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Problem 993.

The ΠΜΕ Journal

Volume 11, Number 2 Spring 2000

The ΠME Journal is published biannually, once in the spring and once in the fall. Each volume consists of ten issues.

Current rates are as follows:

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	\$40 for 5 years
Foreign:	\$25 for 2 years
Back issues:	\$5 each
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TIME AND ITS INVERSE

SCOTT J. BESLIN AND ROBERT LEDET*

1. Introduction. The minute hand on a clock is redundant, since in principle the position of the hour hand determines the time. The angle between the minute hand and the 12 o'clock mark is always twelve times the angle between the hour hand and the position of the preceding hour mark. For example, at 2:24, the angle between the minute hand and 12 o'clock is $\frac{24}{60}(360°) = 144°$, and the angle between the hour hand and the 2 o'clock mark is $\frac{24}{60}(30°) = 12°$. A class discussion of clock arithmetic in an abstract algebra class led to the following intriguing question: *At what times can the positions of the clock hands be switched and the new configuration also represent a possible clock time?* Other teachers may find this problem useful for reviewing the main results about finite groups in a typical undergraduate course on abstract algebra.

Let $\mathbb{R}$ denote the additive group of real numbers and $12\mathbb{Z}$ the subgroup consisting of integers that are multiples of 12. The factor group $G = \mathbb{R}/12\mathbb{Z}$ may be thought of as a "clock" group whose elements correspond to times on a 12-hour clock and whose addition is real "clock addition" modulo 12. Hence if $t \epsilon G, t$ may be expressed as

$$i + x$$

where $i \epsilon \{0, 1, 2, ..., 11\}$ and $0 < x < 1$.

In other words, $t = 0$ corresponds to "12 o'clock," and $i = \lfloor t \rfloor$, the integer floor of t, represents the "hour" part of t as time. For example, the time $t = 2.8$ has $i = 2$ and $x = .8$, and may be realized as 2:48:00 on a 12-hour clock.

Two elements of G are added modulo 12. For example,

$$\begin{aligned}
(3 + .99) &\oplus (8 + .98) \\
&= (3 + 8) + (.99 + .98) \\
&= 11 + 1.97 \\
&= (11 + 1) + .97 \\
&= 0 + .97
\end{aligned}$$

The symbol $\oplus$ represents addition in G.

Re-interpreting this chain of equalities on a 12-hour clock, we have that

$$3 : 59 : 24 \oplus 8 : 58 : 48 = 12 : 58 : 12.$$

In general, if $t = i + x$ and $T = j + y$ are two elements of G, then

$$t \oplus T = (i + x) \oplus (j + y) = \begin{cases} \overline{i + j} + (x + y) & \text{if } x + y < 1 \\ \overline{i + j + 1} + (x + y - 1) & \text{if } x + y \geq 1 \end{cases}$$

Here the vinculum or overbar indicates integer addition modulo 12..

In the remainder of this note, we will investigate an interesting subgroup of G and exhibit its direct sum decomposition. Our investigation begins with the question stated earlier: how many times can the positions of the hour and minute hands of a

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clock be reversed and still define a possible clock time? We choose to call such times "invertible" times, whence the title of this article. For example, the time 12:00:00 is an obvious "fixed point" invertible time.

We will show that the set S of all such invertible times is a finite subgroup of G. We will then determine its structure.

2. Representation. The angles (in radians) with 12 o'clock formed by the hour (h) and minute (m) hands of a clock at time $t = i + x$ are $h = \pi t/6$ and $m = 2\pi(t - \lfloor t \rfloor) = 2\pi(t - i) = 2\pi x$. If the hands are switched, yielding new hour (H) and minute (M) positions which correspond to time $T = j + y$, then $H = \pi T/6$ and $M = 2\pi(T - j) = 2\pi y$. If such a switch results in an accurate time, then $H = m$. So

$$\frac{\pi}{6} T = 2\pi x, \text{ or } \frac{T}{6} = 2x. \tag{1}$$

Similarly, $h = M$; hence

$$\frac{t}{6} = 2y. \tag{2}$$

Using $t = i + x$ and $T = j + y$ in (1) and (2), and solving, we obtain

$$x = \frac{i + 12j}{143} \text{ and } y = \frac{12i + j}{143}. \tag{3}$$

The symmetry in these solutions is understandable since if t is an invertible time, then so is its inverted time T.

Fixed point solutions, when switching the hands produces not just a possible time but the same time, occur when $i = j$. In these cases $x = y = \frac{i}{11}$, from (3). Thus fixed points correspond to times $t = i + \frac{i}{11}$. For example, $2 + \frac{2}{11}$ (approximately 2:10:54 on the clock) is the fixed point solution between 2 o'clock and 3 o'clock. There are obviously eleven fixed point solutions.

For non-fixed point solutions, select $j \epsilon \{0, 1, 2, ..., 11\}$, suppose $i < j$, and apply (3). Indeed,

when $j = 1, i = 0$ yielding one solution from (3);
when $j = 2, i = 0$ or 1, yielding two solutions from (3);
when $j = 3, i = 0, 1,$ or 2, yielding three solutions from (3);
⋮
when $j = 11, i \epsilon \{0, 1, 2, ...10\}$, yielding eleven solutions from (3).

So the total number of solutions in which $i < j$, is $1 + 2 + 3 + ... + 11 = 66$.

Similarly, by symmetry, there are 66 solutions for $i > j$, yielding $66 + 66 + 11 =$ 143 total solutions, fixed and non-fixed.

For instance, consider the non-fixed point solution obtained by letting $j = 1$ and $i = 0$. Then $x = \frac{12}{143}$ and $y = \frac{1}{143}$. Hence, $t = \frac{12}{143}$ hours and $T = 1\frac{1}{143}$ hours. Approximate clock times are 12:05:02 and 1:00:25.

As another example, suppose we wish to find the solution corresponding to the missing blanks

7:___:___ and 5:___:___.

Then $j = 7$ and $i = 5$. Calculating x and y from (3) and then approximating, we get

7 : 28 : 07 and 5 : 37 : 21.

3. The Subgroup S. Amazingly enough, the set S of 143 invertible times forms a finite subgroup of G. Moreover, since $143 = 11 \cdot 13$, the product of two distinct primes, and since G is abelian, the group S must be cyclic!

To see that S is a subgroup, observe from (3) that an element of S may be expressed as an ordered pair $(i, j) \epsilon \{0, 1, 2, \dots, 11\}\{0, 1, 2, \dots, 11\}$; namely, (i, j) represents the time $i + \frac{i+12j}{143}$ hours. The inverted time of pair (i, j) is the pair (j, i). Note that the pair $(0, 0)$ is equivalent to the pair $(11, 11)$. [Hence S has 143 elements.] From the addition $\oplus$ in G described in Section 1, we obtain

THEOREM 1. *Suppose (i, j) and (c, d) are in S. Let $u = j + d + \lfloor \frac{i+c}{12} \rfloor$. Then:*

$$(i, t) \oplus (c, d) = \begin{cases} (\overline{i + c}, u) & if\, u \leq 11; \\ (i + c + 1, u - 12 & if\, \overline{i + c} < 11, (u > 11) \\ (0, u - 11) & if\, \overline{i + c} = 11, (u > 11) \end{cases}.$$

The proof of the theorem is lengthy, but involves writing $i + c$ as $\overline{i + c} + 12 \lfloor \frac{i+c}{12} \rfloor$ and then further using the division algorithm when necessary. We illustrate the technique with four examples.

Example 1. $(2, 5) \oplus (10, 9)$.

Here $u = 5 + 9 + \lfloor \frac{2+10}{12} \rfloor = 15 > 11$ and $\overline{i + c} = 0$. Therefore the sum is (1,3). To see this, note that

$$\begin{aligned} (2, 5) \oplus (10, 9) &= \left(2 + \frac{2 + 5(12)}{143}\right) \oplus \left(10 + \frac{10 + 9(12)}{143}\right) \\ &= \overline{2 + 10} \oplus \frac{2 + 10 + 14(12)}{143} \\ &= 0 \oplus \frac{15(12)}{143} \\ &= 0 \oplus \left(1 + \frac{15(12) - 143}{143}\right) \\ &= 0 \oplus \left(1 + \frac{15(12) - 12(12) + 1}{143}\right) \\ &= (0 \oplus 1) \oplus \frac{12(15 - 12) + 1}{143} \\ &= 1 \oplus \frac{1 + 3(12)}{143} = (1, 3). \end{aligned}$$

In other words, the approximate clock times 2:26:01 and 10:49:31 sum to the approximate time 1:15:32.

Example 2. $(2, 3) \oplus (1, 4)$.

Here $u = 7 \leq 11$, so the sum is $(3, 7)$. Clearly $\left(2 + \frac{2+3(12)}{143}\right) \oplus \left(1 + \frac{1+4(12)}{143}\right) = 3 + \frac{3+7(12)}{143}$.

Example 3. $(5, 6) \oplus (6, 5)$.

The summands are time inverses. Note $u = 11$, and hence the sum is $(11, 11) = (0, 0)$.

Example 4. $(3, 6) \oplus (8, 7)$.

Here $u = 13 > 11$ and $\overline{i + c} = 11$. Observe

$$\left(3 + \frac{3 + 6(12)}{143}\right) \oplus \left(8 + \frac{8 + 7(12)}{143}\right)$$

$$
\begin{aligned}
&= 11 \oplus \frac{11 + 13(12)}{143} \\
&= 11 \oplus \left(\frac{11}{143} + \frac{13(12) - 143}{143} + 1\right) \\
&= (11 \oplus 1) \oplus \left(\frac{11}{143} + \frac{13(12) - 11(12) - 11}{143}\right) \\
&= 0 \oplus \left(\frac{0 + 2(12)}{143}\right) \equiv (0, 2).
\end{aligned}
$$

Since fixed points may be represented as (i, i), it is easy to verify

THEOREM 2. *The set F of eleven fixed points in S is a cyclic subgroup of S.*

At this point, we emphasize that group inverses in S are not necessarily *time* inverses. The group inverse of (i, j) is $(11 - i, 11 - j)$ since $(i, j) \oplus (11 - i, 11 - j) = (11, 11) = (0, 0)$. From (3), the time inverse of (i, j) is (j, i). Hence a non-identity element of S whose group inverse coincides with its time inverse has the form $(i, 11 - i)$ for $i \epsilon \{0, 1, 2, \ldots, 11\}$.

THEOREM 3. *The set $E = \{(i, 11 - i) \mid i = 0, 1, 2, \ldots, 11\} \cup \{(0, 0)\}$ consisting of the 13 elements of S whose group inverses are time inverses is a cyclic subgroup of S.*

Proof. Consider the cyclic subgroup H of S generated by $(11, 0) \epsilon E$. It is easy to verify via Theorem 1 and finite induction that, for $n = 1, 2, 3, \ldots, 12$, $n(11, 0) = (11, 0) \oplus (11, 0) \oplus \ldots \oplus (11, 0)$ (with n summands) is equal to $(11 - (n - 1), n - 1)$. Further, $(0, 11) \oplus (11, 0) \equiv (0, 0)$. Thus $H = E$. □

From elementary abelian group theory (see [1], Sylow's Theorem, or the Fundamental Theorem of Finite Abelian Groups), a cyclic group of order $143 = 11 \cdot 13$ has a unique subgroup of order 11 and a unique one of order 13. In the case of S, these are F and E, respectively. Hence we have

THEOREM 4. *The group S is the internal direct product of its subgroups F and E.*

Thus it may be said that S is generated by its fixed points and by elements whose inverses are time inverses.

Finally, we pose the following problem as a student exercise: Is there a meaningful operation $*$ on S such that $(S, *)$ is a group, and group inverses coincide with time inverses for all elements of S?

Acknowledgements. The authors thank Prof. Bart Braden, Northern Kentucky University, for his helpful suggestions. The authors also acknowledge the editor and the referee for their observations.

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ΠΜΕ Journal, Vol. 11, No. 2, pp 69–74, 2000.

SQUARE ROOTS AND CALCULATORS

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Have you ever wondered just what happens inside a pocket calculator when you press the square root key? What formula does a calculator use to compute $\sqrt{x}$? It is tempting to believe that there a little person who rapidly looks up the desired value in a tiny table and then displays the result. Besides, then we can blame him or her when we get a wrong answer.

Alas! There is no such little person; only some very tiny electronics. And all of this is powered by about 1/1000 of the electrical energy required to light a flashlight. We shall examine how square roots are found in calculators, but first we shall review two square root algorithms, the *long division* method some of us learned in high school and Newton's *divide and average* method which is taught in many calculus classes.

First we calculate $\sqrt{18468.81}$ by the long division method. We group the digits by twos in both directions from the decimal point, obtaining $\sqrt{1\ 84\ 68.81}$. Now find the largest square number that does not exceed the number formed by the leftmost group (or two groups). Here we have $13^2 = 169 < 184 < 196 = 14^2$. We write the 13 in the quotient space above the 184 and subtract the 169 below thus:

$$
\begin{array}{r}
1\ 3 \\
\sqrt{1\ 84\ 68.81} \\
1\ 69 \\
\hline
15
\end{array}
$$

We have found that $130 < \sqrt{18468.81} < 140$.

Letting $n = 18468.81$ and $a = 130$, we seek b such that $(a+b)^2 = n$. The division process so far indicates that

$$n - a^2 = (a + b)^2 - a^2 = 2ab + b^2 = 1568.81.$$

So we now seek a suitable value for b. We find just one digit at a time, so we take b to be just a single digit filling the next decimal place. Thus we look for the largest digit b such that $(130 + b)^2 \le 18468$, that is, we find b such that

$$2ab + b^2 = b(2a + b) \le 1568.$$

Because b is relatively small compared to $a = 130$, we solve the simpler inequality

$$b(2a + 0) = 2ab \le 1568,$$

which should introduce very little error. To that end, bring down the next group 68 of two digits, double the divisor 13 (which is actually 130) and write the resulting 26, leaving space for the missing digit zero (in the 260) as a trial divisor for the 1568, as shown in the display below:

$$
\begin{array}{rr}
 & 1\ 3 \\
 & \sqrt{1\ 84\ 68.81} \\
 & 1\ 69 \\
\hline
26 & 15\ 68 \\
\hline
\end{array}
$$

---

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Now 260 will go into 1568 (or 26 into 156) six times, so we try $b = 6$. Write the 6 in the quotient over the 68, beside the 26 and directly below to use as a multiplier. Thus we arrive at $(2a + b) \times b$. We obtain:

$$\begin{array}{r|l}
 & 1\;3\;6 \\
 & \sqrt{1\,84\,68.81} \\
 & 1\,69 \\
\hline
266 & 15\,68 \\
6 & \\
\hline
\end{array}$$

Now $266 \times 6 = 1596$, which is too large, as happens occasionally because of our trial divisor (260) was smaller than the actual final divisor (266). Thus we must reduce our guess for b from 6 to 5. So replace the three sixes with fives, multiply the 265 by the 5, and subtract to get:

$$\begin{array}{r|l}
 & 1\;3\;5 \\
 & \sqrt{1\,84\,68.81} \\
 & 1\,69 \\
\hline
265 & 15\,68 \\
5 & 13\,25 \\
\hline
 & \;\;2\,43
\end{array}$$

We now know that

$$135^2 < 18468.81 < 136^2,$$

so we take $a = 135$ and search for b as a digit in the tenths place, repeating the process again. We wrote the 265 and 5 in convenient places for multiplying, but their locations are perfect also for adding to obtain the new value of $2a$. Then bring down the next group of two digits, thus:

$$\begin{array}{r|l}
 & 1\;3\;5. \\
 & \sqrt{1\,84\,68.81} \\
 & 1\,69 \\
\hline
265 & 15\,68 \\
5 & 13\,25 \\
\hline
2709 & \;\;2\,43\,81 \\
 & \\
\hline
\end{array}$$

Now 270 divides 2438 nine times, so we try $b = 9$, and the process terminates, as we see below:

$$\begin{array}{r|l}
 & 1\;3\;5.\;9 \\
 & \sqrt{1\,84\,68.81} \\
 & 1\,69 \\
\hline
265 & 15\,68 \\
5 & 13\,25 \\
\hline
2709 & \;\;2\,43\,81 \\
9 & \;\;2\,43\,81 \\
\hline
 & \qquad\quad 0
\end{array}$$

We have found that $\sqrt{18468.81} = 135.9$ by the long division method, a technique that seems quite long and arduous. The interested reader might develop the corresponding

algorithm for finding cube roots using the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Newton's divide-and-average method uses the simple idea that since $2 \times 3 = 6$, then the square root of 6 must lie between 2 and 3 and the best linear approximation to $\sqrt{6}$ is the arithmetic average of 2 and 3, that is, 2.5. Since $(2.5)^2 = 6.25$ and not exactly 6, we divide 6 by 2.5 to find that $2.5 \times 2.4 = 6$. The average of 2.5 and 2.4 is 2.45. Then $(2.45)^2 = 6.0025$. Now divide 6 by 2.45 and average the quotient and divisor to get our next approximation

$$\frac{6/2.45 + 2.45}{2} = \frac{2.448797592 + 2.45}{2} = 2.44948796,$$

and $(2.449489796)^2 = 6.000000261$. We have obtained a very good approximation to $\sqrt{6}$ in just four applications of Newton's method. We write $\sqrt{6} = 2.449490$, correct to seven decimal digits, since the square of 2.449490 equals 6.000000. Similarly, 2.45 is correct to three decimal digits because its square is 6.00. It is true in general that the number of digits of accuracy approximately doubles with each application of Newton's method.

