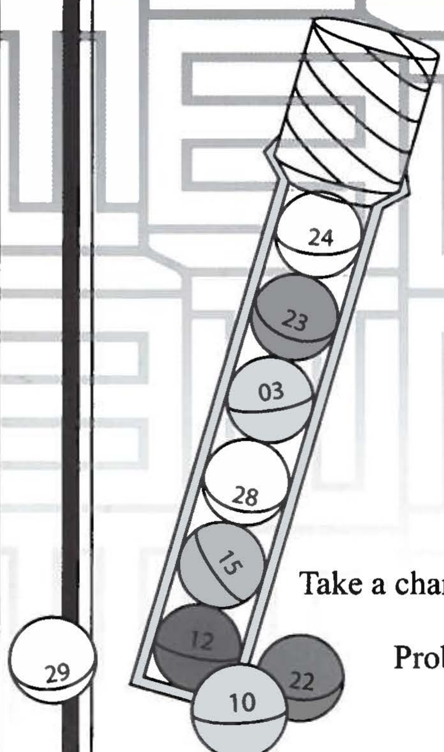

Take a chance on the lottery problem.

Problem 1050

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SYMMETRY IN BIFURCATION DIAGRAMS

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1. Introduction. Dynamical systems is the study of iterated functions. If we start with a function $f:\mathbb{R}\to\mathbb{R}$ and a point $x_0\in\mathbb{R}$, we begin the process by finding $f(x_0)$. We then use this output as an input, and find $f(f(x_0))$. This is the second iterate of x_0 under f, and is denoted $f^2(x_0)$. We continue this process indefinitely:

Given a function $f:\mathbb{R}\to\mathbb{R}$ and a point $x_0\in\mathbb{R}$, the *orbit of x_0 under f* is the sequence

$$x_0,\quad f(x_0)=x_1,\quad f(f(x_0))=f^2(x_0)=x_2,\quad f(f(f(x_0)))=f^3(x_0)=x_3,\ \ldots$$

The goal of dynamical systems is to classify the long run behavior of orbits. We wish to determine if the orbits approach some specific point, go off to infinity, or maybe bounce around randomly - a sign of possible chaotic behavior.

Why do we study dynamical systems? Iteration is natural in a wide variety of applications. Today's weather depends on yesterday's weather, next year's population depends on this year's population, and so on. For more information on applications of dynamical systems, see [1, Lab Visits],[2, Chapter2].

We will be studying the orbits of points under one-parameter families of functions, which we will denote $f_c(x)$, where the c is a real-valued parameter. For example, we will consider $f_c(x)=x^2+c$. For a specific c-value, the orbit of an $x_0\in\mathbb{R}$ could exhibit many different long term behaviors. One special kind of x_0 is a periodic point:

The point x_0 is a *periodic point of period n for f_c* if $f_c^n(x_0)=x_0$ for some natural number n. If n is the smallest natural number for which $f_c^n(x_0)=x_0$, we say x_0 has *prime period n*. If x_0 is a point of period 1, we call x_0 a *fixed point* for f_c.

EXAMPLE: For $f_c(x)=x^2+c$ when $c=-1$, $(1+\sqrt{5})/2$ is a fixed point, and 0 is a point of prime period 2. Why? You can compute for yourself that for $f(x)=x^2-1$, $f((1+\sqrt{5})/2)=(1+\sqrt{5})/2$. Also, $f(0)=-1$ and $f(-1)=(-1)^2-1=0$, so the orbit of 0 is the sequence $0,-1,0,-1,0,-1,\ldots$. Note that 0 is also a point of periods 4, 6, 8, etc., but that it has prime period 2.

Periodic points are classified as attracting, repelling or neutral, depending on the behavior of nearby points. A point x_0 of period n is *attracting* if $|(f^n)'(x_0)|<1$. It is *repelling* if $|(f^n)'(x_0)|<1$ and it is *neutral* if $|(f^n)'(x_0)|=1$.

Devaney [2, pages 39-48] shows that for an attracting periodic point x_0, some neighborhood of nearby points moves closer (under iteration) to the orbit of x_0. He also shows that for a repelling periodic point x_0, some neighborhood of nearby points eventually move away from the orbit of x_0. A variety of behaviors can occur for neutral periodic points.

Calculating $|(f^n)'(x_0)|$, as is required by these definitions, could be a horrible chore, but we have the chain rule to help us out. Let x_0 be a point of period n for f, so that the orbit of x_0 is

$$x_0,x_1=f(x_0),x_2=f^2(x_0),\ldots,x_{n-1}=f^{n-1}(x_0),f^n(x_0)=x_0,x_1,x_2,\ldots$$

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Then since $(f^n)(x_0) = f(f^{n-1}(x_0))$,

$$\begin{aligned}(f^n)'(x_0) &= f'(f^{n-1}(x_0)) \cdot (f^{n-1})'(x_0) \\ &= f'(x_{n-1}) \cdot (f^{n-1})'(x_0) \\ &= f'(x_{n-1}) \cdot f'(f^{n-2}(x_0)) \cdot (f^{n-2})'(x_0) \\ &= f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot (f^{n-2})'(x_0)\end{aligned}$$

The third equality comes from applying the chain rule again to find $(f^{n-1})'$. Continuing this process, we end up with

$$(f^n)'(x_0) = f'(x_{n-1}) \cdot f'(x_{n-2}) \cdots f'(x_1) \cdot f'(x_0)$$

In words, the derivative of the nth iterate of f at x_0 is the product of the derivative of f evaluated at each of the points in the orbit of x_0.

