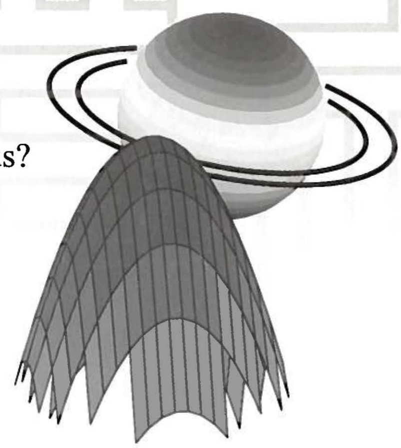
Can you locate the locus?

Problem 1011



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THE PROBABILITY THAT TWO ELEMENTS IN AN EXTRASPECIAL P-GROUP COMMUTE

MARK AFDAHL, GRETCHEN CHRISTIANSON, JILL DIETZ, KIRSTIN KNUDSON, ALFRED FURTH AND SARAH VOLKENANT*

Abstract. We find a formula for the probability that two elements in a central product of groups commute, and use it to compute the probability that two elements in an extraspecial p-group commute.

The paper is organized as follows. In Section 1 below, we give the necessary background information for determining the probability that two elements in a group commute. In Section 2 we show the computations for extraspecial groups of order p^3. In Section 3 we discuss central products of groups. In Section 4, we apply the theory to extraspecial p-groups of any order. Finally, in Section 5 we discuss some computations and open problems.

1. Background Information. Given a group G, we are interested in the probability that two elements picked randomly from G commute. This question is only interesting when G is not abelian!

We will use the following notation in this paper:

- G is a group
- $\mathrm{Pr}(G)$ is the probability that two elements in G commute
- $\mathrm{Cl}_G(x) = \mathrm{Cl}(x) = \{gxg^{-1} \mid g \in G\}$ is the set of conjugates of x in G (where $x \in G$)
- $C(x) = \{g \in G \mid gx = xg\}$ is the centralizer of x in G (where $x \in G$)
- $Z(G)$ is the center of G
- $[G, G]$ is the commutator subgroup of G
- $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in G$

There are two approaches to this question, both of which will be utilized in this paper. First, one can directly compute

$$\mathrm{Pr}(G) = \frac{\#\text{ commuting pairs}}{\#\text{ pairs}}.$$

The number of elements which commute with a given $x \in G$ is equal to $|C(x)|$. Thus the total number of commuting pairs is $\sum_{x \in G} |C(x)|$. The total number of pairs of elements in G is $|G|^2$, thus we get

$$\mathrm{Pr}(G) = \frac{\sum_{x \in G} |C(x)|}{|G|^2}.$$

Second, it is well established that one can use the theory of group actions to get a different formulation for $\mathrm{Pr}(G)$. Below we present a discussion of this based on the work in [3].

Let G act on itself by conjugation. The orbit of an element $x \in G$ under this action is called the class of x and is denoted $\mathrm{Cl}(x)$. The theory of group actions tells us that $|\mathrm{Cl}(x)| = [G : G_x]$, where G_x is the stabilizer of x. Under the action of conjugation, the stabilizer of x equals $C(x)$, the centralizer of x in G.

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If $x, y \in G$ are conjugate elements, then $\mathrm{Cl}(x) = \mathrm{Cl}(y)$. Thus we have

$$\begin{aligned} |\mathrm{Cl}(x)| &= |\mathrm{Cl}(y)| \\ \Rightarrow \quad [G : G_x] &= [G : G_y] \\ \Rightarrow \quad [G : C(x)] &= [G : C(y)] \\ \Rightarrow \quad |C(x)| &= |C(y)|. \end{aligned}$$

Let t_i be the number of conjugates of the element x_i. Then we have

$$\mathrm{Pr}(G) = \frac{\sum_{x \in G} C(x)}{|G|^2} = \frac{\sum_{x_i \in I} t_i |C(x_i)|}{|G|^2}$$

where the right hand sum runs over a set I containing one element from each distinct conjugacy class.

Now $t_i = |\mathrm{Cl}(x_i)| = [G : C(x_i)]$ so we have

$$\mathrm{Pr}(G) = \frac{\sum_{x_i \in I} [G : C(x_i)] |C(x_i)|}{|G|^2}.$$

Finally, LaGrange's theorem tells us that $[G : C(x_i)]|C(x_i)| = |G|$, so the numerator in the fraction above is equal to $m|G|$ where $m = |I|$ is the number of distinct conjugacy classes of elements in G. Let $n = |G|$ then we see that

$$\mathrm{Pr}(G) = \frac{m}{n}.$$

2. Extraspecial Groups of order p^3. If G is a group then the Frattini subgroup of G, denoted $\Phi(G)$, is the intersection of all the maximal subgroups of G. A group G is *extraspecial* if $Z(G) = [G, G] = \Phi(G) \cong \mathbf{Z}/k\mathbf{Z}$ for some $k \in \mathbf{Z}$. There are two non-isomorphic extraspecial groups of order p^3, for each odd prime p. We give them below in terms of their generators and relations.

- $M = \{x, y \mid x^{p^2} = y^p = 1, [x, y] = x^p, x^p \in Z(M)\}$
- $E = \{a, b \mid a^p = b^p = 1, [a, b] = n, n \in Z(E)\}$

PROPOSITION 1. $\mathrm{Pr}(M) = \frac{p^2+p-1}{p^3}$.

Proof. We will use the first method mentioned in Section 1 to compute $\mathrm{Pr}(M)$.

We can use the relations in the group to write every element of M as $x^i y^j$ for some $i = 0, 1, \ldots, p^2 - 1$ and $j = 0, 1, \ldots, p - 1$. Note that the commutator relation implies $y^k x^l = x^{l-klp} y^k$.

We want to count the number of elements in $|C(m)|$ for all $m \in M$. Write $m = x^i y^j$ then an element $x^r y^s \in M$ commutes with m if and only if

$$\begin{aligned} x^i y^j x^r y^s &= x^r y^s x^i y^j \\ \Leftrightarrow x^{i+r-jrp} y^{r+s} &= x^{i+r-sip} y^{j+s}. \end{aligned}$$

This is true if and only if $jrp \equiv sip \bmod p^2$. Thus, $x^r y^s$ commutes with m whenever $is - jr \equiv 0 \bmod p$.

Suppose $i \equiv 0 \bmod p$ and $j \equiv 0 \bmod p$, then $is - jr$ is always equivalent to 0 mod p, so r can be any integer between 0 and $p^2 - 1$, and s can be any integer between 0 and $p - 1$. Thus, p^3 elements commute with m under these conditions. Moreover, there are p elements of the form $x^i y^j$ with $i, j \equiv 0 \bmod p$. (Note that this makes sense since such an element must be in $Z(M)$.)

Suppose $i \equiv 0 \bmod p$ and $j \not\equiv 0 \bmod p$, then $is - jr \equiv -jr \equiv 0 \bmod p$ implies $-j^{-1}jr \equiv r \equiv 0 \bmod p$, while s can be anything. Thus, p^2 elements commute with m under these conditions. Moreover, there are $p(p-1)$ elements of the form $x^i y^j$ with $i \equiv 0 \bmod p$ and $j \not\equiv 0 \bmod p$.

Suppose $i \not\equiv 0 \bmod p$ and $j \equiv 0 \bmod p$, then $is - jr \equiv is \equiv 0 \bmod p$ implies $s \equiv 0 \bmod p$, and r can be anything. Thus, p^2 elements commute with m under these conditions. Moreover, there are $p^2 - p$ elements of the form $x^i y^j$ with $i \not\equiv 0 \bmod p$ and $j \equiv 0 \bmod p$.

Suppose $i \not\equiv 0 \bmod p$ and $j \not\equiv 0 \bmod p$. If $r \equiv 0 \bmod p$ then $is \equiv 0 \bmod p$ implies $s \equiv 0 \bmod p$. If $r \not\equiv 0 \bmod p$ then $is - jr \equiv 0 \bmod p$ implies $s \equiv i^{-1}jr \bmod p$. Thus, $p + (p^2 - p)$ elements commute with m under these conditions. Moreover, there are $(p^2 - p)(p - 1)$ elements of the form $x^i y^j$ with $i \not\equiv 0 \bmod p$ and $j \not\equiv 0 \bmod p$.

Putting all this information together, we see that

$$\begin{aligned} \sum |C(m)| &= [p \cdot p^3] + [p(p-1) \cdot p^2] + [(p^2 - 1) \cdot p^2] + [(p^2 - p)(p - 1) \cdot p^2] \\ &= p^3(p^2 + p - 1). \end{aligned}$$

Thus, $\mathrm{Pr}(M) = \frac{p^3(p^2+p-1)}{p^6} = \frac{p^2+p-1}{p^3}$ and the proposition is proved. □

PROPOSITION 2. $\mathrm{Pr}(E) = \frac{p^2+p-1}{p^3}$.

Proof. We will use the second method mentioned in Section 1 to determine $\mathrm{Pr}(E) = \frac{m}{n}$, where m is the number of distinct conjugacy classes in E, and n is the order of E.

Every element of E can be written uniquely in the form $a^i b^j n^k$ where $i, j, k = 0, 1, \ldots p - 1$. Recall that $n \in Z(E)$ and note that $b^k a^l = a^l b^k n^{-kl}$. We need to determine the conjugacy class of each element in E, then count the number of distinct classes.

Let $x = a^i b^j n^k \in E$. $\mathrm{Cl}(x) = \{gxg^{-1} \mid g \in E\}$ where $g = a^r b^s n^t$ for some $r, s, t = 0, 1, \ldots, p - 1$. Now

$$\begin{aligned} gxg^{-1} &= a^r b^s n^t a^i b^j n^k n^{-t} b^{-s} a^{-r} \\ &= a^r b^s a^i b^j b^{-s} a^{-r} n^k \\ &= a^r b^s a^i b^j a^{-r} b^{-s} n^{k-rs} \\ &= a^r b^s a^{i-r} b^{j-s} n^{k-rs+rj} \\ &= a^{r+i-r} b^{s+j-s} n^{-s(i-r)+k-rs+rj} \\ &= a^i b^j n^{k+rj-is}. \end{aligned}$$

If both $i = 0$ and $j = 0$, then $\mathrm{Cl}(x)$ consists of the single element n^k. Otherwise, the exponent $k + rj - is$ can take on any value between 0 and $p - 1$. In this case, $\mathrm{Cl}(x) = \{x, xn, xn^2, xn^3, \ldots, xn^{p-1}\}$. There are p distinct classes consisting of a single element, and there are $p^2 - 1$ distinct classes consisting of p elements. The total number of distinct conjugacy classes is $p^2 + p - 1$ and we see that $\mathrm{Pr}(E) = \frac{p^2+p-1}{p^3}$. □

3. Central Products of Groups. Let $G_1, G_2, \ldots, G_s$ be normal subgroups of a group G. G is the *central product* of the G_i, written $G = G_1 \circ G_2 \circ \cdots \circ G_s$ if

- $G = G_1 G_2 \cdots G_s$,
- $[G_i, G_j] = 1$ for $i \neq j$,
- $G_i \cap (G_1 G_2 \cdots G_{i-1} G_{i+1} \cdots G_n) \leq Z(G)$

THEOREM 3. *Let $G = G_1 \circ G_2$ be a central product of subgroups of G, then*

$$\mathrm{Pr}(G) = \frac{m_1 + m_2 + (m_1 - z)(m_2 - z) - z}{n^2}$$

where m_i is the number of distinct conjugacy classes in G_i, $z = |Z(G)|$, and $n = |G|$.

We will determine the conjugacy class of each $x \in G$ in each of three cases: $x \in G_1$, $x \in G_2$, $x \in G$ but $x \notin G_1$ and $x \notin G_2$.

CLAIM 4. *If $x \in G_1$ then $\mathrm{Cl}_G(x) = \mathrm{Cl}_{G_1}(x)$.*

Proof. Let $y \in G$, then there exist $g_1 \in G_1$ and $g_2 \in G_2$ such that $y = g_1g_2$. Consider $yxy^{-1} = g_1g_2xg_2^{-1}g_1^{-1} = g_1xg_2g_2^{-1}g_1^{-1} = g_1xg_1^{-1}$ since $[x, g_2] = 1$. Thus, $\mathrm{Cl}_G(x) = \mathrm{Cl}_{G_1}(x)$ and the claim is proved. □

By a similar proof, $\mathrm{Cl}_G(x) = \mathrm{Cl}_{G_2}(x)$ when $x \in G_2$.

Now if $x \in G_i$ then the G-conjugacy class of x consists only of elements in G_i. The claims also show that the number of distinct conjugacy classes corresponding to elements of G which come from either G_1 or G_2 is equal to $m_1 + m_2 - l$, where l is the number of conjugacy classes common to G_1 and G_2. $G_1 \cap G_2 = Z(G)$ and each element of $Z(G)$ is its own conjugacy class. Thus, $l = |Z(G)| = z$.

Now we must count the number of conjugacy classes of elements of G which are a non-trivial product of elements from each of G_1 and G_2.

CLAIM 5. *Let $x \in G$ but $x \notin G_1$ and $x \notin G_2$. The number of distinct conjugacy classes of such elements in G is equal to $(m_1 - z)(m_2 - z)$.*

Proof. Let x be as above. Since $G = G_1G_2$, there exist $g_1 \in G_1$ and $g_2 \in G_2$ such that $x = g_1g_2$. We know that $x \notin G_1$ and $x \notin G_2$, so we must have $g_1, g_2 \notin Z(G)$.

Let y be any element of G, and let $y = h_1h_2$ for some $h_1 \in G_1$ and $h_2 \in G_2$. Consider $yxy^{-1} = h_1h_2g_1g_2h_2^{-1}h_1^{-1} = h_1g_1h_1^{-1}h_2g_2h_2^{-1}$. We can see that for every distinct conjugacy class in G_1 of the form $\{h_1g_1h_1^{-1} \mid h_1 \in G_1\}$, there are $m_2 - z$ conjugacy classes in G of the form given above. Note that since $g_2 \notin Z(G)$, we need to subtract z conjugacy classes from the total number of the form $h_2g_2h_2^{-1}$ in G_2.

Similarly, for every distinct conjugacy class in G_2 of the form $\{h_2g_2h_2^{-1} \mid h_2 \in G_2\}$ there are $m_1 - z$ conjugacy classes in G of the form given above.

We get a total of $(m_1 - z)(m_2 - z)$ conjugacy classes of elements in G which are in neither G_1 nor G_2. Thus the claim is proved. □

The two claims together prove Theorem 3

4. Application to Extraspecial Groups. Every extraspecial p-group is of order p^{2s+1} for some integer $s \geq 1$. The center of an extraspecial p-group is cyclic of order p. Furthermore, such a group is one of the two types listed below [2].

- $G = E_1 \circ E_2 \circ \cdots \circ E_s$ where each E_i is a copy of the extraspecial group E of order p^3,
- $G = E_1 \circ E_2 \circ \cdots \circ E_{s-1} \circ M$ where the $E_i \cong E$ and M is the other extraspecial p-group of order p^3.

THEOREM 6. *If G is an extraspecial p-group of order p^{2s+1}, then* $\Pr(G) = \frac{p^{2s}+p-1}{p^{2s+1}}$.

Proof. We will prove by induction on s that the number of distinct conjugacy classes of elements in G is equal to $p^{2s} + p - 1$, then the result follows. If $s = 1$, then the theorem holds by Propositions 1 and 2. Assume the theorem holds for groups of order p^{2k+1} where $k < s$. We know that $G = E_1 \circ E_2 \circ \cdots \circ E_{s-1} \circ F$ where F is a copy of either M or E, and each $E_i \cong E$. Let $H = E_1 \circ E_2 \circ \cdots \circ E_{s-1}$. From the proof of Theorem 3 we know that the number of distinct conjugacy classes in $G = H \circ F$ is equal to $m_H + m_F + (m_H - z)(m_F - z) - z$ where m_H is the number of distinct conjugacy classes in H, m_F is the number of distinct conjugacy classes in F, and $z = |Z(G)|$. By the inductive hypothesis, $m_H = p^{2(s-1)} + p - 1$. By Propostions 1 and 2, $m_F = p^2 + p - 1$. Thus the number of distinct conjugacy classes in G is equal to $p^{2(s-1)} + p - 1 + p^2 + p - 1 + (p^{2(s-1)} - 1)(p^2 - 1) - p = p^{2s} + p - 1$. Now we see that $\Pr(G) = \frac{p^{2s}+p-1}{p^{2s+1}}$ and the theorem is proved. □

5. Computations and Questions. While it was stipulated that p be an odd prime, it is easy to see that when $p = 2$ the group M is isomorphic to D_8, the dihedral group of order 8. In this case, our computation $\Pr(M) = \frac{2^2+2-1}{2^3} = \frac{5}{8}$ matches that of Gallian in [1]. In general, the probability that two elements in an extraspecial p-group commute is barely better than $\frac{1}{p}$. Certainly, $\lim_{s\to\infty} \Pr(G) = \frac{1}{p}$.

There are many open problems in this area for students to study. Take your favorite non-abelian group or family of groups and start determining conjugacy classes of elements.

A nice extension of the results of this paper might be to finite p-groups that are given as cyclic-by-elementary abelian central extensions. That is, consider a group G given by the following short exact sequence:

$$1 \to N \to G \to V \to 1$$

where N is cyclic of order p and is contained in $Z(G)$, and V is elementary abelian. It turns out that all such groups are central products of extraspecial p-groups with certain abelian p-groups. See [4] for a more complete description.

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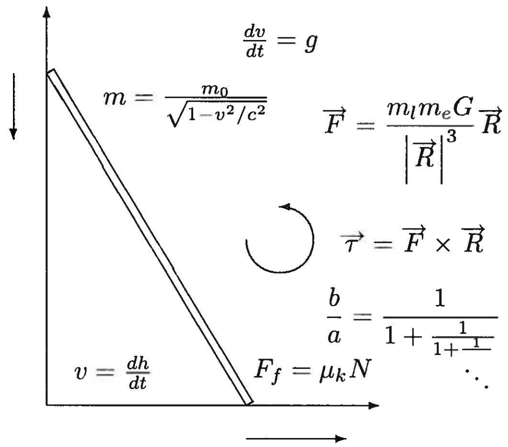

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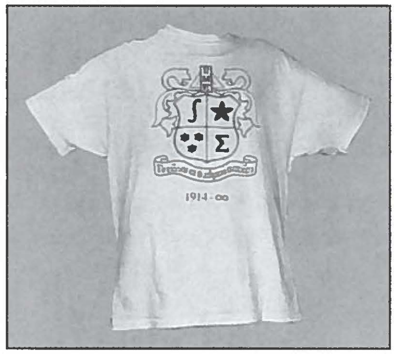

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DOUBLE SUBSTITUTION TECHNIQUE FOR INTEGRATING POLYNOMIALS IN $\sin\theta$ AND $\cos\theta$

MICHAEL J. BOSSÉ AND N. R. NANDAKUMAR*

Abstract. Traditionally, the integral $\int \sin^m\theta\cos^n\theta$ where m and n nonnegative even integers, is evaluated by the repeated use of double angle formulas. In this paper, an algebraic technique is provided which simplifies evaluating these integrals and avoids the conventional approach.

