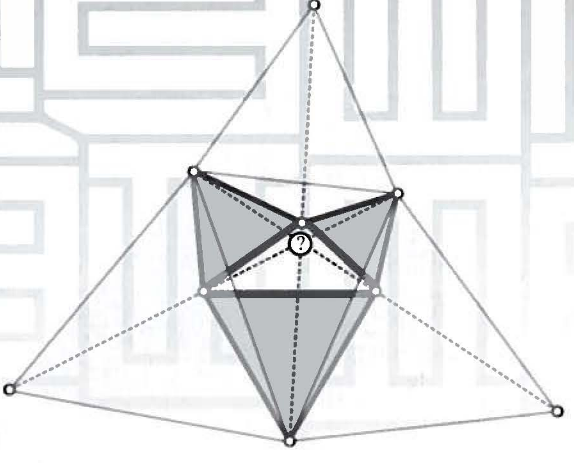

Is there a point to bi-tri-angling?

Problem 1030

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GENERAL FLIP-SHIFT GAMES

JAE GYUN CHEONG, MICHAEL A. JONES AND KEI KANEKO*

Abstract. We examine two puzzles by thinking of the movements as permutations. We then generalize the puzzle as a "flip-shift" game and determine generalizations which yield all permutations of the pieces, as well as some that do not yield all possible permutations.

1. Introduction. Mathematics has often been at the heart of puzzles, including the sliding piece puzzle known as the 14-15 puzzle (Fig. 1). This puzzle has been sold with the 14 and 15 in their correct positions. However, the variation in Fig. 1 was made popular by Sam Loyd, who in the late 1800's offered a cash prize to the first person who could place the square tiles in order. Loyd never had to pay out the cash prize because the pieces of the puzzle in Fig. 1 cannot be put in order by legal moves (sliding a piece into the empty space). (For a history of sliding piece puzzles, see Hordern [1].)

Mathematicians have analyzed this puzzle (including Johnson [2] in 1879) and its variations, *e.g.*, Liebeck [3]. Other puzzles have generated mathematical interest as well, including Rubik's cube. A mathematical generalization of Rubik's cube, called Rubik's tesseract, is analyzed in Velleman [4].

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

FIG. 1. *Sam Loyd's 14-15 Puzzle. Goal: Switch the 14 and 15.*

We begin by examining two puzzles, Xex No. Crunch and the Saturn Puzzler. These puzzles are similar to the 14-15 puzzle in that numbered pieces can be moved and arranged in different orders. However, unlike the 14-15 puzzle, there is no empty space limiting the next move. For this reason, Xex No. Crunch and the Saturn Puzzler are similar to Rubik's cube. There are two possible moves, shifts and flips. We show that all permutations of the pieces are possible in the Saturn Puzzler and Xex No. Crunch puzzles.[1] For certain generalized flip-shift games, the solution technique used for the two specific puzzles can be extended to generate all permutations. However, we prove that not all permutations of the puzzle pieces are possible for all generalized flip-shift games.

2. Two Flip-Shift Puzzles. Xex No. Crunch consists of 20 movable disks arranged along an oval track. The pieces can be rotated or shifted in the left or right direction (Fig. 2). A "shift" changes the location of all the pieces, but preserves their

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[1]The packaging that comes with Xex No. Crunch includes techniques on how to move disks in certain ways to help yield solutions to its variations.

order; specifically, a left shift moves all disks one disk in the counterclockwise direction, while a right shift moves all disks one disk clockwise. The oval track intersects a turnstile that can be used to "flip" the ordering of four of the disks (Fig. 3).

One variation of Xex No. Crunch is to place the 20 disks in numerical order, as pictured in Fig. 2. We do not designate the orientation of the turnstile and consider a question that Sam Loyd would find of interest: Are all 20! permutations of the disks possible? If the answer is "yes," then any initial arrangement of the 20 disks can be transformed through a sequence of flips and shifts to the numerical ordering. Of course, this means that Sam Loyd cannot pre-order the disks into an arrangement that cannot be solved! We show that all 20! permutations are possible.

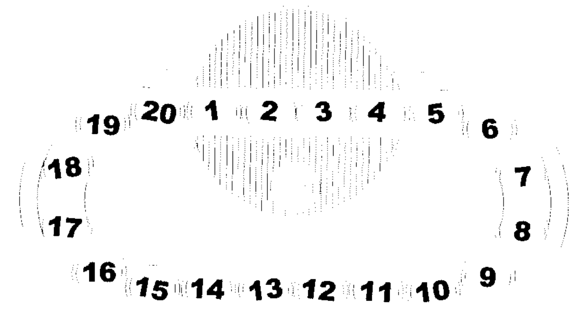

FIG. 2. *Left and right shifts for Xex No. Crunch consist of 'shifting' the disks along the oval track.*

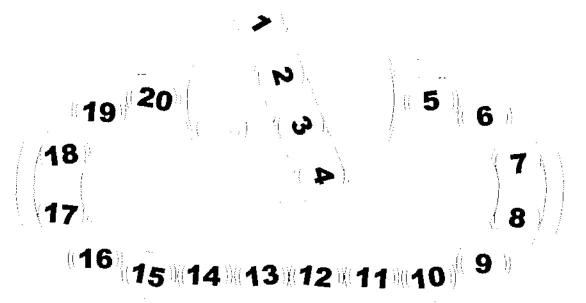

FIG. 3. *Flip for Xex No. Crunch consists of 'flipping' along the turnstile.*

Before explaining how to arrive at all 20! permutations, we develop notation to effectively describe the puzzle mathematically. It is convenient to use two different equivalent notations of permutations to represent the flip and shifts of Xex No. Crunch and generalizations of the puzzle. We are less concerned with the numbering of the disks than the positioning of the disks. Let position 1 be the position on the oval track that is at the left of the turnstile. In Fig. 2, disk 1 is in position 1. Let position 2 be one disk clockwise from position 1. Define positions 3-20 accordingly.

Flip and shifts are merely permutations of the disks on the oval track. We define a flip and left and right shifts according to how these operations affect the disks in

positions 1-20. The flip permutes the disk in position 1 with the disk in position 4, and vice versa, as well as transposes the disks in positions 2 and 3.

DEFINITION 1. *For Xex No. Crunch, the flip permutation is*

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 19 & 20 \\ 4 & 3 & 2 & 1 & 5 & 6 & 7 & \cdots & 19 & 20 \end{pmatrix} = (1\ 4)(2\ 3).$$

DEFINITION 2. *For Xex No. Crunch, the left-shift permutation is*

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 19 & 20 \\ 20 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 18 & 19 \end{pmatrix} = (1\ 20\ 19\ 18\ \cdots\ 3\ 2)$$

and the right-shift permutation is

$$R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 19 & 20 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 20 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ \cdots\ 19\ 20).$$

Realize that $L^{19} = R$ and $R^{19} = L$ and that $R^{20} = L^{20} = F^2 = I$, where I is the identity permutation. Before proving that all 20! permutations of Xex No. Crunch's puzzle pieces are possible, we define a permutation S, the swap permutation, and relate its existence to generating all permutations. The following definition and lemma are valid for all n and are needed later for values other than $n = 20$. However, we do not consider the general definitions of the left and right shift permutations until the next section.

DEFINITION 3. *The swap permutation permutes the disks in positions 1 and 2:*

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n-1 & n \\ 2 & 1 & 3 & 4 & 5 & 6 & \cdots & n-1 & n \end{pmatrix} = (1\ 2).$$

LEMMA 4. *If we can write S as a sequence of flips and shifts, then all $n!$ permutations are possible.*

Proof. All $m!$ permutations of the disks in the first m positions are possible if every permutation of the first m disks, leaving positions $m+1$ through 20 fixed, can be written as a sequence of shifts and flips. We proceed by induction. Realize that S permutes the first two positions; hence, all 2! permutations of the disks in the first 2 positions are possible.

Assume that all $m!$ permutations of the disks in the first m positions are possible. Then, we show that all $(m+1)!$ permutations of the disks in the first $m+1$ positions are possible. Begin by using a left shift to move the disks in positions 2 through $m+1$ into positions 1 through m. By assumption, the disks in positions 1 through m can be arranged in any order by a sequence of shifts and flips, leaving the other disks fixed in their positions. End by using a right shift to move the disks back into positions 2 through $m+1$. There are $m!$ such permutations.

The disk in position 1 by successive use of the swap permutation and the left shift can be placed between any two of the disks between 2 and $m+1$. Indeed, $(SL)^k R^k$ moves the disks in positions 2 through $k+1$ into positions 1 through k, respectively, moves the disk in position 1 into position $k+1$, and leaves the other disks fixed. Thus, we achieve an additional $m \cdot m!$ permutations of the first $m+1$ disks. And, we have accounted for all $(m+1)! = m! + m \cdot m!$ permutations of the disks in the first $m+1$ positions. □

THEOREM 5. *All 20! permutations of the 20 pieces in Xex No. Crunch are possible.*

Proof. By the above lemma, all 20! permutations of the pieces of Xex No. Crunch are possible if the swap permutation can be written as a product of flip and shift permutations.

The product of permutations $FLFRFLFR$ or $(FLFR)^2$ keeps the disks in positions 2-20 in order, but places the disk in position 1 into the 5^{th} position. Notice that

$$\begin{aligned} FLFR &= (1\ 4)(2\ 3)(1\ 20\ 19 \cdots 3\ 2)(1\ 4)(2\ 3)(1\ 2\ 3 \cdots 19\ 20) \\ &= (1\ 3\ 5\ 2\ 4) \end{aligned}$$

and $(FLFR)^2 = (1\ 3\ 5\ 2\ 4)(1\ 3\ 5\ 2\ 4) = (1\ 5\ 4\ 3\ 2)$. Hence, the disk in position 1 has moved to position 5, while all of the other disks remain in their same order. Realize that the disks in positions 2 through 5 have moved into positions 1 through 4, respectively.

To use this operation repeatedly, we must first "put" the disk in position 5 back into position 1. This is achieved by repeated left shifts. And,

$$(FLFR)^2L^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 16 & 17 & 18 & 19 & 20 \\ 1 & 17 & 18 & 19 & 20 & 2 & 3 & \cdots & 12 & 13 & 14 & 15 & 16 \end{pmatrix}.$$

Hence, successive uses of $(FLFR)^2L^4$ keeps all disks except the disk in position 1 in order and marches this disk until it lands behind the disk that was originally in position 2. This takes five iterations because $[(FLFR)^2L^4]^5 = (3\ 2\ 20\ 19\ 18 \cdots 6\ 5\ 4)$. The swap permutation is $(1\ 2) = [(FLFR)^2L^4]^5R$. Since the swap permutation is written as a product of flips and shifts, then all 20! permutations of the pieces of Xex No. Crunch are possible. □

Before we define generalized flip-shift games, there exists another puzzle that consists of permuting pieces by flips and shifts. The Saturn Puzzler is similar to Xex No. Crunch, except that it consists of only 8 "disks" or numbered pieces (Fig. 4).

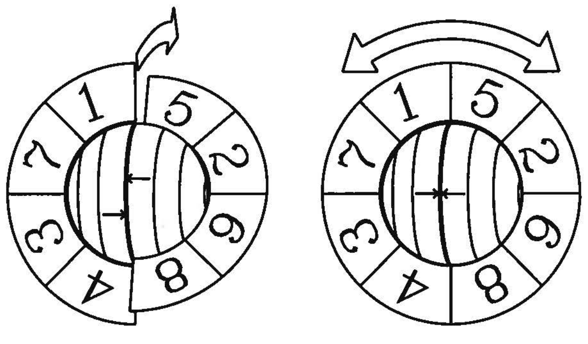

FIG. 4. *Flip (left) and shift (right) for the Saturn Puzzler.*

The movements of the Saturn Puzzler are shifts and flips, as in Xex No. Crunch. Shifts consists of rotating the rings of Saturn around the planet in either the clockwise or counterclockwise directions. The flip is achieved by rotating half of the sphere through a plane that intersects the sphere through a great circle. Both of these movements are pictured in Fig. 4. The mathematical definitions appear below.

DEFINITION 6. *For the Saturn Puzzler, the flip permutation is*

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 5 & 6 & 7 & 8 \end{pmatrix} = (1\ 4)(2\ 3).$$

DEFINITION 7. *For the Saturn Puzzler, the left-shift permutation is*

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} = (1\ 8\ 7\ 6 \cdots 3\ 2)$$

and the right-shift permutation is

$$R = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix} = (1\ 2\ 3\ 4 \cdots 7\ 8).$$

The same technique from Xex No. Crunch can be used to show that all 8! permutations of the pieces of the Saturn Puzzler are possible. As before, we want to show that the swap permutation can be written as a sequence of flips and shifts.

THEOREM 8. *All 8! permutations of the 8 pieces in the Saturn Puzzler are possible.*

Proof. Since the flip size of the Saturn Puzzler and Xex No. Crunch are the same, $(FLFR)^2$ has the same effect. That is, $(FLFR)^2$ keeps the disks in positions 2 through 8 in order, but moves the disk in position 1 into the fifth position. Because there are 8 pieces, it does not take as many iterations of $(FLFR)^2L^4$ to yield the swap permutation. Indeed, $[(FLFR)^2L^4]^2R = (1\ 2)$. □

3. General Flip-Shift Puzzles. The puzzles in the previous section can be generalized to any number of pieces n with flips of any size k where $k < n$. The shift and flip permutations for the general (n, k)-puzzle are defined below.

DEFINITION 9. *Let L_n and R_n be the left-shift and right-shift permutations on n elements, respectively. Define L_n and R_n by*

$$L_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = (1\ n\ n-1 \cdots 3\ 2)$$

and

$$R_n = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & n & 1 \end{pmatrix} = (1\ 2\ 3 \cdots n-1\ n).$$

DEFINITION 10. *Let $F_{n,k}$ be the flip permutation of size k on n elements, then*

$$\begin{aligned} F_{n,k} &= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & k+2 & \cdots & n-1 & n \\ k & k-1 & \cdots & 2 & 1 & k+1 & k+2 & \cdots & n-1 & n \end{pmatrix} \\ &= (1\ k)(2\ k-1) \cdots (s-1\ s+2)(s\ s+1). \end{aligned}$$

Assume that the number of disks is always n and that the flip size is always k. For this reason, we eliminate the subscript notation, *e.g.*, let $R = R_n$, unless it is pertinent. The results that follow often put restrictions on the values of n and k.

3.1. When does the solution technique generalize?. For Xex No. Crunch and the Saturn Puzzler, we used a particular sequence of flips and shifts to yield the swap permutation. We determine below how and when this sequence of flips and shifts yields the swap permutation for general (n, k)-puzzles. The following proposition extends the technique used to solve Xex No. Crunch and the Saturn Puzzler from the previous section. The idea is to be able to move one disk while keeping the other disks in order.

PROPOSITION 11. *For $k = 2s$, the sequence of flips and shifts $(FLFR)^s$ yields the permutation* $(1\ k+1\ k\ k-1 \cdots 3\ 2)$.

Proof. Using the general definitions of the shift and flip permutations, basic multiplication yields

$$FLFR = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-2 & k-1 & k & k+1 & k+2 & \cdots & n \\ 3 & 4 & 5 & \cdots & k & k+1 & 1 & 2 & k+2 & \cdots & n \end{pmatrix}.$$

Notice that it becomes easy to repeat this operation. For example, $(FLFR)^2$ is

$$\begin{pmatrix} 1 & 2 & \cdots & k-4 & k-3 & k-2 & k-1 & k & k+1 & k+2 & \cdots & n \\ 5 & 6 & \cdots & k & k+1 & 1 & 2 & 3 & 4 & k+2 & \cdots & n \end{pmatrix}.$$

Since $k = 2s$ and the operation $FLFR$ only changes the positions of the disks in positions 1 through $k+1$, it follows that $(FLFR)^s$ is

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & k-2 & k-1 & k & k+1 & k+2 & \cdots & n-1 & n \\ k+1 & 1 & 2 & \cdots & k-3 & k-2 & k-1 & k & k+2 & \cdots & n-1 & n \end{pmatrix},$$

or $(1\ k+1\ k\ k-1 \cdots 3\ 2)$. □

By repeated application of sequence of shifts and flips from the above proposition, it is possible to move the disk in position 1 while keeping the other disks in order. This technique yields the swap permutation if successive iterations of $(FLFR)^s L^k$ moves the 1 disk behind the 2 disk. The following theorem indicates the necessary relationship between k and n for this to happen.

THEOREM 12. *The technique used to solve Xex No. Crunch and the Saturn Puzzler can be extended for a flip size of $k = 2s$ and n disks when k and $n-1$ are relatively prime.*

Proof. The permutation $(FLFR)^s$ moves the disk in position 1 into position $k+1$, keeps the disks in positions $k+2$ through n fixed, and moves the disk in position j to $j-1$ for $j = 2$ to $k+1$. We can continue to move the disk that was initially in position 1 by returning it to position 1 using the left shift k times and then repeating $(FLFR)^s$. The permutation $(FLFR)^s L^k (FLFR)^s$ will move the disk originally into position 1 behind the disk that was originally in position $2k+1$, as long as $n > 2k+1$. It follows that $[(FLFR)^s L^k]^{m-1}(FLFR)^s$ moves the disk originally in position 1 behind the disk that was originally in position $mk+1$. Realize that following this permutation by L^k returns the disk that was originally in position 1 back to position 1. Of course, there is an m such that $mk+1 > n$. We are concerned with where the disk originally in position 1 lands relative to the other disks. As the disk originally in position 1 will follow one of the disks in originally in positions 2 through n, we compute $(mk+1)$ modulo $n-1$ to yield which disk the disk originally in position 1 will follow.

To yield the swap permutation, we want to move the disk originally in position 1 behind the disk that was originally in position 2, if possible. Therefore, we want to find an m such that $mk+1$ is congruent to 2 modulo $n-1$. This means that after m iterations of $[(FLFR)^s L^k]^m$ the disk that was originally is position 1 now follows the disk that was originally in position 2, while keeping all of the other disks in their original order.

Finding m such that $mk+1 \equiv 2 \bmod (n-1)$ is equivalent to finding an m such that $mk \equiv 1 \bmod (n-1)$. This is equivalent to finding an m such that there exists an integer a such that $mk = a(n-1)+1$ or $mk - a(n-1) = 1$. By the Euclidean algorithm, $mk - a(n-1) = 1$ implies that the greatest common divisor of k and $n-1$ is 1 (*e.g.*, see [6, p.11]). Equivalently, $[(FLFR)^s L^k]^m R$ can only yield the swap permutation when k and $n-1$ are relatively prime. □

The Euclidean algorithm can be used to determine the minimum m such that $[(FLFR)^s L^k]^m R = (1\ 2)$. These values appear in the following table; m represents the number of forward iterations of $[(FLFR)^s L^k]$ necessary to yield the swap permutation. For example, as discovered in the proof that all permutations of Xex No. Crunch's pieces are possible, the entry in the column 4F ($k = 4$) and the row 20 ($n = 20$) of the table in Fig. 5 is 5. All entries in the "forward" columns of the table in Fig. 5 can be determined by the following algorithm, based on the Euclidean algorithm. When $n-1$ and k are relatively prime, construct the following sequence of remainders until $r_{i+1} = k-1$:

$$\begin{aligned} (n-1) \bmod k &= r_1 \\ [(n-1)+r_1] \bmod k &= r_2 \\ [(n-1)+r_2] \bmod k &= r_3 \\ \vdots \quad &\quad \vdots \\ [(n-1)+r_i] \bmod k &= r_{i+1} \end{aligned}$$

We can compute m by $m = \frac{(i+1)(n-1)+1}{k}$ where $[(FLFR)^s L^k]^m R = (1\ 2)$. This follows since we can add the equalities above to yield:

$$\{(i+1)(n-1) + r_1 + r_2 + \cdots + r_i\} \bmod k \equiv (r_1 + r_2 + \cdots + r_i + r_{i+1}) \bmod k.$$

Canceling r_j for $j = 1$ to i from both sides yields

$$\begin{aligned} (i+1)(n-1) \bmod k &\equiv r_{i+1} \bmod k \\ &= k-1. \end{aligned}$$

Therefore, there exists a b such that $(i+1)(n-1) = bk+k-1$. But, a little rearranging yields $1 = -(i+1)(n-1) + (b+1)k$ which implies that k and $n-1$ are relatively prime. And, our m is $(b+1)$ which can be found by $m = (b+1) = \frac{(i+1)(n-1)+1}{k}$.

