Is there a point to bi-tri-angling?

Problem 1030

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GENERAL FLIP-SHIFT GAMES

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Abstract. We examine two puzzles by thinking of the movements as permutations. We then generalize the puzzle as a “flip-shift” game and determine generalizations which yield all permutations of the pieces, as well as some that do not yield all possible permutations.

1. Introduction. Mathematics has often been at the heart of puzzles, including the sliding piece puzzle known as the 14-15 puzzle (Fig. 1). This puzzle has been sold with the 14 and 15 in their correct positions. However, the variation in Fig. 1 was made popular by Sam Loyd, who in the late 1800’s offered a cash prize to the first person who could place the square tiles in order. Loyd never had to pay out the cash prize because the pieces of the puzzle in Fig. 1 cannot be put in order by legal moves (sliding a piece into the empty space). (For a history of sliding piece puzzles, see Hordern [1].)

Mathematicians have analyzed this puzzle (including Johnson [2] in 1879) and its variations, e.g., Liebeck [3]. Other puzzles have generated mathematical interest as well, including Rubik’s cube. A mathematical generalization of Rubik’s cube, called Rubik’s tesseract, is analyzed in Velleman [4].

We begin by examining two puzzles, Xex No. Crunch and the Saturn Puzzler. These puzzles are similar to the 14-15 puzzle in that numbered pieces can be moved and arranged in different orders. However, unlike the 14-15 puzzle, there is no empty space limiting the next move. For this reason, Xex No. Crunch and the Saturn Puzzler are similar to Rubik’s cube. There are two possible moves, shifts and flips. We show that all permutations of the pieces are possible in the Saturn Puzzler and Xex No. Crunch puzzles. For certain generalized flip-shift games, the solution technique used for the two specific puzzles can be extended to generate all permutations. However, we prove that not all permutations of the puzzle pieces are possible for all generalized flip-shift games.

2. Two Flip-Shift Puzzles. Xex No. Crunch consists of 20 movable disks arranged along an oval track. The pieces can be rotated or shifted in the left or right direction (Fig. 2). A “shift” changes the location of all the pieces, but preserves their

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The packaging that comes with Xex No. Crunch includes techniques on how to move disks in certain ways to help yield solutions to its variations.
order; specifically, a left shift moves all disks one disk in the counterclockwise direction, while a right shift moves all disks one disk clockwise. The oval track intersects a turnstile that can be used to "flip" the ordering of four of the disks (Fig. 3).

One variation of Xex No. Crunch is to place the 20 disks in numerical order, as pictured in Fig. 2. We do not designate the orientation of the turnstile and consider a question that Sam Loyd would find of interest: Are all 20! permutations of the disks possible? If the answer is "yes," then any initial arrangement of the 20 disks can be transformed through a sequence of flips and shifts to the numerical ordering. Of course, this means that Sam Loyd cannot pre-order the disks into an arrangement that cannot be solved! We show that all 20! permutations are possible.

DEFINITION 1. For Xex No. Crunch, the flip permutation is

\[ F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 19 & 20 \\ 4 & 3 & 2 & 1 & 5 & 6 & \cdots & 19 & 20 \end{pmatrix} = (14)(23) \]

DEFINITION 2. For Xex No. Crunch, the left-shift permutation is

\[ L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 19 & 20 \\ 20 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 18 & 19 \end{pmatrix} = (1201918 \ldots 32) \]

and the right-shift permutation is

\[ R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 19 & 20 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 20 & 1 \end{pmatrix} = (1234 \ldots 1920). \]

Realize that \( L^{19} = R \) and \( R^{19} = L \) and that \( R^{20} = L^{20} = F^2 = I \), where \( I \) is the identity permutation. Before proving that all 20! permutations of Xex No. Crunch’s puzzle pieces are possible, we define a permutation \( S \), the swap permutation, and relate its existence to generating all permutations. The following definition and lemma are valid for all \( n \) and are needed later for values other than \( n = 20 \). However, we do not consider the general definitions of the left and right shift permutations until the next section.

DEFINITION 3. The swap permutation permutes the disks in positions 1 and 2:

\[ S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n-1 & n \\ 2 & 1 & 3 & 4 & 5 & 6 & \cdots & n-1 & n \end{pmatrix} = (12). \]

LEMMA 4. If we can write \( S \) as a sequence of flips and shifts, then all \( n! \) permutations are possible.

Proof. All \( n! \) permutations of the disks in the first \( m \) positions are possible if every permutation of the first \( m \) disks, leaving positions \( m+1 \) through 20 fixed, can be written as a sequence of shifts and flips. We proceed by induction. Realize that \( S \) permutes the first two positions; hence, all \( 2! \) permutations of the disks in the first 2 positions are possible.

Assume that all \( m! \) permutations of the disks in the first \( m \) positions are possible. Then, we show that all \( (m+1)! \) permutations of the disks in the first \( m+1 \) positions are possible. Begin by using a left shift to move the disks in positions 2 through \( m+1 \) into positions 1 through \( m \). By assumption, the disks in positions 1 through \( m \) can be arranged in any order by a sequence of shifts and flips, leaving the other disks fixed in their positions. End by using a right shift to move the disks back into positions 2 through \( m+1 \). There are \( m! \) such permutations.

The disk in position 1 by successive use of the swap permutation and the left shift can be placed between any two of the disks between 2 and \( m+1 \). Indeed, \((SL)^k R^k\) moves the disks in positions 2 through \( k+1 \) into positions 1 through \( k \), respectively, moves the disk in position 1 into position \( k+1 \), and leaves the other disks fixed. Thus, we achieve an additional \( m \cdot m! \) permutations of the first \( m+1 \) disks. And, we have accounted for all \( (m+1)! = m! + m \cdot m! \) permutations of the disks in the first \( m+1 \) positions. \( \Box \)
Theorem 5. All 20! permutations of the 90 pieces in Xex No. Crunch are possible. 

Proof. By the above lemma, all 20! permutations of the pieces of Xex No. Crunch are possible if the swap permutation can be written as a product of flip and shift permutations.

The product of permutations $FLFRFLFR$ or $(FLFR)^2$ keeps the disks in positions 2-20 in order, but places the disk in position 1 into the 5th position. Notice that 

$$FLFR = (1 4)(2 3)(1 20 19 \cdots 3 2)(1 4)(2 3)(1 2 3 \cdots 19 20) = (1 3 5 2 4)$$

and $(FLFR)^2 = (1 3 5 2 4)(1 3 5 2 4) = (1 5 4 3 2)$. Hence, the disk in position 1 has moved to position 5, while all of the other disks remain in their same order. Realize that the disks in positions 2 through 5 have moved into positions 1 through 4, respectively.

To use this operation repeatedly, we must first “put” the disk in position 5 back into position 1. This is achieved by repeated left shifts. And,

$$(FLFR)^2L^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 16 & 17 & 18 & 19 & 20 \\ 1 & 17 & 18 & 19 & 20 & 2 & 3 & \cdots & 12 & 13 & 14 & 15 & 16 \end{pmatrix}$$

Hence, successive uses of $(FLFR)^2L^4$ keeps all disks except the disk in position 1 in order and marches this disk until it lands behind the disk that was originally in position 2. This takes five iterations because $[(FLFR)^2L^4]^5 = (3 2 1 20 19 \cdots 6 5 4)$. The swap permutation is $(1 2) = [(FLFR)^2L^4]^5$. Since the swap permutation is written as a product of flips and shifts, then all 20! permutations of the pieces of Xex No. Crunch are possible.

Before we define generalized flip-shift games, there exists another puzzle that consists of permuting pieces by flips and shifts. The Saturn Puzzler is similar to Xex No. Crunch, except that it consists of only 8 “disks” or numbered pieces (Fig. 4).

The movements of the Saturn Puzzler are shifts and flips, as in Xex No. Crunch. Shifts consist of rotating the rings of Saturn around the planet in either the clockwise or counterclockwise directions. The flip is achieved by rotating half of the sphere through a plane that intersects the sphere through a great circle. Both of these movements are pictured in Fig. 4. The mathematical definitions appear below.

Definition 6. For the Saturn Puzzler, the flip permutation is

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 5 & 6 & 7 & 8 \end{pmatrix} = (1 4)(2 3).$$

Definition 7. For the Saturn Puzzler, the left-shift permutation is

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} = (1 8 7 6 \cdots 3 2)$$

and the right-shift permutation is

$$R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix} = (1 2 3 4 \cdots 7 8).$$

The same technique from Xex No. Crunch can be used to show that all 8! permutations of the pieces of the Saturn Puzzler are possible. As before, we want to show that the swap permutation can be written as a sequence of flips and shifts.

Theorem 8. All 8! permutations of the 8 pieces in the Saturn Puzzler are possible.

Proof. Since the flip size of the Saturn Puzzler and Xex No. Crunch are the same, $(FLFR)^2$ has the same effect. That is, $(FLFR)^2$ keeps the disks in positions 2 through 8 in order, but moves the disk in position 1 into the fifth position. Because there are 8 pieces, it does not take as many iterations of $(FLFR)^2L^4$ to yield the swap permutation. Indeed, $(FLFR)^2L^4R = (1 2)$. \[ \square \]

3. General Flip-Shift Puzzles. The puzzles in the previous section can be generalized to any number of pieces $n$ with flips of any size $k < n$. The shift and flip permutations for the general $(n, k)$-puzzle are defined below.

Definition 9. Let $L_n$ and $R_n$ be the left-shift and right-shift permutations on $n$ elements, respectively. Define $L_n$ and $R_n$ by

$$L_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = (1 \ n \ n-1 \cdots 3 2)$$

and

$$R_n = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & n & 1 \end{pmatrix} = (1 2 3 \cdots n-1).$$

Definition 10. Let $F_{n,k}$ be the flip permutation of size $k$ on $n$ elements, then

$$F_{n,k} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & k+2 & \cdots & n-1 & n \\ k & k-1 & \cdots & 2 & 1 & k+1 & k+2 & \cdots & n-1 & n \end{pmatrix} = (1 \ k)(2 \ k-1)(s-1 \ s+1)(s \ s+1).$$

Assume that the number of disks is always $n$ and that the flip size is always $k$. For this reason, we eliminate the subscript notation, e.g., let $R = R_n$ unless it is pertinent. The results that follow often put restrictions on the values of $n$ and $k$. 
3.1. When does the solution technique generalize?. For Xex No. Crunch and the Saturn Puzzler, we used a particular sequence of flips and shifts to yield the swap permutation. We determine below how and when this sequence of flips and shifts yields the swap permutation for general \((n, k)\)-puzzles. The following proposition extends the technique used to solve Xex No. Crunch and the Saturn Puzzler from the previous section. The idea is to be able to move one disk while keeping the other disks in order.

**Proposition 11.** For \(k = 2s\), the sequence of flips and shifts \((FLFR)^s\) yields the permutation \((1k k - 1 \cdots 3 2)\).

**Proof.** Using the general definitions of the shift and flip permutations, basic multiplication yields

\[
FLFR = \begin{pmatrix}
1 & 2 & 3 & \cdots & k - 2 & k - 1 & k & k + 1 & k + 2 & \cdots & n \\
3 & 4 & 5 & \cdots & k & k + 1 & 1 & 2 & 3 & 4 & k + 2 & \cdots & n
\end{pmatrix}.
\]

Notice that it becomes easy to repeat this operation. For example, \((FLFR)^2\) is

\[
\begin{pmatrix}
1 & 2 & \cdots & k - 4 & k - 3 & k - 2 & k - 1 & k & k + 1 & k + 2 & \cdots & n \\
5 & 6 & \cdots & k & k + 1 & 1 & 2 & 3 & 4 & k + 2 & \cdots & n
\end{pmatrix}.
\]

Since \(k = 2s\) and the operation \(FLFR\) only changes the positions of the disks in positions 1 through \(k + 1\), it follows that \((FLFR)^s\) is

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & k - 2 & k - 1 & k & k + 1 & k + 2 & \cdots & n - 1 & n \\
k + 1 & 1 & 2 & \cdots & k - 3 & k - 2 & k - 1 & k & k + 2 & \cdots & n - 1 & n
\end{pmatrix},
\]

or \((1k + 1 k - 1 \cdots 3 2)\).

By repeated application of sequence of shifts and flips from the above proposition, it is possible to move the disk in position 1 while keeping the other disks in order. This technique yields the swap permutation if successive iterations of \((FLFR)^s L^k\) moves the 1 disk behind the 2 disk. The following theorem indicates the necessary relationship between \(k\) and \(n\) for this to happen.

**Theorem 12.** The technique used to solve Xex No. Crunch and the Saturn Puzzler can be extended for a flip size of \(k = 2s\) and \(n\) disks when \(k\) and \(n - 1\) are relatively prime.

**Proof.** The permutation \((FLFR)^s\) moves the disk in position 1 into position \(k + 1\), keeps the disks in positions \(k + 2\) through \(n\) fixed, and moves the disk in position 1 to \(j - 1\) for \(j = 2\) to \(k + 1\). We can continue to move the disk that was initially in position 1 by returning it to position 1 using the left shift \(k\) times and then repeating \((FLFR)^s\). The permutation \((FLFR)^s L^k(FLFR)^s\) will move the disk originally into position 1 behind the disk that was originally in position \(2k + 1\), as long as \(n > 2k + 1\). It follows that \((FLFR)^s L^k(FLFR)\) moves the disk originally in position 1 behind the disk that was originally in position \(mk + 1\). Realize that following this permutation by \(L^k\) returns the disk that was originally in position 1 back to position 1. Of course, there is an \(m\) such that \(mk + 1 > n\). We are concerned with where the disk originally in position 1 lands relative to the other disks. As the disk originally in position 1 will follow one of the disks in positions 2 through \(n\), we compute \((mk + 1)\) modulo \(n - 1\) to yield which disk the disk originally in position 1 will follow.

To yield the swap permutation, we want to move the disk originally in position 1 behind the disk that was originally in position 2, if possible. Therefore, we want to find an \(m\) such that \(mk + 1\) is congruent to 2 modulo \(n - 1\). This means that after \(m\) iterations of \([(FLFR)^s L^k]m\) the disk that was originally position 1 now follows the disk that was originally in position 2, while keeping all of the other disks in their original order.

Finding \(m\) such that \(mk + 1 \equiv 2\) \((\text{mod} \ n - 1)\) is equivalent to finding an \(m\) such that \(mk \equiv 1\) \((\text{mod} \ n - 1)\). This is equivalent to finding an \(m\) such that there exists an integer \(a\) such that \(mk = a(n - 1) + 1\) or \(mk = a(n - 1) - 1\). By the Euclidean algorithm, \(mk - a(n - 1) = 1\) implies that the greatest common divisor of \(k\) and \(n - 1\) is 1 (e.g., see [6, p.11]). Equivalently, \([(FLFR)^s L^k]m\) can only yield the swap permutation when \(k\) and \(n - 1\) are relatively prime.

The Euclidean algorithm can be used to determine the minimum \(m\) such that \([(FLFR)^s L^k]m R = (1 2)\). These values appear in the following table; \(m\) represents the number of forward iterations of \([(FLFR)^s L^k]\) necessary to yield the swap permutation. For example, as discovered in the proof that all permutations of Xex No. Crunch's pieces are possible, the entry in the column \(4F \choose k = 4\) and the row \(20\) \((n = 20)\) of the table in Fig. 5 is 5. All entries in the "forward" columns of the table in Fig. 5 can be determined by the following algorithm, based on the Euclidean algorithm. When \(n - 1\) and \(k\) are relatively prime, construct the following sequence of remainders until \(r_{i + 1} = k - 1:\n
\[
\begin{align*}
(n - 1) & \mod k = r_1 \\
[(n - 1) + r_1] & \mod k = r_2 \\
[(n - 1) + r_2] & \mod k = r_3 \\
& \vdots \\
[(n - 1) + r_i] & \mod k = r_{i + 1}
\end{align*}
\]

We can compute \(m\) by \(m = \frac{-(i + 1)(n - 1)}{k}\) such that \([(FLFR)^s L^k]m R = (1 2)\). This follows since we can add the equalities above to yield:

\[
(i + 1)(n - 1) + r_1 + r_2 + \cdots + r_i \mod k = (r_1 + r_2 + \cdots + r_i + r_{i + 1}) \mod k.
\]

Canceling \(r_j\) for \(j = 1\) to \(i\) from both sides yields

\[
(i + 1)(n - 1) \mod k \equiv r_{i + 1} \mod k = k - 1.
\]

Therefore, there exists a \(b\) such that \((i + 1)(n - 1) = bk + k - 1\). But, a little rearranging yields \(1 = -(i + 1)(n - 1) + (b + 1)k\) which implies that \(k\) and \(n - 1\) are relatively prime. And, our \(m\) is \((b + 1)\) which can be found by \(m = -(b + 1) = \frac{-(i + 1)(n - 1) + 1}{k}\).