To summarize Newton's method, starting with a first approximation r_1 for $\sqrt{n}$, we calculate the second approximation r_2 by the formula

$$r_2 = \frac{n/r_1 + r_1}{2},$$

and r_2 will have twice the accuracy of r_1. Here we have a simple formula that is easily applied and quickly gives a high degree of accuracy. It seems reasonable to conclude that, if one needed to calculate by hand a square root to say ten-place accuracy, Newton's method would be the easier of the two techniques to use. But machines are not people.

Newton's method requires several divisions, and a multiplication or a division requires about 100 times as much computing time in a calculator as an addition or a subtraction uses, so designers of calculators try to eliminate as many multiplications and divisions as possible. In an attempt to learn how calculators find square roots, some years ago I wrote to three well-known calculator manufacturers. One did not reply. Another did reply but warned me that I was asking for an industrial secret and therefore they could not divulge their secret formula. The third manufacturer, Hewlett-Packard, replied that they had recently published an article [4] on that very topic and they sent me a copy of it. The following is the essence of that article.

Curiously, it is the old, long division method that turns out to be simpler for a calculator and is built into its chip. We illustrate the technique by finding $\sqrt{54756}$. We first write 54756 in scientific notation, but with an even exponent, as 5.4756×10^4. We examine only the mantissa 5.4756, which of course lies between 1 and 100. First, subtract squares of successive integers from the mantissa until a negative difference appears, and then restore the immediately preceding value. Thus we have

$$\begin{aligned}
5.4756 - 1^2 &= 4.4756,\\
5.4756 - 2^2 &= 1.4756,\\
5.4756 - 3^2 &= -3.5244.
\end{aligned}$$

So 2 is the first digit of the square root and we restate the next to the last equation

$$5.4756 - 2^2 = 1.4756.$$

To avoid multiplication in calculating these squares, recall that

$$n^2 = 1 + 3 + 5 + \cdots + (2n - 1).$$

Then only additions and subtractions need to be performed if we subtract successive terms of the series for n^2. Thus the calculations become

$$\begin{aligned} 5.4756 - 1 &= 4.4756, \\ 4.4756 - 3 &= 1.4756, \\ 1.4756 - 5 &= -3.5244. \end{aligned}$$

The differences thus calculated are identical to those found earlier. Using the last positive difference, shift the decimal point one group of two places to the right, so the partial root 2 becomes 20 and the remainder is 147.56. As in finding the squares, we utilize another series, specifically

$$(2a + b)b = (2a + 1) + (2a + 3) + (2a + 5) + \cdots + (2a + [2b - 1]),$$

so we subtract successive terms of this series from the remainder until we reach a negative difference, obtaining

$$\begin{aligned} 147.56 - 41 &= 106.56, \\ 107.56 - 43 &= 63.56, \\ 63.56 - 45 &= 18.56, \\ 18.56 - 47 &= -28.44. \end{aligned}$$

The third subtraction is the last one with a nonnegative remainder, so the next digit in the root is 3 and the new remainder is 18.56. Again shift the decimal point two places to the right in the remainder and one place in the root, obtaining a remainder of 1856 and root 230. We repeat the algorithm to find that

$$\begin{aligned} 1856 - 461 &= 1395, \\ 1395 - 463 &= 932, \\ 932 - 465 &= 467, \\ 467 - 467 &= 0, \\ 0 - 479 &= -469. \end{aligned}$$

Hence the third and last digit is 4, and the required square root is $2.34 \times 10^2 = 234$. Of course, the exponent 2 is just half the even exponent 4 obtained by writing the given number in scientific notation.

One delightfully clever modification simplifies the procedure greatly: *multiply everything by 5.* The two series become

$$5n^2 = 5 + 15 + 25 + \cdots + (10n + 5)$$

and

$$5(2a + b)b = (10a + 5) + (10a + 15) + (10a + 25) + \cdots + (10a + [10b - 5]).$$

Now the modified algorithm starts with $5 \times 5.4756 = 27.3780$, and we have

$$\begin{aligned} 27.3780 - 5 &= 22.3780, \\ 22.3780 - 15 &= 7.3780, \\ 7.3780 - 25 &= -17.6220. \end{aligned}$$

The next to the last remainder is restored and the first digit in the root is the 2 in the 25 just subtracted. For successive stages we shift the decimal point in the remainder two places to the right, and to get the required $10a + 5$ for the next series of subtractions, we insert a zero before the terminal 5 in the last minuend 25, obtaining 205. Then we calculate

$$\begin{aligned} 737.80 - 205 &= 532.80, \\ 532.80 - 215 &= 317.80, \\ 317.80 - 225 &= 92.80, \\ 92.80 - 235 &= -142.20. \end{aligned}$$

The 23 from the last minuend gives the first two digits of the root, and the next to the last difference 92.80 becomes the new remainder. Another decimal point shift prepares us for the next iteration. We get

$$\begin{aligned} 9280 - 2305 &= 6975, \\ 6975 - 2315 &= 4660, \\ 4660 - 2325 &= 2335, \\ 2335 - 2335 &= 0, \\ 0 - 2345 &= -2345. \end{aligned}$$

Discard the terminal digit 5 from the last minuend to get the root 234. The remainder, if needed, is the next to the last difference 0. The decimal point is placed in the root as it was earlier.

Hewlett-Packard calculators use this modification of the long division square root algorithm to calculate the mantissa, generally to two places more than necessary, and then round back to the number of places they can display. So, a process that at first glance seems more tedious and time-consuming is actually faster and easier for a machine. In this case, we require just one division (to divide the exponent by 2), only one multiplication (to multiply the mantissa by 5) and then just additions, subtractions, and decimal point shifts, a fast, simple process for a calculator.

It is highly likely that all makes of calculators use the modified long division algorithm for finding square roots. When calculators first appeared with a square root key, I performed an informal test on three calculators, one unit from each of three different brands. If the long division algorithm is used, then calculating $\sqrt{99999999} = 9999.99995$ takes more time than does finding $\sqrt{100.00001} = 10.0000005$ since it requires 10 subtractions to find the digit 9 in a root, but only 1 subtraction to get a zero. Even then it required a quick eye to see the difference in time, but all three calculators did take less time for the latter root than for the former. Newton's method should consume the same length of time no matter what root is sought.

Subsequent articles in the *Hewlett-Packard Journal* gave calculator algorithms for other functions, such as trigonometric and exponential. All such algorithms were carefully chosen to minimize multiplications and divisions, using clever techniques such as the multiplying by 5 that was done here. No, there is no little person sitting inside a chip, just some clever mathematicians and engineers and computer scientists who did some careful thinking nearly half a century ago.

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ΠΜΕ Journal, Vol. 11, No. 2, pp 75–80, 2000.

A NOTE ON THE ALGEBRAIC CONNECTIVITY OF CERTAIN GRAPHS

MASAKAZU NIHEI*

1. Introduction and Definitions. The important role of spectral graph theory in current research is well explained in Fan Chung's recent book [2]. Of particular interest may be applications to modern communication networks.

There are several matrices which encode a graph G; for instance the incidence matrix, the adjacency matrix, or the Laplacian. Spectral graph theory is concerned with answering the question which graph theoretic properties of G can be reconstructed from the spectrum of a matrix associated with G. We illustrate this by considering the Laplacian matrix of G and drawing conclusions concerning the algebraic connectivity of the endline graph of G.

We begin with a few definitions and some notation. We consider finite undirected graphs without loops or multiple edges. We let

$V(G) = \{v_1, \cdots, v_p\}$ and $E(G)$ be the set of vertices and set of edges of a graph, respectively.

Let $A = A(G) = (a_{ij})$ be the adjacency matrix of G, so $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise.

Let $L(G) = D(G) - A(G) = D - A$, where $D = \text{diag}(d_1, \cdots, d_p)$ and $d_i = d(v_i)$ is the degree of vertex v_i, $i = 1, \cdots, p$. Following [4] we will refer to $L(G)$ as the Laplacian matrix of the graph G.

Let $n \geq 2$ and $0 = \alpha_1 \leq \alpha_2 = a(G) \leq \alpha_3 \leq \cdots \leq \alpha_p$ be the eigenvalues of the matrix $L(G)$. It is well known that the second smallest eigenvalue $a(G)$ is zero if and only if G is not connected, [3]. This observation led M. Fiedler to think of $a(G)$ as a quantitative measure of connectivity, [3]. Following him, we call $a(G)$ the *algebraic connectivity* of the graph G.

For example, let P_p and C_p be the circuit graph of order p, $p \geq 2$, and the path of order p, $p \geq 3$ respectively, see Figure 1, Then $a(P_p) = 2(1 - \cos(\pi/p))$ and

FIG. 1.

$a(C_p) = 2(1 - \cos(2\pi/p))$, see [3].

The *edge-connectivity* of the graph G is the minimal number of edges whose removal disconnects the graph G, and is denoted by $\kappa_1(G)$. The *vertex-connectivity* is defined analogously (vertices together with adjacent edges are removed) and is denoted by $\kappa(G)$. Note that it is convenient to put $\kappa(K_p) = p - 1$, where K_p is the complete graph of order p, $p \geq 1$. It is well known, [3], that $a(G) \leq \kappa_1(G)$ for any graph G.

Let G be a graph and $V(G) = \{v_1, \cdots, v_p\}$. We add p new vertices, $\{u_1, \cdots, u_p\}$, to G and p new edges, $\{u_i, v_i\}$, $i = 1, \cdots, p$. Note the u_i are different from any vertex of G and from each other. We obtain a new graph G^+ with $2p$ vertices. Following [5], we call this graph the *endline graph* of the graph G. We also call an edge $\{u_i, v_i\}$ an *endline* of G, see Figure 2. The endline graph of a graph is used, for example, to

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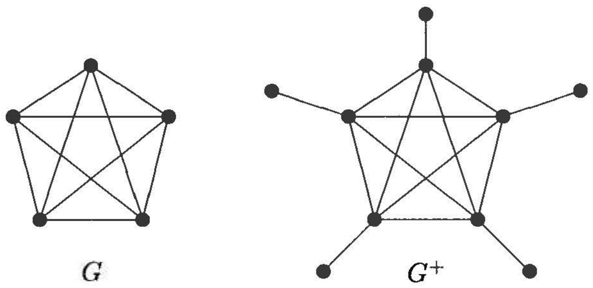

FIG. 2.

study the *middle graph*, see [5] or [6] for more details.

Now, let us calculate the edge-connectivity and the vertex-connectivity of the endline graph of a connected graph. By the definition, we immediately see that $\kappa_1(G^+) = \kappa(G^+) = 1$ for any connected graph G. On the other hand, we easily see that $a(G^+)$ is not always the same value for any connected graph G. Therefore, it may be worth while studying the algebraic connectivity of the endline graph of a connected graph.

The purpose of this paper is to give the algebraic connectivity of the endline graph of a graph.

Terminology not defined here follows that in [1].

2. Results. Let us denote the identity matrix of order p by I_p and the determinant of a square matrix A by $\det A$. The characteristic polynomial of $L(G)$ will be denoted by $\Phi(L(G);\lambda)$. Then we have the following result:

THEOREM 1. *Let G be a graph of order p. Then*

$$\Phi(L(G^+);\lambda) = (\lambda-1)^p\,\Phi(L(G);\lambda-1-1/(\lambda-1)).$$

Proof. By the definition of the endline graph of a graph G, without loss of generality we may write

$$A(G^+) = \begin{bmatrix} 0 & I_p \\ I_p & A(G) \end{bmatrix} \text{ and } D(G^+) = \begin{bmatrix} I_p & 0 \\ 0 & D(G)+I_p \end{bmatrix},$$

where 0 is the zero matrix of order p. Hence

$$\begin{aligned}\Phi(L(G^+);\lambda) &= \det(\lambda I_{2p} - (D(G^+) - A(G^+))) \\ &= \begin{vmatrix} (\lambda-1)I_p & I_p \\ I_p & (\lambda-1)I_p - (D(G)-A(G)) \end{vmatrix} \\ &= \det((\lambda-1)I_p)\det((\lambda-1)I_p - L(G) - I_p((\lambda-1)I_p)^{-1}I_p) \\ &= \det((\lambda-1)I_p)\det((\lambda-1)I_p - L(G) - (1/(\lambda-1))I_p) \\ &= (\lambda-1)^p\det((\lambda-1-1/(\lambda-1))I_p - L(G)) \\ &= (\lambda-1)^p\Phi(L(G);\lambda-1-1/(\lambda-1)).\end{aligned}$$

□

From Theorem 1, we immediately obtain the following result:

COROLLARY 2. *Let G be a graph of order p and let us denote $a(G)$ by a. Then*

$$a(G^+) = (a+2-\sqrt{a^2+4})/2.$$

Proof. Let $n \geq 2$ and $0 = \alpha_1 \leq \alpha_2 = a(G) \leq \alpha_3 \leq \cdots \leq \alpha_p$ be the eigenvalues of the matrix $L(G)$. Then, by Theorem 1, we have

$$\Phi(L(G^+);\lambda) = (\lambda-1)^p\prod_{i=1}^{p}(\lambda-1-1/(\lambda-1)-\alpha_i).$$

Therefore, in order to get the eigenvalues of the matrix $L(G^+)$, we may solve the following equations:

(1) $$x-1-1/(x-1) = \alpha_i, (i = 1,\cdots,p)$$

Let β_i and β_{i+p}, $\beta_i < \beta_{i+p}$, be the solutions of Equation 1. Then, since the function $f(x) = x-1-1/(x-1)$ is a monotone increasing function on the open interval $(-\infty,1)\bigcup(1,\infty)$, we can easily check that if $\alpha_i \leq \alpha_j$, then $\beta_i < 1 < \beta_{i+p} \leq \beta_{j+p}$, see Figure 3.

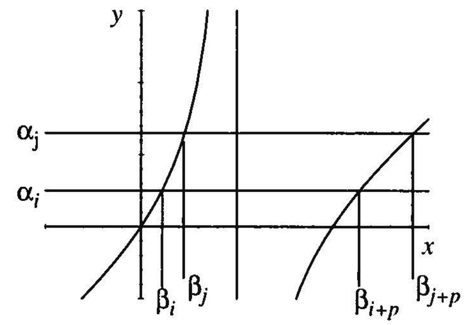

FIG. 3.

Therefore, to get $a(G^+)$, we may solve the following equation:

$$x-1-1/(x-1) = a, a \geq 0.$$

Solving this equation, we obtain

$$x = (a+2\pm\sqrt{a^2+4})/2.$$

Since $a(G^+)$ is the second smallest eigenvalue and 0 is an eigenvalue of $L(G^+)$, we obtain the desired result. □

COROLLARY 3. *If a graph G is connected, then*

(1) $0 < a(G^+) < 1$

(2) $a(G^+) < a(G)$

(3) If $a(H) \leq a(G)$, then $a(H^+) \leq a(G^+)$.

Proof. Since (1) is clear, we need only prove (2) and (3).