EXAMPLE 1. *For the puzzle where $n = 22$ and $k = 8$, the sequence of remainders is*

$$\begin{aligned} 21 \bmod 8 &= 5 \\ 26 \bmod 8 &= 2 \\ 23 \bmod 8 &= 7. \end{aligned}$$

This algorithm terminated after 3 iterations and $m = \frac{(i+1)(n-1)+1}{k} = \frac{3 \cdot 21+1}{8} = 8$. So, 8 is the entry in the table for the 8F column for $n = 22$.

However, it is possible to work "backwards" by moving the disk in position 1 to the left instead of to the right, while keeping the other disks in order. By reversing

$n\backslash k$	4F	4B	6F	6B	8F	8B	10F	10B	12F	12B	14F	14B	16F	16B
8	2	5												
10	7	2												
12	3	8	2	9										
14	10	3	11	2										
16	4	11	X	X	2	13								
18	13	4	3	14	15	2								
20	5	14	16	3	12	7	2	17						
22	16	5	X	X	8	13	19	2						
24	6	17	4	19	3	20	7	16	2	21				
26	19	6	21	4	22	3	X	X	23	2				
28	7	20	X	X	17	10	19	8	X	X	2	25		
30	22	7	5	24	11	18	3	26	17	12	27	2		
32	8	23	26	5	4	27	28	3	13	18	20	11	2	29
34	25	8	X	X	29	4	10	23	X	X	26	7	31	2
36	9	26	6	29	22	13	X	X	3	32	X	X	11	24
38	28	9	31	6	14	23	26	11	34	3	8	29	7	30
40	10	29	X	X	5	34	4	35	X	X	14	25	22	17
42	31	10	7	34	36	5	37	4	24	17	3	38	18	23
44	11	32	36	7	27	16	13	30	18	25	40	3	35	8
46	34	11	X	X	17	28	X	X	X	X	29	16	31	14
48	12	35	8	39	6	41	33	14	4	43	37	10	3	44
50	37	12	41	8	43	6	5	44	45	4	X	X	46	3
52	13	38	X	X	32	19	46	5	X	X	11	40	16	35
54	40	13	9	44	20	33	16	37	31	22	17	36	10	43
56	14	41	46	9	7	48	X	X	23	32	4	51	31	24
58	43	14	X	X	50	7	40	17	X	X	53	4	25	32
60	15	44	10	49	37	22	6	53	5	54	42	17	48	11
62	46	15	51	10	23	38	55	6	56	5	48	13	42	19
64	16	47	X	X	8	55	19	44	X	X	X	X	4	59
66	49	16	11	54	57	8	X	X	38	27	14	51	61	4
68	17	50	56	11	42	25	47	20	28	39	20	47	21	46
70	52	17	X	X	26	43	7	62	X	X	5	64	13	56
72	18	53	12	59	9	62	64	7	6	65	66	5	40	31
74	55	18	61	12	64	9	22	51	67	6	53	20	32	41
76	19	56	X	X	47	28	X	X	X	X	59	16	61	14
78	58	19	13	64	29	48	54	23	45	32	X	X	53	24
80	20	59	66	13	10	69	8	71	33	46	17	62	5	74
82	61	20	X	X	71	10	73	8	X	X	23	58	76	5
84	21	62	14	69	52	31	25	58	7	76	6	77	26	57
86	64	21	71	14	32	53	X	X	78	7	79	6	16	69
88	22	65	X	X	11	76	61	26	X	X	64	23	49	38
90	67	22	15	74	78	11	9	80	52	37	70	19	39	50
92	23	68	76	15	57	34	82	9	38	53	X	X	74	17
94	70	23	X	X	35	58	28	65	X	X	20	73	64	29
96	24	71	16	79	12	83	X	X	8	87	26	69	6	89
98	73	24	81	16	85	12	68	29	89	8	7	90	91	6
100	25	74	X	X	62	37	10	89	X	X	92	7	31	68

FIG. 5. *The minimum number of "moves" to yield the swap permutation.*

the order of operations, $R^k(LFRF)^s$ moves the disk in position 1 behind the disk in position $(-k+1) \bmod (n-1)$. This follows because $[(FLFR)^sL^k][R^k(LFRF)^s] = I$, the identity permutation. The swap permutation can also be achieved by finding the minimum j such that $[R^k(LFRF)^s]^j$ places the disk originally in position 1 into the position behind the disk originally in position 2 while leaving the other disks in their original order. This j can be determined by solving $-jk + 1 \equiv 2 \bmod (n-1)$ for the minimum j. This value can easily be determined since j and m, as described, must add to $n-1$. This follows from k and $n-1$ being relatively prime. The table contains the number of iterations of $[R^k(LFRF)^s]$ necessary to achieve the swap permutation. For example, the entry in column 8B and row $n = 18$ is 2 and indicates that $[R^8(LFRF)^4]^2$ swaps the order of the disks in position 1 and 2. Comparing this to the entry of 8F and row $n = 18$, which is 15, indicates that it is more efficient to use sequences of $[R^k(LFRF)^s]^j$ than $[(FLFR)^sL^k]^m$ to yield the swap permutation.

3.2. When can we guarantee that not all permutations are possible?. Determining whether or not every one of the $n!$ permutations of the n pieces are possible under a shift and a flip of size k often reduces to a question of parity. That is, whether or not the shift and flip permutations are odd or even becomes paramount. First, we review the definition of odd and even permutations. Recall that a transposition is a permutation that transposes two elements and leaves all other elements fixed. Indeed, the swap permutation from the previous section is an example of a transposition.

DEFINITION 13. *A permutation is odd if it can be written as the product of an odd number of transpositions. A permutation is even if it can be written as the product of an even number of transpositions.*

Realize that a permutation cannot be both odd and even. Indeed, although an odd permutation can be written as the product of an odd number of transpositions, it is the case that every such product of transpositions must contain an odd number of transpositions, as indicated in the following theorem.

THEOREM 14. *(Hillman and Alexanderson, [5, p.92]) If a permutation θ is a product of transpositions, $\theta = \alpha_1\alpha_2\cdots\alpha_r = \beta_1\beta_2\cdots\beta_s$, then r and s are both even or both odd.*

Next, we determine the values for which the general flip and shift permutations are odd and even.

PROPOSITION 15. *The shift permutations on n elements are even if n is odd and are odd if n is even.*

Proof. Both shift permutations on n elements can be written as a product of $n-1$ transpositions. Specifically,

$$L_n = (1\ 2)(2\ 3)\cdots(n-2\ n-1)(n-1\ n)$$

and

$$R_n = (1\ 2)(1\ 3)\cdots(1\ n-1)(1\ n).$$

The proposition is proved because $n-1$ is even when n is odd and $n-1$ is odd when n is even. □

PROPOSITION 16. *The flip $F_{n,k}$ is even if k is congruent to 0 or 1 mod 4. Otherwise, $F_{n,k}$ is odd.*

Proof. We consider each of the four possibilities of k mod 4 separately. For ease of presentation, represent

$$F_{n,k} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & k+2 & \cdots & n-1 & n \\ k & k-1 & \cdots & 2 & 1 & k+1 & k+2 & \cdots & n-1 & n \end{pmatrix}$$

by

$$F_{n,k} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix}$$

since all of the elements from $k+1$ to n are fixed under F_k^n.

Let k be equal to $4l$ for some nonnegative integer l. Then, $F_{n,4l}$ is

$$\begin{pmatrix} 1 & 2 & \cdots & 2l & 2l+1 & \cdots & 4l-1 & 4l \\ 4l & 4l-1 & \cdots & 2l+1 & 2l & \cdots & 2 & 1 \end{pmatrix}.$$

The flip $F_{n,4l}$ can be written as the product of $2l$ transpositions; specifically,

$$F_{n,4l} = (1\ 4l)(2\ 4l-1)(3\ 4l-2)\cdots(2l\ 2l+1).$$

Hence, $F_{n,4l}$ is an even permutation.

Let $k = 4l+1$ for some nonnegative integer l. Then, $F_{n,4l+1}$ is

$$\begin{pmatrix} 1 & 2 & \cdots & 2l & 2l+1 & 2l+2 & \cdots & 4l & 4l+1 \\ 4l+1 & 4l & \cdots & 2l+2 & 2l+1 & 2l & \cdots & 2 & 1 \end{pmatrix}.$$

The flip $F_{n,4l+1}$ is an even permutation because it can be written as the product of $2l$ transpositions:

$$F_{n,4l+1} = (1\ 4l+1)(2\ 4l)(3\ 4l-1)\cdots(2l\ 2l+2).$$

If $k = 4l+2$ for some nonnegative integer l, then $F_{n,4l+2}$ is

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & 2l+1 & 2l+2 & \cdots & 4l & 4l+1 & 4l+2 \\ 4l+2 & 4l+1 & 4l & \cdots & 2l+2 & 2l+1 & \cdots & 3 & 2 & 1 \end{pmatrix}.$$

The flip $F_{n,4l+2}$ can be written as a product of $2l+1$ transpositions; indeed,

$$F_{n,4l+2} = (1\ 4l+2)(2\ 4l+1)(3\ 4l)\cdots(2l\ 2l+3)(2l+1\ 2l+2).$$

It follows that $F_{n,4l+2}$ is an odd permutation.

If $k = 4l+3$ for some nonnegative integer l, then $F_{n,4l+3}$ is

$$\begin{pmatrix} 1 & 2 & \cdots & 2l+1 & 2l+2 & 2l+3 & \cdots & 4l+2 & 4l+3 \\ 4l+3 & 4l+2 & \cdots & 2l+3 & 2l+2 & 2l+1 & \cdots & 2 & 1 \end{pmatrix}.$$

Writing $F_{n,4l+3}$ as a product of transpositions yields:

$$F_{n,4l+3} = (1\ 4l+3)(2\ 4l+2)(3\ 4l+1)\cdots(2l\ 2l+4)(2l+1\ 2l+3).$$

And, $F_{n,4l+3}$ is an odd permutation because it can be written as the product of $2l+1$ transpositions. □

THEOREM 17. *All $n!$ permutations are not possible for flip-shift puzzles with flip-size k congruent to 0 or 1 modulo 4 and an odd number of pieces, n.*

Proof. For k congruent to 0 or 1 module 4, the flip permutation is even. Similarly, for n odd, the shift permutations are even. All products of shifts and flips are even permutations. Therefore, none of the odd permutations are possible. □

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ITERATION OF AN EVEN-ODD SPLITTING MAP CAN MAKE INTEGRATION EASIER

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Abstract. We study the dynamics of the map $\mathcal{F}(R(x)) = (R(\sqrt{x}) - R(-\sqrt{x})/(2\sqrt{x})$ on the space of rational functions, in the context of a new method of integration. We give a recursive formula for the iterates of a model family of rational functions, which is closed under the action of $\mathcal{F}$. We give a class of rational functions that are mapped to zero by two iterations of $\mathcal{F}$.

We prove that all polynomials are eventually mapped to even functions by $\mathcal{F}$, and we determine the number of iterations required for a given polynomial. We use power series representation to determine which rational functions are eventually mapped to even functions by $\mathcal{F}$.

1. Introduction. The integration of rational functions is one of the central tasks in calculus. The classical method of partial fractions reduces the problem to that of solving an algebraic equation. If $P(x)$ and $Q(x)$ are polynomials, the evaluation of

$$I = \int_0^\infty \frac{P(x)}{Q(x)}\,dx \tag{1}$$

requires factorization of the denominator

$$\begin{aligned} Q(x) &= (x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_j)^{n_j} \\ &= (x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_k)^{n_k} \times \\ &\quad (x^2+2a_1x+a_1^2+b_1^2)^{m_1}\cdots(x^2+2a_px+a_p^2+b_p^2)^{m_p}. \end{aligned} \tag{2}$$

where $x_1, \ldots, x_j$ are the roots of $Q(x) = 0$, and the factorization is converted to a real form by combining any non-real roots in conjugate pairs.

The difficulty associated with this method is that, as Abel showed, it is impossible to solve the general equation of degree 5 or more by radicals. Exact formulas for the roots of a polynomial are not always available. Therefore an interesting question is to classify the rational functions R for which the integral (1) can be evaluated without factoring the polynomial Q.

The integration of *even rational functions* seems to be an easier problem. Two examples are the classical Wallis formula [5]

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m} \qquad m \in \mathbb{N}, \tag{3}$$

and the evaluation in [1] of

$$\begin{aligned} N_{0,4}(a;m) &= \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \\ &= \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a) \qquad m \in \mathbb{N}, \end{aligned} \tag{4}$$

where

$$P_m(a) = 2^{-2m}\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{5}$$

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The special case

$$N_{0,4}(a;0) = \frac{\pi}{2\sqrt{2(a+1)}} \tag{6}$$

of (4) will be employed in Section 3. See [3] for many more examples.

Recall that the even and odd parts of a function are defined as, respectively,

$$R_e(x) = \frac{R(x)+R(-x)}{2} \quad \text{and} \quad R_o(x) = \frac{R(x)-R(-x)}{2}. \tag{7}$$

We can rewrite $\int_0^\infty R(x)\,dx$ as

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \int_0^\infty R_o(x)\,dx. \tag{8}$$

The first integral on the right has an even integrand, and so is likely to be easier to evaluate than the original. The change of variables $t = x^2$ in the second integral yields the identity

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \frac{1}{2}\int_0^\infty \mathcal{F}(R(x))\,dx, \tag{9}$$

where the map $\mathcal{F}$ is defined by

$$\mathcal{F}(R(x)) = \frac{R(\sqrt{x}) - R(-\sqrt{x})}{2\sqrt{x}}. \tag{10}$$

If the n^{th} iterate $\mathcal{F}^{(n)}(R(x))$ is even for some n, then the integral $\int_0^\infty R(x)\,dx$ reduces to an integral of even functions. In this paper we give necessary and sufficient conditions on the rational function $R(x)$ for this to occur.

The paper is organized as follows. In Section 2 we show that the map $\mathcal{F}$ preserves rationality of the function $R(x)$. Sections 3 and 4 contain examples. In Section 3 we show that $\mathcal{F}$ preserves the family of rational functions

$$R_m(a,x) = \frac{G_m(a)}{x^2 + H_m(a)x + 1}, \quad G_0(a) = 1, \ \ H_0(a) = 2a, \tag{11}$$

and we give recursive formulas for $G_m(a)$ and $H_m(a)$. We also discuss analogous results for the family of rational functions

$$R(x) = \frac{1}{x^3 + ax^2 + bx^3 + 1}, \quad a, b \in \mathbb{R}, \tag{12}$$

where we now include a substitution $x \to -x$ in our mapping function $\mathcal{F}$.

In Section 4 we show that the rational functions of the form

$$R(x) = \frac{xP(x^4) + x^2Q(x^2)}{V(x^4)}, \tag{13}$$

where P, Q, and V are polynomials, are mapped to *even* functions by one iteration of $\mathcal{F}$.

In Section 5 we establish a necessary and sufficient condition for a rational function $R(x)$ to be mapped to an even function by n iterations of $\mathcal{F}$. The condition is that certain coefficients in the power series for $R(x)$ about zero must vanish. In Section 6 we prove that all polynomials are eventually mapped to even functions by $\mathcal{F}$, and we determine the number of iterations required.

2. $\mathcal{F}$ preserves rationality. In this section we prove that the map $\mathcal{F}$ preserves the class of rational functions.

PROPOSITION 1. *If $R(x)$ is a rational function, then $\mathcal{F}(R(x))$ is also rational.*

Proof. Write $R(x) = P(x)/Q(x)$. A direct calculation shows that

$$\mathcal{F}(R(x)) = \frac{P(\sqrt{x})Q(-\sqrt{x}) - P(-\sqrt{x})Q(\sqrt{x})}{Q(\sqrt{x})Q(-\sqrt{x})2\sqrt{x}}. \tag{1}$$

Now observe that $Q(t)Q(-t)$ is an even polynomial in $t = \sqrt{x}$, so it is a polynomial in $t^2 = x$. Similarly $P(t)Q(-t) - P(-t)Q(-t)$ is an odd polynomial in t, so the numerator in (1) is also a polynomial in x, after cancellation with the $\sqrt{x}$ in the denominator. □

3. Examples of the dynamics of $\mathcal{F}$. Consider the rational function

$$R(a,x) = \frac{1}{x^2 + 2ax + 1}, \qquad a \in \mathbb{R}. \tag{1}$$

The even part of $R(a,x)$ is

$$R_e(a,x) = \frac{1+x^2}{x^4 + (2-4a^2)x^2 + 1}. \tag{2}$$

Integrating the even part, we obtain

$$\int_0^\infty \frac{1+x^2}{x^4+(2-4a^2)x^2+1}\,dx = 2\int_0^\infty \frac{1}{x^4+(2-4a^2)x^2+1}\,dx, \tag{3}$$

using the change of variables $x \mapsto 1/x$. The resulting integral can be evaluated using (6) to produce

$$\int_0^\infty R_e(a,x)\,dx = 2N_{0,4}(1-2a^2, 0) = \frac{\pi}{2\sqrt{1-a^2}}. \tag{4}$$

Turning to the odd part of $R(a,x)$, we evaluate the rational function $\mathcal{F}(R(x))$. Direct calculation suggests that the iterates of R under $\mathcal{F}$ have the form

$$\begin{aligned} \mathcal{F}^{(m)}(R(a,x)) &= \frac{G_m(a)}{x^2 + H_m(a)x + 1} \\ &= G_m(a) \times R\left(H_m(a)/2, x\right), \end{aligned} \tag{5}$$

where $G_m(a)$ and $H_m(a)$ are polynomials in a. This is established in the next proposition.

PROPOSITION 2. *The functions $H_m(a)$ and $G_m(a)$ satisfy the recursion formulas*

$$\begin{aligned} H_{m+1}(a) &= 2 - H_m(a)^2, \\ G_{m+1}(a) &= -G_m(a)H_m(a), \\ G_{m+2}(a) &= \frac{G_{m+1}^3(a)}{G_m^2(a)} - 2G_{m+1}(a), \end{aligned} \tag{6}$$

with initial conditions $G_0(a) = 1$ and $H_0(a) = 2a$. In particular, $H_m(a)$ and $G_m(a)$ are polynomials in a.

Proof. The proof is by induction.

Base Case: For $G_0(a) = 1$ and $H_0(a) = 2a$ a direct computation shows that

$$\mathcal{F}^{(1)}(R(x)) = \frac{-2a}{x^2 + (2 - 4a^2)x + 1}, \tag{7}$$

as desired.

We assume that

$$\mathcal{F}^{(m)}(R(x)) = \frac{G_m(a)}{x^2 + H_m(a)x + 1}, \tag{8}$$

and evaluate $\mathcal{F}^{(m+1)}(R(x))$ to obtain the indicated recursion.

Define $R_m(x) = \mathcal{F}^{(m)}(R(x))$ and compute the odd part

$$R_{m,odd}(x) = \frac{1}{2}\left(\frac{G_m(a)}{x^2 + H_m(a)x + 1} - \frac{G_m(a)}{x^2 - H_m(a)x + 1}\right). \tag{9}$$

Substitute $x \to \sqrt{x}$, divide by $\sqrt{x}$ and combine the two fractions to produce

$$\begin{aligned} \mathcal{F}^{(m+1)}(R(x)) &= \frac{-2G_m(a)H_m(a)\sqrt{x}}{(x^2 + (2 - H_m^2(a))x + 1)2\sqrt{x}} \\ &= \frac{-G_m(a)H_m(a)}{(x^2 + (2 - H_m^2(a))x + 1)}. \end{aligned} \tag{10}$$

It follows that $\mathcal{F}^{(m+1)}(R(x))$ has the required form and that the functions $H_m(a)$ and $G_m(a)$ satisfy the recursion stated above.

