



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Have you ever wondered "What is Art?" A theory of neuro-scientists which has been seized enthusiasticly by popular culture is that the left hemisphere of the brain is the "mathematical side", governing logic, counting and sorting, while the right hemisphere is the "artistic side", governing images and creativity. Where in this taxonomy can we place the "elegant mathematical proof" or the "beautiful" mathematical figure?

Has the image at the left side of this page come from the right side of the brain? Is it art? It is titled "Mathematics Kitch" and is an *Escher-Fractal 3D Stereogram,* e.g., a 3D stereogram of the Mandelbrot set based on a tesselation by M.C. Escher.

The ΠME Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.

ΠME Journal, Vol. 11, No. 1, pp 21–22, 1999.

WHEN ARE $A^N - 1$ AND $A^M - 1$ AMICABLE?

FLORIAN LUCA*

For any positive integer k let $\sigma(k)$ be the divisor sum of k. Two positive integers s and t are called *amicable* if $\sigma(s) = \sigma(t) = s + t$. A positive integer s which is self-amicable is called *perfect.*

In [3], we showed that no Fibonacci or Lucas number is perfect. In [4], we showed that no Fermat number, i.e. a number of the form $2^{2^n} + 1$ for some integer $n \geq 0$, can be either perfect or part of an amicable pair.

Let $a > 1$ be a positive integer. In this note, we prove the following:

Theorem 1. *If $a^m - 1$ and $a^n - 1$ are amicable, then $m = n = 1$ and $a - 1$ is perfect.*

Let us remark first that:

Lemma 2. *If $s > 1$ is a positive integer, then $\sigma(s) \leq 3s^2/4$.*

Proof. Since $\sigma(st) \leq \sigma(s)\sigma(t)$ for any two positive integers s and t, it suffices to check the asserted inequality only when s is prime. But if s is prime, then $\sigma(s) = s + 1 \leq 3s^2/4$ with equality if and only if $s = 2$. □

Proof of the Theorem. Since the involved expressions are symmetric in m and n, we shall always assume that $m \leq n$.

Assume first that $m = 1$. We need to show that $n = 1$.

Assume that $n > 1$. Suppose first that $a = 2$. Since $a^n - 1 = 2^n - 1$ and $a - 1 = 1$ are amicable, we get $\sigma(2^n - 1) = \sigma(1) = \sigma(1) = 1$ which forces $2^n - 1 = 1$, hence $n = 1$.

Suppose now that $a > 2$. Since $a^n - 1$ and $a - 1$ are amicable, we get $\sigma(a-1) = a^n + a - 2$. Since by the Lemma, $\sigma(a-1) \leq \frac{3}{4}(a-1)^2$, it follows that $a^n + a - 2 \leq \frac{3}{4}(a-1)^2$. Since $n \geq 2$, we get $a^2 + a - 2 \leq \frac{3}{4}(a-1)^2$, which is equivalent to $a^2 + 10a \leq 11$,which is certainly impossible for $a > 2$.

From now on, we assume $n > 1$. Suppose first that $n = m$. In this case, $a^n - 1$ is perfect. It follows easily that a is odd. Indeed, if a is even, then $a^n - 1$ is an odd perfect number which is congruent to -1 modulo 4. On the other hand, it is well-known that if an odd perfect number exists, then it should be of the form px^2 for some prime $p \equiv 1 \pmod 4$. Hence, a is odd. In this case, $a^n - 1$ is an even perfect number, therefore $a^n - 1 = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Fermat prime. Write

$$2^{p-1}(2^p - 1) = a^n - 1 = (a-1)\frac{a^n - 1}{a - 1}.$$

Since

$$\frac{a^n - 1}{a - 1} \geq \frac{a^2 - 1}{a - 1} = a + 1 > a - 1,$$

it follows easily that the prime $2^p - 1$ divides $(a^n - 1)/(a - 1)$. Hence $(a^n - 1)/(a - 1) = 2^u(2^p - 1)$and $a - 1 = 2^v$ where $u + v = p - 1$. When n is odd, the number $(a^n - 1)/(a - 1)$ is odd as well, hence $u = 0$. It follows that $a = 2^{p-1} + 1$, and

$$2^p - 1 = \frac{a^n - 1}{a - 1} \geq \frac{a^3 - 1}{a - 1} = a^2 + a + 1 = (2^{p-1} + 1)^2 + (2^{p-1} + 1) + 1,$$

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which is impossible for $p \geq 2$. Hence, n is even. Write then $2^{p-1}(2^p - 1) = (a^{n/2} - 1)(a^{n/2} + 1)$. Since $a^{n/2} + 1 > a^{n/2} - 1$ and the greatest common divisor of $a^{n/2} - 1$ and $a^{n/2} + 1$ is 2, it follows that either $a^{n/2} + 1 = 2^{p-2}(2^p - 1)$ and $a^{n/2} - 1 = 2$ or $a^{n/2} + 1 = 2(2^p - 1)$ and $a^{n/2} - 1 = 2^{p-2}$. However, it can be seen immediately that none of the equations $2^{p-2}(2^p - 1) - 2 = 0$ and $2(2^p - 1) - 2^{p-2} = 0$ has any solutions, just because the left sides of either one of the two equations above is larger than 0 for $p \geq 2$.

From now on, assume that $n > m > 1$. If a is even, then $\sigma(a^n - 1) = \sigma(a^m - 1) = a^n + a^m - 2$ is divisible with 2 but not with 4. Hence, it follows that both $a^n - 1$ and $a^m - 1$ should be of the form px^2 for some odd x and some prime $p \equiv 1 \pmod 4$. But since a is even, both such numbers are congruent to -1 modulo 4. Thus, a cannot be even.

Finally, assume that a is odd. In this case, both numbers $a^n - 1$ and $a^m - 1$ are even. In particular, $(a^n - 1)/2$ is a divisor of $a^n - 1$. Hence, $a^n - 1 + a^m - 1 = \sigma(a^n - 1) \geq a^n - 1 + \frac{a^n - 1}{2}$, which implies $a^m - 1 \geq (a^n - 1)/2$. Since $a \geq 3$ (because a is odd) and $n \geq m + 1$, we have

$$a^m - 1 \geq \frac{a^n - 1}{2} \geq \frac{a^{m+1} - 1}{2} = \frac{a - 1}{2}(a^m + \ldots + a + 1) \geq a^m + \ldots + a + 1,$$

which is again impossible. $\square$

We conclude with a few student research problems.

Problem 1. *Find all triples of positive integers (a, m, n) such that $(a^n - 1)/(a - 1)$ and $(a^m - 1)/(a - 1)$ are amicable.*

Problem 2. *Solve the equation*

$$\sigma\left(\frac{a^n - 1}{a - 1}\right) = \frac{a^m - 1}{a - 1}$$

in positive integers a, m, n.

For Problem 2, the reader might want to consult [1] and [2] where similar type of equations were investigated for the function σ replaced with the Euler function ϕ.

Problem 3. *Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence. Find all pairs of positive integers (m, n) such that F_m and F_n are amicable.*

For Problem 3, the reader might want to consult [5] where the above problem was solved for the Fibonacci sequence replaced by the Pell sequence.

Problem 4. *Recall that a* repdigit *number n is a number having only one distinct digit when written in base 10. That is, $n = a \cdot (10^m - 1)/9$ for some $m \geq 1$ and $a \in \{1, \ldots, 9\}$. Recall also that a number n is* multiply perfect *if n divides $\sigma(n)$. Determine all multiply perfect repdigits.*

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THE ARITHMETIC-GEOMETRIC AND LOGARITHMIC MEANS*

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Abstract. We present the Gauss Arithmetic-Geometric Mean and study its relationship with the Logarithmic Mean. In particular we discuss a problem by Vamanamurthy and Vuorinen, [7].

1. Introduction. The Arithmetic-Geometric Mean was discovered and studied by Lagrange and Gauss almost 300 years ago. In this note we consider related problems, some of which are still open.

It is well known that if x and y are positive numbers, then the Arithmetic Mean (A) and the Geometric Mean (G) are defined as follows:

$$A(x, y) = \frac{x + y}{2} \text{ and } G(x, y) = \sqrt{xy}$$

It is less known that these means may be combined to define the *Arithmetic-Geometric Mean* (AG) [7]:

$$AG(x, y) = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n$$

where $x_0 = x$, $y_0 = y$, $x_{n+1} = A(x_n, y_n)$, and $y_{n+1} = G(x_n, y_n)$.

The reader is encouraged to follow Exercise 69, p.608 of [6] to prove that indeed the sequences x_n and y_n converge to the same limit. The truth is that x_n and y_n converge to the same number, say a, very rapidly. So if we could find x and y such that a is an interesting number, like π, AG could be useful to compute that number very accurately. Using Maple software we have established that it takes only 14 iterations to calculate $A(2,5)$ to 10,000 decimal places. Here is an algorithm that was discovered by Eugene Salamin and Richard Brent in 1976 [2] p. 688 that uses AG to compute π :

Set $a_0 = 1$, $g_0 = \frac{1}{\sqrt{2}}$ and $s_0 = \frac{1}{2}$.

For $k = 1, 2, 3, \ldots$ compute

$$\begin{aligned} a_k &= \frac{a_{k-1} + g_{k-1}}{2} \\ g_k &= \sqrt{a_{k-1} \cdot g_{k-1}} \\ c_k &= a_k^2 - g_k^2 \\ s_k &= s_{k-1} - 2^k c_k \\ p_k &= \frac{2a_k^2}{s_k} \end{aligned}$$

It can be proved, [5] that p_k converges quadratically to π i.e.,

number iterations	1	2	3	4	…	n
decimals of π	1	4	9	16	…	n^2

*This paper presents results obtained during the Faculty–Student Reasearch Project with Dr. Jakub Jasinski at the University of Scranton.

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The main purpose of our paper is to introduce the AG and to give a partial solution to an open problem posed by M.K. Vamanurthy and M. Vuorinen [7] p. 165. We will use some definitions from [3], [7], and [2].

DEFINITION 1. Let $x, y > 0$. The *logarithmic mean* (L) is defined as,

$$L(x,y) = \frac{x-y}{\log x - \log y}$$

for $x \neq y$ and $L(x,y) = x$ for $x = y$.

2. *AG* and *L* are continuous. Since $L(x,y)$ is a piecewise defined function and $AG(x,y)$ is defined recursively we would like to take a moment to prove their continuity. This will support our conjecture below.

THEOREM 1. *$A(x,y)$ is a continuous function on $[0,\infty)^2$.*

Proof. Define a function f by

$$f(x,y,z) = \frac{1}{\sqrt{x^2\cos^2 z + y^2 \sin^2 z}}.$$

f is a continuous function on $\mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R}$. Gauss proved [2] p.484 that for $x, y > 0$

$$AG(x,y) = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(x,y,z)dz.$$

For any $(x,y) \neq (0,0)$ and any sequence $(x_n, y_n) \to (x,y)$ the sequence $f(x_n, y_n, z)$ uniformly converges to $f(x,y,z)$ Hence by the interchange theorem (see e.g., [4], Theorem II, p. 461) it follows that the $AG(x,y)$ is continuous at (x,y). To prove that the $AG(x,y)$ is continuous at $(0,0)$ let us pick $\varepsilon > 0$ and let $\delta = \varepsilon$. Assume that $\sqrt{x^2+y^2} < \delta$. We have

$$\begin{aligned} |AG(x,y) - AG(0,0)| &= |AG(x,y)| \\ &< \max\{|x|,\ |y|\} \\ &= \max\{\sqrt{x^2}, \sqrt{y^2}\} \\ &\leq \sqrt{x^2+y^2} = \varepsilon \end{aligned}$$

That concludes that the AG is continuous function for any $x, y \geq 0$. □

THEOREM 2. *L is a continuous function on $(0,\infty)^2$.*

Proof. L is piecewise defined so we shall have two cases:

Case 1 – when the point (x_0, y_0) is on the diagonal $\{(x,x) : x > 0\}$. We must show that for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that whenever $x, y > 0$

$$\sqrt{(x-x_0)^2 + (y-x_0)^2} < \delta \Longrightarrow |L(x,y) - L(x_0,x_0)| < \varepsilon$$

If $x = y$ then the above implication holds with $\delta = \varepsilon$. If $x \neq y$ then by the mean value theorem, [6]

$$|L(x,y) - L(x_0,x_0)| = \left|\frac{x-y}{\log x - \log y} - x_0\right| = \left|\frac{x-y}{\frac{1}{c}(x-y)} - x_0\right| = |c - x_0| \quad (1)$$

where c is between x and y.

$$|c - x_0| = |c - x + x - x_0| \leq |c - x| + |x - x_0| \quad (2)$$

since

$$|x - x_0|, |y - x_0| < \delta \text{ and } |c - x| < |x - y|$$

we obtain

$$|c - x| + |x - x_0| \leq |x - y| + \delta \leq |x - x_0| + |x_0 - y| + \delta \leq 3\delta$$

Therefore $|c - x_0| \leq 3\delta$. Hence by 1 and 2 it is clear that if we take $\delta < \frac{\varepsilon}{3}$ the implication

$$\sqrt{(x-x_0)^2 + (y-x_0)^2} < \delta \Longrightarrow |L(x,y) - L(x_0,x_0)| < \varepsilon$$

holds for all $x, y > 0$.

Case 2 – when the point is not on the diagonal. In this case $L(x,y)$ is continuous as a composition of continuous functions such as x/y, $x-y$, and $\log(x)$. That concludes the proof. □

3. Is $AG_t \geq L$ for some $t \in (0,1)$. It is well known that $A(x,y) \geq G(x,y)$. This can be generalized to a weighted mean inequality

$$w_1 x + w_2 x \geq x^{w_1} \cdot y^{w_2}$$

whenever $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$. To read more about weighted and modified inequalities we suggest [1]. In 1991 B.C. Carlson and M. Vuorinen proved that

$$AG(x,y) \geq L(x,y) \quad (3)$$

More recently inequalities similar to 3 have been studied involving means "modified" in the following way:

DEFINITION 2. For real numbers $t \neq 0$, we define

$$M_t(x,y) = M\left(x^t, y^t\right)^{\frac{1}{t}},$$

where M can be any of the means: G, AG, or L.

For example,

$$AG_t = AG\left(x^t, y^t\right)^{\frac{1}{t}},$$
$$L_t(x,y) = L\left(x^t, y^t\right)^{\frac{1}{t}}.$$

AG_t has many properties of AG. In particular since the function x^t is continuous for $x > 0$ so composed with Theorems 2 and 3 we obtain the following result:

COROLLARY 3. *If $t > 0$ then $AG_t(x,y)$ and $L_t(x,y)$ are continuous functions on $(0,\infty)^2$.*

However, if we modify AG in the inequality 3, its behavior is not entirely clear:

PROBLEM 4. Does there exists a $t \in (0, 1)$ so that AG_t dominates L, i.e. so that $AG_t(x,y) \geq L(x,y)$for all $x, y > 0$?

THEOREM 5. *For positive x and y both $AG_t(x,y)$ and $L_t(x,y)$ are continuous, strictly increasing functions of t.*

Proof. See Theorem 1.2 of [7]. □

THEOREM 6. *$AG_t(x,y) < L(x,y)$ for all $t \in \left(0, \frac{2}{3}\right)$ and $x, y \in \mathbb{R}^+$.*

Proof. From [7] Theorem 3.6 we know that the inequality $AG(x,y) \le L_{\frac{3}{2}}(x,y)$ holds for all x, $y > 0$. From our Definition 2, $L_{\frac{3}{2}}(x,y) = L\left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)^{\frac{2}{3}}$, so we obtain $AG(x,y) \le L\left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)^{\frac{2}{3}}$ and $AG(x,y)^{\frac{3}{2}} \le L\left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)$. Now, let us substitute $p = x^{\frac{3}{2}}$ and $q = y^{\frac{3}{2}}$, then $AG\left(p^{\frac{2}{3}}, q^{\frac{2}{3}}\right)^{\frac{3}{2}} = AG_{\frac{2}{3}}(p,q) \le L(p,q)$. By Theorem 8 $AG_t(x,y) < L(x,y)$ holds for all $t \in \left(0, \frac{2}{3}\right)$. □

Therefore we can exclude any values of t between 0 and $\frac{2}{3}$, since AG_t and L_t are strictly increasing functions of $t \in (0,1)$.