We first prove (2). To prove (2), we need show that

$$a(G) - a(G^+) = (a-2+\sqrt{a^2+4})/2 > 0.$$

By the way, the function $g(x) = (x - 2 + \sqrt{x^2+4})/2$ is a monotone increasing function when $x \geq 0$ and $g(0) = 0$. This implies that $a(G^+) < a(G)$.

Next, noting that $h(x) = (x + 2 - \sqrt{x^2+4})/2$ is a monotone increasing function when $x \geq 0$ and $h(0) = 0$, we easily see that (3) holds. □

Let K_p be the complete graph of order $p \geq 2$. Then, since $a(K_p) = p$, see [3], we have the following formulas:

COROLLARY 4. *Let* $c_p = 2(1 - \cos(2\pi/p))$, *then*

(1) $a(K_p^+) = (p + 2 - \sqrt{p^2+4})/2$, *for* $p \geq 2$, *and*

(2) $a(C_p^+) = (c_p + 2 - \sqrt{c_p^2+4})/2$, *for* $p \geq 3$.

For example, $a(K_5^+) = (7 - \sqrt{29})/2$, see Figure 2, and $a(C_6^+) = (3 - \sqrt{5})/2$, see Figure 4.

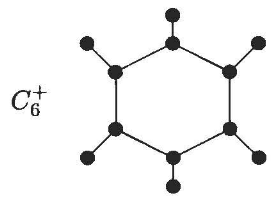

FIG. 4.

Since the function $a(G)$ is nondecreasing for graphs with the same set of vertices (i.e., $a(H) \leq a(G)$ if $E(H) \subseteq E(G)$ and $V(H) = V(E)$, see [3], by Corollary 2 we also have the following:

COROLLARY 5. *Let G be a Hamiltonian graph of order $p \geq 3$ and let* $c_p = 2(1 - \cos(2\pi/p))$, *then*

$$(c_p + 2 - \sqrt{c_p^2+4})/2 \leq a(G^+) \leq (p + 2 - \sqrt{p^2+4})/2.$$

Let G be a graph and $V(G) = \{v_1, \cdots, v_p\}$. We add k new vertices, u_i, and k edges, $\{u_i, v_i\}$, to G, $0 < k < p$, where the u_i are different from any vertex of G and from each other. Then we obtain a new graph G_k^+ with $p + k$ vertices. We call this graph a *partial endline* graph of a graph G, see Figure 5.

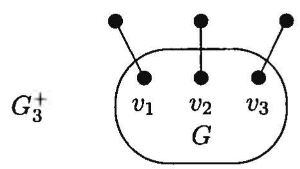

FIG. 5.

Note that the partial endline graph of a graph G depends on the labelling of the vertices of G.

For example, let $G \cong P_3$. Then, according to the labelling of the vertices, $G_1^+ \cong P_4$ or $K_{1,3}$, see Figure 6, where the symbol $G \cong H$ means the two graphs are isomorphic, and $K_{1,r}$ denotes the star graph of order $(r + 1)$.

Now, let us consider the algebraic connectivity of the partial endline graph of a graph. In order to simplify the argument, from now on we suppose that G is labelled.

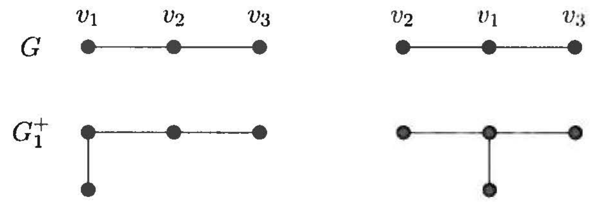

FIG. 6.

FIG. 7.

For example, let G be the labelled graph shown in Figure 7. Then $G_1^+ \cong P_4$ and $G_2^+ \cong P_5$. Hence we have $a(G_1^+) = 2 - \sqrt{2}$ and $a(G_2^+) = 2(1 - \cos(\pi/5))$. This implies that $a(G_2^+) < a(G_1^+)$. In general, the following result holds.

THEOREM 6. *Let G be a graph of order p and k be a positive integer,* $1 < k < p$. *Then*

$$a(G^+) \leq a(G_k{}^+) \leq a(G_{k-1}{}^+) \leq a(G).$$

In oder to prove Theorem 2, we need the following lemma, which is well known as the interlacing theorem (see [3, p.19]).

LEMMA 7. *Let A be a real symmetric matrix with eigenvalues* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$, *and B be one of its principal submatrices; Let B have eigenvalues* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_q$. *Then the inequalities* $\lambda_i \leq \mu_i \leq \lambda_{p-q+i}$, *for* $i = 1, \cdots, q$, *hold.*

Proof. From the definition of the partial endline graph of G, without of generality we may write

$$L(G_k^+) = \left[\begin{array}{c|c} & 0 \\ & \vdots \\ L(G_{k-1}{}^+) & -1 \\ & \vdots \\ & 0 \\ \hline 0\,0 \cdots -1 \cdots 0 & 1 \end{array}\right]$$

with one -1 occurring the k'th row and the other in the k'th column.

This shows that $L(G_{k-1}^+)$ is a principal submatrix of $L(G_k^+)$. Hence, from the above Lemma, we have the desired result. □

Theorem 2 shows that the algebraic connectivity of the partial endline graph of a graph decreases with the number of endlines.

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The Richard V. Andree Awards. The Richard V. Andree Awards are given annually to the authors of the three papers, written by undergraduate students, that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the Pi Mu Epsilon Journal in the past year.

Until his death in 1987, Richard V. Andree was Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon for many years and in a variety of capacities: as President, as Secretary-Treasurer, and as Editor of this Journal.

The awards for papers appearing in 1999 are announced on the next page. The officers and councilors of the Society congratulate the winners on their achievements and wish them well for their futures.

1999 1999

The Richard V. Andree Awards

First Prize: Xialong Ron Yu,
"Curious Numbers",
Pi Mu Epsilon Journal, Vol. 10, No. 10, Spring 1999.

Second Prize: Jack Samuel Calcut III,
"Single Rational Arctangent Identities for Pi",
Pi Mu Epsilon Journal, Vol. 11, No. 1, Fall 1999.

Third Prize: Gina Garza and Natascha Shinkel,
"Which Graphs have Planar Shadow Graphs?",
Pi Mu Epsilon Journal, Vol. 11, No. 1, Fall 1999.

1999 1999

A Letter from the President

Dear Members and Friends:

As you have undoubtedly noticed, the Pi Mu Epsilon Journal got a jump on the new millennium this past fall by changing to a completely new format. We have been most gratified by the positive comments on this new Journal, which is work of the Brigitte Servatius, the new Pi Mu Epsilon Journal Editor. The Journal maintains the traditional features but conforms more closely to the style and layout of other professional mathematical publications. We hope this new format will encourage more students to submit articles to the Journal. This can be a great first step to a career in mathematics and help to develop the communication skills that are important to success in so many areas.

There are a number of other areas that Pi Mu Epsilon is developing to try to make the organization more responsive to the need of the membership. Our web site at

http://www.pme-math.org/

figures prominently in many of these, and we want to make this the primary vehicle for communication for the organization.

I am regularly asked for ideas that will help ensure a vibrant Chapter. I can speak from my limited experience based on some of the successful Chapters that I am familiar with, but things that work well at one institution and for one group of people may not be at all useful at others. Also, I have found over the years that students tend to have much better ideas about interesting activities than I have and that they more actively engage in activities that they have developed.

Beginning this Spring, the Pi Mu Epsilon web site we have a prominent area for Chapter News, which we hope will make us all better aware of some of the successful activities of our Chapters. Advisors and Permanent Faculty Correspondents will be asked to have their Chapter Officers submit a brief (or not so brief) report on Chapter activities at least once each year using a simple form that is located at the web site. Chapters can also use the report provide a link to a local Pi Mu Epsilon web site. If you have other suggestions for ways in which we can make the web site, or any other activity of Pi Mu Epsilon, more useful, please let me or one of the other National Officers know. All of our addresses are listed on the web site.

I would also like to take this opportunity to strongly encourage you to attend the Annual National Meeting of Pi Mu Epsilon this summer. The National Meeting is part of the MathFest at the University of California in Los Angeles, and will be held from August 3 to August 5. Keep in mind that Pi Mu Epsilon will support full transportation expenses for a Student Delegate from each Chapter who gives a talk at the meeting, and half transportation expenses for a Student Delegate who does not present a talk. In addition, we expect to be able to support a portion of the subsistence expenses of Student Delegates who give talks, due to a generous grant from the National Security Agency. More information about the details of funding will soon be sent to each Faculty Advisor, as well as being available at the Pi Mu Epsilon web site. I have found that students attending the MathFest often come back with a much clearer view of the mathematics community and their place in it.

Doug Faires

ΠME Journal, Vol. 11, No. 2, pp 83–85, 2000.

RODICA SIMION
JANUARY 18, 1955 – JANUARY 7, 2000*

RICHARD P. STANLEY†

The mathematical world lost one of its most enthusiastic and dedicated adherents with the tragic death of Rodica Simion on January 7, 2000. Rodica received her B.S. degree from the University of Bucharest in 1974. She came to the U.S. from Romania in 1976 and obtained her Ph.D. at the University of Pennsylvania in 1981 under the direction of Herbert Wilf. Her thesis was entitled "On Compositions of Multisets" and included a very influential result[1] which asserted that certain combinatorially defined polynomials have only real zeros. She taught at Southern Illinois University and Bryn Mawr College before coming to George Washington University (GWU) in 1987. She moved up the career ladder at GWU, culminating in an appointment to Professor in 1997. Just last year she was awarded a prestigious Columbian School Professorship at GWU in recognition for her many contributions to mathematics.

Rodica had a passionate love for mathematics and labored completely selflessly to develop and promote all aspects of the subject, from original research to making deep mathematical results accessible to the general public. Her research remained in the area of combinatorics, where she made many outstanding contributions. As an example of Rodica's research, we mention one pretty result[2] which requires little mathematical background to understand. Let A_n be the set of all permutations $a_1a_2\cdots a_n$ of $1,2,\ldots,n$ with no decreasing subsequence of length 3, i.e., there do not exist $i<j<k$ such that $a_i > a_j > a_k$. For instance,

$$A_3 = \{123, 132, 213, 231, 312\},$$

the only excluded permutation being 321. Similarly let B_n be the set of all permutations $a_1a_2\cdots a_n$ of $1,2,\ldots,n$ such that there do not exist $i<j<k$ satisfying $a_i > a_k > a_j$. For instance,

$$B_n = \{123, 132, 213, 231, 321\},$$

this time the only excluded permutation being 312. It had been known that A_n and B_n had the same number of elements, namely,

$$|A_n| = |B_n| = \frac{1}{n+1}\binom{2n}{n}.$$

(This number is a *Catalan number*, but this fact is not relevant here.) Rodica (together with her long-time collaborator Frank Schmidt) "explained" this seeming coincidence by exhibiting an elegant one-to-one correspondence (bijection) between A_n and B_n. This bijection of Rodica's was just one of many results in the seminal paper in which it appeared. It was the first paper to systematically investigate the theory of "permutations with forbidden patterns," currently a highly active area of research.

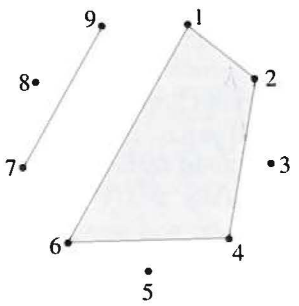
*Partially supported by NSF grant DMS-9500714. I am grateful to Joseph Bonin for providing a wealth of useful information.

†Massachusetts Institute of Technology

[1]See her paper "A multiindexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences," *J. Combinatorial Theory (A)* **36** (1984), 15–22.

[2]See Proposition 19 of "Restricted partitions," *Europ. J. Combinatorics* **6** (1985), 383–406.

A topic which fascinated Rodica (as well as many other mathematicians, including myself) throughout her career was the theory of noncrossing partitions. A *partition* of a set S is a collection of nonempty pairwise disjoint subsets of S whose union is S. For instance, one of the partitions of the set $\{1, 2, \ldots, 9\}$ consists of the subsets $\{1, 2, 4, 6\}$, $\{3\}$, $\{5\}$, $\{7, 9\}$, and $\{8\}$. We can represent this partition geometrically by arranging the elements $1, 2, \ldots, 9$ clockwise around a circle, and drawing polygons whose vertices are the subsets of S defining the partition:



If the polygons do not intersect, as is the case here, then we call the partition *noncrossing*. Noncrossing partitions have a multitude of beautiful properties and unexpected applications. The most basic result is that the number of noncrossing partitions of the set $\{1, 2, \ldots, n\}$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. As an example of the wide applicability of noncrossing partitions, they play a fundamental role in the theory of "free probability" developed by Dan-Virgil Voiculescu and his students. Rodica wrote several fundamental papers on noncrossing partitions and was perhaps the world's leading authority on this topic. She was in the process of writing another paper involving noncrossing partitions at the time of her death.

I wrote one joint paper with Rodica alone[3]. The basic idea for this paper was Rodica's. We were both visiting MSRI (the Mathematical Sciences Research Institute in Berkeley, California) in the fall of 1996 when she walked into my office one day with the question "Did you know that the poset of shuffles is locally rank-symmetric?" I had earlier developed a general theory of certain creatures known as "locally rank-symmetric posets," but few examples were known. Rodica and I spent many stimulating weeks applying the theory of locally rank-symmetric posets to the poset of shuffles.

Altogether Rodica published well over 30 papers. Most of these were original research papers, but a few were expository and exemplify Rodica's strong desire to reveal the beauty of mathematics to as wide an audience as possible. One paper that does not appear in her list of publications but deserves to be a joint paper with me is her write-up[4] of my lecture series given at the Capital City Conference on Combinatorics, held at GWU in 1989. With characteristic modesty Rodica refused to receive any credit for the arduous job of converting my lectures to a survey paper.

The Capital City Conference was actually organized entirely by Rodica and was but one of many events which she helped to arrange. For instance, she was a member of the organizing committee for the Combinatorial Year at MSRI during the 1996–97 academic year. She was a long-standing member of the Permanent Committee for

[3] Flag-symmetry of the poset of shuffles and a local action of the symmetric group, *Discrete Math.* **204** (1999), 369–396.

[4] Some applications of algebra to combinatorics, *Discrete Applied Math.* **34** (1991), 241–277.

the Formal Power Series and Algebraic Combinatorics (FPSAC) conference held each summer in different cities throughout the world. Most recently, she and I were the co-organizers of a Special Session in Memory of Gian-Carlo Rota held January 20–22, 2000, at the annual meeting of the American Mathematical Society in Washington, DC. Gian-Carlo Rota, who died in April of 1999, was perhaps the most influential combinatorialist of his time and was greatly admired by Rodica. It is especially tragic that Rodica passed away less than two weeks before the start of this Special Session. The time scheduled for her own talk was devoted to a series of touching remembrances before an overflow audience.

The research, expository, and organizational activities I have mentioned, together with normal teaching and administrative duties at GWU, would have been full-time work for an ordinary person, but Rodica actually accomplished much more. For instance, around 1990 Rodica became concerned that there wasn't a solid mathematics exhibit anywhere on the east coast. With the help of three of her colleagues and a Master's student in museum studies at GWU she organized an exhibit at GWU, supplying about 25% of the ideas for the content herself, as well as contributing tremendous organizational and fund-raising skills. Meanwhile the Maryland Science Museum in Baltimore had contacted the National Science Foundation about building a mathematics exhibit. To make a long story short, the 6,000 sq ft exhibit *Beyond Numbers* opened at the Maryland Science Museum in 1995, with Rodica putting in the majority of the work. Topics include graph theory, topology (including knot theory), tilings, chaos, minimal surfaces, and much more. The exhibit is still open at the Maryland Science Museum, and a copy is traveling to science museums across the United States[5]. The museum estimates that by the end of 2000 over four million people will have seen the exhibit.

A second program in which Rodica played a major role is the Summer Program for Women in Mathematics, held each summer at GWU for sixteen talented undergraduate women. Rodica was the person most responsible for obtaining funding and for designing the program, which began in 1995. One innovation due to Rodica was basing the program on four short courses in different areas of mathematics that are not typically covered in the undergraduate curriculum. The success of the program is confirmed by the many of its participants who go on to graduate school and who return to the program as counselors.