EXAMPLE: Use your calculator to find the orbit of 0.1 under $f_{-1}(x) = x^2 - 1$. The points in the orbit seem to move closer and closer to 0 and -1. This is evidence that 0 is an attracting point of period 2. Checking this using the chain rule, we find that $|(f_{-1}^2)'(0)| = 0$, and $0 < 1$, which verifies that this periodic point is attracting. Similarly you can check that $|f'((1+\sqrt{5})/2)| = 1+\sqrt{5} > 1$, so $(1+\sqrt{5})/2$ is a repelling fixed point.

One way of putting together a complete picture of the behavior of the orbits of a one-parameter family of functions (like $x^2 + c$) is to construct what we will call a bifurcation diagram. This is a picture with the parameter c on the horizontal axis and the coordinate x on the vertical axis. We plot a point at (c_0, x_0) if x_0 is a periodic point for f_{c_0}. For a complete introduction to bifurcation diagrams, see [3].

The $n = k$ *bifurcation diagram* for f_c has a point plotted at (c_0, x_0) if x_0 is a point of (not necessarily prime) period k for f_{c_0}. This point is marked with a thick dot if x_0 is an attracting periodic point, and with a thin dot if x_0 is a repelling periodic point. (Our pictures are drawn by a program called Orbits. The windows version of this program uses colors to indicate whether a periodic point is attracting or repelling, and whether the derivative at that point is positive or negative. You can download this program for free, and see color versions of the pictures in this paper, at http://faculty.gvsu.edu/sorensej/symmetry.)

EXAMPLE: Figure 1 shows the $n = 2$ bifurcation diagram for $f_c(x) = x^2 + c$.

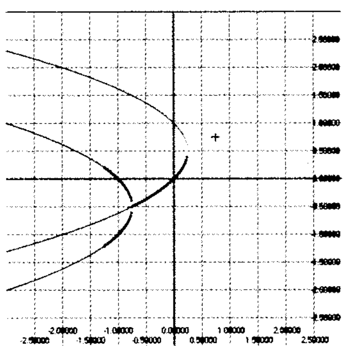

FIG. 1.

Note that when $c = -1$, there is a thin dot at $x = (1+\sqrt{5})/2 \approx 1.618$ since this is a repelling fixed point, and a thick dot at $x = 0$ and $x = -1$ since this is an attracting 2-cycle.

2. Symmetry in Bifurcation Diagrams.

EXAMPLE: Figure 2 shows the $n = 1$ and $n = 2$ bifurcation diagrams for $f_c(x) = c\sin(x)$. It is here that we begin to notice interesting symmetries. The $n = 1$ picture appears to be symmetric about the c-axis, while the $n = 2$ picture is also symmetric about the x-axis.

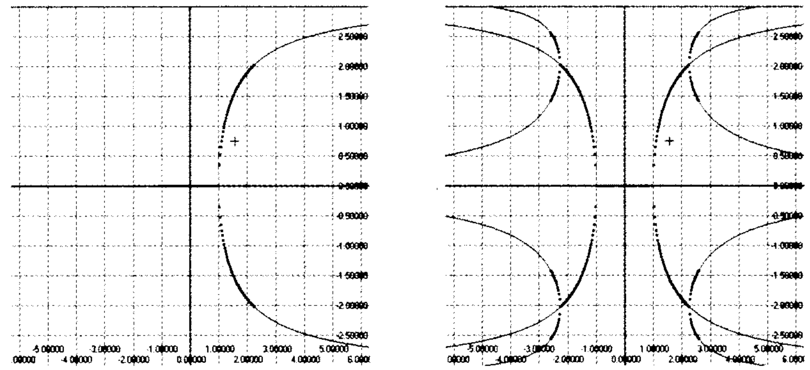

FIG. 2. *Bifurcation diagrams for* $f_c(x) = c\sin(x)$

What properties of a family of functions produce symmetries like these? Are there other symmetries that arise for other functions? These questions were the beginning of our investigation. We began by defining the symmetries that can occur.

Consider a one-parameter family of functions f_c. If $f_c(-x) = f_c(x)$ for all c and all x, then we call f_c *even for* x. If $f_c(-x) = -f_c(x)$ for all c and all x, then we call f_c *odd for* x. If $f_{-c}(x) = f_c(x)$ for all c and all x, then we call f_c *even for* c. If $f_{-c}(x) = -f_c(x)$ for all c and all x, then we call f_c *odd for* c.

EXAMPLES: The family $f_c(x) = x^2 + c$ is even for x (and neither odd nor even for c). The family $f_c(x) = c\sin(x)$ is both odd for x and odd for c.

We also considered the case where the function is symmetric about the line $y = x$, but found that this was a dead end:

Note that if the family of functions $f_c(x)$ is symmetric about the line $y = x$, then every x in the domain of f_c is a point of period two for all values of c.

Proof. If f_c is symmetric about the line $y = x$, then $f_c = f_c^{-1}$. Now consider $f_c^2(x) = f_c(f_c(x))$. Since f_c is its own inverse, we have $f_c^2(x) = f_c(f_c^{-1}(x)) = x$ by the definition of inverse. Therefore, every x is a point of period two for all values of c. (Notice that some x may also be fixed points.) □

A bifurcation diagram can exhibit three types of symmetry: We say that a bifurcation diagram is *symmetric about the c-axis* if whenever x_0 is a point of period n for f_{c_0}, then $-x_0$ is as well. In symbols, we require that if $f_{c_0}^n(x_0) = x_0$, then $f_{c_0}^n(-x_0) = -x_0$. We saw above that the diagram for $f_c(x) = c\sin(x)$ exhibits symmetry about the c-axis.