For the rest of a problem i.e., where $t \in \left[\frac{2}{3}, 1\right)$ we have the following conjecture.

CONJECTURE 7. *For every number $t \in \left[\frac{2}{3}, 1\right)$ there exists a constant $m > 0$ such that if $y > mx > 0$ then $AG_t(x,y) < L(x,y)$.*

Our conjecture is based on computer calculations. To illustrate our results we present "matrices" where x and y play the role of the indices and if $AG_t(x,y) \ge L(x,y)$ then we have a white dot, otherwise we have a black dot. Now we are ready to take a look at the matrices for $t = 0.6667$, 0.67, and 0.7, Figures 1–3.

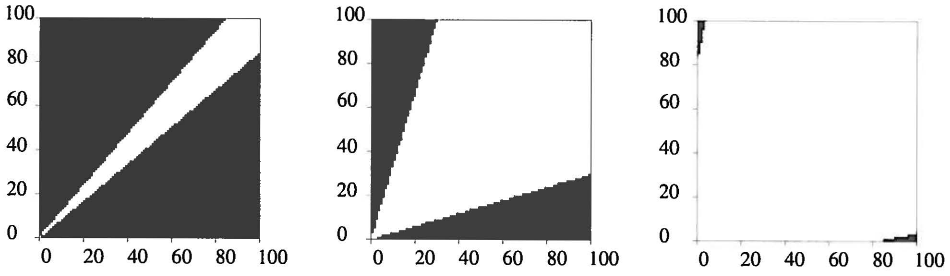

FIG. 1.

As we can see from these graphs, that the slope m of the line separating the vertical black region from the white area is increasing rapidly as we increase t. We almost cannot see the black area when $t = 0.7$ because the slope is approximately $m = 87$. We created another Maple program that would calculate the slopes for other values of t and here are the results:

t	slope m
0.68	13
0.7	87
0.75	1.16×10^{4}
0.8	1.40×10^{8}
0.85	7.10×10^{20}
0.9	1.10×10^{107}
0.95	3.4×10^{10210}

(4)

As we can see from table 4, the slope m increases, and increases very rapidly. Since both functions that we are working with are continuous functions of t, x, and y we conjecture that there is no such $t \in (0,1)$ that will satisfy $AG_t(x,y) \ge L(x,y)$ for all $x, y \in (0,\infty)$.

Acknowledgement. I want to dedicate this paper and say special thanks to Dr. Jakub Jasinski for all of his time, effort, help, and assistance during this research.

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Anatoly Shtekhman is a native of Vinnitsa, Ukraine. In 1995 he moved to the U.S. where, in 1996, he became a math major at the University of Scranton. This paper is the result of a junior research project under the direction of Dr. Jakub Jasinski.

A Note on Infinite Harmony.

$$\sum_{i=1}^{n} \frac{1}{i} = \infty$$

The divergence of the Harmonic Series may be considered a "classic" among the basic results of calculus. There are many different proofs of this fact and it might be a nice exercise for students to find their own proofs before looking up the standard proof in the their Calculus Book. Here is a proof by ANDREW CUSUMANO of Great Neck, New York:

We start with the observation that

$$\frac{1}{x-c} + \frac{1}{x+c} = \frac{2x}{x^2-c^2} > \frac{2}{x}$$

from which we easily deduce that the sum of any string of $2k+1$ terms of the harmonic series centered about $1/(2k+1)$ is

$$\left[\frac{1}{k+1} + \cdots + \frac{1}{2k}\right] + \frac{1}{2k+1} + \left[\frac{1}{2k+2} + \cdots + \frac{1}{3k+1}\right] > k\left(\frac{2}{2k+1}\right) + \frac{1}{2k+1} = 1$$

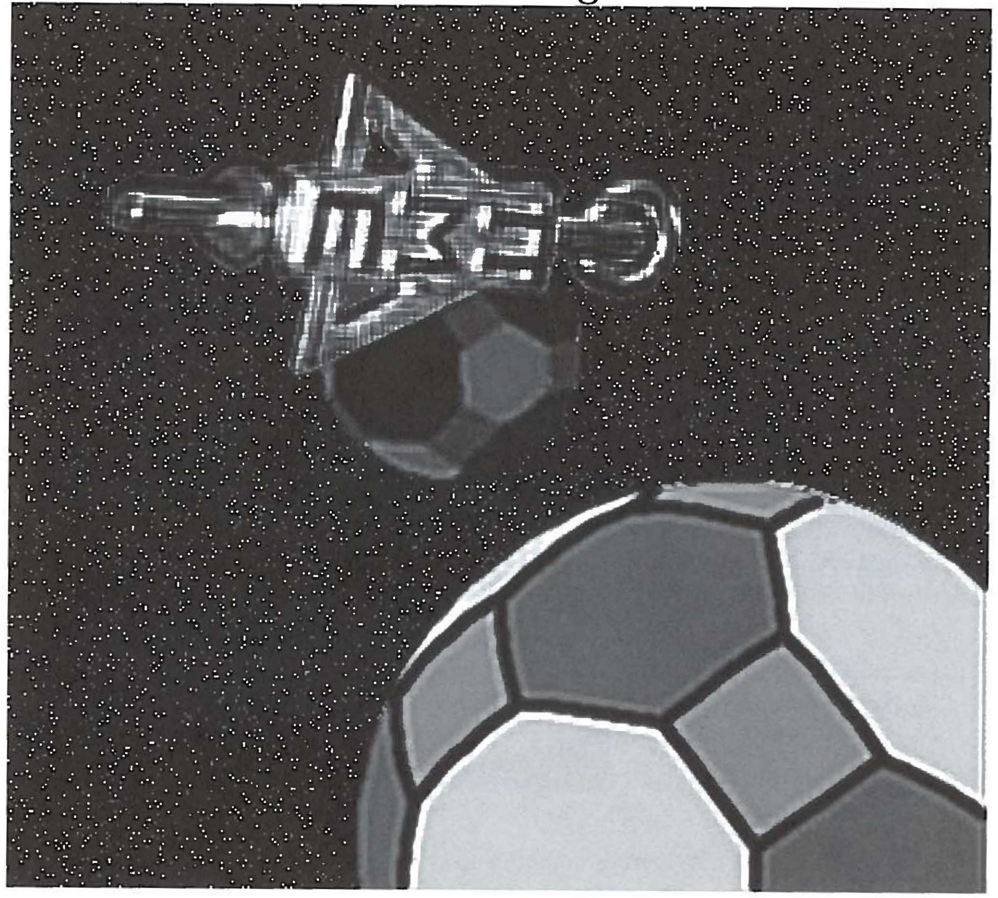
The following $3(2k+1) = 2(3k+1)+1$ terms will be centered about the term

$$\frac{1}{(3k+2)+(3k+1)} = \frac{1}{3(2k+1)}$$

and so will also sum up to a number greater than 1, and so on. It follows immediately that the harmonic series diverges.

The asymptotics of these estimates could be investigated.

Go where no man has gone before ...



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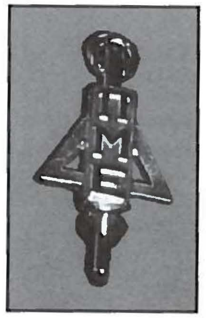

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COLORFUL PATHS IN GRAPHS

EUGENE SPIEGEL*

A proper coloring of a graph is a coloring of the vertices of the graph in which vertices which are connected by an edge have different colors. The chromatic number of the graph is the smallest number of colors needed to have a proper coloring. For example, the complete graph on n vertices has chromatic number n, while a cycle on n vertices has chromatic number 2 if n is even and 3 if n is odd. Certainly the most renown theorem on the chromatic number of a graph is the 1976 theorem of K. Appel and W. Haken [1, 2] which states that the chromatic number of a planar graph is at most 4. A graph is planar if it can be drawn in the plane without any two edges crossing. The result of Appel and Haken answered in the affirmative the question, posed by F. Guthrie in 1852, of whether it is possible to color the countries of any map with just four colors. Not all graphs are planar. For example, neither the complete graph on 5 vertices K_5, nor $K_{3,3}$, the graph having three utilities and three houses as vertices and edges only between utilities and houses, is planar.

In the following G will denote a finite simple graph with vertex set $V(G)$, edge set $E(G)$, and chromatic number $\chi(G)$. A sequence, $v_1, e_1, v_2, e_2, \ldots, v_n$, with $v_i \in V(G)$, $v_i \neq v_j$ for $i \neq j$, and $e_i \in E(G)$, an edge from v_i to v_{i+1}, is called a path of length n. Sometimes we will omit the edges in the description of this path. A well known result of Gallai [3] tells us that there is a path of length $\chi(G)$ in G.

Suppose that G has been properly colored using the colors $\{1, 2, \ldots, \chi(G)\}$. We will call the path $v_1, v_2, \ldots, v_n$ colorful if each vertex has been colored a different color. In this note we show that there are many colorful paths of length $\chi(G)$ in G. More precisely, with S_n denoting the symmetric group on $\{1, 2, \ldots, n\}$, we prove

THEOREM 1. *Suppose G is colored using the colors $\{1, 2, \ldots, \chi(G)\}$ and $\sigma \in S_{\chi(G)}$. Then there exists a path $v_1, e_1, v_2, \ldots, v_{\chi(G)}$ in G with v_i of color $\sigma(i)$ for $1 \leq i \leq \chi(G)$.*

To verify the theorem we show that the given coloring can be altered to obtain a coloring in which the result easily holds, and then show that the alteration process does not change whether a particular path is colorful or not. Further, it is sufficient to verify the result for σ the identity element of $S_{\chi(G)}$.

Let us call the ordered partition $A_1, A_2, \ldots, A_n$ of $V(G)$ a coloring partition, if, for each i, no two distinct elements of A_i are connected by an edge of G. By coloring the elements of A_i color i we then have a proper coloring of G. Conversely, any proper coloring, C, of G gives rise to a coloring partition of $V(G)$, where A_i consists of the vertices in the coloring which have been colored i. Refer to this latter partition as the partition of C.

We say the coloring partition $A_1, A_2, \ldots, A_n$ is special if, for $1 < i$ and $v \in A_i$, there exists a $w \in A_{i-1}$ with v and w connected by an edge of G. In general, a coloring partition need not be special. However, if the partition of a coloring is special, then we easily obtain a colorful path, $v_1, v_2, \cdots, v_n$. Indeed, we can select v_n to be any element of A_n and if $v_i \in A_i$ has been selected, let v_{i-1} be an element of A_{i-1} which is edge-connected to v_i.

The following describes a procedure for associating a special partition to a given coloring partition. Suppose that $A_1, A_2, \ldots, A_n$ is a coloring partition of $V(G)$. We

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first move each element of A_2 which is not edge-connected to any element of A_1 into A_1. This results in a new coloring partition $B_1, B_2, \ldots, B_n$. Then move each element of B_3 which is not edge-connected to any element of B_2 into B_2. Again a new partition arises. Continue this until each element of the highest numbered set in the current partition is not edge-connected to any element of the previous set.

We then obtain a partition, say, $C_1, C_2, \ldots, C_m$, with $m \leq n$. We continue repeating this entire process until we have arrived at a coloring partition $D_1, D_2, \ldots, D_r$ in which no additional changes can take place in n consecutive allowable moves. It must then be the case that for, $2 \leq i \leq r$, each element of D_i is edge-connected to an element of D_{i-1}. This partition is then a special partition of $V(G)$. Call this special partition the induced partition of $A_1, A_2, \ldots, A_n$. Of course if $n = \chi(G)$, then $r = \chi(G)$, since the induced partition is still a proper coloring of G. In general, however, $\chi(G)$ can be smaller than r. (If G is the chain $v_1, v_2, \ldots, v_n$ and we color v_i with color i then $r = n$. But, what happens if we reorder the colors?) The proof of the theorem now follows immediately from the following lemma.

LEMMA 2. *Let G be a finite graph, C a proper coloring of $V(G)$ using $\chi(G)$ colors , and C' the induced partition of the coloring C. Suppose $v_1, v_2, \ldots, v_{\chi(G)}$ is a path in G. Then this path is a colorful path for C with $C(v_i) = i$ if and only if the path is a colorful path for C' with $C'(v_i) = i$.*

Proof. Let $A_i = \{v \in V(G) \mid C(v) = i\}$ and $B_i = \{v \in V(G) \mid C'(v) = i\}$. Then $A_1, A_2, \ldots, A_{\chi(G)}$ and $B_1, B_2, \ldots, B_{\chi(G)}$ are each colorful partitions of $V(G)$ with the latter one the induced partition of the former one. Suppose that $v_1, v_2, \ldots, v_{\chi(G)}$ is a colorful path for C with $C(v_i) \in A_i$. If $i > 1$, v_i is edge connected to an element of A_{i-1}. In every step of the process to obtain the induced partition C' of C, it follows that v_i always remains in the i'th set of each intermediate partition. Hence $v_i \in B_i$ and $v_1, v_2, \ldots, v_{\chi(G)}$ is a colorful path for C'.

Conversely, suppose $v_1, v_2, \ldots, v_{\chi(G)}$ is a colorful path for C' with $C'(v_i) \in B_i$. Suppose $v_i \in A_{t(i)}$. As v_i and v_{i+1} are edge-connected we have $t(i) \neq t(i+1)$. Also, we can not have $t(i) > t(i+1)$ since $B_1, B_2, \ldots, B_{\chi(G)}$ is obtained by the induction process from the partition $A_1, A_2, \ldots, A_{\chi(G)}$. Hence $t(i) < t(i+1)$ and $1 \leq t(1) < t(2) < \cdots < t(\chi(G)) \leq \chi(G)$. We conclude that, for each i, $t(i) = i$ and $v_1, v_2, \ldots, v_{\chi(G)}$ is a colorful path for C with $C(v_i) = i$. $\square$

When we have a coloring of G with $\chi(G)$ colors, the theorem finds different chains in G for distinct elements of $S_{\chi(G)}$. This observation, which is stated in the following corollary, can then be considered an extension of Gallai's result.

COROLLARY 1. *If G has chromatic number $\chi(G)$, then there are at least $(\chi(G))!$ different chains in G of length $\chi(G)$.*

The interested reader can find material on the chromatic number of a graph in many books on combinatorics or graph theory.

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BISECTING A TRIANGLE

ANTHONY TODD*

1. The Problem. The problem of bisecting a shape into equal areas (or volumes) has been studied for quite some time. Direct applications of the solutions that exist in two and three dimensions have been found in the study of image enhancement and hydrostatics. In both fields it is desirable to know which lines (or planes) bisect a given figure. We wish to do a similar inquiry in two dimensions for triangles with the added condition that the perimeter is also bisected by the line which bisects the area.