1. Introduction. Let $P(\sin, \cos\theta)$ be a trigonometric polynomial in $\sin\theta$ and $\cos\theta$. Let our assigned task be to evaluate $\int P(\sin\theta, \cos\theta)\,d\theta$ where each term in the integrand is of the form of a constant times $\sin^m\theta\cos^n\theta$ where m and n are positive integers. For instance

$$\int \sin^6\theta\cos^4\theta\,d\theta \quad \text{or} \quad \int [\sin^4\theta\cos^3\theta + 3\sin^6\theta\cos^4\theta - \sin^4\theta + 2\cos^3\theta]\,d\theta.$$

In any term within the integrand, if one of m or n is an odd integer, the integration is relatively simple. For example, say $m = 2k+1$, then using the Pythagorean identity $\sin^2\theta = 1 - \cos^2\theta$ with the substitution $\cos\theta = z$ the term will be reduced to a polynomial in z. If m and n are even, the traditional method repeatedly employs double angle formulas to simplify the integral. This brief investigation considers an algebraic technique which can readily simplify these integrals and avoid the repeated use of double angle formulas.

2. Technique. This technique begins with the substitutions. Let

$$u = \cos\theta + i\sin\theta \quad \text{and} \quad v = \cos\theta - i\sin\theta.$$

With this substitution it is easy to see that $du = iu\,d\theta$ and $dv = -iv\,d\theta$. Therefore,

$$\cos\theta = \frac{u+v}{2} \quad \text{and} \quad \sin\theta = \frac{u-v}{2i}.$$

Since $uv = \cos^2\theta + \sin^2\theta = 1$, the integral can be written as

$$\int P(\sin\theta, \cos\theta)\,d\theta = -i\int Q(u)\,du + i\int R(v)\,dv + C\int d\theta,$$

where Q and R are polynomials in one variable and C is a constant. Notably, if in each term of the polynomial $P(\sin\theta, \cos\theta)$, m and n of $\sin\theta$ and $\cos\theta$ are even, then it is easy to recognize that $Q = R$, since the substitution for $\sin\theta$ and $\cos\theta$ will be symmetric in u and v. The following example illustrates when P has one term with $m = 4$ and $n = 2$ that $Q = R$.

$$\begin{aligned}\cos^2\theta\sin^4\theta &= \left(\frac{u+v}{2}\right)^2\left(\frac{u-v}{2i}\right)^4\\ &= 2^{-6}(u^2+2uv+v^2)(u^4-4u^3v+6u^2v^2-4uv^3+v^4)\\ &= 2^{-6}(u^2+2+v^2)(u^4-4u^2+6-4v^2+v^4)\end{aligned}$$

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$$\begin{aligned}
&= 2^{-6}(u^6 - 4u^4 + 6u^2 - 4 + v^2 \\
&\qquad +2u^4 - 8u^2 + 12 - 8v^2 + 2v^4 \\
&\qquad\qquad +u^2 - 4 + 6v^2 - 2v^4 + v^6) \\
&= 2^{-6}(u^6 - 2u^4 - u^2 + 4 - v^2 - 2v^4 + v^6) \\
&= 2^{-6}[(u^6 - 2u^4 - u^2) + (v^6 - 2v^4 - v^2) + 4] \\
&= Q(u) + R(v) + C.
\end{aligned}$$

Hence, $Q = R$.

Thus, when m and n are even, substitutions can be made which will allow each term of the polynomial in $\sin\theta$ and $\cos\theta$ to be written in $\cos\theta$ (or $\sin\theta$ if desired) using Pythagorean identities. Since, as noted previously, $\cos\theta = \frac{u+v}{2}$, a substitution can be made to eliminate the use of double angle formulas. The following specific example demonstrates (although not in the general case) that since $\cos^m\theta = \left(\frac{u+v}{2}\right)^m$, by expanding the right hand side using binomial theorem we obtain a straight forward expression which is easy to integrate. Since

$$\begin{aligned}
\cos^4\theta &= \left(\frac{u+v}{2}\right)^4 \\
&= \frac{1}{16}[u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4] \\
&= \frac{1}{16}[u^4 + 4u^2 + 6 + 4v^2 + v^4]
\end{aligned}$$

we have

$$\begin{aligned}
\int \cos^4\theta\, d\theta &= \frac{1}{16}\int (u^4 + 4u^2)\, d\theta + \frac{1}{16}\int (v^4 + 4v^2)\, d\theta + \frac{3}{8}\int d\theta \\
&= \frac{1}{16}\int \frac{u^4 + 4u^2}{iu}\, du + \frac{1}{16}\int \frac{v^4 + 4v^2}{-iv}\, dv + \frac{3}{8}\int d\theta \\
&= \frac{1}{16i}\int (u^3 + 4u)\, du - \frac{1}{16i}\int (v^3 + 4v)\, d\theta + \frac{3}{8}\int d\theta \\
&= \frac{1}{16i}\left[\frac{u^4}{4} + \frac{4u^2}{2}\right] - \frac{1}{16i}\left[\frac{v^4}{4} + \frac{4v^2}{2}\right] + \frac{3}{8}\theta + K \\
&= \frac{1}{32}\left[\frac{u^4 - v^4}{2i}\right] + \frac{1}{16}\left[\frac{4u^2 - 4v^2}{2i}\right] + \frac{3}{8}\theta + K.
\end{aligned}$$

Since by DeMoivre's theorem $u^m = \cos m\theta + i\sin m\theta$ and $v^m = \cos m\theta - i\sin m\theta$, we obtain

$$\int \cos^4\theta\, d\theta = \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + K.$$

Similarly, with this new substitution the integral $\int \cos^m\theta\, d\theta$ is readily evaluated without the conventional and repeated use of double angle formulas.

3. Extending the Technique. A natural extension of this method arises from first assuming that

$$P(x,y) = \sum_{i+j=0}^{N} c_{ij}x^{2i}y^{2j}.$$

When exponents are even, by using Pythagoras' identity, we can write $\sin^{2i}\theta\cos^{2j}\theta$ as a polynomial in $\cos\theta$ only. Thus the polynomial $P(\sin\theta,\cos\theta)$ can be written as

$$P(\sin\theta,\cos\theta) = \sum_{j=0}^{N} a_j\cos^{2j}\theta.$$

This leads to

$$\int P(\sin\theta,\cos\theta)\, d\theta = \sum_{j=0}^{N} a_j\int \cos^{2j}\theta\, d\theta,$$

where each term in the sum on the right hand side is in the form of a constant times $\cos^{2j}\theta$. By using the substitution above, and observing that $uv = 1$ and $\binom{2j}{k} = \binom{2j}{2j-k}$, we obtain

$$\begin{aligned}
\cos^{2j}\theta &= \left(\frac{u+v}{2}\right)^{2j} \\
&= 2^{-2j}\left[\sum_{k=0}^{j-1}\binom{2j}{k}u^{2(j-k)} + \binom{2j}{j} + \sum_{k=0}^{j-1}\binom{2j}{k}v^{2(j-k)}\right]
\end{aligned}$$

Thus

$$\begin{aligned}
\int \cos^{2j}\theta\, d\theta &= 2^{-2j}\left[\sum_{k=0}^{j-1}\binom{2j}{k}\int \frac{u^{2(j-k)-1}}{i}du + \int \binom{2j}{j}d\theta \right.\\
&\qquad\left. - \sum_{k=0}^{j-1}\binom{2j}{k}\int \frac{v^{2(j-k)-1}}{i}dv\right] \\
&= 2^{-2j}\left[\sum_{k=0}^{j-1}\binom{2j}{k}\frac{u^{2(j-k)}}{2(j-k)i} - \sum_{k=0}^{j-1}\binom{2j}{k}\frac{v^{2(j-k)}}{2(j-k)i} + \binom{2j}{j}\theta\right] \\
&= 2^{-2j}\left[\sum_{k=0}^{j-1}\binom{2j}{k}\frac{u^{2(j-k)} - v^{2(j-k)}}{2(j-k)i} + \binom{2j}{j}\theta\right].
\end{aligned}$$

Again, by DeMoivre's theorem it follows that $\frac{u^{2(j-k)} - v^{2(j-k)}}{2i} = \sin[2(j-k)\theta]$. Hence we obtain

$$\int \cos^{2j}\theta\, d\theta = 2^{-2j}\left[\sum_{k=0}^{j-1}\binom{2j}{k}\sin[2(j-k)\theta] + \binom{2j}{j}\theta\right] + K,$$

where K is the constant of integration.

Associating the first technique with its extension leads to the following observation: To evaluate $\int P(\sin\theta,\cos\theta)\, d\theta$ where P is a polynomial in $\sin\theta$ and $\cos\theta$, and each term in the integrand is of the form $\sin^m\theta\cos^n\theta$ with even non-negative integers m and n, the integrand can be reduced to a polynomial in $\cos^2\theta$ by using Pythagorean identity. Since the integration is linear, the rule

$$\int \cos^{2j}\theta\, d\theta = 2^{-2j}\left[\sum_{k=0}^{j-1}\binom{2j}{k}\sin[2(j-k)\theta] + \binom{2j}{j}\theta\right] + K,$$

can be employed to evaluate the integral. Therefore, if the above conditions are met, one rule can be used to integrate any such trigonometric polynomial, regardless of its seeming complexity.

4. Conclusion. Although there is nothing novel in this discussion, Pythagorean identities together with algebraic techniques of substitution demonstrate utility in integrating trigonometric polynomials. With today's concern for individualizing instruction according student learning style, this technique may prove pedagogically beneficial to some who currently struggle with more traditional techniques. Additionally, this investigation bridges algebra with trigonometry in the realm of calculus. College calculus students are no less in need of making connections across mathematical disciplines in order to observe and cognitively internalize topical interconnections in mathematics.

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RESULTS INVOLVING CONTINUITY OF THE DERIVATIVE AT A POINT AND ON AN INTERVAL

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1. Motivation. When calculus is applied to other areas of study, the standard derivative of a real-valued function of a real variable is often assumed to be continuous. Some simple examples show that the derivative of a function may fail to be continuous, even when it exists everywhere on an interval. Furthermore, certain pathological functions have a derivative at a point, and yet behave in a way that would indicate that the graph of the function has no unique tangent line at that point. This paper will characterize continuity of the derivative by reformulating the standard definition of derivative, and also prove further results using this new definition.

Here is an example which shows that a function may be differentiable everywhere and have a discontinuous derivative.

Example 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), x \in \mathbb{R} \setminus \{0\} \\ 0 \quad\quad , x = 0 \end{cases}$$

f is continuous on $\mathbb{R} \setminus \{0\}$, being the composition of continuous functions, and using the squeeze theorem it is easy to show that f is continuous at 0, so f is continuous everywhere. Differentiating f on $\mathbb{R} \setminus \{0\}$, we use the chain rule and the product rule to obtain $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. Differentiating f at 0 using the definition of derivative yields $f'(0) = 0$. The derivative is discontinuous at 0, however, since $f'(0) = 0$, while

$$\lim_{n\to\infty} f'\left(\frac{1}{(2n+1)\pi}\right) = \lim_{n\to\infty} \frac{1}{(2n+1)\pi} \sin((2n+1)\pi) - \cos((2n+1)\pi) = 1.$$

This shows that the derivative of a function may be discontinuous at a point, while the derivative exists everywhere in a neighborhood of that point. The reader should keep in mind the result of Baire's which states that when a derivative exists everywhere, the set of its discontinuities is at most first category.(See [2])

Here is an example which shows that the existence of the derivative of a function at a point gives little information about how the function behaves near that point.

Example 2. The function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x^2, x \text{ is rational} \\ 0 \;, x \text{ is irrational} \end{cases}$$

is continuous at 0, but discontinuous everywhere else, since the rational numbers are dense in $\mathbb{R}$. One can visualize this function as a parabola joined with its tangent line at $x = 0$, which is the x–axis. It is then easy to see that $g'(0) = 0$. This shows that the existence of the derivative of a function at a point implies very little about the behavior of the function near that point.

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2. Redefining the Derivative. Now let us formulate a modified definition of the derivative.

DEFINITION 1. *For an open interval $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$, let*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f^*(x) = \lim_{\substack{(h,k) \to (0,0) \\ h \neq k}} \frac{f(x+h) - f(x+k)}{h-k}$$

If $f^*(a)$ exists, it must be equal to $f'(a)$, since

$$\lim_{\substack{(h,k) \to (0,0) \\ h \neq k}} \frac{f(a+h) - f(a+k)}{h-k} = c \implies \lim_{\substack{(h,0) \to (0,0) \\ h \neq k}} \frac{f(a+h) - f(a)}{h} = c.$$

Note that the second expression is the definition of $f'(a)$. We refer to f^* as the "strong derivative of f."

If we interpret these definitions graphically, we may consider the traditional derivative as approximating the slope of the line tangent to the graph of f at $(x, f(x))$ by secant lines through $(x, f(x))$. The strong derivative, in contrast, approximates the slope of the tangent line through $(x, f(x))$ by every secant line near $(x, f(x))$, including those which pass through $(x, f(x))$.

The following theorem characterizes the continuity of f' in terms of the existence of f^*, provided f' exists everywhere.

THEOREM 2. *Let f be defined on an interval $I = (a,b) \subseteq \mathbb{R}$, and suppose $f'(x)$ exists for all x in I. Then f^* exists at $c \iff f'$ is continuous at c.*

Proof. $(\implies)$

Let a_n be a sequence such that $\lim_{n\to\infty} a_n = c$, and $f'(a_n)$ exists for all n. For each a_n choose $b_n \neq a_n$ such that $|a_n - b_n| < 1/n$, and

$$\left| \frac{f(a_n) - f(b_n)}{a_n - b_n} - f'(a_n) \right| < \frac{1}{n}.$$

Let $h_n = a_n - c$, and let $k_n = b_n - c$. Then $\lim_{n\to\infty} h_n = \lim_{n\to\infty} k_n = 0$ and

$$\lim_{n\to\infty} f'(a_n) = \lim_{n\to\infty} \frac{f(a_n) - f(b_n)}{a_n - b_n} = \lim_{n\to\infty} \frac{f(c+h_n) - f(c+k_n)}{h_n - k_n} = f^*(c) = f'(c).$$

Since this is true for every sequence a_n that converges to c, $\lim_{x\to c} f'(x) = f'(c)$.

$(\impliedby)$ Let $\varepsilon > 0$ be given. We have that f' is continuous at c, so there is a $\delta > 0$ such that $|f'(x) - f'(c)| < \varepsilon$ whenever $|x - c| < \delta$. Let $|h| < \delta, |k| < \delta$, and without loss of generality, assume $h < k$. The mean value theorem implies that there exists $y \in (c+h, c+k)$ such that $\dfrac{f(c+h) - f(c+k)}{h-k} = f'(y)$. Since $|h|, |k| < \delta$, we have $|c - y| < \delta$, so

$$\left| \frac{f(c+h) - f(c+k)}{h-k} - f'(c) \right| = |f'(y) - f'(c)| < \varepsilon.$$

This implies that

$$\lim_{\substack{(h,k) \to (0,0) \\ h \neq k}} \frac{f(c+h) - f(c+k)}{h-k} = f'(c),$$

which shows that $f^*(c)$ exists. □

Note that in the $(\implies)$ portion of the proof, we made no use of the fact that f' exists everywhere. We may as well have assumed that a is a cluster point of the set $\{x : f'(x) \in \mathbb{R}\}$ without changing the proof significantly. The $(\impliedby)$ proof, in contrast, uses the mean value theorem, which makes the assumption that f' exists everywhere in I quite necessary. It would be interesting to find a weaker condition that implies the same result.

To express the following corollary, we need the concept of category. A set $A \subseteq \mathbb{R}$ is *nowhere dense* if, for all $x \in \mathbb{R} \setminus A$, there exists a neighborhood U of x, such that $U \subseteq (\mathbb{R} \setminus A)$. A set $E \subseteq \mathbb{R}$ is said to be of *first category* if E is the union of countably many nowhere dense sets. A set is said to be of second category if it is not of first category. A trivial example of a set of first category is the set rational numbers, being a countable set. In rough terms, a first category set may be considered "small" and second category sets may be considered "large." Every second category set is dense in $\mathbb{R}$, and every interval in $\mathbb{R}$ is of second category. Finally, a function is of first class if it is the limit of a sequence of continuous functions. For a further discussion of category, see [2].

COROLLARY 3. *If f' exists everywhere on an open interval I, f^* must exist on a set which is the complement of a set of first category.*

Proof. Baire's Theorem on Functions of First Class (see [2]) implies that if f' exists everywhere on an open interval, the set A of discontinuities of f' is at most first category. By theorem 1, f^* will exist on $I \setminus A$, which is the set of points where f' is continuous. □

Example 2 revisited. To see that the existence of the strong derivative implies more about the behavior of a function than does the existence of the traditional derivative, let us return to the second example from section 1: the function

$$g(x) = \begin{cases} x^2, x \text{ is rational} \\ 0\ , x \text{ is irrational} \end{cases}$$

$g'(0) = 0$, but since g is discontinuous everywhere except 0, g' does not exist on any open interval around 0, so we cannot use the theorem proved above to deduce any information about $g^*(0)$.

We know that if $g^*(0)$ exists, $g^*(0) = 0$, since $g'(0) = 0$. Consider the sequences $a_n = 1/n$, $b_n = 1/n - 1/(\pi n^2)$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, so

$$g^*(0) = \lim_{n\to\infty} \frac{g(a_n) - g(b_n)}{a_n - b_n}$$

if $g^*(0)$ exists. However, $g(a_n) = 1/{a_n}^2$ for all $n \in \mathbb{N}$, since a_n is always rational, and $g(b_n) = 0$ for all $n \in \mathbb{N}$, since b_n is always irrational. Then

$$g^*(0) = \lim_{n\to\infty} \frac{g(a_n) - g(b_n)}{a_n - b_n} = \lim_{n\to\infty} \frac{1/n^2}{1/(\pi n^2)} = \pi.$$

This contradicts the fact that $g^*(0) = 0$, so $g^*(0)$ does not exist.

This example is interesting in that it shows that the existence of the derivative f' at a point a implies very little about how the function f behaves near a. In the next section we show that the existence of $f^*(a)$ implies a great deal about how f behaves near a.

3. Further Results. Given a function f, the existence of $f^*(a)$ implies that f must be Lipschitz in some neighborhood of a.