We say that a bifurcation diagram is *symmetric about the x-axis* if whenever x_0 is a point of period n for f_{c_0}, then x_0 is also a point of period n for f_{-c_0}. In symbols, we require that if $f_{c_0}^n(x_0) = x_0$, then $f_{-c_0}^n(x_0) = x_0$. The bifurcation diagram for $f_c(x) = x^2 + c^2$ exhibits x-axis symmetry.

We say that a bifurcation diagram is *symmetric about the origin* if whenever x_0 is a point of period n for f_{c_0}, then $-x_0$ is a point of period n for f_{-c_0}. In symbols, we require that if $f_{c_0}^n(x_0) = x_0$, then $f_{-c_0}^n(-x_0) = -x_0$. The bifurcation diagram for $f_c(x) = c\cos(x)$ exhibits symmetry about the origin.

In order to prove our results on symmetries of bifurcation diagrams, we need the following preliminary results:

THEOREM 1. *If f is an odd, differentiable function, then f' is an even function. Similarly, if f is an even, differentiable function, f' is odd.*

Proof. Assume f is an odd, differentiable function. So $f(-x) = -f(x)$ for all x. Taking the derivatives of both sides of this equation (and using the chain rule on the left hand side) we obtain

$$f'(-x) \cdot -1 = -f'(x)$$
$$f'(-x) = f'(x)$$

So f' satisfies our definition of an even function.

The other proof is similar. □

LEMMA 2. *If f is an odd function, then $f^n(-x) = -f^n(x)$ for all x.*

If f is an even function, then $f^n(-x) = f^n(x)$ for all x.

The proof of Lemma 2 is straightforward and is left to the reader.

In our symmetry proofs below, we will be proving that the bifurcation diagrams are *completely symmetric*, by which we mean that not only do the corresponding periodic points exist, but the corresponding derivatives are equal. This would mean that corresponding point would be the same "color", and that it would be attracting, repelling or neutral exactly as the original point is.

THEOREM 3. *For any family of functions f_c which is odd for* x, *the bifurcation diagram is completely symmetric about the* c*-axis.*

Proof. Suppose $f_c^n(x_0) = x_0$. To show complete symmetry about the c-axis we must show that $f_c^n(-x_0) = -x_0$, and that $(f_c^n)'(x_0) = (f_c^n)'(-x_0)$.

First, we show that $f_c^n(-x_0) = -x_0$. Since $f_c(x)$ is odd for x, by Lemma 2 we have that $-f_c^n(x) = f_c^n(-x)$, for all c and for all x. Therefore,

$$\begin{aligned} f_c^n(-x_0) &= -f_c^n(x_0) \\ &= -x_0 \end{aligned}$$

Thus, $f_c^n(-x_0) = -x_0$.

Now, we show $(f_c^n)'(x_0) = (f_c^n)'(-x_0)$. Repeated application of the chain rule gives us

$$(f_c^n)'(-x_0) = f_c'(f_c^{n-1}(-x_0)) \cdots f_c'(f_c(-x_0)) \cdot f_c'(-x_0).$$

Since $f_c(x)$ is odd, by the lemma above we have

$$(f_c^n)'(-x_0) = f_c'(-f_c^{n-1}(x_0)) \cdots f_c'(-f_c(x_0)) \cdot f_c'(-x_0).$$

We know by the lemma above that the derivative of an odd function is even, therefore

$$(f_c^n)'(-x_0) = f_c'(f_c^{n-1}(x_0)) \cdots f_c'(f_c(x_0)) \cdot f_c'(x_0).$$

This represents the chain rule applied to $(f_c^n)'(x_0)$. Therefore,

$$(f_c^n)'(-x_0) = (f_c^n)'(x_0).$$

Thus, the bifurcation diagram of any function which is odd for x is completely symmetric about the c-axis. □

THEOREM 4. *For any family of functions f_c which is even for* x *and odd for* c, *the bifurcation diagram is completely symmetric about the origin.*

Proof. We will begin by using induction to show that when f_c is even for x and odd for c, then $f_{-c}^n(x) = -f_c^n(x)$ for all x, c and n.

This is true when $n = 1$ since f_c is odd for c.

Now we assume that $f_{-c}^k(x) = -f_c^k(x)$.

$$\begin{aligned} f_{-c}^{k+1}(x) &= f_{-c}(f_{-c}^k(x)) && \text{by the definition of iterate.} \\ &= f_{-c}(-f_c^k(x)) && \text{by the assumption,} \\ &= f_{-c}(f_c^k(x)) && \text{because } f_c \text{ is even for } x. \\ &= -f_c(f_c^k(x)) && \text{because } f_c \text{ is odd for } c. \\ &= -f_c^{k+1}(x) && \text{by regrouping.} \end{aligned}$$

Therefore $f_{-c}^n(x) = -f_c^n(x)$ for all n.

Now suppose $f_c^n(x_0) = x_0$. To show complete symmetry about the origin, we must show $f_{-c}^n(-x_0) = -x_0$ and $(f_c^n)'(x_0) = (f_{-c}^n)'(-x_0)$.