Rodica had a consistently upbeat, cheerful, and caring personality. She was the kind of person who could light up a room as soon as she entered. She never had a harsh word for anyone and worked tirelessly to counsel any person in need of advice or guidance. She was exceptionally modest about her own accomplishments and would artfully deflect any praise directed toward her. She will be dearly missed by her countless friends throughout the world.

Richard P. Stanley, MIT, Cambridge MA, 02139-4307. rstan@math.mit.edu

Richard P. Stanley is professor of mathematics at Massachusetts Institute of Technology. A summary of his work can be found in his two volume opus "Enumerative Combinatorics". Volume I appeared in 1986 and volume II in 1999.

[5] See www.mdsci.org for a schedule.

From the Right Side

©Rodica Simion, 1999.

The late Rodica Simion, Professor of Mathematics at George Washington University, painted this picture to congratulate Richard Stanley on the publication of his new book "Enumerative Combinatorics". Some of the design elements are typical objects in combinatorial geometry: a Ferris diagram, a Young lattice, and a polytope with its face poset. She has placed these objects among a colorful array of blooming flowers which delightfully spill over the margin, an effect that could not be reproduced here. It reminded me of a beautiful sentence in Eduard Mörike's novel "Mozart auf der Reise nach Prag" which, foreshadowing Mozart's early death, speculates that the earth was, in reality, not able to bear the abundance spilling from his being. Rodica and Mozart have more than a short life in common.

The ΠME Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.

ΠME Journal, Vol. 11, No. 2, pp 87–96, 2000.

TAXICAB ANGLES AND TRIGONOMETRY

KEVIN THOMPSON AND TEVIAN DRAY *

Abstract. A natural analogue to angles and trigonometry is developed in taxicab geometry. This structure is then analyzed to see which, if any, congruent triangle relations hold. A nice application involving the use of parallax to determine the exact (taxicab) distance to an object is also discussed.

1. Introduction. Taxicab geometry, as its name might imply, is essentially the study of an ideal city with all roads running horizontal or vertical. The roads must be used to get from point A to point B; thus, the normal Euclidean distance function in the plane needs to be modified. The shortest distance from the origin to the point (1,1) is now 2 rather than $\sqrt{2}$. So, taxicab geometry is the study of the geometry consisting of Euclidean points, lines, and angles in $\mathbb{R}^2$ with the taxicab metric

$$d((x_1,y_1),(x_2,y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

A nice discussion of taxicab geometry was given by Krause [1, 2], and some of its properties have been discussed elsewhere, including taxicab conic sections [3, 4, 5, 6], and the taxicab isometry group [7].

In this paper we will explore a slightly modified version of taxicab geometry. Instead of using Euclidean angles measured in radians, we will mirror the usual definition of the radian to obtain a taxicab radian (a t-radian). (A similar approach was used by Euler [8] to discuss the value of π for a class of generalized circles which includes the taxicab circle.) Using this definition, we will define taxicab trigonometric functions and explore the structure of the addition formulas from trigonometry. As applications of this new type of angle measurement, we will explore the existence of congruent triangle relations and illustrate how to determine the distance to a nearby object by performing a parallax measurement.

Henceforth, the label taxicab geometry will be used for this modified taxicab geometry; a subscript e will be attached to any Euclidean function or quantity.

2. Taxicab Angles. There are at least two common ways of defining angle measurement: in terms of an inner product and in terms of the unit circle. For Euclidean space, these definitions agree. However, the taxicab metric is not an inner product since the natural norm derived from the metric does not satisfy the parallelogram law. Thus, we will define angle measurement on the unit taxicab circle which is shown in Figure 1.

DEFINITION 1. *A* t-radian *is an angle whose vertex is the center of a unit (taxicab) circle and intercepts an arc of (taxicab) length 1. The taxicab measure of a taxicab angle θ is the number of t-radians subtended by the angle on the unit taxicab circle about the vertex.*

It follows immediately that a taxicab unit circle has 8 t-radians since the taxicab unit circle has a circumference of 8. For reference purposes the Euclidean angles $\pi/4$, $\pi/2$, and π in standard position now have measure 1, 2, and 4, respectively. The following theorem gives the formula for determining the taxicab measures of some other Euclidean angles.

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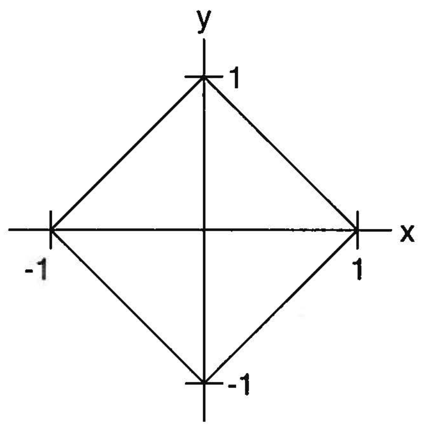

FIG. 1. *The taxicab unit circle.*

THEOREM 2. *An acute Euclidean angle ϕ_e in standard position has a taxicab measure of*

$$\theta = 2 - \frac{2}{1+\tan_e \phi_e} = \frac{2\sin_e \phi_e}{\sin_e \phi_e + \cos_e \phi_e}$$

Proof. The taxicab measure θ of the Euclidean angle ϕ_e is equal to the taxicab distance from (1,0) to the intersection of the lines $y = -x+1$ and $y = x\tan_e \phi_e$. The x-coordinate of this intersection is

$$x_0 = \frac{1}{1+\tan_e \phi_e},$$

and thus the y-coordinate of P is $y_0 = -x_0 + 1$. Hence, the taxicab distance from (1,0) to P is

$$\theta = 1 - x_0 + y_0 = 2 - \frac{2}{1+\tan_e \phi_e}.$$

□

DEFINITION 3. *The reference angle of an angle ϕ is the smallest angle between ϕ and the x-axis.*

Theorem 2 can easily be extended to any acute angle lying entirely in a quadrant.

COROLLARY 4. *If an acute Euclidean angle ϕ_e with Euclidean reference angle ψ_e is contained entirely in a quadrant, then the angle has a taxicab measure of*

$$\begin{aligned}\theta &= \frac{2}{1+\tan_e \psi_e} - \frac{2}{1+\tan_e(\phi_e+\psi_e)}\\ &= \frac{2\sin_e \phi_e}{(\cos_e(\phi_e+\psi_e)+\sin_e(\phi_e+\psi_e))(\cos_e \psi_e + \sin_e \psi_e)}\end{aligned}$$

This corollary implies the taxicab measure of a Euclidean angle in non-standard position is not necessarily equal to the taxicab measure of the same Euclidean angle in standard position. Thus, although angles are translation invariant, they are not

rotation invariant. This is an important consideration when dealing with any triangles in taxicab geometry.

In Euclidean geometry, a device such as a cross staff or sextant can be used to measure the angular separation between two objects. The characteristics of a similar device to measure taxicab angles would be very strange to inhabitants of a Euclidean geometry; measuring the taxicab size of the same Euclidean angle in different directions would usually yield different results. Thus, to a Euclidean observer the taxicab angle measuring device must fundamentally change as it is pointed in different directions. Of course this is very odd to us since our own angle measuring devices do not appear to change as we point them in different directions.

The taxicab measure of other Euclidean angles can also be found. Except for a few cases, these formulas will be more complicated since angles lying in two or more quadrants encompass corners of the unit circle.

LEMMA 5. *The taxicab measure of any Euclidean right angle is 2 t-radians.*

Proof. Without loss of generality, let θ be an angle encompassing the positive y-axis. As shown in Figure 2, split θ into two Euclidean angles α_e and β_e with reference angles $\pi/2 - \alpha_e$ and $\pi/2 - \beta_e$, respectively. Using Theorem 2, we see that

$$\begin{aligned}\theta &= \frac{2\sin_e \alpha_e}{\cos_e \alpha_e + \sin_e \alpha_e} + \frac{2\sin_e \beta_e}{\cos_e \beta_e + \sin_e \beta_e}\\ &= \frac{2\sin_e \alpha_e}{\cos_e \alpha_e + \sin_e \alpha_e} + \frac{2\cos_e \alpha_e}{\sin_e \alpha_e + \cos_e \alpha_e}\\ &= 2\end{aligned}$$

since $\alpha_e + \beta_e = \pi/2$. □

We now state the taxicab version of the familiar result for the length of an arc from Euclidean geometry and note that its proof is obvious since all distances along a taxicab circle are scaled equally as the radius is changed. This result will be used when we turn to congruent triangle relations and the concept of parallax.

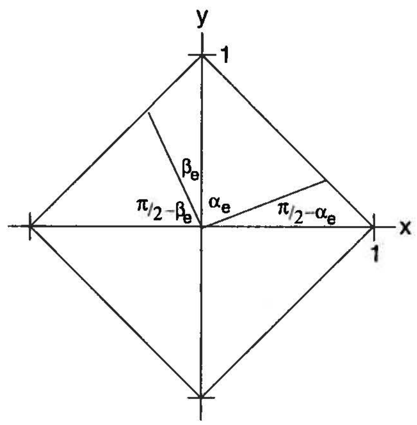

FIG. 2. *Taxicab right angles are precisely Euclidean right angles.*

THEOREM 6. *The length s of the arc intercepted on a (taxicab) circle of radius r by the central angle with taxicab measure θ is given by $s = r\theta$.*

From the previous theorem we can easily deduce the following result.

COROLLARY 7. *Every taxicab circle has 8 t-radians.*

3. Taxicab Trigonometry. We now turn to the definition of the trigonometric functions sine and cosine in taxicab geometry. From these definitions familiar formulas for the tangent, secant, cosecant, and cotangent functions can be defined and results similar to those below can be obtained.

DEFINITION 8. *The point of intersection of the terminal side of a taxicab angle θ in standard position with the taxicab unit circle is the point* $(\cos_t \theta, \sin_t \theta)$.

It is important to note that the taxicab sine and cosine values of a taxicab angle do not agree with the Euclidean sine and cosine values of the corresponding Euclidean angle. For example, the angle 1 t-radian has equal taxicab sine and cosine values of 0.5. The range of the cosine and sine functions remains $[-1,1]$, but the period of these fundamental functions is now 8. It also follows immediately (from the distance function) that $|\sin_t \theta| + |\cos_t \theta| = 1$. In addition, the values of cosine and sine vary (piecewise) linearly with θ:

$$\cos_t \theta = \begin{cases} 1 - \frac{1}{2}\theta\,, & 0 \le \theta < 4 \\ -3 + \frac{1}{2}\theta\,, & 4 \le \theta < 8 \end{cases}\,, \qquad \sin_t \theta = \begin{cases} \frac{1}{2}\theta\,, & 0 \le \theta < 2 \\ 2 - \frac{1}{2}\theta\,, & 2 \le \theta < 6 \\ -4 + \frac{1}{2}\theta\,, & 6 \le \theta < 8 \end{cases}$$

Table 3.1 gives useful straightforward relations readily derived from the graphs of the sine and cosine functions which are shown in Figure 3. The structure of the graphs of these functions is similar to that of the Euclidean graphs of sine and cosine. Note that the smooth transition from increasing to decreasing at the extrema has been replaced with a corner. This is the same effect seen when comparing Euclidean circles with taxicab circles.

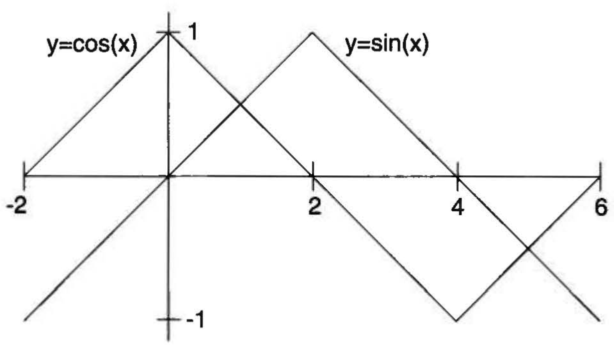

FIG. 3. *Graphs of the taxicab sine and cosine functions.*

$\sin_t(-\theta) = -\sin_t \theta$	$\sin_t(\theta+2) = \cos_t \theta$
$\cos_t(-\theta) = \cos_t \theta$	$\cos_t(\theta-2) = \sin_t \theta$
$\sin_t(\theta-4) = -\sin_t \theta$	$\sin_t(\theta+8k) = \sin_t \theta, k \in \mathbb{Z}$
$\cos_t(\theta-4) = -\cos_t \theta$	$\cos_t(\theta+8k) = \cos_t \theta, k \in \mathbb{Z}$

TABLE 3.1
Basic Taxicab Trigonometric Relations

As discussed below, and just as in the standard taxicab geometry described in [2], SAS congruence for triangles does not hold in modified taxicab geometry. Thus, the routine proofs of sum and difference formulas are not so routine in this geometry. The first result we will prove is for the cosine of the sum of two angles. The formula

given for the cosine of the sum of two angles only takes on two forms; the form used in a given situation depends on the locations of α and β. The notation $\alpha \in I$ will be used to indicate α is an angle in quadrant I and similarly for quadrants II, III, and IV.

THEOREM 9. $\cos_t(\alpha+\beta) = \pm(-1+|\cos_t \alpha \pm \cos_t \beta|)$ *where the signs are chosen to be negative when α and β are on different sides of the x-axis and positive otherwise.*

Proof. Without loss of generality, assume $\alpha, \beta \in [0,8)$, for if an angle θ lies outside $[0,8)$, $\exists k \in \mathbb{Z}$ such that $(\theta+8k) \in [0,8)$ and use of the identity $\cos_t(\theta+8k) = \cos_t \theta$ will yield the desired result upon use of the following proof.

All of the subcases have a similar structure. We will prove the subcase $\alpha \in II, \beta \in III$. In this situation $6 \le \alpha+\beta \le 10$ and we take the negative signs on the right-hand side of the equation. Thus,

$$\begin{aligned} 1 - |\cos_t \alpha - \cos_t \beta| &= 1 - |1 - \frac{1}{2}\alpha - (-3 + \frac{1}{2}\beta)| \\ &= 1 - |4 - \frac{1}{2}(\alpha+\beta)| \\ &= \begin{cases} -3 + \frac{1}{2}(\alpha+\beta), & \text{for } 6 \le \alpha+\beta < 8 \\ 5 - \frac{1}{2}(\alpha+\beta), & \text{for } 8 \le \alpha+\beta \le 10 \end{cases} \\ &= \cos_t(\alpha+\beta) \end{aligned}$$

□

COROLLARY 10. $\cos_t(2\alpha) = -1 + 2|\cos_t \alpha|$.

The curious case structure in Theorem 9 is due to the odd combinations of quadrants that determine which sign to choose. The reason for the sign change when α and β are on different sides of the x-axis lies in the fact that a corner of the cosine function is being crossed (i.e. different pieces of the cosine function are being used) to obtain the values of the cosine of α and β. Table 3.2 summarizes which form of $\cos_t(\alpha+\beta)$ should be used when.

	α	β
	same	quadrant
$\cos_t(\alpha+\beta) = -1 + \lvert\cos_t \alpha + \cos_t \beta\rvert$	I	II
	III	IV
$\cos_t(\alpha+\beta) = 1 - \lvert\cos_t \alpha - \cos_t \beta\rvert$	I	III
	I	IV
	II	III
	II	IV

TABLE 3.2
Forms of $\cos_t(\alpha+\beta)$ and Regions of Validity

We can use Theorem 9 and the relations in Table 3.1 to establish a pair of corollaries.