DEFINITION 1. *A line which bisects the area and perimeter of a given triangle shall be called a B-line.*

THEOREM 2. *Given ΔABC, if any two of the following statements hold about a line λ which intersects ΔABC, then the third statement also holds:*

1. *λ is concurrent with the incenter I of ΔABC (recall that I is the point of concurrence of the three interior angle bisectors).*
2. *λ bisects the perimeter of ΔABC.*
3. *λ bisects the area of ΔABC.*

Verification of Theorem 2 can be obtained by constructing $\Delta CY'I$, $\Delta CIX'$, $\Delta AIY'$, ΔAIB, $\Delta BX'I$ as seen in Figure 1, and utilizing the fact that the incenter is

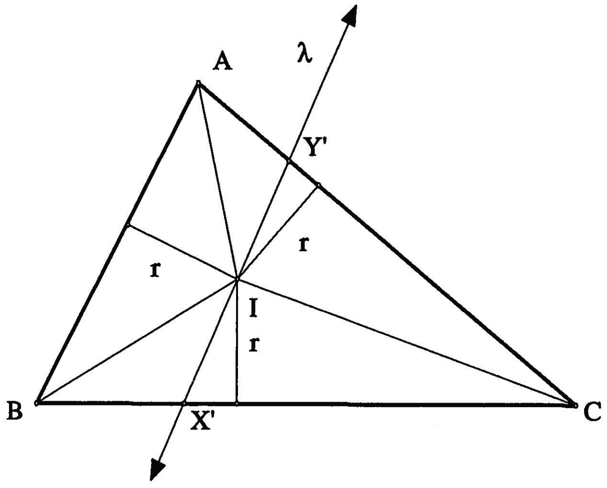

FIG. 1.

equidistant from the edges of the triangle. Let us examine the properties of a B-*line* in relationship to the triangle and the proportion of its angles.

2. Bisecting Envelopes.

2.1. Area Bisectors. In order to further examine the solutions to our triangle problem let us return our attention to the lines that bisect the area of a triangle. We shall explore the ideas of Dunn and Pretty in their article, *Halving a Triangle* [DP, pp. 105-108].

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We will begin with the definition of an envelope.

DEFINITION 3. *A curve which is tangent to every member of a family of curves is called an envelope for that family of curves.*

Let ΔABC be given. We can choose our coordinate system to be the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$ to describe the plane in which ΔABC lies. Let us assign the origin of this coordinate system to be A with B and C being at $(0, c)$ and $(b, 0)$ Solving for the

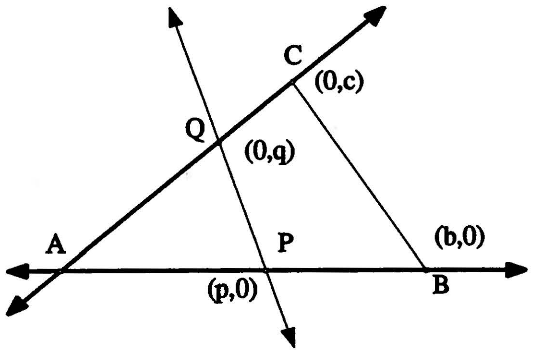

FIG. 2.

envelope for the lines which bisect the area of a triangle yields $8xy = bc$, which is an equation for a hyperbola. Thus any line tangent to this hyperbola within the proper domain will bisect the area of that triangle.

We now note that there are three such hyperbolas for every triangle which are asymptotic to the extended sides of the triangle. These three hyperbolas are tangent to one another at the medians of ΔABC. (For convenience (and fun) we shall call this figure a *tri-perbola*). Thus, all lines tangent to the tri-perbola of ΔABC bisect the area of ΔABC. Figure 3 shows the tri-perbola for an equilateral triangle with its

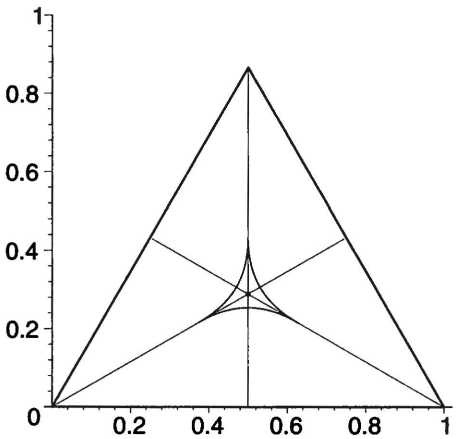

FIG. 3.

medians and Figure 4 shows the tri-perbola for a 3-4-5 right triangle (the incenter I is denoted by the circle).

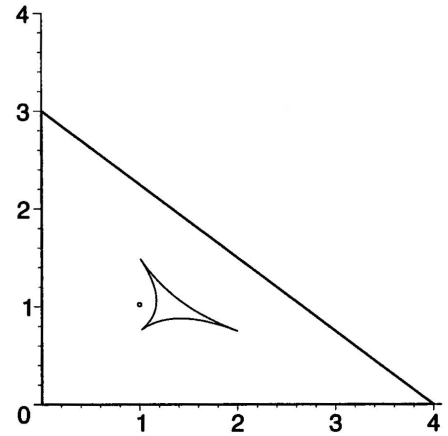

FIG. 4.

An immediate consequence of this discussion is that a line tangent to the tri-perbola and concurrent with the incenter is then also a B-*line*. We now have in our power the ability to determine the number of B-*lines* that exist for a given triangle.

THEOREM 4. *Suppose we have ΔABC, the hyperbolic envelope (tri-perbola) of area bisectors for ΔABC, and the incenter I of ΔABC.*

1. *If I lies outside of the hyperbolic envelope of ΔABC or at the point of tangency of the envelope to a median of ΔABC, then there exists exactly one B-line for ΔABC.*
2. *If I lies on the hyperbolic envelope of ΔABC except at the point of tangency of the envelope to a median of ΔABC, then there exist exactly two B-lines for ΔABC.*
3. *If I lies inside of the hyperbolic envelope of ΔABC, then there exist exactly three B-lines for ΔABC.*

The proof can be seen as one keeps track of the number of times that a line tangent to the envelope sweeps out a given area as you move its point of tangency around the envelope.

2.2. Perimeter Bisectors. Let us now turn our attention to the envelope for the family of lines which bisect the perimeter of a triangle. Following a similar coordinate system as described earlier, the envelope for perimeter bisectors from a vertex is then defined by the following equation:

$$y = s - 2\sqrt{sx} + x, \tag{1}$$

where s is the semiperimeter of ΔABC.

(An interesting note about equation (1) is that this equation is the curve that is hinted at in string art. See Figure 5.)

The envelope for the perimeter bisectors has a form similar to the area bisectors only in the case of the equilateral triangle (see Figure 6). We must carry the construction of the envelope one step further. While it is true that Figure 6 shows the envelopes of perimeter bisectors, in a non-isosceles triangle these curves do not always intersect, as in Figure 7. In order to close the figure they must be connected

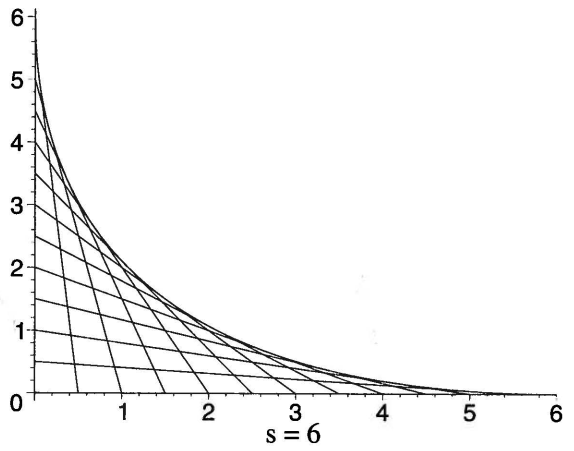

FIG. 5.

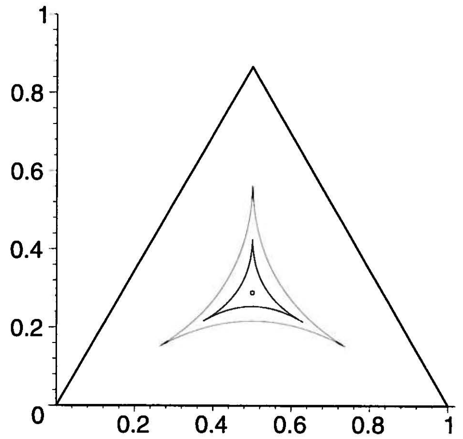

FIG. 6.

by the splitters (the lines which extend from the vertex of a triangle and bisect the perimeter of a triangle) of the triangle as seen in Figure 8. In order to complete the picture we shall show the envelopes for a 3-4-5 right triangle which has only one B-*line* solution. In Figure 9 is plotted the incenter (denoted as a small circle) and the one B-*line* solution.

3. Picture Time. We know the maximum number of B-*line* solutions that any triangle could have is three. It would appear that the number of two B-*line* solution triangles are small in comparison to the one and three B-*line* solution triangles since they occur only when I is concurrent with the curve which defines one of our two envelopes. What we do not know is what the triangles look like that produce one, two, or three B-*line* solutions.

In Figure 10, is the plot of the number of B-*line* solutions given angle measure-

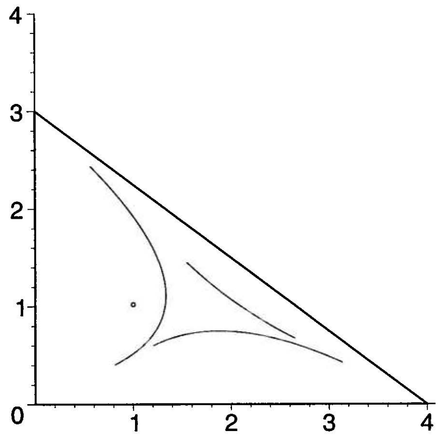

FIG. 7.

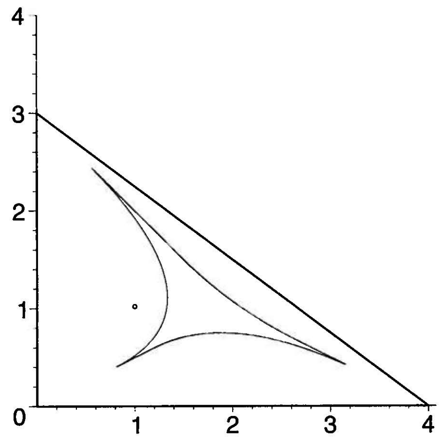

FIG. 8.

ments α and γ (in degrees). We see from Figure 10, as stated earlier, that there are a maximum of three solutions with the majority of triangles having only one solution. The long extensions of the surface that represents three B-*line* solutions follows the lines that define isosceles triangles ($\alpha = \gamma$, $\alpha = \pi - 2\gamma$, and $\gamma = \pi - 2\alpha$). Note that Figure 10 also shows that the solutions for the two B-*line* triangles is in fact the boundary of the three B-*line* solution surface.

With the equations to the boundary of the three B-*line* solutions, we then are able to show what the triangles look like that have one, two, or three B-*line* solutions. Since we have the solutions to the angle measures, given a side $\overline{AC}$ of length 1, we can solve for the position of B to yield one, two, or three B-*line* solutions. To do this we need only to use the case for two solutions since we know that they form the boundary between one and three B-*line* solutions.

Let $\overline{AC}$ be of length 1 with A at (0, 0) and C at (1, 0). The vertex B then lies

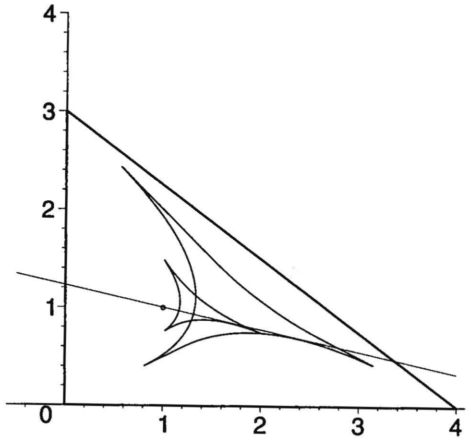

FIG. 9.

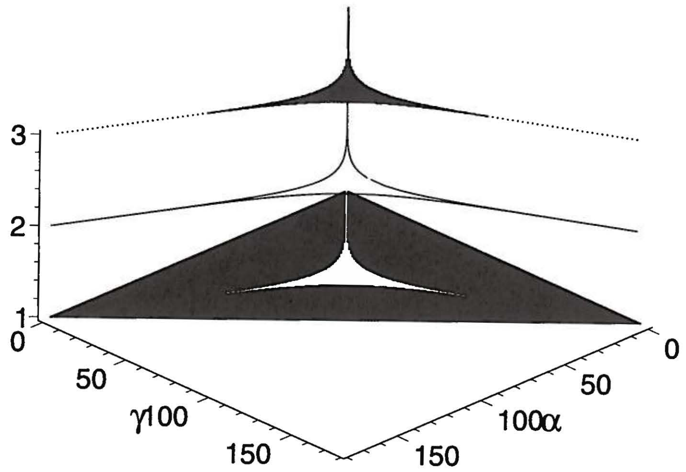

FIG. 10.

at the intersection of the lines

$$y = (\tan\alpha)x \text{ and } y = -(\tan\gamma)x + (\tan\gamma),$$

where (α, γ) is an element of the boundary curves defining two B-*line* solutions.

Solving for the point of intersection of these two lines gives (in parametric form)

$$\left[\frac{\tan\gamma}{\tan\alpha + \tan\gamma}, \frac{\tan\gamma\tan\alpha}{\tan\alpha + \tan\gamma}\right].$$

Thus given side $\overline{AC}$ and vertex B above $\overleftrightarrow{AC}$, if B lies in the shaded region under the outer arc and above the two inner arcs as seen in Figure 11, ΔABC has three

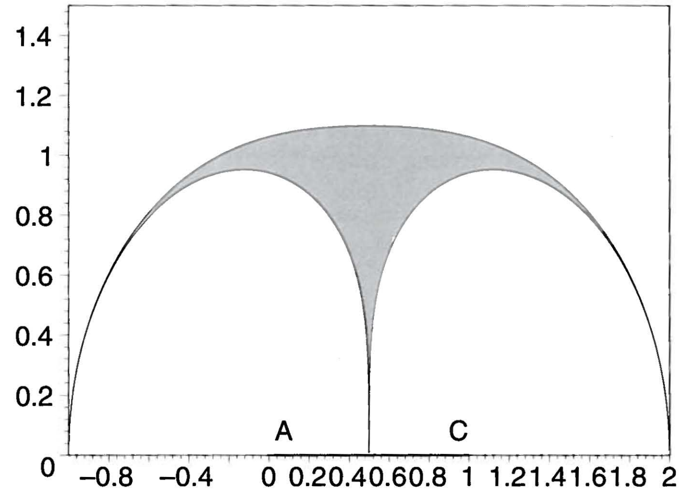

FIG. 11.

B-*line* solutions. If B lies on one of the three arcs (the boundary of the shaded region) then there are two B-*line* solutions for ΔABC. Lastly, if B lies anywhere else not in nor on the figure formed by the three arcs, then there is only one B-*line* solution for ΔABC.

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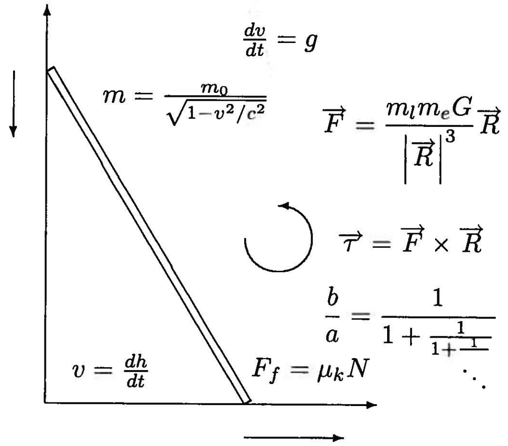
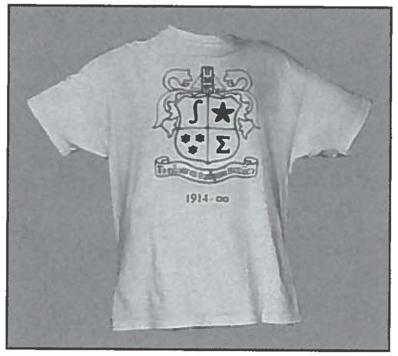
Problem. A painter is high atop a 40 foot ladder leaning against a 60 foot house at an angle such that the ratio of the lengths of the legs of the right triangle formed is the golden section. If the base of the ladder is moving away from the house at at the speed of light divided by Avagadro's number, what are the thoughts of the mathematician watching the ensuing disaster?



Solution. She hopes that none of the paint will get on her new ΠΜΕ Tee Shirt.