First, we show that $f_{-c}^n(-x_0) = -x_0$. Since $f_c(x)$ is even for x, we know by Lemma 2 that $f_c^n(-x) = f_c^n(x)$ for all c and for all x. Thus,

$$\begin{aligned} f_{-c}^n(-x_0) &= f_{-c}^n(x_0) \\ &= -f_c^n(x_0) \\ &= -x_0 \end{aligned}$$

Now, we show $(f_c^n)'(x_0) = (f_{-c}^n)'(-x_0)$. Repeated application of the chain rule along with the facts that f_c is even for x and odd for c give us

$$\begin{aligned} (f_{-c}^n)'(-x_0) &= f_{-c}'(f_{-c}^{n-1}(-x_0)) \cdots f_{-c}'(f_{-c}(-x_0)) \cdot f_{-c}'(-x_0) \\ &= f_{-c}'(f_{-c}^{n-1}(x_0)) \cdots f_{-c}'(f_{-c}(x_0)) \cdot f_{-c}'(-x_0) \\ &= f_{-c}'(-f_c^{n-1}(x_0)) \cdots f_{-c}'(-f_c(x_0)) \cdot f_{-c}'(-x_0) \end{aligned}$$

We know by the lemma above that the derivative of a function which is even for x is odd for x, therefore

$$\begin{aligned} (f_{-c}^n)'(-x_0) &= -f_{-c}'(f_c^{n-1}(x_0)) \cdots - f_{-c}'(f_c(x_0)) \cdot -f_{-c}'(x_0) \\ &= f_c'(f_c^{n-1}(x_0)) \cdots f_c'(f_c(x_0)) \cdot f_c'(x_0) \end{aligned}$$

This represents the chain rule applied to $(f_c^n)'(x_0)$. Therefore,

$$(f_{-c}^n)'(-x_0) = (f_c^n)'(x_0).$$

Thus, the bifurcation diagram of any function $f_c(x)$ which is even for x and odd for c is completely symmetry about the origin. □

THEOREM 5. *For any family of functions, $f_c(x)$, which is even for* c, *the bifurcation diagram is completely symmetric about the* x*-axis.*

Proof. Since f_c is even for c, we have that $f_{-c} = f_c$ for all c. This is saying that f_{-c} and f_c are the same function. So if we assume that $f_c^n(x_0) = x_0$, we automatically know that $f_{-c}^n(x_0) = x_0$ and $(f_c^n)'(x_0) = (f_{-c}^n)'(x_0)$. Thus, the bifurcation diagram for any family of functions, f_c, which is even for c is completely symmetric about the x-axis. □

THEOREM 6. *For any family of functions f_c which is odd for* x *and odd for* c, *the bifurcation diagram for even iterates is completely symmetric about the* x*-axis.*

Theorem 6 is saying that the $n = 2, 4, 6, \ldots$ bifurcation diagrams will be symmetric about the x-axis, but the $n = 1, 3, 5, \ldots$ bifurcation diagrams may not be.

Proof. Suppose $f_c^{2n}(x_0) = x_0$. To show complete symmetry, we must show $f_{-c}^{2n}(x_0) = x_0$ and $(f_c^{2n})'(x_0) = (f_{-c}^{2n})'(x_0)$.

First, we show that $f_{-c}^{2n}(x_0) = x_0$.

$$\begin{aligned} f_{-c}^{2n}(x_0) &= f_{-c}(f_{-c}(f_{-c}^{2n-2}(x_0))) \\ &= -f_c(-f_c(f_{-c}^{2n-2}(x_0))) && \text{since the function is odd for } c. \\ &= -(-f_c(f_c(f_{-c}^{2n-2}(x_0)))) && \text{since the function is odd for } x. \\ &= f_c(f_c(f_{-c}^{2n-2}(x_0))). \end{aligned}$$

We continue this process of forming pairs of iterates and bringing the minus signs out, using the fact that the function is odd for c and for x. Since there are an even number of iterates, all the minus signs will cancel, leaving us with $f_{-c}^{2n}(x_0) = f_c^{2n}(x_0) = x_0$.

Now, we will show that $(f_c^{2n})'(x_0) = (f_{-c}^{2n})'(x_0)$. Consider $(f_{-c}^{2n})'(x_0)$. By the chain rule, we say

$$(f_{-c}^{2n})'(x_0) = f'_{-c}(f_{-c}^{2n-1}(x_0)) \cdot (f_{-c}^{2n-1})'(x_0).$$

This can be further expanded to

$$\begin{aligned}(f_{-c}^{2n})'(x_0) &= f'_{-c}(f_{-c}^{2n-1}(x_0)) \cdots f'_{-c}(f_{-c}(x_0)) \cdot f'_{-c}(x_0)\\ &= f'_{-c}(-f_c^{2n-1}(x_0)) \cdots f'_{-c}(-f_c(x_0)) \cdot f'_{-c}(x_0)\end{aligned}$$

We know by Lemma 1 that the derivative of an odd function (for x) is even (for x), therefore

$$\begin{aligned}(f_{-c}^{2n})'(x_0) &= f'_{-c}(f_c^{2n-1}(x_0)) \cdots f'_{-c}(f_c(x_0)) \cdot f'_{-c}(x_0)\\ &= -f'_c(f_c^{2n-1}(x_0)) \cdots - f'_c(f_c(x_0)) \cdot -f'_c(x_0)\end{aligned}$$

Since we have $2n$ terms, we have an even number of negative signs, and we are left with

$$(f_{-c}^{2n})'(x_0) = f'_c(f_c^{2n-1}(x_0)) \cdots f'_c(f_c(x)) \cdot f'_c(x_0).$$

This represents the chain rule applied to $(f_c^{2n})'(x_0)$. Therefore,

$$(f_{-c}^{2n})'(x_0) = (f_c^{2n})'(x_0).$$

Thus, the bifurcation diagram for any family of functions $f_c(x)$ which is both odd for c and for x has even iterates completely symmetric about the x-axis. □

Notice that what has been proven in the theorems above is that if x_0 is a point of prime period n, then the point which is its image under the relevant symmetry must not only be of period n, but of *prime* period n. To understand why this is true, consider x_0 and y_0 to be two points which are symmetric to each other according to the theorems. Assume that x_0 is of prime period k, and that y_0 has period k, but has prime period $j < k$. Then, by the symmetry, x_0 must also be a point of period j, but this contradicts our assumption, so all points which correspond to each other under the relevant symmetry have the same prime period.