Since $H_m(a) = -G_{m+1}(a)/G_m(a)$, the second recursion for $G_m(a)$ now follows from the recursion for $H_m(a)$. The fact that $G_{m+2}(a)$ is necessarily a polynomial can be proved by induction. □

We summarize our discussion in a theorem:

THEOREM 3. *The family of functions*

$$R(a, x) := \frac{G_m(a)}{x^2 + H_m(a)x + 1}, \qquad a \in \mathbb{R}, \tag{11}$$

is closed under the action of $\mathcal{F}$, and the following integral formula holds:

$$\int_0^\infty \frac{G_m(a)\,dx}{x^2 + H_m(a)x + 1} = \frac{\pi G_m(a)}{\sqrt{1 - H_m^2(a)}} + \frac{1}{2}\int_0^\infty \frac{G_{m+1}(a)\,dx}{x^2 + H_{m+1}(a)x + 1}, \tag{12}$$

where $H_m(a)$ and $G_m(a)$ are as defined in (6).

For rational functions of the form $R(a, x) = 1/(x^2 + 2ax + 1)$ we are able to determine which rational functions are eventually mapped to even functions. Simply, the solution a to $H_m(a) = 0$ will give particular rational functions $R(a, x)$ that map to an even function after m applications of $\mathcal{F}$, because when $H_m(a) = 0$ the resulting rational function $\mathcal{F}^{(m-1)}(R(a, x))$ is even.

Turning our attention to rational functions of the form $R(x) = 1/(x^3 + ax^2 + bx^3 + 1)$ we can obtain similar, more complicated, recursive equations for the coefficients. There $\mathcal{F}$ must be slightly modified in order for iterations of $\mathcal{F}$ to preserve the structure of $R(x)$. If we add a second substitution $x \to -x$ to $\mathcal{F}$, then this modified map $\mathcal{G}$ allows similar results.

So far we have note considered the convergence of our integrals. It is shown in [4] that $\int_0^\infty 1/(x^3 + ax^2 + 2bx + 1)\,dx$ converges if a and b satisfy the condition

$4a^3 - 18ab + 4b^3 + 27 > 0$. It is an open question whether one can find conditions on a for convergence of $\int_0^\infty \mathcal{F}^{(m)}\left(1/(x^2 + 2ax + 1)\right) dx$. Similarly, it is an open question whether one can find conditions on a and b for the convergence of $\int_0^\infty \mathcal{G}^{(m)}\left(1/(x^3 + ax^2 + 2bx + 1)\right) dx$.

4. Mapping rational to even functions by $\mathcal{F}^{(m)}$. If a rational function $R(x)$ is mapped to an even function by applying $\mathcal{F}$ finitely many times, then $R(x)$ can be integrated in finitely many steps, provided one has an efficient algorithm for the integration of even functions. The beginnings of such an algorithm are described in [3].

Recall that R is an even function if and only if $\mathcal{F}(R(x)) = 0$. We now describe a family of rational functions $R(x)$ that become even after one application of $\mathcal{F}$. In other words, $\mathcal{F}^{(2)}(R(x)) = 0$.

THEOREM 4. *Let P, Q, and V be polynomials in x, and consider the rational function*

$$R(x) = \frac{xP(x^4) + x^2Q(x^2)}{V(x^4)}. \tag{1}$$

$\mathcal{F}(R(x))$ is an even function.

Proof. The odd part of $R(x)$ simplifies to

$$\begin{aligned} &\left(\frac{xP(x^4) + x^2Q(x^2)}{V(x^4)}\right)_{odd} \\ &= \frac{xP(x^4) + x^2Q(x^2)}{V(x^4)} - \frac{(-x)P((-x)^4) + (-x)^2Q((-x)^2)}{V((-x)^4)} \\ &= \frac{2xP(x^4)}{V(x^4)}. \end{aligned} \tag{2}$$

The substitution $x \to \sqrt{x}$ and division by $2\sqrt{x}$ result in

$$\frac{2\sqrt{x}P((\sqrt{x})^4)}{2\sqrt{x}Q((\sqrt{x})^4)} = \frac{P(x^2)}{V(x^2)}, \tag{3}$$

and this function is even. □

Observe that the resulting even function is independent of Q.

Example. Consider the case $P(x) = bx + c$, $Q(x) = dx + e$ and $V(x) = x^2 + 2ax + 1$. We wish to evaluate the integral

$$\int_0^\infty \frac{x(bx^4 + c) + x^2(dx^2 + e)}{x^8 + 2ax^4 + 1}\,dx = \int_0^\infty \frac{bx^5 + dx^4 + ex^2 + cx}{x^8 + 2ax^4 + 1}\,dx. \tag{4}$$

The even part reduces to the integral

$$\int_0^\infty R_e(x)dx = \int_0^\infty \frac{dx^4 + ex^2}{x^8 + 2ax^4 + 1}\,dx = \frac{(d + e)\pi}{2^{3/2}(1 + a)^{1/2}(4 + \sqrt{8(1 + a)})^{1/2}}\,dx. \tag{5}$$

$\mathcal{F}$ converts the odd part to an even function, which can be integrated:

$$\int_0^\infty \mathcal{F}(R(x))dx = \int_0^\infty \frac{bx^2 + c}{x^4 + 2ax^2 + 1}\,dx = \frac{(b + c)\pi}{2^{3/2}(1 + a)^{1/2}}. \tag{6}$$

For both integrals we have used equations from [3],

$$\int_0^\infty \frac{bx^5+dx^4+ex^2+cx}{x^8+2ax^4+1}\,dx = \int_0^\infty R_e(x)\,dx + \frac{1}{2}\int_0^\infty \mathcal{F}(R(x))\,dx \tag{7}$$
$$= \frac{2\pi(d+e)+\pi\left(4+\sqrt{8(1+a)}\right)^{1/2}(b+c)}{2^{3/2}(1+a)^{1/2}\left(4+\sqrt{8(1+a)}\right)^{1/2}}.$$

5. The power series condition for $\mathcal{F}^{(n)}(R)=0$. The power series expansion $R(x)=\sum_{j=-\infty}^{\infty}a_jx^j$ about $x=0$ of a rational function has only a finite number of non-zero terms with negative powers of z [6, Section 5.6]. The coefficients satisfy a periodicity condition: there exists an $m\in\mathbb{N}$ such that $a_j=a_{j+m}$ for all j. This property of a rational function is key to finding a necessary and sufficient condition on $R(x)$ to ensure that some $\mathcal{F}^{(n)}(R(x))$ is even.

THEOREM 5. *Consider the rational function*

$$R(x)=\sum_{j=-m}^{\infty}a_jx^j. \tag{1}$$

The n^{th} iterate $\mathcal{F}^{(n)}(R(x))$ is even if and only if $a_{2^{n+1}j+2^{n+1}-1}=0$ for all integers $j\geq -m$. First we state a simple Lemma.

LEMMA 6. *The power series for the n^{th} iterate of $R(x)$ is given by*

$$\mathcal{F}^{(n)}(R(x))=\frac{1}{2^n}\sum_{j=0}^{\infty}a_{2^nj+2^n-1}x^j. \tag{2}$$

Proof. Apply $\mathcal{F}$ to equation (1) and use induction. □

Now we can prove Theorem 5. We use induction on the number of iterations.

Proof. Base Case: $n=1$. First assume $\mathcal{F}^{(1)}(R(x))$ is even; then

$$\mathcal{F}^{(1)}(R(x))=\mathcal{F}^{(1)}\left(\sum_{j=-m}^{\infty}a_jx^j\right)$$
$$=\frac{1}{2}\sum_{j=-m}^{\infty}a_{2j+1}\frac{(\sqrt{x})^{2j+1}}{\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}}\sum_{j=-m}^{\infty}a_{2j+1}x^{j+\frac{1}{2}}$$
$$=\frac{1}{2}\sum_{j=-m}^{\infty}a_{2j+1}x^j$$
$$=\frac{1}{2}\sum_{j=-m}^{\infty}a_{4j+3}x^{2j+1}+\frac{1}{2}\sum_{j=-m}^{\infty}a_{4j+1}x^{2j}. \tag{3}$$

But we know that $\mathcal{F}^{(1)}(R(x))$ is even, which can only occur if the odd part of $\mathcal{F}^{(1)}(R(x))$ vanishes. Therefore $a_{2^2j+3}=0$ for all $j\in\mathbb{Z}$, as required.

Now suppose $a_{4j+3}=0$ for all $j\in\mathbb{N}$. The function $R(x)$ can be expressed in the form

$$R(x)=\sum_{j=-m}^{\infty}a_jx^j \tag{4}$$
$$=\sum_{j=-m}^{\infty}a_{4j}x^{4j}+\sum_{j=-m}^{\infty}a_{4j+1}x^{4j+1}+\sum_{j=-m}^{\infty}a_{4j+2}x^{4j+2}+\sum_{j=-m}^{\infty}a_{4j+3}x^{4j+3}.$$

But we know that

$$\sum_{j=-m}^{\infty}a_{4j+3}x^{4j+3}=0, \tag{5}$$

and

$$\mathcal{F}^{(1)}\left(\sum_{j=-m}^{\infty}a_{4j+2}x^{4j+2}\right)=0,\qquad \mathcal{F}^{(1)}\left(\sum_{j=-m}^{\infty}a_{4j}x^{4j}\right)=0, \tag{6}$$

because the arguments are even functions. So the substitution $x\to\sqrt{x}$ and division by $2\sqrt{x}$ lead to

$$\mathcal{F}^{(1)}(R(x))=\mathcal{F}^{(1)}\left(\sum_{j=-m}^{\infty}a_{4j+1}x^{4j+1}\right)$$
$$=\frac{1}{2\sqrt{x}}\sum_{j=-m}^{\infty}a_{4j+1}(\sqrt{x})^{4j+1}$$
$$=\frac{1}{2}\sum_{j=-m}^{\infty}a_{4j+1}x^{2j}, \tag{7}$$

which is even. This establishes the base case.

Assume Theorem (5) holds for some n, and suppose $\mathcal{F}^{(n+1)}(R(x))$ is even. Now by Lemma 5.1,

$$\mathcal{F}^{(n+1)}(R(x))=\frac{1}{2^{n+1}}\sum_{j=-m}^{\infty}a_{2^{n+1}j+2^{n+1}-1}x^j \tag{8}$$
$$=\frac{1}{2^{n+1}}\left(\sum_{j=-m}^{\infty}a_{2^{n+2}j+2^{n+1}-1}x^{2j}+\sum_{j=-m}^{\infty}a_{2^{n+2}j+2^{n+2}-1}x^{2j+1}\right).$$

But $\mathcal{F}^{(n+1)}(R(x))$ is even, so each coefficient $a_{2^{n+2}j+2^{n+2}-1}$ must be 0.

Now suppose $a_{2^{n+2}j+2^{n+2}-1}=0$ for all $j\geq -m$. By our lemma,

$$\mathcal{F}^{(n)}(R(x))=\frac{1}{2^n}\sum_{j=-m}^{\infty}a_{2^nj+2^n-1}x^j \tag{9}$$
$$=\frac{1}{2^n}\sum_{j=-m}^{\infty}a_{2^{n+2}j+2^n-1}x^{4j}+\frac{1}{2^n}\sum_{j=-m}^{\infty}a_{2^{n+2}j+2^{n+1}-1}x^{4j+1}$$
$$+\frac{1}{2^n}\sum_{j=-m}^{\infty}a_{2^{n+2}j+3(2^n)-1}x^{4j+2}+\frac{1}{2^n}\sum_{j=-m}^{\infty}a_{2^{n+2}j+2^{n+2}-1}x^{4j+3}.$$

Also,

$$\mathcal{F}\Big(\frac{1}{2^n}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^n-1}x^{4j}\Big)=0\,,\quad \mathcal{F}\Big(\frac{1}{2^n}\sum_{j=-m}^{\infty} a_{2^{n+2}j+3(2^n)-1}x^{4j+2}\Big)=0, \tag{10}$$

because both sums in 10 are even and by assumption $a_{2^{n+2}j+2^{n+2}-1}=0$. Therefore

$$\begin{aligned}\mathcal{F}^{(n+1)}(R(x)) &= \mathcal{F}\left(\frac{1}{2^n}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+1}-1}x^{4j+1}\right)\\ &= \frac{1}{2^{n+1}}\sum_{j=-m}^{\infty} a_{2^{n+2}j+2^{n+1}-1}x^{2j}\,,\end{aligned}\tag{11}$$

which is even, as required. □

This result is useful because now we can take any rational function, express it as a power series and determine whether it will *ever* result in an even function. It also implies that $R(\sqrt{x})-R(-\sqrt{x})=iR(-i\sqrt{x})-iR(i\sqrt{x})$ if and only if $\mathcal{F}$ maps $R(x)$ to an even function. Now we have a closed form test for whether a rational function will be mapped to an even function under $\mathcal{F}$ on the next iteration.

6. A special property of mapping polynomials. By definition, $\mathcal{F}$ maps every even function to 0. The converse is also true: if $\mathcal{F}(R(x))=0$, then $R(x)$ is even. An interesting open problem is to classify all functions R for which there exists an integer n such that $\mathcal{F}^{(n)}(R(x))=0$.

All polynomials are eventually mapped to 0 by repeated application of the map $\mathcal{F}$. Further, the number of iterations required can be exactly determined from the exponents present in the polynomial.

THEOREM 7. *Let $P(x)$ be a polynomial. Then there exists a non-negative integer n such that*

$$\mathcal{F}^{(n)}(P(x))=0\,. \tag{1}$$

Proof. Let $P(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m$. Any monomial of even degree is mapped to zero after one iteration of $\mathcal{F}$:

$$\mathcal{F}(x^{2k})=\frac{x^{2k}-x^{2k}}{2\sqrt{x}}=0\,. \tag{2}$$

So it suffices to look at powers of x of the form $x^{2^m p-1}$, where m is a positive integer and p is odd, and to show that there is a positive integer n such that $\mathcal{F}^{(n)}(x^{2^m p-1})=0$. Notice that

$$\mathcal{F}^{(1)}\left(x^{2^m p-1}\right)=x^{2^{m-1}p-1}\,,\qquad \mathcal{F}^{(2)}\left(x^{2^m p-1}\right)=x^{2^{m-2}p-1}\,, \tag{3}$$

and so on. Iterating this procedure yields

$$\mathcal{F}^{(m)}\left(x^{2^m p-1}\right)=x^{p-1}\,. \tag{4}$$

Notice that x^{p-1} is an even power of x. Therefore

$$\mathcal{F}^{(m+1)}\left(x^{2^m p-1}\right)=\mathcal{F}\left(x^{p-1}\right)=0\,. \quad □ \tag{5}$$

COROLLARY 8. *Let $P(x)=a_1x^{2^{k_1}p_1-1}+a_2x^{2^{k_2}p_2-1}+\cdots+a_nx^{2^{k_n}p_n-1}$, where p_i is odd, and $1\le i\le n$. Define*

$$k=\max(k_i)\,,\qquad 1\le i\le n\,.$$

Then

$$\mathcal{F}^{(k+1)}\left(P(x)\right)=0\,,$$

and this is the first iterate that vanishes.

Proof. This is the situation of Theorem (7) with $m=k$. □

7. Open Questions. Finding other particular classes of functions such as (1) for the second, third and n^{th} iterations of $\mathcal{F}$ would be very useful for recognizing which rational functions $R(x)$ eventually map to even functions. Currently, our general Theorem 5 can tell us this by looking at the power series expansion, but specific cases would also be interesting.

Unfortunately, many rational functions will never map to even functions. Describing the behavior of these rational functions under the map $\mathcal{F}$ becomes complicated. In the case of the reciprocal of the quadratic, applying our mapping function results in a standard formula (5) for the iterates. An idea related to the integrability over $[0,\infty)$ is to use these recursion formulas to classify the behavior of the zeroes of a rational function under iteration of $\mathcal{F}$; we have started to consider this.

We have used Mathematica and Maple to study the behavior of $\mathcal{F}$ on rational functions, to find fixed points of $\mathcal{F}$, to find periodic points of $\mathcal{F}$, and to measure the length of their orbits. We have also found functions that are pre-periodic under $\mathcal{F}$. The fact that a large class of functions have periodic behavior under $\mathcal{F}$ allows us to map our functions back to themselves resulting in expressing these functions as a sum of other even functions.

The fixed points of $\mathcal{F}$ have recently been found [2], and they are of the form

$$R(x)=\frac{x^{m-1}}{x^m-1}\,, \tag{1}$$

where $m\in\mathbb{N}$ and m is odd. Unfortunately, these fixed points are *not* integrable on $[0,\infty)$.

Finally, we continue to study the dynamics of $\mathcal{F}$ on the space of rational functions, following [2]. It is an open question whether the fixed points of $\mathcal{F}$ are attracting or repelling, and how one might define the multiplier of $\mathcal{F}$.

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HYPERPERFECT NUMBERS

JOHN C. M. NASH

Abstract. A new formula for hyperperfect numbers is demonstrated and new examples of hyperperfect numbers are given.

A number n is called *perfect* it is equal to the sum of its devisors. For example, $6 = 1 + 2 + 3$ is perfect. In [1], p. 49, this concept was generalized n to be k-hyperperfect if $n = 1 + k\Sigma d_i$ where the d_i are the proper divisors of n, $1 < d_1 < n$. The following theorem can be used to generate examples of hyperperfect numbers.

Theorem 1. *If $n = p^k\left(p^{k+1} - (p-1)\right)$ and $p^{k+1} - (p-1)$ is prime, p prime, then n is $p-1$-hyperperfect.*

Proof:

$$\begin{aligned}
1 + (p-1)\Sigma d_i &= 1 + (p-1)[p + p^2 + \ldots + p^k + (p^{k+1} - (p-1))(1 + p + \ldots + p^{k-1})] \\
&= 1 + (p-1)\left[\frac{p^{k+1} - p}{p-1} + \left(p^{k+1} - (p-1)\right)\frac{p^k - 1}{p-1}\right] \\
&= 1 + p^{k+1} - p + p^k p^{k+1} + p - 1 - p^k(p-1) - p^{k+1} \\
&= p^k\left(p^{k+1} - (p-1)\right). \quad \square
\end{aligned}$$

Examples: $5^4(5^5-4)$, $5^6(5^7-4)$, and $5^{14}(5^{15}-4)$ are 4-hyperperfect. $301, 49 \times 337$, and $7^5(7^6-6)$ are 6-hyperperfect. $11^2(11^3-10)$ and $11^{16}(11^{17}-10)$ are 10-hyperperfect. $13^3(13^4-12)$, $13^4(13^5-12)$, and $13^5(13^6-12)$ are 12-hyperperfect.

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From the Right Side

The Klein Bottle Elves

You will probably never meet one of the creatures represented here face to face. This is a rare look at some of the dreaded Gremlins of Academia.

Eric Hemmingsen, former chair of the Mathematics Department at Syracuse University, was famous for drawing these mysterious figures on dinner napkins, when napkins were made of stiff paper. Now, at long last, he has confessed that he often had a particular dean in mind. That might explain why the elf is often drawn when he has tied himself into knots and is picking his own pocket, or why his head has non-trivial homology.

The dean is long gone, and paper napkins are soft, but the Mathematics Library at S.U. is named the Eric Hemmingsen Library. It is located right in the mathematics department and is by virtue of its collection, as well as its location, a most valuable resource. Did the Klein Bottle Elves help to keep the library in the math building?

The ΠME Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.

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FIBONACCI PERIODS IN SYMMETRIC GROUPS

CHRISTOPHER NOONAN* AND ALAN KOCH†

Abstract. The Fibonacci Sequence is generalized so that the entries are not integers but permutations of a finite set. For any choice of initial conditions the resulting sequence is periodic. The concept of period is defined and computed for all S_n, $n \leq 6$. Additional properties are also shown, and many questions concerning Fibonacci periods are posed.