DEFINITION 4. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is Lipschitz on I if there exists $c > 0$ such that for all $x, y \in I$, $|f(x) - f(y)| \leq c|x - y|$.*

Lipschitz functions are very well behaved: they are uniformly continuous, absolutely continuous, and differentiable almost everywhere (see [1]). Suppose $f : \mathbb{R} \to \mathbb{R}$, and $f^*(a) = L \in \mathbb{R}$. To see that f is Lipschitz near a, note that

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h) - f(a+k)}{h-k} = L$$

implies that for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\left|\frac{f(a+h) - f(a+k)}{h-k} - L\right| < \varepsilon, \text{ whenever } \sqrt{h^2+k^2} < \delta_\varepsilon$$

Which implies

$$-\varepsilon + L < \frac{f(a+h) - f(a+k)}{h-k} < \varepsilon + L, \text{ whenever } \sqrt{h^2+k^2} < \delta_\varepsilon$$

Letting $\varepsilon < \left|\frac{L}{2}\right|$, we have

$$-\frac{L}{2} < -\varepsilon + L < \frac{f(a+h) - f(a+k)}{h-k} < \varepsilon + L < \frac{3L}{2},$$

so that

$$\left|\frac{f(a+h) - f(a+k)}{h-k}\right| < \left|\frac{3L}{2}\right|,$$

which implies

$$|f(a+h) - f(a+k)| < \left|\frac{3L}{2}\right| |(a+h) - (a+k)|, \text{ whenever } \sqrt{h^2+k^2} < \delta_\varepsilon$$

This shows that f is Lipschitz in some neighborhood of a. Since every Lipschitz function is the difference of two increasing functions, this implies that f is differentiable (in the traditional sense) almost everywhere in a neighborhood of a (see [1], p. 104). The reader not familiar with measure theory might like to know that this implies that f^* exists on a dense set in I if f' exists everywhere on I.

Note that in the ($\Longrightarrow$) portion of the proof of Theorem 1, we made no use of the fact that f' exists everywhere. Thus, we have the stronger statement

COROLLARY 5. *If $f^*(a)$ exists, then $\lim_{x\to a} f'(x) = f'(a)$*

Seeing that the existence of f^* at a point implies that f is Lipschitz on an interval, it is natural to ask if f being Lipschitz on an interval implies that f^* exists somewhere. In fact, there is a Lipschitz function f such that f^* exists nowhere, but such a function is very difficult to construct directly, and the author would be very interested in a construction of such a function using elementary methods, with no reference to measure theory or category.

Example 3. Here we construct a Lipschitz function f, such that f^* exists nowhere. In our construction of f, we will use the fact that there exists a set, A, which is a Lebesgue measurable subset of $[-1, 1]$, such that for all open $U \subset [-1, 1]$, both $A \cap U$ and $U \setminus A$ have positive measure. For a construction of A, see [3]. Let $F = \int_0^x \chi_A \, dt$, where $\chi_A(x) = 1$ for $x \in A$, $\chi_A(x) = 0$ elsewhere. Then F is Lipschitz, since

$$|F(y) - F(x)| = \left|\int_x^y \chi_A \, dt\right| \leq |y - x|.$$

Also, $F'(x) = \chi_A(x)$ for almost all $x \in (-1, 1)$, by ([1], 107). This implies that for any open interval U in $[-1, 1]$, there exist $a, b \in U$ such that $F'(a) = 0$ and $F'(b) = 1$, since otherwise, the set

$$\left\{x : \frac{d}{dx}\int_0^x \chi_A \, dt \neq \chi_A(x)\right\}$$

has positive measure (see [1]). Thus,

$$\lim_{x\to a} F'(x)$$

does not exist for any $a \in [-1, 1]$, so $f^*(a)$ does not exist for any $a \in [-1, 1]$. As a corollary to this, we know that F' cannot exist everywhere on an interval in $[-1, 1]$, since F' would then be continuous on a set of second category, so F^* would exist on a set of second category, by Theorem 1. This inspires another observation:

Observation. Let $0 < c < 1$. There is no measurable set $A \subset [0, 1]$ such that $m(I \cap A) = cm(I)$ for every interval $I \subseteq [0, 1]$

Proof. Let A be such a set. Then $\int_0^x \chi_A \, dt = cx$, so

$$\frac{d}{dx}\int_0^x \chi_A \, dt = c$$

for all $x \in (0, 1)$, a contradiction. □

We can use the function F, constructed in example 3, to construct a function which has a strong derivative at only one point, which shows that the existence of the strong derivative at a point does not imply the existence of the strong derivative elsewhere. Consider $f = xF$. Then

$$\frac{hF(h) - kF(k)}{h-k} = \frac{h(F(h) - F(k)) + (h-k)F(k)}{h-k} = \frac{h(F(h) - F(k))}{h-k} + F(k),$$

so we can compute $f^*(0)$ as

$$\begin{aligned}
\left|\lim_{\substack{(h,k)\to(0,0)\\ h\neq k}} \frac{hF(h) - kF(k)}{h-k}\right| &\leq \left|\lim_{\substack{(h,k)\to(0,0)\\ h\neq k}} \frac{h(F(h) - F(k))}{h-k}\right| + \left|\lim_{\substack{(h,k)\to(0,0)\\ h\neq k}} F(k)\right| \\
&\leq \left|\lim_{\substack{(h,k)\to(0,0)\\ h\neq k}} h\right| + \left|\lim_{\substack{(h,k)\to(0,0)\\ h\neq k}} F(k)\right| \\
&= 0,
\end{aligned}$$

since $\lim_{k\to 0} F(k) = 0$, so $f^*(0) = 0$. But $f^*(a)$ does not exist for any $a \neq 0$, since, for $a \neq 0$, $f'(a) = F(a) + aF'(a)$, so for all $a \neq 0$, for all $\delta > 0$, there are $x, y \in (a-\delta, a+\delta)$ such that $f'(x) = F(x)$, while $f'(y) = F(y) + y$, so, if $f^*(a)$ exists, then

$$\lim_{x\to a} f'(x) = \lim_{x\to a} F(x) + x = \lim_{x\to a} F(x),$$

which is a contradiction, since $a \neq 0$. It is interesting to note that multiplying F by x produces a function f for which f^* exists at 0, just as multiplying $x \sin\left(\frac{1}{x}\right)$ by x yields a function whose derivative exists at 0.

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The Richard V. Andree Awards. The Richard V. Andree Awards are given annually to the authors of the three papers, written by undergraduate students, that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the Pi Mu Epsilon Journal in the past year.

Until his death in 1987, Richard V. Andree was Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon for many years and in a variety of capacities: as President, as Secretary-Treasurer, and as Editor of this Journal.

The awards for papers appearing in 2000 are announced on the next page. Because the papers differ in nature to such a degree that a ranking was impossible - one paper written by a high school student, one by a team of students, one resulting from a senior thesis- the awards are ordered alphabetically by title. The officers and councilors of the Society congratulate the winners on their achievements and wish them well for their futures.

2000 2000

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Mathew J. Hale,
"Circular Functions of Multiple Integral Angles",
Pi Mu Epsilon Journal, Vol. 11, No. 3, Fall 2000.

Nitish Dass, Jonathan McGrath, Erin Urbanski,
"The Determinant of a (m,n) Pretzel",
Pi Mu Epsilon Journal, Vol. 11, No. 3, Fall 2000.

Brett Alan Enge,
"The Search for Tri-Operate Fields",
Pi Mu Epsilon Journal, Vol. 11, No. 3, Fall 2000.

2000 2000

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ERROR BOUNDS INVOLVING ALMOST-BINOMIAL APPROXIMATIONS OF HYPERGEOMETRIC PROBABILITIES

ABU I. M. JALAL*

1. Introduction. Problems involving sums of several Hypergeometric random variables generally involve complex calculations which are beyond the grasp of most undergraduate students, and even beyond what a trained statistician would be comfortable with, armed with only a standard scientific calculator. Straightforward approximations have been developed using the Poisson distribution, the Binomial distribution, and a variant of the Binomial distribution, the Almost-Binomial distribution. The purpose of this paper is to perform an error-bound based comparison of the Almost-Binomial approximations with the Binomial approximations.

We will use the following notation:

1. $X \sim$ Hypergeometric (N_1, N_2, k), means that X has pdf (probability density function)

$$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{k-x}}{\binom{N}{k}},$$

$x \leq k, x \leq N_1$, and $k - x \leq N_2$. For this distribution $E(X) = k\frac{N_1}{N}$ and $\mathrm{Var}(X) = k\,\frac{N_1}{N}\,\frac{N_2}{N}\,\frac{(N-K)}{(N-1)}$, where $N = N_1 + N_2$.

2. $X \sim$ Poisson (λ), means that X has pdf $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, 3, \ldots$. For this distribution $E(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$.
3. $X \sim$ Binomial (n,p), means that X has pdf $f(x) = \binom{n}{x}p^x(1-p)^{n-x}, x = 0, 1, 2, \ldots, n$. For this distribution $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$.
4. $X \sim$ Almost-Binomial (n,p), means that X has pdf

$$f(x) = \begin{cases} \binom{n}{x}p^x(1-p)^{n-x}, & x = 0, 1, 2, \ldots, [n] \\ 1 - \sum_{x=0}^{[n]}\binom{n}{x}p^x(1-p)^{n-x}, & x = [n]+1, \end{cases}$$

where $[n] =$ the greatest integer less than or equal to n. The Almost-Binomial distribution is a generalization of the Binomial distribution where n is allowed to take on non-integer values. For this distribution $E(X) \approx np$ and $\mathrm{Var}(X) \approx np(1-p)$. (Thompson, 1999).

For example, $X \sim$ Almost-Binomial $(n = 3.4, p = 0.2)$. The pdf of the Almost-Binomial $(3.4, 0.2)$ is

$$f(x) = \begin{cases} \binom{3.4}{x}0.2^x(1-0.2)^{3-x}, & x = 0, 1, 2, 3, \\ 1 - \sum_{x=0}^{3}\binom{3.4}{x}0.2^x(1-0.2)^{3-x}, & x = 4 \end{cases}$$

We get the following probability values: Note that$E(X) = 0.68001(np = 0.68)$

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x	F(x)
0	0.46828
1	0.39804
2	0.11941
3	0.01393
4	0.00034

TABLE 1.1
Almost-Binomial (3.4, 0.2)

and $\mathrm{Var}(X) = 0.544076(np(1-p) = 0.544)$.

Suppose, $Y = X_1+X_2+X_3+\ldots+X_r$, where $X_1, X_2, X_3, \ldots, X_r$ are independent and $X_i \sim$ Hypergeometric $(N_{i1}, N_{i2}, k_i), N_{i1} + N_{i2} = N_i$. Then,

$$E(Y) = k_1\frac{N_{11}}{N_1} + k_2\frac{N_{21}}{N_2} + k_3\frac{N_{31}}{N_3} + \ldots + k_r\frac{N_{r1}}{N_r} = \sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}$$

$$\begin{aligned}\mathrm{Var}(Y) &= k_1\frac{N_{11}}{N_1}\frac{N_{12}}{N_1}\frac{(N_1-k_1)}{(N_1-1)} + k_2\frac{N_{21}}{N_2}\frac{N_{22}}{N_2}\frac{(N_2-k_2)}{(N_2-1)} \\ &+ k_3\frac{N_{31}}{N_3}\frac{N_{32}}{N_3}\frac{(N_3-k_3)}{(N_3-1)} + \ldots + k_r\frac{N_{r1}}{N_r}\frac{N_{r2}}{N_r}\frac{(N_r-k_r)}{(N_r-1)} \\ &= \sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}\frac{N_{i2}}{N_i}\frac{(N_i-k_i)}{(N_i-1)}.\end{aligned}$$

2. Approximation Methods. In this section, we will discuss various approximations to probabilities involving sums of independent Hypergeometric random variables. All four schemes are method-of-moments based.

1. **Poisson Approximations:** Set

$$\lambda = \sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}.$$

We can find individual probability approximations by plugging in the values of λ and x into the pdf of the Poisson Distribution.

2. **Binomial Approximations (1):** Set

$$n = \sum_{i=1}^{r} k_i \text{ and } p = \frac{\sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}}{\sum_{i=1}^{r} k_i}.$$

We can compute individual probability approximations by plugging in the values of n and p into the pdf of the Binomial Distribution.

3. **Almost-Binomial Approximations:** Set

$$np = \sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i} \text{ and } np(1-p) = \sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}\frac{N_{i2}}{N_i}\frac{(N_i-k_i)}{(N_i-1)}.$$

Solving for n and p gives,

$$n = \frac{\left(\sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}\right)^2}{\sum_{i=1}^{n} k_i\frac{N_{i1}}{N_1}\left(1 - \frac{N_{i2}}{N_i}\frac{(N_i-k_i)}{(N_i-1)}\right)}$$

and

$$p = 1 - \frac{\sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}\frac{N_{i2}}{N_i}\frac{(N_i-k_i)}{(N_i-1)}}{\sum_{i=1}^{n} k_i\frac{N_{i1}}{N_i}}.$$

We can plug in these values into the pdf of the Almost-Binomial Distribution and get approximations for the probabilities. Lorry Lenvoy (1996) introduced this method to approximate probabilities involving a single Hypergeometric random variable. She compared it to the Normal distribution and Binomial approximation (1), recommending the use of Almost-Binomial approximation for situations where $\frac{k}{N} \leq 0.2$ and $\frac{N_1}{N} \leq 0.3$.

4. **Binomial Approximations (2):** Shift the non-integer n from the Almost-Binomial approximations to the closest integer value n^*. Then, we set

$$n^*p^* = \sum_{i=1}^{r} k_i\frac{N_{i1}}{N_i}$$

to obtain the value of p^*. Again, we can compute individual probability approximations by plugging in the values of n^* and p^* into the pdf of Binomial distribution. Sandiford (1962) proposed this technique to approximate probabilities involving a single Hypergeometric random variable.

We present an example. Suppose, X_1, X_2 are X_3 independent. $X_1 \sim$ Hypergeometric $(4, 11, 4), X_2 \sim$ Hypergeometric $(3, 9, 3)$, and $X_3 \sim$ Hypergeometric $(3, 10, 3)$. Let $Y = X_1 + X_2 + X_3$. We get the table of probabilities shown in Table 2.

We should note that the performance of the Almost-Binomial approximation relative to all the other approximations here should come as no surprise. The first and second moments of the approximating Almost-Binomial most closely match those of the sum of the Hypergeometric random variables.

3. Theoretical Bounds on the Errors:. At this point we could continue with a series of representative examples to compare the Almost-Binomial approximation with the other approximations. Sandiford (1962) and Lendvoy (1996) are both collections of examples. However, the purpose of this paper is a more mathematical comparison based on theoretical bounds on the errors in the approximations.

The theoretical bounds on the errors we give below are based on work done in a more general setting that involves errors in approximating probabilities concerning sums of independent Bernoulli random variables. ($X \sim$ Bernoulli (p), if $P(X = 0) = 1 - p$ and $P(X = 1) = p$).

THEOREM 1. *If $X \sim$ Hypergeometric (N_1, N_2, k), where $k \leq N_1 \wedge N_2$, then the distribution of X is equivalent to that of a sum of k independent Bernoulli random variables.*

Y	Exact	Binomial(1) (0.2509,10)	Almost Binomial (0.3947, 6.3562)	Poisson (2.509)	Binomial(2) (04181.6)
0	.038731	.055641	.041118	.081352	.038823
1	.170829	.186363	.170443	.204109	.167368
2	.303619	.280887	.297682	.256052	.300639
3	.282076	.250878	.281895	.214143	.288015
4	.149241	.147049	.154249	.134319	.155206
5	.046252	.059102	.047404	.067401	.044607
6	.0083564	.0164961	.0069879	.0281845	.005342
7	.0008487	.0031572	.0002319	.0101020	0
8	.0000453	.0003965	0	.003168	0
9	.0000012	.0000295	0	.0008832	0
10	0	0	0	.0002216	0
Total|error|		0.1122437	**0.0170828**	0.2607799	0.0239916

TABLE 2.1
A comparison of true probabilities for Y with approximate probabilities

Proof. See Vatutin and Mikhailov (1982). □ Although in complex cases it may be impossible to get exact solutions for the p's of the Bernoulli random variables, one can get approximate solutions by having a computer algebra system factor the probability generating function of the Hypergeometric random variable. In simple cases one can solve exactly for the p's. We give an example: If $X \sim$ Hypergeometric ($N_1 = 6,\ N_2 = 10,\ k = 2$), we get the following table for the pdf of $X, f(x)$:

x	$f(x)$
0	0.375
1	0.5
2	0.125

TABLE 3.1
Hypergeometric (6, 10, 2)

It is easy to show that this distribution is equivalent to the sum of two independent Bernoulli variables, one with $p = 0.25$, and the other with $p = 0.5$.

The significance of Theorem 1 for us is that we can use the error bounds derived for sums of independent Bernoulli variables for our situation. For the results that follow, $X_1, X_2, X_3, \ldots, X_s$ are independent Bernoulli (p_i) random variables,

$$X = \sum_{i=1}^{s} X_i, \qquad \bar{p} = \sum_{i=1}^{s} p_i/s, n = \frac{(\sum_{i=1}^{s} p_i)^2}{\sum_{i=1}^{s} p_i^2}, \qquad p = \frac{\sum_{i=1}^{s} p_i^2}{\sum_{i=1}^{s} p_i},$$

and $[n]$ denotes the greatest integer $\leq n$. Let n^* be the closest integer to n and let $p^* = s\bar{p}/n^*$. Let $\tilde{P}$ be a random variable with

$$P(\tilde{P} = p_i) = \frac{p_i}{\sum_{i=1}^{s} p_i}, \qquad i = 1, 2, 3, \ldots, s.$$

Suppose that $A \subset \{0, 1, 2, \ldots, [n]\}$. Let

$$\mathrm{Bin1}(A) = \sum_{x \epsilon A} \binom{s}{x} (\bar{p}^x (1 - \bar{p})^{s-x},$$

$$\mathrm{Bin2}(A) = \sum_{x \epsilon A} \binom{n^*}{x} (p^*)^x (1 - p^*)^{n^* - x}, \text{ and}$$

$$\mathrm{AlBin}(A) = \sum_{x \epsilon A} \binom{n}{x} (p)^x (1 - p)^{n-x}.$$

THEOREM 2. Bin1(A) *satisfies*

$$|P(X \in A) - \mathrm{Bin1(A)}| \leq \frac{(1 - \bar{p}^{s+1} - (1 - \bar{p})^{s+1})}{(s+1)\bar{p}(1 - \bar{p})} \sum_{i=1}^{s} (p_i - \bar{p})^2.$$

Proof. See Ehm (1991). □

THEOREM 3. Bin2(A) *satisfies*

$$|P(X \in A) - \mathrm{Bin2(A)}| \leq$$
$$\frac{4}{1 - p^*} \mathrm{Var}(\tilde{p}) + \frac{1}{1 - p^*} \frac{p|n - n^*|}{n^*} + (k - [n] - 1) \frac{P(X \geq [n] + 2)}{1 - p} \frac{1}{[n] + 1}.$$

Proof. See Barbour, Holst, and Janson (1992) (pp. 188 - 191) and Thompson (1999). □

THEOREM 4. AlBin(A) *satisfies*

$$|P(X \in A) - \mathrm{AlBin(A)}| \leq \frac{4}{1 - p} \mathrm{Var}(\tilde{p}) + (k - [n] - 1) \frac{P(X \geq [n] + 2)}{1 - p} \frac{1}{[n] + 1}.$$

Proof. See Thompson (1999). □

Before applying these results to the Hypergeometric set up we note that the third term on the right in Theorem 3 (and the second term on the right in Theorem 4) tend to be either 0 or so extraordinarily small that they have essentially no impact on the bound. In the comparisons we make later, we take these terms to be 0.