Our paper will conclude with a different type of symmetry theorem. As our work progressed, we began to consider which types of symmetry were impossible for certain families of functions. This theorem says that for a function which is odd for c, the only way we can have x-axis symmetry is if 0 is the only periodic point. The end of this proof uses Sarkovskii's Theorem [2, Chapter 11], a beautiful result which gives an ordering of periodic points. One consequence of this theorem is that if a function has no points of prime period 2, then the only periodic points for the function are fixed points.

THEOREM 7. *For any family of functions f_c which is odd for* c, *the bifurcation diagram is symmetric about the* x-*axis for all iterates if and only if when $f_c^n(x_0) = x_0$ it follows that $x_0 = 0$.*

Proof. Let $f_c(x)$ be a family of functions which is odd for c, and assume that the bifurcation diagram is symmetric about the x-axis. We begin by considering fixed points. If x_0 is a point such that $f_{c_0}(x_0) = x_0$, then by symmetry we know that $f_{-c_0}(x_0) = x_0$. But since f_c is odd for c, we also have that $f_{-c_0}(x_0) = -f_{c_0}(x_0) = -x_0$. Thus we have that $x_0 = -x_0$, which can only occur when $x_0 = 0$. Hence, the only fixed points that can occur are when $x_0 = 0$.

Because of this, $f_c(x)$ can only cross or touch the line $y = x$ at $x = 0$. Furthermore, $f_c(x)$ can only cross or touch the line $y = -x$ at $x_0 = 0$, because if $f_c(x)$ crosses this line, then $f_{-c}(x) = -f_c(x)$ crosses the line $y = x$ at the same x value, which would create another fixed point. Because the function is odd for c, $f_c(x) = -f_{-c}(x)$ for all c. Choosing c = 0 gives the formula $f_0(x) = -f_0(x)$, hence $f_0(x)$ is the zero function. Then, because we want a family of functions which is continuous for c, we must have $f_c(x)$ such that $-|x| < f_c(x) < |x|$ for all $x \neq 0$ and for all c, which is equivalent to all of the functions being trapped between the lines $y = x$ and $y = -x$. Observe that for all c at $x = 0$ we have $f_c(x) = 0$.

Consider some initial value $x_0 \neq 0$ Then $-|x_0| < f_c(x_0) < |x_0|$ which is equivalent to $|f_c(x_0)| < |x_0|$. This property holds for all x including $f_c(x_0)$, so we have that $|f_c^2(x_0)| \leq |f_c(x_0)|$. (The possibility of equality is now included because it may occur, but only if $f_c(x_0) = 0$.) Combining these two inequalities, we get that $|f_c^2(x_0)| \leq |f_c(x_0)| < |x_0|$. Thus $|f_c^2(x_0)| < |x_0|$, and hence we never get $f_c^2(x_0) = x_0$. Therefore, there can be no points of prime period two for any value of c. By Sarkovskii's Theorem, since there are no points of prime period two for any c value, there are only points of period 1. We have shown that the fixed points only occur at $x = 0$, thus the bifurcation diagram can only be symmetric about the x-axis in this trivial case. □

3. Conclusion. There are many more questions to be asked about symmetry and bifurcation diagrams. After proving the above results about how symmetries of the family of functions lead to symmetries of the bifurcation diagrams, we began to ask the inverse question: Do symmetries of the bifurcation diagram imply the existence of symmetries in the original function? It turns out that this question is much more difficult, due in part to the fact that two different functions can both produce the same bifurcation diagrams. We can say some things - the contrapositives of the above results give us theorems like "If the bifurcation diagram for f_c is *not* symmetric about the c-axis, then the family of functions f_c is *not* odd for x." Yet this is not a complete answer to our questions, and there is more work to be done.

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THREE CONNECTIONS TO CONTINUED FRACTIONS

EZRA BROWN*

Introduction. It is often the case that seemingly unrelated parts of mathematics turn out to have unexpected connections. In this paper, we explore three puzzles and see how they are related to continued fractions, an area of mathematics with a distinguished history within the world of number theory.

Puzzle 1: A Mistake. A typesetting error produced $g-1$ instead of g^{-1}, and a student was quick to point out that, for the problem at hand, it didn't matter. What is the value of g?

Puzzle 2: A Whole Lot of Cows. About 22 centuries ago, Archimedes wrote a letter in which he challenged his fellow mathematicians to determine the size of a certain herd of cattle. To do this involves solving the equation $x^2 - 410286423278424y^2 = 1$ in nonzero integers x and y. What is the connection, and why does this equation even have such a solution?

Puzzle 3: A Mystery. I once read a book on number theory that contained a tantalizing problem whose solution eluded me for years. In this book, it stated that the expression

$$2+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cdots}}}}$$

was equal to a certain real number θ. What is θ and why is this expression equal to θ?

The quantity g in Puzzle 1 produces a continued fraction, the solution to Puzzle 2 uses a continued fraction, and the expression in Puzzle 3 *is* a continued fraction.

Don't know what a continued fraction is? Don't worry—you'll find out.

A TEXnical Error and Simple Continued Fractions. In typing out a recent problem set in TEX, I inadvertently typed `$g-1$` (which typesets as $g-1$) instead of `$g^{-1}$` (which typesets as g^{-1}). One student was quick to point out that, for the problem at hand, it didn't matter. Knowing this and knowing that the answer is positive, what is g?