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ΠΜΕ Journal, Vol. 11, No. 1, pp 39-44, 1999.

PARTITIONS AND YOUNG'S LATTICES

DAVID WARREN*

An earlier paper in this journal [2] explored the relationships between partitions and their Young lattices. The purpose of this article is to expand that work. We begin with a review of partitions, lattices, and Hasse diagrams.

A *partition* of a natural number N is a finite sequence of natural numbers n_1, n_2, $\ldots$, n_m in non-increasing order such that $\sum_{i=1} mn_i = N$. We will represent partitions by Ferrer's diagrams as shown in the first example. See [1] for details.

Example 1. Ferrer's diagram for the partition $(4,2,1)$ *of seven is shown in Figure 1.*

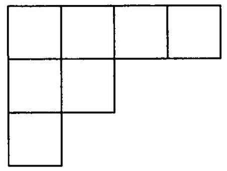

FIG. 1.

A *lattice* is a partially ordered set that has the property that any two elements x and y have a least upper bound (join) and a greatest lower bound (meet). If we order the set of all partitions of a set P by containment, the resulting poset is a lattice called *Young's lattice.* We will represent the resulting lattices as a Hasse diagram.

The bottom element in a Hasse diagram is assigned a rank of one. The *rank* of a subpartition in Young's lattice is the number of squares in Ferrer's diagram of that partition.

In [2] it was shown that Young's lattice for the partition (n) is a single chain as shown in Figure 2, and Young's lattice for the partition $(n,2)$ (with $n>2$) is shown

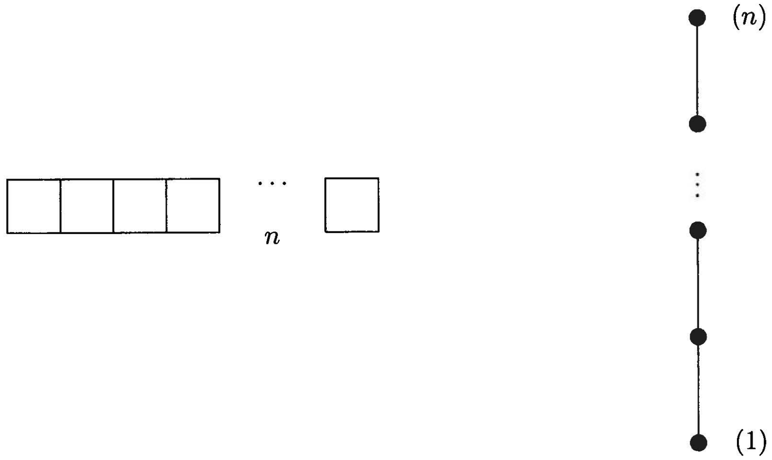

FIG. 2.

in Figure 3.

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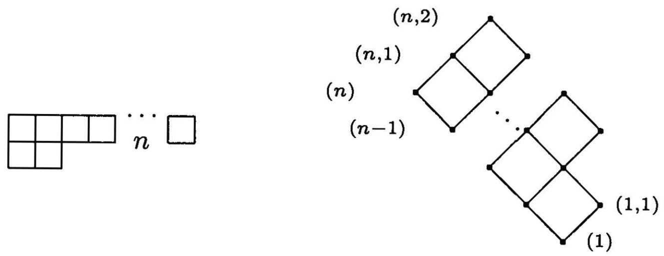

FIG. 3.

THEOREM 1. *Young's lattice for the partition (m, n), where $m > n$, is the lattice shown in Figure 4.*

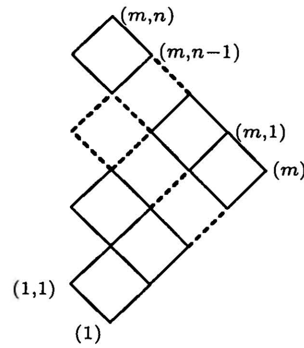

FIG. 4.

Proof. Let n be a fixed integer. The proof is by induction on n. From [2] we know that Figure 4 is Young's lattice for the partitions $(m, 1)$ and $(m, 2)$. Assume that the portion of Figure 5 with the solid lines is Young's lattice for (m, n) with $n < m - 1$.

The only subpartitions of $(m, n + 1)$ that are not subpartitions of (m, n) are $(k, n+1)$ for $n+1 \leq k \leq m$. Since $(k, n+1) \subseteq (k+1, n+1)$ and $(k, n) \subseteq (k, n+1)$ for $n + 1 \leq k \leq m$, to draw Young's lattice for $(m, n + 1)$ we need only add the dashed lines shown in Figure 5 to Young's lattice for (m, n). Therefore, Figure 4 does represent Young's lattice for (m, n). □

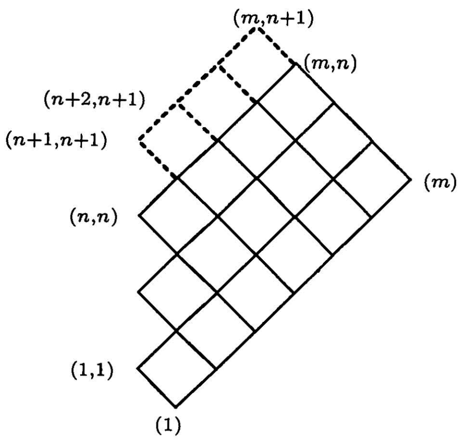

FIG. 5.

The following theorem is another extension of a theorem found in [2].

THEOREM 2. *Young's lattice for the partition $(m, 1, 1, \ldots, 1)$ with n rows is as shown in the lattice in Figure 6.*

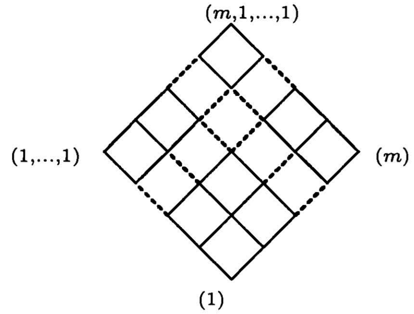

FIG. 6.

Once we found the general forms of partitions with two-dimensional Young's lattices, we began looking at the conjugates of these partitions and their representative lattices. The *conjugate partition* of a given partition is the partition obtained by switching the rows and columns of Ferrer's diagram for the given partition. To switch between a lattice and its conjugate we developed different numbering grids. The two dimensional lattices are obtained from three general types of partitions. The first is a partition with exactly one row or its conjugate with exactly one column in Ferrer's diagram. Since Young's lattices of these partitions are single chains, they can be drawn on any of the following grids.

The second type of partition is one with exactly two rows and any number of columns or its conjugate with exactly two columns and any number of rows. In this case, we need two different labelings (grids) when converting between a lattice and its conjugate.

First, consider the partition with exactly two rows. We will label the spine of the lattice as the line connecting the subpartitions of the form $(n, n-1)$. The origin of the lattice is the position of the subpartition (1). Labeling from the origin, each time we move one position up and to the right, we add one to the first row of the subpartition. Each time we move one position up and to the left, we add one to the second row of the subpartition (as long as in the partition (x, y), $x \geq y$) as shown in Figure 7.

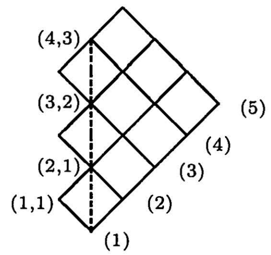

FIG. 7.

Now we will consider the conjugate, the partition with exactly two columns. The spine of this lattice is the line connecting the subpartitions of the form $(2, 2, \ldots, 2, 1)$ with n rows. The origin of this lattice is also the position of the subpartition (1). Labeling from the origin, each time we move one position up and to the left we add another row to the subpartition. Each time we move one position up and to the right, we add one to the highest row which does not have a 2 (two squares) in it as shown in Figure 8.

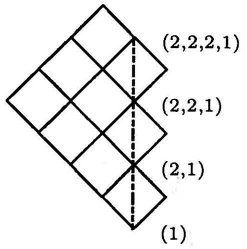

FIG. 8.

The third type of partition which has a two dimensional lattice is a partition of the form $(m,1,1,\ldots,1)$ with n rows. With this lattice, we only need to define one grid system, but this grid is different than either of the previous two. The spine of this lattice is the line connecting the subpartitions with the number of rows and number of columns being equal (i.e.: $(2,1)$, $(3,1,1)$, $(4,1,1,1)$, etc.). Again, the origin is the position of the subpartition (1). Labeling from the origin each time we move one position up and to the right, we add one to the first row of the subpartition. Each time we move one position up and to the left, we add another row to the subpartition. This can be seen in Figures 9 and 10.

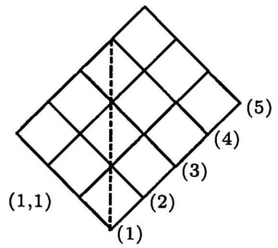

FIG. 9.

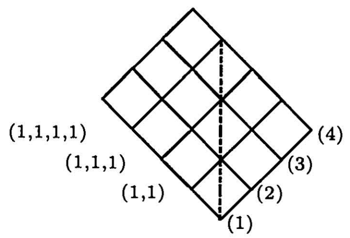

FIG. 10.

Once we defined the grids needed to convert between conjugates, we developed an algorithm to convert a specific subpartition into its conjugate.

Consider the partition (x,y) where $x \geq y$. The first step is to take the first entry, x, and form an x-tuple with a one in each position. This gives us the partition $(\overset{1}{1},\overset{2}{1},\overset{3}{1},\ldots\overset{x}{1})$. Since each position in a partition represents a row, and columns change

to rows in the conjugate, we can see that this step gives us the correct number of entries in the conjugate partition.

The second step is to take the second entry, y, and add one to each of the first y entries of the x-tuple. This gives us the partition $\overset{1}{2},\overset{2}{2},\overset{3}{2},\ldots\overset{y-1}{2},\overset{y}{2},\overset{y+1}{1},\ldots\overset{x}{1}$. This step takes the y elements in the second row of the original partition and changes them into the y elements in the second column of the conjugate. Here again we get the correct number of entries in the conjugate partition.

Note that this pattern holds for every position in an n-tuple partition.

With this algorithm, we can determine the conjugate of any subpartition in a lattice. Using the appropriate grid, we can then develop the conjugate lattice.

THEOREM 3. *The conjugate of a particular lattice is found by reflection about the spine.*

As we were trying to find all the general forms of partitions and their lattices, we ran into a little problem when we looked at the partition $(3,2,1)$ As it turned out, this is the basic partition with a three dimensional lattice. The lattice for this partition is shown in Figure 11.

Working with this partition, we noticed a general pattern that led us to the following theorem.

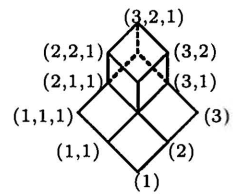

FIG. 11.

THEOREM 4. *Young's lattice for the partition* $(m,2,1,\ldots,1)$ *with* n *rows is shown in Figure 12.*

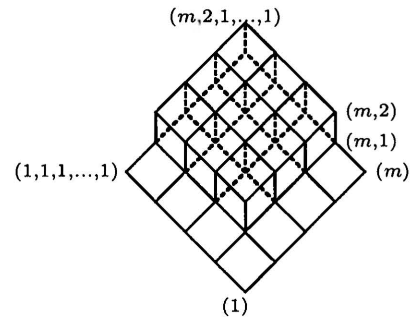

FIG. 12.

Proof. We first show that Young's lattice of the partition $(m,2,1)$ is displayed in Figure 13 by the thicker lines. We proceed by induction on m. By Figure 11, this is true for $(3,2,1)$. Now assume that the lattice for the partition $(m,2,1)$ is as shown in Figure 13 by the thicker lines. The only subpartitions of $(m+1,2,1)$ that are not subpartitions of $(m,2,1)$ are $(m+1)$, $(m+1,1)$, $(m+1,2)$, $(m+1,1,1)$, and $(m+1,2,1)$. As pictured by the dashed lines in Figure 13, these subpartitions connect, as desired, to the lattice for $(m,2,1)$.

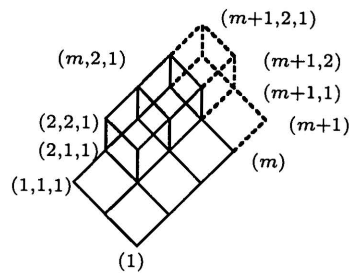

FIG. 13.

Next we prove that the lattice for the partition $(m,2,\underbrace{1,\ldots,1}_{k})$ is as shown in Figure 12 by inducting on k, the number of ones in the partition. In the previous paragraph we have shown this for $k=1$. Assume that the diagram in Figure 12 is correct for $k>1$. The subpartitions of the partition $(m,2,\underbrace{1,\ldots,1}_{k+1})$ that are not subpartitions for $(m,2,\underbrace{1,\ldots,1}_{k})$ are $(1,1,\underbrace{1,\ldots,1}_{k+1})$, $(j,1,\underbrace{1,\ldots,1}_{k+1})$, and $(j,2,\underbrace{1,\ldots,1}_{k+1})$ for $2 \le j \le m$. Now $(1,\underbrace{1,\ldots,1}_{k+1}) \subseteq (1,1,\underbrace{1,\ldots,1}_{k+1}) \subseteq (2,1,\underbrace{1,\ldots,1}_{k+1})$, and for $2 \le j < m$, $(j,1,\underbrace{1,\ldots,1}_{k+1}) \subseteq (j+1,1,\underbrace{1,\ldots,1}_{k+1})$, $(j,1,\underbrace{1,\ldots,1}_{k+1}) \subseteq (j,2,\underbrace{1,\ldots,1}_{k+1})$, and $(j,2,\underbrace{1,\ldots,1}_{k+1}) \subseteq (j+1,2,\underbrace{1,\ldots,1}_{k+1})$, Thus the connections displayed in Figure 14 are correct and the theorem is proved. □

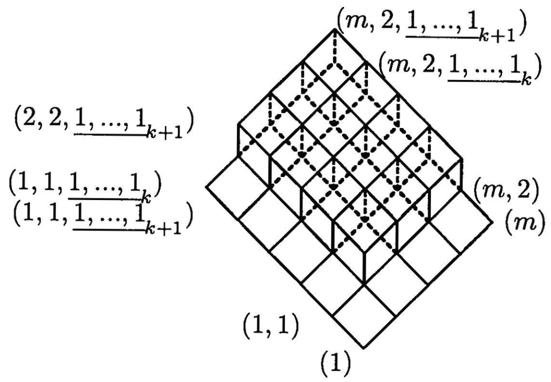

FIG. 14.

REFERENCES

[1] BOGART, KENNETH P., "Introductory Combinatorics", Harcourt Brace Jovanovich, 1990.
[2] HURST, CARISSA, *Young's Lattices,* Pi Mu Epsilon Journal, Spring 95, 92-95.

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CRANKS, COMPUTERS, AND FERMAT'S LAST THEOREM

JOHN A. ZUEHLKE*

In 1995 Andrew Wiles proved Fermat's Last Theorem [2] which states that the equation

$$x^n + y^n = z^n$$

has no solutions for x, y, z, and n positive integers with $n > 2$.

Yet to this day, in Internet articles and letters to math department professors, people still claim to have found positive integers that actually satisfy the Fermat equation for $n > 2$. Some of these people have the temerity to state that their triplet satisfies the Fermat equation for infinitely many values of n.

Let's suppose that someone claims to have found a counter-example to Fermat's Last Theorem on their computer with one of the terms being

$$1234567891011121314^{123456789101112131 5}.$$

Clearly you know this person is wrong. But now you have a decision to make. Do you respond by saying:

a) It is time for a computer upgrade.
b) All semi-stable elliptic curves over $\mathbb{Q}$ are modular.
c) So what do you know about Hecke algebras, Euler systems, and finite flat group schemes?
d) Here are some simple inequalities that show your "counter-example" is incorrect.