Note that, $\mathrm{Var}(\tilde{p})$ depends on $\sum_{i=1}^{s} p_i, \sum_{i=1}^{s} p_i^2$, and $\sum_{i=1}^{s} p_i^3$. It can be written as

$$\mathrm{Var}(\tilde{p}) = \frac{\sum_{i=1}^{s} p_i^3}{\sum_{i=1}^{s} p_i} - \left(\frac{\sum_{i=1}^{s} p_i^2}{\sum_{i=1}^{s} p_i} \right)^2.$$

Next, suppose $Y_1, Y_2, Y_3, \ldots, Y_r$ are independent Hypergeometric random variables. Let ,

$$\mu_1 = \sum E(Y_i) = \sum_{i=1}^{r} k_i \frac{N_{i1}}{N_i},$$

$$\mu_2 = \sum \mathrm{Var}(Y_i) = \sum_{i=1}^{r} k_i \frac{N_{i1}}{N_i} \frac{N_{i2}}{N_i} \frac{(N_i - k_i)}{(N_i - 1)}$$

$$\mu_3 = \sum E(Y_i - \mu_i)^3 = \sum_{i=1}^{r} k_i \frac{N_{i1}}{N_i} \frac{N_{i2}}{N_i} \frac{(N_{i2} - N_{i1})}{N_i} \frac{(N_i - k_i)}{(N_e - 1)} \frac{(N_i - 2k_i)}{(N_i - 2)}$$

(Johnson, Kotz and Kemp, 1993, p. 250).

Set $s = \sum_{i=1}^{r} k_i$. Equating moments with our sum of Bernoulli's we get (from Johnson, Kotz and Kemp, 1993, p. 131)

$$\mu_1 = \sum_{i=1}^{s} p_i, \qquad \mu_2 = \sum_{i=1}^{s} p_i(1-p_i), \qquad \text{and} \mu_3 = \sum_{i=1}^{s} p_i(1-p_i)(1-2p_i).$$

Through some algebraic manipulation we get

$$\frac{4}{1-p}\mathrm{Var}(\tilde{p}) = 2 + \frac{2\mu_3}{\mu_2} - \frac{4\mu_2}{\mu_1}.$$

4. A Single Hypergeometric Random Variable. For a single Hypergeometric random variable,

$$\frac{4}{1-p}\mathrm{Var}(\tilde{p}) - 2 + \frac{2\left(k\frac{N_1}{N}\frac{N_2}{N}\frac{(N_2-N_1)}{N}\frac{(N-k)}{(N-1)}\frac{(N-2k)}{(N-2)}\right)}{k\frac{N_1}{N}\frac{N_2}{N}\frac{(N-k)}{(N-1)}} - \frac{4\left(k\frac{N_1}{N}\frac{N_2}{N}\frac{(N-k)}{(N-1)}\right)}{k\frac{N_1}{N}}$$
$$= \frac{4(N_1-1)(k-1)}{(N-1)(N-2)}.$$

For Binomial(1) approximations, the error bound simplifies down to [See Ehm (1991)]

$$\frac{k}{k+1}\left(1-\left(\frac{N_1}{N}\right)^{k+1}-\left(\frac{N_2}{N}\right)^{k+1}\right)\frac{k-1}{N_1}.$$

From the results we see that the theoretical bounds for the Binomial(1) approximation and the Almost-Binomial are about the same when $N_1 \approx \frac{1}{4}N$. If N_1 is dramatically less than $\frac{1}{4}N$, we expect the Almost-Binomial bound to be much less that Binomial(1) bound. We can look at the following example:

Suppose $N_1 = 1,000, N_2 = 999,000$, and $k = 1,000$. Then, the bound on the error of the Almost-Binomial approximation is 0.00000399. On the other hand, the bound on the error of the Binomial (1) approximation is 0.0000959.

5. Sums of Independent Identically Distributed Hypergeometric Random Variables. Suppose $X_1, X_2, X_3, \ldots, X_r$ are r independent and identical Hypergeometric (N_1, N_2, k) random variables. Then,

$$\frac{4}{1-p}\mathrm{Var}(\tilde{p}) = 2 + \frac{2r\mu_3}{r\mu_2} - \frac{4r\mu_2}{r\mu_1} = 2 + \frac{2\mu_3}{\mu_2} - \frac{4\mu_2}{\mu_1},$$

the same as the single variable case.

Ehm's error bound is $(1-\tilde{p}^{rk+1} - (1-\tilde{p})^{rk+1})\frac{rk}{(rk+1)}\left(\frac{k-1}{N-1}\right)$

Let's consider the following example:

Suppose X_1, X_2, X_3, X_4, X_5 are 5 identical and independent Hypergeometric random variables with $N_{11} = N_{21} = N_{31} = N_{41} = N_5 = 1000, N_{12} = N_{22} = N_{32} = N_{42} = N_{52} = 999,000$, and $k_1 = k_2 = k_3 = k_4 = k_5 = 1000$. $X = \sum_{i=1}^{5} X_i$. Then, the bound on the error of the Almost-Binomial approximations is 0.00000399. On the other hand, the bound on the error of the Binomial (1) approximations is 0.0099881.

X	Exact	Almost Binomial (2502.5, 2/1001)
0	0.006704330	0.006704330
1	0.033588761	0.033588761
2	0.084106393	0.084106393
3	0.140345702	0.140345702
4	0.175572614	0.175572614
5	0.175642913	0.175642913
6	0.146369094	0.146369093
7	0.104507492	0.104507492
8	0.065264876	0.065264876
9	0.036214711	0.036214711
10	0.018078355	0.018078355
Total |Error|		**0.000000001**

TABLE 5.1
A comparison of true probabilities for X with Almost-Binomial approximations

We can look at the true and approximate probabilities for the significant values to demonstrate that the claimed accuracy of the Almost-Binomial approximations is correct.

Note that the bound on the error and the Almost-Binomial approximations can both be easily found with a standard scientific calculator. Whereas, it is necessary to resort to a computer program to find the exact probability values.

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A FAMILY OF POLYNOMIAL FUNCTIONS

THOMAS KOSHY*

1. Introduction. This short article investigates a class of polynomials $K_n(x)$ with integral coefficients. They are defined by

$$K_n(x) = K_{n-1}(x) + xK_{n-2}(x) \tag{1}$$

where

$$K_1(x) = 1 \text{ and } K_2(x) = x.$$

The first ten members of this family are:

$$\begin{aligned}
K_1(x) &= 1\\
K_2(x) &= x\\
K_3(x) &= 2x\\
K_4(x) &= x^2 + 2x\\
K_5(x) &= 3x^2 + 2x\\
K_6(x) &= x^3 + 5x^2 + 2x\\
K_7(x) &= 4x^3 + 7x^2 + 2x\\
K_8(x) &= x^4 + 9x^3 + 9x^2 + 2x\\
K_9(x) &= 5x^4 + 16x^3 + 11x^2 + 2x\\
K_{10}(x) &= x^5 + 14x^4 + 25x^3 + 13x^2 + 2x
\end{aligned}$$

The polynomials $K_n(x)$ have several interesting properties:

- The degree of $K_n(x)$ is $\lfloor n/2 \rfloor$, so $K_{2n}(x)$ and $K_{2n+1}(x)$ have the same degree, where $\lfloor x \rfloor$ denotes the floor of x.
- The leading coefficient of $K_n(x)$ is

$$K_n(x) = \begin{cases} 1 & \text{if n is even} \\ \lfloor (n+1)/2 \rfloor & \text{otherwise.} \end{cases}$$

- $x | K_n(x)$ for every $n \geq 2$.
- The coefficient of x is always 2.

When $n = 1$, (1) yields $K_n(1) = K_{n-1}(1) + K_{n-2}(1)$, where $K_1(1) = 1 = K_2(1)$. This is precisely the recursive definition of the nth Fibonacci number F_n, so $K_n(1) = F_n$.

Fibonacci numbers can also be defined explicitly by Binet's formula **[1]**:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the solutions of the quadratic equation $x^2 = x + 1$.

Since $K_n(1) = F_n$, it follows that the sum of the coefficients in every polynomial $K_n(x)$ is a Fibonacci number. In other words, every row sum in the array of coefficients in Figure 1 is a Fibonacci number.

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n\j	0	1	2	3	4	5
1	1					
2	1					
3	2					
4	1	2				
5	3	2				
6	1	5	2			
7	4	7	2			
8	1	9	9	2		
9	5	16	11	2		
10	1	14	25	13	2	

$\leftarrow$ row sum $= 13 = F_7$ (row 7)

Figure 1

Let $K(n,j)$ denote the element in row n and column j, where $n \geq 1$ and $j \geq 0$. It can be defined recursively as follows:

$$K(n,0) = \begin{cases} 1 & \text{if n is even} \\ \lfloor (n+1)/2 \rfloor & \text{otherwise,} \end{cases}$$

$$K(n,j) = \begin{cases} K(n-1,j-1) + K(n-2,j) & \text{if n is even} \\ K(n-1,j) + K(n-2,j) & \text{otherwise,} \end{cases}$$

where $n \geq 3$ and $j \leq \lfloor (n-2)/2 \rfloor$. $K(n,j)$ may be considered 0 if $j > \lfloor (n-2)/2 \rfloor$.

2. Jacobsthal Polynomials. There is a close relationship between the polynomials $K_n(x)$ and the Jacobsthal polynomials $J_n(x)$, as we will see shortly. The Jacobsthal polynomials are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \tag{2}$$

where $J_1(x) = 1 = J_2(x)$ **[2]**. The first ten members of the Jacobsthal family are:

$$\begin{aligned} J_1(x) &= 1 \\ J_2(x) &= 1 \\ J_3(x) &= x+1 \\ J_4(x) &= 2x+1 \\ J_5(x) &= x^2+3x+1 \\ J_6(x) &= 3x^2+4x+1 \\ J_7(x) &= x^3+6x^2+5x+1 \\ J_8(x) &= 4x^3+10x^2+6x+1 \\ J_9(x) &= x^4+10x^3+15x^2+7x+1 \\ J_{10}(x) &= 5x^4+20x^3+21x^2+8x+1 \end{aligned}$$

Notice that $J_n(1) = F_n$.

3. A Polynomial Expansion of $J_n(x)$. The coefficients of the Jacobsthal polynomials lie on the rising diagonals of the left-justified Pascal's triangle, but in the reverse order, as Figure 2 shows. Using this observation, we can derive the following explicit formula for $J_n(x)$.

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

coefficients of $J_6(x)$

Figure 2

THEOREM 1.

$$J_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{\lfloor n/2 \rfloor + j}{\lfloor (n-1)/2 \rfloor - j} x^{\lfloor (n-1)/2 \rfloor - j}$$

This formula can be established by showing that $J_n(x)$ satisfies the initial conditions and the recurrence relation (1).

For example,

$$\begin{aligned} J_8(x) &= \sum_0^3 \binom{4+j}{3-j} x^{3-j} \\ &= \binom{4}{3}x^3 + \binom{5}{2}x^2 + \binom{6}{1}x^1 + \binom{7}{0}x^0 \\ &= 4x^3 + 10x^2 + 6x + 1 \end{aligned}$$

4. Binet's Formula for $J_n(x)$. Let r and s be the solutions of the characteristic equation $t^2 - t - x = 0$ of the recurrence relation (2) **[3]**. Then $r = \dfrac{1+\sqrt{1+4x}}{2}$, $s = \dfrac{1-\sqrt{1+4x}}{2}$, $r+s = 1$, $rs = -x$, and $r - s = \sqrt{1+4x}$. Using the techniques of solving recurrence relations **[3]**, it can be shown that $J_n(x)$ can also be defined by the Binet's formula

$$J_n(x) = \frac{r^n - s^n}{\sqrt{1+4x}}, \tag{3}$$

where $n \geq 1$.

The next theorem shows the link between the polynomials $K_n(x)$ and $J_n(x)$.

THEOREM 2.

$K_n(x) = x[J_{n-1}(x) + J_{n-2}(x)]$, *where* $n \geq 2$.

Proof. Since $K_n(x)$ satisfies the same recurrence relation as $J_n(x)$, it follows that $K_n(x) = Ar^n + Bs^n$, where the expressions A and B are to be determined subject to the initial conditions $K_1(x) = 1$ and $K_2(x) = x[3]$. These two conditions yield the equations

$$\begin{aligned} Ar + Bs &= 1 \\ Ar^2 + Bs^2 &= x. \end{aligned}$$

Solving this system, we get $A = \dfrac{x-s}{r\sqrt{1+4x}}$ and $B = \dfrac{r-x}{s\sqrt{1+4x}}$. Therefore

$$\begin{aligned}K_n(x) &= \frac{x-s}{r\sqrt{1+4x}}\cdot r^n + \frac{r-x}{s\sqrt{1+4x}}\cdot s^n\\ &= \frac{(x-s)r^{n-1}+(r-x)s^{n-1}}{\sqrt{1+4x}}\\ &= \frac{x(r^{n-1}-s^{n-1})-(rs)(r^{n-2}-s^{n-2})}{\sqrt{1+4x}}\\ &= x[J_{n-1}(x)+J_{n-2}(x)],\end{aligned}$$

by formula (3). □

Thus, to find any polynomial $K_n(x)$, it suffices to multiply the sum of the consecutive members $J_{n-1}(x)$ and $J_{n-2}(x)$ of the Jacobsthal family by x, where $n \geq 3$.

For example,

$$\begin{aligned}K_9(x) &= x[J_8(x)+J_7(x)]\\ &= x[(4x^3+10x^2+6x+1)+(x^3+6x^2+5x+1)]\\ &= 5x^4+16x^3+11x^2+2x\end{aligned}$$

Since $K_m(1) = F_m = J_m(x)$, Theorem 4.1 yields the familiar Fibonacci recurrence formula.

COROLLARY 3.

$$F_n = F_{n-1}+F_{n-2}, \quad n \geq 3.$$

5. A Polynomial Expansion for $K_n(x)$. Theorems 3.1 and 4.1 can be employed to derive a polynomial formula for $K_n(x)$.

Case 1 Let $n = 2k+1$ be odd. By Theorem 4.1,

$$\begin{aligned}J_{n-1}(x)+J_{n-2}(x) &= \sum_0^{k-1}\binom{k+j}{k-j-1}x^{k-j-1}+\sum_0^{k-1}\binom{k+j-1}{k-j-1}x^{k-j-1}\\ &= \sum_0^{k-1}\left[\binom{k+j}{k-j-1}+\binom{k+j-1}{k-j-1}\right]x^{k-j-1}\\ &= \sum_0^{k-1}\frac{k+3j+1}{2j+1}\binom{k+j-1}{k-j-1}x^{k-j-1}\end{aligned}$$

Case 2 Let $n = 2k$ be even. Then:

$$\begin{aligned}J_{n-1}(x)+J_{n-2}(x) &= \sum_0^{k-1}\binom{k+j-1}{k-j-1}x^{k-j-1}+\sum_0^{k-2}\binom{k+j-2}{k-j-2}x^{k-j-2}\\ &= \sum_0^{k-1}\binom{k+j-1}{k-j-1}x^{k-j-1}+\sum_1^{k-1}\binom{k+j-2}{k-j-1}x^{k-j-1}\\ &= \sum_1^{k-1}\left[\binom{k+j-1}{k-j-1}+\binom{k+j-2}{k-j-1}\right]x^{k-j-1}+x^{k-1}\\ &= x^{k-1}+\sum_1^{k-1}\frac{k+3j-1}{2j}\binom{k+j-2}{k-j-1}x^{k-j-1}\end{aligned}$$

Thus we have the following result.

THEOREM 4.

$$K_n(x)/x = \begin{cases} x^{k-1}+\displaystyle\sum_1^{k-1}\frac{k+3j-1}{2j}\binom{k+j-2}{k-j-1}x^{k-j-1} & \textit{if } n = 2k \textit{ is even}\\ \displaystyle\sum_0^{k-1}\frac{k+3j+1}{2j+1}\binom{k+j-1}{k-j-1}x^{k-j-1} & \textit{if } n = 2k+1 \textit{ is odd.}\end{cases}$$

For example,

$$\begin{aligned}K_6(x)/x &= x^2+\sum_1^2\frac{2+3j}{2j}\binom{1+j}{2-j}x^{2-j}\\ &= x^2+\frac{5}{2}\binom{2}{1}x+\frac{8}{4}\binom{3}{0}\\ &= x^2+5x+2\end{aligned}$$

therefore $K_6(x) = x^3+5x^2+2x$

Likewise,

$$K_7(x)/x = \sum_0^2\frac{4+3j}{2j+1}\binom{2+j}{2-j}x^{2-j} = 4x^2+7x+2,$$

so

$$K_7(x) = 4x^3+7x^2+2x.$$

Since $K_n(1) = F_n$, the next result follows from Theorem 5.1.

COROLLARY 5.

$$\text{1)} \quad F_{2n} = 1+\sum_1^{n-1}\frac{n+3j-1}{2j}\binom{n+j-2}{n-j-1} \tag{4}$$

$$\text{1)} \quad F_{2n+1} = \sum_0^{n-1}\frac{n+3j+1}{2j}\binom{n+j-1}{n-j-1} \tag{5}$$

For example,

$$\begin{aligned}F_{11} &= \sum_0^4\frac{6+3j}{2j+1}\binom{4+j}{4-j}\\ &= \frac{6}{1}\binom{4}{4}+\frac{9}{3}\binom{5}{3}+\frac{12}{5}\binom{6}{2}+\frac{15}{7}\binom{7}{1}+\frac{18}{9}\binom{8}{0}\\ &= 6+30+36+15+2 = 89\end{aligned}$$

and $F_{12} = 1+\displaystyle\sum_1^5\frac{5+3j}{2j}\binom{4+j}{5-j} = 144.$

Lucas numbers L_n are very closely related to Fibonacci numbers. They are defined by $L_n = L_{n-1} + L_{n-2}$, where $L_1 = 1$ and $L_2 = 3$ **[1]**.