Now $g-1 = g^{-1} = \frac{1}{g}$, so $g = 1+\frac{1}{g}$. If we multiply both sides by g and transpose, we are led to the equation $g^2-g-1=0$. This has two solutions, namely $g = \frac{1\pm\sqrt{5}}{2}$; but $g>0$, so the answer is $g = \frac{1+\sqrt{5}}{2}$. End of story — or is it? Let's look a bit deeper.

Substituting this value for g in the expression on the right, we see that $g = 1+\cfrac{1}{1+\cfrac{1}{g}}$; do it again and we get $g = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{g}}}$; if we continue this process, we

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$$g = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \tag{1}$$

— that is, if the expression on the right makes sense.

Turns out, it *does* make sense: it is what we call a *simple continued fraction*, or *scf* for short. The three dots indicate that the pattern repeats forever, so that we have an *infinite* scf. In order to understand what this is, we need to talk about the finite ones first.

A *finite simple continued fraction* is an expression of the form

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \dots + \cfrac{1}{b_{k-1} + \cfrac{1}{b_k}}}}},$$

where the b_i's are integers and $b_i \geq 1$ for $i \geq 1$. This notation is not easy to use, so we customarily write $x = \langle b_0, b_1, b_2, \dots, b_k \rangle$ to represent the above scf. We'll call the b_i's *partial quotients* of x. Now since b_i is a positive integer for $i \geq 1$, we see that $0 \leq x - b_0 < 1$ and so $b_0 = [x]$, the greatest integer in x. Let us write $x_0 = x$ and $x_{n+1} = \dfrac{1}{x_n - b_n}$ for $n \geq 1$; the numbers x_i are called the *complete quotients* of x. It turns out that $b_n = [x_n]$ for all n, and it is but a short step to the following theorem.

THEOREM 1. *(a) Every positive rational number has exactly two representations as a finite scf, differing only in the last place—if $x = \langle b_0, b_1, b_2, \dots, b_k \rangle$ with $b_k > 1$, then the other finite scf representing x is $\langle b_0, b_1, b_2, \dots, b_k - 1, 1 \rangle$.*

(b) Every finite scf represents a rational number.

Part (a) is a consequence of Euclid's Algorithm for finding greatest common divisors; here is an example:

$$\begin{aligned} &\frac{355}{113} = 3 + \frac{16}{113}; \quad \frac{113}{16} = 7 + \frac{1}{16}, \text{and so} \\ &\frac{355}{113} = \langle 3, 7, 16 \rangle. \end{aligned} \tag{2}$$

Part (b) follows from the fact that simplifying a finite scf involves only finitely many arithmetic operations involving integers and rationals, so that the end result is a rational number. Here is an example:

$$\langle 2, 1, 2, 1, 1, 4, 1, 1, 6 \rangle = \frac{1264}{465} \tag{3}$$

Checking the arithmetic in the above examples is a good idea; writing out their decimal equivalents might be revealing.

If $x = \langle b_0, b_1, b_2, \dots, b_k \rangle$, then for $n \leq k$, the theorem tells us that the scf $C_n = \langle b_0, b_1, b_2, \dots, b_n \rangle$ is a rational number called the n^{th} *convergent* to x. This brings up a problem with continued fractions: how do we calculate the convergents? Dealing with an 8-deep fraction is tedious at best, so is there a shortcut? In fact, there is.

THEOREM 2. *Let $b_0, b_1, \dots$ be real numbers with $b_i \geq 1$ for $i \geq 1$. Define the numbers P_n and Q_n as follows:*

$$\begin{aligned} P_{-1} &= 1, \quad Q_{-1} = 0, \\ P_{-2} &= 0, \quad Q_{-2} = 1, \\ P_n &= b_n P_{n-1} + P_{n-2}, \qquad n \geq 0; \\ Q_n &= b_n Q_{n-1} + Q_{n-2}, \qquad n \geq 0. \end{aligned}$$

Then the successive convergents to the scf $\langle b_0, b_1, b_2, \dots, b_k \rangle$ are $C_n = P_n/Q_n$, for $n \geq 1$.

For example, for that number g defined in Equation 1, the numbers P_n and Q_n, for $n \geq 0$, are $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$ and $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$, respectively — the Fibonacci numbers. Their ratios approximate the number $g = (1 + \sqrt{5})/2$, commonly known as the golden mean ("g" as in "golden").

Here is a sketch of the proof of Theorem 2: For the base case, notice that $C_0 = b_0 = b_0/1$ and indeed $P_0 = b_0 = b_0 \cdot 1 + 0 = b_0 P_{-1} + P_{-2}$ and $Q_0 = 1 = b_0 \cdot 0 + 1 = b_0 Q_{=1} + Q_{-2}$. Then, with the induction hypothesis in hand — namely, that the above formulas are true for all $n \leq k$ and for all scf's — we notice that

$$\langle b_0, b_1, b_2, \dots, b_{k+1} \rangle = \langle b_0, b_1, b_2, \dots, b_k + \frac{1}{b_{k+1}} \rangle$$

and apply the formulas to the right-hand side.

The convergents exhibit some curious behavior. For example, the successive convergents in Equation 3 begin as follows:

$$\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{6}, \frac{87}{32}, \dots$$

Notice that

$$\begin{aligned} 19 \cdot 32 - 87 \cdot 7 &= -1, \\ 11 \cdot 7 - 19 \cdot 4 &= 1, \\ 8 \cdot 4 - 11 \cdot 3 &= -1, \end{aligned}$$

and in general, it appears that the following is true:

THEOREM 3. *If P_n and Q_n are as in Theorem 2, then $P_n Q_{n+1} - P_{n+1} Q_n = (-1)^{n+1}$ for $n \geq 0$.*

Exercise: use the formulas to prove it.