**Example 1.** For the puzzle where \(n = 22\) and \(k - 8\), the sequence of remainders is

\[
\begin{array}{c}
21 \\
26 \\
23
\end{array}
\]

This algorithm terminated after 3 iterations and \(m = \frac{-(i + 1)(n - 1)}{k} = \frac{321}{8} = 8\). So, \(8\) is the entry in the table for the \(8F\) column for \(n = 22\).

However, it is possible to work "backwards" by moving the disk in position 1 to the left instead of to the right, while keeping the other disks in order. By reversing...
the order of operations, \( R^n(LFRF)^y \) moves the disk in position 1 behind the disk in position \((-k+1)\) mod \((n-1)\). This follows because \([(FLFR)^k L^k][R^n(LFRF)^y] = I \), the identity permutation. The swap permutation can also be achieved by finding the minimum \(j\) such that \([R^n(LFRF)]^j\) places the disk originally in position 1 into the position behind the disk originally in position 2 while leaving the other disks in their original order. This \(j\) can be determined by solving \(-jk + 1 \equiv 2 \) mod \((n-1)\) for the minimum \(j\). This value can easily be determined since \(k\) and \(m\), as described, must add to \(n-1\). This follows from \(k\) and \(n-1\) being relatively prime. The table contains the number of iterations of \([R^n(LFRF)]^j\) necessary to achieve the swap permutation. For example, the entry in column 8B and row \(n=8\) is \(2\) and indicates that \([R^n(LFRF)]^2\) swaps the order of the disks in position 1 and 2. Comparing this to the entry of 8F and row \(n=8\) is \(15\), which is 15, indicates that it is more efficient to use sequences of \([R^n(LFRF)]^j\) than \([(FLFR)^k L^k]^z\) to yield the swap permutation.

### 3.2. When can we guarantee that not all permutations are possible?

Determining whether or not every one of the \(n!\) permutations of the \(n\) pieces are possible under a shift and a flip of size \(k\) often reduces to a question of parity. That is, whether or not the shift and flip permutations are odd or even becomes paramount. First, we review the definition of odd and even permutations. Recall that a transposition is a permutation that transposes two elements and leaves all other elements fixed. Indeed, the swap permutation from the previous section is an example of a transposition.

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**DEFINITION 13.** A permutation is odd if it can be written as the product of an odd number of transpositions. A permutation is even if it can be written as the product of an even number of transpositions.

Realize that a permutation cannot be both odd and even. Indeed, although an odd permutation can be written as the product of an odd number of transpositions, it is the case that every such product of transpositions must contain an odd number of transpositions, as indicated in the following theorem.

**THEOREM 14.** (Hillman and Alexanderson, [5, p. 92]) If a permutation \( \theta \) is a product of transpositions, \( \theta = \alpha_1 \alpha_2 \cdots \alpha_r = \beta_1 \beta_2 \cdots \beta_s \), then \( r \) and \( s \) are both even or both odd.

Next, we determine the values for which the general flip and shift permutations are odd and even.

**PROPOSITION 15.** The shift permutations on \( n \) elements are even if \( n \) is odd and are odd if \( n \) is even.

**Proof.** Both shift permutations on \( n \) elements can be written as a product of \( n-1 \) transpositions. Specifically, \( \theta = (12)(3 \cdots n-1) \) and \( \phi = (12)(3 \cdots n-1) \).

The proposition is proved because \( n-1 \) is even when \( n \) is odd and \( n-1 \) is odd when \( n \) is even.

**PROPOSITION 16.** The flip \( F_{n,k} \) is even if \( k \) is congruent to 0 or 1 mod 4. Otherwise, \( F_{n,k} \) is odd.

**Proof.** We consider each of the four possibilities of \( k \) mod 4 separately. For ease of presentation, represent

\[
F_{n,k} = \begin{pmatrix}
1 & 2 & \cdots & k-1 & k+1 & k+2 & \cdots & n-1 & n \\
-k & -1 & \cdots & 2 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

by

\[
F_{n,k} = \begin{pmatrix}
1 & 2 & \cdots & k-1 & k \\
-k & -1 & \cdots & 2 & 1
\end{pmatrix}
\]

since all of the elements from \( k+1 \) to \( n \) are fixed under \( F_{n,k} \).

Let \( k \) be equal to 4\( l \) for some nonnegative integer \( l \). Then, \( F_{n,4l} \) is

\[
F_{n,4l} = \begin{pmatrix}
1 & 2 & \cdots & 2l & 2l+1 & \cdots & 4l-1 & 4l \\
4l & 4l-1 & \cdots & 2l+2 & 2l+1 & \cdots & 2 & 1
\end{pmatrix}
\]

The flip \( F_{n,4l} \) can be written as the product of \( 2l \) transpositions; specifically,

\[
F_{n,4l} = (4l)(4l-1)(3l+3)(2l-1)\cdots (2l+1) \cdot (2l+2) \cdot (2l+1) \cdots \cdot 2 \cdot 1
\]

Hence, \( F_{n,4l} \) is an even permutation.

Let \( k = 4l+1 \) for some nonnegative integer \( l \). Then, \( F_{n,4l+1} \) is

\[
F_{n,4l+1} = \begin{pmatrix}
1 & 2 & \cdots & 2l & 2l+1 & 2l+2 & \cdots & 4l+1 & 4l+1 \\
4l+1 & 4l & \cdots & 2l+2 & 2l+1 & 2l & \cdots & 2 & 1
\end{pmatrix}
\]
The flip $F_{n,4l+1}$ is an even permutation because it can be written as the product of 2l transpositions:

$$F_{n,4l+1} = (1\ 4l+1)(2\ 4l)(3\ 4l-1)\cdots(2l\ 2l+2).$$

If $k = 4l + 2$ for some nonnegative integer $l$, then $F_{n,4l+2}$ is written as:

$$F_{n,4l+2} = (1\ 2\ 3\ \cdots\ 2l+1\ 2l+2\ \cdots\ 4l\ 4l+1\ 4l+2).$$

The flip $F_{n,4l+2}$ can be written as a product of $2l+1$ transpositions; indeed,

$$F_{n,4l+2} = (1\ 4l+2)(2\ 4l+1)(3\ 4l)\cdots(2l\ 2l+3)(2l+1\ 2l+2).$$

It follows that $F_{n,4l+2}$ is an odd permutation.

If $k = 4l + 3$ for some nonnegative integer $l$, then $F_{n,4l+3}$ is written as:

$$F_{n,4l+3} = (1\ 2\ \cdots\ 2l+1\ 2l+2\ 2l+3\ \cdots\ 4l\ 4l+2\ 4l+3).$$

Writing $F_{n,4l+3}$ as a product of transpositions yields:

$$F_{n,4l+3} = (1\ 4l+3)(2\ 4l+2)(3\ 4l+1)\cdots(2l\ 2l+4)(2l+1\ 2l+3).$$

And, $F_{n,4l+3}$ is an odd permutation because it can be written as the product of $2l+1$ transpositions.

\textbf{Theorem 17.} All $n!$ permutations are not possible for flip-shift puzzles with flip-size $k$ congruent to 0 or 1 modulo 4 and an odd number of pieces, $n$.

\textit{Proof.} For $k$ congruent to 0 or 1 modulo 4, the flip permutation is even. Similarly, for $n$ odd, the shift permutations are even. All products of shifts and flips are even permutations. Therefore, none of the odd permutations are possible.

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\textbf{REFERENCES}


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**Abstract.** We study the dynamics of the map $F(R(x)) = (R(\sqrt{2}) - R(-\sqrt{2})/(2\sqrt{2})$ on the space of rational functions, in the context of a new method of integration. We give a recursive formula for the iterates of a model family of rational functions, which is closed under the action of $F$. We give a class of rational functions that are mapped to zero by two iterations of $F$.

We prove that all polynomials are eventually mapped to even functions by $F$, and we determine the number of iterations required for a given polynomial. We use power series representation to determine which rational functions are eventually mapped to even functions by $F$.

1. Introduction. The integration of rational functions is one of the central tasks in calculus. The classical method of partial fractions reduces the problem to that of solving an algebraic equation. If $P(x)$ and $Q(x)$ are polynomials, the evaluation of

$$I = \int_0^\infty \frac{P(x)}{Q(x)} \, dx$$

requires factorization of the denominator

$$Q(x) = (x - x_1)^{n_1} (x - x_2)^{n_2} \cdots (x - x_k)^{n_k}$$

where $x_1, \ldots, x_k$ are the roots of $Q(x) = 0$, and the factorization is converted to a real form by combining any non-real roots in conjugate pairs.

The difficulty associated with this method is that, as Abel showed, it is impossible to solve the general equation of degree 5 or more by radicals. Exact formulas for the roots of a polynomial are not always available. Therefore an interesting question is to classify the rational functions $R$ for which the integral (1) can be evaluated without factoring the polynomial $Q$.

The integration of even rational functions seems to be an easier problem. Two examples are the classical Wallis formula [5]

$$\int_0^\infty \frac{dx}{x^2 + 1} = \pi \left(\frac{2m}{m+1}\right)$$

and the evaluation in [1] of

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \pi \left(\frac{2m+1}{2a+1}\right)^{1/2} P_m(a)$$

where

$$P_m(a) = 2^{-2m} \sum_{k=0}^m 2^k \binom{2m - 2k}{m-k} (m+k) (a+1)^k.$$
The special case
\[
N_0(a;0) = \frac{\pi}{2\sqrt{2(a+1)}},
\]
of (4) will be employed in Section 3. See [3] for many more examples.
Recall that the even and odd parts of a function are defined as, respectively,
\[
R_e(x) = \frac{R(x) + R(-x)}{2} \quad \text{and} \quad R_o(x) = \frac{R(x) - R(-x)}{2}.
\]
We can rewrite \( \int_0^\infty R(x) \, dx \) as
\[
\int_0^\infty R(x) \, dx = \int_0^\infty R_e(x) \, dx + \int_0^\infty R_o(x) \, dx.
\]
The first integral on the right has an even integrand, and so is likely to be easier
than the original. The change of variables \( t = x^2 \) in the second integral yields
the identity
\[
\int_0^\infty R(x) \, dx = \int_0^\infty R_e(x) \, dx + \frac{1}{2} \int_0^\infty F(R(x)) \, dx,
\]
where the map \( F \) is defined by
\[
F(R(x)) = \frac{R(\sqrt{t}) - R(-\sqrt{t})}{2\sqrt{t}}.
\]
If the \( n \)th iterate \( F^{(n)}(R(x)) \) is even for some \( n \), then the integral \( \int_0^\infty R(x) \, dx \) reduces
to an integral of even functions. In this paper we give necessary and sufficient
conditions on \( R(x) \) for this to occur.

The paper is organized as follows. In Section 2 we show that the map \( F \) preserves
rationality of the function \( R(x) \). Sections 3 and 4 contain examples. In Section 3 we
prove that all polynomials are eventually mapped to even functions by \( F \), and we
also discuss analogous

\[ R(x) = \frac{G_m(a)}{x^2 + H_m(a)x + 1}, \quad G_0(a) = 1, \quad H_0(a) = 2a, \]
and we give recursive formulas for \( G_m(a) \) and \( H_m(a) \). We also discuss analogous
results for the family of rational functions
\[ R(x) = \frac{1}{x^3 + ax^2 + bx + 1}, \quad a, b \in \mathbb{R}, \]
where we now include a substitution \( x \to -x \) in our mapping function \( F \).
In Section 4 we show that the rational functions of the form
\[ R(x) = \frac{P(x^4) + x^2Q(x^2)}{V(x^2)}, \]
where \( P, Q, \) and \( V \) are polynomials, are mapped to even functions by one iteration
of \( F \).

In Section 5 we establish a necessary and sufficient condition for a rational function
\( R(x) \) to be mapped to an even function by \( n \) iterations of \( F \). The condition is that
certain coefficients in the power series for \( R(x) \) about zero must vanish. In Section 6
we prove that all polynomials are eventually mapped to even functions by \( F \), and we
determine the number of iterations required.

2. \( F \) preserves rationality. In this section we prove that the map \( F \) preserves
the class of rational functions.

**Proposition 1.** If \( R(x) \) is a rational function, then \( F(R(x)) \) is also rational.

**Proof.** Write \( R(x) = P(x)/Q(x) \). A direct calculation shows that
\[
F(R(x)) = \frac{P(\sqrt{t})Q(\sqrt{t}) - P(-\sqrt{t})Q(-\sqrt{t})}{Q(\sqrt{t})Q(-\sqrt{t})2\sqrt{t}}.
\]
Now observe that \( Q(t)Q(-t) \) is an even polynomial in \( t = \sqrt{x} \), so it is a polynomial
in \( t^2 = x \). Similarly \( P(t)Q(-t) - P(-t)Q(t) \) is an odd polynomial in \( t \), so the
numerator in (1) is also a polynomial in \( x \), after cancellation with the \( \sqrt{x} \) in the
denominator.

3. Examples of the dynamics of \( F \). Consider the rational function
\[
R(a,x) = \frac{1}{x^2 + 2ax + 1}, \quad a \in \mathbb{R}.
\]
The even part of \( R(a,x) \) is
\[
R_e(a,x) = \frac{x^2}{x^4 + (2 - 4a^2)x^2 + 1}.
\]
Integrating the even part, we obtain
\[
\int_0^\infty R_e(a,x) \, dx = 2N_o(a)(1 - 2a^2,0) = \frac{\pi}{2\sqrt{1 - a^2}}.
\]
Turning to the odd part of \( R(a,x) \), we evaluate the rational function \( F(R(x)) \). Direct
calculation suggests that the iterates of \( R \) under \( F \) have the form
\[
F^{(n)}(R(a,x)) = \frac{G_m(a)}{x^2 + H_m(a)x + 1} = \frac{G_m(a) \times R(H_m(a)/2,x)}{x},
\]
where \( G_m(a) \) and \( H_m(a) \) are polynomials in \( a \). This is established in the next proposition.

**Proposition 2.** The functions \( H_m(a) \) and \( G_m(a) \) satisfy the recursion formulas

\[
H_{m+1}(a) = 2 - H_m(a)^2, \quad G_{m+1}(a) = -G_m(a)H_m(a),
\]

\[
G_{m+2}(a) = \frac{G_{m+1}(a)}{G_m(a)} - 2G_{m+1}(a),
\]
with initial conditions \( G_0(a) = 1 \) and \( H_0(a) = 2a \). In particular, \( H_m(a) \) and \( G_m(a) \)
are polynomials in \( a \).

**Proof.** The proof is by induction.
It allows similar results. There determine which rational functions are eventually mapped to even functions. Simply, from the recursion for $G_m(a)$ we can obtain similar, more complicated, recursive equations for the coefficients.

Substitute as desired. We assume that

$$F^{(m)}(R(x)) = \frac{G_m(a)}{x^2 + H_m(a)x + 1},$$

and evaluate $F^{(m+1)}(R(x))$ to obtain the indicated recursion. Define $R_m(x) = F^{(m)}(R(x))$ and compute the odd part

$$R_{m,odd}(x) = \frac{G_m(a)}{x^2 + H_m(a)x + 1} - \frac{G_m(a)}{x^2 - H_m(a)x + 1}.$$

Substitute $x \to \pm \sqrt{x}$ and combine the two fractions to produce

$$F^{(m+1)}(R(x)) = \frac{-2G_m(a)H_m(a)\sqrt{x}}{(x^2 + (2 - H_m(a))x + 1)/2\sqrt{x}}$$

$$= \frac{-G_m(a)H_m(a)}{(x^2 + (2 - H_m(a))x + 1)}.$$

It follows that $F^{(m+1)}(R(x))$ has the required form and that the functions $H_m(a)$ and $G_m(a)$ satisfy the recursion stated above.

Since $H_m(a) = -G_{m+1}(a)/G_m(a)$, the second recursion for $G_m(a)$ now follows from the recursion for $H_m(a).$ The fact that $G_{m+2}(a)$ is necessarily a polynomial can be proved by induction. ∎

We summarize our discussion in a theorem:

**Theorem 3. The family of functions**

$$R(a,x) := \frac{G_m(a)}{x^2 + H_m(a)x + 1}, \quad a \in \mathbb{R},$$

is closed under the action of $F$, and the following integral formula holds:

$$\int_0^\infty \frac{G_m(a)}{x^2 + H_m(a)x + 1} dx = \frac{\pi G_m(a)}{\sqrt{1 - H_m(a)}} + \frac{1}{2} \int_0^\infty \frac{G_m(a)}{x^2 + H_m(a)x + 1} dx,$$

where $H_m(a)$ and $G_m(a)$ are as defined in (6).