Answer "a" might be correct, but it is not enlightening. Answers "b" and "c" are good choices if the person you are dealing with is an expert in Algebraic Number Theory. However if you are not dealing with such an expert answer "d" is probably the best response.

THEOREM 1. *If $x^n + y^n = z^n$ for x, y, z, n positive integers with $n \ge 2$, then:*

$$x > n \quad \textit{and} \quad y > n \quad \left(\textit{so } z > \sqrt[n]{2}\,n\right), \tag{1}$$

$$z > \frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}. \tag{2}$$

Statement (1) is due to Gruenert [1].

Proof. Proof of (1). If $x^n + y^n = z^n$ we can assume $x \ge y$. Let $z = x + u$ for some positive integer u. Then $x^n + y^n = z^n = x^n + nx^{n-1}u + \ldots + u^n$, so $y^n > nx^{n-1}u$, and

$$y > n\left(\frac{x}{y}\right)^{n-1} u.$$

Since $x \ge y$, this implies $y > n$ and $x > n$. So $z^n > 2n^n$ or $z > \sqrt[n]{2}\,n$.

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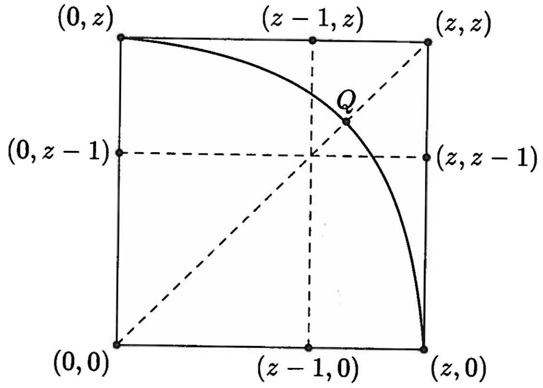
Proof of (2). If $x^n + y^n = z^n$ for x, y, z, n positive integers with $n \geq 2$, fix z and consider the graph of the function $y = \sqrt[n]{z^n - x^n}$



in the first quadrant. The function strictly decreases for $x \in [0, z]$ and the graph passes through the points

$$(0, z), \quad Q = \left(\frac{z}{\sqrt[n]{2}}, \frac{z}{\sqrt[n]{2}}\right), \quad \text{and} \quad (z, 0).$$

It is clear from the graph that if $z - 1 < \frac{z}{\sqrt[n]{2}}$, there can be no integer solutions. This implies (2). □

It is an interesting exercise to show that $\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1} > \sqrt[n]{2}n$ for $n > 1$, that $\sqrt[n]{2}n$ is asymptotic to n, and $\dfrac{\sqrt[n]{2}}{\sqrt[n]{2}-1}$ is asymptotic to $\dfrac{n}{\log 2}$.

The inequalities in the Theorem do not rule out all possible "counter-examples", but they can be useful in many cases. In particular, the next time someone tells you that they found a counter-example on their computer with one of the terms being

$$141312111098765432 1^{151312111098765432 1},$$

first show them the inequalities and then tell them that it is time for a computer upgrade.

REFERENCES

[1] J. A. Gruenert. *Wenn $n > 1$, so Gibt es Unter den Ganzen Zahlen von 1 bis n Nicht Zwei Werte von x und y, Fuer Welche, Wenn z Einen Ganzen Wert Bezeichnet, $x^n + y^n = z^n$ ist.* Archiv Math. Phys., **26**, 1856, 119–120.

[2] A. Wiles. *Modular Curves and Fermat's Last Theorem,* Annals of Math., **141**, 1995, 443–551.

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John Zuehlke received his BA in mathematics from Princeton University in 1992. After graduation he worked as a high school mathematics and computer science teacher, and as a computer programmer. He is currently a Ph.D. candidate at Columbia University. His research interests include Diophantine Equations and Transcendental Number Theory.

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PROBLEM DEPARTMENT

EDITED BY CLAYTON W. DODGE

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed to arrive by July 1, 2000. Solutions by students are given preference.

Problems for Solution.

966. *Proposed by Count Juan Mower, Big Twenty Township, Maine.*

Although there are several solutions to this base eleven addition alphametic in which 7 divides $SEVEN$ or where 8 divides $EIGHT$, there is only one in which 5 divides $FIVE$. Find that solution:

$$FIVE + SEVEN + EIGHT = TWENTY.$$

Curiously, in that unique solution, 5 divides $EIGHT$, too.

967. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let N be a natural number greater than 1 with d distinct positive prime divisors. If p and q are the largest and smallest of these divisors, then prove that

$$\log_p N \leq d \leq \log_q N.$$

968. *Proposed by Doru Popescu Anastasiu, Liceul Radu Greceanu, Slatina, Romania.*

Determine all real numbers x and y such that

$$16x^2 + 21y^2 - 12xy - 4x - 6y + 1 = 0.$$

969. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find $y(x)$ if

$$e^{-x}\frac{d^2y}{dx^2} + ye^x = 0.$$

970. *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that

$$\int_0^{\pi/4} \frac{\cos(x)\ln(\sin(x))}{\sin(x)\cos(2x)}dx = -\frac{\pi + \ln 2}{4\sqrt[4]{2}}B\left(\frac{1}{4}, \frac{1}{2}\right)$$

and

$$\int_0^{\pi/4} \frac{\cos(x)\ln(\cos(2x))}{\cos^{3/4}(2x)}dx = -\frac{\pi}{2\sqrt{2}}B\left(\frac{1}{4}, \frac{1}{2}\right)$$

where $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ is the Beta function.

971. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Find an integer-sided obtuse triangle with acute angles in the ratio 7/5.

972. *Proposed by Paul S. Bruckman, Berkeley, California.*

Given three non-collinear points A, B, and C in the complex plane, determine I, the incenter of triangle ABC as a "weighted average" of these points.

973. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington, Pennsylvania.*

Prove that $a_{n+1} = 2a_n + a_{n-1}$, given that $a_0 = 0$ and

$$a_n = \binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + 2^3\binom{n}{7} + \cdots$$

974. *Proposed by Kenichiro Kashihara, Sagamihara, Kanagawa, Japan.*

Given any positive integer n, the *Pseudo-Smarandache function* $Z(n)$ is the smallest integer m such that n divides

$$\sum_{k=1}^{m} k.$$

a) Solve the Diophantine equation $Z(x) = 8$.

b) Show that for any positive integer p the equation $Z(x) = p$ has solutions.

*c) Show that the equation $Z(x) = Z(x+1)$ has no solutions.

*d) Show that for any given positive integer r there exists an integer s such that the absolute value of $Z(s) - Z(s+1)$ is greater than r.

975. *Proposed by Doru Popescu Anastasiu, Liceul Radu Greceanu, Slatina, Romania.*

For any given fixed positive integer n, determine the positive integers $x_1, x_2, \ldots, x_n$ such that

$$x_1 + 2(x_1 + x_2) + 3(x_1 + x_2 + x_3) + \ldots + n(x_1 + x_2 + \ldots + x_n) = \frac{2n^3 + 3n^2 + 7n}{6}.$$

976. *Proposed by Rajindar S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

If $x + y + z + t = \pi$, prove that

$$\tan(x+y)\tan(z+t) > 27\cot x \cot y \cot z \cot t.$$

977. *Proposed by Rajindar S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

If A, B, and C are the angles of a triangle, then prove that

$$\cot(A/2) + \cot(B/2) + \cot(C/2) > \cot(A) + \cot(B) + \cot(C).$$

978. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

In the array below place sixteen digits to form eight not necessarily distinct squares without using the digit zero. The answer is unique.

***979.** *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Dedicated to Professor M. V. Subbarao on the occasion of his 78^{th} birthday.

Do there exist an infinite number of triples of consecutive positive integers such that one of them is prime, another is a product of two primes, and the third is a product of three primes. Two such examples are 6, 7, 8 and 77, 78, 79.

Correction. Frank Battles of Massachusetts Maritime Academy, Buzzards Bay, Massachusetts, reported that the web address given in the solution to Problem 914 [Fall 1998, page 744] should have a dot "." instead of a slash "/" between "index" and "html." He further stated that the alphametic solver at that website could solve neither Problem 745 ($ENID + DID = DINE$) nor 940. The correct address is http://www.ceng.metu.edu.tr/~selcuk/alphametic/index.html.

Solutions.

924. [Fall 1997, Fall 1998] *Proposed by George Tsapakidis, Agrino, Greece.*

Find an interior point of a triangle so that its projections on the sides of the triangle are the vertices of an equilateral triangle.

III. *Comment and solution by* **Paul Yiu**, *Florida Atlantic University, Boca Raton, Florida.* Solution I, given by W. H. Peirce, computes the barycentric coordinates of the desired point P as

$$\left(\frac{a^2\sin(A\pm 60°)}{D_\pm \sin(A)}, \frac{b^2\sin(B\pm 60°)}{D_\pm \sin(B)}, \frac{c^2\sin(C\pm 60°)}{D_\pm \sin(C)}\right),$$

where $D_\pm = a^2 + b^2 + c^2 \pm 4\sqrt{3}H$, H being the area of the triangle. If we homogenize these coordinates, we find surprisingly simple descriptions of the point P. There are two such points $P_\pm$, with homogeneous barycentric coordinates $a\sin(A\pm 60°) : b\sin(B\pm 60°) : c\sin(C\pm 60°)$. Then one recognizes $P_\pm$ as the isogonal conjugates of the points

$$F_\pm = \frac{a}{\sin(A\pm 60°)} : \frac{b}{\sin(B\pm 60°)} : \frac{c}{\sin(C\pm 60°)},$$

which are the *isogonal centers* of the triangle. [If one erects equilateral triangles outwardly (respectively inwardly) on the sides of triangle ABC, the three lines from each vertex of triangle ABC to the third vertex of the equilateral triangle erected on the opposite side are concurrent at the isogonal center F_+ (respectively F_-)].

Denote by O and K the circumcenter and the symmedian point of triangle ABC. It is easy to see that the points $P_\pm$ divide the segment OK harmonically, in the ratio $a^2+b^2+c^2 : \pm 4\sqrt{3}H$.

Solution. Without restriction to the interior of the triangle, there are two such points, one of which always lies outside the triangle and the other is an interior point if and only if every angle of the triangle is less than 120°. These points are the *isodynamic* points of the triangle, the two points common to the three Apollonian circles of the triangle. The *Apolonian circle* for side AB of triangle ABC has diameter XX', where X and X' are the points on the line BC such that CX and CX' bisect angle C. It is well known that that circle is the locus of all points P in the plane such that $AP : BP = b : a$. It is clear that the three Apolonian circles for the three sides of a triangle pass through two common points, the *isodynamic points* $I_\pm$ of the triangle. Kimberling [1] has given the trilinear coordinates of the isodynamic points as $I_\pm = \sin(A\pm\pi/3) : \sin(B\pm\pi/3) : \sin(C\pm\pi/3)$.

If P is a point in the plane of the triangle, its *pedal* triangle (the triangle whose vertices are the projections of P on the sides of the given triangle) has side lengths $AP\sin(A)$, $BP\sin(B)$, and $CP\sin(C)$. This triangle is equilateral if and only if

$$AP : BP : CP = \frac{1}{\sin A} : \frac{1}{\sin B} : \frac{1}{\sin C} = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

Thus $AP : BP = b : a$, $BP : CP = c : b$, and $CP : AP = a : c$ and the desired result follows.

Reference:

1. C. Kimberling, *Central points and central lines in the planes of a triangle*, Math. Mag. 67 (1994) 163-186.

940. [Fall 1998] *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

In the following base ten alphametic determine the maximum value for $MONEY$:

$$DAD + SEND = MONEY.$$

Solution by **Daniel Hermann**, *student, Angelo State U., San Angelo, Texas.*

We must have $M=1$, $O=0$, and $S=9$. Further, $D+E(+1\text{perhaps}) = N+10$ and $D+E+1 \le 7+8+1 = 16$. So we try $N=6$. From the tens column we have $A+N+1 = 17$ or 18, which is not possible with the unused digits. Thus we try $N=5$. Since A cannot be chosen large enough to provide a carry into the hundreds column, then D and E are 8 and 7 or 7 and 8. From the units column, then $Y=6$ or 4 and there is a carry to the tens column. Thus $1+A+5=E$. We must have $A=2$, so $E=8$, $D=7$, and $Y=4$. This yields a maximum of 10584 for $MONEY$. Specifically, $727+9857=10584$. There are two smaller solutions, $727+9637=10364$ and $636+9846=10482$.

Also solved by **Charles D. Ashbacher,** Charles Ashbacher Technologies, Hiawatha, IA, **Frank P. Battles,** Massachusetts Maritime Academy, Buzzards Bay, **Karl Bittenger,** Austin Peay State University, Clarksville, TN, **Scott H. Brown,** Auburn University at Montgomery, AL, **Paul S. Bruckman,** Berkeley, CA, Mark Evans, Louisville, KY, **Victor G. Feser,** University of Mary, Bismarck, ND, **Stephen I. Gendler,** Clarion University of Pennsylvania, **Richard I. Hess,** Rancho Palos Verdes, CA, **Carl Libis,** Granada Hills, CA, **Yoshinobu Murayoshi,** Okinawa, Japan, **Michael R. Richardson, Jr.,** Austin Peay State University, Clarksville, TN, **H.-J. Seiffert,** Berlin, Germany, **Kevin P. Wagner,** University of South Florida, Saint Petersburg, **Katie Wibby,** Alma College, MI, **Rex H. Wu,** Brooklyn, NY, **Adrien Chun Yiu Au Yeung,** Stanford University, Palo Alto, CA, and the **Proposer.**

941. [Fall 1998] *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, Pennsylvania.*

Let $a_1 = 1$, $a_2 = k > 2$, and for $n > 2$, $a_n = ka_{n-1} - a_{n-2}$.

a) Show that the general term a_n is given by

$$a_n = \frac{B^n - B^{-n}}{B - B^{-1}}, \text{ where } B = \frac{k+\sqrt{k^2-4}}{2}.$$

b) Find a suitable expression for the sum S_n of the first n terms.

Solution by **Kandasamy Muthuvel**, *University of Wisconsin Oshkosh, Oshkosh, Wisconsin.*

a) Since $(2B-k)^2 = k^2-4$, then $k = B+B^{-1}$, so $B^{-1} = (k-\sqrt{k^2-4})/2$. Now, by mathematical induction, we have

$$a_1 = \frac{B-B^{-1}}{B-B^{-1}} = 1, \text{ and } a_2 = \frac{B^2-B^{-2}}{B-B^{-1}} = B+B^{-1} = k,$$

so the formula for a_n is true for $n=1$ and $n=2$. Next suppose the formula is true for $n = m-2$ and $n = m-1$. Then we have

$$\begin{aligned} ka_{m-1} - a_{m-2} &= \frac{k(B^{m-1}-B^{-(m-1)})}{B-B^{-1}} - \frac{B^{m-2}-B^{-(m-2)}}{B-B^{-1}} \\ &= \frac{(B+B^{-1})(B^{m-1}-B^{-(m-1)})}{B-B^{-1}} - \frac{B^{m-2}-B^{-(m-2)}}{B-B^{-1}} \\ &= \frac{B^m - B^{-m}}{B-B^{-1}} = a_m. \end{aligned}$$

b) We have that

$$\begin{aligned} S_n &= \sum_{i=1}^n a_i = \sum_{i=1}^n \frac{B^i - B^{-i}}{B-B^{-1}} = \frac{1}{B-B^{-1}}\left(\sum_{i=1}^n B^i - \sum_{i=1}^n B^{-i}\right) \\ &= \frac{1}{B-B^{-1}}\left(\frac{B^{n+1}-B}{B-1} - \frac{B^{-(n+1)}-B^{-1}}{B^{-1}-1}\right) = \frac{B}{B^2-1}\left(\frac{B^{n+1}-B}{B-1} - \frac{B^{-n}-1}{1-B}\right) \\ &= \frac{B^{n+2}-B^2+B^{-n+1}-B}{(B+1)(B-1)^2} = \frac{(B^{n+1}-1)(B^n-1)}{(B+1)(B-1)^2B^{n-1}}. \end{aligned}$$

Also solved by **Frank P. Battles,** Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman,** Berkeley, CA, **Kenneth B. Davenport,** Frackville, PA, **Charles R. Diminnie,** Angelo State University, San Angelo, TX, **Russell Euler** and **Jawad Sadek,** Northwest Missouri State University, Maryville, **George P. Evanovich,** Saint Peter's College, Jersey City, NJ, **Stephen I. Gendler,** Clarion University of Pennsylvania, **Richard I. Hess,** Rancho Palos Verdes, CA, **Joe Howard,** New Mexico Highlands University, Las Vegas, **Carl Libis,** Granada Hills, CA, **David E. Manes,** SUNY College at Oneonta, NY, **Yoshinobu Murayoshi,** Okinawa, Japan, **William H. Peirce,** Rangeley, ME, **Shiva K. Saksena,** University of North Carolina at Wilmington, **H.-J. Seiffert,** Berlin, Germany, **Kevin P. Wagner,** University of South Florida, Saint Petersburg, **Rex H. Wu,** Brooklyn, NY, and the **Proposer.**

Another form for S_n is $(1 - a_{n+1} + a_n)/(2-k)$, correct since $k \ne 2$.