1. Introduction. One of the best known sequences that is usually defined recursively is the Fibonacci sequence. This sequence dates back to the thirteenth century and was originally used by Leonardo Fibonacci to study rabbit populations. The Fibonacci sequence, denoted $\{f_n\}$, is given by

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \end{aligned}$$

Since that time there have been numerous applications of the Fibonacci sequence. In 1960 D.D. Wall [6] considered the Fibonacci sequence modulo m. He observed that, mod m, the Fibonacci sequence would cycle, that is it would repeat itself over and over. For example, if $m = 4$ the sequence is

$$0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, 1 \ldots$$

Notice that $f_6 \equiv f_0$ and $f_7 \equiv f_1 \pmod 4$. Since the next term is always the sum of the two previous ones, once the initial two terms arise again the sequence must cycle. He said that the *period* of the Fibonacci sequence is 6 mod 4, which is written as $k(4) = 6$.

The period of the Fibonacci sequence mod p for p a prime has been studied both theoretically (for example [1],[2]) as well as computationally (for example [5]). Some of the properties of $k(p)$ are

1. If $p > 2$ then $k(p)$ is even. (In the case $p = 2$ we have $k(2) = 3$.)
2. If $p \equiv \pm 1 \pmod{10}$ then $k(p)$ divides $p - 1$.
3. If $p \equiv \pm 3 \pmod{10}$ then $k(p)$ divides $2(p+1)$ but not $p + 1$.
4. If $p \equiv \pm 3 \pmod{10}$ then $k(p)$ divides 4.

Proofs of these facts originally appeared in [6]. See also [4] for proofs using linear algebra. The period when p is a prime has been computed up to $p = 415,993$ ([2]).

A natural generalization of this problem is to change the initial conditions from 0 and 1. Perhaps surprisingly, this change usually does not change the length of the period mod p, although for p congruent to $\pm 1 \pmod{10}$ there are sometimes choices for initial conditions where the period is exactly one-half of $k(p)$[4, Th. 3.8c].

Of course, there is no reason why the notion of a "Fibonacci sequence" cannot be extended to other number systems. It is quote easy to extend the concept to any algebraic structure that has both an additive and a multiplicative identity, for example fields and matrices. If we drop the usual initial conditions, all that is needed to have a Fibonacci sequence is a set with a binary operation. In this paper, we will construct Fibonacci sequences where the terms are not real numbers but permutations, and

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instead of adding the terms we will compose them. Much like when working mod m, these permutations will also cycle, and the lengths of these periods can be computed.

While some preliminary results are given in section 4, the purpose is not to completely answer all of the questions concerning Fibonacci periods over symmetric groups. Instead, it is to inform the reader of the concept, describe a few basic facts about these periods, and provide unsolved problems that are accessible to any second-year undergraduate student.

The authors would like to thank M. Jean McKemie for her input during the preparation of this paper.

2. Symmetric Groups. We will briefly describe what a symmetric group is. The reader who is already familiar with symmetric groups and their properties may skip to the next section.

Let S_n be the set of all permutations of the set $\{1, 2, 3, \ldots, n\}$. In other words, an element $\sigma \in S_n$ is a bijection of the set. We will represent σ using *cycle notation.* For example, if $\sigma \in S_6$ is given by

$$\begin{array}{lll} \sigma(1) = 3 & \sigma(2) = 6 & \sigma(3) = 4 \\ \sigma(4) = 1 & \sigma(5) = 5 & \sigma(6) = 2 \end{array}$$

then we write $\sigma = (134)(26)$. To determine how σ permutes an element of the set, simply find the element, and if it is not adjacent to the right-hand parenthesis, σ maps this element to the element listed to its right. If it is adjacent to a right-hand parenthesis, then σ maps this element to the left-most element in the same set of parentheses. If the element does not appear at all, it is fixed.

As another example, if $\tau = (152)(367)(89)$, then $\tau(1) = 5$, $\tau(5) = 2$, and $\tau(2) = 1$. The triple (152) is called a *cycle* as repeated applications of τ cycle through these elements. Since it has length three is is called a 3-cycle. Similarly, $\tau(3) = 6$, $\tau(6) = 7$, and $\tau(7) = 3$. Also, τ interchanges 8 and 9 (and so (89) is a 2-cycle). Finally, $\tau(4) = 4$ since 4 does not appear in the cycle notation.

If σ and τ are permutations of the set $\{1, 2, 3, \ldots, n\}$, then so is the composition $\sigma\tau$, thus $\sigma\tau \in S_n$. We refer to this as the *product* of σ and τ. This can be computed using cycle notation by reading right to left. For example, if $\sigma = (124)$ and $\tau = (1324)$, then $\sigma\tau = (124)(1324)$. If we wish to find $\sigma\tau(3)$, the cycle on the right says that $\tau(3) = 2$. The cycle on the left gives $\sigma(2) = 4$. Thus $\sigma\tau(3) = 4$. Below is a table of $\sigma\tau(i)$ for $i = 1, 2, 3$, and 4.

$$\begin{aligned} \sigma\tau(1) &= \sigma(\tau(1)) = \sigma(3) = 3 \\ \sigma\tau(2) &= \sigma(\tau(2)) = \sigma(4) = 1 \\ \sigma\tau(3) &= \sigma(\tau(3)) = \sigma(2) = 4 \\ \sigma\tau(4) &= \sigma(\tau(4)) = \sigma(1) = 2 \end{aligned}$$

This tells us that $\sigma\tau = (124)(1324) = (1342)$.

Any element can be written as a product of 2-cycles. (A 2-cycle is also called a *transposition.*) For example, $(12345) = (15)(14)(13)(12)$. While the decomposition into transpositions is not unique (notice that $(12345) = (15)(14)(13)(12)(12)(12)$), the number of transpositions in the decomposition for an element is always even or always odd ([3, Th. 2.2.15]). An element $\sigma \in S_n$ is called *even* if it can be written as a product of an even number of transpositions. If it can be written as product of an odd number of transpositions, then σ is called *odd*. Given σ and τ we can determine

whether $\sigma\tau$ is even or odd – it works like addition of numbers:

σ	τ	$\sigma\tau$
Even	Even	Even
Even	Odd	Odd
Odd	Even	Odd
Odd	Odd	Even

It is not too hard to show that the multiplication works in this manner: the element $\sigma\tau$ can be decomposed into transpositions by decomposing σ and τ individually. Thus if σ decomposes into t_1 transpositions and σ_2 decomposes into t_2 transpositions, then one decomposition of $\sigma\tau$ uses $t_1 + t_2$ transpositions.

The permutation that fixes every element will be denoted e. This element has the unique property that $e\sigma = \sigma e = \sigma$ for all $\sigma \in S_n$.

3. Fibonacci Sequences. It makes sense to define the Fibonacci sequence $\{\sigma_n\}$ by the recurrence relation

$$\sigma_n = \sigma_{n-1}\sigma_{n-2}$$

However, there is no natural choice of initial conditions. While $e \in S_n$ is analogous to 0, there is no element in S_n analogous to 1. It is for this reason that we will study Fibonacci sequences with different choices of initial conditions. We will see that different choices of initial conditions will give us different period lengths, and that there are many more period lengths for S_n than there are in the mod p problem. The notation $k(S_n, \sigma_0, \sigma_1)$ will be the length of the period with initial conditions $\sigma_0, \sigma_1 \in S_n$.

Example. The following calculations show that $k(S_3, (12), (123)) = 6$:

$$\begin{array}{ll} \sigma_0 = (12) & \sigma_4 = (23)(12) = (132) \\ \sigma_1 = (123) & \sigma_5 = (12)(132) = (13) \\ \sigma_2 = (12)(123) = (23) & \sigma_6 = (132)(13) = (12) \\ \sigma_3 = (123)(23) = (12) & \sigma_7 = (13)(12) = (123) \end{array}$$

To illustrate how the initial conditions can make a difference in the period, notice that $k(S_3, e, (12)) = 2$:

$$\begin{array}{ll} \sigma_0 = e & \sigma_2 = (12)(12) = e \\ \sigma_1 = (12) & \sigma_3 = (12)e = (12) \end{array}$$

Some natural questions are:

1. Given n, what choice of initial conditions gives the largest period?
2. Given n, what is the maximum value of $k(S_n, \sigma_0, \sigma_1)$?
3. More generally, what are some properties of $k(S_n, \sigma_0, \sigma_1)$?

Let us denote by $k(S_n)$ the longest period using any initial conditions in S_n.

Those familiar with abstract algebra may ask: why use the group S_n rather than any other? By Cayley's Theorem [3, 2.1.16], every finite group of order n can be viewed as an subgroup of S_n. In other words, if we know $k(S_n)$ then for any group of order n we have an upper bound: $k(G) \leq k(S_n)$. Furthermore, $k(G, \sigma_0, \sigma_1) = k(S_n, \sigma_0, \sigma_1)$, where $\sigma_0, \sigma_1 \in G \leq S_n$, so to answer the question over S_n is to answer the question for any two elements picked from any finite group.

4. Results. So what is known about $k(S_n, \sigma_0, \sigma_1)$?

1. $k(S_2) = 3$, $k(S_3) = 8$, $k(S_4) = 18$, $k(S_5) = 96$, and $k(S_6) = 216$.
   These have been computed with the help of two MAPLE procedures that are given in the appendix.
2. $k(S_n) \leq (n!)^2$.
   There are $(n!)^2$ possible ordered pairs of elements in S_n. It is impossible for an ordered pair to appear twice without completing a period.
3. For all i, $k(S_n, \sigma_0, \sigma_1) = k(S_n, \sigma_i, \sigma_{i+1})$.
   In other words, if you pick any two consecutive terms in a period, then the period they generate has the same length as the original period. (In fact, it generates the exact same period!)
4. If either σ_0 or σ_1 is an odd permutation (or both), then $k(S_n, \sigma_0, \sigma_1)$ is a multiple of 3.
   This can be shown by considering the three cases. Consider the table below, where "E" and "O" represent even and odd permutations.

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	$\cdots$
Case 1	O	O	E	O	O	E	O	O	E	O	$\cdots$
Case 2	O	E	O	O	E	O	O	E	O	O	$\cdots$
Case 3	E	O	O	E	O	O	E	O	O	E	$\cdots$

   Notice that, in the first case, if the sequence repeats starting with σ_i then σ_i and σ_{i+1} must both be odd. This only occurs when i is a multiple of 3. The other two cases are similar.
   In the case where σ_0 and σ_1 are both even, of course, σ_i is even for all i, so no such conclusion can be drawn.
5. $k(S_n, \sigma, \tau) = k(S_n, \tau, \sigma)$ if $n \leq 4$.
   This again was determined using the MAPLE procedures. The result is false for $n \geq 5$: $k(S_5, (12345), (123)) = 12$ and $k(S_5, (123), (12345)) = 14$. Clearly, equality will always hold if $\sigma\tau = \tau\sigma$.

5. Questions. Finally, here are some unsolved problems.

1. Is there a formula for $k(S_n)$? There does not seem to be an obvious pattern in the results above. There may be a nicer pattern if only $k(S_p)$ is considered only when p is a prime – the study of Fibonacci sequences modulo m was simplified in the case where m was prime.
2. Is there a better upper bound for $k(S_n)$ than $(n!)^2$? In the examples computed above, the actual $k(S_n)$ was nowhere near this bound. In fact, for $n \leq 6$, $k(S_n) \leq n^3$.
   Generally, there are clearly better bounds for $k(S_n)$ than $(n!)^2$. For example, since S_{n-1} is contained in S_n in a natural way, any period starting with two elements in S_n is bounded by $k(S_{n-1})$. By the third result, if there are two consecutive terms in a Fibonacci period that come from S_{n-1}, then this period is also bounded by S_n. Using the bound $k(S_{n-1}) \leq ((n-1)!)^2$ gives

$$k(S_n) \leq \max\{(n!)^2 - ((n-1)!)^2, k(S_{n-1})$$
$$\leq \max\{((n-1)!)^2 (n^2-1), ((n-1)!)^2\} = ((n-1)!)^2 (n^2-1)$$

3. If $k(S_n, \sigma_0, \sigma_1) > 3$, must $k(S_n, \sigma_0, \sigma_1)$ be even? Evidence from $n \leq 6$ seems to suggest that this is so. By [7, Cor. 5], if there is a period of odd length greater than three then the period cannot contain the identity element. The converse, however, is not true, as no pair of initial conditions $\sigma_0, \sigma_1 \in S_6$ give an odd period of length greater than three.

Why would knowing $k(S_n)$ be important? One application is in the field of cryptology. Every element $\sigma \in S_{26}$ can be thought of as a rearrangement of the alphabet. If $\sigma_0, \sigma_1 \in S_{26}$, we can encode a message as follows: to encode the i^{th} letter, use the rearrangement given by σ_i, the i^{th} term in the Fibonacci sequence given by these initial conditions. This polyalphabetic substitution could be very hard to break if $k(S_{26}, \sigma_0, \sigma_1)$ were extremely large.

To demonstrate this cryptological application, here is an example of the idea applied only to the vowels. Suppose we want to send the message: *AOAUEEE*. Number the vowels $A = 1$, $E = 2$, $I = 3$, $O = 4$, and $U = 5$. Pick $\sigma_0 = (13452)$ and $\sigma_1 = (1243)$. The encryption is shown in the following table

Letter	Number	σ_i	Encoded Number	Encoded Letter
A	1	(13542)	3	I
O	4	(1243)	3	I
A	1	(45)	1	A
U	5	(12453)	3	I
E	2	(1253)	5	U
E	2	(145)(23)	3	I
E	2	(14352)	1	A

so the encoded message is *IIAIUIA*. Why is this difficult to break? For starters, a frequency count technique does not work – each letter gets coded differently. Since $k(S_5, (13452), (1243)) = 54$, the pattern will not reemerge until the 55^{th} character. Also, if the breaker has a little bit of information, it cannot be used to get more – for example if someone knew that the first letter was A all that person would know is that $\sigma_0(1) = 3$ – very little knowledge is obtained about σ_0 itself. In practice it is a good idea to pick σ_0 and σ_1 so that the $\sigma_i's$ do not have many fixed points. One way to do this is to let σ_0 be a 26-cycle and let σ_1 be some power of σ_0. In most cases $k(S_{26}, \sigma_0, \sigma_0^i) = 84$. Picking σ_1 to be a power of σ_0 decreases the number of usable combinations, and hence a person attempting to crack the code has fewer things to check. However the number of ways of picking σ_0 is still 25!, which is greater than 1.5×10^{25}.

Appendix: Maple Procedures. This first procedure computes the period given the two initial conditions.

```
> with(group):
> period:=proc(a,b) local i,sigma;
> sigma[0]:=a;
> sigma[1]:=b;
> for i from 2 to length(mulperms(a,b))^2 while
sigma[i-1]<>sigma[0] or mulperms(sigma[i-1],sigma[i-2])<>sigma[1] do
> sigma[i]:=mulperms(sigma[i-1],sigma[i-2]);
> od;
> (i-1);
> end;
```

This second procedure computes the maximum period when σ_0 is taken from "list" and σ_1 is any element of S_n. If "list" consists of one of each cycle type in S_n, then maxperiod will give $k(S_n)$. To use this, you will also need the "period" procedure from above.

```
> maxperiod:=proc(n,list) local i,j,s,max,temp,top;
> max:=0;
> s:=elements(permgroup(n,{[[1,2]],[[seq(i,i=1..n)]]}));
> top:=nops(list);
> for i from 1 to top do
> for j from 1 to n!  do
> temp:=period(list[i],s[j]);
> if temp>max then max:=temp fi;
> od;
> print(i,max);
> od;
> max;
> end;
```

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THE SUM OF THE K'TH POWERS OF THE FIRST N POSITIVE INTEGERS

A. SALEH-JAHROMI* AND JULIEN DOUCET†

Many different methods for calculating the sums of the k'th powers have been developed and many articles have been published in this area. For some history of the subject, and for a selection of these articles, we mention [1], [2], [3], [4], [5], [6], [7], and [8]. In this paper we give a new approach to computing the sum, which is

$$S_k(n)=\sum_{i=1}^{n} i^k = 1^k+2^k+3^k+4^k+5^k+\ldots+(n-1)^k+n^k,\ \text{where } k\geq 0. \quad (1)$$

This sum often occurs in computing areas [9] and in analyzing the number of operations needed to solve linear equations by Gauss elimination [10]. We will derive a formula for $S_k(n)$ which is recursive in k.

For the base if the recursion, we note that $S_0(n)=n$. For illustration, we compute $S_1(n)$ in terms of $S_0(n)$. We start with the formula

$$(1+x)^2=1+2x+x^2.$$

Substituting values for x in this formula, the following table can be created:

$$\begin{array}{ll}
x=1, & (1+1)^2=1+2(1)+1^2\\
x=2, & (1+2)^2=1+2(2)+2^2\\
x=3, & (1+3)^2=1+2(3)+3^2\\
x=4, & (1+4)^2=1+2(4)+4^2\\
x=5, & (1+5)^2=1+2(5)+5^2\\
\vdots & \vdots\\
x=n, & (1+n)^2=1+2(n)+n^2
\end{array}$$

By adding the left and right-hand sides of these equalities and simplifying, we obtain

$$(1+n)^2=n+2(1+2+3+4+5+\ldots+n)+1.$$

Rearranging this and combining it with (1) gives

$$\begin{aligned}S_1(n)&=\sum_{i=1}^{n} i = 1+2+3+4+5+\ldots+n\\ &=\frac{(1+n)^2-(1+n)}{2}=\frac{[1+S_0(n)]^2-[1+S_0(n)]}{2}\end{aligned} \quad (2)$$

Now, to compute $S_k(n)$ we start with

$$\begin{aligned}(1+x)^{k+1}&=\sum_{i=0}^{k+1}\binom{k+1}{i}x^i\\ &=\binom{k+1}{0}1+\binom{k+1}{1}x+\binom{k+1}{2}x^2+\cdots+\binom{k+1}{k-1}x^{k-1}+\binom{k+1}{k}x^k+x^{k+1}\end{aligned} \quad (3)$$

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As we did in determining $S_1(n)$, we substitute values for x and create the following table:

$$
\begin{aligned}
(1+1)^{k+1} &= \tbinom{k+1}{0}1 + \tbinom{k+1}{1}1 + \tbinom{k+1}{2}1^2 + \ldots + \tbinom{k+1}{k}1^k + 1^{k+1}\\
(1+2)^{k+1} &= \tbinom{k+1}{0}1 + \tbinom{k+1}{1}2 + \tbinom{k+1}{2}2^2 + \ldots + \tbinom{k+1}{k}2^k + 2^{k+1}\\
(1+3)^{k+1} &= \tbinom{k+1}{0}1 + \tbinom{k+1}{1}3 + \tbinom{k+1}{2}3^2 + \ldots + \tbinom{k+1}{k}3^k + 3^{k+1}\\
&\vdots\\
(1+n)^{k+1} &= \tbinom{k+1}{0}1 + \tbinom{k+1}{1}n + \tbinom{k+1}{2}n^2 + \ldots + \tbinom{k+1}{k}n^k + n^{k+1}
\end{aligned}
$$

Again we add the left and right-hand sides of the equalities and in this case we obtain:

$$
\begin{aligned}
(1+n)^{k+1} = &\tbinom{k+1}{0}n +\\
&\tbinom{k+1}{1}(1+2+3+\ldots+n) +\\
&\tbinom{k+1}{2}(1^2+2^2+3^2+\ldots+n^2) + \ldots +\\
&\tbinom{k+1}{k-1}(1^{k-1}+2^{k-1}+3^{k-1}+\ldots+n^{k-1}) +\\
&\tbinom{k+1}{k}(1^k+2^k+3^k+\ldots+n^k) + 1
\end{aligned}
$$

So we get

$$
\begin{aligned}
[1+S_0(n)]^{k+1} &= 1 + \tbinom{k+1}{0}S_0(n) + \tbinom{k+1}{1}S_1(n) + \tbinom{k+1}{2}S_2(n) + \cdots + \tbinom{k+1}{k}S_k(n)\\
&= 1 + \sum_{i=0}^{k-1}\tbinom{k+1}{i}S_i(n) + \tbinom{k+1}{k}S_k(n)
\end{aligned}
$$

Then

$$
S_k(n) = \frac{[1+S_0(n)]^{k+1} - \left[1+\sum_{i=0}^{k-1}\binom{k+1}{i}S_i(n)\right]}{k+1} \tag{4}
$$

If $k=1$, then equation (4) becomes $S_i(n) = \frac{(1+n)^2-(1+n)}{2} = \frac{n(1+n)}{2}$ implying the validity of equation (4).