For rational functions of the form $R(a,x) = 1/(x^2 + 2ax + 1)$ we are able to determine which rational functions are eventually mapped to even functions. Simply, the solution $a$ to $H_m(a) = 0$ will give particular rational functions $R(a,x)$ that map to an even function after $m$ applications of $F$, because when $H_m(a) = 0$ the resulting rational function $F^{(m-1)}(R(a,x))$ is even.

Turning our attention to rational functions of the form $R(x) = 1/(x^2 + ax^2 + bx + c)$ we can obtain similar, more complicated, recursive equations for the coefficients. There $F$ must be slightly modified in order for iterations of $F$ to preserve the structure of $R(x)$. If we add a second substitution $x \to -x$ to $F$, then this modified map $G$ allows similar results.

So far we have noted the convergence of our integrals. It is shown in [4] that

$$\int_0^\infty \frac{1}{(x^2 + ax^2 + bx + c)} dx$$

converges if $a$ and $b$ satisfy the condition $4a^3 - 18ab + 4b^3 + 27 > 0$. It is an open question whether one can find conditions on $b$ for convergence of

$$\int_0^\infty \frac{1}{(x^2 + 2ax + b + c)} dx.$$
For both integrals we have used equations from [3],

\[
\int_0^\infty \frac{bx^n + dx^3 + cx^3 + cx^3}{e^x + 1} \, dx = \int_0^\infty R(x) \, dx + \frac{1}{2} \int_0^\infty F(R(x)) \, dx
\]

\[
2\pi (d + e) + \pi \left( 4 + \sqrt{8(2\alpha + 1)} \right)^{1/2} (b + c)
\]

\[
= \frac{2^{1/2} (1 + \alpha)^{1/2} (4 + \sqrt{8(1 + \alpha)})^{1/2}}{2(1 + \alpha)}.
\]

5. The power series condition for \(F(n)(R) = 0\). The power series expansion

\( R(x) = \sum_{j=m}^{\infty} a_j x^j \) about \( x = 0 \) of a rational function has only a finite number of non-zero terms with negative powers of \( x \) [6, Section 5.6]. The coefficients satisfy a periodicity condition: there exists an \( m \in \mathbb{N} \) such that \( a_j = a_{j+m} \) for all \( j \). This property of a rational function is key to finding a necessary and sufficient condition on \( R(x) \) to ensure that some \( F(n)(R(x)) \) is even.

**Theorem 5.** Consider the rational function

\[
R(x) = \sum_{j=m}^{\infty} a_j x^j.
\]

The \( n \)th iterate \( F(n)(R(x)) \) is even if and only if \( a_{2j+1} = 0 \) for all integers \( j \geq -m \). First we state a simple Lemma.

**Lemma 6.** The power series for the \( n \)th iterate of \( R(x) \) is given by

\[
F(n)(R(x)) = \frac{1}{2^n} \sum_{j=-m}^{\infty} a_{2j+2n-1} x^{2j}.
\]

**Proof.** Apply \( F \) to equation (1) and use induction. \( \Box \)

Now we can prove Theorem 5. We use induction on the number of iterations.

**Proof.** Base Case: \( n = 1 \). First assume \( F(1)(R(x)) \) is even, then

\[
F(1)(R(x)) = \frac{1}{2} \sum_{j=-m}^{\infty} a_{2j+2} x^{2j+1}.
\]

Now suppose \( a_{2j+3} = 0 \) for all \( j \in \mathbb{N} \). The function \( R(x) \) can be expressed in the form

\[
R(x) = \sum_{j=m}^{\infty} a_j x^j = \sum_{j=-m}^{\infty} a_{2j+3} x^{2j+3}.
\]

But we know that \( F(n)(R(x)) \) is even, which can only occur if the odd part of \( F(n)(R(x)) \) vanishes. Therefore \( a_{2j+3} = 0 \) for all \( j \in \mathbb{Z} \), as required.
Also,
\[ \mathcal{F}\left(\frac{1}{2^n} \sum_{j=-m}^{\infty} a_{2^n+j+1} x^{j+1}\right) = 0, \]
\[ \mathcal{F}\left(\frac{1}{2^n} \sum_{j=-m}^{\infty} a_{2^n+j+1} x^{j+1}\right) = 0, \]
because both sums in 10 are even and by assumption \(2^{n+j+1} \cdot \pi = 1\). Therefore
\[ \mathcal{F}^{(n+1)}(R(x)) = \mathcal{F}\left(\frac{1}{2^n} \sum_{j=-m}^{\infty} a_{2^n+j+1} x^{j+1}\right) = \mathcal{F}\left(\frac{1}{2^{n+1}} \sum_{j=-m}^{\infty} a_{2^n+j+1} x^{j+1}\right), \]
which is even, as required. \( \square \)

This result is useful because now we can take any rational function, express it as a power series and determine whether it will ever result in an even function. It also implies that \(R(\sqrt{2}) - R(-\sqrt{2}) = iR(-i\sqrt{2}) - iR(i\sqrt{2}) \) if only if \( \mathcal{F} \) maps \( R(x) \) to an even function. Now we have a closed form test for whether a rational function will be mapped to an even function under \( \mathcal{F} \) on the next iteration.

6. A special property of mapping polynomials. By definition, \( \mathcal{F} \) maps every even function to 0. The converse is also true: if \( \mathcal{F}(R(x)) = 0 \), then \( R(x) \) is even. An interesting open problem is to classify all functions \( R \) for which there exists an integer \( n \) such that \( \mathcal{F}^{(n)}(R(x)) = 0 \).

All polynomials are eventually mapped to 0 by repeated application of the map \( \mathcal{F} \). Further, the number of iterations required can be exactly determined from the exponents present in the polynomial.

Theorem 7. Let \( P(x) \) be a polynomial. Then there exists a non-negative integer \( n \) such that
\[ \mathcal{F}^{(n)}(P(x)) = 0. \]
Proof. Let \( P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \). Any monomial of even degree is mapped to zero after one iteration of \( \mathcal{F} \):
\[ \mathcal{F}(x^{2k}) = \frac{x^{2k} - x^{-2k}}{2\sqrt{x}} = 0. \]
So it suffices to look at powers of \( x \) of the form \( x^{2p} - 1 \), where \( p \) is a positive integer and \( p \) is odd, and to show that there is a positive integer \( n \) such that \( \mathcal{F}^{(n)}(x^{2p} - 1) = 0 \). Notice that
\[ \mathcal{F}^{(1)}(x^{2p} - 1) = x^{2p} - 1, \]
\[ \mathcal{F}^{(2)}(x^{2p} - 1) = x^{2p} - 1, \]
and so on. Iterating this procedure yields
\[ \mathcal{F}^{(m)}(x^{2p} - 1) = x^{2p} - 1, \]
Notice that \( x^{p-1} \) is an even power of \( x \). Therefore
\[ \mathcal{F}^{(m+1)}(x^{2p} - 1) = \mathcal{F}(x^{p-1}) = 0. \]
\( \square \)

Corollary 8. Let \( P(x) = a_0 x^{2p_1} + a_2 x^{2p_2} + \cdots + a_n x^{2p_n} - 1 \), where \( p_i \) is odd, and \( 1 \leq i \leq n \). Define
\[ k = \max(k_i), \quad 1 \leq i \leq n. \]
Then
\[ \mathcal{F}^{(k+1)}(P(x)) = 0, \]
and this is the first iterate that vanishes.
Proof. This is the situation of Theorem 7 with \( m = k \). \( \square \)

7. Open Questions. Finding other particular classes of functions such as (1) for the second, third and \( n^{th} \) iterations of \( \mathcal{F} \) would be very useful for recognizing which al rational functions \( R(x) \) eventually map to even functions. Currently, our general Theorem 5 can tell us this by looking at the power series expansion, but specific cases would also be interesting.

Unfortunately, many rational functions will never map to even functions. Describing the behavior of these rational functions under the map \( \mathcal{F} \) becomes complicated. In the case of the reciprocal of the quadratic, applying our mapping function results in a standard formula (5) for the iterates. An idea related to the integrability over \([0, \infty)\) is to use these recursion formulas to classify the behavior of the zeroes of a rational function under iteration of \( \mathcal{F} \); we have started to consider this.

We have used Mathematica and Maple to study the behavior of \( \mathcal{F} \) on rational functions, to find fixed points of \( \mathcal{F} \), to find periodic points of \( \mathcal{F} \), and to measure the length of their orbits. We have also found functions that are pre-periodic under \( \mathcal{F} \). The fact that a large class of functions have periodic behavior under \( \mathcal{F} \) allows us to map our functions back to themselves resulting in expressing these functions as a sum of other even functions.

The fixed points of \( \mathcal{F} \) have recently been found [2], and they are of the form
\[ R(x) = \frac{x^{m-1}}{x^m - 1}, \]
where \( m \in \mathbb{N} \) and \( m \) is odd. Unfortunately, these fixed points are not integrable on \([0, \infty)\).

Finally, we continue to study the dynamics of \( \mathcal{F} \) on the space of rational functions, following [2]. It is an open question whether the fixed points of \( \mathcal{F} \) are attracting or repelling, and how one might define the multiplier of \( \mathcal{F} \).

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HYPERPERFECT NUMBERS

JOHN C. M. NASH

Abstract. A new formula for hyperperfect numbers is demonstrated and new examples of hyperperfect numbers are given.

A number n is called perfect if it is equal to the sum of its divisors. For example, 6 = 1 + 2 + 3 is perfect. In [1], p. 49, this concept was generalized n to be k-perfect if n = 1 + n_{k-1} where the d_{k} are the proper divisors of n, 1 < d_{k} < n. The following theorem can be used to generate examples of hyperperfect numbers.

THEOREM 1. If n = p^{k} (p^{k+1} - (p - 1)) and p^{k+1} - (p - 1) is prime, p prime, then n is p - 1-perfect.

Proof:

1 + (p - 1)\Sigma d_{k} = 1 + (p - 1)[p + p^{2} + \ldots + p^{k} + (p^{k+1} - (p - 1))(1 + p + \ldots + p^{k-1})]

= 1 + (p - 1) \left[ \frac{p^{k+1} - p}{p - 1} + \frac{(p^{k+1} - (p - 1))p^{k}}{p - 1} \right]

= 1 + p^{k+1} - p + p^{k+1} + 1 - p^{k+1} - (p - 1) - p^{k+1}

= p^{k+1} - (p - 1). \Box

Examples: 5^{3}(5^{3} - 4), 5^{3}(5^{3} - 4), and 5^{3}(5^{3} - 4) are 4-perfect. 301, 49 \times 337, and 7^{3}(7^{3} - 6) are 6-perfect. 11^{2}(11^{2} - 10) and 11^{2}(11^{2} - 10) are 10-perfect. 13^{3}(13^{3} - 12), 13^{3}(13^{3} - 12), and 13^{3}(13^{3} - 12) are 12-perfect.

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Dr. Nash was born in Boston, MA.Received his Ph.D. in 1985 from Rutgers in Mathematics. Taught at Marshall University, Huntington, West Virginia 1986-1987. He lives with his father Dr. John Forbes Nash, Jr., of Princeton University, a pioneer of game theory and winner of the Nobel Prize in Economics.
The Klein Bottle Elves
You will probably never meet one of the creatures represented here face to face. This is a rare look at some of the dreaded Gremlins of Academia.

Eric Hemmingsen, former chair of the Mathematics Department at Syracuse University, was famous for drawing these mysterious figures on dinner napkins, when napkins were made of stiff paper. Now, at long last, he has confessed that he often had a particular dean in mind. That might explain why the elf is often drawn when he has tied himself into knots and is picking his own pocket, or why his head has non-trivial homology.

The dean is long gone, and paper napkins are soft, but the Mathematics Library at S.U. is named the Eric Hemmingsen Library. It is located right in the mathematics department and is by virtue of its collection, as well as its location, a most valuable resource.

Did the Klein Bottle Elves help to keep the library in the math building?

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FIBONACCI PERIODS IN SYMMETRIC GROUPS
CHRISTOPHER NOONAN* AND ALAN KOCH†

Abstract. The Fibonacci Sequence is generalized so that the entries are not integers but permutations of a finite set. For any choice of initial conditions the resulting sequence is periodic. The concept of period is defined and computed for all $S_n$, $n \leq 6$. Additional properties are also shown, and many questions concerning Fibonacci periods are posed.

1. Introduction. One of the best known sequences that is usually defined recursively is the Fibonacci sequence. This sequence dates back to the thirteenth century and was originally used by Leonardo Fibonacci to study rabbit populations. The Fibonacci sequence, denoted $(f_n)$, is given by

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

Since that time there have been numerous applications of the Fibonacci sequence. In 1960 D.D. Wall [6] considered the Fibonacci sequence modulo $m$. He observed that, mod $m$, the Fibonacci sequence would cycle, that is it would repeat itself over and over. For example, if $m = 4$ the sequence is

$$0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, 1, \ldots$$

Notice that $f_6 \equiv f_0$, and $f_7 \equiv f_1$ (mod 4). Since the next term is always the sum of the two previous ones, once the initial two terms arise again the sequence must cycle. He said that the period of the Fibonacci sequence is 6 mod 4, which is written as $k(4) = 6$.

The period of the Fibonacci sequence mod $p$ for $p$ a prime has been studied both theoretically (for example [1], [2]) as well as computationally (for example [5]). Some of the properties of $k(p)$ are

1. If $p > 2$ then $k(p)$ is even. (In the case $p = 2$ we have $k(2) = 3$.)
2. If $p \equiv \pm 1 \pmod{10}$ then $k(p)$ divides $p - 1$.
3. If $p \equiv \pm 3 \pmod{10}$ then $k(p)$ divides $2(p + 1)$ but not $p + 1$.
4. If $p \equiv \pm 3 \pmod{10}$ then $k(p)$ divides 4.

Proofs of these facts originally appeared in [6]. See also [4] for proofs using linear algebra. The period when $p$ is a prime has been computed up to $p = 415,993$ (22).

A natural generalization of this problem is to change the initial conditions from 0 and 1. Perhaps surprisingly, this change usually does not change the length of the period mod $p$, although for $p$ congruent to $\pm 1 \pmod{10}$ there are sometimes choices for initial conditions where the period is exactly one-half of $k(p)$ [4, Th. 3.8c].

Of course, there is no reason why the notion of a “Fibonacci sequence” cannot be extended to other number systems. It is quite easy to extend the concept to any algebraic structure that has both an additive and a multiplicative identity, for example fields and matrices. If we drop the usual initial conditions, all that is needed to have a Fibonacci sequence is a set with a binary operation. In this paper, we will construct Fibonacci sequences where the terms are not real numbers but permutations, and

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instead of adding the terms we will compose them. Much like when working mod m, these
permutations will also cycle, and the lengths of these periods can be computed.

While some preliminary results are given in section 4, the purpose is not to
completely answer all of the questions concerning Fibonacci periods over symmetric
groups. Instead, it is to inform the reader of the concept, describe a few basic facts
about these periods, and provide unsolved problems that are accessible to any second­
year undergraduate student.

The authors would like to thank M. Jean McKennie for her input during the
preparation of this paper.

2. Symmetric Groups. We will briefly describe what a symmetric group is.
The reader who is already familiar with symmetric groups and their properties may
skip to the next section.

Let \( S_n \) be the set of all permutations of the set \( \{1, 2, 3, \ldots, n\} \). In other words,
an element \( \sigma \in S_n \) is a bijection of the set. We will represent \( \sigma \) using cycle notation.
For example, if \( \sigma \in S_5 \) is given by
\[
\sigma(1) = 3 \quad \sigma(2) = 6 \quad \sigma(3) = 4 \\
\sigma(4) = 1 \quad \sigma(5) = 5 \quad \sigma(6) = 2
\]
then we write \( \sigma = (134)(26) \). To determine how \( \sigma \) permutes an element of the set,
simply find the element, and if it is not adjacent to the right­hand parenthesis, \( \sigma \)
maps this element to the element listed to its right. If it is adjacent to a right­hand
parenthesis, then \( \sigma \) maps this element to the left­most element in the same set of
parentheses. If the element does not appear at all, it is fixed.