942. [Fall 1998] *Proposed by John S. Spracker, Western Kentucky University, Bowling Green, Kentucky.*

Calculate the following "sums" of the form $\sum_{n=1}^{\infty} a_n$ for each given sequence $\{a_n\}$ and given addition $\oplus$. To deal with non-associative operations define $S_1 = a_1$ and $S_{n+1} = S_n \oplus a_{n+1}$ for $n > 0$. [*Editor's note: Several solvers kindly pointed out that the original formula was incorrectly printed as $S_{n+1} = S_n \oplus a_n$. If the originally published formula is used, the values of a_n will vary somewhat, but the limits are unchanged.*]

a) On R^+ let $a \oplus b = 1/a + 1/b$ and take $a_n = n$. Then $S_1 = 1$, $S_2 = 1 + 1/2 = 3/2$, $S_3 = 2/3 + 1/3 = 1, \ldots$.

b) On R^+ let $a \oplus b = ab/(a+b)$ and take $a_n = 1/n$.

c) On R^+ let $a \oplus b = \sqrt{ab}$ and take $a_n = 1/n$.

d) On R let $a \oplus b = \cos(a+b)$ and take $a_n = 2\pi n$.

Solution by **Richard I. Hess**, *Rancho Palos Verdes, California.*

a) We have $S_4 = 1/1 + 1/4 = 5/4$, and in general, $S_n = 1$ for odd n and $S_n = (n+1)/n$ for even n. Thus $S = \lim_{n\to\infty} S_n = 1$.

b) Here $S_1 = 1$, $S_2 = (1)(1/2)/(1+1/2) = 1/3$, $S_3 = (1/3)(1/3)/(1/3+1/3) = 1/6$, $S_4 = (1/6)(1/4)/(1/6+1/4) = 1/10$, and in general, $S_n = 2/(n(n+1))$. Therefore, $S = 0$.

c) Now $S_1 = 1$, $S_2 = \sqrt{(1)(1/2)} = 1/\sqrt{2}$, $S_3 = \sqrt{(1/\sqrt{2})(1/3)} < 1/\sqrt{3}$, and in general, $S_n < 1/\sqrt{n}$. Again, $S = 0$.

d) In this case, $S_1 = 2\pi$, $S_2 = \cos(2\pi + 4\pi) = 1$, $S_3 = \cos(1 + 6\pi) = \cos(1)$, $S_4 = \cos(\cos(1))$, $S_5 = \cos(\cos(\cos(1)))$, etc. For $0 < x < 1$ we have $0 < \cos x < x$, so the S_n form a bounded decreasing sequence and S exists and is the number x such that $\cos x = x$. That is,

$$S = 0.73908513321516064165531208767387340401 34 \ldots.$$

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Charles R. Diminnie,** Angelo State University, San Angelo, TX, **Carl Libis,** Granada Hills, CA, **David E. Manes,** SUNY College at Oneonta, NY, **H.-J. Seiffert,** Berlin, Germany, **Kevin P. Wagner,** University of South Florida, Saint Petersburg, **Rex H. Wu,** Brooklyn, NY, and the **Proposer.**

943. [Fall 1998] *Proposed by Paul S. Bruckman, Berkeley, California.*

Let $\alpha = (1+\sqrt{5})/2$ and let F_n denote the n'th Fibonacci number, so that $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n > 0$. For $n = 1, 2, \ldots$ define

$$U_n = \alpha^n \prod_{k=1}^{n} \frac{F_{2k}}{F_{2k+1}} \text{ and } V_n = \alpha^n \prod_{k=1}^{n} \frac{F_{2k-1}}{F_{2k}}.$$

Prove that $U = \lim_{n\to\infty} U_n$ and $V = \lim_{n\to\infty} V_n$ exist. If possible, evaluate U and V in closed form.

Solution by **Rex H. Wu**, *Brooklyn, New York.*

Since $\alpha = (1+\sqrt{5})/2$, then $\alpha^{-1} = (\sqrt{5}-1)/2$ and also, $F_{2k}/F_{2k+1} \le \alpha^{-1}$. We know furthermore that $F_n = [\alpha^n - (-\alpha^{-1})^n]/\sqrt{5}$.

Observe that U_n is strictly decreasing, with a maximum at $n = 1$, or $U_1 = \alpha/2$. If there is a nonzero lower bound, then $\lim_{n\to\infty} U_n = U$ exists and, as we shall see, the limit of V_n also exists. Now we look for a lower bound.

$$U_n = \alpha^n \prod_{k=1}^{n} \frac{F_{2k}}{F_{2k+1}} = \prod_{k=1}^{n} \alpha \frac{\alpha^{2k} - (-\alpha^{-1})^{2k}}{\alpha^{2k+1} - (-\alpha^{-1})^{2k+1}} = \prod_{k=1}^{n} \alpha \frac{\alpha^{2(2k+1)} - \alpha^2}{\alpha^{2(2k+1)} + 1}$$

$$= \prod_{k=1}^{n} \frac{(\alpha^{2(2k+1)} - \alpha^2)(\alpha^{2(2k+1)} - 1)}{(\alpha^{2(2k+1)} + 1)(\alpha^{2(2k+1)} - 1)} = \prod_{k=1}^{n} \frac{\alpha^{8k+4} - \alpha^{4k+4} - \alpha^{4k+2} + \alpha^2}{\alpha^{8k+4} - 1}$$

$$> \prod_{k=1}^{n} \frac{\alpha^{8k+4} - \alpha^{4k+4} - \alpha^{4k+2} + \alpha^2}{\alpha^{8k+4}} = \prod_{k=1}^{n} \left(1 - \frac{1}{\alpha^{4k}} - \frac{1}{\alpha^{4k+2}} + \frac{1}{\alpha^{8k+2}}\right)$$

$$> \prod_{k=1}^{n} \left(1 - \frac{1}{\alpha^{4k}} - \frac{1}{\alpha^{4k+2}}\right) = \prod_{k=1}^{n} \left(1 - \frac{1+\alpha^{-2}}{(\alpha^4)^k}\right).$$

Since $1 + 1/\alpha^2 \approx 1.382$ and $\alpha^4 \approx 6.854$, we can conclude that

$$U_n > \prod_{k=1}^{n} \left(1 - \frac{1}{4^k}\right) > 1 - \sum_{k=1}^{n} \frac{1}{4^k} = \frac{2}{3}.$$

All of the above just showed $\alpha/2 \ge U_n > 2/3$. With the fact that U_n is strictly decreasing, $\lim_{n\to\infty} U_n = U$ exists and is greater than zero. To show the limit of V_n exists, note that V_n is strictly decreasing. If we multiply U_n and V_n, we get

$$U_n V_n = \alpha^{2n} \frac{F_1}{F_{2n+1}} = \frac{\alpha^{2n}}{F_{2n+1}} = \frac{\alpha^{2n}}{[\alpha^{2n+1} - (-\alpha^{-1})^{2n+1}]/\sqrt{5}} = \frac{\sqrt{5}}{\alpha + \alpha^{-(4n+1)}}.$$

Then $\lim_{n\to\infty} U_n V_n = \sqrt{5}/\alpha$. Since $\alpha/2 \ge \lim_{n\to\infty} U_n = U > 2/3$, then $(3\sqrt{5})/(2\alpha) > \lim_{n\to\infty} V_n = V \ge (2\sqrt{5})/(\alpha^2)$.

Addendum: Another way to prove the existence of V is to use the fact that $\lim_{n\to\infty}(1+a)(1+a^2)(1+a^3)\ldots(1+a^n)$ exists (and is nonzero) if $0 \le a < 1$. This is true since V_n is strictly increasing and

$$V_n = \prod_{k=1}^{n} \alpha \frac{F_{2k-1}}{F_{2k}} = \prod_{k=1}^{n} \frac{\alpha^{2k} + \alpha/\alpha^{2k-1}}{\alpha^{2k} - 1/\alpha^{2k}} < \prod_{k=1}^{n} \frac{\alpha^{2k} + \alpha/\alpha^{2k-1}}{\alpha^{2k}} = \prod_{k=1}^{n} \left(1 + \alpha^{-(4k-2)}\right).$$

Also solved by **Richard I. Hess**, Rancho Palos Verdes, CA, and the **Proposer.**

944. [Fall 1998] *Proposed by David Iny, Baltimore, Maryland.*

Evaluate

$$\int_0^{\infty} \frac{dx}{x + e^x}.$$

Solution by **Frank P. Battles**, *Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

Denote by I the given integral, which clearly exists, and recall that, for non-negative integral n,

$$\int_0^{\infty} x^{n-1} e^{-x} dx = \Gamma(n) = (n-1)!,$$

so that

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \text{ for } a > 0.$$

For $x \in [0, \infty)$, we have $xe^{-x} < 1$, whence

$$I = \int_0^{\infty} \frac{e^{-x}dx}{1 + xe^{-x}} = \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n x^n e^{-(n+1)x} dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} x^n e^{-(n+1)x} dx$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(n+1)^{n+1}} \approx 0.8063956162.$$

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Kenneth B. Davenport,** Frackville, PA, **Russell Euler** and **Jawad Sadek,** Northwest Missouri State University, Maryville, **Robert C. Gebhardt,** Hopatcong, NJ, **Richard I. Hess** (who gave a 42 decimal place answer), Rancho Palos Verdes, CA, **Joe Howard,** New Mexico Highlands University, Las Vegas, and the **Proposer.**

***945.** [Fall 1998] *Proposed by the late Jack Garfunkel, Flushing, New York.*

Let A, B, C be the angles of a triangle and A' ,B' , C' those of another triangle with $A \geq B \geq C$, $A > C$, $A' \geq B' \geq C'$, and $A' > C'$. Prove or disprove that if $A - C \geq 3(A' - C')$, then $\sum \cos(A/2) \leq \sum \sin(A')$.

Solution by **Paul S. Bruckman,** *Berkeley, California.*

We shall show that the coefficient 3 is unnecessary, that the stated conclusion is true whenever $A - C \geq A' - C'$ and the rest of the hypothesis is true. Let U, V, W be arbitrary numbers and define $R(U, V, W) = \cos(U/2) + \cos(V/2) + \cos(W/2)$ and $S(U, V, W) = \sin(U) + \sin(V) + \sin(W)$.

We prove the following identity by using trigonometric formulas on its product term to replace the product of two sine factors by a sum and then the resulting two products of a sine and a cosine by sums,

$$(1) \quad S(U,V,W) = \sin(U+V+W) + 4\sin(\frac{U+V}{2})\sin(\frac{V+W}{2})\sin(\frac{W+U}{2}).$$

Note that

$$R(A,B,C) = \cos\left(\frac{\pi - B - C}{2}\right) + \cos\left(\frac{\pi - C - A}{2}\right) + \cos\left(\frac{\pi - A - B}{2}\right)$$
$$= S\left(\frac{B+C}{2}, \frac{C+A}{2}, \frac{A+B}{2}\right)$$
$$= \sin(A+B+C)+$$
$$= \sin\pi + 4\sin\left(\frac{\pi + A}{2}\right)\sin\left(\frac{\pi + B}{2}\right)\sin\left(\frac{\pi + C}{2}\right) = S(A,B,C)$$

by identity 1.

Let $S = S(A, B, C)$ and $S' = S'(A', B', C')$. We need to show that $S \leq S'$ under the revised hypotheses. Set $A - C = 6D$, so that $0 < D < \pi/6$. Then there is a θ with $-1 \leq \theta \leq 1$ such that $A = \pi/3 + (3 - \theta)D$, $B = \pi/3 + 2\theta D$, and $C = \pi/3 - (3 + \theta)D$.

Let $F(D, \theta) = S(A, B, C)$. It suffices to show that $F_D(D, \theta) < 0$ for all permissible D and θ. Since F is analytic in either variable, we have

$$F_D(D,\theta) = (3-\theta)\cos(A) + 2\theta\cos(B) - (3+\theta)\cos(C)$$

and

$$F_{DD}(D,\theta) = -(3-\theta)^2\sin(A) - 4\theta^2\sin(B) - (3+\theta)^2\sin(C).$$

Clearly, $F_{DD} < 0$ in its permissible domain. For any fixed θ , since F_D is a decreasing function of D, it suffices to show that $\lim_{D \to 0^+} F_D(D, \theta) \leq 0$. We have

$$\lim_{D \to 0^+} F_D(D,\theta) = (3 - \theta + 2\theta - 3 - \theta)\cos(\pi/3)) = 0.$$

This implies that $F_D(D, \theta) < 0$ for all permissible D and θ. Therefore, $S \leq S'$.

946. [Fall 1998] *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, Pennsylvania.*

Let M be a point inside (outside) triangle ABC if $\angle A$ is acute (obtuse) and let $m\angle MBA + m\angle MCA = 90°$.

a) Prove that $(BC \cdot AM)^2 = (AB \cdot CM)^2 + (CA \cdot BM)^2$.

b) Show that the Pythagorean theorem is a special case of the formula of part (a).

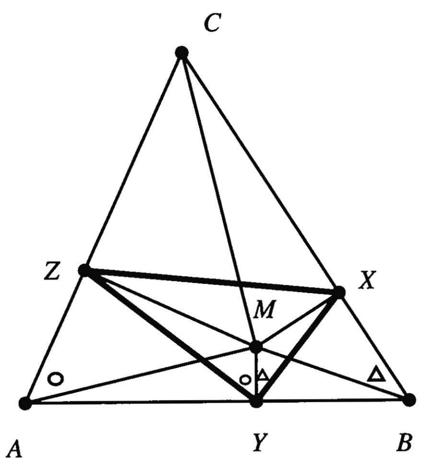
Solution by the Proposer.

We consider the case where $\angle A$ is acute. From point M drop perpendiculars MX, MY, and MZ onto sides BC, AB, and CA respectively, as shown in the figure. Now M, X, B, and Y lie on a circle with diameter $d = BM$. Then

$$(2) \qquad XY = d\sin B = BM\sin B$$

and similarly,

$$(3) \qquad ZX = CM\sin C \text{ and } YZ = AM\sin A.$$



Recall that the law of sines, as applied to triangle ABC, states

$$(4) \qquad \frac{\sin A}{BC} = \frac{\sin B}{CA} = \frac{\sin C}{AB}$$

Since $\angle MBY = \angle MXY$ and $\angle MCZ = \angle MXZ$, as shown in the figure by the spotted angles and the triangle-marked angles, then $\angle YXZ = \angle MBY + \angle MCZ = 90°$. Hence $(YZ)^2 = (XY)^2 + (ZX)^2$, into which we substitute equations 2 and 3, and then 4 and simplify to get

$$(AM\sin A)^2 = (BM\sin B)^2 + (CM\sin C)^2$$

and

$$(BC \cdot AM)^2 = (CA \cdot BM)^2 + (AB \cdot CM)^2.$$

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Murray S. Klamkin,** University of Alberta, Canada, and **William H. Peirce,** Rangeley, ME.