Since $F_{2n} = F_n L_n$ **[2]**, it follows that the sum on the right-hand side of (4) has non-trivial factors when $n \geq 3$. For example,

$$1 + \sum_1^{13} \frac{13+3j}{2j} \binom{12+j}{13-j} = 377 = F_{14} = 13 \cdot 29 = F_7 L_7.$$

Besides, since $F_{2n+1} = F_n^2 + F_{n+1}^2$ it follows that the sum in (5) is the sum of two (Fibonacci) squares. For example,

$$\sum_0^4 \frac{6+3j}{2j+1} \binom{4+j}{4-j} = F_{11} = 89 = 25 + 64 = F_5^2 + F_6^2.$$

We now turn our attention to constructing a generating function for $K_n(x)$.

6. A Generating Function for $K_n(x)$. The polynomials $K_n(x)$ can be realized as coefficients in a power series expansion. To see this, first notice that

$$\frac{1}{1-t-xt^2} + \frac{1}{(1-rt)(1-st)} = \frac{A}{1-rt} + \frac{B}{1-st}$$

where $A = \frac{r}{\sqrt{1+4x}}$ and $B = -\frac{s}{\sqrt{1+4x}}$, using partial fractions. So

$$\frac{1}{1-t-xt^2} = A\sum_0^\infty r^n t^n + B\sum_0^\infty s^n t^n = \sum_0^\infty \frac{(r^{n+1} - s^{n+1})t^n}{\sqrt{1+4x}}$$
$$= \sum_0^\infty J_{n+1} t^n, \quad \text{by (3)}$$

Then

$$\frac{t}{1-t-xt^2} = \sum_0^\infty J_{n+1} t^{n+1} = \sum_1^\infty J_n(x) t^n$$
$$= \sum_0^\infty J_n(x) t^n, \quad \text{since } J_0(x) = 0.$$

Therefore,

$$\frac{t^2+t^3}{1-t-xt^2} = \sum_2^\infty [J_{n-1}(x) + J_{n-2}(x)] t^n$$
$$= \sum_2^\infty [K_n(x)/x] t^n, \quad \text{by Theorem 4.1.}$$

That is,

$$\frac{x(1+t)t^2}{1-t-xt^2} = \sum_2^\infty K_n(x) t^n.$$

In other words,

$$f(t) = \frac{t + (x-1)t^2}{1-t-xt^2} = \sum_1^\infty K_n(x) t^n.$$

Thus, the function $f(t)$ generates the polynomials $K_n(x)$ as coefficients of t^n, where $n \geq 1$.

In particular, the function

$$g(t) = \frac{t}{1-t-t^2} = \sum_1^\infty F_n t^n$$

generates the Fibonacci numbers.

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Thomas Khoshy received his Ph.D. from Boston University and has been on the faculty of Framingham State College since 1971.

In Memoriam.

Josephine Gardner Good. It is with deepest sympathies that we report the passing of Josephine Gardner Good, the beloved wife of Richard A. Good. Dr. Richard (Dick) Good served Pi Mu Epsilon from 1975 to 1993 as Secretary-Treasurer and Councilor. Dick is Professor Emeritus of Mathematics of the University of Maryland and lives in Hyattsville, Maryland. Dr. Josephine Good, or "Jo" to her friends, received a Ph.D. with a joint major in nutrition and biochemistry from the University of Wisconsin at Madison. She served on the faculties of the University of Rhode Island and Oregon State University before marrying Dick in 1946. When Dick became involved with Pi Mu Epsilon, Jo would attend the Summer Meetings with him. She loved square dancing, bridge, and sewing. In fact, there was a time when she had made Dick so many shirts that he taught an entire semester without wearing the same shirt twice!

The 2000 National Pi Mu Epsilon Meeting

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held in Los Angeles, CA from August 3-4, 2000. As in the past, the meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections.

The J. Sutherland Frame Lecturer was **John H. Ewing** from the American Mathematical Society. His presentation was entitled "The Mathematics of Computers".

Student Presentations. The following student papers were presented at the meeting. An asterisk(*) after the name of the presenter indicates that the speaker received a best paper award.

What Does a 40% Chance of Rain Really Mean?
Katie Fleming
Ohio Xi - Youngstown State University

Statistical Analysis of Mastery Learning
Jodie Matulja
Ohio Xi - Youngstown State University

Mark McGwire Meets Mathematicians
David Gerberry
Ohio Xi -Youngstown State University

A Risky Algorithm: The Relative Risk vs. the Odds Ratio
John Slanina
Ohio Xi - Youngstown State University

Virtual Visualizations
Robert Shuttleworth
Ohio Xi - Youngstown State University

Properties of Positive Semi-Definite Operators
Anthony D. VanHoy
North Carolina Delta - East Carolina University

Fractal Tilings with Radial Symmetry Adam Roberts
Ohio Nu - The University of Akron

Irregular Sierpinski Triangles
Matthew Palmer
Ohio Nu - The University of Akron

Manifolds: They're Not Just For Cars
Duane K. Farnsworth
Ohio Omicron - Mount Union College

Fundamental Groups and the Manifolds in Your Cereal Bowl
Judy Maendel*
Ohio Omicron - Mount Union College

The Best Seat in the House
Sarah Grove
Ohio Xi - Youngstown State University

Separated at Birth?
Ryan Siskind
Ohio Xi - Youngstown State University

Convergence of Infinite Series
Sara LaLumia*
Ohio Xi - Youngstown State University

Roulette with a Twist
Marie Artesse
California Nu - Sonoma State University

Trigonometric Functions of Matrices
Rachael Floit
Illinois Eta - Augustana College

Understanding the Finite Element Method for Solving Differential Equations
Jonathan Moussa
Massachusetts Alpha - Worcester Polytechnic Institute

Statistics in the World of Civil Services
Yakov Kronrod*
Massachusetts Alpha - Worcester Polytechnic Institute

Application of Sampling Techniques in a Gambling Survey
Bethany Bray
Michigan Zeta - University of Michigan at Dearborn

Searching for a Perfect Voting System
Joel Lepak*
Ohio Xi - Youngstown State University

The Irrationality of ϵ and π
Thomas Wakefield*
Ohio Xi - Youngstown State University

Zero Sum Rado Number for $X_1 + X_2 + C = X_3$
Kathryn Rendall
Wisconsin Delta - St. Norbert College

Investigating the Irregularity Strength of Trees
David Kravitz
Delaware Alpha - University of Delaware

Finite Division Rings are Fields
Todd Horne
New York Beta - Hunter College

Symmetric Exponential Equations
David Kurzynski
Wisconsin Delta - St. Norbert College

Energizer Fractions: They Keep Going and Going and ...
Erin M. Bergman*
Wisconsin Delta - St. Norbert College

Pollution Dispersion in Large Indoor Places
Jeffrey Housman
California Nu - Sonoma State University

Pricing the American Call Option
Michelle Swenson
Nebraska Alpha - University of Nebraska at Lincolm

An Introduction to Traffic Flow
Lori McMenamin
Michigan Zeta - University of Michigan at Dearborn

RSA Cipher System: What Is It and Why Is It So Safe?
Hai He
New York Beta - Hunter College

Braess' Paradox in Computer Networks
Abhiram Shandilya
Texas Zeta - Angelo State University
Women in the History of Mathematics
Elizabeth Evans
South Carolina Alpha - University of South Carolina
Figure It Out: The Mathematics Behind Figure Skating Scores
Abby Mroczenski
Wisconsin Delta - St. Norbert College
Radial Checkers: A New Twist To An Old Game
Heather A. Olm
Wisconsin Delta - St. Norbert College
Sam Loyd's Fifteen Puzzle: The Even, The Odd and The Solvable
Stacy A. May
Illinois Zeta - Southern Illinois University at Edwardsville
Fun with Flexagons
Jeffrey Dumont
Pennsylvania Tau - Lafayette College
Which is the Right Path For Me
Rosemary Tomase
Wisconsin Delta - St. Norbert College
Continuity of the Derivative at a Point and on an Interval
John T. Griesmer*
Ohio Delta - Miami University
An Adaptation of the Improved Euler's Method for 2-Dimensional Hamiltonian Systems
Dzuan K. Nguyen
Nebraska Alpha - University of Nebraska at Lincoln
The SI Realization of Forces at the Nano-Newton Level
Laura A. Feeney
Ohio Delta - Miami University

Call For Papers.

The next ΠME meeting will take place in Madison, Wisconsin, August 2–4, 2001. See the ΠME webpage (http://www.pme-math.org/) for application deadlines and forms. See also the MAA webpage (http://www.maa.org/meetings/mathfest01_frontpage.html) for other activities in the Badger State.

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CATASTROPHIC CANCELLATION ON THE HIGH SEAS

AMY LANGVILLE*

Captain Bob is the captain of a large cruise ship which is currently leaving the port of Baltimore, Maryland. The first destination on this Mediterranean Cruise is Athens, Greece. The crew will be at sea for seven consecutive days until its first stop. Captain Bob, a recent Weight Watchers member, is in a predicament. Each week he must attend a Weight Watchers meeting, updating the group on his diet and weight loss progress. Weight Watchers has agreed to allow Captain Bob to conduct this week's meeting over the phone. Captain Bob's problem lies in the fact that his cruise ship has no personal scales and none are available at the port in Athens. The ship's engineer has persuaded Captain Bob to weigh himself using the large scales which all major ports have to weigh the ship and its cargo. The engineer explains that when they reach Athens, and after the passengers have disembarked, they could simply weigh the ship with the captain aboard. The captain could step off the ship, and the ship alone could then be weighed. A simple subtraction of the weight of the ship and the captain together minus the weight of the ship alone would give the captain's weight. The scales are known to be accurate to six digits. By the engineer's reasoning that should be plenty enough to accurately capture the captain's weight.

On the seventh day, after all the passengers had disembarked from the ship, the captain went through with the procedure. The engineer reported the weight of the ship plus the weight of the captain to be $1.000004 \cdot 10^8$ pounds and the weight of the ship to be $1.000001 \cdot 10^8$ pounds. After the simple subtraction, the captain must then weigh $.000003 \cdot 10^8 = 300$ pounds.

The captain was shocked, "300 lbs! Just last week I weighed 200 lbs!"

"But this is a cruise ship and the cook did serve filet mignon and cheesecake all week," countered the engineer.

Still in disbelief, the captain postponed his call to the Weight Watchers club until he got to the bottom of this mystery. Since the ship had docked for the day in Athens, the captain decided to take the engineer's latest piece of advice and visit the numerical analysis professor of a nearby university. By lunch, Captain Bob had caught up with one of the world's most renowned numerical analysts, Dr. Socrates.

"Ahh, I see. Such a common problem–the problem of catastrophic cancellation," Dr. Socrates remarked.

"Great! So you can explain this to me," said the captain excitedly.

"Sure, but it may take some calculations and fiddling." And so begins the professor's explanation of catastrophic cancellation.

"I'll explain exactly what catastrophic cancellation refers to in just a bit, but first, here's an intuitive explanation for it. It happens when we are subtracting two numbers of the same sign and the two numbers are in error. Specifically, when the two operands are in error and the result of the subtraction is much[1] smaller than the two operands, we can encounter this problem because then the result of the subtraction is of the same magnitude as the error. Scales, just like computers, do not always represent measurements exactly. In your situation, Captain, you used a scale that was accurate to 6 digits to represent your approximate weight and you subtracted the weight of the ship from the weight of the ship plus yourself, two numbers of equal

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[1]The mathematical meaning of "much" will be clarified in the next section.

magnitude. Therefore, catastrophic cancellation, the loss of accuracy in a subtraction, has occurred. Your weight as recorded on the scale may be completely wrong.

"Let's be more specific. A scale gives an approximation to your actual weight. Naturally, with any approximation, we are interested in how close the approximation is to the actual value. We call $w(x)$ the approximation of the actual value, x. In your case, Captain Bob, we will let

$w(b)$ = approximate weight of both the ship and the captain $= 1.000004 \cdot 10^8$,
$w(s)$ = approximate weight of the ship $= 1.000001 \cdot 10^8$.
Then $w(b) - w(s)$ = approximate weight of the captain
$= 1.000004 \cdot 10^8 - 1.000001 \cdot 10^8 = .000003 \cdot 10^8$.

By the engineer's calculation, you weigh 300 pounds. The engineer did the subtraction properly; there is no error in the subtraction operation. Yet you claim the resulting answer from the subtraction is still in error. You are right. The problem lies in the last digit of the subtraction. While it may be true that $4 - 1 = 3$, we are not sure that the 4 and the 1 are accurate. The scale you used only guarantees 6 digits of accuracy. The leading 5 zeros in the mantissa of the result are accurate. However, the 3 may not be accurate. Therefore, we have no idea whether the result of 300 pounds is accurate.

"Consider the weight of the ship plus Captain Bob. The seventh digit, 4, may be inaccurate. Suppose the next time we weigh the ship plus the captain the scale reads $1.000003 \cdot 10^8$. Then $w(b) - w(s) = .2000000 \cdot 10^3 = 200$ pounds. You, Captain, would be much happier with this answer. Yet every digit is still in error. Now suppose we do the procedure one more time. The ship plus the captain weighs $1.000007 \cdot 10^8$ this time. Then you would weigh 600 pounds. Preposterous? Yes. But this teaches you to compare the magnitude of the error in the operands with the magnitude of the result."

"So I should just call Weight Watchers and tell them that I don't know my weight this week. I weigh so much less than the ship that catastrophic cancellation has occurred," said Captain Bob.

"Yes, and if your next stop is Rome, I know they have personal scales there."

"Thanks for all your help, Dr. Socrates," said the captain with relief.

"Sure!"

The captain departed for Rome thoroughly satisfied with Dr. Socrates' explanation of his phony weight, yet Dr. Socrates continued to ponder the precise nature of catastrophic cancellation. In fact, later that day Dr. Socrates sat at his desk and revised the notes he planned to present to his class that evening. Some excerpts from his lecture on "Catastrophic Cancellation and the Captain Bob Story" follow.

An Analysis of Catastrophic Cancellation. The absolute error of a measurement is the difference between the measured value and the actual value. The absolute error in the weight of the ship is $|w(s) - s|$. In two different scenarios an absolute error of .1 units might have contrasting meanings. For example, in measuring the distance from a point in Athens to a point in Rome an absolute error of .1 inches would be laudable, while in a heart surgery procedure, which requires a .2 inch incision, an absolute error of .1 inches would be unacceptable[2].

To remedy this problem, we need to consider relative error, that is, we talk about the error relative to the magnitude of one of the values involved. There are two ways to make an absolute error relative. One type of relative error starts with the absolute

[2]The idea for this example is due to Carl Meyer.

error and makes it relative by dividing by the actual value, resulting in $|\frac{w(s)-s}{s}|$. The other type of relative error divides the absolute error by the approximated value, giving $|\frac{w(s)-s}{w(s)}|$. Both relative errors normalize the absolute error. Therefore, the .1 inch absolute error in the measurement of the distance from Athens to Rome (646 miles) gives a relative error of $\frac{.1}{40930560} = .0000000244$, a minuscule value and thus an excellent result. On the other hand, the .1 inch absolute error in the heart surgery gives a relative error of $\frac{.1}{.2} = .5$. The error in the size of the incision is half the size of the actual incision! Looking at relative error as opposed to absolute error can often tell us whether an error is acceptable in a particular context.

Let's take this notion of relative error and get back to Captain Bob. Recall that $w(s)$ represents the approximate weight of the ship and s represents the actual, yet unknown, weight of the ship. Let's use the first type of relative error. Let δ_s represent this relative error, i.e., $\delta_s = |\frac{w(s)-s}{s}|$. Similarly, δ_b represents the relative error associated with the ship plus the captain. Thus, $\delta_b = |\frac{w(b)-b}{b}|$. With any measuring device, we would like to know just how "good" its approximations are, compared to the actual values. Let's assume that the scale is calibrated and its relative errors do not exceed U. Therefore, $|\delta_b| \leq U$ and $|\delta_s| \leq U$. In Captain Bob's case, $U = .5 \cdot 10^{-6}$. Hence the scale guarantees 6 digits of accuracy.

Now we have all the machinery in place to analyze the relative quality of an approximation such as the captain's weight. Specifically, how well does $w(b) - w(s)$ approximate $b - s$? Consider the relative error associated with the captain's weight:

$$\begin{aligned}\frac{|w(b) - w(s) - (b - s)|}{|b - s|} &= \frac{|(w(b) - b) - (w(s) - s)|}{|b - s|} \\ &\leq \frac{|b| \cdot \delta_b + |s| \cdot \delta_s}{|b - s|} \\ &\leq \frac{|b| + |s|}{|b - s|} \cdot U = A \cdot U,\end{aligned}$$

where

$$A = \frac{|b| + |s|}{|b - s|}.$$

A is called the amplification factor or the condition number of the subtraction. A condition number for a mathematical problem indicates how much the input error is amplified in the final result. Here the input error is U, the error in the operands. The error in the captain's weight can be as large as $A \cdot U$. From the formula for A, we observe that the condition number for subtraction is large when the result of the subtraction, $b - s$, has much smaller magnitude than the individual operands, b and s. If this is the case, the subtraction is ill-conditioned. This is what happens in the captain's example. In that example, $|b - s|$ is on the order of 10^2 and $|b| + |s|$ is on the order of 10^8, hence $|b - s| \ll |b| + |s|$, since the captain's weight is so small in comparison to the weight of the ship. Therefore, A is large, approximately 10^6, and the relative error associated with the captain's weight could be large. This means the error that occurs when weighing the ship can be amplified by as many as six orders of magnitude in the subtraction that approximates the weight of the captain. This explains the loss of digits of accuracy in the resulting answer of 300 pounds.

Catastrophic cancellation is not limited to Captain Bob's scale. The same considerations apply when we subtract two numbers, x and y, on a computer, such as a calculator or PC. The mere process of entering the numbers can cause errors. For instance, a computer might represent $\frac{1}{3}$ as .3333. To account for this rounding, let $fl(x)$ denote the floating point representation of a number x. For example, $fl(\frac{1}{3}) = .3333$. Since only 4 digits are used in the decimal representation of $\frac{1}{3}$, $U = .5 \cdot 10^{-4}$, where U represents the unit roundoff of the computer. In a floating point subtraction, $fl(x) - fl(y)$, the relative error associated with this difference is

$$\frac{|fl(x) - fl(y) - (x - y)|}{|x - y|} \leq A \cdot U,$$

where

$$A = \frac{|x| + |y|}{|x - y|}$$

is the amplification factor.