Equation (4) is an iterative formula. Knowing $S_k(n)$ we can determine $S_{k+1}(n)$, in terms of $S_0(n), S_1(n), S_2(n), \ldots, S_k(n)$. Clearly $S_k(n)$ is a polynomial of degree $k+1$ in n, and can be written as

$$
S_k(n) = \frac{1}{k+1}k^{k+1} + \frac{1}{2}n^k + \text{a polynomial of degree } k-1,
$$

where the coefficients must sum to 1. The series terminates at n or n^2 according as k is even or odd, except for $S_1(n)$. This formula is particularly appropriate for computers since it is a generlization process for $S_{k+1}(n)$. Using MAPLE it is possible to calulate equation (4), for any positive k and n, in the following form:

```
S_k(n)=((((1+n)^(k+1))-1)-Sum('sum('i'^'j','i'=1..n)'*binomial
((k+1),'j'),\'j'=0..(k-1)))/(k+1);
```

Acknowledgement: We would like to thank professor Michael C. Berg of the Mathematics Department, and professor D.C. Moebs of the Physics Department of Loyola Marymount University for reading this paper and making suggestions.

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Dear Ghost of Gauss,

You are the master of all mathematical wisdom. I have studied and computed and cannot determine the answer to this question. Please, with your towering mathematical perspective, settle this question for me: "May I, or may I not, wear a ΠΜΕ *Lapel Pin on my* ΠΜΕ *Tee Shirt"?*

Frustrated and Anxious.

Dear F & A,

In general, no. The Tee shirt's lack of lapels is not an insurmoutable difficulty, nor are the cannons of good taste. Yet one further condition is needed. The ΠΜΕ lapel pin must be *yours*. Going around, taking other people's ΠΜΕ lapel pins and sticking them on your tee shirt will not make you popular in the mathematical community.

I have spoken.

P.S. The gold clad keypins are available at the national office at the price of $12 each. To purchase a keypin, write to the secretary-treasurer:

Robert M. Woodside
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FACTORIAL AS A CONTINUOUS FUNCTION

SEAN EFFINGER-DEAN*

Experimenting with the program Derive, I came upon an interesting phenomenon. When I asked Derive to graph the function $y = x!$, the program presented a smooth line, as if factorial were a continuous function. But, I could not see how something like .7! could even be determined, since factorial, in my mind, was only defined for whole numbers!

However, triangular numbers are defined as $1 + 2 + 3 + \ldots + n$ where n is a whole number. Gauss, as a schoolboy, found that this is the same as $n(n+1)/2$ which, as you will notice, makes sense for *all* numbers, not just whole numbers. My goal was established then: to find a *continuous* function that was equal to factorial for all whole numbers. After days of work (and many failed attempts), I finally discovered a very interesting solution to my question, which, I have since learned, is a classic result known as Stirling's formula. Be aware that my goal was not to find a proof for this formula; it was simply to investigate and, possibly, discover something about factorial. Therefore, some of the series I have used are not convergent, and must be considered "formal" series instead. Here is what I did to reach my solution, step by step.

Starting with $n! = 1 \cdot 2 \cdot 3 \cdots n$, we take logarithms and use Taylor series to get:

$$\ln(n!) = \sum_{i=1}^{n} \ln(i) = \sum_{i=1}^{n}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(i-1)^k}{k}$$

These terms are rearranged, (even though most of these series are not convergent.)

$$\ln(n!) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{i=1}^{n}(i-1)^k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{i=1}^{n-1} i^k$$

We denote $\sum_{i=1}^{n} i^k$ by $f_k(n)$, so, setting $u = n-1$, we have

$$\ln(n!) = f_1(u) - \frac{1}{2}f_2(u) + \frac{1}{3}f_3(u) - \frac{1}{4}f_4(u) + \cdots$$

Using formulas for $f_k(u)$, see for example the previous article,[1] we get

$$\ln n! = \left[\frac{u^2}{2} + \frac{u}{2}\right] - \left[\frac{u^3}{6} + \frac{u^2}{4} + \frac{u}{12}\right] + \left[\frac{u^4}{12} + \frac{u^3}{6} + \frac{u^2}{12}\right] - \left[\frac{u^5}{20} + \frac{u^4}{8} + \frac{u^3}{12} - \frac{u}{120}\right] + \cdots$$

Hmmmmm... let's take the derivative!

$$\frac{d\ln n!}{dn} = \left[u + \frac{1}{2}\right] - \left[\frac{u^2}{2} + \frac{u}{2} + \frac{1}{12}\right] + \left[\frac{u^3}{3} + \frac{u^2}{2} + \frac{u}{6}\right] - \left[\frac{u^4}{4} + \frac{u^3}{2} + \frac{u^2}{4} - \frac{1}{120}\right] + \cdots$$

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[1] Editorial Note: The author derived the formulas for the sums of the k'th powers of the integers quite differently from methods in previous article. He establishes the recursion

$$f_k(u) = k\left(\int_0^u f_{k-1}(t)dt - c_k u\right).$$

and reports on interesting patterns for the value of the constant c as k increases.

Regrouping by first terms, second terms, etc....

$$\frac{d\ln n!}{dn} = [u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots] + \frac{1}{2}[1 - u + u^2 - \ldots] - \frac{1}{12}[1 - 2u + 3u^2 - \ldots] + \cdots$$

These are all Taylor expansions!

$$\frac{d\ln n!}{dn} = \ln(n) + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots$$

where we have replaced $(u+1)$ by n. Integration yields

$$\ln n! = b + n\ln(n) - n + \frac{\ln(n)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \cdots$$

where b is simply a constant. Hence we get:

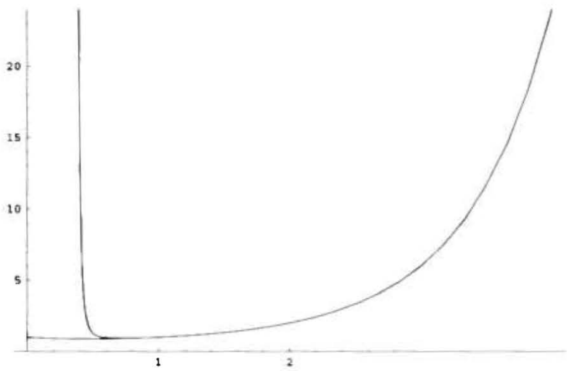
$$n! = (B)n^n\sqrt{n}\exp\left(-n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \cdots\right)$$

where $B = e^b$. Using large integral values for n to approximate B, we find that $B =$ 2.5066282746310005.... This constant doesn't look familiar, but some experimentation with a calculator seems to show that $B = \sqrt{2\pi}$.

My formula turns out then to be what I'm told is Stirling's formula!

$$n! = n^n\sqrt{2\pi n}\exp\left(-n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \cdots\right)$$

The graph below shows plots both of *Mathematica*'s $y = x!$ and of my result, which appear to correspond nicely after about $x = 0.6$.



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Editor's Comment. This paper is a brilliant example of how far a student can get using modern tools with mathematical insight. Sean rediscovered a classical result several years before he is scheduled to see it in a good undergraduate curriculum, where it would most likely be stated without proof. The reader is encouraged to compare Sean's approach with a combinatorial proof of Stirling's formula (see e.g. Bogart, Introductory Combinatorics, Harcourt Brace) in order to justify the use of formal power series.

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SOME PROPERTIES OF THE EQUATION $S(X) = K$

REX H. WU*

In 1979, Florentin Smarandache introduced a number theoretic function. For any positive integer n, the Smarandache function $S(n)$ is defined as $S(n) = k$ if k is the smallest positive integer such that n divides $k!$. Since then, some interesting properties have been discovered about this function. Just one example, for $x > 4$, the expression

$$\pi(x) = -1 + \sum_{k=2}^{x}\left[\frac{S(k)}{k}\right],$$

where $[x]$ is the greatest integer function, gives the exact number of primes less than or equal to x, [1].

In this note, we will look at some elementary properties associated with the equation $S(x) = k$.

First, let's see how we can solve the equation $S(x) = k$. Suppose

$$k = p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}\cdots p_j^{\beta_j} \text{ and}$$
$$(k-1)! = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_j^{\alpha_j}p_{j+1}^{\alpha_{j+1}}\cdots p_t^{\alpha_t}.$$

Then $k! = p_1^{\alpha_1+\beta_1}p_2^{\alpha_2+\beta_2}p_3^{\alpha_3+\beta_3}\cdots p_j^{\alpha_j+\beta_j}\,p_{j+1}^{\alpha_{j+1}}\cdots p_t^{\alpha_t}$, for some prime p_i and nonnegative integers α_i and β_i; with $i = 1, 2, 3, \ldots, j, \ldots, t$. Here, j is used as the number of prime factors of k. Note that if p_i is a prime that divides k but not $(k-1)!$, then take $\alpha_i = 0$. If x_0 were a solution to $S(x) = k$, then $x_0 \mid k!$. Furthermore, $x_0 \nmid (k-1)!$. Obviously x_0 contains some factor $p_i^{\gamma_i}$, where $\alpha_i < \gamma_i \le \alpha_i + \beta_i$, for some $i = 1, 2, 3, \ldots, j$. So we have our first conclusion.

THEOREM 1. *x_0 is a solution to $S(x) = k$ if and only if $x_0 = MNQ$, where*

$$M = \prod_{i \in I} p_i^{\lambda_i},$$

where I can be any nonempty subset of $\{1, 2, 3, \ldots, j\}$ and $1 \le \lambda_i \le \beta_i$;

$$N = \prod_{i \in I} p_i^{\alpha_i},$$

where, again, if p_i is a prime that divides k but not $(k-1)!$, then take $\alpha_i = 0$; and Q is any factor of $(k-1)!/N$.

Proof. We have

$$MN = \prod_{i \in I} p_i^{\alpha_i + \lambda_i}.$$

Since $\alpha_i < \alpha_i + \lambda_i \le \alpha_i + \beta_i$, we know $MN \mid k!$ but $MN \nmid (k-1)!$. For N, using the highest exponent α_i so that $p_i^{\alpha_i} \mid (k-1)!$ is essential. Otherwise, MN may divide $(k-1)!$ and rendering MN not a solution. Furthermore, if Q divides $(k-1)!/N$, then

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MNQ divides $M(k-1)!$ which in turn divides $k!$. Therefore, MNQ is a solution to $S(x) = k$.

Observe that using all the nonempty subsets of $\{ 1, 2, 3, \ldots, j \}$ for M would generate all the factors of k. In combination with all the factors of $(k-1)!/N$, we have all the solutions to $S(x) = k$.

Suppose there is a solution x_0 to $S(x) = k$, we are going to show x_0 is of the form MNQ. Let $x_0 = p_1^{\varepsilon_1} p_2^{\varepsilon_2} p_3^{\varepsilon_3} \cdots p_m^{\varepsilon_m}$. Note that $S(x_0) = k = \max\{S(p_i^{\varepsilon_i})\}$ for $i = 1, 2, 3, \ldots, m$; with $m \leq t$. Obviously, if p_i is not a factor of k, $S(p_i^{\varepsilon_i}) < k$. Even if p_i is a factor of k, if $\varepsilon_i \leq \alpha_i$, then $S(p_i^{\varepsilon_i}) < k$ because $S(p_i^{\varepsilon_i}) \mid (k-1)!$. If $\varepsilon_i > \alpha_i + \beta_i$, $S(p_i^{\varepsilon_i}) > k$. Therefore, we have $p_i \mid k$ and $\alpha_i < \varepsilon_i \leq \alpha_i + \beta_i$. Notice that there can be more than one such p_i's such that $S(p_i^{\varepsilon_i}) = \max\{S(p_i^{\varepsilon_i})\}$ if k has more than one prime factor. This shows if x_0 were a solution, then x_0 contains $MN = \prod_{i=1}^{j} p_i^{\varepsilon_i}$, for some subset I of $\{1, 2, 3, \ldots, j\}$ and $\alpha_i < \varepsilon_i \leq \alpha_i + \beta_i$. Also notice that any multiples of MN, say MNQ, is a solution to $S(x) = k$, provided $MNQ \mid k!$. The question is what can Q be?

Obviously, $MNQA = k!$, for some integer A. $QA = k!/MN = (k/M)((k-1)!/N)$. From the previous expression, Q can be any factor of $(k-1)!/N$. What if Q contains a prime factor p_q such that p_q is also a factor of k? Then p_q must have an exponent $\varepsilon_q \leq \alpha_q$, in which case $p_q^{\varepsilon_q}$ is a factor of $(k-1)!/N$. Otherwise, $S(p_q^{\varepsilon_q}) = k$ if $\alpha_q < \varepsilon_q \leq \alpha_q + \beta_q$, but then this factor would be part of MN. Or $S(p_q^{\varepsilon_q}) > k$ if $\varepsilon_q > \alpha_q + \beta_q$. Therefore, we can only have $Q \mid ((k-1)!/N)$. □

An example would best illustrate this theorem. Let's solve $S(x) = 12$. Here $k = 12 = 2^2 \times 3$, $(k-1)! = 11! = 2^8 \times 3^4 \times 5^2 \times 7 \times 11$. Let's look at the number of solutions instead of each individual solution. Obviously, the number of solutions for any particular M is $\tau(Q)$, where $\tau(n)$ is the number of factors for the positive integer n. If $n = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, then $\tau(n) = (\alpha_0 + 1)\ (\alpha_1 + 1)\ (\alpha_2 + 1) \cdots (\alpha_n + 1)$.

M	N	$Q = 11!/N$	$\tau(Q)$ = number of solutions
2	2^8	factors of $3^4 \times 5^2 \times 7 \times 11$	60
3	3^4	factors of $2^8 \times 5^2 \times 7 \times 11$	108
2^2	2^8	factors of $3^4 \times 5^2 \times 7 \times 11$	60
2×3	$2^8 \times 3^4$	factors of $5^2 \times 7 \times 11$	12
$2^2 \times 3$	$2^8 \times 3^4$	factors of $5^2 \times 7 \times 11$	12

Adding up the last column gives a total of 252 solutions.

While the above theorem works, it gets cumbersome if k gets large. Let's explore a little bit and look for a simpler method. We will also switch our attention to look for the number of solutions rather than all the specific solutions to $S(x) = k$.

THEOREM 2. *x_0 is a solution to $S(x) = k$ if and only if $x_0 \mid k!$ and $k \nmid (k!/x_0)$.*

Proof. Suppose x_0 is a solution to $S(x) = k$, then by definition, $x_0 \mid k!$ and for any $n < k$, $x_0 \nmid n!$. It suffices to show the case $n = k-1$, since if $x_0 \nmid (k-1)!$ then $x_0 \nmid n!$ for any $n < k-1$. Therefore, for $n = k-1$, $x_0 \nmid n!$ implies $kx_0 \nmid k!$ or $k \nmid (k!/x_0)$.

Let's say $x_0 | k!$ but $k \nmid (k!/x_0)$. Since $k \nmid (k!/x_0)$ is equivalent to $kx_0 \nmid k!$, or $x_0 \nmid (k-1)!$. Obviously, if $x_0 \nmid (k-1)!$, then $x_0 \nmid n!$ for any $n \leq k-1$. This is the very definition of the Smarandache function. Therefore, $S(x_0) = k$. □

Theorems 1 and 2 are actually equivalent. To see if MNQ is a solution or not, all we need to do is to see if k divides $k!/(MNQ)$ or not. Suppose p_m is one of the primes used in MN, i.e. $1 \leq m \leq j$, then $MNQ = p_m^{\alpha_m + \lambda_m} A$, for some integer A and $1 \leq \lambda_m \leq \beta_m$. So $k!/(MNQ) = p_m^{\beta_m - \lambda_m} B$ for some integer B. Obviously,

$k \nmid (k!/(MNQ))$ because k has a factor $p_m^{\beta_m}$ and $p_m^{\beta_m} \nmid p_m^{\beta_m - \lambda_m} B$.

COROLLARY 3. *If k is prime, then there are $\tau(k!)/2$ solutions to $S(x) = k$.*

Proof. Let's pair up the divisors of $k!$ such that the product of each pair is $k!$, i.e., if $x_0 \mid k!$, then x_0 is paired up with $k!/x_0$. k is prime implies $k \nmid (k-1)!$. Then either $k \mid x_0$ or $k \mid (k!/x_0)$ but not both. If $k \nmid x_0$, then $k!/x_0$ is a solution to $S(x) = k$. Otherwise, x_0 is. This shows exactly half of the factors of $k!$ are solutions to $S(x) = k$ if k is prime. □

Once we know theorem 2, we can look for the number of solutions to $S(x) = k$ with ease. Let's denote $\omega(k)$ the number of solutions to $S(x) = k$.

COROLLARY 4. *There are $\omega(k) = \tau(k!) - \tau((k-1)!)$ solutions to $S(x) = k$.*

Proof. According to Theorem 2, this is to look for the number of factors of $k!$ that are not divisible by k.

Let's look at the factors of $k!$, in particular, we are interested in the ones that are not divisible by k. To look for those, we will find out the number of factors that are divisible by k, i.e., factors of the form kA, for some integer A. Since $kA \mid k!$, we have $A \mid (k-1)!$. There is a total of $\tau((k-1)!)$ such A's. Since there are $\tau(k!)$ factors of $k!$, there are $\omega(k) = \tau(k!) - \tau((k-1)!)$ factors that are not divisible by k. □

Corollary 2 gives another proof to corollary 1. If k is prime, then $k = p_0$ is a prime different from all the primes less than or equal to $(k-1)$. If there are $\tau((k-1)!)$ factors for $(k-1)!$, then, $\tau(k!) = \tau(k(k-1)!) = \tau(k)\tau((k-1)!) = 2\tau((k-1)!)$, which is the same as $\omega(k) = \tau(k!)/2$.

Now let's look at the first 15 values for $\omega(k)$. Note that $\omega(12)$ confirms the result we obtained using theorem 1.

k	$\tau(k!)$	$\omega(k)$
1	1	1
2	2	1
3	4	2
4	8	4
5	16	8
6	30	14
7	60	30
8	96	36
9	160	64
10	270	110
11	540	270
12	792	252
13	1584	792
14	2592	1008
15	4032	1440

Pay attention to the $\tau(k!)$'s and $\omega(k)$'s where $\omega(k) = \tau(k!)/2 = \tau((k-1)!)$. Also look at the corresponding k. A pattern seems to arise. The k's are prime except when it is 4. One may wonder if this pattern would be true for all.

Before we go onto proving the above statement, we need to utilize a function, $E(n,p)$, which gives the largest exponent of a prime p such that $p^{E(n,p)} \mid n!$.

$$E(n,p) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i} \right]$$

gives the numerical value of $E(n,p)$, where $[x]$ is the greatest integer function. In particular, if $n = p^m$,

$$E(n,p) = 1 + p + p^2 + \cdots + p^{m-1} = \frac{p^m - 1}{p - 1}.$$

Also observe that

(i) if $n = Np$, for some positive integer N, then $N = \frac{n}{p} \leq E(n,p)$ with equality only when $N < p$ and

(ii) if $p_1 > p_2$, then $E(n,p_1) \leq E(n,p_2)$.

LEMMA 5. *If $n = Qp^m$ for some prime p and some integers Q and m, then $E(n,p) = Q\,E(p^m,p) + E(Q,p)$.*

Proof

$$\begin{aligned} E(n,p) &= \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right] \\ &= \sum_{i=1}^{\infty} \left[\frac{Qp^m}{p^i}\right] \\ &= \sum_{i=1}^{m} Qp^{m-i} + \sum_{i=m+1}^{\infty} [Qp^{m-i}] \\ &= Q(1 + p + p^2 + \cdots + p^{m-1}) + \sum_{i=1}^{\infty} \left[\frac{Q}{p^i}\right] \\ &= Q\,E(p^m,p) + E(Q,p) \qquad \square \end{aligned}$$

THEOREM 6. *If there are $\tau(k!)/2$ solutions to $S(x) = k$, then k is prime or $k = 4$.*

Proof. Here, $\omega(k) = \tau(k!) - \tau((k-1)!) = \tau(k!)/2$. Or equivalently, if $\tau(k!) = 2\tau((k-1)!)$ then k is prime or $k = 4$. If we could show that k is a composite number other than 4 implies $\tau(k!) \neq 2\tau((k-1)!)$ and we are done.