Infinite scf's first turn up in Europe in the sixteenth and seventeenth centuries in the work of Bombelli (1526-1573) and Cataldi (1548-1626), with Wallis (1616-1703) and Huyghens (1629-1695) first working out the theory ([2], Chaps. 2 and 3). They have broad applications in number theory, both to the approximation of irrational numbers by rationals (the Mystery) and to the solution of certain quadratic equations in integers (the Cows). We may define the *infinite simple continued fraction* $\langle b_0, b_1, b_2, \dots \rangle$ by

$$x = \langle b_0, b_1, b_2, \dots \rangle = \lim_{k \to \infty} \langle b_0, b_1, b_2, \dots, b_k \rangle,$$

provided this limit exists. If $b_k \geq 1$ for all $k \geq 1$, the limit does exist:

THEOREM 4. *If $b_0, b_1, b_2, \ldots$ is a sequence of numbers such that $b_i \geq 1$ for $i \geq 1$, and if the numbers P_n and Q_n are defined as above, then:*

(a) $\dfrac{P_0}{Q_0} < \dfrac{P_2}{Q_2} < \dfrac{P_4}{Q_4} < \ldots$, *and* $\dfrac{P_1}{Q_1} > \dfrac{P_3}{Q_3} > \dfrac{P_5}{Q_5} > \ldots$;

(b) $\dfrac{P_{2n}}{Q_{2n}} < \dfrac{P_{2n+1}}{Q_{2n+1}}$ *for all $n \geq 0$; and*

(c) the limit $\lim_{k\to\infty}\langle b_0, b_1, b_2, \ldots, b_k\rangle$ exists.

Proof. Let us sketch the proof. To prove (a), use the results from Theorem 2 and Theorem 3: for example, begin one of the base cases by observing that $b_2 \geq 1$ since

$$0 < \frac{P_2}{Q_2} = \frac{b_2P_1 + P_0}{b_2Q_1 + Q_0};$$

then, clear of fractions and apply some algebra. The proof of (b) is simply a restatement of Theorem 3, paying special attention to the parity of the subscripts. Now, by (a), the convergents P_{2n}/Q_{2n} form an increasing sequence; by (b) this sequence is bounded above by any odd convergent. By the completeness of the real numbers, the sequence $\{P_{2n}/Q_{2n}\}$ converges to their least upper bound x. It turns out that x is also the limit of the sequence of odd convergents, and to show this is not hard. □

Now, let's try computing the scf for an irrational number — say, $\sqrt{19}$. We know it is infinite, but let's just see what happens. To begin, we notice that $[\sqrt{19}] = 4$, so that $\sqrt{19} = 4 + \sqrt{19} - 4$, and the algorithm proceeds as follows.

$$\begin{aligned}
\sqrt{19} &= 4 + (\sqrt{19} - 4),\\
\frac{1}{\sqrt{19}-4} &= \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3},\\
\frac{3}{\sqrt{19}-2} &= \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5},\\
\frac{5}{\sqrt{19}-3} &= \frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2},\\
\frac{2}{\sqrt{19}-3} &= \frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5},\\
\frac{5}{\sqrt{19}-2} &= \frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3},\\
\frac{3}{\sqrt{19}-4} &= \sqrt{19}+4 = 8 + (\sqrt{19}-4),\\
\frac{1}{\sqrt{19}-4} &= \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3},
\end{aligned}$$

and hey, look, it repeats: $\sqrt{19} = \langle 4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8\ldots\rangle$, which we abbreviate as $\sqrt{19} = \langle 4, \overline{2, 1, 3, 1, 2, 8}\rangle$. It is also true that $\sqrt{2} = \langle 1, \overline{2}\rangle$, and rational approximations to $\sqrt{2}$ were known almost 4 millennia ago. Calculate the scf for several values of $\sqrt{D}$, where D is any positive non-square integer, and you will be convinced that the following theorem is true:

THEOREM 5. *There exists a least positive integer k, called the period length, such that the scf expansion of $\sqrt{D}$ is given by $\sqrt{D} = \langle N, \overline{b_1, \ldots, b_{k-1}, 2N}\rangle$.*

This result dates from the seventeenth century and grew out of correspondence between Pierre Fermat (1601-1665) and the English mathematician William Brouncker (1620-1684). In fact, these so-called *periodic continued fractions* are precisely those that represent quadratic irrationalities — and this is a theorem due to, of all people, that startling prodigy Évariste Galois (1811-1832):

THEOREM 6 (Galois). *The simple continued fraction $x = \langle b_0, b_1, b_2, \ldots\rangle$ represents a quadratic surd — i.e. an irrational root of a quadratic equation with rational coefficients — if and only if x is periodic.*

Computer algebra systems often have built-in functions to find the scf expansion of quadratic surds. But it is fun to write your own; try it and see. And now, on to the cattle!