Now it should be clear why subtraction of floating point numbers of almost equal magnitude should be avoided. The result of the subtraction has a magnitude similar to the error in the operands. In this case, the subtraction is ill-conditioned and the relative error can be large. The computed difference can be completely wrong. Unfortunately, catastrophic cancellation cannot be blamed the next time the bathroom scale registers an unsightly number, unless, of course, the Captain Bob procedure is used.

Acknowledgements. I thank Ilse Ipsen, Carl Meyer and Michael Shearer for their helpful comments and suggestions which improved the presentation of the paper.

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Amy Langville is a doctoral student in the Operations Research Program at North Carolina State University. She spends most of her time now studying iterative methods for finding the stationary distribution of Stochastic Automata Networks.

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A CONDITION FOR THE EXISTENCE OF INFINITELY MANY FERMAT PRIMES

FLORIAN LUCA*

Abstract. In this note, we give a condition for the existence of infinitely many Fermat primes in terms of the irrationality of the series of all the reciprocals of the numbers n for which the regular polygon with n sides can be constructed with ruler and compass.

Let $(a_n)_{n\geq 0}$ be a strictly increasing sequence of positive integers. Assume that the series

$$\sum_{n\geq 0} \frac{1}{a_n} \tag{1}$$

is convergent. The first natural question which arises is whether the sum of the series given by (1) is a rational or an irrational number. There are several papers in the literature (see, for example, [1], [2] and [3]) in which various criteria are given for the sum of the series (1) to be irrational.

In this short note, we investigate the irrationality character of a series of the form (1) in which the sequence $(a_n)_{n\geq 0}$ is connected with the Fermat primes. Recall that for $n \geq 0$, the number $F_n = 2^{2^n} + 1$ is called the nth Fermat number. It is known that F_n is prime for $0 \leq n \leq 4$ and that F_n is composite for $5 \leq n \leq 30$. It is also known that F_n is composite for some other values of n beyond 30, and it is now a popular belief that there are only finitely many Fermat primes, and maybe even that the only Fermat primes are the first five of them.

In this note, we give a criterion for the existence of only finitely many Fermat primes in terms of the rationality of a sum of the type (1). For any positive integer n, let $\phi(n)$ be the Euler function of n. Let $\mathcal{C}$ be the set of all the positive integers n such that $\phi(n)$ is a power of 2. We use the notation $\mathcal{C}$ from the word *constructible*, since by a well-known theorem of Gauss, if $n \geq 3$, then the regular polygon with n sides can be constructed with the ruler and the compass if and only if $n \in \mathcal{C}$.

We have the following result.

THEOREM 1. *The sum of the series*

$$(1) \qquad \sum_{n\in\mathcal{C}} \frac{1}{n}$$

is rational if and only if there are only finitely many Fermat primes.

We also recall that the irrationality of the series (1) when $a_n = F_n$ for all $n \geq 0$ follows as a special case from a result in [4].

Proof. Let $\mathcal{C}_1 \subset \mathcal{C}$ be the subset of all the odd numbers in $\mathcal{C}$. Let also $0 \leq m_1 < m_2 < \ldots < m_k < \ldots$ be the set (possibly finite) of all the non-negative integers n such that F_n is prime. Clearly,

$$\mathcal{C} = \{2^t n \mid n \in \mathcal{C}_1 \text{ and } t \geq 0\}, \tag{3}$$

and

$$\mathcal{C}_1 = 1 \cup \{n \mid n = F_{m_{i_1}} \cdot F_{m_{i_2}} \cdot \ldots \cdot F_{m_{i_t}}\ t \geq 1 \text{ and } 1 \leq i_1 < i_2 < \ldots < i_t\}. \tag{4}$$

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Hence, it is easy to see that

$$\sum_{n\in\mathcal{C}}\frac{1}{n}=\sum_{t\geq 0}\sum_{n\in\mathcal{C}_1}\frac{1}{2^t n}=2\sum_{n\in\mathcal{C}_1}\frac{1}{n}=2\prod_{k\geq 1}\Big(1+\frac{1}{F_{m_k}}\Big). \tag{5}$$

It therefore suffices to investigate the irrationality of the product

$$\prod_{k\geq 1}\Big(1+\frac{1}{F_{m_k}}\Big). \tag{6}$$

If there exist only finitely many Fermat primes, then the product (6) consists of only finitely many factors; hence, it is rational.

Assume now that there are infinitely many Fermat primes. We denote by α the product appearing in formula (6) and for $n \geq 1$ we set

$$x_n=\prod_{k=1}^{n}(1+F_{m_k})$$

and

$$y_n=\prod_{k=1}^{n}F_{m_k}.$$

We first notice that $\alpha < 2$. Indeed,

$$\alpha<\prod_{n\geq 0}\Big(1+\frac{1}{F_n}\Big)<\prod_{n\geq 0}\Big(1+\frac{1}{2^{2^n}}\Big)=\sum_{m\geq 0}\frac{1}{2^m}=2.$$

We now find an upper bound for the error of approximating α by $\frac{x_n}{y_n}$. Notice that

$$0<\alpha-\frac{x_n}{y_n}=\frac{x_n}{y_n}\Big(\prod_{k\geq n+1}\Big(1+\frac{1}{F_{m_k}}\Big)-1\Big)<2\big(e^{\sum_{k\geq n+1}\frac{1}{F_{m_k}}}-1\big). \tag{7}$$

However,

$$\sum_{k\geq n+1}\frac{1}{F_{m_k}}<\sum_{j\geq 2^{m_{n+1}}}\frac{1}{2^j}=\frac{1}{2^{2^{m_{n+1}}-1}}<\frac{3}{F_{m_{n+1}}}. \tag{8}$$

Thus, from formulae (7) and (8) we get

$$0<\alpha-\frac{x_n}{y_n}<2\big(e^{\frac{3}{F_{m_{n+1}}}}-1\big)<\frac{12}{F_{m_{n+1}}}. \tag{9}$$

For the rightmost inequality (9) we used the fact that $e^x < 1+2x$ for $x \in (0,\ 1)$. Assume now that α is rational and write it as $\alpha = \frac{a}{b}$ where a and b are coprime positive integers. From inequality (9), it follows that

$$0<ay_n-bx_n<\frac{12by_n}{F_{m_{n+1}}}. \tag{10}$$

At this point, we distinguish two cases:

Case 1 *There exist infinitely many n's with $m_{n+1} > m_n + 1$.*
Suppose that n satisfies the above property. In this case,

$$F_{m_{n+1}}>2^{2^{m_n+2}}>(2^{2^{m_n+1}}-1)^2=\Big(\prod_{j=0}^{m_n}F_j\Big)^2\geq y_n^2. \tag{11}$$

With such n's, inequality (10) becomes

$$0<ay_n-bx_n<\frac{12by_n}{y_n^2}=\frac{12b}{y_n}. \tag{12}$$

If n is large enough, then the right hand side of inequality (12) becomes smaller than 1, which is impossible because $ay_n - bx_n$ is a positive integer. This case is therefore settled.

Case 2 *There exists k_0 such that F_n is prime for all $n > k_0$.*

In this case, there are only finitely many Fermat numbers which are not prime. Since the product

$$\prod_{n\geq 0}\Big(1+\frac{1}{F_n}\Big) \tag{13}$$

and the product (6) differ only by finitely many rational factors, it suffices to prove that the product given by formula (13) is irrational. We keep the previous notations with the convention that $m_k = k$ for all $k \geq 0$. With this notation, inequality (10) becomes

$$0<ay_n-bx_n<\frac{12by_n}{F_{n+1}}. \tag{14}$$

Since

$$y_n=\prod_{k=0}^{n}F_k=2^{2^{n+1}}-1<F_{n+1},$$

it follows that

$$0<ay_n-bx_n<12b. \tag{15}$$

Let $s = 12b$. By inequality (15) and the pigeon hole principle, it follows that there exist infinitely many pairs $(n,\ t)$ with $n > 0$ and $1 \leq t < s$ such that the equation

$$ay_n-bx_n=ay_{n+t}-bx_{n+t} \tag{16}$$

is satisfied for all such pairs. Equation (16) can be rewritten as

$$bx_n\Big(\prod_{j=1}^{t}(F_{n+j}+1)-1\Big)=ay_n\Big(\prod_{j=1}^{t}F_{n+j}-1\Big). \tag{17}$$

In particular, equality (17) implies that

$$F_n+1 \mid x_n \text{ and } x_n \mid ay_n\Big(\prod_{j=1}^{t}F_{n+j}-1\Big). \tag{18}$$

Since $2^{2^n-1}+1 \mid F_n+1$, it follows that

$$2^{2^n-1}+1 \le a\cdot \gcd(2^{2^n-1}+1,\ y_n)\cdot \gcd(2^{2^n-1}+1,\ \prod_{j=1}^{t} F_{n+j}-1)$$

or

$$2^{2^n-1}+1 \le a\prod_{k=0}^{n} \gcd(2^{2^n-1}+1,\ F_k)\cdot \gcd(2^{2^n-1}+1,\ \prod_{j=1}^{t} F_{n+j}-1). \tag{19}$$

It is well-known that if a, m and n are positive integers with a even, then

$$\gcd(a^m+1,\ a^n+1)=\begin{cases} a^{\gcd(m,n)}+1, & \text{if } \mu_2(m)=\mu_2(n),\\ 1, & \text{otherwise.}\end{cases} \tag{20}$$

In the above formula, for a positive integer k we have denoted by $\mu_2(k)$ the order at which the prime number 2 appears in the prime factor decomposition of k. From the above formula (20), we deduce easily that

$$\gcd(2^{2^n-1}+1,\ F_k)=\begin{cases} 3, & \text{if } k=0,\\ 1, & \text{if } k>0.\end{cases} \tag{21}$$

Hence, inequality (19) becomes

$$2^{2^n-1}+1 \le 3a\cdot \gcd(2^{2^n-1}+1,\ \prod_{j=1}^{t} F_{n+j}-1). \tag{22}$$

Let $D=\gcd(2^{2^n-1}+1,\ \prod_{j=1}^{t} F_{n+j}-1)$. Since $D \mid 2^{2^n-1}+1$, it follows that $2^{2^n}\equiv -2 \pmod D$. Hence, $2^{2^{n+j}}\equiv 2^{2^j} \pmod D$ for all $j>0$, therefore $F_{n+j}\equiv F_j \pmod D$ for all $j>0$. However, since

$$D \mid \prod_{j=1}^{t} F_{n+j}-1,$$

it follows that

$$D \mid \prod_{j=1}^{t} F_j - 1.$$

In particular,

$$D \le \prod_{j=1}^{t} F_j - 1 < \prod_{j=0}^{t} F_j = 2^{2^{t+1}}-1 < 2^{2^s}. \tag{23}$$

Combining inequalities (22) and (23), it follows that

$$2^{2^n-1}+1 < 3a2^{2^s},$$

which is impossible because n can assume infinitely many values.

This disposes of Case 2 and concludes the proof of Theorem 1. □

Remark In many instances, the irrationality of a product of the form (6) (or (13)) can be inferred from an old criterion of V. Brun (see [2]). Unfortunately, the convergents $\frac{x_n}{y_n}$ of the products (6) or (13) do not satisfy the hypothesis from Brun's criterion. However, one can use Brun's criterion to prove the following result.

THEOREM 2. *Let a, b and q be three integers such that $q\ge 2$, $a>b>0$, and $(a,\ b)\ne(2,\ 1)$ and let $(m_k)_{k\ge 1}$ be any strictly increasing sequence of positive integers. Then, the infinite product*

$$\prod_{k\ge 0}\Big(1+\frac{1}{a^{q^{m_k}}+b^{q^{m_k}}}\Big)$$

is irrational.

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[4] GOLOMB, J.W., *A Special Case. On the sum of the reciprocals of the Fermat numbers and related irrationalities,* Can. J. of Math. Vol. 15, pp 475-478, 1963.

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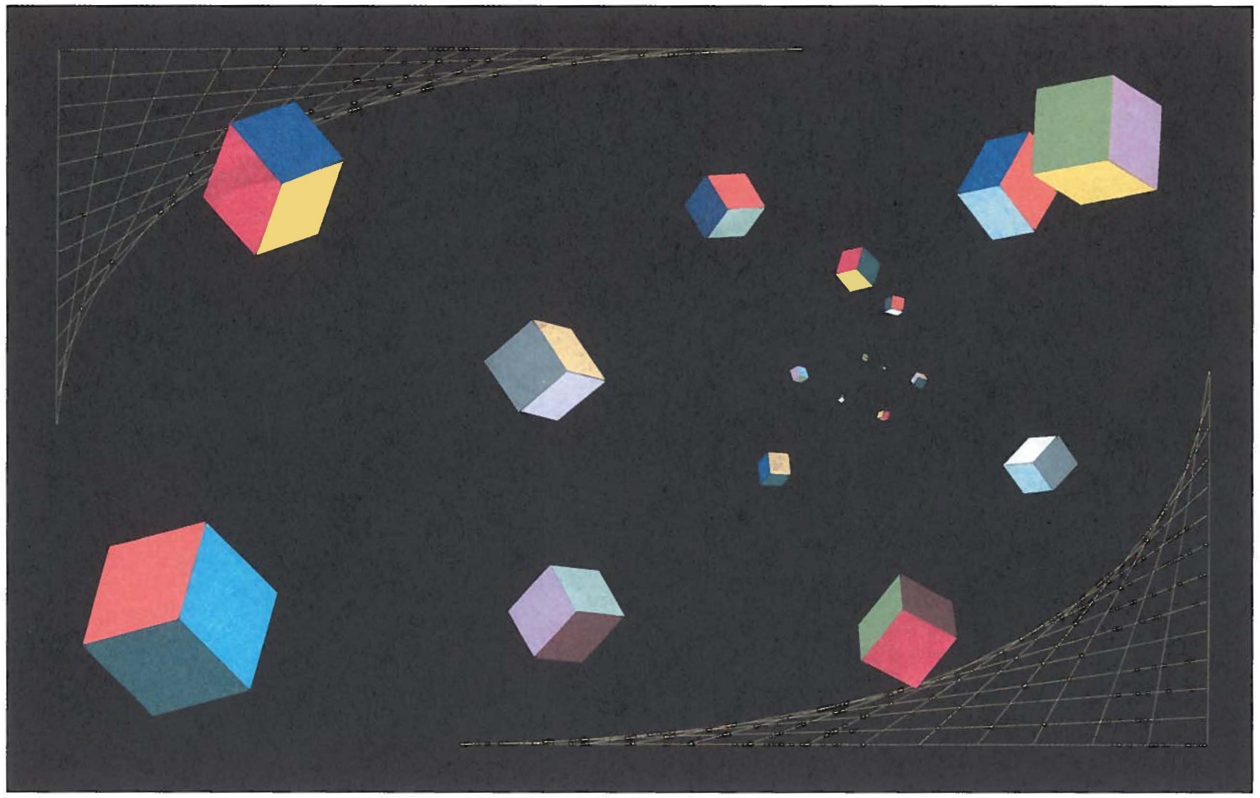
Florian Luca is a researcher at the Mathematical Institute of the UNAM in Morelia, México. His hobbies are his wife Raluca, travelling and learning foreign languages.

From the Right Side



Asteroid Space. As a mathematics major at the Courant Institute, while studying epicycloids and hypocycloids, Rex. H. Wu created Asteroid Space. Mathematical form is complemented by the artist's choice of color, size and arrangement.

Rex H. Wu is a physician who still loves mathematics and finds time to contribute to our Problems Department after taking care of patients and teaching residents at the NYU Downtown Hospital. In fact, working on problem 971 of this Journal inspired him to "A Proof without Words" on the law of tangents, which will appear in the *College Mathematics Journal.* For Dr. Wu, art and mathematics always go together.

The ΠME *Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.*

ΠME Journal, Vol. 11, No. 4, pp 215–227, 2001.

PROBLEM DEPARTMENT

EDITED BY CLAYTON W. DODGE

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. **Please note my new e-mail address: dodge@maine.edu.** *Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by December 1, 2001. Solutions identified as by students are given preference.*

Problems for Solution.

1007. *Proposed by the editor.*

As children, my siblings and I would eat great quantities of peanut butter. A favorite treat was (and still is) peanut butter on a banana. (Peel the banana first! Then put on the peanut butter.) Thus solve this base ten alphametic

$$PEANUT = BUTTER + BANANA.$$

1008. *Proposed by Ice B. Risteski, Skopje, Macedonia.*

There exist polynomials with integer coefficients that are irreducible over the field of rational numbers but are reducible over the field of residues with respect to any prime modulus p. Prove that $f(x) = x^4 - 10x^2 + 1$ is such a polynomial.

1009. *Proposed by Ice B. Risteski, Skopje, Macedonia.*

a) Prove that if the polynomials $f(x)$ and $g(x)$ with integer coefficients are relatively prime over the field $\mathbb{Z}_p$ of residues with respect to the prime modulus p and at least one of the leading coefficients is not divisible by p, then these polynomials are relatively prime over the field of rational numbers.

b) Show by way of an example that for any prime p the converse assertion does not hold.

1010. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Show that

$$\left|\sum_{n=0}^{\infty} \frac{e^n}{(n+1)^{n+1}} + \frac{1}{2} - e\right| < \frac{1}{2}$$

1011. *Proposed by Maureen Cox and Albert W. White, St. Bonaventure University, St. Bonaventure, New York.*

Find a closed form expression for

$$\sum_{n=1}^{\infty} \frac{n^2 - n - 1}{(n+1)!}.$$

1012. *Proposed by William Chau, New York, New York.*

Find each perfect number p such that the product of the proper divisors of p is equal to p.

1013. *Proposed by Wu Wei Chao, He Nan Normal University, Xin Xiang City, He Nan Province, China.*

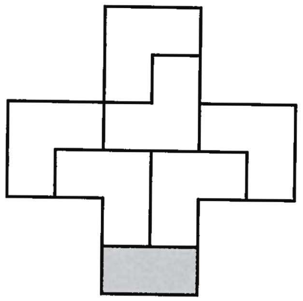
Observe that $77 \times 88 = 6776$, $77 \times 858 = 66066$, $777 \times 858 = 666666$, $7777 \times 8558 = 66555566$, $707 \times 8558 = 6050506$, etc. Prove that there exist infinitely many triples of palindromic natural numbers x, y, z such that $xy = z$.

1014. *Proposed by Miguel Amengual Covas, Santanyi - Mallorca, Spain.*

Given in $\mathbb{R}^3$ an elliptic paraboloid, find the locus of the centers of the spheres which cut the paraboloid in two circles.