COROLLARY 11. $\sin_t(\alpha+\beta) = \pm(-1+|\sin_t \alpha \pm \cos_t \beta|)$ *where the signs are chosen according to Table 3.3.*

Proof. First, note $\sin_t \theta = \cos_t(\theta-2)$. As with the cosine addition formula, all cases are proved similarly. We will assume $\alpha \in I$ and $\beta \in IV$. We have $\alpha-2$ and β in the same quadrant, and thus

$$\sin_t(\alpha+\beta) = \cos_t((\alpha+\beta)-2)$$

$$= \cos_t((\alpha - 2) + \beta)$$
$$= -1 + |\cos_t(\alpha - 2) + \cos_t \beta|$$
$$= -1 + |\sin_t \alpha + \cos_t \beta|$$

□

COROLLARY 12. $\sin_t(2\alpha) = -1 + 2|\cos_t(\alpha - 1)|$

Proof. $\sin_t(2\alpha) = \cos_t(2\alpha - 2) = \cos_t(2(\alpha - 1)) = -1 + 2|\cos_t(\alpha - 1)|$ □

	α	β
$\sin_t(\alpha+\beta) = -1 + \|\sin_t \alpha + \cos_t \beta\|$	I	III
	I	IV
	II	II
	IV	IV
$\sin_t(\alpha+\beta) = 1 - \|\sin_t \alpha - \cos_t \beta\|$	I	I
	I	II
	II	III
	II	IV
	III	III
	III	IV

TABLE 3.3
Forms of $\sin_t(\alpha+\beta)$ and Regions of Validity

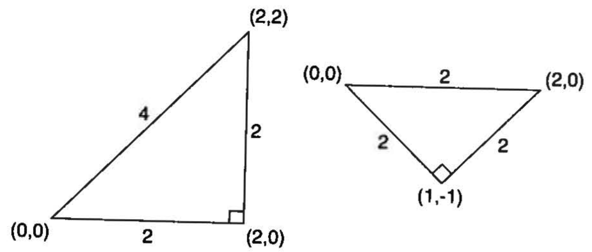

FIG. 4. *Triangles satisfying ASASA that are not congruent.*

4. Congruent Triangles. In Euclidean geometry we have many familiar conditions that ensure two triangles are congruent. Among them are SAS, ASA, and AAS. In modified taxicab geometry the only condition that ensures two triangles are congruent is SASAS. One example eliminates almost all of the other conditions.

Consider the two triangles shown in Figure 4. The triangle formed by the points (0,0), (2,0), and (2,2) has sides of lengths 2, 2, and 4 and angles of measure 1, 1, and 2 t-radians. The triangle formed by the points (0,0), (2,0), and (1,-1) has sides of length 2 and angles of measure 1, 1, and 2. These two triangles satisfy the ASASA condition but are not congruent. This also eliminates the ASA, SAS, and AAS conditions as well the possibility for a SSA or AAA condition.

The triangle formed by the points (0,0), (0.5,1.5), and (1.5, 0.5) has sides of length 2 and angles 1, 1.5, and 1.5 t-radians. Thus, it satisfies the SSS condition with the second triangle in the previous example. However, the angles of these triangles and not congruent. Hence, the SSS and SSSA conditions fail.

The last remaining condition, SASAS, actually does hold. Its proof relies on the fact that even in this geometry the sum of the angles of a triangle is a constant 4 t-radians, which in turn relies on the fact that, given parallel lines and a transversal, alternate interior angles are congruent. We begin by noting that opposite angles are congruent. This leads immediately to the following result.

LEMMA 13. *Given two parallel lines and a transversal, the alternate interior angles are congruent.*

Proof. Using Figure 5 translate α along the transversal to become an angle opposite β. By the note above, α and β are congruent. □

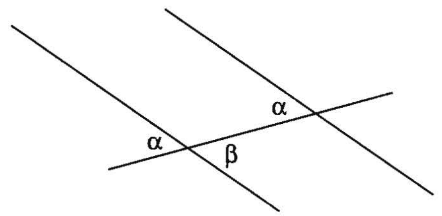

FIG. 5. *Alternate interior angles formed by parallel lines and a transversal are congruent.*

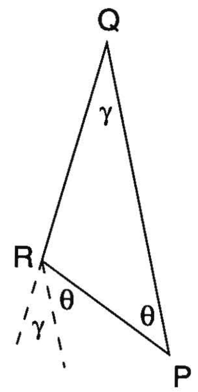

FIG. 6. *The sum of the angles of a taxicab triangle is always 4 t-radians.*

THEOREM 14. *The sum of the angles of a triangle in modified taxicab geometry is 4 t-radians.*

Proof. Given the triangle in Figure 6, we can translate the angle γ from Q to R and by the congruence of alternate interior angles conclude the sum of the angles of the triangle is 4 t-radians. □

Therefore, given two triangles having all three sides and any two angles congruent, the triangles must be congruent. However, as we have seen, this is the only congruent triangle relation in taxicab geometry.

5. Parallax. Parallax, the apparent shift of an object due to the motion of the observer, is a commonly used method for estimating the distance to a nearby object. The method of stellar parallax was used extensively to find the distances to nearby stars in the 19^{th} and early 20^{th} centuries. We now wish to explore the method and results of parallax in taxicab geometry and examine how these differ from the Euclidean method and results. We will discover that the taxicab method yields the same formula commonly used in the Euclidean case with the exception that the taxicab formula is exact.

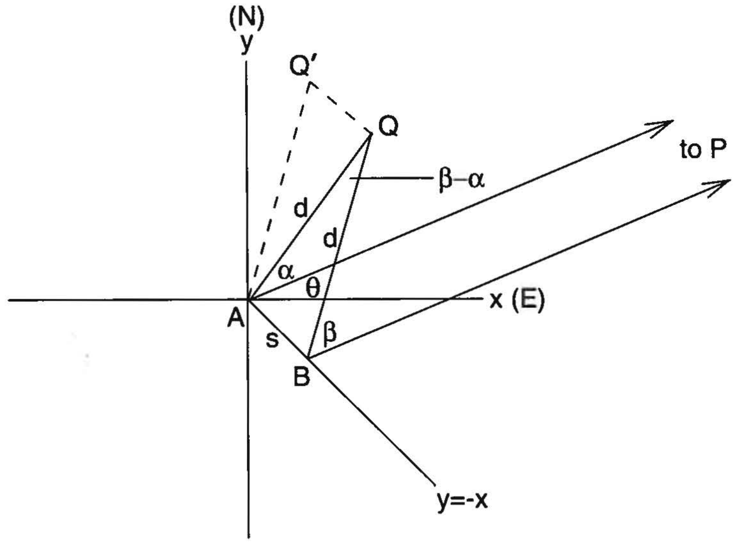

FIG. 7. *A parallax diagram in taxicab geometry.*

Suppose that as a citizen of Modified Taxicab-land you wish to find the distance to a nearby object Q in the first quadrant, and that there is also a distant reference object P "at infinity" essentially in the same direction as Q with reference angle θ (Figure 7). The distant reference object should be far enough away so that it appears stationary when you move small distances. We may assume without loss of generality that the object Q does not lie on either axis, for if it did, we could move a small distance to get the object in the interior of the first quadrant.

Initially standing at A, measure the angle α between Q and P using the taxicab equivalent of a cross staff or a sextant. Now, for reasons to be apparent later, you should move a small distance (relative to the distance to the object) in such a way that the distance to the object does not change. This can be accomplished by moving in either of two directions, and, provided you move only a small distance, the object remains in the interior of the first quadrant. Furthermore, exactly one of these directions results in the angle between Q and P being increased, so that the situation depicted in Figure 7 is generic.

You have therefore moved from A to B in one of the following directions: NW, NE, SW, or SE. Now measure the new angle β between the two objects. With this information we can now find the taxicab distance to the object Q. Construct the point Q' such that $\overline{QQ'}$ is parallel to $\overline{AB}$ and $\ell(\overline{QQ'}) = \ell(\overline{AB}) = s$. The angle $\measuredangle PAQ'$ has measure β since it is merely a translation of $\measuredangle QBP$. Thus, $\measuredangle QAQ'$ has measure $\beta - \alpha$. Now, the lengths of $\overline{AQ}$ and $\overline{BQ}$ are equal since the direction of movement from A to B was shrewdly chosen so that the distance to the object remained constant. Since translations do not affect lengths, this implies AQ and AQ' have equal lengths. Hence, the points Q and Q' lie on a taxicab circle of radius d centered at A. Using the formula for the length of a taxicab arc in Theorem 6,

$$d = \frac{s}{\beta - \alpha} \tag{1}$$

where s and d are taxicab distances and $\beta - \alpha$ is a taxicab angle. This formula is identical to the Euclidean distance *estimation* formula with d and s Euclidean distances and $\beta - \alpha$ an Euclidean angle. However, as we shall now see, the commonly used Euclidean version is truly an approximation and not an exact result. This realization is necessary to logically link the commonly used Euclidean formula and the taxicab formula.

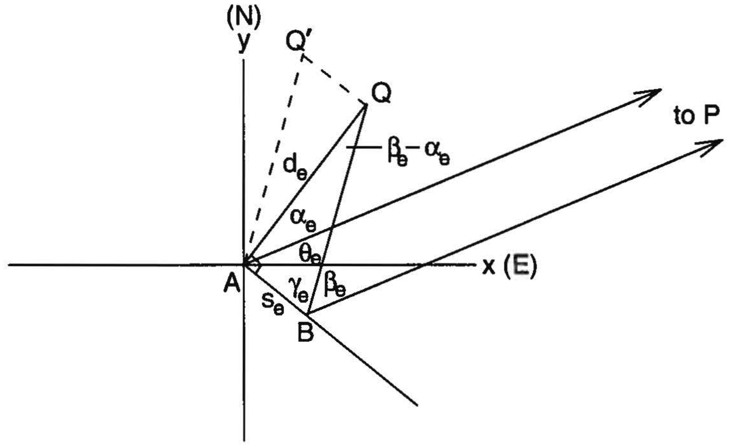

FIG. 8. *A parallax diagram in Euclidean geometry.*

Using Figure 7 but with all distances and angles now Euclidean, we isolate $\triangle QAB$. Using the law of sines and the fact that $\gamma = 3\pi/4 - (\beta_e + \theta_e)$, we have

$$d_e = \frac{s_e(\cos_e(\beta_e + \theta_e) + \sin_e(\beta_e + \theta_e))}{\sqrt{2}\sin_e(\beta_e - \alpha_e)} \tag{2}$$

This formula can be simplified by moving from A to B in a direction perpendicular to the line of sight to Q from A rather than in one of the four prescribed directions above. In this case $m(\measuredangle QBA) = \pi/2 - (\beta_e - \alpha_e)$ (Figure 8). Thus, the law of sines gives the Euclidean parallax formula

$$d_e = \frac{s_e}{\tan_e(\beta_e - \alpha_e)}$$

If we now apply the approximation $\tan_e(\beta_e - \alpha_e) \approx (\beta_e - \alpha_e)$ for small angles we obtain the commonly used Euclidean parallax formula

$$d_e \approx \frac{s_e}{\beta_e - \alpha_e},$$

which is not exact.

It is interesting to note the quite different movement requirements in the two geometries needed to obtain the best possible approximations of the distance to the object. This difference lies in the methods of keeping the distance to the object as constant as possible. In the Euclidean case, moving small distances on the line tangent to the circle of radius d centered at the object (i.e. perpendicular to the radius of this circle) essentially leaves the distance to the object unchanged. In taxicab geometry, moving in one direction along either $y = x$ or $y = -x$ keeps the distance to the object exactly unchanged.

We are now in a position to justify the link between results (1) and (2). Since the line segment $\overline{QQ'}$ of taxicab length s lies on a taxicab circle, $s = \sqrt{2}s_e$. The distance

d to the object is given by $d = d_c(\cos_e(\alpha_e + \theta_e) + \sin_e(\alpha_e + \theta_e))$ since the Euclidean angle between the line of sight $\overline{AQ}$ and the x-axis is $(\alpha_e + \theta_e)$. Using Corollary 4 with $\phi = (\beta_e - \alpha_e)$ and $\psi = (\alpha_e + \theta_e)$, the taxicab measure of $\beta - \alpha$ is given by

$$\beta - \alpha = \frac{2\sin_e(\beta_e - \alpha_e)}{(\cos_e(\beta_e + \theta_e) + \sin_e(\beta_e + \theta_e))(\cos_e(\alpha_e + \theta_e) + \sin_e(\alpha_e + \theta_e))}.$$

Using these substitutions, formula (1) becomes formula (2).

6. Conclusion. With this natural definition of angles in taxicab geometry, some of the same difficulties arise as with Euclidean angles in taxicab geometry. Congruent triangles are few and far between. Only with the strictest requirements, namely that all three sides and two angles are congruent, are we able to conclude that two triangles must be congruent.

In addition to creating a natural definition of angles and trigonometric functions, we have also unwittingly created an environment in which a parallax method can be used to determine the exact distance to a nearby object rather than just an approximation. This is not too surprising a result since there exist directions in which one can travel without the distance to an object changing. With this result, and taxicab and Euclidean angle measuring instruments, the exact Euclidean distance to the object can now be found (up to measurement error of course). The trick is to build your own taxicab cross staff or sextant.

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ΠME Journal, Vol. 11, No. 2, pp 97–99, 2000.

ON THE SUMS $\sum_{I=1}^{N} \lceil I/P \rceil^M$ AND $\sum_{I=1}^{N} \lfloor I/P \rfloor^M$

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Abstract. This note gives a demonstration of how the Bernoulli polynomials can be used to derive analytical expressions for the sums $\sum_{i=1}^{n} \lceil i/p \rceil^m$ and $\sum_{i=1}^{n} \lfloor i/p \rfloor^m$, where m is a non-negative integer.

1. Introduction. Given a real number x, denote by $\lceil x \rceil$ the least integer that is no less than x, and denote by $\lfloor x \rfloor$ the greatest integer that is no larger than x. The functions $\lceil x \rceil$ and $\lfloor x \rfloor$ are often referred to as the ceiling- and floor values of x, respectively. Recently, the sums $\sum_{i=1}^{n} \lceil i/p \rceil$ and $\sum_{i=1}^{n} \lfloor i/p \rfloor$, where n and p are arbitrary positive integers have been studied. Sivakumar, Dimopoulos and Lu [8] derived analytical expressions for these sums, and a simplified derivation was given by Tuenter [9], using the well-known formula $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. It turns out that the latter approach is readily generalized, and it is relatively straightforward to derive expressions for the sums of the title, where m is an arbitrary positive integer, using the known formulae for the sums of powers of the integers, as one can find in, for instance [4, p. 269]. The derivation given here is different, and might be used in an undergraduate course on number theory to give another application for, and to illustrate the elegance of the Bernoulli polynomials.

2. Bernoulli polynomials and Power sums. The Bernoulli polynomials [1], discovered by Jacob Bernoulli (1654–1705), and christened as such by Leonard Euler [7], arise in the study of the sums of powers of the first natural numbers:

$$\sigma_m(n) \equiv 0^m + 1^m + \cdots + n^m,$$

and the desire to find closed forms expressions for these sums. Bernoulli was not the first to study the problem of summing powers; before him Johann Faulhaber of Ulm (1580–1635) had already published [3] the formulae for the sums of powers up to and including the exponent 17, and indicated that he had computed the formulae for sums of powers as far the exponent 25. These formulae were derived by a process of intense computation, heuristic reasoning and induction [5, 7]. Bernoulli was the first, however to study the sums of powers in a more structured manner *and* document how he arrived at his results. A modern day account of how Bernoulli approached the problem and arrived at the polynomials that bear his name, is given in [2, pp. 278–283]. The first few Bernoulli polynomials, as can be found in for instance [6], are given by

$$\begin{aligned} B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \end{aligned}$$

and share the property

$$B_m(x+1) - B_m(x) = mx^{m-1}.$$

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Indeed, apart from the constant term, this property uniquely defines the Bernoulli polynomials. Although the value of the constant term plays no role in the subsequent derivations, we mention that is determined by the requirement that the integral of each of the Bernoulli polynomials over the interval $[0,1]$ is 0. An expression for the sum of powers in terms of the Bernoulli polynomials is now easily determined as

$$\sum_{x=0}^{n-1} x^{m-1} = \frac{1}{m}\sum_{x=0}^{n-1}\left[B_m(x+1) - B_m(x)\right] = \frac{1}{m}\left[B_m(n) - B_m(0)\right],$$

and gives the familiar $\sigma_0(n) = n+1$, $\sigma_1(n) = \frac{1}{2}n(n+1)$, $\sigma_2(n) = \frac{1}{6}n(n+1)(2n+1)$, and $\sigma_3(n) = \frac{1}{4}n^2(n+1)^2$.