As another example, if \( \tau = (152)(367)(89) \), then \( \tau(1) = 3, \tau(5) = 2, \) and \( \tau(2) = 1 \).
The triple \((152)\) is called a cycle as repeated applications of \( \tau \) cycle through these
elements. Since it has length three is called a 3­cycle. Similarly, \( \tau(3) = 6, \tau(6) = 7, \)
and \( \tau(7) = 3 \). Also, \( \tau \) interchanges 8 and 9 (and so \( (89) \) is a 2­cycle). Finally, \( \tau(4) = 4 \)
since 4 does not appear in the cycle notation.

If \( \sigma \) and \( \tau \) are permutations of the set \( \{1, 2, 3, \ldots, n\} \), then so is the composition
\( \sigma \tau \), thus \( \sigma \tau \in S_n \). We refer to this as the product of \( \sigma \) and \( \tau \). This can be computed
using cycle notation by reading right to left. For example, if \( \sigma = (124) \) and \( \tau = (1324) \),
then \( \sigma \tau = (124)(1324) \). If we wish to find \( \sigma \tau(3) \), the cycle on the right says that
\( \tau(3) = 2 \). The cycle on the left gives \( \sigma(2) = 4 \). Thus \( \sigma \tau(3) = 4 \). Below is a table of
\( \sigma \tau(i) \) for \( i = 1, 2, 3, \) and 4.
\[
\begin{align*}
\sigma \tau(1) &= \sigma(\tau(1)) = \sigma(3) = 3 \\
\sigma \tau(2) &= \sigma(\tau(2)) = \sigma(4) = 1 \\
\sigma \tau(3) &= \sigma(\tau(3)) = \sigma(2) = 4 \\
\sigma \tau(4) &= \sigma(\tau(4)) = \sigma(1) = 2
\end{align*}
\]
This tells us that \( \sigma \tau = (124)(1324) = (1342) \).

Any element can be written as a product of 2­cycles. (A 2­cycle is also called a
transposition.) For example, \((12345) = (15)(14)(13)(12)(15) \). While the decomposition
into transpositions is not unique (notice that \((12345) = (15)(14)(13)(12)(15) \)), the number of
transpositions in the decomposition for an element is always even or
always odd (§ 2.2.15). An element \( \sigma \in S_n \) is called even if it can be written as
a product of an even number of transpositions. If it can be written as product of
an odd number of transpositions, then \( \sigma \) is called odd. Given \( \sigma \) and \( \tau \) we can determine
whether \( \sigma \tau \) is even or odd it works like addition of numbers:

\[
\begin{array}{ccc}
\sigma & \tau & \sigma \tau \\
\text{Even} & \text{Even} & \text{Even} \\
\text{Even} & \text{Odd} & \text{Odd} \\
\text{Odd} & \text{Even} & \text{Odd} \\
\text{Odd} & \text{Odd} & \text{Even}
\end{array}
\]

It is not too hard to show that the multiplication works in this manner: the element
\( \sigma \tau \) can be decomposed into transpositions by decomposing \( \sigma \) and \( \tau \) individually. Thus
if \( \sigma \) decomposes into \( t_1 \) transpositions and \( \tau_2 \) decomposes into \( t_2 \) transpositions, then
one decomposition of \( \sigma \tau \) uses \( t_1 + t_2 \) transpositions.

The notation that fixes every element will be denoted \( e \). This element has
the unique property that \( \sigma e = e \sigma = \sigma \) for all \( \sigma \in S_n \).

3. Fibonacci Sequences. It makes sense to define the Fibonacci sequence \( \{a_n\} \)
by the recurrence relation
\[
a_n = a_{n-1} a_{n-2}
\]
However, there is no natural choice of initial conditions. While \( e \in S_n \) is analogous
to 0, there is no element in \( S_n \) analogous to 1. It is for this reason that we will
study Fibonacci sequences with different choices of initial conditions. We will see
that different choices of initial conditions will give us different period lengths, and
that there are more period lengths for \( S_n \) than there are in the mod p problem.
The notation \( k(S_n, a_0, \sigma_1) \) will be the length of the period with initial conditions
\( a_0, \sigma_1 \in S_n \).

Example. The following calculations show that \( k(S_3, (12), (123)) = 6 \):
\[
\begin{align*}
a_0 &= (12) \\
\sigma_1 &= (13) \\
\sigma_2 &= (123) \\
\sigma_3 &= (123)(12)
\end{align*}
\]
To illustrate how the initial conditions can make a difference in the period, notice
that \( k(S_3, e, (12)) = 2 \):
\[
\begin{align*}
a_0 &= e \\
\sigma_1 &= (12) \\
\sigma_2 &= (12)(12) = e
\end{align*}
\]
Some natural questions are:
1. Given \( n \), what choice of initial conditions gives the largest period?
2. Given \( n \), what is the maximum value of \( k(S_n, a_0, \sigma_1) \)?
3. More generally, what are some properties of \( k(S_n, a_0, \sigma_1) \)?

Let us denote by \( k(S_n) \) the longest period using any initial conditions in \( S_n \).

Those familiar with abstract algebra may ask: why use the group \( S_n \) rather than
any other? By Cayley’s Theorem [§ 3.2.1.16], every finite group of order \( n \) can be viewed
as a subgroup of \( S_n \). In other words, if we know \( k(S_n) \), then for any group of order
\( n \) we have an upper bound: \( k(G) \leq k(S_n) \). Furthermore, \( k(G, a_0, \sigma_1) = k(S_n, a_0, \sigma_1) \),
where \( a_0, \sigma_1 \in G \leq S_n \), so to answer the question over \( S_n \) is to answer the question
for any two elements picked from any finite group.
4. Results. So what is known about $k(S_n, \sigma_0, \sigma_1)$?

1. $k(S_2) = 3$, $k(S_3) = 5$, $k(S_4) = 18$, $k(S_5) = 96$, and $k(S_6) = 216$.

These have been computed with the help of two MAPLE procedures that are given in the appendix.

2. $k(S_n) \leq (n!)^2$.

There are $(n!)^2$ possible ordered pairs of elements in $S_n$. It is impossible for an ordered pair to appear twice without completing a period.

3. For all $i$, $k(S_{n+i}, \sigma_0, \sigma_1) = k(S_n, \sigma_0, \sigma_1)$.

In other words, if you pick any two consecutive terms in a period, then the period they generate has the same length as the original period. (In fact, it generates the exact same period!)

4. If either $\sigma_0$ or $\sigma_1$ is an odd permutation (or both), then $k(S_n, \sigma_0, \sigma_1)$ is a multiple of 3.

This can be shown by considering the three cases. Consider the table below, where "E" and "O" represent even and odd permutations.

<table>
<thead>
<tr>
<th>$\sigma_0$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_3$</th>
<th>$\sigma_4$</th>
<th>$\sigma_5$</th>
<th>$\sigma_6$</th>
<th>$\sigma_7$</th>
<th>$\sigma_8$</th>
<th>$\sigma_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>Case 2</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
</tr>
<tr>
<td>Case 3</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
<td>E</td>
<td>O</td>
</tr>
</tbody>
</table>

Notice that, in the first case, if the sequence repeats starting with $\sigma_1$ then $\sigma_1$ and $\sigma_{i+1}$ must both be odd. This only occurs when $i$ is a multiple of 3. The other two cases are similar.

In the case where $\sigma_0$ and $\sigma_1$ are both even, of course, $\sigma_i$ is even for all $i$, so no such conclusion can be drawn.

5. $k(S_n, \sigma, \tau) = k(S_n, \tau, \sigma)$ if $n \leq 4$.

This again was determined using the MAPLE procedures. The result is false for $n \geq 5$: $k(S_5, (12345),(12345)) = 12$ and $k(S_5, (12345)) = 14$.

Clearly, equality will always hold if $\sigma = \tau$.

5. Questions. Finally, here are some unsolved problems.

1. Is there a better upper bound for $k(S_n, \sigma_0, \sigma_1)$?

2. Is there a better upper bound for $k(S_n)$ than $(n!)^2$? In the examples computed above, the actual $k(S_n)$ was nowhere near this bound. In fact, for $n \leq 6$, $k(S_n) \leq n^2$.

Generally, there are clearly better bounds for $k(S_n)$ than $(n!)^2$. For example, since $S_{n-1}$ is contained in $S_n$ in a natural way, any period starting with two elements in $S_{n-1}$ is bounded by $k(S_{n-1})$. By the third result, if there are two consecutive terms in a Fibonacci period that come from $S_{n-1}$, then this period is also bounded by $S_n$. Using the bound $k(S_{n-1}) \leq ((n-1)!)^2$ gives

$$k(S_n) \leq \max\{(n!)^2 - ((n-1)!)^2, k(S_{n-1})\} \leq \max\{((n-1)!)^2 - ((n-1)!)^2, ((n-1)!)^2 \} = (n!)^2 - (n-1)^2$$

3. If $k(S_n, \sigma_0, \sigma_1) > 5$, must $k(S_n, \sigma_0, \sigma_1)$ be even? Evidence from $n \leq 6$ seems to suggest that this is so. By [7, Cor. 5], if there is a period of odd length greater than three then the period cannot contain the identity element. The converse, however, is not true, as no pair of initial conditions $\sigma_0, \sigma_1 \in S_6$ give an odd period of length greater than three.

Why would knowing $k(S_n)$ be important? One application is in the field of cryptology. Every element $\sigma \in S_6$ can be thought of as a rearrangement of the alphabet. If $\sigma_0, \sigma_1 \in S_6$, we can encode a message as follows: to encode the $i$th letter, use the rearrangement given by $\sigma_i$, the $i$th term in the Fibonacci sequence given by these initial conditions. This polyalphabetic substitution could be very hard to break if $k(S_{26}, \sigma_0, \sigma_1)$ were extremely large.

To demonstrate this cryptological application, here is an example of the idea applied only to the vowels. Suppose we want to send the message: AOAUFE.

Number the vowels $A = 1, E = 2, I = 3, O = 4$, and $U = 5$. Pick $\sigma_0 = (13524)$ and $\sigma_1 = (1243)$. The encryption is shown in the following table.

<table>
<thead>
<tr>
<th>Letter</th>
<th>Number</th>
<th>$\sigma_i$</th>
<th>Encoded Number</th>
<th>Encoded Letter</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>(13524)</td>
<td>3</td>
<td>I</td>
</tr>
<tr>
<td>O</td>
<td>4</td>
<td>(1243)</td>
<td>3</td>
<td>I</td>
</tr>
<tr>
<td>U</td>
<td>5</td>
<td>(12453)</td>
<td>3</td>
<td>I</td>
</tr>
<tr>
<td>E</td>
<td>2</td>
<td>(14523)</td>
<td>3</td>
<td>I</td>
</tr>
<tr>
<td>E</td>
<td>2</td>
<td>(14523)</td>
<td>1</td>
<td>A</td>
</tr>
</tbody>
</table>

so the encoded message is IIAUIIA. Why is this difficult to break? For starters, a frequency count technique does not work — each letter gets coded differently. Since $k(S_6,(13524),(12453)) = 54$, the pattern will not reemerge until the 55th character. Also, if the breaker has a little bit of information, it cannot be used to get more for example if someone knew that the first letter was $A$ all that person would know is that $\sigma_0(1) = 3$ — very little knowledge is obtained about $\sigma_0$ itself. In practice it is a good idea to pick $\sigma_0$ and $\sigma_1$ so that the $\sigma_i$'s do not have many fixed points. One way to do this is to let $\sigma_0$ be a 26-cycle and let $\sigma_1$ be some power of $\sigma_0$. In most cases $k(S_6, \sigma_0, \sigma_0) = 64$. Picking $\sigma_1$ to be a power of $\sigma_0$ decreases the number of usable combinations, and hence a person attempting to crack the code has fewer things to check. However the number of ways of picking $\sigma_0$ is still 25!, which is greater than $1.5 \times 10^{25}$.

Appendix: Maple Procedures. This first procedure computes the period given the two initial conditions:

```maple
> with(group):
> period:=proc(a,b) local i,sigma;
> sigma[0]:=a;
> sigma[1]:=b;
> for i from 2 to length(mulperms(a,b))^2 while sigma[i-1]=sigma[0] or mulperms(sigma[i-1],sigma[i-2])<sigma[1] do
> sigma[1]:=mulperms(sigma[i-1],sigma[i-2]);
> od;
> end;
```

This second procedure computes the maximum period when $\sigma_0$ is taken from "list" and $\sigma_1$ is any element of $S_n$. If "list" consists of one of each cycle type in $S_n$, then maxperiod will give $k(S_n)$. To use this, you will also need the "period" procedure from above.
maxperiod:=proc(n,list) local i,j,s,max,temp,top;
max:=0;
s:elements(permgroup(n,[[[1,2]],[[seq(i,i=1 .. n)]]]));
top:=nops(list);
for i from 1 to top do
for j from 1 to n! do
    temp:=period(list[i],s[j]);
    if temp > max then max:=temp fi;
    od;
    print(i,max);
od;
max;
end;

REFERENCES


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THE SUM OF THE k′TH POWERS OF THE FIRST N POSITIVE INTEGERS

A. Saleh-Iahromi* and Julien Doucet†

Many different methods for calculating the sums of the k′th powers have been developed and many articles have been published in this area. For some history of the subject, and for a selection of these articles, we mention [1], [2], [3], [4], [5], [6], [7], and [8]. In this paper we give a new approach to computing the sum, which is

\[ S_k(n) = \sum_{i=1}^{n} i^k = 1^k + 2^k + 3^k + 4^k + 5^k + \ldots + (n-1)^k + n^k, \text{ where } k \geq 0. \]  

This sum often occurs in computing areas [9] and in analyzing the number of operations needed to solve linear equations by Gauss elimination [10]. We will derive a formula for \( S_k(n) \) which is recursive in \( k \).

For the base if the recursion, we note that \( S_0(n) = n \). For illustration, we compute \( S_1(n) \) in terms of \( S_0(n) \). We start with the formula

\[ (1 + x)^2 = 1 + 2x + x^2. \]

Substituting values for \( x \) in this formula, the following table can be created:

\[
\begin{array}{c}
X = 1, & (1 + 1)^2 = 1 + 2(1) + 1^2 \\
X = 2, & (1 + 2)^2 = 1 + 2(2) + 2^2 \\
X = 3, & (1 + 3)^2 = 1 + 2(3) + 3^2 \\
X = 4, & (1 + 4)^2 = 1 + 2(4) + 4^2 \\
X = 5, & (1 + 5)^2 = 1 + 2(5) + 5^2 \\
\vdots \\
X = n, & (1 + n)^2 = 1 + 2(n) + n^2 \\
\end{array}
\]

By adding the left and right-hand sides of these equalities and simplifying, we obtain

\[(1 + n)^2 = n + 2(1 + 2 + 3 + 4 + 5 + \ldots + n) + 1. \]

Rearranging this and combining it with (1) gives

\[ S_k(n) = \frac{(1 + n)^{k+1} - \sum_{i=1}^{n+1} i^{k+1} - (1 + n)}{2} = \frac{1 + S_k(n)^2 - 1 + S_k(n)}{2} \]

Now, to compute \( S_k(n) \) we start with

\[(1 + x)^{k+1} = \sum_{i=0}^{k+1} (k+1)_i x^i = (k+1)_0 + (k+1)_1 x + (k+1)_2 x^2 + \ldots + (k+1)_{k+1} x^{k+1}, \]
As we did in determining $S_1(n)$, we substitute values for $x$ and create the following table:

\[
(1 + 1)^{k+1} = \binom{k+1}{0} + \binom{k+1}{1} \cdots + \binom{k+1}{k+1}
\]

\[
(1 + 2)^{k+1} = \binom{k+1}{0} + (k+1) \binom{k+1}{1} + (k+1) \binom{k+1}{2} \cdots + \binom{k+1}{k+1} \binom{k+1}{k+1} + k \binom{k+1}{k}
\]

\[
(1 + 3)^{k+1} = \binom{k+1}{0} + (k+1) \binom{k+1}{1} + (k+1) \binom{k+1}{2} \cdots + \binom{k+1}{k+1} \binom{k+1}{k} + 3 \binom{k+1}{k+1} + 2k \binom{k+1}{k}
\]

\[
(1 + n)^{k+1} = \binom{k+1}{0} + (k+1) \binom{k+1}{1} \cdots + (k+1) \binom{k+1}{k+1} + n^2 \binom{k+1}{k} + \cdots + n^{k-1} \binom{k+1}{k} + n^k + n^{k+1}
\]

Again we add the left and right-hand sides of the equalities and in this case we obtain:

\[
(1 + n)^{k+1} = \binom{k+1}{0} n + \binom{k+1}{1} (1^2 + 2^2 + \cdots + n^2) + \cdots + \binom{k+1}{k} (1^k + 2^k + \cdots + n^k)
\]

\[
= \binom{k+1}{0} n + \binom{k+1}{1} S_2(n) + \cdots + \binom{k+1}{k} S_k(n)
\]

So we get

\[
[1 + S_0(n)]^{k+1} = 1 + (k+1) S_0(n) + \binom{k+1}{2} S_2(n) + \cdots + (k+1) S_k(n)
\]

\[
= 1 + \sum_{i=0}^{k-1} \binom{k+1}{i} S_i(n)
\]

Then

\[
S_k(n) = \frac{[1 + S_0(n)]^{k+1} - \sum_{i=0}^{k-1} \binom{k+1}{i} S_i(n)}{k + 1}
\]

If $k = 1$, then equation (4) becomes

\[
S_k(n) = \frac{(1+n)^2 - (1+n)}{2} = \frac{n(n+1)}{2}
\]

implying the validity of equation (4).