1015. *Proposed by Richard I. Hess, Rancho Palos Verdes, California, and Robert T. Wainwright, New Rochelle, New York.*

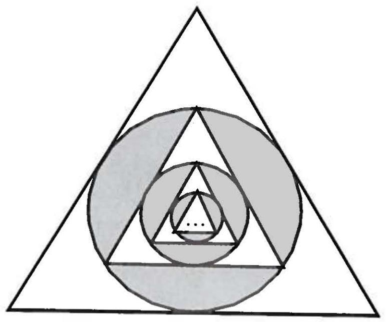
As shown in the accompanying figure, the X pentomino can be 90% covered with six congruent tiles. (The shaded area is not covered by these tiles.) Design a tile so that three of them cover at least 85% of the X pentomino. Any of the tiles may be turned over, but they must not overlap each other or the border.



1016. *Proposed by Brian Reid, student, County College of Morris, Randolph, New Jersey.*

A regular n-gon is inscribed in a circle of radius r. Then a circle is inscribed in that n-gon and a similar n-gon is inscribed in that circle, and so on forever. The accompanying figure shows the situation for $n = 3$.

a) Find the ratio of the sum of the areas lying inside each circle and outside its inscribed n-gon for $n = 3$ to the area of the original circle. This area is shaded in the figure.



b) Find the limit of that ratio as $n \to \infty$.

1017. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Consider Pascal's triangle with the rows numbered $0, 1, 2, \ldots$. If the sum of all the elements above the n'th row is a prime, characterize the number of elements in row $n - 1$.

1018. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

For a fixed number k, $0 < k < 1$, at each toss of a fair coin a gambler bets the fraction k of the money he has at the moment. In the long run, what percentage of the tosses must he win in order to break even?

1019. *Proposed by Kenneth B. Davenport, Frackville, Pennsylvania.*

Eduoard Lucas showed $(1^5 - 3^5 + 5^5 - \cdots + (-1)^{n+1}(2n-1)^5)/(1 - 3 + 5 - \ldots + (2n-1))$ is always a square number for every positive integer n but never a fourth power. Show that

$$\frac{(1^7 - 3^7 + 5^7 - \cdots + (-1)^{n+1}(2n-1)^7) - 28(1^3 - 3^3 + 5^3 - \cdots + (-1)^{n+1}(2n-1)^3)}{1 - 3 + 5 - \ldots + (2n-1)}$$

is always a cube, but almost never a sixth power.

1020. *Proposed by M. V. Subbarao, University of Alberta, Edmonton, Alberta, Canada.*

Dedicated to friend and colleague Murray S. Klamkin on his 80th birthday. [*Klamkin ably edited this Problem Department for 10 years until 1968 - ed.*]

Let $p_1, p_2, \ldots, p_r$ be r distinct odd primes and let a be any fixed integer. You are given that $(p_1 + a)(p_2 + a) \cdots (p_r + a) - 1$ is divisible by $(p_1 + a - 1)(p_2 + a - 1) \cdots (p_r + a - 1)$, which is trivially true for $r = 1$. Can it hold for any $r > 1$? If so, give a specific example. A \$100 award will be given for the first received valid example.

Remark 1. For $a = 0$, this is a known unsolved problem of D. H. Lehmer, Bull. Amer. Math. Soc. 38 (1932) 745-751. For $a = 1$, this also is an unsolved problem of mine in *A Companion to a Lehmer Problem,* Colloq. Math. Debrecen 52 (1998) 683-698.

Remark 2. One can also consider the more general problem obtained by replacing the primes $p_1, p_2, \ldots, p_r$ by their arbitrary powers $p_1^{b_1}, p_2^{b_2}, \ldots p_r^{b_r}$. My conjecture here is that at least for the cases $a = 0$ or 1, we must have $r = 1$. See my joint paper with V. Sivaramaprasad, *Some analogues of a Lehmer problem,* Rocky Mountain J. Math. 15 (1985) 609-629.

Solutions.

980. [Spring 2000] *Proposed by the editor.*

The addition alphametic

$$HALF + HALF = WHOLE$$

has unique solutions in both bases 7 and 8. Of course, in any base $WHOLE$ must be an even number. It is curious that in base 9 there are three solutions, two of which have $HALF$ even. Find that base 9 solution in which $HALF$ is an odd number.

Solution by **Frank P. Battles**, *Massachusetts Maritime Academy, Buzzards Bay, Massachusetts*

Clearly $W = 1$, so we must have $H = 8$ with a carry from the previous column. Now $L = 0$ with no carry from the units column, so $F \leq 4$. Since $A + A > 9$, then $A \geq 5$. Because O cannot be 1, then $A \neq 5$, so $A = 6$ or 7.

If $A = 6$, then $O = 3$ and F must be odd to make $HALF$ odd. Since 1 and 3 are already used, we cannot have $A = 6$. Thus $A = 7$. Then $O = 5$ and F must be even to make $HALF$ odd. Since 0 is already used and $F = 4$ requires $E = 8$, then we must have $F = 2$ and $E = 4$. Thus our unique solution is $8702 + 8702 = 18504$.

Also solved by **Charles D. Ashbacher**, Charles Ashbacher Technologies, Hiawatha, IA, **Scott H. Brown**, Auburn University, AL, **Paul S. Bruckman**, Berkeley, CA, **Joshua Duncan**, Jacksonville University, FL, **Mark Evans**, Louisville, KY, **Stephen I. Gendler**, Clarion University of Pennsylvania, **Richard I. Hess**, Rancho Palos Verdes, CA, **Carl Libis**, Richard Stockton College of NJ, Pomona, **Yoshinobu Murayoshi**, Okinawa, Japan, **William H. Peirce**, Rangeley, ME, **H.-J. Seiffert**, Berlin, Germany, **Robert A. Stump**, Richmond, VA, and the **Proposer**.

Editorial comment: Our long-retired kindly old friend, Professor Euclide Paracelso Bombasto Umbugio, eminent numerologist of Guayazuela, asked me to propose this alphametic for him. He has been enjoying his retirement in seclusion and felt that attaching his name to the problem would bring too much attention to him and cause many interruptions of his research into the distribution of even primes. When it was pointed out to him that the telephones Guayazuela installed 20 years ago still had not been connected up, he agreed to let me acknowledge his authorship.

Four of the listed solvers failed at first to see the twist in this problem. There are three base 9 solutions, $HALF = 8602$, 8702, and 8703, two of which terminate in an even digit and only one in an odd digit. The test for oddness in base 9, however, is not the oddness of the last digit, but rather the oddness of the sum of the digits. Therefore, 8602 and 8703 in base 9 represent even numbers, and 8702 is the odd one. Professor Umbugio sends his congratulations to those solvers who obtained the correct solution unaided. (He said it took him only four tries to find the right answer.)

Hess pointed out that the unique base 7 and base 8 solutions are respectively $HALF = 6502$ and 7503.

981. [Spring 2000] *Proposed by Cecil Rousseau, The University of Memphis, Memphis, Tennessee.*

Show that the set $\{\lfloor\sqrt{2}\rfloor, \lfloor 2\sqrt{2}\rfloor, \lfloor 3\sqrt{2}\rfloor, \ldots, \lfloor n\sqrt{2}\rfloor, \ldots\}$, where n is a natural number and $\lfloor x\rfloor$ is the greatest integer in x, contains infinitely many powers of 3.

Solution by **Paul S. Bruckman**, *Berkeley, California.*

Given any positive integer n, let $m = m(n) = \lfloor \log_3(n\sqrt{2})\rfloor$. Then

$$3^m < n\sqrt{2} < 3^{m+1}. \tag{1}$$

Let $a_n = (n\sqrt{2} - 3^m)/(2 \cdot 3^m)$, $n = 1, 2, \ldots$. Then $0 < a_n < 1$ for all n. Also the sequence $\{a_n\}$ is dense in (0,1). In particular, $0 < a_n < 1/(2 \cdot 3^m)$ for infinitely many n. Equivalently, for infinitely many n, $3^m < n\sqrt{2} < 3^m + 1$. Therefore, there exist infinitely many n such that $\lfloor n\sqrt{2}\rfloor = 3^m$.

Also solved by **Angelo State Problem Group,** Angelo State University, San Angelo, TX, **Mark Evans,** Louisville, KY, **Richard I. Hess,** Rancho Palos Verdes, CA, **Murray S. Klamkin,** University of Alberta, Canada, **Rex H. Wu,** Brooklyn, NY, and the **Proposer.**

982. [Spring 2000] *Proposed by Charles Ashbacher, Charles Ashbacher Technologies, Hiawatha, Iowa.*

In his book "Comments and Topics on Smarandache Notions and Problems", K. Kashihara defines for any positive integer n, the *Smarandache Inferior Square Part* SISP(n) to be the largest square less than or equal to n and the *Smarandache Superior Square Part* SSSP(n) to be the smallest square greater than or equal to n. Now define

$$s_n = \sqrt[n]{\mathrm{SSSP}(0) + \cdots + \mathrm{SSSP}(n)}$$

and

$$t_n = \sqrt[n]{\mathrm{SISP}(0) + \cdots + \mathrm{SISP}(n)}$$

a) Find the value of $\lim_{n\to\infty}(s_n - t_n)$.

b) Find the value of $\lim_{n\to\infty} s_n/t_n$.

Solution by **Mark Evans**, *Louisville, Kentucky.*

We have that $n = (\sqrt{n})^2 < \mathrm{SSSP}(n) < (\sqrt{n}+1)^2 = n + 2n^{1/2} + 1$. It follows that

$$\sqrt[n]{\frac{n^2}{2}} < \sqrt[n]{\frac{n(n+1)}{2}} = \sqrt[n]{0+1+2+\cdots+n} \le s_n$$

and

$$s_n < \sqrt[n]{\int_0^n (x + 2\sqrt{x} + 1)dx} = \sqrt[n]{\frac{n^2}{2} + \frac{3}{2}n^{3/2} + n}$$
$$= \sqrt[n]{\frac{n^2}{2}\left(1 + \frac{8}{3\sqrt{n}} + \frac{2}{n}\right)} = \sqrt[n]{\frac{n^2}{2}(1+\epsilon)}$$

where $\epsilon \to 0$ as $n \to \infty$. Thus $s_n \to (n^2/2)^{1/n}$ as $n \to \infty$. Similarly $t_n \to (n^2/2)^{1/n}$ as $n \to \infty$. So we have

a) $\lim_{n\to\infty}(s_n - t_n) = 0$ and

b) $\lim_{n\to\infty}(s_n/t_n) = 1$.

Also solved by **Paul S. Bruckman**, Berkeley, CA, **Richard I. Hess**, Rancho Palos Verdes, CA, **Robert A. Stump**, Richmond, VA, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

983. [Spring 2000] *Proposed by Rex H. Wu, Brooklyn, New York.*

Evaluate the integrals

a) and $\displaystyle\int_0^{\pi/2} \ln(\frac{1+\sin x}{1+\cos x})$

b) $\displaystyle\int_0^{\pi/2} \ln(\frac{1+\cos x+\sin x}{1+\cos x})$.

Solution by **Murray S. Klamkin**, *University of Alberta, Edmonton, Alberta, Canada.*

a) Let

$$I = \int_0^a \ln(b + f(x))dx.$$

On replacing x by $a - x$, we get that

$$\int_0^a \ln(b + f(x))dx = -\int_a^0 \ln(b + f(a-x))dx = \int_0^a \ln(b + f(a-x))dx$$

It follows that

$$\int_0^{\pi/2} \ln(\frac{1+\sin x}{1+\cos x})dx = \int_0^{\pi/2} \ln(1+\sin x)dx - \int_0^{\pi/2} \ln(1+\cos x)dx$$

since $\cos x = \sin(\pi/2 - x)$.

b) By letting $x = a - t$ in the integral

$$J = \int_0^a \ln(\frac{b + f(x) + f(a-x)}{b + f(x)})dx$$

where $[f(x)]^2 + [f(x-a)]^2 = b^2$, we obtain an equal integral. Adding these two integrals together, we get

$$2J = \int_0^a \ln 2dx = a\ln 2$$

The given integral corresponds to the case where $f(x) = b\cos x$, $a = \pi/2$, and $b = 1$, so

$$\int_0^{\pi/2} \ln(\frac{1 + \cos x + \sin x}{1 + \cos x})dx = \frac{\pi \ln 2}{4}.$$

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Diana Beck**, University of Wisconsin - Superior, **Paul S. Bruckman**, Berkeley, CA, **Kenneth B. Davenport**, Frackville, PA, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Ovidiu Furdui**, Western Michigan University, Kalamazoo, MI, **Robert C. Gebhardt**, Hopatcong, NJ, **Joe Howard, Portales**, New Mexico, **Yoshinobu Murayoshi**, Okinawa, Japan, **N. R. Nandakumar**, Delaware State University, Dover, **Shiva K. Saksena**, University of North Carolina at Wilmington, **H.-J. Seiffert**, Berlin, Germany, **Trey Smith**, Angelo State University, San Angelo, TX, **J. Ernest Wilkins**, Jr., Clark Atlanta University, GA, **Yongzhi Yang**, University of St. Thomas, St. Paul, MN, and the **Proposer**.

984. [Spring 2000] **Proposed by Peter A. Lindstrom, Batavia, New York.**
Test for convergence the infinite series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!e^n}.$$

Solution by **H.-J. Seiffert**, *Berlin, Germany.*

This series is divergent. More generally, we show that if a is a fixed real number, then the infinite series

$$\sum_{n=1}^{\infty} \frac{n^{n-a}}{n!e^n}. \tag{1}$$

converges if $a > 1/2$ and diverges if $a \le 1/2$.

From Stirling's formula we have

$$\sqrt{2\pi n}n^n e^{-n} < n! < \sqrt{2\pi n}n^n e^{-n} e^{1/12n}, n \in N.$$

Since $e^{-x} > 1 - x$ for all real x, it follows that

$$\frac{1}{\sqrt{2\pi}n^{a+1/2}}\left(1 - \frac{1}{12n}\right) < \frac{n^{n-a}}{n!e^n} < \frac{1}{\sqrt{2\pi}n^{a+1/2}}, n \in N. \tag{2}$$

Since, as is well known,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$ and diverges for $p \le 1$, inequality (2) establishes the desired result for series (1) and hence the divergence of the given series.

Also solved by **Paul S. Bruckman**, Berkeley, CA, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Mark Evans**, Louisville, KY, **Ovidiu Furdui**, Western Michigan University, Kalamazoo, MI, **Robert C. Gebhardt**, Hopatcong, NJ, **GVSU Math/Stat Problem Solving Group**, Grand Valley State University, Allendale, MI, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard, Portales**, New Mexico, **Murray S. Klamkin**, University of Alberta, Canada, **Yoshinobu Murayoshi**, Okinawa, Japan, **Shiva K. Saksena**, University of North Carolina at Wilmington, **Dennis P. Walsh**, Middle Tennessee State University, Murfreesboro, **J. Ernest Wilkins**, Jr., Clark Atlanta University, GA, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

Two incorrect solutions were received.

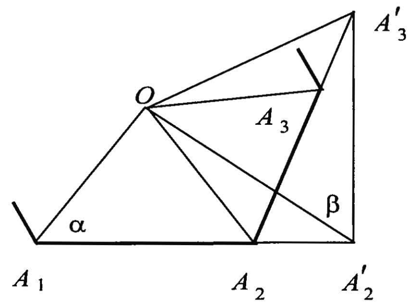
Howard pointed out that this proposal appears as problem 3, p. 432 of G. Klambauer, "Aspects of Calculus," Springer-Verlag, 1986.

985. [Spring 2000] *Proposed by Ayoub B. Ayoub, Penn State Abington College, Abington, Pennsylvania.*

Extend the sides A_1A_2 and A_2A_3 of a regular n-gon $A_1A_2A_3 \cdots A_n$ to A_2' and A_3' respectively such that $A_2A_2' = A_3A_3'$ and $m\angle A_2A_2'A_3' = 90°$. Show that $m\angle A_1OA_2' = 90°$, where O is the center of the n-gon.

Solution by **Yoshinobu Murayoshi**, *Okinawa, Japan.*

Draw lines from O to A_2, A_2', and A_3', and let $\alpha = \angle OA_1A_2'$ and $\beta = \angle OA_2'A_3'$, as shown in the figure.



We see that $\triangle OA_1A_2' \cong \triangle OA_2A_3'$ and $OA_2' \cong OA_3'$ by construction. (A rotation about O carries one triangle into the other.) Then $\triangle OA_2'A_3'$ is isosceles and $\angle OA_3'A_2' = \beta$. Since $\angle OA_2A_1 = \angle OA_2A_3 = \alpha$, then $\angle A_2'A_2A_3' = 180° - 2\alpha$. Now

$$\angle OA_2'A_1 = 90° - \beta = \angle OA_3'A_2 = \angle OA_3'A_2' - \angle A_2'A_3'A_2 = \beta - [90° - (180° - 2\alpha)],$$

which reduces to $\alpha = \beta$. Then $\angle OA_2'A_1 = 90° - \beta = 90° - \alpha$, so $A_2'OA_1 = 90°$.

Also solved by **Paul S. Bruckman**, Berkeley, CA, **Mark Evans**, Louisville, KY, **Richard I. Hess**, Rancho Palos Verdes, CA, **H.-J. Seiffert**, Berlin, Germany, **Rex H. Wu** (two solutions), Brooklyn, NY, and the **Proposer**.

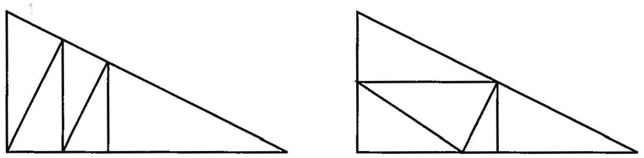
Bruckman and Seiffert each pointed out that this proof holds only for $n \ge 5$. For $n = 4$ Seiffert observed that $A_2' = A_2$ and $A_3' = A_3$ and O is the intersection of the diagonals of the square, so the theorem is true in that case, too. It makes no sense to try $n < 4$.

986. [Spring 2000] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find a triangle in the plane that can be dissected into five triangles all similar to itself.

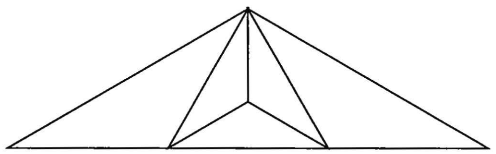
I. Composite of solutions by **Paul S. Bruckman**, *Berkeley, California,* **Ovidiu Furdui**, *Western Michigan University, Kalamazoo, MI,* **S. Gendler** *and* **J. Gendler**, *Clarion University of Pennsylvania,* **Richard I. Hess**, *Rancho Palos Verdes, California,* **Yoshinobu Murayoshi**, *Okinawa, Japan,* **Robert A. Stump**, *Richmond, VA, and* **Rex H. Wu**, *Brooklyn, New York.*

Any right triangle can be so dissected in many ways. Two examples appear in the figure.