3. Bernoulli polynomials and the sums $\sum\lfloor i/p\rfloor^m$ and $\sum\lceil i/p\rceil^m$. One can derive closed form expressions for the sums of the title in almost exactly the same manner as was done for the sum of powers of the integers. Let $n = qp + r$, where $0 \le r < p$, so that q is the quotient and r is the remainder in the division of n by p. Now use the property of the Bernoulli polynomials, take $\lfloor i/p\rfloor$ in lieu of x, split the summation into two separate sums, and cancel the common terms to obtain:

$$\sum_{i=0}^{pq-1} \lfloor i/p\rfloor^{m-1} = \frac{1}{m}\sum_{i=0}^{pq-1}\left[B_m\left(\lfloor i/p\rfloor + 1\right) - B_m\left(\lfloor i/p\rfloor\right)\right] = \cdots = \frac{p}{m}\left[B_m\left(q\right) - B_m\left(0\right)\right].$$

Now add the sum of the remainder $\sum_{i=pq}^{pq+r} \lfloor i/p\rfloor^{m-1} = (r+1)q^{m-1}$, and one obtains the desired closed form expression

$$\sum_{i=0}^{n} \lfloor i/p\rfloor^{m-1} = \frac{p}{m}\left[B_m(q) - B_m(0)\right] + (r+1)q^{m-1}.$$

It is now possible to derive a closed form expression for the ceiling sum by using the result for the floor sum, the relation $\lceil x\rceil = 1 + \lfloor x\rfloor$ (for non-integer values of x), and the binomial theorem. However, a moment's reflection will make one realize that the two sums have a similar structure, and that it is more straightforward to simply repeat the method of proof for the floor sum, and derive

$$\sum_{i=1}^{n} \lceil i/p\rceil^{m-1} = \frac{p}{m}\left[B_m(q+1) - B_m(1)\right] + r(q+1)^{m-1}.$$

The first few of these sums are tabulated below, and the expressions for the sums $\sum\lfloor i/p\rfloor$ and $\sum\lceil i/p\rceil$ are, of course, in agreement with the results derived by Sivakumar et al. [8].

$$\sum_{i=1}^{n} \lfloor i/p\rfloor = \frac{1}{2}q(n-p+r+2)$$

$$\sum_{i=1}^{n} \lceil i/p\rceil = \frac{1}{2}(q+1)(n+r)$$

$$\sum_{i=1}^{n} \lfloor i/p\rfloor^2 = \frac{1}{6}(n-r)(2q-1)(q-1) + (r+1)q^2$$

$$\sum_{i=1}^{n} \lceil i/p\rceil^2 = \frac{1}{6}(q+1)(2qn+n+4rq+5r)$$

$$\sum_{i=1}^{n} \lfloor i/p\rfloor^3 = \frac{1}{4}(n-r)q(q-1)^2 + (r+1)q^3$$

$$\sum_{i=1}^{n} \lceil i/p\rceil^3 = \frac{1}{4}(q+1)^2(qn+3rq+4r)$$

4. Discussion. As was mentioned in the introduction, the sums of the title can also be expressed in terms of the power sums over the integers:

$$\sum_{i=0}^{n} \lfloor i/p\rfloor^m = p\,\sigma_m(q-1) + (r+1)q^m \quad\text{and}\quad \sum_{i=1}^{n} \lceil i/p\rceil^m = p\,\sigma_m(q) + r(q+1)^m,$$

where the first expression is valid for all non-negative integers m, and the second for all positive integers m. This is not difficult to show by breaking up the sums into segments of length p. It also follows directly by substitution of the expressions for $B_m(q) - B_m(0)$, and $B_m(q+1) - B_m(1)$, derived in the previous section. The formulation in terms of the Bernoulli polynomials has the advantage of being more practical in the environment of computer algebra packages, as most of these have the Bernoulli polynomials built in.

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List of Referees

The editor wishes to acknowledge the substantial contributions made by the following mathematicians who refereed manuscripts for the Pi Mu Epsilon Journal during the past year.

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Call for Referees. If you are willing to serve as a referee in the future, please email your name, affiliation, mailing address, and the mathematical areas you feel competent to referee to bservat@wpi.edu (Brigitte Servatius).

ΠME Journal, Vol. 11, No. 2, pp 101–112, 2000.

PROBLEM DEPARTMENT

EDITED BY CLAYTON W. DODGE

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed to arrive by December 1, 2000. Solutions by students are given preference.

Problems for Solution.

980. *Proposed by the editor.*

The addition alphametic

$$HALF + HALF = WHOLE$$

has unique solutions in both bases 7 and 8. Of course, in any base $WHOLE$ must be an even number. It is curious that in base 9 there are three solutions, two of which have $HALF$ even. Find that base 9 solution in which $HALF$ is an odd number.

981. *Proposed by Cecil Rousseau, The University of Memphis, Memphis, Tennessee.*

Show that the set

$$\{\lfloor\sqrt{2}\rfloor, \lfloor 2\sqrt{2}\rfloor, \lfloor 3\sqrt{2}\rfloor, \cdots, \lfloor n\sqrt{2}\rfloor, \cdots\},$$

where n is a natural number and $\lfloor x\rfloor$ is the greatest integer in x, contains infinitely many powers of 3.

982. *Proposed by Charles Ashbacher, Charles Ashbacher Technologies, Hiawatha, Iowa.*

In his book "Comments and Topics on Smarandache Notions and Problems", K. Kashihara defines for any positive integer n, the *Smarandache Inferior Square Part,* $\mathrm{SISP}(n)$, to be the largest square less than or equal to n and the *Smarandache Superior Square Part,* $\mathrm{SSSP}(n)$ to be the smallest square greater than or equal to n. Now define $s_n = \sqrt[n]{\mathrm{SSSP}(0) + \cdots + \mathrm{SSSP}(n)}$ and $t_n = \sqrt[n]{\mathrm{SISP}(0) + \cdots + \mathrm{SISP}(n)}$.

a) Find the value of $\lim\limits_{n\to\infty} s_n - t_n$.

b) Find the value of $\lim\limits_{n\to\infty} \dfrac{s_n}{t_n}$.

983. *Proposed by Rex H. Wu, Brooklyn, New York.*

Evaluate the integrals

a) $\displaystyle\int_0^{\pi/2} \ln\left(\frac{1+\sin(x)}{1+\cos(x)}\right)dx$ and

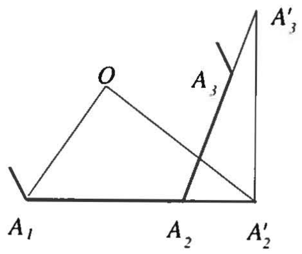
b) $\displaystyle\int_0^{\pi/2} \ln\left(\frac{1+\cos(x)+\sin(x)}{1+\cos(x)}\right)dx.$

984. *Proposed by Peter A. Lindstrom, Batavia, New York.*
Test for convergence the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{n^n}{n!e^n}\right).$$

985. *Proposed by Ayoub B. Ayoub, Penn State Abington College, Abington, Pennsylvania.*

Extend the sides A_1A_2 and A_2A_3 of a regular n-gon $A_1A_2A_3\ldots A_n$ to A_2' and A_3' respectively such that $A_2A_2' = A_3A_3'$ and $m\angle A_2A_2'A_3' = 90°$. Show that $m\angle A_1OA_2' = 90°$, where O is the center of the n-gon.



986. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find a triangle in the plane which can be dissected into five triangles all similar to itself.

987. *Proposed by Kenneth P. Davenport, Frackville, Pennsylvania.*

For a given positive integer n find for what positive integers $b > n$ and a there is a solution to the Diophantine equation

$$1 + 2 + \cdots + n = b + (b+1) + \cdots + (b+a).$$

988. *Proposed by Kenneth P. Davenport, Frackville, Pennsylvania.*

For what values of n is this sum the square of an integer:

$$1^3 - 2^3 + 3^3 - \cdots + (-1)^{n+1}n^3.$$

989. *Proposed by Joel Brenner, Palo Alto, California.*

a) In the set of all primes find the density of the primes p such that the greatest common divisor of all the divisors of $p-1$ is 1. Note that a statistical experiment would lead to a wrong answer since three of the first six primes have this property.

b) In the set of all positive integers find the density of those integers $n > 1$ such that the greatest common divisor of all the divisors of $n-1$ is 1.

990. *Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Identify all triangles ABC such that $\cos^2 A + \cos^2 B + \cos^2 C = 1$.

991. *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

Eight people play rounds of golf in 2 foursomes at a time. Thus, for example, one round might have the foursomes $ABCD$ and $EFGH$. They desire to have each pair of players playing together in a foursome exactly the same number of times.

a) Is this possible in six rounds?
b) Is it possible in 7 rounds?
c) Explain why your answers to the above questions differ.

992. *Proposed by Mark Evans, Louisville, Kentucky.*

Consider three statistical distributions f, g, and h such that, for $0 < k < 1$,

$$h = kf + (1-k)g.$$

a) Express the variance of h as a function of k, the variances of f and g, and the means of f and g.

b) Use the expression derived in (a) to show that the variance of h equals the variance of f when $f = g$.

*c) Explain the results of (a).

993. *Proposed by Les Wood, Forest City, Maine.*

Determine which stacks in less space, logs or split wood. Assume the logs are uniformly perfect cylinders of radius r and constant length. Assume these logs are split with no waste into perfect quarters, that is, their cross sections are circular sectors of central angle 90°.

Corrections. Cecil Rousseau of The University of Memphis pointed out that the denominator in the first integral in Problem 970 [Fall 1999, p. 47] should be under a radical. That is, the correct integral is

$$\int_0^{\pi/4}\frac{\cos(x)\ln(\sin(x))}{\sqrt{\sin(x)\cos(2x)}}\,dx.$$

Rex H. Wu found three errors in his solution to Problem 943 [Fall 1999]. There should not be an α in front of the fraction in the very last product in the displayed equation at the bottom of page 52. About 1/3 of the way down page 53 the phrase "V_n is strictly decreasing" should read "V_n is strictly increasing." Finally, at the very end of the solution, about 2/3 of the way down page 53, the right side of the last displayed inequality, from the $<$ sign on, should be replaced by

$$=\prod_{k=1}^{n}\left(1+\frac{\alpha^2+1}{\alpha^{4k}-1}\right) < \prod_{k=1}^{n}\left(1+\left(\frac{\alpha^2+1}{\alpha^4}\right)^k\right).$$

In the figure for the solution to Problem 946 [Fall 1999, page 55] reorder the letters A, B, C, X, Y, Z respectively by B, C, A, Z, X, Y.

Solutions.

953. [Spring 1999] *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

Since we want to enjoy our cake with a minimum amount of guilt, find the solution to the following base 10 alphametic that yields the minimum value for $ICING$.

$$ICING + CAKE = YUMMY.$$

Solution by **Rex H. Wu,** *Brooklyn, New York.*

Since $I \neq 0$, then try $I = 1$ and $Y = 2$. To minimize C, $C = 5$. Then $U = 0$ since $U \neq I$. So far we have $(I, Y, C, U) = (1, 2, 5, 0)$. Now (G, E) can be one of $\{(3,9), (9,3), (4,8), (8,4)\}$. Again, to minimize G, we take $(G, E) = (3, 9)$ or $(4, 8)$. Then N can be one of $\{3, 4, 6, 7, 8, 9\}$.

If $N = 3$, then $G = 4$ and $E = 8$. And $N + K + 1 = K + 4 \equiv M \pmod{10}$. Then A, K, and M have to be taken from $\{6, 7, 9\}$, which is not possible.

If $N = 4$, then $G = 3$ and $E = 9$. We also have $N+K+1 = K+5 \equiv M(\text{mod } 10)$. Here the choices for A, K, and M are $\{6,7,8\}$. Again it is impossible.

Next try $N = 6$. Then (G, E) can be $(3,9)$ or $(4,8)$. If $(G, E) = (3,9)$, then $N + K + 1 = K + 7 \equiv M(\text{mod } 10)$. Again it is impossible to assign 4, 7 and 8 to A, K and M. If $(G, E) = (4,8)$, then $N + K + 1 = K + 7 \equiv M(\text{mod } 10)$. Once more, there is no way we can assign 3, 7 and 9 to A, K and M.

Finally, if $N = 7$, then $(G, E) = (3,9)$, $K = 8$, $M = 6$ and $A = 4$.

Therefore $ICING = 15173$, $CAKE = 5489$ and $YUMMY = 20662$.

Editorial note: There is one other solution $37318 + 7906 = 45224$ to this alphametic. In either solution, the values of G and E can be interchanged, as can also the values of N and K.

Also solved by **Charles D. Ashbacher**, Charles Ashbacher Technologies, Hiawatha, IA, **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Mark Evans**, Louisville, KY, **Stephen I. Gendler**, Clarion University of Pennsylvania, **Richard I. Hess**, Rancho Palos Verdes, CA, **Yoshinobu Murayoshi**, Okinawa, Japan, **H.-J. Seiffert**, Berlin, Germany, **Kevin P. Wagner**, University of South Florida, Largo, and the **proposer**.

954. [Spring 1999] *Proposed by Florian Luca, Syracuse University, Syracuse, New York.*

For any real number y let $[y]$ be the largest integer less than or equal to y. Suppose M is a set of positive integers with the following property: if $x > 1$ is an element of M, then both $[x\ln(x)]$ and $[\sqrt{x}]$ are elements of M. Show that, if M contains a positive integer greater than 3, then M contains all positive integers.

Solution by **Richard I. Hess**, *Rancho Palos Verdes, California.*

Since $[3\ln 3] = 3$, $[2\ln 2] = 1$, and $[\sqrt{3}] = [\sqrt{2}] = 1$, then possible sets M are $\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{1,2,3\}$, and N, the set of all natural numbers. We show no other sets are possible.

If M contains an integer $x > 3$, then repeated application of $[\sqrt{x}]$ (possibly 0 times) will get you to one of the numbers 4 through 15. Application of the two given functions, perhaps several times, on any of the numbers 5 through 15 will get you to 4. For example, $[5\ln 5] = 8$, $[8\ln 8] = 16$, and $[\sqrt{16}] = 4$, so both 5 and 8 lead to 4. It is easy to check that all other numbers from 6 to 15 also lead to 4. Clearly, numbers 16 or larger lead eventually to 4 through 15 by means of $[\sqrt{x}]$. Thus, if M contains any number greater than 3, it contains the number 4.

Suppose there is a number $m > 3$ not in M. Then M does not contain any integer in any of the intervals

$$\begin{aligned} I_1 &= [m^2, (m+1)^2 - 1], \\ I_2 &= [m^4, (m+1)^4 - 1], \ldots, \\ I_k &= [m^{2^k}, (m+1)^{2^k} - 1], \ldots. \end{aligned}$$

Otherwise, repeated application of $[\sqrt{x}]$ would produce m.

Consider $x_0 = 4$, $x_1 = [4\ln 4], \ldots, x_{n+1} = [x_n \ln x_n]$. We show that the x_n cannot miss all the intervals I_k. That is, if, for a given k, n is the largest integer such that $x_n \le m^{2^k}$ then $x_{n+1} \le [m^{2^k} \ln m^{2^k}]$, so we show that, when k is large enough,

$$m^{2^k} \ln m^{2^k} < (m+1)^{2^k} - 1,$$

and x_{n+1} lies in I_k. To that end we show

$$2^k \ln m = \ln m^{2^k} < \frac{(m+1)^{2^k} - 1}{m^{2^k}} = (1 + \frac{1}{m})^{2^k} - \frac{1}{m^{2^k}}.$$

Since $\ln m < m$ and $(1 + 1/m)^m > 2$, it suffices to show that

$$2^k m \le 2^{2^k/m} = (2^{1/m})^{2^k}.$$

Since 2^k is a factor on the left and an exponent on the right and $2^{1/m} > 1$, then the inequality clearly holds for large enough k. Hence x_{n+1} is in the interval I_k and therefore m is in M.

Also solved by **Rex H. Wu**, Brooklyn, NY, and the **proposer**.

955. [Spring 1999] *Proposed by Peter A. Lindstrom, Batavia, New York.*

Let G be a finite geometric series whose terms are all positive integers. If G has a sum that is a prime number, then prove that the first term is 1 and the number of terms of the series is a prime.