Equation (4) is an iterative formula. Knowing $S_k(n)$ we can determine $S_{k+1}(n)$, in terms of $S_0(n), S_1(n), S_2(n), \ldots, S_k(n)$. Clearly $S_k(n)$ is a polynomial of degree $k + 1$ in $n$, and can be written as

\[
S_k(n) = \frac{1}{k+1} k^{k+1} + \frac{1}{2} n^k + \text{a polynomial of degree } k - 1,
\]

where the coefficients must sum to 1. The series terminates at $n$ or $n^2$ according as $k$ is even or odd, except for $S_1(n)$. This formula is particularly appropriate for computers since it is a generalization process for $S_{k+1}(n)$. Using MAPLE it is possible to calculate equation (4), for any positive $k$ and $n$, in the following form:

\[
S_k(n) = \sum((1+n)\binom{k+1}{k+1} - \sum(\text{sum}(\text{sum}(\text{sum}(\text{sum}(...)\binom{k+1}{j+1})),j=0,k),k)) / (k+1)
\]

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**REFERENCES**


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_Dear Ghost of Gauss,_

_You are the master of all mathematical wisdom. I have studied and computed and cannot determine the answer to this question. Please, with your towering mathematical perspective, settle this question for me: “May I, or may I not, wear a IIME Lapel Pin on my IIME Tee Shirt”?_  

_Frustrated and Anxious._

_Dear F & A,_

_In general, no. The Tee shirt’s lack of lapels is not an insurmountable difficulty, nor are the canons of good taste. Yet one further condition is needed. The IIME lapel pin must be yours. Going around, taking other people’s IIME lapel pins and sticking them on your tee shirt will not make you popular in the mathematical community._

_I have spoken._

_P.S. The gold clad keypins are available at the national office at the price of $12 each. To purchase a keypin, write to the secretary-treasurer:_

Robert M. Woodside  
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East Carolina University  
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FACTORIAL AS A CONTINUOUS FUNCTION
SEAN EFFINGER-DEAN*

Experimenting with the program Derive, I came upon an interesting phenomenon. When I asked Derive to graph the function $y = x!$, the program presented a smooth line, as if factorial were a continuous function. But, I could not see how something like $0.7!$ could even be determined, since factorial, in my mind, was only defined for whole numbers!

However, triangular numbers are defined as $1 + 2 + 3 + \ldots + n$ where $n$ is a whole number. Gauss, as a schoolboy, found that this is the same as $n(n + 1)/2$, which, as you will notice, makes sense for all numbers, not just whole numbers. My goal was established then: to find a continuous function that was equal to factorial for all whole numbers. After days of work (and many failed attempts), I finally discovered a very interesting solution to my question, which, I have since learned, is a classic result known as Stirling's formula. Be aware that my goal was not to find a proof for this formula; it was simply to investigate and, possibly, discover something about factorial. Therefore, some of the series I have used are not convergent, and must be considered "formal" series instead. Here is what I did to reach my solution, step by step.

Starting with $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, we take logarithms and use Taylor series to get:

$$\ln(n!) = \sum_{i=1}^{n} \ln(i) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^{k+1} (i-1)^k}{k}$$

These terms are rearranged, (even though most of these series are not convergent.)

$$\ln(n!) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{i=1}^{n} (i-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left( \sum_{i=1}^{n-1} i^k + n^k \right)$$

We denote $\sum_{i=1}^{n} i^k$ by $f_k(n)$, so, setting $u = n - 1$, we have

$$\ln(n!) = f_1(u) + \frac{1}{2} f_2(u) + \frac{1}{3} f_3(u) + \frac{1}{4} f_4(u) + \ldots$$

Using formulas for $f_k(u)$, see for example the previous article,¹ we get

$$\ln(n!) = \left[ \frac{u^2}{2} + \frac{u}{2} + \left( \frac{u^3}{6} + \frac{u^2}{4} + \frac{u}{12} \right) + \left( \frac{u^4}{20} + \frac{u^3}{8} + \frac{u^2}{12} - \frac{u}{120} \right) \right] + \ldots$$

Hmmmmm... let's take the derivative!

$$\frac{d}{dn} \ln(n!) = \left[ u + \frac{1}{2} - \frac{u^2}{2} + \frac{u}{12} \right] + \left[ \frac{u^3}{3} + \frac{u^2}{2} + \frac{u}{6} \right] - \left[ \frac{u^4}{4} + \frac{u^3}{2} + \frac{u^2}{4} - \frac{1}{120} \right] + \ldots$$

¹Saratoga Springs High School

Editorial Note: The author derived the formulas for the sums of the $k$'th powers of the integers quite differently from methods in previous article. He establishes the recursion

$$f_k(u) = k \int_0^u f_{k-1}(t) dt - c_k u$$

and reports on interesting patterns for the value of the constant $c$ as $k$ increases.
These are all Taylor expansions!

\[ \frac{d \ln n!}{dn} = [\ln u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots] + \frac{1}{12} [1 - 2u + 3u^2 - \cdots] + \cdots \]

where we have replaced \((u + 1)\) by \(n\). Integration yields

\[ \ln n! = n \ln n - n + \ln n + \frac{1}{2} \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{180n^7} + \cdots \]

where \(b\) is a constant. Hence we get:

\[ n! = (b)n^n \sqrt{n} \exp \left( -n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{180n^7} + \cdots \right) \]

where \(B = e^b\). Using large integral values for \(n\) to approximate \(B\), we find that \(B = 2.5066282746310005\). This constant doesn't look familiar, but some experimentation with a calculator seems to show that \(B = \sqrt{3}\).

My formula turns out then to be what I'm told is Stirling's formula!

The graph below shows plots both of Mathematica's \(y = x!\) and of my result, which appear to correspond nicely after about \(x = 0.6\).

\[ \frac{d \ln n!}{dn} = [\ln u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots] + \frac{1}{12} [1 - 2u + 3u^2 - \cdots] + \cdots \]

These are all Taylor expansions!

\[ \frac{d \ln n!}{dn} = (\ln n) + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{1260n^5} - \frac{1}{180n^7} + \cdots \]

where we have replaced \((u + 1)\) by \(n\). Integration yields

\[ \ln n! = b + n \ln (n) - n + \ln (n) + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{1260n^5} - \frac{1}{180n^7} + \cdots \]

where \(b\) is simply a constant. Hence we get:

\[ n! = (B)n^n \sqrt{n} \exp \left( -n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{180n^7} + \cdots \right) \]

SOME PROPERTIES OF THE EQUATION \(S(x) = K\)

REX H. WU*

In 1979, Florentin Smarandache introduced a number theoretic function. For any positive integer \(n\), the Smarandache function \(S(n)\) is defined as \(S(n) = k\) if \(k\) is the smallest positive integer such that \(n\) divides \(k!\). Since then, some interesting properties have been discovered about this function. Just one example, for \(x > 4\), the expression

\[ n(x) = -1 + \sum_{k=2}^{x} \left[ \frac{S(k)}{k} \right] \]

where \([x]\) is the greatest integer function, gives the exact number of primes less than or equal to \(x\), \([1]\).

In this note, we will look at some elementary properties associated with the equation \(S(x) = k\).

First, let's see how we can solve the equation \(S(x) = k\). Suppose

\[ k = p_1^{a_1} p_2^{a_2} \cdots p_j^{a_j} \]

and

\[ (k - 1)! = p_1^{b_1} p_2^{b_2} \cdots p_j^{b_j} \]

Then \(k! = p_1^{b_1 + a_1} p_2^{b_2 + a_2} \cdots p_j^{b_j + a_j} \cdots p_j^{a_j} \cdots p_1^{a_1}\), for some prime \(p_i\) and nonnegative integers \(a_i\) and \(b_i\); with \(i = 1, 2, 3, \ldots, j\). Here, \(j\) is used as the number of prime factors of \(k\). Note that if \(p_i\) is a prime that divides \(k\) but not \((k - 1)\), then take \(a_i = 0\). If \(x_0\) were a solution to \(S(x) = k\), then \(x_0 \mid k!\). Furthermore, \(x_0 \mid (k - 1)!\). Obviously \(x_0\) contains some factor \(p_i^{\alpha_i}\), where \(\alpha_i < \gamma_i \leq \alpha_i + \beta_i\), for some \(i = 1, 2, 3, \ldots, j\). So we have our first conclusion.

**Theorem 1.** \(x_0\) is a solution to \(S(x) = k\) if and only if \(x_0 = MNQ\), where

\[ M = \prod_{i \in I} p_i^{\alpha_i}, \]

where \(I\) can be any nonempty subset of \(\{1, 2, 3, \ldots, j\}\) and \(1 \leq \alpha_i \leq \beta_i;\)

\[ N = \prod_{i \in I} p_i^{\beta_i}, \]

where, again, if \(p_i\) is a prime that divides \(k\) but not \((k - 1)\), then take \(a_i = 0\); and \(Q\) is any factor of \((k - 1)!/N\).

**Proof.** We have

\[ MN = \prod_{i \in I} p_i^{\alpha_i + \lambda_i}. \]

Since \(\alpha_i < \alpha_i + \beta_i \leq \alpha_i + \beta_i\), we know \(MN \mid k!\) but \(MN \nmid (k - 1)!\). For \(N\), using the highest exponent \(\alpha_i\) so that \(p_i^{\alpha_i} \mid (k - 1)!\) is essential. Otherwise, \(MN\) may divide \((k - 1)!\) and rendering \(MN\) not a solution. Furthermore, if \(Q\) divides \((k - 1)!/N\), then

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Editor's Comment. This paper is a brilliant example of how far a student can go. Effinger-Dean is currently a senior at Saratoga Springs High School. His main interests include mathematics, physics, and music, especially musical theater.

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Editor's Comment. This paper is a brilliant example of how far a student can go. Effinger-Dean is currently a senior at Saratoga Springs High School. His main interests include mathematics, physics, and music, especially musical theater.
MNQ divides $M(k - 1)!$ which in turn divides $k!$. Therefore, $MNQ$ is a solution to $S(x) = k$.

Observe that using all the nonempty subsets of $\{1, 2, \ldots, j\}$ for $M$ would generate all the factors of $k$. In combination with all the factors of $(k - 1)!/N$, we have all the solutions to $S(x) = k$.

Suppose there is a solution $x_0$ to $S(x) = k$, we are going to show $x_0$ is of the form $MNQ$. Let $x_0 = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$. Note that $S(x_0) = k = \max\{S(p_i^{a_i})\}$ for $i = 1, 2, \ldots, n$. With $m = k - 1, t$. Obviously, if $p_i$ is not a factor of $k, S(p_i^{a_i}) < k$. Even if $p_i$ is a factor of $k, \text{if } \alpha_i < \beta_i, \text{then } S(p_i^{a_i}) < k$ because $S(p_i^{a_i}) < k$. If $\alpha_i > \beta_i$, $S(p_i^{a_i}) > k$. Therefore, we have $p_i | k$ and $\alpha_i < \beta_i$. Notice that there can be more than one such $p_i$ such that $S(p_i^{a_i}) = \max\{S(p_i^{a_i})\}$ if $i$ has more than one prime factor. This shows if $x_0$ were a solution, then $x_0$ contains $MNQ = \prod_{i=1}^k p_i^{a_i}$ for some subset $I$ of $\{1, 2, \ldots, j\}$ and $\alpha_i < \beta_i$ for all $i$. Also notice that any multiples of $MN$, say $MNQ$, is a solution to $S(x) = k$, provided $MNQ | k$. The question is what can $Q$ be?

Obviously, $MNQ/k!$, for some integer $A$. $QA = k! MNQ(k/MNQ) = k! (k/MNQ) (k - 1)! / N$. From the previous expression, $Q$ can be any factor of $(k - 1)! / N$. If $Q$ contains a prime factor $p_i$ such that $p_i$ is also a factor of $k$, then $p_i$ must have an exponent $\epsilon_i = \alpha_i$, in which case $p_i^{\alpha_i}$ is a factor of $(k - 1)!$. Otherwise, $S(p_i^{\alpha_i}) k!$. Hence, $p_i^{\alpha_i} < \alpha_i + \beta_i$, but then this factor would be part of $MNQ$. Or $S(p_i^{\alpha_i}) > k$ if $\epsilon_i > \alpha_i + \beta_i$. Therefore, we can only have $Q$ $\{((k - 1)! / N)\}$. D

An example would best illustrate this theorem. Let’s solve $S(x) = 12$. Here $k = 12 = 2^2 3$, $(k - 1)! = 11! = 3^4 5^2 7 11$. Let’s look at the number of solutions instead of each individual solution. Obviously, the number of solutions for any particular $M$ is $\tau(Q)$, where $\tau(n)$ is the number of factors for the positive integer $n$. If $p_i^{\alpha_i} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, then $\tau(n) = (\alpha_0 + 1) (\alpha_1 + 1) (\alpha_2 + 1)$.  

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
<th>$Q = 11! / N$</th>
<th>$\tau(Q)$ = number of solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2^2$</td>
<td>factors of $3^4 \times 5^2 \times 7 \times 11$</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>$3^3$</td>
<td>factors of $2^3 \times 5^2 \times 7 \times 11$</td>
<td>108</td>
</tr>
<tr>
<td>22</td>
<td>$2^2$</td>
<td>factors of $3^4 \times 5^2 \times 7 \times 11$</td>
<td>60</td>
</tr>
<tr>
<td>$2 \times 3$</td>
<td>$2^2 \times 3^1$</td>
<td>factors of $3^2 \times 5^2 \times 7 \times 11$</td>
<td>12</td>
</tr>
<tr>
<td>$2 \times 3$</td>
<td>$2^2 \times 3^4$</td>
<td>factors of $3^2 \times 5^2 \times 7 \times 11$</td>
<td>12</td>
</tr>
</tbody>
</table>

Adding up the last column gives a total of 252 solutions.

While the above theorem works, it gets cumbersome if $k$ gets large. Let’s explore a little bit and look for a simpler method. We will also switch our attention to look at the number of solutions rather than all the specific solutions to $S(x) = k$.

**Theorem 2.** $x_0$ is a solution to $S(x) = k$ if and only if $x_0 \not| k!$ and $k \not| (k! / x_0)$.

**Proof.** Suppose $x_0$ is a solution to $S(x) = k$, then by definition, $x_0 | k!$ and for any $n < k$, $x_0 | n!$. It suffices to show the case $n = k - 1$. Since if $x_0 | (k - 1)!$, then $x_0 | n!$ for all $n < k - 1$. Therefore, for $n = k - 1$, $x_0 | n!$ implies $kx_0 | k! \not| k! / x_0$. Since $k \not| k! / x_0$, $k | (k! / x_0)$.

Let’s say $x_0 | k!$ but $k \not| (k! / x_0)$. Since $k \not| (k! / x_0)$ is equivalent to $k \not| x_0 | (k! - 1)$. Obviously, if $x_0 | (k! - 1)$, then $x_0 | n!$ for any $n \leq k - 1$. This is the very definition of the Smarandache function. Therefore, $S(x_0) = k$. D

**Theorems 1 and 2 are actually equivalent.** To see if $MNQ$ is a solution or not, all we need to do is to see if $k$ divides $k! / (MNQ)$ or not. Suppose $p_m$ is one of the primes used in $MNQ$, i.e. $1 \leq m \leq j$, then $MNQ = p_m^{a_m} k A$, for some integer $A$ and $1 \leq a_m \leq \beta_m$. So $k! / (MNQ) = p_m^{a_m - \beta_m} B$ for some integer $B$. Obviously, $k \not| (k! / (MNQ))$ because $k$ has a factor $p_m^{a_m}$ and $p_m^{a_m} \not| p_m^{a_m - \beta_m}$. B.