Again, let's write k and $(k-1)!$ in their canonical prime factorization forms, $k = p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}\cdots p_j^{\beta_j}$ and $(k-1)! = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_j^{\alpha_j}p_{j+1}^{\alpha_{j+1}}\cdots p_t^{\alpha_t}$. Then, $E(k,p_i) = \alpha_i + \beta_i$ and $E(k-1,p_i) = \alpha_i$.

Next, we will look at the inequality $\beta_i \leq \frac{\alpha_i}{2j}$, for some positive integer $i \leq j$. In particular, we are interested in whether $\tau(k!) = 2\tau((k-1)!)$ or not when $\beta_i \leq \frac{\alpha_i}{2j}$ and when $\beta_i > \frac{\alpha_i}{2j}$.

If we rewrite $k = Qp_i^{\beta_i}$, then from the lemma, we have $E(k,p_i) = E(Qp_i^{\beta_i},p_i) = Q\,E(p_i^{\beta_i},p_i) + E(Q,p_i)$. Furthermore, $\alpha_i = E(k,p_i) - \beta_i = Q\,E(p_i^{\beta_i},p_i) + E(Q,p_i) - \beta_i$. Suppose $\beta_i > \alpha_i/2j$, then a substitution for α_i and some rearrangements give $2j + 1 > (Q\,E(p_i^{\beta_i},p_i) + E(Q,p_i))/\beta_i$. And finally,

(1)
$$2j + 1 > Q\,\frac{p_i^{\beta_i} - 1}{\beta_i(p_i - 1)} + \frac{E(Q,p_i)}{\beta_i}.$$

Case (I). k has only one prime factor, $k = p^\beta$ with $\beta > 1$.

From the assumption, we have $j = 1$ and $Q = 1$. From Equation (1) we have

$$3 > \frac{p^\beta - 1}{\beta(p-1)}.$$

Note that $E(Q,p)/\beta = E(1,p)/\beta = 0$. There are only a few cases that this inequality is true, namely, $(p,\beta) = (2,2),(2,3)$ and $(3,2)$, corresponding to $k = 4, 8$ and 9, respectively. By assumption $k \neq 4$. It is easy to check that $\tau(8!) \neq 2\tau(7!)$ and $\tau(9!) \neq 2\tau(8!)$.

Case (II). k has 2 distinct prime factors, $k = p_1^{\beta_1}p_2^{\beta_2}$ with $\beta_i > 0$.

Here, $j = 2$ and without loss of generality, $Q = p_1^{\beta_1}$. Again, we have

$$2j + 1 = 5 > Q\,\frac{p_2^{\beta_2} - 1}{\beta_2(p_2 - 1)} + \frac{E(Q,p_2)}{\beta_2}.$$

The inequality is true if $(Q,p_2,\beta_2) = (2,3,2),(2,p_2,1),(2^2,p_2,1)$ for some prime $p_2 \geq 3$ and $(3,p_2,1)$ for some prime $p_2 \geq 5$.

For $(Q,p_2,\beta_2) = (2,3,2)$, we have $k = 18$. A little computation shows that $\tau(18!) \neq 2\tau(17!)$.

For $(Q,p_2,\beta_2) = (2,p_2,1)$, or $k = 2p_2$, if $\tau((k-1)!) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_t+1)$, then $\tau(k!) = (\alpha_1+1+1)(\alpha_2+1+1)(\alpha_3+1)\cdots(\alpha_t+1)$. If $\tau(k!) = 2\tau((k-1)!)$, we have $(\alpha_1+1+1)(\alpha_2+1+1) = 2(\alpha_1+1)(\alpha_2+1)$. Simplifying the last equation gives $2 = \alpha_1\alpha_2$. Therefore $(\alpha_1,\alpha_2) = (1,2)$ or $(2,1)$. From observation (ii), we know $(\alpha_1,\alpha_2) = (1, 2)$ is not possible. It is also impossible for $(\alpha_1,\alpha_2) = (2, 1)$ because $\alpha_1 = 2 \neq E(2p_2 - 1, 2)$ for any prime p_2.

The argument is identical for $(Q,p_2,\beta_2) = (3,p_2,1)$. When we reach $(\alpha_1,\alpha_2) = (2, 1)$, we have $\alpha_1 = 2 = E(3p_2 - 1, 3)$. Here, we have $3p_2 - 1$=6, 7 or 8. But then there is no $p_2 > 3$ satisfying this condition.

Similarly, for $(Q,p_2,\beta_2) = (2^2,p_2,1)$, we have $4 = (\alpha_1 - 1)\alpha_2$ after equating $\tau((2^2p_2)!) = 2\tau((2^2p_2 - 1)!)$. Solving $4 = (\alpha_1 - 1)\alpha_2$ to get $(\alpha_1,\alpha_2) = (2, 4)$, $(3, 2)$ and $(5, 1)$. Again, by observation (ii), $(\alpha_1,\alpha_2) = (2, 4)$ is impossible. $(\alpha_1,\alpha_2) = (3, 2)$ is also impossible because $\alpha_1 = 3 = E(2^2p_2 - 1, 2)$ implies $2^2p_2 - 1 = 4$ or 5. But there is no such a p_2. For $(\alpha_1,\alpha_2) = (5, 1)$, $\alpha_1 = 5 \neq E(2^2p_2 - 1, 2)$ for any p_2.

Case (III). k has 3 distinct prime factors, $k = p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}$ with $\beta_i > 0$.

Here, $j = 3$ and $Q = p_1^{\beta_1}p_2^{\beta_2} \geq 2 \times 3 = 6$. The inequality

$$2j + 1 = 7 > Q\,\frac{p_3^{\beta_3} - 1}{\beta_3(p_3 - 1)} + \frac{E(Q,p_3)}{\beta_3}$$

holds if $(Q,p_3,\beta_3) = (6,p_3,1)$ for some prime $p_3 \geq 5$. That is $k = 2 \times 3 \times p_3$. Again, setting $\tau(k!) = 2\tau((k-1)!)$ yields $2(\alpha_1 + \alpha_2 + \alpha_3 + 3) = \alpha_1\alpha_2\alpha_3$. From observation (i), we know $\alpha_1 = E(6p_3, 2) > 3p_3 \geq 15$, $\alpha_2 = E(6p_3, 3) > 2p_3 \geq 10$ and $\alpha_3 = E(6p_3,p_3) \geq 6$. It is easy to verify that under these conditions $\alpha_1\alpha_2\alpha_3 > 2(\alpha_1 + \alpha_2 + \alpha_3 + 3)$.

Case (IV). k has 4 or more distinct prime factors.

We have $j \geq 4$ and

$$Q = \prod_{i=1}^{j-1} p_i^{\beta_i} \geq 2 \times 3 \times 5 = 30.$$

Since $2j + 1 < Q$, we have

$$2j + 1 < Q\,\frac{p_j^{\beta_j} - 1}{\beta_j(p_j - 1)} + \frac{E(Q,p_j)}{\beta_j}.$$

We have just shown $\tau(k!) \neq 2\tau((k-1)!)$ for the special cases when $\beta_i > \alpha_i/2j$. Now let's show $\tau(k!) < \sqrt{e}\,\tau((k-1)!) < 2\tau((k-1)!)$ for any other composite number $k \neq 4$ if $\beta_i \leq \alpha_i/2j$. Here $e = 2.71828\ldots$ is the Euler number.

$\beta_i \leq \alpha_i/2j$ implies $\beta_i < (\alpha_i + 1)/2j$. We also know

$$\tau(k!) = (\alpha_1 + \beta_1 + 1)(\alpha_2 + \beta_2 + 1)\cdots(\alpha_j + \beta_j + 1)(\alpha_{j+1} + 1)\cdots(\alpha_t + 1)$$

and

$$\tau((k-1)!) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1)(\alpha_{j+1} + 1)\cdots(\alpha_t + 1).$$

Since all factors after the $(j+1)$'st term are the same, it suffices if we just look at $(\alpha_1+\beta_1+1)(\alpha_2+\beta_2+1)\cdots(\alpha_j+\beta_j+1)$ and $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_j+1)$. Subsequently, we have

$$\begin{aligned}(\alpha_1 + 1 + \beta_1)\cdots(\alpha_j + 1 + \beta_j) &< \left(\alpha_1 + 1 + \frac{\alpha_1 + 1}{2j}\right)\cdots\left(\alpha_j + 1 + \frac{\alpha_j + 1}{2j}\right)\\ &= (\alpha_1 + 1)\cdots(\alpha_j + 1)\left(1 + \frac{1}{2j}\right)^j.\end{aligned}$$

Since $\lim\left(1 + \frac{1}{2j}\right)^j = \sqrt{e}$ and it is a strictly increasing function, we have

$$\begin{aligned}(\alpha_1 + 1 + \beta_1)(\alpha_2 + 1 + \beta_2)\cdots(\alpha_j + 1 + \beta_j) &< \sqrt{e}(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1)\\ &< 2(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_j + 1).\end{aligned}$$

In other words, $\tau(k!) < \sqrt{e}\tau((k-1)!) < 2\tau((k-1)!)$. □

Combining corollary 1 and theorem 3, we have:

THEOREM 7. *There are $\tau(k!)/2$ solutions to $S(x) = k$ if and only if k is prime or $k = 4$.*

COROLLARY 8. *$\tau(k!) = 2\tau((k-1)!)$ if and only if k is prime or $k = 4$.*

Proof. This follows from Theorem 7 if we let $\tau(k!)/2 = \tau(k!) - \tau((k-1)!)$. □

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[1] SEAGULL, L., *Smarandache function and the number of primes up to x*, Mathematical Spectrum. Vol. 28 No. 3, p 53. 1995.

Rex H. Wu, NYU Downtown Hospital, Department of Medicine, 170 William Street, New York, NY 10038, rexhwu@aol.com

Rex H. Wu is a physician at NYU Downtown Hospital who saw the airplane crashing into the World Trade Center on September 11, 2001. He couldn't imagine people would do such vicious acts. Only blocks away, he and his colleagues treated a few hundred victims at NYU Downtown Hospital that day. He also volunteered on-site the next couple days. He wishes to express his deepest sorrow to all the innocent lives lost during the attack. And his greatest respect goes to all the heroes on ground zero.

CLAYTON W. DODGE RECEIVES THE C. C. MACDUFFEE AWARD FOR DISTINGUISHED SERVICE

On August 4, 2001, at the annual meeting of Pi Mu Epsilon in Madison, Wisconsin, the C. C. MacDuffee Award for Distinguished Service was presented to Clayton W. Dodge, Professor Emeritus of Mathematics at the University of Maine. The location of this presentation was of historic significance, because Cyrus C. MacDuffee, seventh president of Pi Mu Epsilon, was Professor at the University of Wisconsin. The Award, established in 1966, honors the memory of this superb teacher and algebraist, whose dedication and service profoundly influenced our society. Previous award recipients are J. Sutherland Frame, Richard V. Andree, John S. Gold, Francis Regan, J. C. Eaves, Houston Karnes, Richard Good, Milton D. Cox, and Eileen L. Poiani.

Professor Clayton Dodge was an active student at Miss Blakeslee's Kindergarten in Malden, Massachusetts, and his later education was "all downhill from there". In 1949, he graduated from Browne and Nichols School in Cambridge, Massachusetts, spent a semester at Harvard and eventually graduated from the University of Maine in 1956, majoring in mathematics with minors in electrical engineering and psychology.

He labored to teach arithmetic, algebra, and science for a whole six months at Brecksville Junior-Senior High School in Ohio and joyfully returned to teach at the University of Maine as an instructor of mathematics. In 1960 he received a master's degree in mathematics under Howard Eves, who inspired him to work in geometry and problems. He did graduate work in mathematics at Brown University in 1960-1961.

For two years in the early 1960's, he assisted Howard Eves in editing the Elementary Problems Department of the American Mathematical Monthly. Later, he served on the University of Maine Problems Group for the seven years that it edited that department. In 1981 he assumed the editorship of the Problem Department of the Pi Mu Epsilon Journal.

With the current issue, Clayton Dodge has completed a remarkable 20 years as Problems Editor of this Journal. Starting with problem #462, (Spring 1980, Volume7 No2) problem proposals were sent to Clayton Dodge, while Leon Bankoff was still problems editor. Transition from Leon Bankoff to Clayton Dodge took place over the period of a year. With the spring 1982 issue his apprenticeship had ended. All problem proposals and solutions were received, handled with care, formulated, polished, checked and corrected by Clayton Dodge, until problem #1006, which was the last problem whose solution was to be sent to the by now so familiar address. The Fall 2000 issue was the start of a new transition. More than half of all the problems published thus far in this Journal have gone through the hands of Clayton Dodge.

He has written five published textbooks, two others that were duplicated for use in his classes and has written several articles primarily on pedagogy, geometry, and calculators. A strong advocate of the use of calculators and computers by students, he wrote text material and taught several courses in their use, emphasizing the understanding of their workings so as to maximize their usefulness and make their results meaningful, see for example [1]. When color came to computers, because there was a great lack of appropriate software, he wrote software for graphing functions in both 2 and 3 dimensions, for demonstrating basic concepts of the calculus, and for grade books, software that gained wide acceptance during the DOS years.

Since retirement he has helped build houses for the local chapter of Habitat for Humanity and he serves on its board of directors. He sings in a choir and an oratorio society, and he has taken up the sport of scuba diving in warm tropical waters. He dabbles in stained glass and enjoys working around the house.

For a mathematical project, he is editing notes for a book on the arbelos, written by the late Victor Thebault of France and the late Leon Bankoff of Los Angeles, for 60 years a practicing dentist and PMEJ problems editor from 1968 to 1981. The arbelos is the figure formed as follows. Draw two mutually tangent circles, external to one another and not necessarily the same size. Surround these circles by another circle just tangent to them both. These circles all share a common diametral line. Cut the figure along this line and throw away one half, including the line. The figure that remains, looking like a bent two-tined fork, is the arbelos, also known as the shoemaker's knife. It may also be described as a triangle whose sides are semicircles and whose angles are all zero degrees.

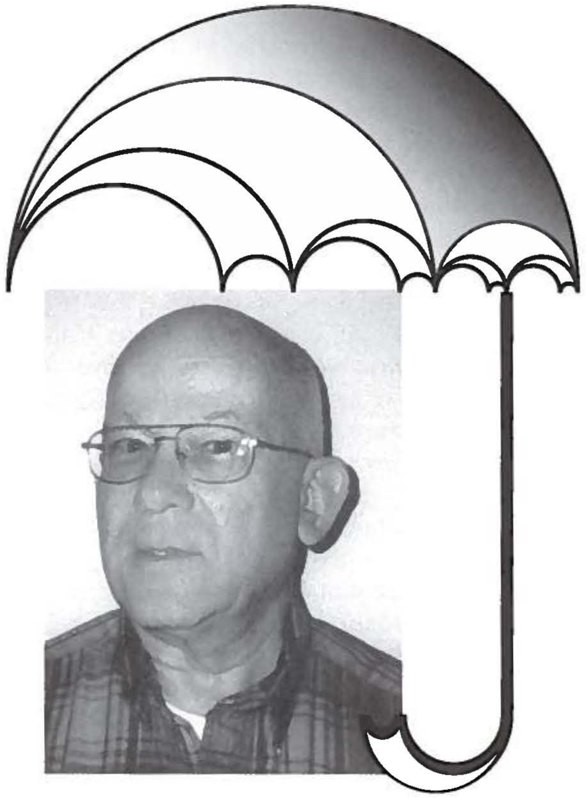

FIG. 1. *Professor Dodge never goes anywhere without his arbelos.*

We hope that after the transition to the new problems editors is complete, Clayton Dodge will quickly complete his arbelos task, the impatient reader may enjoy a preview in [2].

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[1] CLAYTON W. DODGE, *Square roots and calculators,* ΠME Journal, Vol. 11, No. 2, pp 69–74, 2000.

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ΠME Journal, Vol. 11, No. 5, pp 273–287, 2001.

PROBLEM DEPARTMENT

EDITED BY MICHAEL MCCONNELL, JON A. BEAL, AND CLAYTON W. DODGE

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Michael McConnell, 840 Wood Street, Mathematics Department, Clarion University, Clarion, PA 16214, or sent by email to mmcconnell@clarion.edu. Electronic submissions using LaTeX are encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by July 1, 2002. Solutions identified as by students are given preference.

Problems for Solution.

1021. *Proposed by Tom Moore, Bridgewater State College, Bridgewater, Massachusetts*

Student solutions solicited

Let $D(n)$ be the sum of the (base 10) digits of the positive integer n. Are there twin primes p and $p+2$ such that $D(p) = D(p+2)$?

1022. *Proposed by William Chau, Middletown, New Jersey*

Find an ordered pair (n, m) where n and m are composite numbers such that $n! = m^2$, or prove that there is none.

1023. *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, New York*

If $U_1 = 16$ and $U_{n+1} = U_n + 8n + 12$, find

$$\sum_{n=0}^{\infty} \frac{1}{U_{n+1}}$$

1024. *Proposed by Clayton W. Dodge, University of Maine, Orono. Maine*

Find the largest positive integer b and an integer c such that

$$\sqrt{2002 + b\sqrt{c}} + \sqrt{2002 - b\sqrt{c}} = 64.$$

1025. *Proposed by Ayoub B. Ayoub, Pennsylvania State University–Abington College, Abington, Pennsylvania*

Find the following in simplest form;

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{x+k}$$

1026. *Proposed by Ayoub B. Ayoub, Pennsylvania State University–Abington College, Abington, Pennsylvania*

Express the value of the following $(n+1) \times (n+1)$ determinant as a product involving linear factors of x_i's and α_i's.

$$\begin{vmatrix} (x_1+\alpha_1)^n & (x_1+\alpha_2)^n & \dots & (x_1+\alpha_{n+1})^n \\ (x_2+\alpha_1)^n & (x_2+\alpha_2)^n & \dots & (x_2+\alpha_{n+1})^n \\ \vdots & \vdots & & \vdots \\ (x_{n+1}+\alpha_1)^n & (x_{n+1}+\alpha_2)^n & \dots & (x_{n+1}+\alpha_{n+1})^n \end{vmatrix}$$

1027. *James Chew, North Carolina Agricultural and Technical State University, Greensboro, North Carolina*

Student solutions solicited

Let a jar contain 1 green marble and 9 red marbles, thoroughly mixed. One marble is randomly drawn, and its color is noted. A second jar contains 2 green marbles and 8 red marbles. One marble is drawn from the second jar and again the color is noted. The next jar contains 3 green marbles and 7 red marbles. One marble is drawn from the third jar and again the color is noted. Repeat this process until a fifth marble has been drawn from the jar containing 5 green and 5 red marbles. Let $X =$ the number of green marbles drawn. Calculate $P(X = i), i = 0, 1, 2, \ldots, 5$.

A local newspaper gives probabilities of rain for the next 5 days as: 10% , 20% , 30% , 40% ,50% . Use the marbles-in-the-jar model to determine the probability of getting a) exactly two days of rain, b) at least two days of rain.

1028. *Proposed by Editors.*

As a modification of #1027, explain how to modify the model in problem 1027 so that the assumption of independence is removed. Based on your new model, determine the probability of getting a) exactly two days of rain, b) at least two days of rain.