Solution by **Skidmore College Problem Group**, *Skidmore College, Saratoga Springs, New York.*

A counterexample is the series 4 + 6 + 9 whose sum 19 is a prime but whose initial term is not 1. Similarly, a series of just one term violates the theorem. We shall show that the theorem is true if the ratio r is a positive integer and the number of terms is greater than 1, but first we prove the following generalization. Let G be a finite geometric series of $n > 1$ positive integral terms and whose ratio r is the positive rational number s/t in lowest terms. If G has a sum that is a prime number, then the first term is t^{n-1} and the number of terms of the series is a prime.

Let $G = a + ar + ar^2 + \cdots + ar^{n-1}$, so that

$$G \cdot t^{n-1} = at^{n-1} + ast^{n-2} + +as^{n-1} = a(t^{n-1} + st^{n-2} + \cdots + s^{n-1}).$$

Since G is a prime and $(s,t) = 1$, and t divides all terms up to the last one, then t must divide that term, too, so divides a. Divide by t and repeat the argument $n - 1$ times to show that t^{n-1} divides a. If a contains any other prime factor, then G also contains that factor and is not prime because the sum of the terms in the parentheses is greater than 1. Hence, $a = t^{n-1}$ and G equals the quantity in parentheses.

If $n = uv$ is composite, then the terms in parentheses can be grouped in v groups of u terms each and thereby factors with the first group of u terms as one factor. Since u and v are each greater than 1, each factor is greater than 1 and the value in the parentheses is composite. This is impossible since G is prime. Thus n is prime and our generalization is established.

If r is a positive integer and $n > 1$, then $t = 1$ and the corrected theorem follows.

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman**, Berkeley, CA, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Joyce Gendler** and **Stephen I. Gendler**, Clarion, PA, **Richard I. Hess**, Rancho Palos Verdes, CA, **Murray S. Klamkin**, University of Alberta, Canada, **Henry S. Lieberman**, Waban, MA, **H.-J. Seiffert**, Berlin, Germany, **Kenneth M. Wilke**, Topeka, KS, **Rex H. Wu**, Brooklyn, NY, **Monte J. Zerger**, Adams State College, Alamosa, CO, and the **proposer**.

Editorial note: Most solvers explicitly assumed the series had more than one term and that the ratio was a positive integer. Seiffert also provided the same counterexample as in the featured solution.

956. [Spring 1999] *Proposed by Charles Ashbacher, Decisionmark, Cedar Rapids, Iowa.*

For any positive integer n, the value of the *Smarandache function* $S(n)$ is the smallest positive integer m such that n divides $m!$. Thus, for example, $S(1) = 1$, $S(2) = 2$, $S(6) = 3$, and $S(8) = 4$. Let p be an odd prime. Prove that the following summation diverges:

$$\sum_{k=1}^{\infty} \frac{1}{S(p^k)}.$$

Solution by **Kevin P. Wagner**, *student, University of South Florida, Largo, Florida.*

Actually the summation diverges whenever p is any natural number n. For any natural numbers n and k, n^k divides $(nk)!$ And therefore $S(n^k) \leq nk$. Then

$$\sum_{k=1}^{\infty} \frac{1}{S(n^k)} \geq \sum_{k=1}^{\infty} \frac{1}{nk} = \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges whenever n is nonzero.

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Mark Evans,** Louisville, KY, **Stephen I. Gendler,** Clarion University of Pennsylvania, **Grand Valley State University Problem Solving Group,** Allendale, MI, **Richard I. Hess,** Rancho Palos Verdes, CA, **Murray S. Klamkin,** University of Alberta, Canada, **Carl Libis,** Antioch College, Yellow Springs, OH, **H.-J. Seiffert,** Berlin, Germany, **Rex H. Wu,** Brooklyn, NY, and the **proposer**.

957. [Spring 1999] *Proposed by the late Jack Garfunkel, Flushing, New York.*

Triangle ABC is inscribed in a circle. The angle bisectors of ABC are drawn and extended to the circle to points A', B', C'. Triangle $A'B'C'$ is drawn. Prove that $s/r \geq s'/r'$ where s, s', r, r' are respectively the semiperimeters and inradii of triangles ABC and $A'B'C'$.

Solution by **Murray S. Klamkin,** *University of Alberta, Edmonton, Alberta, Canada.*

Note first that by replacing r by F/s, etc., where F is the area of ABC, the inequality becomes the isoperimetric one $F'/s'^2 \geq F/s^2$. Also, $A' = (B+C)/2$, $B' = (C+A)/2$, and $C' = (A+B)/2$.

Much more general results are given in [1]. It is shown there that if ABC and $A'B'C'$ are two triangles such that $A' = w_1A + w_2B + w_3C$, $B' = w_1B + w_2C + w_3A$, and $C' = w_1C + w_2A + w_3B$, where w_1, w_2, w_3, are nonnegative weights whose sum is 1, then

(1) $$F'/s'^2 \geq F/s^2,$$
(2) $$r'/R' \geq r/R,$$
(3) $$s'/R' \geq s/R,$$
(4) $$F'/R'^2 \geq F/R^2.$$

If, additionally, $A'B'C'$ has the same circumradius R as ABC, then (2), (3), and (4) become $r' \geq r$, $s' \geq s$, and $F' \geq F$. There is equality in these inequalities if and only if ABC is equilateral.

The given inequality corresponds to the special case $w_1 = 0$, $w_2 = w_3 = 1/2$.

Reference 1. M. S. KLAMKIN, *Notes on inequalities involving triangles or tetrahedrons,* Publ. Electrotechn. Fak. Ser. Mat. Fiz. Univ. Beograd, No. 330 (1970)4-7.

Also solved by **Paul S. Bruckman,** Edmonds, WA, **Yoshinobu Murayoshi,** Okinawa, Japan, **Rex H. Wu,** Brooklyn, NY, and the **proposer**.

958. [Spring 1999] *Proposed by George Tsapakidis, Agrinio, Greece.*

In a triangle ABC the length of the bisector AD is equal to the length of the median AM, both drawn from the same vertex A. Prove that triangle ABC is isosceles.

I. *Solution by* **Murray S. Klamkin,** *University of Alberta, Alberta, Ontario, Canada.*

The contrapositive theorem is that if $AB \neq AC$, then $AD \neq AM$ and is contained in the more general result that $h_a \leq t_a \leq m_a$ with equality if and only if $AB = AC$. For the sake of completeness, we include the proof.

Let E be the foot of the altitude. Assuming, without loss of generality, that $c \leq b$, then we show that the order of the feet on BC is $[EDMC]$. If B is non-acute, then clearly E is first. For B acute, $BAE = 90° - B \leq A/2$ since $B \geq C$. This gives the order $[BED]$. Since D divides BC in the ratio c/b, then $BD = ac/(b+c) \leq a/2$ and we have the order $[BEDMC]$. This, together with the Pythagorean theorem, gives the desired results.

II. *Solution by* **Yoshinobu Murayoshi,** *Okinawa, Japan.* Since

$$AM = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2} \text{ and } BM = \frac{\sqrt{bc(b+c+a)(b+c-a)}}{b+c},$$

then $(AM)^2 = (AD)^2$ reduces to $(b-c)^2[2(b+c)^2 - a^2] = 0$. Since $b + c > a$, then we must have $b = c$ and the triangle is isosceles.

Also solved by **Miguel Amengual Covas,** Cala Figuera, Mallorca, Spain, **Dipendra Bhattacharya** and **Stephen I. Gendler,** Clarion University of Pennsylvania, **Paul S. Bruckman,** Berkeley, CA, **Mark Evans,** Louisville, KY, **Richard I. Hess,** Rancho Palos Verdes, CA, **Henry S. Lieberman,** Waban, MA, **William H. Peirce,** Rangeley, ME, **H.-J. Seiffert,** Berlin, Germany, **Kevin P. Wagner,** University of South Florida, Largo, **Kenneth M. Wilke,** Topeka, KS, **Rex H. Wu,** Brooklyn, NY, and the **proposer**.

959. [Spring 1999] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find in closed form the sum

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

I. *Solution by* **Karthik Gopalratnam,** *student, Angelo State University, San Angelo, Texas.*

For positive integral n the binomial theorem yields

$$(1+x)^n = \sum_{k=1}^{n} \binom{n}{k} x^k.$$

Differentiating both sides with respect to x, we get

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$

Now set $x = 1$ to obtain

$$\sum_{k=1}^{n} k\binom{n}{k} = n(1+1)^{n-1} = n2^{n-1}.$$

II. *Solution by* **Kenneth M. Wilke,** *Topeka, Kansas.*

Since

$$k\binom{n}{k} = \frac{k \cdot n!}{(n-k)!k!} = \frac{n(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1},$$

we have

$$\sum_{k=1}^{n} k\binom{n}{k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n2^{n-1}.$$

III. *Solution by* **George P. Evanovich,** *South River, New Jersey.*

We have that

$$S = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{n-k} = \sum_{j=0}^{n-1}(n-j)\binom{n}{j}.$$

Now take the average of the first and last expressions to obtain

$$S = \frac{1}{2}\sum_{k=1}^{n}(k+n-k)\binom{n}{k} = \frac{1}{2}n\sum_{k=1}^{n}\binom{n}{k} = \frac{n}{2}2^{n} = n2^{n-1}.$$

Also solved by **Miguel Amengual Covas,** Cala Figuera, Mallorca, Spain, **Frank P. Battles,** Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman,** Berkeley, CA, **Pat Costello,** Eastern Kentucky University, Richmond, **Kenneth B. Davenport,** Frackville, PA, **Rob Downes,** Plainfield, NJ, **Richard I. Hess,** Rancho Palos Verdes, CA, **Murray S. Klamkin,** University of Alberta, Canada, **Carl Libis,** (two solutions) Antioch College, Yellow Springs, OH, **Henry S. Lieberman,** Waban, MA, **Peter A. Lindstrom,** Batavia, NY, **William H. Peirce,** Rangeley, ME, **Shiva K. Saksena,** University of North Carolina at Wilmington, **H.-J. Seiffert,** Berlin, Germany, **Skidmore College Problem Group,** Saratoga Springs, NY, **Kevin P. Wagner,** University of South Florida, Largo, **Rex H. Wu,** Brooklyn, NY, **Monte J. Zerger,** Adams State College, Alamosa, CO, and the **proposer.**

960. [Spring 1999] *Proposed by Timothy Sipka, Alma College, Alma, Michigan.*

A *triangular number* is any number of the form $n(n+1)/2$, where n is a positive integer. Prove that the units digit of any triangular number is 0, 1, 3, 5, 6, or 8.

I. *Solution by* **Kenneth M. Wilke,** *Topeka, Kansas.*

Let $n(n+1)/2 = k$. Then $(2n+1)^2 = 8k+1$. Since odd squares in base ten terminate only in 1, 5, or 9 (never 3 or 7), then $8k+1 \equiv 1$, 5, or 9 (mod 10) and $k \equiv 0$ or 5, 3 or 8, 1 or 6 (mod 10) respectively. That is, $n(n+1)/2$ ends only in 0, 1, 3, 5, 6, or 8.

II. *Solution by* **Monte J. Zerger,** *Adams State College, Alamosa, Colorado.*

Let $T(n) = n(n+1)/2$. Because

$$T(20+n) \equiv T(20-n) \equiv T(n) \pmod{20},$$

it is sufficient to check that the units digit of each of the first ten triangular numbers is one of 0, 1, 3, 5, 6, and 8. These values are actually assumed by $T(4)$, $T(1)$, $T(2)$, $T(5)$, $T(3)$, and $T(7)$ respectively.

Also solved by **Charles D. Ashbacher,** Charles Ashbacher Technologies, Hiawatha, IA, **Paul S. Bruckman,** Berkeley, CA, **Mark Evans,** Louisville, KY, **Stephen I. Gendler,** Clarion University of Pennsylvania, **Daniel Hermann,** Angelo State University, San Angelo, TX, **Richard I. Hess,** Rancho Palos Verdes, CA, **Danner Hodgson,** Belmont University, Nashville, TN, **Carl Libis,** Antioch College, Yellow Springs, OH, **Henry S. Lieberman,** Waban, MA, **Peter A. Lindstrom,** Batavia, NY, **William H. Peirce,** Rangeley, ME, **Mike Pinter,** Belmont University, Nashville, TN, **H.-J. Seiffert,** Berlin, Germany, **Skidmore College Problem Group,** Saratoga Springs, NY, **Kevin P. Wagner,** University of South Florida, Largo, **Rex H. Wu,** Brooklyn, NY, and the **proposer.**

961. [Spring 1999] *Proposed by Charles Ashbacher, Charles Ashbacher Technologies, Hiawatha, Iowa.*

Given any positive integer n, the value of the *Pseudo-Smarandache function* $Z(n)$ is the smallest positive integer m such that n exactly divides

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}.$$

Thus $Z(1) = 1$, $Z(2) = 3$, $Z(3) = 2$, $Z(4) = 7$, etc.

a) Prove there is an infinite family of integers n such that $3 \cdot Z(n) = n$.

b) Prove that there are an infinite number of pairs (m, n) such that $m \cdot Z(n) = n \cdot Z(m)$.

I. *Solution by* **Paul S. Bruckman,** *Berkeley, California.*

a) We show that if $n = 3p$, where p is a prime and $p \equiv -1 \pmod 6$, then $Z(n) = p = n/3$. Let $T(n) = n(n+1)/2$ for all n, and suppose p is a prime and $p = 6u-1$. Then $T(p) = p(p+1)/2 = 3pu$, so that $3p$ divides $T(p)$. Note that $T(p-1) = p(p-1)/2 = p(3u-1)$ is not divisible by 3 and $T(n)$ is not divisible by p for any smaller positive n. Therefore $Z(3p) = p$, which shows that $3Z(3p) = 3p$. Since there are infinitely many primes p with $p \equiv -1 \pmod 6$, we see there are infinitely many n with $n = 3Z(n)$.

b) An obvious and trivial solution is provided by setting $m = n$. We may also generate infinitely many solutions by setting $m = 3p$ and $n = 3q$, where p and q are arbitrary primes satisfying the conditions of part (a).

There are infinitely many other solutions. If p is an odd prime of the form $4k+3$, then $T(p-1) = (4k+2)(4k+3)/2 = (2k+1)(4k+3)$ and $T(p) = (4k+3)(4k+4)/2 = (4k+3)(2k+2)$, so $Z(2p) = p$. (For primes of the form $4k+1$, $Z(2p) = 2p-1$.) So, if p and q are two primes each congruent to 3 modulo 4, then $m = 2p$ and $n = 2q$ yield $mZ(n) = nZ(m) = 2pq$.

These are not the only solutions. In general we require that $m/Z(m) = n/Z(n)$. This ratio need not be an integer. For example, $Z(12) = 8$ and $Z(3) = 2$, so that $3Z(12) = 12Z(3)$.

II. *Solution by* **Murray S. Klamkin,** *University of Alberta, Edmonton, Alberta, Canada.*

First, part (b) follows trivially from part (a).

a) Let p be an odd prime. By Dirichlet's theorem, there are an infinite number of integers m such that $2pm-1$ is a prime. It now follows that $Z(p[2pm-1]) = 2pm-1$, so that $Z(p[2pm-1])/(p[2pm-1]) = 1/p$.

Also solved by **Stephen I. Gendler,** Clarion University of Pennsylvania, **H.-J. Seiffert,** Berlin, Germany, **Kevin P. Wagner,** University of South Florida, Largo, **Kenneth M. Wilke,** Topeka, KS, **Rex H. Wu,** Brooklyn, NY, and the **proposer**.

962. [Spring 1999] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Shoelace clock. You are given a shoelace, some matches, and a pair of scissors. The shoelace burns like a fuse when lit at either end and takes exactly 60 minutes to burn. The burn rate may vary from one point on the shoelace to another, but it has a symmetry property in that the burn rate a distance x from the left end is the same as the burn rate the same distance x from the right end.

a) Find the shortest time interval you can measure.

b) Find the shortest time interval you can measure if you have two such laces that are identical.

c) Repeat part b if the two laces, which still burn for 60 minutes each, are not identical and not symmetric.