**Corollary 3.** If $k$ is prime, then there are $\tau(k! / 2)$ solutions to $S(x) = k$.

**Proof.** Let’s pair up the divisors of $k!$ such that the product of each pair is $k!$, i.e., if $x_0 | k!$, then $x_0$ is paired up with $k! / x_0$. $x_0$ is prime implies $k \not| (k - 1)$. Then either $k \not| x_0$ or $k | (k! / x_0)$ but not both. If $k \not| x_0$, then $k! / x_0$ is a solution to $S(x) = k$. Otherwise, $x_0$ is. This shows exactly half of the factors of $k!$ are solutions to $S(x) = k$ if $k$ is prime. D

Once we know theorem 2, we can look for the number of solutions to $S(x) = k$ with ease. Let’s denote $\omega(k)$ the number of solutions to $S(x) = k$.

**Corollary 4.** There are $\omega(k) = \tau(k! / 2)$ solutions to $S(x) = k$.

**Proof.** According to Theorem 2, this is to look for the number of factors of $k!$ that are not divisible by $k$.

Let’s look at the factors of $k!$, in particular, we are interested in the ones that are not divisible by $k$. To look for those, we will find out the number of factors that are divisible by $k$, i.e., factors of the form $k A$, for some integer $A$. Since $k A | k!$, we have $A | (k - 1)!$. There is a total of $\tau((k - 1)!)$ such $A$’s. Since there are $\tau(k!)$ factors of $k!$, there are $\omega(k) = \tau(k! / 2)$ factors that are not divisible by $k$. D

Corollary 2 gives another proof to corollary 1. If $k$ is prime, then $k \not| p_0$ is a prime different from all the primes less than or equal to $(k - 1)$. If there are $\tau((k - 1)!)$ factors for $(k - 1)!$, then $\tau(k! / 2)$ is the same as $\omega(k) = \tau(k! / 2)$.

Now let’s look at the first 15 values for $\omega(k)$. Note that $\omega(12)$ confirms the result we obtained using theorem 1.

$$
\begin{array}{ccc}
\omega(k) & \tau(k! / 2) & \omega(k) \\
1 & 1 & 1 \\
2 & 2 & 1 \\
3 & 4 & 2 \\
4 & 8 & 4 \\
5 & 16 & 8 \\
6 & 30 & 14 \\
7 & 60 & 30 \\
8 & 96 & 36 \\
9 & 160 & 64 \\
10 & 270 & 110 \\
11 & 540 & 270 \\
12 & 792 & 232 \\
13 & 1584 & 792 \\
14 & 2592 & 1008 \\
15 & 4032 & 1440 \\
\end{array}
$$

Pay attention to the $\omega(k)$’s and $\tau(k!)$’s where $\omega(k) = \tau(k! / 2) = \tau((k - 1)!)$). Also look at the corresponding $k$. A pattern seems to arise. The $k$’s are prime except when $k = 4$. One may wonder if this pattern would be true for all.

Before we go onto proving the above statement, we need to utilize a function, $E(n, p)$, which gives the largest exponent of a prime $p$ such that $p^\alpha | n!$.

$$
E(n, p) = \sum_{i=1}^n \left\lfloor \frac{n}{p^i} \right\rfloor
$$


Note that \( E(Q, p_1) / \beta = E(1, p_1) / \beta = 0 \). There are only a few cases that this inequality is true, namely, \( (p, \beta) = (2, 2), (2, 3) \) and \( (3, 2) \), corresponding to \( k = 4, 8 \) and 9, respectively. By assumption \( k \neq 4 \). It is easy to check that \( \tau(8!) \neq 2 \tau(7!) \) and \( \tau(9!) \neq 2 \tau(8!) \).

Case (II). \( k \) has 2 distinct prime factors, \( k = p_1^{\beta_1} p_2^{\beta_2} \) with \( \beta_i > 0 \).

Here, \( j = 2 \) and without loss of generality, \( Q = p_1^{\beta_1} \). Again, we have

\[
2j + 1 > 5 > \frac{p_1^{\beta_1} - 1}{\beta_1(p_2 - 1)} + \frac{E(Q, p_2)}{\beta_2}
\]

The inequality is true if \((Q, p_2, \beta_2) = (2, 3, 2), (2, p_2, 1), (2^2, p_2, 1)\) for some prime \( p_2 \geq 3 \) and \((3, p_2, 1)\) for some prime \( p_2 \geq 5 \).

For \((Q, p_2, \beta_2) = (2, 3, 2)\), we have \( k = 18 \). A little computation shows that \( \tau(18!) \neq 2 \tau(17!) \).

For \((Q, p_2, \beta_2) = (2, p_2, 1)\), or \( k = 2p_2\), if \( \tau(k(k-1)) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1) \) then \( \tau(k(k-1)) = (a_1 + 1)(a_2 + 1)(a_3 + 1) \cdots (a_r + 1) \). If \( \tau(k(k-1)) = 2 \tau(k(k-1)) \), we have \( (a_1 + 1)(a_2 + 1) = 2(a_1 + 1)(a_2 + 1) \). Simplifying the last equation gives \( 2 = a_1 a_2 \). Therefore \((a_1, a_2) = (1, 2)\) or \((2, 1)\). From observation (ii), we know \((a_1, a_2) = (1, 2)\) is not possible. It is also impossible for \((a_1, a_2) = (2, 1)\) because \( a_2 = 2 \neq E(2p_2 - 1, 2) \) for any prime \( p_2 \).

The argument is identical for \((Q, p_2, \beta_2) = (3, p_2, 1)\). When we reach \((a_1, a_2) = (2, 1)\), we have \( a_1 = 2 = E(3p_2 - 1, 3) \). Here, we have \( 3p_2 - 1 \neq 5, 7 \) or 8. But then there is no \( p_2 > 3 \) satisfying this condition.

Similarly, for \((Q, p_2, \beta_2) = (2^2, p_2, 1)\), we have \( 4 = (a_1 + 1) a_2 \) after equating \( \tau((2^2p_2 - 1)) = 2 \tau((2^2p_2 - 1)) \). Solving \( 4 = (a_1 + 1) a_2 \) to get \((a_1, a_2) = (2, 4), (3, 2)\) and \((5, 1)\). Again, by observation (ii), \((a_1, a_2) = (2, 4)\) is impossible. \((a_1, a_2) = (3, 2)\) is also impossible because \( a_1 = 3 = E(2p_2 - 2, 1) \) implies \( 2p_2 - 1 = 4 \neq 5 \). But there is no such a \( p_2 \).

Case (III). \( k \) has 3 distinct prime factors, \( k = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \) with \( \beta_i > 0 \).

Here, \( j = 3 \) and \( Q = p_1^{\beta_1} p_2^{\beta_2} 
\geq 2 \times 3 \times 6 = 36 \). The inequality

\[
2j + 1 > 7 > \frac{p_1^{\beta_1} - 1}{\beta_1(p_2 - 1)} + \frac{E(Q, p_3)}{\beta_3}
\]

holds if \((Q, p_3, \beta_3) = (6, p_3, 1)\) for some prime \( p_3 \geq 5 \). That is \( k = 2 \times 3 \times p_3 \). Again, setting \( \tau(k(k-1)) = 2 \tau(k(k-1)) \) yields \( 2(a_1 + a_2 + a_3 + 1) \). From observation (i), we know \( a_1 E(6p_3, 2) > 3p_3 \geq 15 \), \( a_2 = E(6p_3, 3) > 3p_3 \geq 10 \) and \( a_3 = E(6p_3, p_3) \geq 6 \). It is easy to verify that under these conditions \( a_1 a_2 a_3 > 2(a_1 + a_2 + a_3 + 1) \). Case (IV). \( k \) has 4 or more distinct prime factors.

We have \( j \geq 4 \) and

\[
Q = \prod_{i=1}^{j-1} p_i^{\beta_i} \geq 2 \times 3 \times 5 = 30.
\]

Since \( 2j + 1 < Q \), we have

\[
2j + 1 < \frac{p_1^{\beta_1} - 1}{\beta_1(p_2 - 1)} + \frac{E(Q, p_3)}{\beta_3}.
\]
We have just shown that \(\tau(k)\) is not equal to \(\sqrt{k} \tau((k-1))\) for the special cases when \(\beta_l > \alpha_l/2j\). Now let’s show that \(\tau(k!) < \sqrt{e} \tau((k-1)!) < 2\tau((k-1)!)/k\) for any other composite number \(k \neq 4\) if \(\beta_l \leq \alpha_l/2j\). Here \(e = 2.71828\ldots\) is the Euler number.

\[
\tau(k!) = (\alpha_1 + \beta_1 + 1)(\alpha_2 + \beta_2 + 1)\cdots(\alpha_j + \beta_j + 1)(\alpha_{j+1} + 1)\cdots(\alpha_l + 1)
\]

and

\[
\tau((k-1)!) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1)(\alpha_{j+1} + 1)\cdots(\alpha_l + 1).
\]

Since all factors after the \((j+1)\) th term are the same, it suffices if we just look at \((\alpha_1 + \beta_1 + 1)(\alpha_2 + \beta_2 + 1)\cdots(\alpha_j + \beta_j + 1)\) and \((\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1)\). Subsequently, we have

\[
(\alpha_1 + 1 + \beta_1)\cdots(\alpha_j + 1 + \beta_j) < \left(\frac{\alpha_1 + 1 + \alpha_j + 1}{2j}\right)^j = (\alpha_1 + 1)\cdots(\alpha_j + 1)^{1/j}
\]

\[
= (\alpha_1 + 1)(\alpha_1 + 1)^{1/j} = (\alpha_1 + 1)^{1/j + 1}
\]

Since \(\lim \left(1 + \frac{1}{k}\right)^j = \sqrt{e}\) and it is a strictly increasing function, we have

\[
(\alpha_1 + 1 + \beta_1)(\alpha_2 + 1 + \beta_2)\cdots(\alpha_j + 1 + \beta_j) < e(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1) < 2(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1).
\]

In other words, \(\tau(k!) < \sqrt{e} \tau((k-1)!) < 2\tau((k-1)!)/k\).

Combining corollary 1 and theorem 3, we have:

**THEOREM 7.** There are \(\tau(k)/2\) solutions to \(S(x) = k\) if and only if \(k\) is prime or \(k = 4\).

**COROLLARY 8.** \(\tau(k!) = 2\tau((k-1)!)/k\) if and only if \(k\) is prime or \(k = 4\).

**Proof.** This follows from Theorem 7 if we let \(\tau(k)/2 = \tau(k) - \tau((k-1)!)/k\).

**REFERENCES**


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Rex H. Wu is a physician at NYU Downtown Hospital who saw the airplane crashing into the World Trade Center on September 11, 2001. He couldn’t imagine people would do such vicious acts. Only blocks away, he and his colleagues treated a few hundred victims at NYU Downtown Hospital that day. He also volunteered on-site the next couple days. He wishes to express his deepest sorrow to all the innocent lives lost during the attack. And his greatest respect goes to all the heroes on ground zero.

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**CLAYTON W. DODGE RECEIVES THE C. C. MACDUFFEE AWARD FOR DISTINGUISHED SERVICE**

On August 4, 2001, at the annual meeting of Pi Mu Epsilon in Madison, Wisconsin, the C. C. MacDuffee Award for Distinguished Service was presented to Clayton W. Dodge, Professor Emeritus of Mathematics at the University of Maine. The location of this presentation was of historic significance, because Cyrus C. MacDuffee, seventh president of Pi Mu Epsilon, was Professor at the University of Wisconsin. The Award, established in 1966, honors the memory of this superb teacher and algebraist, whose dedication and service profoundly influenced our society. Previous award recipients are J. Sutherland Frame, Richard V. Andree, John S. Gold, Francis Regan, J. C. Eaves, Houston Karnes, Richard Good, Milton D. Cox, and Eileen L. Polani.

Professor Clayton Dodge was an active student at Miss Blakelee’s Kindergarten in Malden, Massachusetts, and his later education was “all downhill from there”. In 1949, he graduated from Browne and Nichols School in Cambridge, Massachusetts, spent a semester at Harvard and eventually graduated from the University of Maine in 1956, majoring in mathematics with minors in electrical engineering and psychology. He labored to teach arithmetic, algebra, and science for a whole six months at Brecksville Junior-Senior High School in Ohio and joyfully returned to teach at the University of Maine as an instructor of mathematics. In 1960 he received a master’s degree in mathematics under Howard Eves, who inspired him to work in geometry and problems. He did graduate work in mathematics at Brown University in 1960-1961.

For two years in the early 1960’s, he assisted Howard Eves in editing the Elementary Problems Department of the American Mathematical Monthly. Later, he served on the University of Maine Problems Group for the seven years that it edited that department. In 1981 he assumed the editorship of the Problem Department of the Pi Mu Epsilon Journal.

With the current issue, Clayton Dodge has completed a remarkable 20 years as Problems Editor of this Journal. Starting with problem #462, (Spring 1980, Volume 7 No. 2) problem proposals were sent to Clayton Dodge, while Leon Bankoff was still problems editor. Transition from Leon Bankoff to Clayton Dodge took place over the period of a year. With the spring 1982 issue his apprenticeship had ended. All problem proposals and solutions were received, handled with care, formulated, polished, checked and corrected by Clayton Dodge, until problem #1006, which was the last problem whose solution was to be sent to the by now so familiar address. The Fall 2000 issue was the start of a new transition. More than half of all the problems published thus far in this Journal have gone through the hands of Clayton Dodge.

He has written five published textbooks, two others that were duplicated for use in his classes and has written several articles primarily on pedagogy, geometry, and calculators. A strong advocate of the use of calculators and computers by students, he wrote text material and taught several courses in their use, emphasizing the understanding of their workings so as to maximize their usefulness and make their results meaningful, see for example [1]. When color came to computers, because there was a great lack of appropriate software, he wrote software for graphing functions in both 2 and 3 dimensions, for demonstrating basic concepts of the calculus, and for grade books, software that gained wide acceptance during the DOS years.

Since retirement he has helped build houses for the local chapter of Habitat for Humanity and he serves on its board of directors. He sings in a choir and an oratorio.
society, and he has taken up the sport of scuba diving in warm tropical waters. He dabbles in stained glass and enjoys working around the house.

For a mathematical project, he is editing notes for a book on the arbelos, written by the late Victor Thebault of France and the late Leon Bankoff of Los Angeles, for 60 years a practicing dentist and PMEJ problems editor from 1968 to 1981. The arbelos is the figure formed as follows. Draw two mutually tangent circles, external to one another and not necessarily the same size. Surround these circles by another circle just tangent to them both. These circles all share a common diametral line. Cut the figure along this line and throw away one half, including the line. The figure that remains, looking like a bent two-tined fork, is the arbelos, also known as the shoemaker’s knife. It may also be described as a triangle whose sides are semicircles and whose angles are all zero degrees.

![Arbelos](image)

We hope that after the transition to the new problems editors is complete, Clayton Dodge will quickly complete his arbelos task, the impatient reader may enjoy a preview in [2].

REFERENCES


Express the value of the following \((n+1)\times(n+1)\) determinant as a product involving linear factors of \(x_i\)'s and \(\alpha_i\)'s.

\[
\begin{vmatrix}
(x_1 + \alpha_1)^n & (x_1 + \alpha_2)^n & \ldots & (x_1 + \alpha_{n+1})^n \\
(x_2 + \alpha_1)^n & (x_2 + \alpha_2)^n & \ldots & (x_2 + \alpha_{n+1})^n \\
\vdots & \vdots & \ddots & \vdots \\
(x_{n+1} + \alpha_1)^n & (x_{n+1} + \alpha_2)^n & \ldots & (x_{n+1} + \alpha_{n+1})^n \\
\end{vmatrix}
\]

**1027. James Chew, North Carolina Agricultural and Technical State University, Greensboro, North Carolina**

Student solutions solicited

Let a jar contain 1 green marble and 9 red marbles, thoroughly mixed. One marble is randomly drawn, and its color is noted. A second jar contains 2 green marbles and 8 red marbles. One marble is drawn from the second jar and again the color is noted. Repeat this process until a fifth marble has been drawn from the jar containing 5 green and 5 red marbles. Let \(X\) = the number of green marbles drawn. Calculate \(P(X = i), i = 0, 1, 2, \ldots, 5\).

A local newspaper gives probabilities of rain for the next 5 days as: 100%, 20%, 30%, 40%, 50%. Use the marbles-in-the-jar model to determine the probability of a) exactly two days of rain, b) at least two days of rain.

**1028. Proposed by Editors.**

As a modification of #1027, explain how to modify the model in problem 1027 so that the assumption of independence is removed. Based on your new model, determine the probability of getting a) exactly two days of rain, b) at least two days of rain.