***1029.** *Proposed by Ice B. Risteski, Skopje, Macedonia.*

If P and Q denote the linear differential operators

$$P = \sum_{i=0}^{m} p_i(x)D^i, \qquad Q = \sum_{j=0}^{n} q_j(x)D^j, \quad (D = \frac{d}{dx})$$

show that

$$QP = \sum_{s=0}^{m+n} r_s(x)D^s,$$

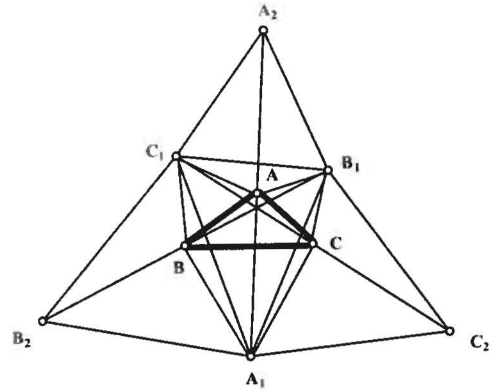
where

$$r_s(x) = \sum_{j=\max(0,s-m)}^{n} \left\{ \left[\sum_{i=\max(0,s-j)}^{\min(s,m)} \binom{j}{s-i} p_i^{(i+j-s)}(x) \right] q_j(x) \right\}$$

1030. *Proposed by Ayoub B. Ayoub, Pennsylvania State University Abington College, Abington, Pennsylvania*

On the sides of an arbitrary triangle ABC, three equilateral triangles, A_1BC, AB_1C, and ABC_1 are drawn outward. Then on the sides of the triangle $A_1B_1C_1$, another three equilateral triangles $A_2B_1C_1$, $A_1B_2C_1$, and $A_1B_1C_2$ are drawn outward relative to the triangle $A_1B_1C_1$. Show that each set of points $\{A_2, A, A_1\}$,

$\{B_2, B, B_1\}$, and $\{C_2, C, C_1\}$ lie on a straight line and that the three lines meet in one point.



1031. *Proposed by Andrew Cusumano, Great Neck, New York*

Notice that

$$\sqrt[3]{9+\sqrt{80}} + \sqrt[3]{9-\sqrt{80}} = 3$$

and

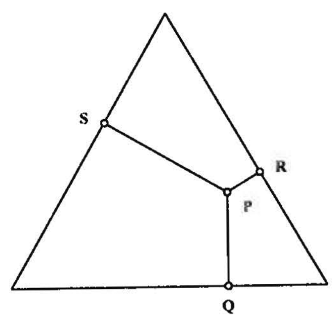
$$\sqrt[3]{161+\sqrt{25920}} + \sqrt[3]{161-\sqrt{25920}} = 7.$$

Generalize this by showing that

$$\sqrt[3]{\frac{x^3-3x}{2} + \frac{(x^2-1)\sqrt{x^2-4}}{2}} + \sqrt[3]{\frac{x^3-3x}{2} - \frac{(x^2-1)\sqrt{x^2-4}}{2}} = x.$$

1032. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey*

Consider an equilateral triangle with sides of length 1 unit, as shown below. From an arbitrary interior point P, draw perpendiculars PQ, PR and PS. Find the sum of the lengths of PQ, PR, and PS.



1033. *Proposed by Kenneth B. Davenport Frackville, Pennsylvania*

Student solutions solicited

Show that

$$\frac{2\sin(2\theta) - 3\sin(\theta)}{1-\cos(\theta) - 2\cos(2\theta)} = -\tan\left(\frac{\theta}{2}\right)$$

for all values of θ where both sides are defined.

Solutions.

994. [Fall 2000] *Proposed by the editor.*

Although the alphametic $BRENNER = (JOEL)^2$ has no solution in base ten, there is a number M such that $BRENNER$ is the square of a positive integer x in every base greater than or equal to M. Furthermore, the same four digits are used for B, R, E, and N in each such base. Find these digits, the value of M, and the digits of x, the square root of $BRENNER$.

I. Solution by **William H. Peirce**, *Rangeley, Maine.*

In a computer search of over 36,000 cases in bases ten or greater, eighteen instances were found where $BRENNER$ was the square of a 4-digit number. In each such $BRENNER$ in a given base M where a carry occurred in squaring x, the number formed by that $BRENNER$ was not a square in base $M+1$. It is easy to see that, in fact, since $BRENNER$ is to be a square in every base larger than M, no carry can occur in the squaring; carries will not occur when the base becomes large enough. In the six cases where $BRENNER$ was a square and no carry occurred in the squaring, $BRENNER$ was a square in all larger bases.

Thus if the four-digit number x is $abcd$ in base M and if there are no carries in the squaring, we must have $B=a^2$, $R=2ab$, $E=2ac+b^2$, $N=2ad+2bc$, $N=2bd+c^2$, $E=2cd$, and $R=d^2$. We must therefore have $2ab=d^2$, $2ac+b^2=2cd$, and $2ad+2bc=2bd+c^2$, whose unique solution in positive integers is $b=c=d=2a$. Then we have, for any base $M \geq 12a^2+1$,

$$x^2=(a,2a,2a,2a)^2=(a^2,4a^2,8a^2,12a^2,12a^2,8a^2,4a^2)=BRENNER.$$

The smallest solution is $x^2=(1,2,2,2)^2=(1,4,8,12,12,8,4)=BRENNER$ in all bases $M \geq 13$.

II. Comment by **Kenneth M. Wilke**, *Topeka, Kansas.*

Note that (1,2,2,2) in base M is equal to (1,5,9,7) in base $M-1$, so we can say that we do have the pseudo-solution $(BRENNER)=(1,4,8,12,12,8,4)$ in base M is the square of $(JOE'L)=(1,5,9,7)$ in base $M-1$, where, unfortunately, $E \neq E'$.

III. Comment by **Mark Evans**, *Louisville, Kentucky.*

I found the following solutions to the equation $(JOEL)^2=BRENNER$:

$$\begin{aligned}(4,1,7,6)^2&=(16,10,7,12,12,7,10) \text{ in base } 26,\\(3,51,27,8)^2&=(15,6,27,44,44,27,6) \text{ in base } 58,\\(6,5,52,26)^2&=(36,68,52,35,35,52,68) \text{ in base } 76,\\(5,81,68,27)^2&=(35,57,68,53,53,68,57) \text{ in base } 84, \text{ and}\\(8,41,24,81)^2&=(70,21,24,60,60,24,21) \text{ in base } 109.\end{aligned}$$

Also solved by **Paul S. Bruckman**, Sacramento, CA, **Richard I. Hess**, Rancho Palos Verdes, CA, **H.-J. Seiffert**, Berlin, Germany, **Kenneth M. Wilke**, Topeka, KS, and **the Proposer**.

995. [Fall 2000] *Proposed by Peter A. Lindstrom, Batavia, New York.*

a) Consider the *geometric-arithmetic* recursive sequence f given by

$$f(1)=a, f(2)=ar+d, \text{ and } f(i)=rf(i-1)+d \text{ for } i \geq 2,$$

where a, d, and r are nonzero constants, $r \neq 1$, and i is an integer. Express $\sum_{i=1}^{n} f(i)$ in closed form.

b) Consider the *arithmetic-geometric* recursive sequence g given by

$$g(1)=a, g(2)=r(a+d), \text{ and } g(i)=r(g(i-1)+d) \text{ for } i \geq 2,$$

where a, d, and r are nonzero constants, $r \neq 1$, and i is an integer. Express $\sum_{i=1}^{n} g(i)$ in closed form.

Solution by **Ovidui Furdui**, *student, Western Michigan University, Kalamazoo, Michigan.*

a) From the recursion formula we see that

$$\sum_{i=2}^{n} f(i)=r\sum_{i=2}^{n} f(i-1)+d(n-1)=r\sum_{i=1}^{n-1} f(i)+d(n-1),$$

from which it follows that

$$(1-r)\sum_{i=1}^{n} f(i)=f(1)-rf(n)+d(n-1).$$

It is easy to observe and prove by mathematical induction that

$$f(n)=ar^{n-1}+d+dr+dr^2+\cdots+dr^{n-2}=ar^{n-1}+d\frac{r^{n-1}-1}{r-1}.$$

We combine these latter two equations to find that

$$\sum_{i=1}^{n} f(i)=\frac{1}{1-r}\left[a-r\left(ar^{n-1}+d\frac{r^{n-1}-1}{r-1}\right)+d(n-1)\right],$$

which reduces to

$$\sum_{i=1}^{n} f(i)=\frac{1}{1-r}\left[a-ar^n-dr\frac{r^{n-1}-1}{r-1}+d(n-1)\right].$$

b) If one replaces d by rd in the definition formulas of part (a), one obtains the formulas for part (b). Hence the solution is found to be

$$\sum_{i=1}^{n} g(i)=\frac{1}{1-r}\left[a-ar^n-dr^2\left(\frac{r^{n-1}-1}{r-1}\right)+dr(n-1)\right].$$

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Brian Bradie**, Christopher Newport University, Newport News, VA, **Paul S. Bruckman**, Sacramento, CA, **William Chau**, East Brunswick, NJ, **Kenneth B. Davenport**, Frackville, PA, **Amelia Dunst**, **Becky Lindstrom**, **James Luterek**, **Krista McConnaughey** and **Pamela Patrie**, SUNY College at Fredonia, NY, **Russell Euler** and **Jawad Sadek**, Northwest Missouri State University, Maryville, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Mark Evans**, Louisville, KY, **Robert C. Gebhardt**, Hopatcong, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, New Mexico, **Koopa Tak-Lun Koo**, Boston College, MA, **Carl Libis**, Richard Stockton College of NJ, Pomona, **Peter A. Lindstrom**, Batavia, NY, **David E. Manes**, SUNY College at Oneonta, **William H. Peirce**, Rangeley, ME, **Shiva K. Saksena**, University of North Carolina at Wilmington, **H.-J. Seiffert**, Berlin, Germany, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

996. [Fall 2000] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

If $P_i(x)$ is the Legendre polynomial, given by $P_0(x) = 1$ and for positive integral n,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

show that

$$nP_n(\cos x) = \sum_{m=1}^{n} \cos(mx) P_{n-m}(\cos x).$$

Solution by **Paul S. Bruckman**, *Sacramento, California.*

The generating function of the Legendre polynomials is

$$(1) \qquad \theta^{-1} = \sum_{n=0}^{\infty} P_n(x)t^n, \text{ where } \theta = \theta(t,x) = (1-2tx+t^2)^{1/2}.$$

We also note that

$$\frac{1-t\cos x}{1-2t\cos x+t^2} = \frac{1-t\cos x}{\theta^2(t,\cos x)} = \frac{1}{2}\left(\frac{1}{1-te^{ix}}+\frac{1}{1-te^{-ix}}\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(e^{inx}+e^{-inx})t^n = \sum_{n=0}^{\infty} t^n \cos nx.$$

Therefore, letting $\varphi = \varphi(t,x) = \theta(t,\cos x)$, we see that $\varphi^{-1} = \sum_{n=0}^{\infty} t^n P_n(\cos x)$ and

$$(1-t\cos x)\varphi^{-3} = \sum_{n=0}^{\infty} t^n \cos nx \sum_{n=0}^{\infty} t^n P_n(\cos x) = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} P_{n-m}(\cos x)\cos mx.$$

On the other hand, by differentiating $\varphi^{-1} = \sum_{n=0}^{\infty} t^n P_n(\cos x)$ with respect to t, we obtain

$$(t\cos x - t^2)\varphi^{-3} = \sum_{n=0}^{\infty} nt^n P_n(\cos x).$$

Note that

$$(t\cos x - t^2)\varphi^{-3} + \varphi^{-1} = (t\cos x - t^2 + 1 - 2t\cos x + t^2)\varphi^{-3} = (1-t\cos x)\varphi^{-3}$$
$$= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} P_{n-m}(\cos x)\cos mx.$$

By comparison of coefficients we get

$$(2) \qquad (n+1)P_n(\cos x) = \sum_{m=0}^{n} P_{n-m}(\cos x)\cos mx.$$

Now subtracting $P_n(\cos x)$ from both sides of (2), which is the term for $m = 0$ in the right side of (2), we obtain the desired identity.

Also solved by **Brian Bradie**, Christopher Newport University, Newport News, VA, **Ovidui Furdui**, Western Michigan University, Kalamazoo, **H.-J. Seiffert**, Berlin, Germany, and the **Proposer**.

997. [Fall 2000] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Evaluate the integral

$$\int_4^8 \frac{\ln(9-x)dx}{\ln(9-x)+\ln(x-3)}.$$

I. Solution by **Sophie Trawalter**, *student, University of North Carolina at Wilmington, Wilmington, North Carolina.*

Let I denote the given integral. Making the successive substitutions $x = y + 6$, so $dx = dy$, and then $u = -y$, so $du = -dy$, we find that

$$I = \int_{-2}^{2} \frac{\ln(3-y)dy}{\ln(3-y)+\ln(3+y)} = \int_{-2}^{2} \frac{\ln(3+u)du}{\ln(3+u)+\ln(3-u)}.$$

Now the integral I must equal the average of these two integrals. That is,

$$I = \frac{1}{2}\int_{-2}^{2} dx = 2.$$

II. Solution by **Kristen Klingensmith**, **Danielle Quinn**, **Thomas Renken**, **James Slayton**, *and* **Sheri Webber**, *jointly, students, SUNY Fredonia, Fredonia, New York.*

From its graph, the integrand appears to be symmetric about the point $(6, 1/2)$. So make the substitution $y = x - 6$, obtaining the first integral shown in Solution I above. Let $f(y)$ denote the new integrand less $1/2$. We show that f is an odd function. Thus

$$f(-y) = \frac{\ln(3+y)}{\ln(3+y)+\ln(3-y)} - \frac{1}{2} = \frac{\ln(3+y)+\ln(3-y)-\ln(3-y)}{\ln(3+y)+\ln(3-y)} - \frac{1}{2}$$
$$= 1 - \frac{\ln(3-y)}{\ln(3-y)+\ln(3+y)} - \frac{1}{2} = -f(y).$$

Hence f is symmetric about the origin and therefore the integral is equal to 2, the area of a rectangle with base 4 and height $1/2$.

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Brian Bradie**, Christopher Newport University, Newport News, VA, **Paul S. Bruckman**, Sacramento, CA, **William Chau**, East Brunswick, NJ, **Brian Clester**, Perry, GA, **Jos Luis Diaz-Barrero**, Universitat Politécnica de Catalunya, Barcelona, Spain, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **Russell Euler** and **Jawad Sadek**, Northwest Missouri State University, Maryville, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Mark Evans**, Louisville, KY, **Ovidui Furdui**, Western Michigan University, Kalamazoo, **Christian K. Hansen**, Eastern Washington University, Cheney, WA, **Richard I. Hess**, Rancho Palos Verdes, CA, **Gerald A. Heuer**, Concordia College, Moorhead, MN, **Joe Howard**, Portales, New Mexico, **Koopa Tak-Lun Koo**, Boston College, MA, **Benjamin Landon**, University of Central Florida, Orlando, **Peter A. Lindstrom**, Batavia, NY, **David E. Manes**, SUNY College at Oneonta, **David Urman**, California State University, Sacramento, CA, **J. Ernest Wilkins**, Jr., Clark Atlanta University, GA, and the **Proposer**.

Bradie, Evans, and Hess each independently cut the interval of integration in half and then used the idea of Solution I to show that, after an appropriate substitution in the second integral, the two half-integrands added to the constant 1 over half the interval, thus arriving at the correct answer 2.

998. [Fall 2000] *Proposed by David Iny, Baltimore, Maryland.*
For nonnegative integers k and n, let

$$J_{kn} = \frac{1}{(1+k)^2}\binom{n}{0} - \frac{1}{(2+k)^2}\binom{n}{1} + \cdots + \frac{(-1)^n}{(n+k+1)^2}\binom{n}{n}.$$

a) Determine the value of b_k such that the limit L_k exists, where

$$L_k = \lim_{n\to\infty}[(n+1)(n+2)\cdots(n+k+1)J_{kn} - b_k \ln(n+1)].$$

b) Evaluate L_k using your value of b_k and the definition of Euler's constant γ given by

$$\gamma = \lim_{n\to\infty}\left[\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \ln n\right] = 0.577\ldots$$

c) Using your results of parts (a) and (b), evaluate, if it exists,

$$\lim_{k\to\infty}\left(\frac{L_k}{k!} + \ln k\right).$$

Solution by **H.-J. Seiffert**, *Berlin, Germany.*
If $\Gamma(x)$ denotes the gamma function, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, so that $\Gamma(n+1) = n!$ for positive integral n, then (see R. L. Graham, D. E. Knuth, and O. Patashnik, "Concrete Mathematics," 2nd ed., Addison-Wesley, 1994, p. 188, eqn. 5.41)

$$\sum_{j=0}^{n}\frac{(-1)^j}{j+x}\binom{n}{j} = \frac{n!\Gamma(x)}{\Gamma(n+x+1)}, \quad x>0.$$

Differentiating with respect to x and multiplying the result by -1 yields

$$\sum_{j=0}^{n}\frac{(-1)^j}{(j+x)^2}\binom{n}{j} = \frac{n!\Gamma(x)}{\Gamma(n+x+1)}(\psi(n+x+1) - \psi(x)), \quad x>0.$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Taking $x = k+1$, we find

$$J_{kn} = \frac{n!k!}{(n+k+1)!}(H_{n+k+1} - H_k),$$

where $H_m = \sum_{r=1}^{m} 1/r$ is the m'th harmonic number; empty sums have the value zero.

a) We have

$$\begin{aligned}
&(n+1)(n+2)\cdots(n+k+1)J_{kn} - b_k\ln(n+1)\\
&= \frac{(n+k+1)!}{n!}J_{kn} - b_k\ln(n+1)\\
&= k!(H_{n+k+1} - H_k) - b_k\ln(n+1)\\
&= k![H_{n+k+1} - \ln(n+k+1) - H_k] + k!\ln\left(\frac{n+k+1}{n+1}\right) + (k! - b_k)\ln(n+1).
\end{aligned}$$

Hence, the limit L_k exists only when $b_k = k!$.

b) If $b_k = k!$, then $L_k = k!(\gamma - H_k)$.

c) We have

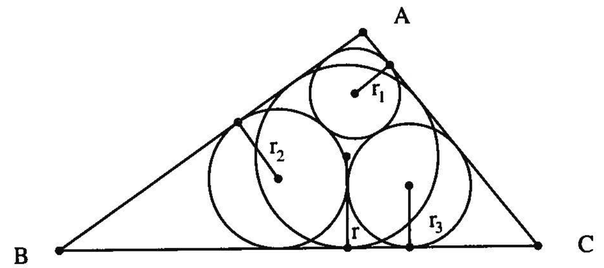
$$\lim_{k\to\infty}\left(\frac{L_k}{k!} + \ln k\right) = \lim_{k\to\infty}(\gamma - H_k + \ln k) = \gamma - \gamma = 0.$$

Also solved by **Paul S. Bruckman**, Sacramento, CA, **Ovidui Furdui**, Western Michigan University, Kalamazoo, and the **Proposer**.

999. [Fall 2000] *Proposed by the late Jack Garfunkel, Flushing, New York.*
Prove that

$$r \le \frac{(r_1+r_2+r_3)(3+\sqrt{3})}{9},$$

with equality when $r_1 = r_2 = r_3$, where r is the inradius of triangle ABC and r_1, r_2, and r_3 are the radii of the mutually tangent circles in the *Malfatti configuration*, shown in the accompanying figure.



Solution by **Miguel Amengual Covas**, *Cala Figuera, Mallorca, Spain.*
Using the relation

$$r = \frac{\sqrt{r_1r_2r_3}(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}+\sqrt{r_1+r_2+r_3})}{\sqrt{r_1r_2}+\sqrt{r_2r_3}+\sqrt{r_3r_1}}$$

(Example 2.3 in H. Fukagawa and D. Pedoe, "Japanese Temple Problems", The Charles Babbage Research Centre, 1989), and the known inequality

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9 \text{ with } x=\sqrt{r_1}, y=\sqrt{r_2}, z=\sqrt{r_3},$$

which is equivalent to

$$\frac{\sqrt{r_1r_2r_3}}{\sqrt{r_1r_2}+\sqrt{r_2r_3}+\sqrt{r_3r_1}} \le \frac{\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}}{9},$$

we have

$$(1) \qquad r \le \frac{(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3})(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}+\sqrt{r_1+r_2+r_3})}{9}.$$

Finally, using Cauchy's inequality $(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2)$ with $a=b=c=1$ and x, y, z as above, we see that

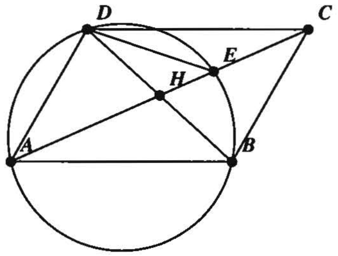
$$(2) \qquad (\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3})^2 \le 3(r_1+r_2+r_3).$$

By combining (1) and (2) we have the desired result. It is easy to see that equality holds if and only if $r_1 = r_2 = r_3$.