II. Composite of solutions by **Monte J. Zerger**, *Adams State College, Alamosa, Colorado, and the* **Proposer**.

The figure below shows a $30° - 30° - 120°$ triangle so dissected.



987. [Spring 2000] *Proposed by Kenneth P. Davenport, Frackville, Pennsylvania.*

For a given positive integer n find for what positive integers $b > n$ and a there is a solution to the Diophantine equation

$$1 + 2 + +n = b + (b+1) + +(b+a).$$

Solution by **Rex H. Wu**, *Brooklyn, New York.*

We have $n(n+1)/2 = [b + (b+a)](a+1)/2$, so $n(n+1) = (a+1)(2b+a)$. From the last expression, it is obvious that $(a+1) \mid n(n+1)$ and $2b = n(n+1)/(a+1) - a$. Since b >n, we must have

$$\frac{n(n+1)}{a+1} - a > 2n$$

whence

$$a < \frac{\sqrt{8n^2+1} - (2n+1)}{2}.$$

Lastly, since $2b = n(n+1)/(a+1) - a$, we must have $2 \mid n(n+1)/(a+1) - a$. Therefore

$$b = \frac{n(n+1)}{2(a+1)} - \frac{a}{2}.$$

In conclusion, given a positive integer n, we can characterize the positive integers a and b with $b > n$ as $(a+1) \mid n(n+1)$, $a < [\sqrt{8n^2+1} - (2n+1)]/2$, and $2 \mid n(n+1)/(a+1) - a$.

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman**, Berkeley, CA, **Mark Evans**, Louisville, KY, **Robert C. Gebhardt**, Hopatcong, NJ, and **Richard I. Hess**, Rancho Palos Verdes, CA.

988. [Spring 2000] *Proposed by Kenneth P. Davenport, Frackville, Pennsylvania.*

For what values of n is this sum the square of an integer:

$$1^3 - 2^3 + 3^3 + \cdots + (-1)^{n+1}n^3.$$

Solution by **GVSU Math/Stat Problem Solving Group**, *Grand Valley State University, Allendale, Michigan.*

We show that $n = 2m(m-1)+1$ for any natural number m. Let S_n denote the stated sum. Since $S_n < 0$ for n even, it cannot then be the square of an integer. Suppose then that n is odd. Then

$$\begin{aligned} S_n &= 1^3 + 3^3 + \cdots + n^3 - [2^3 + 4^3 + \cdots + (n-1)^3] \\ &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 - 2[2^3 + 4^3 + \cdots + (n-1)^3] \\ &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 - 16[1^3 + 2^3 + \cdots + ((n-1)/2)^3] \\ &= \left(\frac{n(n+1)}{2}\right)^2 - 4\left(\frac{n-1}{2}\cdot\frac{n+1}{2}\right)^2 \\ &= \left(\frac{n+1}{2}\right)^2 (2n-1). \end{aligned}$$

We see that $2n-1$ must be a perfect square, say k^2, so $n = (k^2+1)/2$, where k is odd, say $k = 2m-1$. Then, for any positive integer m we have

$$n = \frac{k^2+1}{2} = \frac{(2m-1)^2+1}{2} = \frac{4m^2-4m+2}{2} = 2m(m-1)+1.$$

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman**, Berkeley, CA, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Mark Evans**, **Louisville**, KY, **Robert C. Gebhardt**, Hopatcong, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, **Portales**, New Mexico, **Murray S. Klamkin**, University of Alberta, Canada, **Peter A. Lindstrom**, Batavia, NY, **Yoshinobu Murayoshi**, Okinawa, Japan, **William H. Peirce**, Rangeley, ME, **Shiva K. Saksena**, University of North Carolina at Wilmington, **Harry Sedinger**, St. Bonaventure University, NY, **H.-J. Seiffert**, Berlin, Germany, **Skidmore College Problem Group**, Saratoga Springs, NY, **Robert A. Stump**, Richmond, VA, **J. Ernest Wilkins**, Jr., Clark Atlanta University, GA, **Rex H. Wu**, Brooklyn, NY, **Monte J. Zerger**, Adams State College, Alamosa, CO, and the **Proposer**.

989. [Spring 2000] *Proposed by Joel Brenner, Palo Alto, California.*

a) In the set of all primes find the density of the primes p such that the greatest common divisor of all the divisors of $p-1$ is 1. Note that a statistical experiment would lead to a wrong answer since three of the first six primes have this property.

b) In the set of all positive integers find the density of those integers $n > 1$ such that the greatest common divisor of all the divisors of $n-1$ is 1.

I. Solution to Part (a) by **Rex H. Wu**, *Brooklyn, New York.*

a) Only Fermat primes, primes of the form $p = 2^q + 1$, have the property that the greatest common divisor (gcd) of all divisors of $p - 1 \neq 1$. Thus the number of Fermat primes less than or equal to $2^m + 1$ is certainly less than or equal to m. Since the number of primes less than or equal to x, $\pi(x)$, is given by $\pi(x) \approx x/\ln x$, then the proportion of primes p with gcd of $p-1$ to all primes p, for $p \le 2^m + 1$ is given by

$$\delta(m) \le \frac{m}{\frac{2^m+1}{\ln(2^m+1)}} = \frac{m\ln(2^n+1)}{2^m+1} \approx \frac{m^2\ln 2}{2^m+1} \to 0$$

as $m \to \infty$. Therefore, the density of all primes p such that the gcd of all divisors of $p-1$ is not 1 is $1 - \lim \delta(m) = 1 - 0 = 1$.

II. Solution to Part (b) by **Paul S. Bruckman**, *Berkeley, California.*

Note that the problem is incorrectly stated. The words "all the divisors" should read "all the proper divisors" since 1 is a divisor of every number.

b) For any given prime q and any natural number n, let $f(q,n)$ denote the number of natural numbers less than n that are powers of q. Then $f(q,n) \le \log_q n = (\ln n)/\ln q$. Then the number of numbers $k < n$ for which the gcd of all the proper divisors of k is 1 is given by $\sum f(q,n)$, where the summation is taken over all primes $q \le n$. The density of these numbers k in the natural numbers, where q is a prime, is thus

$$\delta = \lim_{n\to\infty} \sum_{q\le n} \frac{f(q,n)}{n} = \lim_{n\to\infty} \sum_{q\le n} \frac{\ln n}{n \ln q} \le \lim_{n\to\infty} \frac{\ln n}{n} \sum_{q \le n} \frac{1}{\ln q}$$
$$\le \lim_{n\to\infty} \frac{\ln n}{n} \cdot (2\ln n) = 2 \lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$$

by two applications of L'Hopital's rule. The density we want is the complement of this density, thus it is 1; that is, almost all positive integers have 1 for the gcd of all their proper divisors.

Also solved by **Paul S. Bruckman** (Part (a)), and the **Proposer.**

990. [Spring 2000] *Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Identify all triangles ABC such that $\cos^2 A + \cos^2 B + \cos^2 C = 1$.

Solution by **Charles R. Diminnie**, *Angelo State University, San Angelo, Texas.*

Since $C = 180° - A - B$, then

$$\begin{aligned}
&\cos^2 A + \cos^2 B + \cos^2 C = \cos^2 A + \cos^2 B + \cos^2(A+B)\\
&= \cos^2 A + \cos^2 B + \cos^2 A \cos^2 B - 2\cos A \cos B \sin A \sin B + \sin^2 A \sin^2 B\\
&= \cos^2 A + \cos^2 B + \cos^2 A \cos^2 B - 2\cos A \cos B \sin A \sin B + (1-\cos^2 A)(1-\cos^2 B)\\
&= 1 + 2[\cos^2 A \cos^2 B - \cos A \cos B \sin A \sin B]\\
&= 1 + 2\cos A \cos B \cos(A+B) = 1 - 2\cos A \cos B \cos C.
\end{aligned}$$

It follows that the left side is one if and only if $\cos A \cos B \cos C = 0$, that is, if and only if one of the angles of the triangle is a right angle. Thus the given equation is true if and only if the triangle is a right triangle.

Also solved by **Miguel Amengual Covas**, Cala Figuera, Mallorca, Spain, **Scott H. Brown**, Auburn University at Montgomery, AL, **Paul S. Bruckman**, Berkeley, CA, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Mark Evans**, Louisville, KY, **Ovidiu Furdui**, Western Michigan University, Kalamazoo, MI, **Robert C. Gebhardt**, Hopatcong, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard, Portales**, New Mexico, **Murray S. Klamkin**, University of Alberta, Canada, **Yoshinobu Murayoshi**, Okinawa, Japan, **William H. Peirce**, Rangeley, ME, **Shiva K. Saksena**, University of North Carolina at Wilmington, **H.-J. Seiffert**, Berlin, Germany, **Trey Smith**, Angelo State University, San Angelo, TX, **Rex H. Wu**, Brooklyn, NY, and the **Proposer.**

991. [Spring 2000] *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

Eight people play rounds of golf in two foursomes at a time. Thus, for example, one round might have the foursomes $ABCD$ and $EFGH$. They desire to have each pair of players playing together in a foursome exactly the same number of times.

a) Is this possible in six rounds?

b) Is it possible in seven rounds?

c) Explain why your answers to the above questions differ.

I. Solution by **Mark Evans**, *Louisville, Kentucky.*

The following array of seven rounds puts the golfers in foursomes so that each golfer is paired with each other golfer exactly three times. Since there are six possible pairs in any foursome, each round generates twelve pairings. Since each player must be paired the same number of times with each of the other seven players, the total number of pairings must be a multiple of 7. Since 12 is not a multiple of 7, then the number of rounds must be a multiple of 7. Hence it is impossible with six or fewer rounds. Our example demonstrates that seven rounds suffice.

Round	Foursome 1	Foursome 2
1	ABCD	EFGH
2	ABEF	CDGH
3	ABGH	CDEF
4	ACEG	BDFH
5	ACFH	BDEG
6	ADEH	BCFG
7	ADFG	BCEH

II. Comment by **Richard I. Hess**, *Rancho Palos Verdes, California.*

Mark the seven points B, C, D, E, F, G, H in order and equally spaced on the circumference of a circle and draw a triangle connecting the points B, C, and E. Leaving the vertex labels fixed on the circle, rotate the triangle about the center of the circle and read the vertices at each position to obtain the foursomes that contain A for the seven rounds thus: $ABCE$, $ACDF$, $ADEG$, $AEFH$, $ABFG$, $ACGH$, and $ABDH$.

Also solved by **Paul S. Bruckman**, Berkeley, CA, **Charles R. Diminnie** and **Trey Smith**, Angelo State University, San Angelo, TX, **Richard I. Hess**, and the **Proposer.**

992. [Spring 2000] *Proposed by Mark Evans, Louisville, Kentucky.*

Consider three statistical distributions f, g, and h such that, for $0 < k < 1$,

$$h = kf + (1-k)g.$$

a) Express the variance of h as a function of k, the variances of f and g, and the means of f and g.

b) Use the expression derived in (a) to show that the variance of h equals the variance of f when $f = g$.

*c) Explain the results of (a).

Solution by **Paul S. Bruckman**, *Berkeley, California.*

Let $(\sigma_h)^2$, $(\sigma_f)^2$, and $(\sigma_g)^2$ denote the variances of h, f, and g, respectively. Also let $\rho = \rho_{fg}$ denote the correlation coefficient of f and g. If m_{fg}, m_f, and m_g represent the means of fg, f, and g, respectively, then we have

$$\rho = \frac{m_{fg} - m_f m_g}{\sigma_f \sigma_g}.$$

It then follows by expansion and definition that

$$(1) \qquad (\sigma_h)^2 = k^2(\sigma_f)^2 + 2k(1-k)\rho(\sigma_f\sigma_g) + (1-k)^2(\sigma_g)^2.$$

If $f = g$, we note that $\rho = 1$ (f and g are perfectly correlated); in this case, since $\sigma_f = \sigma_g$, we obtain

$$(\sigma_h)^2 = k^2(\sigma_f)^2 + 2k(1-k)(\sigma_f)^2 + (1-k)^2(\sigma_f)^2 = (k+1-k)^2(\sigma_f)^2,$$

or

$$(\sigma_h)^2 = (\sigma_f)^2.$$

The formula in (1) is a natural consequence of applying the definitions of variance and of the correlation coefficient, and hardly requires explanation.

Also solved by **Richard I. Hess**, Rancho Palos Verdes, CA, and the **Proposer**.

993. [Spring 2000] *Proposed by Les Wood, Forest City, Maine.*

Determine which stacks in less space, logs or split wood. Assume the logs are uniformly perfect cylinders of radius r and constant length. Assume these logs are split with no waste into perfect quarters, that is, their cross sections are circular sectors of central angle 90°.

Solution by **Rex H. Wu**, *Brooklyn, New York.*

I assume we are stacking the logs and the split logs in an infinite 3-D space. Since logs are cylindrical, we can just consider the cross sections, namely, a circle and a quarter circle. And we will pack the 2-D space with circles and quarter circles to see which takes less space.

The most compact way of packing 2-D space with circles is to have every circle touching six other circles, shown in Figure 993a. Figure 993b is an enlargement of

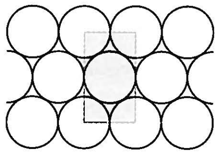


FIG. 993A.

the shaded area in Figure 993a and contains the area of two whole circles of radius r. A little calculation shows the width to be $2r$ and the height to be $2r\sqrt{3}$. Therefore

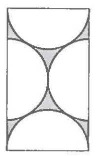


FIG. 993B.

the average space occupied by a circle packed this way is $(2r)(2r\sqrt{3})/2 = 2r^2\sqrt{3} \approx 3.46410r^2$.

A compact way of packing the 2-D space with quarter circles is shown in Figure 993c. Figure 993d shows a basic packing unit which contains the area of half

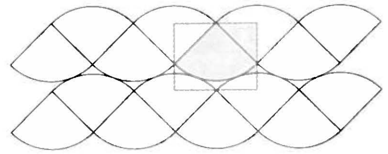


FIG. 993C.

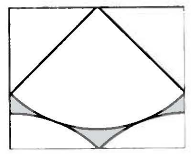


FIG. 993D.

a circle of radius r. Its width is $r\sqrt{2}$ and its height is $r[\sqrt{7/2} - \sqrt{1/2}]$. The average space occupied by a circle in this packing with quarter circles is $r^2(\sqrt{2})[\sqrt{7/2} - \sqrt{1/2}]/(1/2) = 2(\sqrt{7}-1)r^2 \approx 3.29150r^2$.

Therefore, splitting the logs into quarters will save us nearly 5% of the space required by full logs.

Also solved by **Richard I. Hess**, Rancho Palos Verdes, CA, and the **Proposer**.

Editorial comment: When Les was interviewed in his wood lot, he commented that the packing of Figure 993c becomes increasingly expeditious as the number of congruent sectors into which each log is cut escalates, and there is no desolate space whatsoever in the limit as that number becomes unfathomable. In fact, this basic maneuver of dissecting a circle of radius r into n sectors and accumulating the sectors on top of one another in an alternating fabrication and then suffering n to appreciate without demarcation shows commencing geometry students that the area of a circle is equivalent to that of a rectangle whose dimensions are r by πr, which is half the circumference of the circle. He added, "If 'tweren't for that pedagogy, I wouldn't trouble m'self to split 'em 'tall."

MATH ACROSTIC

The MATHACROSTIC in this issue has been contributed by Dan Hurwitz.

Clue	Numbers
a. In the large	126 114 171 060 012 147
b. A local coordinate system	081 178 030 151 053
c. Bases for systems of logarithms	064 046 155 124 094 017 086
d. Between right and straight	175 076 116 167 008 039
e. See m	015 121 004 043
f. Cantor symbol	057 168 108 018 038
g. Member of Brouwer's school	005 084 122 042 170 106 027 156 162 140 070 097
h. Parallel lines do this	104 154 129 011 164 160 073 181 115 179 048
i. It is as smooth as its inverse	148 161 029 127 158 113 176 049 055 001 021 075 089 135
j. Congruent maps are this	031 010 119 163 063 172 136 072
k. Map onto	182 083 102 092 128 139 020
l. Number of binary trees	040 101 177 059 077 150 035
m. Followed by E., a way to determine convergence of a series	051 024 079 091 022 143
n. Partitions should be this	056 165 133 069 146 118 095 007 025 078
o. Error whose size is denoted by β	037 003 045 117 100 144 087
p. He made an early attempt to calculate the earth's circumference	152 174 131 071 110 023 009 041 093 125 062 169
q. Next to the big toe on the first foot in Greenlandic	109 050 033 067 013 134
r. A Prime Number Theorist	047 159 149 065 130 180 019 058
s. A circuit zero	026 173 088
t. Distribution with expected value $(b-a)/n$	090 142 052 057 082 028 120
u. His questions in a letter have been examined for centuries	074 105 061 036 111 141 080 002
v. Mathematicians do it	016 107 166 145 085 123 068 096 138 044 103
w. Geometry teachers' frequent symbol	132 098 014 066 034
x. Statistical diagram showing distribution of paired random variables	112 006 099 137 153 054 037

001i	002u	003o	004e	005g	006x	007n	008d	009p	010j		011h	012a	013q	014w
	015e	016v	017c		018f	019r		020k	021i	022m		023p	024m	025n
026s	027g	028t		029i	030b	031j	032x	033q		034w	035l	036u		037o
038f	039d		040l	041p	042g	043e	044v	045o	046c	047r		048h	049i	
050q	051m	052t	053e	054x		055i	056n	057f	058l	059r	060a	061u	062p	
063j	064c		065r	066w		067q	068v	069n	070g	071p		072j	073h	074u
075i	076d	077l	078n		079n	080u	081b	082t	083k	084g	085v	086c		087o
088s		089i	090t	091m	092k	093p	094c	095n	096v		097g	098w	099x	100o
	101l	102k	103v		104h	105u	106g		107v	108f	109q		110p	111u
112x	113i	114a	115h	116d	117o		118n	119j	120t	121e	122g	123v	124c	125p
126a		127i	128k	129h		130r	131p	132w	133n	134q	135i	136j	137x	138v
139k	140g	141u	142t	143m		144o	145v	146n	147a	148i		149r	150l	151b
152p		153x	154h		155c	156g		157t	158i	159r	160h	161i	162g	163j
	164h	165n	166v	167d	168f	169p	170g	171a	172j		173s	174p	175d	176i
	177l	178b	179h		180r	181h	182k							

The solution to the MATHACROSTIC in last issue was taken from "Algebraic Topology", a classic text by W. S. Massey:

By using the fundamental group, topological problems about spaces and continuous maps can sometimes be reduced to purely algebraic problems about groups and homomorphisms. This is the basic strategy of algebraic topology.

Charles R. Diminni was the first solver, immediately followed by Jeanette Bickley and Paul S. Bruckman.

Contents