*1029. Proposed by Ice B. Risteski, Skopje, Macedonia.*

If \(P\) and \(Q\) denote the linear differential operators

\[
P = \sum_{i=0}^{m} p_i(x)D^i, \quad Q = \sum_{j=0}^{n} q_j(x)D^j, \quad (D = \frac{d}{dx})
\]

show that

\[
QP = \sum_{s=0}^{m+n} r_s(x)D^s,
\]

where

\[
r_s(x) = \sum_{j=\text{max}(0,s-m)}^{n} \left\{ \sum_{i=\text{max}(0,s-j)}^{\text{min}(s,m)} \binom{s}{j-s} p_i(x)q_j(x) \right\}
\]

**1030. Proposed by Ayoub B. Ayoub, Pennsylvania State University Abington College, Abington, Pennsylvania**

On the sides of an arbitrary triangle \(ABC\), three equilateral triangles, \(A_1BC\), \(AB_1C\), and \(ABC_1\) are drawn outward. Then on the sides of the triangle \(A_1B_1C_1\), another three equilateral triangles \(A_2B_1C_1\), \(A_1B_2C_1\), and \(A_1B_1C_2\) are drawn outward relative to the triangle \(A_1B_1C_1\). Show that each set of points \(\{A_2, A, A_1\}\), \(\{B_2, B, B_1\}\), and \(\{C_2, C, C_1\}\) lie on a straight line and that the three lines meet in one point.

**1031. Proposed by Andrew Cusumano, Great Neck, New York**

Notice that

\[
\sqrt{9 + \sqrt{80}} + \sqrt{9 - \sqrt{80}} = 3
\]

and

\[
\sqrt{161 + \sqrt{25920}} + \sqrt{161 - \sqrt{25920}} = 7.
\]

Generalize this by showing that

\[
\sqrt{\frac{x^3 - 3x}{2}} + \frac{\sqrt{2x^2 - 4}}{2} + \frac{\sqrt{2x^2 - 4} - (x^2 - 1)\sqrt{2x^2 - 4}}{2} = x.
\]

**1032. Proposed by Robert C. Gebhardt, Hopatcong, New Jersey**

Consider an equilateral triangle with sides of length 1 unit, as shown below. From an arbitrary interior point \(P\), draw perpendiculars \(PQ, PR, \text{and } PS\). Find the sum of the lengths of \(PQ, PR, \text{and } PS\).

**1033. Proposed by Kenneth B. Davenport Frackville, Pennsylvania**

Student solutions solicited

Show that

\[
2 \sin(2\theta) - 3 \sin(\theta) = -\left(\frac{\tan \left(\frac{\theta}{2}\right)}{1 - \cos \left(\frac{\theta}{2}\right) - 2 \cos(2\theta)}\right)
\]

for all values of \(\theta\) where both sides are defined.
Solutions.

994. [Fall 2000] Proposed by the editor.

Although the alphanumeric BRENNER = (JOEL)$^2$ has no solution in base ten, there is a number $M$ such that BRENNER is the square of a positive integer $x$ in every base greater than or equal to $M$. Furthermore, the same four digits are used for $B$, $R$, $E$, and $N$ in each such base. Find these digits, the value of $M$, and the digits of $x$, the square root of BRENNER.


In a computer search of over 36,000 cases in bases ten or greater, eighteen instances were found where BRENNER was the square of a 4-digit number. In each such BRENNER in a given base $M$ where a carry occurred in squaring $x$, the number formed by that BRENNER was not a square in base $M + 1$. It is easy to see that, in fact, BRENNER is to be a square in every base larger than $M$, no carry can occur in the squaring; carries will not occur when the base becomes large enough. In the six cases where BRENNER was a square and no carry occurred in the squaring, BRENNER was a square in all larger bases.

Thus if the 4-digit number $x$ is abcd in base $M$ and if there are no carries in the squaring, we must have $B = a^2$, $R = 2ab$, $E = 2ac + b^2$, $N = 2ad + 2bc$, $N' = 2bd + c^2$, $E' = 2ad$, and $R = d^2$. We must therefore have $2ab = d^2$, $2ac + b^2 = 2ad$, and $2ad + 2bc = 2bd + c^2$, whose unique solution in positive integers is $b = c = d = 2a$. Then we have, for any base $M \geq 12a^2 + 1$,

$$x^2 = (a, 2a, 2a, 2a)^2 = (a^2, 4a^2, 8a^2, 12a^2, 12a^2, 8a^2, 4a^2)^2 = BRENNER.$$  

The smallest solution is $x^2 = (1, 2, 2, 2)^2 = (1, 4, 8, 12, 12, 8, 4) = BRENNER$ in all bases $M \geq 13$.

II. Comment by Kenneth M. Wilke, Topeka, Kansas.

Note that $(1,2,2,2)$ in base $M$ is equal to $(1,5,9,7)$ in base $M - 1$, so we can say that we do have the pseudo-solution (BRENNER) = $(1,4,8,12,12,8,4)$ in base $M$ is the square of the (JOEL) = $(1,5,9,7)$ in base $M - 1$, where, unfortunately, $E \neq E'$.  

III. Comment by Mark Evans, Louisville, Kentucky.

I found the following solutions to the equation (JOEL)$^2 = BRENNER$:

$$(4, 1, 7, 6)^2 = (16, 10, 7, 12, 12, 7, 10), (3, 51, 27, 8)^2 = (15, 6, 27, 44, 44, 27, 6), (6, 5, 52, 26)^2 = (36, 68, 52, 35, 35, 52, 68), (5, 81, 68, 27)^2 = (35, 57, 68, 53, 53, 68, 57), (8, 41, 24, 81)^2 = (70, 21, 24, 60, 60, 24, 21)$$


a) Consider the geometric-arithmetic recursive sequence $f$ given by

$$f(1) = a, f(2) = ar + d, \text{ and } f(i) = rf(i - 1) + d \text{ for } i \geq 2,$$

where $a$, $d$, and $r$ are nonzero constants, $r \neq 1$, and $i$ is an integer. Express $\sum_{i=1}^{n} f(i)$ in closed form.

b) Consider the arithmetic-geometric recursive sequence $g$ given by

$$g(1) = a, g(2) = r(a + d), \text{ and } g(i) = r(g(i - 1) + d) \text{ for } i \geq 2,$$

where $a$, $d$, and $r$ are nonzero constants, $r \neq 1$, and $i$ is an integer. Express $\sum_{i=1}^{n} g(i)$ in closed form.

Solution by Ovidiu Purdui, student, Western Michigan University, Kalamazoo, Michigan.

a) From the recursion formula we see that

$$\sum_{i=1}^{n} f(i) = \frac{r}{1 - r} \left[ a - r \left( \frac{a r^{n-1} + d r^{n-1} - 1}{r - 1} \right) \right] + d(n - 1),$$

from which it follows that

$$1 - r \sum_{i=1}^{n} f(i) = f(1) - r f(n) + d(n - 1).$$

It is easy to observe and prove by mathematical induction that

$$f(n) = ar^{n-1} + d + dr + dr^2 + \cdots + dr^{n-2} = ar^{n-1} + d \frac{r^{n-1} - 1}{r - 1}.$$

We combine these latter two equations to find that

$$\sum_{i=1}^{n} f(i) = \frac{1}{1 - r} \left[ a - ar^n - dr^{n-1} - d(n - 1) \right],$$

which reduces to

$$\sum_{i=1}^{n} f(i) = \frac{1}{1 - r} \left[ a - ar^n - dr^{n-1} - d(n - 1) \right].$$

b) If one replaces $d$ by $rd$ in the definition formulas of part (a), one obtains the formulas for part (b). Hence the solution is found to be

$$\sum_{i=1}^{n} g(i) = \frac{1}{1 - r} \left[ a - ar^n - dr^n - d \frac{r^{n-1} - 1}{r - 1} + dr(n - 1) \right].$$

996. [Fall 2000] Proposed by Ice B. Ristaski, Skopje, Macedonia.
If \( P_n(x) \) is the Legendre polynomial, given by \( P_0(x) = 1 \) and for positive integral \( n, \)
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,
\]
show that
\[
nP_n(cos x) = \sum_{m=0}^{n} \cos(mx)P_n(mcos x).
\]

Solution by Paul S. Bruckman, Sacramento, California.
The generating function of the Legendre polynomials is
\[
\theta^{-1} = \sum_{n=0}^{\infty} P_n(x) t^n, \text{ where } \theta = \theta(t,x) = (1 - 2tx + t^2)^{1/2}.
\]
We also note that
\[
\int \frac{1 - t \cos x}{1 - 2t \cos x + t^2} \frac{dt}{\theta(t,x)} = \frac{1 - t \cos x}{\theta(t,x)} = \frac{1}{2} \left( \frac{1}{1 - te^{-i \omega}} + \frac{1}{1 - te^{i \omega}} \right)
\]
\[
= \frac{1}{2} \sum_{n=0}^{\infty} (e^{i \omega} + e^{-i \omega})t^n = \sum_{n=0}^{\infty} t^n \cos nx.
\]
Therefore, letting \( \phi = \phi(t,x) = \theta(t,x) \cos x \), we see that \( \phi^{-1} = \sum_{n=0}^{\infty} t^n P_n(cos x) \) and
\[
(1 - t \cos x)\phi^{-3} = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} P_n(mcos x) \cos nx.
\]
On the other hand, by differentiating \( \phi^{-1} = \sum_{n=0}^{\infty} t^n P_n(cos x) \) with respect to \( t \), we obtain
\[
(t \cos x - t^2)\phi^{-3} = \sum_{n=0}^{\infty} nt^n P_n(cos x).
\]
Note that
\[
(t \cos x - t^2)\phi^{-3} + \phi^{-1} = (t \cos x - t^2 + 1 - 2t \cos x + t^2)\phi^{-3} = (1 - t \cos x)\phi^{-3}
\]
\[
= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} P_n(mcos x) \cos nx.
\]
By comparison of coefficients we get
\[
(n + 1)P_n(cos x) = \sum_{m=0}^{n} P_n(mcos x) \cos nx.
\]
Now subtracting \( P_n(cos x) \) from both sides of (2), which is the term for \( m = 0 \) in the right side of (2), we obtain the desired identity.

Also solved by Brian Bracide, Christopher Newport University, Newport News, VA, Ovidiu Furdui, Western Michigan University, Kalamazoo, H.-J. Seiffert, Berlin, Germany, and the Proposer.

Evaluate the integral
\[
\int_{-\infty}^{\infty} \frac{\ln(9 - x)dx}{\ln(9 - x) + \ln(x - 3)}.
\]
I. Solution by Sophie Trawalter, student, University of North Carolina at Wilminton, Wilmington, North Carolina.
Let \( I \) denote the given integral. Making the successive substitutions \( x = y + 6 \), so \( dx = dy \), and then \( u = -y \), so \( du = -dy \), we find that
\[
I = \int_{-\infty}^{\infty} \frac{\ln(3 - y)dy}{\ln(3 - y) + \ln(3 + y)} = \int_{-\infty}^{\infty} \frac{\ln(3 + u)du}{\ln(3 + u) + \ln(3 - u)}.
\]
Now the integral \( I \) must equal the average of these two integrals. That is,
\[
I = \frac{1}{2} \int_{-\infty}^{\infty} dx = 2.
\]
II. Solution by Kristen Klingensmith, Danielle Quinn, Thomas Renken, James Slayton, Sherly Webber, jointly, students, SUNY Fredonia, Fredonia, New York.
From its graph, the integrand appears to be symmetric about the point \((6, \frac{1}{2})\).
So make the substitution \( y = x - 6 \), obtaining the first integral shown in Solution I above. Let \( f(y) \) denote the new integrand less \( 1/2 \). We show that \( f \) is an odd function. Thus
\[
f(-y) = \frac{\ln(3 + y) - \ln(3 - y)}{\ln(3 + y) + \ln(3 - y)} = \frac{1}{2} \frac{\ln(3 + y) + \ln(3 - y) - \ln(3 - y) - \ln(3 + y)}{\ln(3 + y) + \ln(3 - y) - \ln(3 - y) - \ln(3 + y)}
\]
\[
= -f(y).
\]
Hence \( f \) is symmetric about the origin and therefore the integral is equal to \( 2 \), the area of a rectangle with base 4 and height \( 1/2 \).
For nonnegative integers $k$ and $n$, let

$$J_{kn} = \frac{1}{(1+k)^2} \binom{n}{0} - \frac{1}{(2+k)^2} \binom{n}{1} + \cdots + \frac{(-1)^n}{(n+k+1)^2} \binom{n}{n}.$$ 

a) Determine the value of $b_k$ such that the limit $L_k$ exists, where

$$L_k = \lim_{n \to \infty} [(n+1)(n+2) \cdots (n+k+1)J_{kn} - b_k \ln(n+1)].$$

b) Evaluate $L_k$ using your value of $b_k$ and the definition of Euler's constant $\gamma$ given by

$$\gamma = \lim_{n \to \infty} \left[ \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \ln n \right] = 0.577\ldots$$

c) Using your results of parts (a) and (b), evaluate, if it exists,

$$\lim_{k \to \infty} \left( \frac{L_k}{k!} + \ln k \right).$$

Solution by H.-J. Seiffert, Berlin, Germany.

If $\Gamma(x)$ denotes the gamma function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, so that $\Gamma(n+1) = n!$ for positive integral $n$, then (see R. L. Graham, D. E. Knuth, and O. Patashnik, "Concrete Mathematics," 2nd ed., Addison-Wesley, 1994, p. 188, eqn. 5.41)

$$\sum_{j=0}^n \frac{(-1)^j}{j+x} \binom{n}{j} = \frac{n! \Gamma(x)}{\Gamma(n+x+1)}, \quad x > 0.$$

Differentiating with respect to $x$ and multiplying the result by $-1$ yields

$$\sum_{j=0}^n \frac{(-1)^j}{(j+x)^2} \binom{n}{j} = \frac{n! \Gamma(x)}{\Gamma(n+x+1)} (\psi(x+n+1) - \psi(x)), \quad x > 0,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Taking $x = k+1$, we find

$$J_{kn} = \frac{n!}{(n+k+1)!} (H_{n+k+1} - H_k),$$

where $H_m = \sum_{r=1}^m 1/r$ is the $m$'th harmonic number; empty sums have the value zero.

a) We have

$$(n+1)(n+2) \cdots (n+k+1)J_{kn} - b_k \ln(n+1)$$

$$= \frac{(n+k+1)!}{n!} J_{kn} - b_k \ln(n+1)$$

$$= k!(H_{n+k+1} - H_k) - b_k \ln(n+1)$$

$$= k!(H_{n+k+1} - \ln(n+k+1) - H_k) + k! \ln \left( \frac{n+k+1}{n+1} \right) + (k! - b_k) \ln(n+1).$$

Hence, the limit $L_k$ exists only when $b_k = k!$.

b) If $b_k = k!$, then $L_k = k!(\gamma - H_k)$.“

PROBLEM DEPARTMENT

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998. [Fall 2000] Proposed by David Iny, Baltimore, Maryland.

For nonnegative integers $k$ and $n$, let

$$J_{kn} = \frac{1}{(1+k)^2} \binom{n}{0} - \frac{1}{(2+k)^2} \binom{n}{1} + \cdots + \frac{(-1)^n}{(n+k+1)^2} \binom{n}{n}.$$ 

a) Determine the value of $b_k$ such that the limit $L_k$ exists, where

$$L_k = \lim_{n \to \infty} [(n+1)(n+2) \cdots (n+k+1)J_{kn} - b_k \ln(n+1)].$$

b) Evaluate $L_k$ using your value of $b_k$ and the definition of Euler's constant $\gamma$ given by

$$\gamma = \lim_{n \to \infty} \left[ \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \ln n \right] = 0.577\ldots$$

c) Using your results of parts (a) and (b), evaluate, if it exists,

$$\lim_{k \to \infty} \left( \frac{L_k}{k!} + \ln k \right).$$

Solution by H.-J. Seiffert, Berlin, Germany.