Also solved by **Paul S. Bruckman**, Sacramento, CA, **Yoshinobu Murayoshi**, Okinawa, Japan, and the **Proposer**.

1000. [Fall 2000] *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, New York.*

Let $ABCD$ be a parallelogram with $\angle A = 60°$. Let the circle through A, B, and D intersect AC again at E and let AC and BD meet at H. See the figure.



Let $[PQR]$ denote the area of triangle PQR. Show that

a) $[DHE] \cdot (AC)^2 = [ADH] \cdot (DB)^2$,

b) $[ADE] - [DEC] = 2[DHE]$, and

c) $2(HE) \cdot (AC) = (DB)^2$.

I. Solution to parts (a) and (c) by **Ovidui Furdui**, *student, Western Michigan University, Kalamazoo, Michigan.*

We note that the size of angle A is irrelevant.

a) We observe that AC and DB bisect one another, so that $AH = AC/2$ and $DH = HB = DB/2$. Also $(HE)(AH) = (DH)(HB)$ because AE and DB are intersecting chords of the circle. Then $HE = (DH)(HB)/(AH) = (DB)^2/4(AH)$. Since the two triangles DHE and ADH have the same altitude from vertex D, we have $[DHE]/[ADH] = (HE)/(AH) = (DB)^2/4(AH)^2 = (DB)^2/(AC)^2$, which yields the desired equation.

c) From part (a) we have $HE = (DB)^2/4(AH)$, so $2(HE)(AC) = 4(HE)(AH) = (DB)^2$.

II. Solution to part (b) by **Brian Bradie**, *Christopher Newport University, Newport News, Virginia.*

b) Since $[ADH] = [DHC] = [DHE] + [DEC]$, then $[ADE] - [DEC] = [ADH] + [DHE] - [DEC] = 2[DHE]$.

Also solved by **Miguel Amengual Covas**, Cala Figuera, Mallorca, Spain, **Brian Bradie**, Christopher Newport University, **Newport News**, VA, **Scott H. Brown**, Auburn University of Montgomery, AL, **Paul S. Bruckman**, Sacramento, CA, **William Chau**, East Brunswick, NJ, **Ovidui Furdui**, Western Michigan University, Kalamazoo, **Joe Howard**, Portales, New Mexico, **Koopa Tak-Lun Koo**, Boston College, MA, **Henry S. Lieberman**, Waban, MA, **David E. Manes**, SUNY College at Oneonta, **Yoshinobu Murayoshi**, 2 solutions, Okinawa, Japan, **William H. Peirce**, Rangeley, ME, **Kenneth M. Wilke**, Topeka, KS, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

1001. [Fall 2000] *Proposed by David Tselnik, Fargo, North Dakota.*

The *Euler numbers* E_n, for $n = 0, 1, 2, \ldots$, are defined by

$$\operatorname{sech} x = \frac{1}{\cosh x} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n,$$

so that $E_n = 0$ for all odd n, $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, etc. Prove the following relations:

a) $\displaystyle\sum_{j=0}^{2m} \binom{4m}{2j} |E_{2j}| = 2\sum_{k=0}^{m} \binom{4m}{4k} E_{4k}$ for $m = 1, 2, 3, \ldots$,

b) $\displaystyle\sum_{j=0}^{2m+1} \binom{4m+2}{2j} |E_{2j}| = 2\sum_{k=0}^{m} \binom{4m+2}{4k} E_{4k}$ for $m = 0, 1, 2, \ldots$,

c) $\displaystyle\sum_{j=0}^{2m} \binom{4m}{2j} |E_{2j}| = -2\sum_{k=1}^{m} \binom{4m}{4k-2} E_{4k-2}$ for $m = 1, 2, 3, \ldots$, and

d) $\displaystyle\sum_{j=0}^{2m+1} \binom{4m+2}{2j} |E_{2j}| = -2\sum_{k=0}^{m} \binom{4m+2}{4k+2} E_{4k+2}$ for $m = 0, 1, 2, \ldots$.

Solution by **Paul S. Bruckman**, *Sacramento, California.*

a) For convenience we define $e_n = [1 + (-1)^n]/2$ as the characteristic function of the even integers. Since $\cosh x = \sum_{n=0}^{\infty} e_n x^n/n!$, it follows by convolution with the series for $\operatorname{sech} x$ that

$$(1) \qquad \sum_{k=0}^{n} e_{n-k} \binom{n}{k} E_k = 0, \quad n = 1, 2, \ldots$$

The functions $\cosh x$ and $\operatorname{sech} x$, however, are even, which implies that

$$(2) \qquad \sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0, \quad n = 1, 2, \ldots$$

It is also known that $|E_{2k}| = (-1)^k E_{2k}$. Note that (2) implies

$$\sum_{j=0}^{2m} \binom{4m}{2j} E_{2j} = \sum_{j=0}^{2m+1} \binom{4m+2}{2j} E_{2j} = 0, \quad m = 1, 2, \ldots$$

Now we have

$$\begin{aligned} 2\sum_{k=0}^{m} \binom{4m}{4k} E_{4k} &= 2\sum_{k=0}^{2m} \binom{4m}{2k} E_{2k} e_k \\ &= \sum_{j=0}^{2m} \binom{4m}{2j} E_{2j} + \sum_{j=0}^{2m} (-1)^j \binom{4m}{2j} E_{2j} \\ &= 0 + \sum_{j=0}^{2m} \binom{4m}{2j} |E_{2j}|, \end{aligned}$$

which is part (a).

b) If $m \geq 0$, then we have

$$\begin{aligned} 2\sum_{k=0}^{m} \binom{4m+2}{4k} E_{4k} &= 2\sum_{k=0}^{2m} \binom{4m+2}{2k} E_{2k} e_k \\ &= \sum_{j=0}^{2m} \binom{4m+2}{2j} E_{2j} + \sum_{j=0}^{2m} (-1)^j \binom{4m+2}{2j} E_{2j} \\ &= -E_{4m+2} + 0 + \sum_{j=0}^{2m} \binom{4m+2}{2j} |E_{2j}| \end{aligned}$$

$$= -E_{4m+2} + \sum_{j=0}^{2m+1} \binom{4m+2}{2j} |E_{2j}| - |E_{4m+2}|$$

$$= -E_{4m+2} + \sum_{j=0}^{2m+1} \binom{4m+2}{2j} |E_{2j}| + E_{4m+2},$$

which reduces to part (b).

c) Next, let $o_n = 1 - e_n = [1-(-1)^n]/2$ be the characteristic function of the odd integers. Note that

$$-2\sum_{k=1}^{m} \binom{4m}{4k-2} E_{4k-2} = -2\sum_{j=1}^{2m-1} \binom{4m}{2j} E_{2j} o_j$$

$$= \sum_{j=1}^{2m-1} \binom{4m}{2j} |E_{2j}| - \sum_{j=1}^{2m-1} \binom{4m}{2j} E_{2j}$$

$$= \sum_{j=0}^{2m} \binom{4m}{2j} |E_{2j}| - \sum_{j=0}^{2m} \binom{4m}{2j} E_{2j} - |E_0| + E_0 - |E_{4m}| + E_{4m}$$

$$= \sum_{j=0}^{2m} \binom{4m}{2j} |E_{2j}| - 0,$$

which is part (c), for $m \geq 1$.

d) Finally, if $m \geq 0$, we have that

$$-2\sum_{k=0}^{m} \binom{4m+2}{4k+2} E_{4k+2} = -2\sum_{j=0}^{2m+1} \binom{4m+2}{2j} E_{2j} o_j$$

$$= \sum_{j=0}^{2m+1} \binom{4m+2}{2j} |E_{2j}| - \sum_{j=0}^{2m+1} \binom{4m+2}{2j} E_{2j}$$

$$= \sum_{j=0}^{2m+1} \binom{4m+2}{2j} E_{2j} - 0,$$

which is part (d).

Also solved by **Ovidui Furdui**, Western Michigan University, Kalamazoo, and the **Proposer**.

1002. [Fall 2000] *Proposed by L. Seagull, Glendale Community College, Glendale, Arizona.*

Let n be a composite integer greater than or equal to 48. Prove that between n and $S(n)$ there exist at least five primes, where $S(n)$ is the Smarandache function: for any positive integer n, $k = S(n)$ if k is the smallest positive integer such that n divides $k!$. Then, for example, $S(3) = 3$ and $S(8) = 4$.

Comment by H.-J. Seiffert, Berlin, Germany.

This result was posed by the same proposer and proved by N. J. Kuenzi and B. Prelipp in Problem 4541, "School Science and Mathematics," vol. 96, no. 7, 1996, p. 392.

Also solved by **Paul S. Bruckman**, Sacramento, CA, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

1003. [Fall 2000] *Proposed by I. M. Radu, Bucharest, Romania.*

Show that between $S(n)$ and $S(n+1)$, where $S(n)$ is the Smarandache function, there exists at least one prime number. See Problem 1002 for the definition of the Smarandache function.

I. Comment by **Paul S. Bruckman**, *Sacramento, California.*

As it stands, the conjecture is false. We find the following counterexamples for $n \leq 100$: $n = 2, 3, 4, 5, 9, 14, 15, 20, 21, 27, 32, 35, 51, 54, 55, 63, 65$, and 99. For example, $S(54) = 9, S(55) = 11$, and $S(56) = 7$ and there are no primes between 9 and 11 or between 11 and 7. It may be conjectured that the conjecture is false for infinitely many n, although this has not been established. It is not clear what the proposer had in mind.

Editorial comment. The proposer assumed weak inequalities in the comments he made about his conjecture, which this editor overlooked, so the prime 11 would count for him in both cases $n = 54$ and $n = 55$. He stated that he checked and found the conjecture true for all n up to 4800. Your editor gets 5 nights without his teddy bear for not checking the wording more carefully.

II. Disproof by **Rex H. Wu**, *Brooklyn, New York.*

Assuming weak inequalities were intended in the proposal, all the counterexamples up to 1,000,000 are given on pages 52 and 53 in the book "An Introduction to the Smarandache Function" by Charles Ashbacher, which can be downloaded at `http://www.gallup.unm.edu/~smarandache/Ashbacher-SmFu.pdf`.
They are $S(224) = 8$, $S(225) = 10$; $S(2057) = 22$ and $S(2058) = 21$; $S(265225) = 206$ and $S(265226) = 202$; and $S(845637) = 302$ and $S(845638) = 298$.

1004. [Fall 2000] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find the minimum value of $f_n = x_1 + x_2 + \cdots + x_n$ if the x_k are all nonnegative and

$$\sum_{k=1}^{n} \cos^2 x_k = 1.$$

Solution by **William H. Peirce**, *Rangeley, Maine.*

Clearly the x_k can be restricted to angles in the first quadrant. Both the constraint and the function f_n are symmetric functions of the x_k, and when this symmetry exists, any internal extremum of f_n occurs at a point where the x_k are all equal to, say, x. Therefore $n\cos^2 x = 1$, so $x = \arccos(1/\sqrt{n})$ and $f_n = n\arccos(1/\sqrt{n})$ is a candidate for an extremum of f_n. It is easily verified that any perturbation of the x_k from x produces a larger value for f_n, so we have indeed found the minimum for any $n > 2$.

If $n = 2$, then x_1 and x_2 can be any angles whose sum is $\pi/2$, so $f_2 = \pi/2$. If $n = 1$, then $x_1 = 0$. We see that in all cases, then, we have $f_n = n\arccos(1/\sqrt{n})$.

Also solved by **Brian Bradie**, Christopher Newport University, Newport News, VA, **Paul S. Bruckman**, Sacramento, CA, **Russell Euler** and **Jawad Sadek**, Northwest Missouri State University, Maryville, **Mark Evans**, Louisville, KY, **Ovidui Furdui**, Western Michigan University, Kalamazoo, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, New Mexico, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

1005. [Fall 2000] *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, Pennsylvania.*

Prove that, if $n > 2$ is an odd number,

$$\sum_{k=1}^{(n-1)/2} \sin\frac{4k\pi}{n} = \sin\frac{4\pi}{n} + \sin\frac{8\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} < 0.$$

I. Solution by **Kim Thielke**, *student, Alma College, Alma Michigan.*

For any natural number m it is easy to show that

$$\sum_{k=1}^{m} \sin kx = \frac{\cos(x/2) - \cos((2m+1)x/2)}{2\sin(x/2)}.$$

Now let n be an odd number greater than or equal to 3. We find that

$$\sum_{k=1}^{(n-1)/2} \sin(4k\pi/n) = \frac{\cos(2\pi/n) - \cos(2\pi)}{2\sin(2\pi/n)} = \frac{\cos(2\pi/n) - 1}{2\sin(2\pi/n)}.$$

Since $n \geq 3$, the denominator is positive and the numerator negative. The theorem follows.

II. Solution by **J. Ernest Wilkins, Jr.**, *Clark Atlanta University, Atlanta, Georgia.*

Let $n = 2p+1$, where p is a positive integer and define $x = 4\pi/n$, $y = 2\pi - x/2 = px$, and $z = \exp(ix)$. Then the indicated sum S is the imaginary part of

$$\sum_{k=1}^{p} z^k = \frac{z^p - 1}{1 - z^{-1}} = \frac{e^{iy} - 1}{1 - e^{-ix}}.$$

Therefore,

$$2S = \frac{\sin(x) + \sin(y) - \sin(x+y)}{1 - \cos(x)}.$$

Because $x = 4\pi - 2y$, it is clear that $\sin x = -\sin 2y = -2\sin y \cos y$, that $\sin(x+y) = -\sin y$, and that $(1 - \cos x)S = (1 - \cos y)\sin y$. It follows that

$$S = \frac{1}{2}\tan\frac{y}{2}.$$

We conclude that $S < 0$ because $0 < x \leq 4\pi/3$ and $2\pi/3 \leq y/2 < \pi$.

Also solved by **Brian Bradie**, Christopher Newport University, Newport News, VA, **Paul S. Bruckman**, Sacramento, CA, **Kenneth B. Davenport**, Frackville, PA, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Mark Evans**, Louisville, KY, **Ovidui Furdui**, Western Michigan University, Kalamazoo, **Robert C. Gebhardt**, Hopatcong, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, New Mexico, **Gerald A. Heuer**, Concordia College, Moorhead, MN, **David E. Manes**, SUNY College at Oneonta, **Yoshinobu Murayoshi**, Okinawa, Japan, **William H. Peirce**, Rangeley, ME, **H.-J. Seiffert**, Berlin, Germany, and the **Proposer**.

1006. [Fall 2000] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

a) How many aces can be served in one game of tennis?

b) How many consecutive aces can be served in one game of tennis?

c) You and I are playing a set of tennis. In the last 8 points you have served 7 aces and I have served 1. What is our score?

d) In a tennis match you have just served aces on 6 consecutive points. What is the score?

Solution by **Skidmore College Problem Group**, *students, Saratoga Springs, New York.*

a) ∞. In principle, a game at deuce could see an infinite number of alternating aces and points lost by the server.

b) 5. Since tie-breakers are not referred to as games, a single game has one player serving. If this player trails 0-40, 5 consecutive aces will win the game. In every other circumstance, less than 5 consecutive aces will have this effect.

c) 6-6. Player A has just served 5 consecutive aces (cf. b) to make the score 6-6. Player B then served an ace to start the tie-breaker, and then A served two aces, making the tie-breaker score 1-2 and B will serve next.

d) 0-1. Player A must have served two aces in the tie-breaker to end the last set. Who serves the first game of this set depends only on who served the last game of the last set, not on who served the tie-breaker. So, if B served that game, then A serves the first game of this set. If A now serves 4 aces to beat B at love, the stated conditions have been satisfied.

Also solved by **Mark Evans**, Louisville, KY, **William H. Peirce**, Rangeley, ME, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

Corrections.. In the Spring 2001 issue please make these corrections. On page 217, problem 1019, in each of the two denominators the term $(2n-1)$ should be multiplied by $(-1)^{n+1}$. In the solution to problem 983, in the last displayed line on Page 219, please append "$= 0$" after the difference of the integrals.

MATH A CROSTIC

The MATHACROSTIC in this issue has been contributed by Dan Hurwitz.

a. Belief in third powers — 085 153 009 114 139 194

b. Doing Gauss-Jordan Steps (2 wds.) — 017 095 151 122 037 175 048 203 006 167 099

c. Thumbless C.S. conversion (2 wds.) — 144 129 115 200 049 038 100 179 054 018 166 154

d. British philosopher/mathematician (1861–1947) — 171 036 011 105 050 150 029 001 098

e. One-to-one correspondence with the natural numbers — 059 023 184 013 121 079 195 168 148 141 064

f. A prime date, when available (hyph.) — 093 146 039 138 196 008 033 162 060 069 186

g. Function on larger domain — 102 073 178 062 108 051 112 088 140

h. Empty — 131 118 181 174

i. Ten per cent — 076 022 133 182

j. Proof introductory clause — 136 090 067 096 071 053 159 061 128 077 192 123 027 147 172 082

k. Planes including a given point — 028 086 198 104 163

l. Round the clock calculations (2 wds.) — 149 016 176 160 058 134 010 002 042 068 019 092 109 087 197 081 157 012 032 185

m. They depart from the general pattern — 070 173 043 097 164 024 126 189

n. Fixed point subscripts — 142 066 190 034 083 007 158

o. Juxtaposed — 020 080 101 165 063 201 046 117 107 035 188 155

p. Doable procedures are this — 170 124 145 057 094 005 074 045 116

q. Needed at Monte Carlo (hyp.) — 025 204 106 127 040 152 047 072 161 065 137 119 015 003 193 030

r. Singular example of exponential growth (3 wds.) — 125 052 169 143 103 177 078 021 091 110 075 084

s. Translated Euclid into Arabic — 044 120 055 135 202 113 180

t. Switching circuits state — 026 056 111

u. Found on inner product spaces — 004 130 187 199

v. Properly contained in the set of all sets — 183 041 156 132 191 014 089 031

				001d	002l		003q	004u	005p	006b	007n	008f		009a
010l		011o	012l		013e	014v	015q		016l	017b		018c	019l	020o
021r	022i	023e	024m		025q	026t	027j	028k	029d	030q	031v	032l	033f	034n
	035o	036d	037b		038c	039f	040q	041v	042l	043m	044s	045p	046o	
047q	048b	049c	050d	051g		052r	053j		054c	055s	056t	057p	058l	059e
060f	061j	062g		063o	064e	065q		066n		067j	068l	069f		070m
071j		072q	073g	074p	075r	076i	077j		078r	079e	080o	081l		082j
083n	084r	085a	086k		087l	088g		089v	090j	091r	092l	093f		094p
095b	096j	097m	098d		099b	100c	101o	102g	103r	104k	105d	106q		107o
108g		109l	110r	111t	112g	113s	114a	115c	116p		117o	118h	119q	120s
121e	122b		123j	124p		125r	126m	127q	128j		129c	130u	131h	132v
133i	134l	135s	136j	137q	138f	139a		140g	141e	142n	143r		144c	145p
	146f	147j	148c	149l	150d		151b	152q	153a	154c	155o		156v	157l
	158n	159j	160l	161q	162f	163k	164m	165o	166c	167b	168e		169r	170p
	171d	172j	173m	174h	175b		176l	177r	178g		179c	180s	181h	182i
	183v	184c	185l	186f		187u	188o	189m	190n	191v	192j	193q		194a
195e	196f	197l	198k	199u	200c	201o	202s	203b	204q					

Last month's mathacrostic was taken from "Indiscrete Thoughts" by Gian-Carlo Rota.

The full text of the quote is:

> "Mathematicians have to attend (secretly) physics meetings in order to find out what is going on in their fields. [Physicists have the P.R., the savoir-faire, and the chutzpah to write readable, or at least legible accounts of subjects that are not yet obsolete, something few mathematicians would dare do, fearing expulsion from the A.M.S.]"

Jeanette Bickley was the first solver.

Contents

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