947. [Fall 1998] *Proposed by Paul S. Bruckman, Berkeley, California.*

In the card game of hearts, a regular deck of 52 cards is dealt to four players. An assigned player leads off, and tricks are taken by rules that need not concern us here. Each heart-suit card is assigned a value of 1 point, and the queen of spades has a value of 13 points; thus, the total value of each hand is 26 points. Your score for any hand is the sum of the points in the tricks you have taken. If one player, however, takes *all* 26 points in any hand, then that player is awarded 0 points and each of the other players is burdened with 26 points. The object of the game is to accumulate the fewest points. Hands continue to be played until at least one player has 100 or more points, at which time the player with the fewest points is declared the winner of the game. Ties are possible. Suppose the winner's total gain after a game is the total of the differences between his score and that of each other player. At $1 a point, what is the winner's maximum possible total gain per game?

Solution by **Katazyrna Potocka, Catherine Holl,** *and* **Kevin Weis,** *students, The College of New Jersey, Ewing, New Jersey.* In order to maximize the winner's gains, it is necessary to maximize the points of the other players while keeping the winner's points to a minimum. Ideally, then, if player A is to be the winner over players B, C, and D, the next to the last hand should leave the scores (0, 99, 99, 99) for (A, B, C, D), which is impossible since there is a total of 26 points per round. It is possible to achieve the scores (0, 97, 97, 92) by arriving at (0, 78, 78, 78) and having two hands with scores (0, 13, 6, 7) and (0, 6, 13, 7). In the last round, if A takes all 26 points, the final scores will be (0, 123, 123, 118) and A will win $364. Another scenario would be for each loser to have 99 points at the end of the next to the last round, so that the scores would have to be (15, 99, 99, 99). The final scores could then become (15, 125, 125, 125) if again A takes all 26 points. Here A would win only $3 \times (125 - 15) = \$330$. That is, a higher penultimate score for A leads to lower winnings. So the maximum winnings would be $364.

Also solved by **Mark Evans**, Louisville, KY, **Victor G. Feser**, University of Mary, Bismarck, ND, **Stephen I. Gendler**, Clarion University of Pennsylvania, **Grand Valley State University Problem Solving Group**, Allendale, MI, **Richard I. Hess**, Rancho Palos Verdes, CA, **Carl Libis**, Granada Hills, CA, **Tracy MacLake**, Alma College, MI, **Harry Sedinger**, St. Bonaventure University, NY, **Jeremy TerBush**, Alma College, MI, **Allison Topham**, Alma College, MI, **Kevin P. Wagner**, University of South Florida, Saint Petersburg, **Rex H. Wu**, Brooklyn, NY, **Kathleen Zellen**, Alma College, MI, and the **Proposer.**

948. [Fall 1998] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.* All six faces of a cube 4 inches on a side are painted red. Then the cube is chopped into 64 smaller 1-inch cubes. The "inside" faces are left unpainted. The 64 small cubes are put into a box and one is drawn at random, and tossed. Find the probability that when it comes to rest its upper face will be red.

I. *Solution by the* **Skidmore College Problem Group,** *Skidmore College, Saratoga Springs, New York.*

There are 8 little cubes which lie totally inside the large cube and hence have no red faces, 24 (the center 4 cubes on each of the 6 faces) with just one red face, 24 (the center 2 cubes on each of the 12 edges) with just 2 red faces, and 8 (the 8 corner cubes) with three red faces. The probability of obtaining a red face is thus

$$\frac{8}{64} \cdot \frac{0}{6} + \frac{24}{64} \cdot \frac{1}{6} + \frac{24}{64} \cdot \frac{2}{6} + \frac{8}{64} \cdot \frac{3}{6} = \frac{1}{4}$$

II. *Solution by* **Amy Kuiper,** *student, Alma College, Alma, Michigan.*

This is a two-step process, drawing and then rolling. You need only look, however, at the probability of selecting a painted face out of the total number of faces. Since there are 64 cubes, there are 384 total faces. There are 16 faces per side times 6 sides to the original cube, so 96 faces are painted. Therefore, the probability of rolling a painted face is $96/382 = 1/4$.

III. *Generalization by* **Grand Valley State University Problem Solving Group,** *Grand Valley State University, Allendale, Michigan.*

This problem may be generalized to an $n \times n \times n$ cube chipped into n^3 small cubes. Then $(n-2)^3$ small cubes have 0 red sides, $6(n-2)^2$ have 1 red side, $12(n-2)$ have 2, and 8 have 3 red sides, yielding a total of $6n^2$ red sides out of $6n^3$ total sides. The probability of rolling a red face is therefore $6n^2/6n^3 = 1/n$, which yields $1/4$ for the 4 by 4 by 4 cube.

All solvers used one of the methods in the featured solutions. Numerals following each name indicate which method(s) that solver used. Also solved by **Alma College Problem-Solving Group,** I, MI, **Charles D. Ashbacher,** I, Charles Ashbacher Technologies, Hiawatha, IA, **Frank P. Battles,** I, Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman,** I, Berkeley, CA, **George P. Evanovich,** I, Saint Peter's College, Jersey City, NJ, **Mark Evans,** II, Louisville, KY, **Victor G. Feser,** I, University of Mary, Bismarck, ND, **Stephen I. Gendler,** II, Clarion University of Pennsylvania, **Nitin Goil,** I, Northwest Missouri State University, Maryville, MO, **Grand Valley State University Problem Solving Group,** I, Allendale, MI, **Richard I. Hess,** II, Rancho Palos Verdes, CA, **Paul R. Krueger,** I, Alma College, MI, **Peter A. Lindstrom,** I, Batavia, NY, **Kevin Metz,** I, Alma College, MI, **Yoshinobu Murayoshi,** I, II, Okinawa, Japan, **Katazyrna Potocka, Catherine Holl,** and **Kevin Weis,** I, The College of New Jersey, Ewing, **Mike Reed,** I, II, Alma College, MI, **Shiva K. Saksena,** I, University of North Carolina at Wilmington, **Harry Sedinger,** I, St. Bonaventure University, NY, **H.-J. Seiffert,** I, Berlin, Germany, **Jamie Shirely,** I, II, Alma College, MI, **Skidmore College Problem Group,** II, Saratoga Springs, NY, **Jeremy TerBush,** I, Alma College, MI., **Kevin P. Wagner,** II, University of South Florida, Saint Petersburg, **Dana Weston,** I, Alma College, MI, **Rex H. Wu,** I, Brooklyn, NY, and the Proposer.

949. [Fall 1998] *Proposed by Charles Ashbacher, Decisionmark, Cedar Rapids, Iowa.*

In a collection of problems edited by Dumitrescu and Seleacu [1] a positive integer is said to be a *Smarandache pseudo-odd(even) number* if some permutation of its digits is odd(even). For example, 12345678 is both Smarandache pseudo-even and pseudo-odd since 12456783 is odd. A positive integer is said to be a *Smarandache pseudo-multiple* of the positive integer k if some permutation of its digits is divisible by k.

a) Prove that if a positive integer is chosen at random, the probability that it is Smarandache pseudo-odd is 1.

b) Prove that if a positive integer is chosen at random, the probability that it is Smarandache pseudo-even is 1.

c) Prove that if a positive integer is chosen at random, the probability that it is a Smarandache pseudo-multiple of 3 is 1/3.

d) Prove that if a positive integer is chosen at random, the probability that it is a Smarandache pseudo-multiple of 5 is 1.

Reference:

1. C. DUMITRESCU AND V. SELEACU, *Some Notions and Questions in Number Theory,* Erhus University Press, 1994.

Solution by **Stephen I. Gendler**, *Clarion University, Clarion, Pennsylvania.*

Any base ten numeral of n digits has the probability $(9/10)^n$ that it does not contain the digit d. As $n \to \infty$, $(9/10)^n \to 0$. Hence we have the following lemma.

LEMMA. *The probability that a number does not contain a specific digit d is 0 and hence the probability it does contain that digit is 1.*

a, b, d) An integer is pseudo-odd if it contains at least one odd digit, it is pseudo-even if it contains at least one even digit, and it is a pseudo-multiple of 5 if it contains at least one 0 or 5. By the lemma the probability of any of these three events is 1.

c) The sum of the digits of a number is 0, 1, or 2 (mod 3), each with equal probability 1/3. A number is pseudo-divisible by 3 if and only if it is divisible by 3 if and only if the sum of its digits is 0 (mod 3), so that probability is 1/3.

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Mark Evans,** Louisville, KY, **Richard I. Hess,** Rancho Palos Verdes, CA,, **Harry Sedinger** and **Doug Cashing,** St. Bonaventure University, NY, **Rex H. Wu,** Brooklyn, NY, and the **Proposer.**

951. [Fall 1998] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

An ant crawls along the surface of a *dicube,* a $1 \times 1 \times 2$ rectangular block.

a) If the ant starts at a corner, where is the point farthest from it? (It is not the opposite corner!)

b) Find two points that are farthest apart from each other on the surface of the dicube.

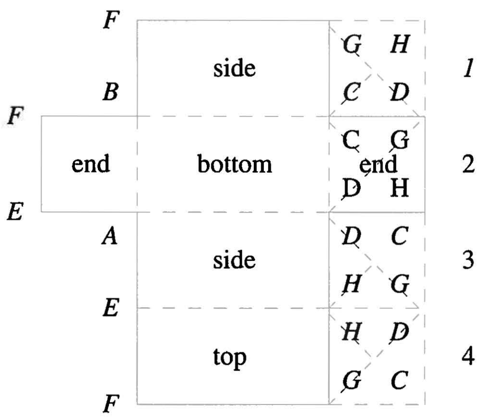


Figure for Problem 951a by Meagan Tripp

Solution to part (a) by **Meagan Tripp**, *Alma College, Alma Michigan.* Let us start from point A in the dicube $ABCDEFGH$, where G is the opposite corner. Cut the dicube open as shown in the figure, showing the various positions, 1, 2, 3, and 4, of the opposite end $CDGH$ in relation to each of the four sides. Thus we can see all possible direct paths to any point on the opposite end. We know that the distance from A to G, $2\sqrt{2}$ according to position 1 or position 4, is not the farthest. Since the corners H (see position 4), D (position 2), and C (position 1) and the edges GC, CD, DH, and HG are closer to A than G is, we are left to consider points along the diagonal DG. Since in positions 1 and 4 angle AGD is a right angle, points near G on diagonal DG are farther from A than $2\sqrt{2}$. Using positions 2 and 3 we find that distance AG is $\sqrt{10}$, which is greater than $2\sqrt{2}$, verifying that points along DG that are close to G are indeed further from A than G is. The distance AP from A to a point P on diagonal DG, as shown in the flap end 2, is $\sqrt{(2+x)^2+x^2}$, where x is the distance from D to the projection of P on DH. As x increases, the distance AP increases from $2\sqrt{2}$. Similarly, the distance AP, using position 1, is $\sqrt{+2(1-x)^2}$ since $GP = (1-x)\sqrt{2}$. Here, as x increases from 0 to 1, AP decreases from 10. The point of longest distance from A along DG is the point where these two distances AP are equal, which occurs at $x = 0.75$. From positions 1 and 4 we see that points near P but off the diagonal DG are closer to A than P is. Therefore, the point on the surface of the dicube furthest from A lies 1/4 of the way from the opposite vertex G along the end diagonal GD. That distance is $\sqrt{2.75^2 + .75^2} \approx 2.85044$.

II. Solution to part (b) by **Rex H. Wu**, *Brooklyn, New York.*

Observation: Let $[a,b]$ be an interval and x_0 a point in $[a,b]$. Let $f(x)$ be an increasing function and $g(x)$ a decreasing function on $[a,b]$ such that $f(x_0) = g(x_0)$. Then on the interval $[a,b]$ $\min\{f,g\} = f(x)$ for $x < x_0$ and $\min\{f,g\} = g(x)$ for $x \geq x_0$. Furthermore, $\max\{\min\{f,g\}\} = f(x_0)$. That is, the smaller of f and g has its largest value at the point where the two curves meet. The centers of the two 1×1 squares are exactly 3 units apart traveling on the surface of the dicube. Likewise, consider a rectangle that is the intersection of a horizontal plane and the dicube. Any point on the unit edge of the rectangle has just one other point on that rectangle that is 3 units away, traveling on the edges of the rectangle, the point symmetrically located in the opposite edge. No point on the rectangle is further away. Two points more than 3 units apart, then, cannot lie in one horizontal plane.

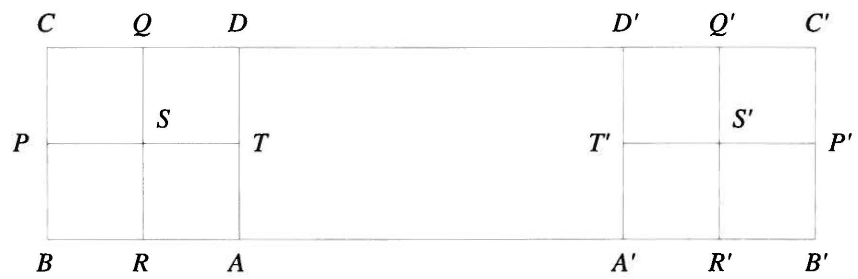


FIG. 1.

Label the vertices of the dicube A, B, C, D, A' , B' , C' , D' so that $ABCD$ is one square, $A'B'C'D'$ is the other square, and AA', BB', CC', DD' are edges each two units long. See Figure 1, which shows the ends and one side of the sliced-open dicube. Let the points we are seeking be M and N, with M on the $ABCD$ square. Divide that square into four $\frac{1}{2}$ by $\frac{1}{2}$ subsquares. Point M must lie in or on one of those subsquares. We may assume without loss of generality that it is the subsquare containing A. Label the vertices of the subsquares as shown in the figure.

There are three cases to consider. We start with Case 1, in which point N is in the subsquare $A'R'S'D'$. Using Figure 1, draw four circular arcs $R'V'$, $S'U'$, RV, and SU, centered at points R, S, R', and S', each of radius 3. Figure 2 shows those arcs in just that portion of Figure 1 in which the two subsquares containing M and N lie.

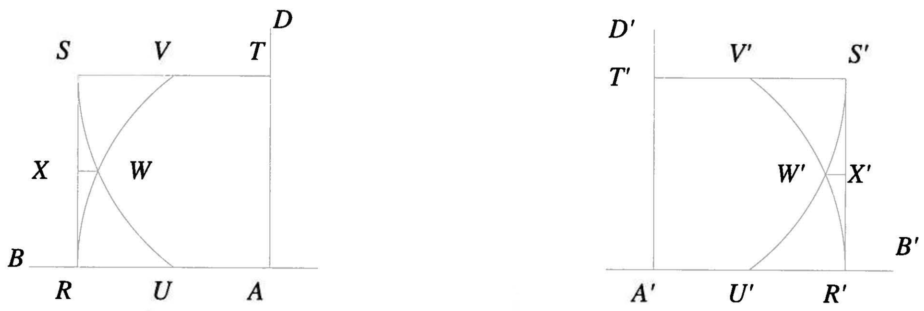


FIG. 2.

(Note that the actual distance RU, for example, is less than 0.042 units, although for the sake of clarity it is drawn much larger in the figure.) If M and N lie inside or on those circles, they are no more than 3 units apart. Hence we must take at least one of them, say M, outside a circle, that is, within a circular "triangle," say RSU. Then N must lie within the "pentagon" $R'S'V'W'U'$.