If $\Gamma(x)$ denotes the gamma function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, so that $\Gamma(n+1) = n!$ for positive integral $n$, then (see R. L. Graham, D. E. Knuth, and O. Patashnik, "Concrete Mathematics," 2nd ed., Addison-Wesley, 1994, p. 188, eqn. 5.41)

$$\sum_{j=0}^n \frac{(-1)^j}{j+x} \binom{n}{j} = \frac{n! \Gamma(x)}{\Gamma(n+x+1)}, \quad x > 0.$$

Differentiating with respect to $x$ and multiplying the result by $-1$ yields

$$\sum_{j=0}^n \frac{(-1)^j}{(j+x)^2} \binom{n}{j} = \frac{n! \Gamma(x)}{\Gamma(n+x+1)} (\psi(x+n+1) - \psi(x)), \quad x > 0,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Taking $x = k+1$, we find

$$J_{kn} = \frac{n!}{(n+k+1)!} (H_{n+k+1} - H_k),$$

where $H_m = \sum_{r=1}^m 1/r$ is the $m$'th harmonic number; empty sums have the value zero.

a) We have

$$(n+1)(n+2) \cdots (n+k+1)J_{kn} - b_k \ln(n+1)$$

$$= \frac{(n+k+1)!}{n!} J_{kn} - b_k \ln(n+1)$$

$$= k!(H_{n+k+1} - H_k) - b_k \ln(n+1)$$

$$= k!(H_{n+k+1} - \ln(n+k+1) - H_k) + k! \ln \left( \frac{n+k+1}{n+1} \right) + (k! - b_k) \ln(n+1).$$

Hence, the limit $L_k$ exists only when $b_k = k!$.

b) If $b_k = k!$, then $L_k = k!(\gamma - H_k)$.”

999. [Fall 2000] Proposed by the late Jack Garfunkel, Flushing, New York.

Prove that

$$\frac{r_1 + r_2 + r_3}{3 + \sqrt{3}} \leq 9$$

with equality when $r_1 = r_2 = r_3$, where $r$ is the inradius of triangle $ABC$ and $r_1$, $r_2$, and $r_3$ are the radii of the mutually tangent circles in the Malfatti configuration, shown in the accompanying figure.

Solution by Miguel Amengual Covas, Cala Piguera, Mallorca, Spain.

Using the relation

$$\sum_{i=1}^3 \left( \frac{1}{x_i} + \frac{1}{y_i} + \frac{1}{z_i} \right) = \frac{n!}{\Gamma(n+x+1)} \prod_{i=1}^3 (x_i y_i z_i),$$

which is equivalent to

$$\frac{1}{\sqrt{r_1 r_2 r_3}} \leq \frac{1}{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}},$$

we have

$$r \leq \frac{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}}{9} \left( \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 r_2 + r_1 r_3 + r_2 r_3} \right).$$

Finally, using Cauchy's inequality $(ax + by + cz)^2 \leq (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ with $a = b = c = 1$ and $x$, $y$, $z$ as above, we see that

$$\frac{1}{\sqrt{r_1 r_2 r_3}} \leq 3 \left( \frac{r_1 + r_2 + r_3}{9} \right).$$

By combining (1) and (2) we have the desired result. It is easy to see that equality holds if and only if $r_1 = r_2 = r_3$.

Also solved by Paul S. Bruckman, Sacramento, CA, Ovidiul Furdui, Western Michigan University, Kalamazoo, and the Proposer.
1000. [Fall 2000] Proposed by Albert White, St. Bonaventure University, St. Bonaventure, New York.

Let $ABCD$ be a parallelogram with $\angle A = 60^\circ$. Let the circle through $A$, $B$, and $D$ intersect $AC$ again at $E$ and let $AC$ and $BD$ meet at $H$. See the figure.

![Parallelogram and Circle](image)

Let $[PQR]$ denote the area of triangle $PQR$. Show that

a) $[DHE] \cdot (AC)^2 = [ADH] \cdot (DB)^2$,

b) $[ADE] - [DEC] = 2[DHE]$, and

c) $2[H(E) \cdot (AC)] = (DB)^2$.

I. Solution to parts (a) and (c) by Ovidui Furdui, student, Western Michigan University, Kalamazoo, Michigan.

We note that the size of angle $A$ is irrelevant.

a) We observe that $AC$ and $DB$ bisect one another, so that $AH = AC/2$ and $DH = HB = DB/2$. Also $HE = (DB)^2/(AH)$ because $AE$ and $DB$ are intersecting chords of the circle. Then $HE = (DH)\cdot(HB)/(AH) = (DB)^2/(4AH)$. Since the two triangles $DHE$ and $ADH$ have the same altitude from vertex $D$, we have $[DHE]/[ADH] = (HE)/(AH) = (DB)^2/(4AH)^2 = (DB)^2/(AC)^2$, which yields the desired equation.

c) From part (a) we have $HE = (DB)^2/(4AH)$, so $2[H(E)\cdot(AC)] = 4[HE]\cdot(AH) = (DB)^2$.

II. Solution to part (b) by Brian Bradie, Christopher Newport University, Newport News, Virginia.

b) Since $[ADH] = [DHC] - [DEC]$, then $[ADE] - [DEC] = [ADH] + [DHE] - [DEC] = 2[DHE]$.


The Euler numbers $E_n$, for $n = 0, 1, 2, \ldots$, are defined by

$$\text{sech } z = \frac{1}{\cosh z} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}$$

so that $E_n = 0$ for all odd $n$, $E_0 = 1$, $E_2 = -2$, $E_4 = 5$, $E_5 = -61$, etc. Prove the following relations:

\[a) \sum_{j=0}^{2m} \binom{4m}{2j} E_{2j} = 2 \sum_{k=0}^{m} \frac{4m}{4k} E_{4k} \text{ for } m = 1, 2, 3, \ldots; \]
\[b) \sum_{j=0}^{2m} \binom{4m+2}{2j} E_{2j} = 2 \sum_{k=0}^{m} \frac{(4m+2)}{4k} E_{4k} \text{ for } m = 0, 1, 2, \ldots; \]
\[c) \sum_{j=0}^{2m} \binom{4m}{2j} E_{2j} = -2 \sum_{k=0}^{m} \frac{4m}{4k-2} E_{4k-2} \text{ for } m = 1, 2, 3, \ldots; \]
\[d) \sum_{j=0}^{2m} \binom{4m+2}{2j} E_{2j} = -2 \sum_{k=0}^{m} \frac{(4m+2)}{4k+2} E_{4k+2} \text{ for } m = 0, 1, 2, \ldots. \]

Solution by Paul S. Bruckman, Sacramento, California.

a) For convenience we define $E_n = (1 + (-1)^n)/2$ as the characteristic function of the even integers. Since $\cosh z = \sum_{n=0}^{\infty} E_n x^n/n!$, it follows by convolution with the series for $\text{sech } x$ that

\[(1) \quad \sum_{k=0}^{n} \frac{E_{n-k}}{k!} E_k = 0, \quad n = 1, 2, \ldots \]

The functions $\cosh x$ and $\text{sech } x$, however, are even, which implies that

\[(2) \quad \sum_{k=0}^{n} \frac{E_{n-k}}{k!} E_k = 2 \quad n = 1, 2, \ldots \]

It is also known that $E_{2k} = (-1)^k E_{2k}$. Note that (2) implies

\[\sum_{j=0}^{2m} \binom{4m}{2j} E_{2j} = \sum_{j=0}^{2m+1} \binom{4m+2}{2j} E_{2j} = 0, \quad m = 1, 2, \ldots \]

Now we have

\[2 \sum_{k=0}^{m} \frac{2m}{2k} E_{2k} = 2 \sum_{k=0}^{m} \frac{2m}{2k} E_{2k+4k} \]
\[= 2 \sum_{k=0}^{m} \binom{4m}{2j} E_{2j} + 2 \sum_{j=0}^{m} \binom{4m+2}{2j} E_{2j} \]
\[= 0 + \sum_{j=0}^{2m} \binom{4m+2}{2j} E_{2j}, \]

which is part (a).

b) If $m > 0$, then we have

\[\sum_{k=0}^{m} \binom{4m+2}{2k} E_{2k} = 2 \sum_{k=0}^{m} \binom{4m}{2k} E_{4k+2k} \]
\[= \sum_{k=0}^{m} \binom{4m+2}{2j} E_{2j} + 2 \sum_{j=0}^{m} \binom{4m+2}{2j} E_{2j} \]
\[= -E_{2m+2} + 0 + \sum_{j=0}^{2m} \binom{4m+2}{2j} E_{2j}, \]
which reduces to part (b).

c) Next, let \( c_n = 1 - c_0 = \frac{1 - (-1)^n}{2} \) be the characteristic function of the odd integers. Note that

\[
-2 \sum_{k=1}^{\infty} \left( \frac{4m}{4k - 2} \right) E_{4k-2} = -2 \sum_{j=1}^{2m-1} \left( \frac{4m}{2j} \right) E_{2j} c_j
\]

\[
= \sum_{j=1}^{2m-1} \left( \frac{4m}{2j} \right) E_{2j} - \sum_{j=1}^{2m-1} \left( \frac{4m}{2j} \right) E_{2j} = 0
\]

which is part (c), for \( m \geq 1 \).

d) Finally, if \( m \geq 0 \), we have that

\[
-2 \sum_{k=0}^{\infty} \left( \frac{4m+2}{4k+2} \right) E_{4k+2} = -2 \sum_{j=0}^{2m+1} \left( \frac{4m+2}{2j} \right) E_{2j} c_j
\]

\[
= \sum_{j=0}^{2m+1} \left( \frac{4m+2}{2j} \right) E_{2j} - \sum_{j=0}^{2m+1} \left( \frac{4m+2}{2j} \right) E_{2j} = 0
\]

which is part (d).

Also solved by Ovidui Furdui, Western Michigan University, Kalamazoo, and the Proposer.


Let \( n \) be a composite integer greater than or equal to 48. Prove that between \( n \) and \( S(n) \) there exist at least five primes, where \( S(n) \) is the Smarandache function: for any positive integer \( n, k = S(n) \) if \( k \) is the smallest positive integer such that \( n \) divides \( k \). Then, for example, \( S(3) = 3 \) and \( S(8) = 4 \).

Comment by H.-J. Seiffert, Berlin, Germany.

This result was posed by the same proposer and proved by N. J. Kuenzi and B. Prellipp in Problem 4541, "School Science and Mathematics," vol. 96, no. 7, 1996, p. 392.

Also solved by Paul S. Bruckman, Sacramento, CA, Rex H. Wu, Brooklyn, NY, and the Proposer.

1003. [Fall 2000] Proposed by I. M. Radu, Bucharest, Romania.

Show that between \( S(n) \) and \( S(n+1) \), where \( S(n) \) is the Smarandache function, there exists at least one prime number. See Problem 1002 for the definition of the Smarandache function.

I. Comment by Paul S. Bruckman, Sacramento, California.

As it stands, the conjecture is false. We find the following counterexamples for \( n \leq 100: n = 2, 3, 4, 5, 7, 14, 15, 20, 21, 27, 32, 35, 51, 54, 55, 63, 65, 99 \) and 99. For example, \( S(54) = 9, S(55) = 11, \) and \( S(56) = 7 \) and there are no primes between 9 and 11 or between 11 and 7. It may be conjectured that the conjecture is false for infinitely many \( n \), although this has not been established. It is not clear what the proposer had in mind.

Editorial comment. The proposer assumed weak inequalities in the comments he made about his conjecture, which this editor overlooked, so the prime 11 would count for him in both cases.

II. Disproof by Rex H. Wu, Brooklyn, New York.

Assuming weak inequalities were intended in the proposal, all the counterexamples up to \( 1,000,000 \) are given on pages 52 and 53 in the book "An Introduction to the Smarandache Function" by Charles Ashbacher, which can be downloaded at http://www.gallup.unm.edu/~smarandache/Smf.pdf. There are \( S(224) = 8, S(225) = 10, S(2057) = 22 \) and \( S(205225) = 206 \) and \( S(845637) = 302 \) and \( S(845638) = 298 \).


Find the minimum value of \( u_n = x_1 + x_2 + \cdots + x_n \) if the \( x_k \) are all nonnegative and

\[
\sum_{k=1}^{n} \cos^2 x_k = 1.
\]

Solution by William H. Peirce, Rangeley, Maine.

Clearly the \( x_k \) can be restricted to angles in the first quadrant. Both the constraint and the function \( u_n \) are symmetric functions of the \( x_k \), and when this symmetry exists, any internal extremum of \( u_n \) occurs at a point where the \( x_k \) are all equal to, say, \( x \). Therefore \( n \cos^2 x = 1 \), so \( x = \arccos(1/\sqrt{n}) \) and \( u_n = n \arccos(1/\sqrt{n}) \) is a candidate for an extremum of \( u_n \). It is easily verified that any perturbation of the \( x_k \) from \( x \) produces a larger value for \( u_n \), so we have indeed found the minimum for any \( n \geq 2 \).

If \( n = 2 \), then \( x_1 = x_2 = 0 \). We see that in all cases, then, we have \( u_n = n \arccos(1/\sqrt{n}) \).

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA, Paul S. Bruckman, Sacramento, CA, Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Mark Evans, Louisville, KY, Ovidiul Furdui, Western Michigan University, Kalamazoo, Richard I. Hess, Rancho Palos Verdes, CA, Joe Howard, Portales, New Mexico, Rex H. Wu, Brooklyn, NY, and the Proposer.


Prove that if \( n > 2 \) is an odd number,

\[
\sum_{k=1}^{(n-1)/2} \sin \frac{4k\pi}{n} = \sin \frac{4\pi}{n} + \sin \frac{8\pi}{n} + \cdots + \sin \frac{2(n-1)\pi}{n} < 0.
\]
I. Solution by Kim Thielke, student, Alma College, Alma Michigan.
For any natural number \( m \) it is easy to show that
\[
\sum_{k=1}^{m} \sin(kx) = \frac{\cos(x/2) - \cos((2m + 1)x/2)}{2\sin(x/2)}.
\]
Now let \( n \) be an odd number greater than or equal to 3. We find that
\[
\sum_{k=1}^{(n-1)/2} \sin(4k\pi/n) = \frac{\cos(2\pi/n) - \cos(2\pi)}{2\sin(2\pi/n)} = \frac{\cos(2\pi/n) - 1}{2\sin(2\pi/n)}.
\]
Since \( n \geq 3 \), the denominator is positive and the numerator negative. The theorem follows.

II. Solution by J. Ernest Wilkins, Jr., Clark Atlanta University, Atlanta, Georgia.
Let \( n = 2p + 1 \), where \( p \) is a positive integer and define \( x = 4\pi/n \), \( y = 2\pi - x/2 = \pi x \), and \( z = \exp(ix) \). Then the indicated sum \( S \) is the imaginary part of
\[
\sum_{k=1}^{p} z^k = \frac{e^\pi - 1}{1 - e^{-x}}.
\]
Therefore,
\[
2S = \sin(x) + \sin(y) - \sin(x + y).
\]
Because \( x = 4\pi - 2y \), it is clear that \( \sin x = -\sin 2y = -2\sin y \cos y \), that \( \sin(x + y) = -\sin y \), and that \( (1 - \cos x)S = (1 - \cos y)\sin y \). It follows that
\[
S = \frac{1}{2} \tan \frac{y}{2}.
\]
We conclude that \( S < 0 \) because \( 0 < x \leq 4\pi/3 \) and \( 2\pi/3 < y/2 < \pi \).


a) How many aces can be served in one game of tennis?
b) How many consecutive aces can be served in one game of tennis?
c) You and I are playing a set of tennis. In the last 8 points you have served 7 aces and I have served 1. What is our score?
d) In a tennis match you have just served aces on 6 consecutive points. What is the score?

Solution by Skidmore College Problem Group, students, Saratoga Springs, New York.
a) \( \infty \). In principle, a game at deuce could see an infinite number of alternating aces and points lost by the server.
The MATHACROSTIC in this issue has been contributed by Dan Hurwitz.

a. Belief in third powers
b. Doing Gauss-Jordan Steps
   (2 wds.)
c. Thumbless C.S. conversion
   (2 wds.)
d. British philosopher/mathematician
   (1861-1947)
e. One-to-one correspondence
   with the natural numbers
f. A prime date, when
   available (hyph.)
g. Function on larger domain
h. Empty
i. Ten per cent
j. Proof introductory clause
k. Planes including a given point
l. Round the clock calculations
   (2 wds.)
m. They depart from the general pattern
n. Fixed point subscripts
o. Juxtaposed
p. Doable procedures are this
q. Needed at Monte Carlo
   (hyp.)
r. Singular example of
   exponential growth (3 wds.)
s. Translated Euclid into Arabic

t. Switching circuits state
u. Found on inner product spaces
v. Properly contained in the set of all sets

Last month's mathacrostic was taken from "Indiscrete Thoughts" by Gian-Carlo Rota.

The full text of the quote is:

"Mathematicians have to attend (secretly) physics meetings in order to find out what is going on in their fields. [Physicists have the P.R., the savoir-faire, and the chutzpah to write readable, or at least legible accounts of subjects that are not yet obsolete, something few mathematicians would dare do, fearing expulsion from the A.M.S.]

Jeanette Bickley was the first solver.
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