Solution by **Rex H. Wu,** *Brooklyn, New York.*

I will solve part (c) first.

c) Label the two laces AB and CD. Light each end of AB and also one end of CD, say C, at the same instant. When AB burns out, put out the fire on CD. If we burn the remaining segment from both ends at the same time, it will take $60/4 = 15$ minutes.

a) Cut the lace AB at its midpoint to get two identical 30-minute laces AM and BN, which are identical but not symmetrical. Lay the laces beside one another so A and B align and M and N align. Light ends A and M at the same instant. When it burns out 15 minutes later, cut lace BN at that same exact point, obtaining two non-identical 15-minute laces. Now we can apply part (c) to measure $15/4 = 3.75$ minutes.

b) Suppose we have three laces that each burn in the same time t, two laces AB and CD that are identical but not symmetrical, and a third lace UV. Line up laces AB and CD so A and C align and B and D align. Now light ends A, U, and V at the same time. At the instant UV burns out, we put out the fire on AB. Call its remaining segment MB and cut CD at the point N that aligns with M. Now we have three laces MB, CN, and ND that each burn time t/2, and with the two laces MB and ND identical. We are back to the situation we began with. The physicists among us will soon find they are burning atoms or quarks while the mathematicians among us can carry out the process ad infinitum. This method can measure time intervals in the form of $t/2^n$.

To solve the case of two identical and symmetrical laces, then, we can cut each lace at the mid-point, obtaining four identical segments. Throw away any one of them and apply the method of the preceding paragraph. There is no minimum time that can be measured.

Also solved by the **proposer**.

Editorial note: These puzzles are from the proposer's booklet Shoelace Clock Puzzles, prepared for the Gathering for Gardner, January, 1998.The proposer credits Carl Morris of Harvard University for the original idea.

963. [Spring 1999] *Proposed by Peter A. Lindstrom, Batavia, New York.*

Consider the functions

$$f(x) = \sin(\cos x) + \cos x \text{ and } g(x) = \sin(\cos x) - \cos x$$

on the interval $0 \le x \le \pi$. Without using the calculus,

a) show that their graphs are each symmetric about the point $(\pi/2, 0)$.

b) show that f is always decreasing, so that $f(\pi) \le f(x) \le f(0)$.

c) show that g is always increasing, so that $g(0) \le g(x) \le g(\pi)$.

I. *Solution to parts* (a) *and* (b) *by* **Kevin P. Wagner,** *University of South Florida, Largo, Florida.*

a) Clearly $f(\pi/2) = g(\pi/2) = 0$, $\cos(\pi - x) = -\cos x$ and $\sin(-x) = -\sin x$. Hence $f(\pi - x) = f(x)$ and $g(\pi - x) = g(x)$, so f and g are both symmetric about the point $(\pi/2, 0)$.

b) Let $0 \le x \le y \le \pi$. Then $1 \ge \cos x \ge \cos y \ge -1$. If $-1 \le u \le v \le 1$, then $\sin u \le \sin v$. Therefore, $\sin(\cos x) \ge \sin(\cos y)$, so $f(x) \ge f(y)$ and f is decreasing on the interval $[0, \pi]$.

II. *Solution to part* (c) *by* **Shiva K. Saksena,** *University of North Carolina at Wilmington, North Carolina.*

c) Let $0 \le x < y \le \pi/2$, so that $0 \le 2x < 2y \le \pi$. It thus suffices to show that $g(2y) - g(2x) > 0$. To that end we have that

$$\begin{aligned}
&g(2y) - g(2x)\\
&= \sin(\cos 2y) - \cos 2y - \sin(\cos 2x) + \cos 2x\\
&= [\sin(\cos 2y) - \sin(\cos 2x)] - (\cos 2y - \cos 2x)\\
&= 2\cos[(\cos 2y + \cos 2x)/2]\sin[(\cos 2y - \cos 2x)/2] + 2\sin(y+x)\sin(y-x)\\
&= 2\cos[\cos(y+x)\cos(y-x)]\sin[-\sin(y+x)\sin(y-x)] + 2\sin(y+x)\sin(y-x)\\
&> -2\cos[\cos(y+x)\cos(y-x)][\sin(y+x)\sin(y-x)] + 2\sin(y+x)\sin(y-x)\\
&= 2\sin(y+x)\sin(y-x)\{-\cos[\cos(y+x)\cos(y-x)] + 1\} > 0
\end{aligned}$$

since $\sin(-u) > -u$ when $u > 0$ and since $0 < y - x < y + x < \pi$. Therefore, $g(x)$ is increasing on $[0, \pi]$.

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Mark Evans,** Louisville, KY, **Richard I. Hess,** parts (a) and (b), Rancho Palos Verdes, CA, **Shiva K. Saksena,** University of North Carolina at Wilmington, **Rex H. Wu,** Brooklyn, NY, and the **proposer**.

***964.** [Spring 1999] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

There are n_k balls of color k for $k = 1, 2, \ldots, r$. The total number of balls is $n_1 + n_2 + \cdots + n_r = 2^m$, where m is a positive integer.

a) In how many ways can these balls be separated into unordered color pairs?

b) Find the probability of selecting a particular color pair.

I. *Solution to Part* (a) *by* **Mark Evans,** *Louisville, Kentucky.*

The basic approach is to construct an upper triangular square matrix of order r for each partition of the balls into pairs. In this matrix row k and column k each represent color k and a number s in the i, j position represents s pairs of colors i and j. For instance, the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

represents the partition into the pairs 11, 22, 23, 33, 34, 44, 44 of the following 14 balls: two of color 1, three of color 2, four of color 3, and five of color 4. Clearly the number of balls of color k is equal to the sum of the elements in row k plus those in column k.

A computer program and spreadsheet was designed to handle up to 100 colors in theory, however the intensity of the calculation rapidly becomes prohibitive. Some solutions include

956 ways for 1, 2, 2, 3, 4, 4 balls of each of 6 colors,
913 ways for 5, 6, 7, 10 balls of each of 4 colors,
1065 ways for 2, 3, 4, 5, 6 balls of each of 5 colors,
73 ways for 2, 2, 2, 2, 2 balls of each of 5 colors, and
58 ways for 2, 3, 4, 5 balls of each of 4 colors.

Editorial note. Upon request the problem department editor will send to any reader a copy of Evans' program and complete output for the last listed case.

II. *Solution to Part* (b) *by the* **Problem Department Editor**.

Since we wish to pick just one unordered color pair, this part apparently is independent of Part (a). With $2m$ balls, there are $(2m)(2m-1)/2$ ways to select an unordered pair. Of these, there are $n_j n_k/2$ ways to pick a pair of distinct colors j and k and there are $n_k(n_k-1)/2$ ways to choose a pair of color k. Hence their probabilities are respectively

$$\frac{n_j n_k}{2m(2m-1)} \text{ and } \frac{n_k(n_k-1)}{2m(2m-1)}.$$

965. [Spring 1999] *Proposed by David Iny, Baltimore, Maryland.*

Evaluate the integral

$$\int_0^\infty \frac{e^{-x}}{1+x}dx.$$

Solution by **Andrew Ostergaard,** *high school student, Hopatcong, New Jersey.*

The exponential integral of x, $\mathrm{Ei}(x)$, is given by

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^t}{t}dt = -\gamma - \ln x + \left(\frac{x}{1\cdot 1!} - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!}\cdots\right),$$

where γ is Euler's constant, given by

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right) = 0.5572156\ldots.$$

Now

$$\int_0^\infty \frac{e^{-x}}{1+x}dx = \int_1^\infty \frac{e^{-(u-1)}}{u}du = e\int_1^\infty \frac{e^{-u}}{u}du = e\mathrm{Ei}(1) = 0.596347\ldots.$$

Also solved by **Paul S. Bruckman,** Berkeley, CA, **Kenneth B. Davenport,** Frackville, **PA**, **Richard I. Hess,** Rancho Palos Verdes, CA, **Murray S. Klamkin,** University of Alberta, Canada, and the **proposer**.

The 1999 National Pi Mu Epsilon Meeting

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics society was held in Providence, RI. from July 31 to August 2 1999. As in the past, the meeting was held in conjunction with the national meeting of the Mathematical association of America's Student Sections.

The J. Sutherland Frame Lecturer was **V. Frederick Rickey** from the US Military Academy at West Point. His presentation was entitled "The Creation of the Calculus: Who, What, When, Where, Why".

Student Presentations. The following student papers were presented at the meeting. An asterisk (*) after the name of the presenter indicates that the speaker received a best paper award.

Constructing Graphs from Digraphs,
Mark Crawford and Oscar Neal
Michigan Epsilon – Western Michigan University

Number of Spanning Trees of a $1\times n$ Grid Graph
Melissa Desjarlais
Michigan Theta – Alma College

Construction of Bond Lattices for Trees, Unions of Trees, and other Graphs
Ben Goodwin
Arkansas Beta – Hendrix College

Seeing the Trees Through the Forest
Tammylynne Johnson
Wisconsin Epsilon – Carthage College

Characteristics of Graphs with Linked and Disjoint Cycles
Jennifer Hespen, Trent Lalonde, Katherine Sharrow and Nathan Thomas
New York Omicron – Clarkson University

Strong Selectivity, Monochromatic, and Zero-Sum Solutions to Equations
Kate Rendall
Wisconsin Delta – St. Norbert College

Normalizability of the Moore-Penrose Inverse
Don Hixon
South Dacota Alpha – University of South Dakota

Methods of Solutions of Linear Equations
Anna Pietrusinska
New Jersey Gamma – Rutgers University

Rings of Integer Valued Polynomials
Sanjai Kumar Gupta*
North Carolina Beta – University of North Carolina at Chapel Hill

Divisibility Tests for Large Numbers
Michael Nasvadi
Ohio Epsilon – Kent State University

How Does a Bouncy Ball Bounce?
Teresa Selee*
Ohio Xi – Yougstown State University

The Goat and the Silo
Tom Wakefield
Ohio-Xi – Youngstown State University

Invariant Subspace Problem
Emilie Wiesner
Virginia Theta – Washington and Lee University

The Running of the Bulls; When Will it End?
Robert Shuttleworth
Ohio Xi – Youngstown State University

Making Music: the Scientific Sonification and Vizualization of data
Libby Wiebel*
Wisconsin Delta – St. Norbert College

Mathematical Modeling of Warfare
Ben Jantson*
Ohio Xi – Youngstown State University

Is that Body a Perfect 10?
Kim Ramsey and Sarah Clippinger
Ohio Delta- Miami University

Using Integration to Measure the Volume of Oil Tanks
Matthew Palmer
Ohio Nu – The University of Akron

Sound Investments
Ben Keck
Ohio Xi – Youngstown State University

Mathematics at the Market
Sara LaLumia*
Ohio Xi – Youngstown State University

Leak Detection in Pressurized Pipe Lines
Brian Ball and Paul Dostert
Massachusetts Alpha – WPI, Virginia Delta -JMU

The Mathematics Behind Microscopic Temperature Sensors
John Slanina
Ohio Xi – Youngstown State University

Helicity and Writhing for Non-closed Curves
David Futer
Pennsylvania Alpha – University of Pennsylvania

An Introduction to Fractals and some Real World Applications
Scott Fallstrom
Washington Zeta – Eastern Washington University

An Exploration of Life in Two Dimensions
Pace Petty
Texas Delta – Stephen F. Austin State University

A Subset of R^2 that Intersects Every Circle Exactly 3 times
Ben Byer
Ohio Delta – Miami University

The Mother Worms Blanket Problem
Robin Driesner*
Illinois Zeta – Southern Illinois University at Edwardsville

The Perona-Malik Model in Computer Vision
James Tripp
Virginia Alpha – University of Richmond

Chaotic Attractions near Forbidden Symmetry
Jeffrey Dumont*
Pennsylvania Tau – Lafayette College

The Cinese Remainder Theorem
Judy Maendel
Ohio Omicron – Mount Union College

The Moebius Problem
Duane Farnsworth
Ohio Omicron – Mount Union College

The Structure of an Odd Perfect Number
Matthew Konicki
Virginia Zeta – Mary Washington College

Waring's Problem in Number Theory
Katarsyna Potocka
New Jersey Theta – The college of New Jersey

Carmichael, Pseudo-primes, and Sigma-Phi Theta
Kevin Weis
New Jersey Theta – The College of New Jersey

The use of Three Branches of Mathematics on the General Term of the Pell Sequence
Louis Richard Camara
Florida Epsilon – University of South Florida

Call For Papers.

The next ΠME meeting will take place in Los Angeles, California, August 3–5, 2000. See the ΠME webpage (http://www.pme-math.org/) for application deadlines and forms. See also the MAA webpage (http://www.maa.org/meetings/mathfest00) for other activities in the Golden State.

MATH ACROSTIC

The MATHACROSTIC in this issue has been contributed by Dan Hurwitz.

a. Fermat wrote in the margin of this book — 096 113 017 166 028 100 053 131 178 075 004 063 156 189 037 047 168

b. A maximal base determined by nets — 112 104 052 192 162 184 061 140 020 012 115

c. Assigned quantities — 079 182 030 085 123 147 071 155 136

d. They help visualize conditional probabilities (2 words) — 127 016 110 188 158 024 141 081 048 059 035 092

e. Σωκρατες did this — 066 124 031 187 109

f. English Number Theorist (187-1947) — 033 130 060 146 077

g. Classmates probabilities (2 words) — 098 111 180 038 025 144 154 034 074 089

h. He proved the Prime Number Theorem — 170 069 080 008 054 036 177 027

i. Based on predecessor — 067 014 107 191 169 043 057 117 160

j. A property of Pascal's Triangle — 056 088 163 064 194 134 148 005

k. Slight amplitude variation — 151 186 091 084 172 129

l. Approximated π with 355/113 circa 1573 — 073 108 002 185

m. A conjecture about ζ — 011 116 065 183 078 145 042 018 138 049 171 001 090 122 101 026 150

n. Exists between congruent triangles — 193 040 152 120 164 006 072 019

o. Path analysis used for resource allocation — 058 176 149 165 041 102 013 125

p. Used by Gerbert to replace counters — 050 133 118 039 093 070

q. Author of "Théorie Analytique des Probabilités". — 105 126 139 173 015 179 003

r. Made an error of the first kind — 087 128 097 082 174 062 135 023

s. To stand in the way of a function extension — 044 137 195 153 051 095 161 032

t. One group having order 24 — 132 099 142 021 009 046 157 103 181

u. Did definitive work on associative systems — 086 007 119 167

v. Interesting property of 370 (5 words) — 094 055 076 143 045 190 121 083 010 029 114 068 175 159 106 022

001m	002l	003q		004a	005j	006n	007u	008h	009t	010v	011m	012b	013o	014i
	015q	016d	017a	018m	019n	020b	021t	022v		023r	024d	025g	026m	027h
028a	029v		030c	031e	032s	033f	034g	035d	036h	037a	038g	039p	040n	
041o	042m	043i	044s		045v	046t	047a	048d		049m	050p	051s	052b	053a
	054h	055v	056j	057i	058o		059d	060f	061b	062r	063a	064j	065m	066e
067i	068v		069h	070p	071c	072n	073l	074g	075a	076v	077f		078m	079c
080h		081d	082r	083v	084k	085c	086u	087r	088j		089g	090m	091k	092d
093p		094v	095s	096a	097r	098g	099t	100a	101m		102o	103t	104b	105q
106v	107i		108l	109e	110d		111g	112b	113a	114v	115b	116m	117i	118p
119u	120n		121v	122m	123c	124e		125o	126q	127d	128r	129k		130f
131a	132t	133p	134j	135r	136c		137s	138m		139q	140b	141d	142t	143v
	144g	145m	146f		147c	148j	149o	150m	151k	152n	153s	154g	155c	
156a	157t	158d		159v	160i	161s	162b	163j	164n		165o	166a	167u	
168a	169i	170h	171m	172k	173q		174r	175v	176o	177h	178a	179q	180g	181t
182c	183m		184b	185l	186k		187e	188d	189a	190v	191i	192b	193n	194j
195s														

The solution to the MATHACROSTIC in last issue was taken from "Dynamics and Bifurcations", by Jack Hale and Huseyin Kocak:

To facilitate qualitative analysis, geometric concepts such as vector field, orbit, equilibrium point, and limit set are included in this discussion. The next topic is the notion of stability of an equilibrium point and the role of linear approximation in determining stability.

Jeanette Bickley and Charles R. Diminni where the first solvers to submit solutions.

Contents