Figures 1 and 2 show the orientation of the squares when rectangle $ADD'A'$ is left attached to them. Figure 3 shows the orientation of the subsquares of Figure 2 when rectangle $ABB'A'$ is left attached to them. If M lies inside "quadrilateral" $RUWX$ or if N lies inside $R'U'W'X'$, then the distance MN is less than 3 units. Hence M lies in "triangle" SWX and N must lie in "quadrilateral" $S'X'W'V'$. From Figure 3 we see that M must lie within about 0.01 of S and N must lie close to $S'V'$. Figure 2 shows that such points are less than 3 units apart. Hence N does not lie in subsquare $A'R'S'T'$.

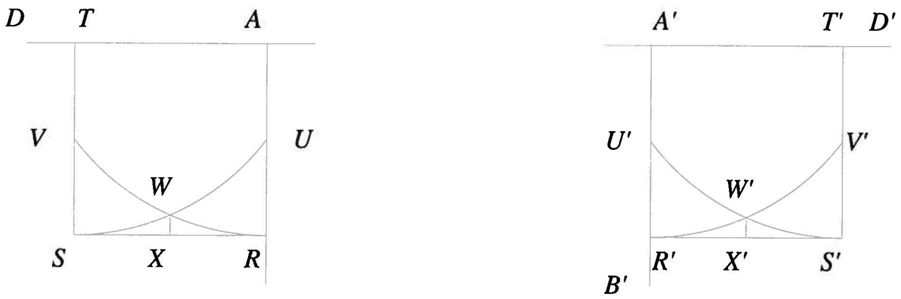


FIG. 3.

Case 2 has point N in the subsquare $D'Q'S'T'$ of Figure 1. We draw circular arcs of radius 3 in these subsquares, arc $V'W'$ centered at R and arc UV centered at Q', as shown in Figure 4. Figure 5 shows the three other orientations of these two subsquares. By an argument similar to that of Case 1, we see that the optimum locations for M and N are at R and 1/6 unit from Q' toward D'. The distance MN then is

$$MN = \sqrt{3^2 + (1/6)^2} \approx 3.0046.$$

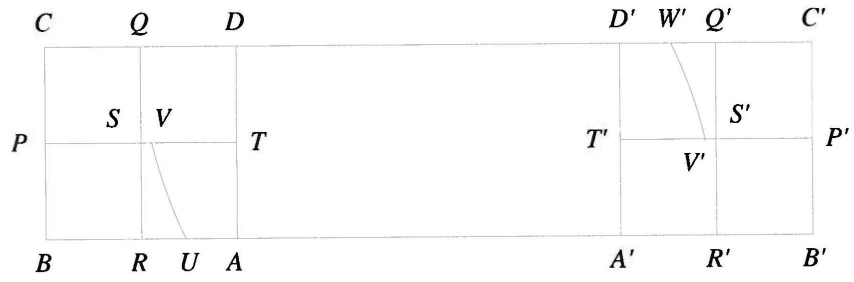


FIG. 4.

In Case 3, point N lies in the subsquare $C'P'S'Q'$. Consider points M_1 and N_1 with distances w, x, y, and z from the sides of their subsquares, as shown in Figure 6. Note the distances D_1, D_2, D_3, and D_4, which show some of the possible routes the ant might take. Note that the top two orientations duplicate the bottom two. To maximize the horizontal distances at 3 units, distances D_1 and D_2 show that we must have $w = x$ and $y = z$. Furthermore, $x, y \in [0, \frac{1}{2}]$.

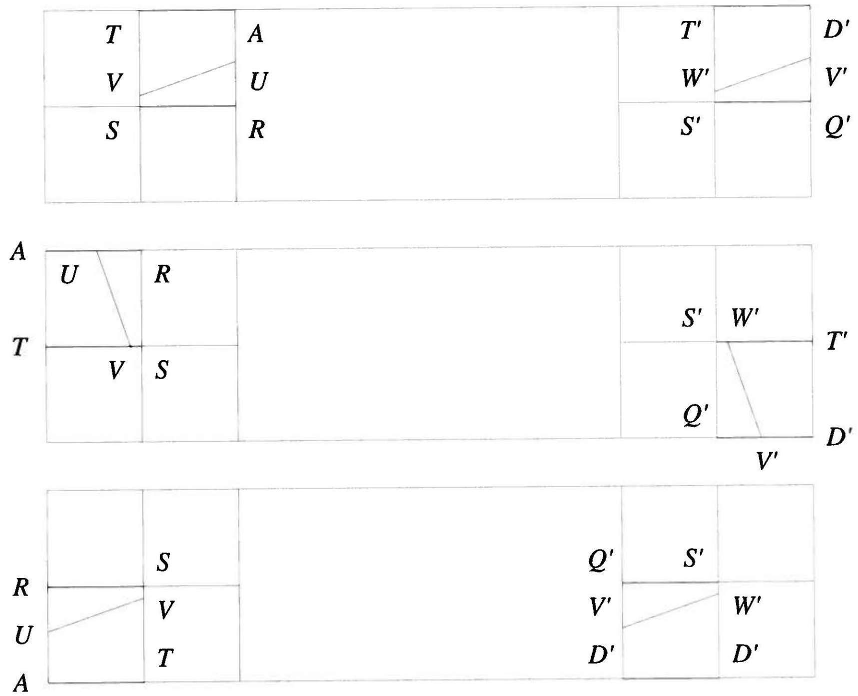


FIG. 5.

We have the distances

$$D_1^2 = (1 - 2y)^2 + 3^2,$$
$$D_2^2 = (2x)^2 + 3^2,$$

$$D_3^2 = (3/2 + x - y)^2 + (5/2 + y - x)^2, \text{ and}$$
$$D_4^2 = (7/2 + x - y)^2 + ((1/2) - x + y)^2.$$

Using our Observation, we see that the optimum distance occurs when $D_1 = D_2$, when $x + y = \frac{1}{2}$. Thus M and N will lie on the diagonals through A and C' respectively.

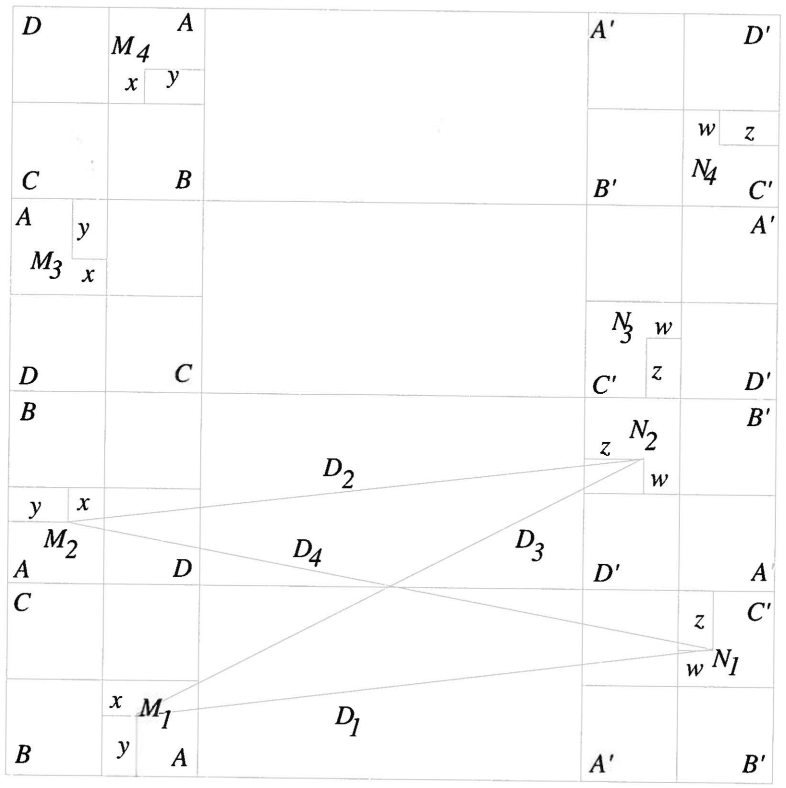


FIG. 6. *(by Rex Wu)*

Substitute $x = \frac{1}{2} - y$ into the expression for D_3 to get $D_3^2 = 8y^2 + 8$.

Again use our Observation to set $D_1 = D_3$ to get $2y^2 + 2y - 1 = 0$, so that $y = (\sqrt{3} - 1)/2$ and $x = 1 - \sqrt{3}/2$.

Finally,

$$D_1^2 = (1 - 2y)^2 + 3^2 = (2 + \sqrt{3})^2 + 3^2 = 16 - 4\sqrt{3},$$

so that $D_1 = \sqrt{16 - 4\sqrt{3}} \approx 3.011942358$.

We can use this same technique in a $1 \times 1 \times n$ box to get that the final coordinates for M and N are

$$(x, y) = \left(\frac{n - \sqrt{n^2 - 1}}{2}, \frac{1 - n + \sqrt{n^2 - 1}}{2} \right)$$

Also solved by **Rex H. Wu,** part (a), Brooklyn, NY, and the **Proposer.** Two incorrect solutions were also received. The proposer has written an unpublished paper entitled "Kotani's Ant Problem," which generalizes the problem to a $1 \times a \times b$ box with rather interesting results.

952. [Fall 1998] *Proposed by Peter A. Lindstrom, Batavia, New York.*

Let A, B, C denote the measures of the angles and a, b, c the lengths of the opposite sides of a triangle. Show that

$$\sin A \sin B + \sin B \sin C + \sin C \sin A$$
$$= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)(bc+ca+ab)}{4a^2b^2c^2}.$$

I. *Solution by* **Kevin P. Wagner**, *student, University of South Florida, Saint Petersburg, Florida.*

By the laws of sines and cosines, we have $(\sin A)/a = (\sin B)/b = (\sin C)/c$ and $\cos C = (a^2 + b^2 - c^2)/2ab$, so that

$$\sin^2 C = 1 - \left(\frac{a^2 + b^2 - c^2}{2ab} \right)^2 = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4a^2b^2}.$$

Substituting this value into the right side RHS of the desired equation yields

$$\text{RHS} = \frac{ab + ac + bc}{c^2} \sin^2 C = \frac{ab \sin^2 C}{c^2} + \frac{a \sin^2 C}{c} + \frac{b \sin^2 C}{c} = \text{LHS}$$

by using the law of sines.

II. *Solution by* **Grand Valley State University Problem Group**, *Grand Valley State University, Allendale, Michigan.*

The area K of the triangle is given by $K = (1/2)bc \sin A$, so that

$$\sin A = \frac{2K}{bc}$$

and two similar expressions for $\sin B$ and $\sin C$. Recall Heron's formula,

$$K^2 = \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{16}$$

Now we have

$$\sin A \sin B + \sin B \sin C + \sin C \sin A$$
$$= \frac{4K^2}{a^2bc} + \frac{4K^2}{ab^2c} + \frac{4K^2}{abc^2}$$
$$= \frac{4K^2(ab + bc + ac)}{a^2b^2c^2}$$
$$= \frac{(a+b+c)(a+b-c)(b+c-a)(b+c-a)(ab+bc+ac)}{4a^2b^2c^2}$$

Also solved by **Miguel Amengual Covas,** Cala Figuera, Mallorca, Spain, **Scott H. Brown,** Auburn University at Montgomery, AL, **Paul S. Bruckman,** Berkeley, CA, **William Chau,** AT&T Laboratories, Middletown, NJ, **Chantel Cleghorn, Hardin Dunham,** and **Daniel Hermann,** Angelo State University, San Angelo, TX, **Erin Cooper,** Alma College, MI, **Russell Euler** and **Jawad Sadek,** Northwest Missouri State University, Maryville, **George P. Evanovich,** Saint Peter's College, Jersey City, NJ, **Mark Evans,** Louisville, KY, **Richard I. Hess,** Rancho Palos Verdes, CA, **Joe Howard,** New Mexico Highlands University, Las Vegas, **Edward John Koslowska,** Southwest Texas Junior College, Eagle Pass, **Henry S. Lieberman,** Waban, MA, **David E. Manes,** SUNY College at Oneonta, NY, **Yoshinobu Murayoshi,** Okinawa, Japan, **William H. Peirce,** Rangeley, ME, **Shiva K. Saksena,** University of North Carolina at Wilmington, **H.-J. Seiffert,** Berlin, Germany, **Rex H. Wu,** Brooklyn, NY, **Monte J. Zerger,** Adams State College, Alamosa, CO, and the **Proposer.**

Late solution to Problem 934 by Andrew Ostergaard, student, Hopatcong, NJ.

MATH ACROSTIC

The MATHACROSTIC in this issue has been contributed by Gerald Liebowitz.

a. An ancient people — 183 166 047 094 176 132 194 150

b. Rose lover — 225 153 084 163

c. Author of "direct method" — 187 030 024 174 073 180 042 055

d. Type of integral — 012 027 167 039 082 178 171 103

e. $B^2 - 4AC$ — 089 006 031 044 216 017 204 192 147 161 209 224

f. Parabola (right?) — 202 033 164 050 179 064 029 231 121 072 014 197 099 213 115

g. Memo — 043 059 157 127

h. Staff reduction method — 205 058 070 170 079 111 093 066 128

i. See word x. — 092 087 134 199 156 025 222

j. $x = 3$ when $t = 0$, (with k.) — 227 219 207 011 074 019 191

k. See j. — 041 002 162 212 220 230 154 201 193

l. Not easily perturbed. — 049 020 195 226 155 215

m. With "to", perfectly. — 010 131 100 141

n. Ephermeral publication. — 102 186 097 069 057 107 114 159 003 125 140 129 053 081 117

o. Spread of fluid. — 181 076 061 149 165 120 101 083 110

p. Name associated with Poincarè. — 068 056 124 106 008 130 048 189 211

q. Attack verbally. — 091 142 022 184 091 032 126

r. $\omega_j/2\pi$. — 160 060 036 013 105 188 221 118 135 071 138

s. One to whom all has value. — 172 182 210 228 109 009 158 086 052 045 123 067 116 095 144

t. Solution. — 196 143 175 151

u. Where to search for a missing tangent plane. — 005 185 145 037 122 136 098 075 198 133 218 088 206

v. Risk measuring adverb. — 004 040 139 080 152 200 113 015 090 104 028

w. See x. — 001 112 063 208 038 096 217

x. Jacobian result. (with i. and w.) — 229 035 046 007 168 051 203 018

y. Iteration paths. — 148 078 169 021 214 054

z. Traveling Salesmen stop here — 177 146 108 023

*. Non-tangled smooth subset. — 223 119 077 173 026 085 137 190 034 016 065

001w	002k		003n	004v	005u	006e	007x	008p	009s	010m	011j	012d		013r
014f	015v	016*	017e	018x	019j	020l	021y	022q	023z		024c	025i	026*	027d
028v	029f	030c	031e		032q	033f	034*	035x	036r	037u	038w	039d	040v	
041k	042c	043g	044e	045s	046x	047a	048p		049l	050f	051x	052s		053n
054y		055c	056p	057n	058h	059g	060r		061o	062q	063w	064f	065*	
066h	067s	068p	069n	070h		071r	072f	073c	074j	075u	076o	077*	078y	079h
080v	081n		082d	083o	084b	085*	086s		087i	088u	089e		090v	091q
092i	093h	094a		095s	096w	097n		098u	099f	100m		101o	102n	103d
104v	105r	106p	107n	108z		109s	110o		111h	112w	113v	114n		115f
116s	117n	118r	119*	120o	121f	122u	123s	124p		125n	126q	127g		128h
129n	130p	131m		132a	133u	134i	135r	136u		137*	138r		139v	140n
141m		142q	143t	144s	145u	146z	147e		148y	149o		150a	151t	152v
153b	154k	155l	156i	157g	158s		159n	160r		161e	162k		163b	164f
165o	166a	167d	168x	169y	170h	171d	172s	173*		174c	175t	176a	177z	178d
	179f	180c	181o		182s	183a	184q		185u	186n	187c	188r		189p
190*		191j	192e	193k	194a	195l	196t		197f	198u	199i	200v	201k	202f
203x	204e	205h	206u	207j	208w	209e		210s	211p		212k	213f	214y	215l
216e	217w	218u	219j	220k	221r	222i		223*	224e	225b	226l	227j	228s	229x
230k	231f													

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