Archimedes and the Cattle. Archimedes was the greatest mathematician in antiquity, and one measure of his greatness was that he thought BIG. He moved the earth with a big stick, and it turns out that such a stick needed to be about 10^{35} millimeters long. He filled the universe with grains of sand and counted the grains (roughly 10^{51} of them), then imagined an even bigger universe full of sand and counted those grains (about 10^{63} grains). Finally, in a letter to his friend Eratosthenes he posed the problem of finding the size of a certain herd of cattle, which size pitifully dwarfs all of the previous numbers. (Maybe he wrote the letter — and maybe he didn't; more about that later.) Here are the details:

Archimedes asks us to find the numbers W, X, Y and Z of white, black, brown, and spotted bulls, and the numbers w, x, y and z of white, black, brown, and spotted cows, subject to the following nine conditions:

$$\begin{array}{llll}
1. & W = (1/2 + 1/3)X + Z, & 2. & X = (1/4 + 1/5)Y + Z,\\
3. & Y = (1/6 + 1/7)W + Z, & 4. & w = (1/3 + 1/4)(X + x),\\
5. & x = (1/4 + 1/5)(Y + y), & 6. & y = (1/5 + 1/6)(Z + z),\\
7. & z = (1/6 + 1/7)(W + w), & 8. & W + X \text{ is a square, and}\\
9. & Y + Z \text{ is a triangular number.} & &
\end{array} \tag{4}$$

These amount to seven linear equations and two nonlinear equations in eight unknowns. so that it is not obvious that a solution even exists. Solving the linear equations 1 through 7 is fairly straightforward, and with a computer algebra system such as *Mathematica* we can do it in the blink of an eye. It turns out that the first seven variables are rational multiples of z with common denominator 5439213; if we put $z = 5439213v$, we obtain integer values for all eight variables, namely

$$\begin{array}{llll}
W = 10366482v, & X = 7460514v, & Y = 7358060v, & Z = 4149387v,\\
w = 7206360v, & x = 4893246v, & y = 3515820v, & z = 5439213v,
\end{array} \tag{5}$$

where v is an integer-valued parameter. At a minimum, Archimedes now has about 50 million head of cattle.

To satisfy condition 8, we want $W + X = 17826996v$ to be a square. Since $17826996 = 4 \cdot 4456749$, with the latter factor square-free, this will occur if we put $v = 4456749s^2$. This yields $W = 46200808287018s^2$ with similarly magnified values for the other seven variables. At this point, with about 225 trillion head of cattle in the herd, Archimedes has clearly overrun the planet—and he still must satisfy condition 9, namely that $Y + Z$ must be triangular.

The triangular numbers are $1, 3, 6, 10, 15, \ldots$ and have the general form $n(n+1)/2$, so this means that $Y + Z = 51285802909803s^2 = n(n+1)/2$ for some integer n. Multiplying this equation by 8 and adding 1 yields the equation

$$410286423278424s^2 + 1 = (2n+1)^2.$$

So, if we set $t = 2n+1$, we conclude that satisfying conditions 1-9 amounts to solving the equation

$$t^2 - 410286423278424s^2 = 1, \tag{6}$$

for nonzero integers s and t.

It is apparent madness to prove that s exists, let alone ever find it. But it does, and we can, by means of continued fractions. It turns out that the convergents to the scf expansion of an irrational number are excellent approximations to that number. In particular, we have the following corollary to Theorem 3:

THEOREM 7. *If $\alpha = \langle b_0, b_1, b_2, \ldots\rangle$ is the scf for α, and if P_n and Q_n are as in Theorem 2, then $|P_n/Q_n - \alpha| < 1/Q_n^2$.*

For example, $|355/113 - \pi| = 0.00000026676\ldots < 0.0000783\ldots = 1/113^2$. Using this result, Lagrange (1736-1813) was able to prove that the scf for $\sqrt{D}$ encodes all of the solutions to the equation $x^2 - Dy^2 = 1$:

THEOREM 8 (Lagrange). *Let D be a positive non-square integer, let $N = [\sqrt{D}]$, and let $\sqrt{D} = \langle N, \overline{b_1, \ldots, b_{k-1}, 2N}\rangle$, where k is the period length. If P_n and Q_n are as in Theorem 2, and m is a positive integer, then*

$$P_{mk}^2 - DQ_{mk}^2 = \begin{cases} -1, & \text{if } mk \text{ is odd;} \\ 1, & \text{if } mk \text{ is even.} \end{cases}$$

Furthermore, if the integers x and y satisfy $x^2 - Dy^2 = \pm 1$, then there exists an integer m such that $x = P_{mk}$ and $y = Q_{mk}$, where k is as above.

For a proof, see [12], Section 13.4 or [7], Section 14.5. As an example, we found that $\sqrt{19} = \langle 4, \overline{2, 1, 3, 1, 2, 8}\rangle$, with period length $k = 6$; a short calculation reveals that $P_6 = 170, Q_6 = 39$, and $170^2 - 19 \cdot 39^2 = 28900 - 19 \cdot 1521 = 28900 - 28899 = 1$.

We now see that Lagrange's Theorem will enable us to find a solution to Equation 6, so we need the scf expansion of $\sqrt{410286423278424}$. Now the period length of this scf is 203254, but *Mathematica* happily computes both this scf and — using the formulas from Theorem 2 — the values of s and t. The final value for $W = 46200808287018s^2$ is, as stated in the literature [4], a 206545-digit number. Written out in full, at 80 characters a line and 72 lines a page, it runs to 37 pages; it took about 35 seconds to calculate the scf, and about 5 minutes to find the value of W. The total number of cattle turns out to be

$$7760271406486818269530232 \cdots 8973723406626719455081800,$$

with the $\cdots$ representing 206495 missing digits. Lots of milking to be done here!

And now, let's solve that mystery.

Generalized Continued Fractions, and the Mystery Solved. More generally, we can look at so-called *generalized continued fractions*, i.e. expressions of the form

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots + \cfrac{a_k}{b_k + \ldots}}}},$$

where the a_i's and b_i's are numbers. Again, this notation is unwieldy, so we write this as

$$x = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+\cdots}\frac{a_k}{b_k+\cdots}. \tag{7}$$

If $a_k > 0$ and $b_k \geq 1$ for all k, then the above expression converges to a real number. Using induction, we can prove that the successive convergents to Equation 7 are P_n/Q_n, for $n \geq 1$, where

$$\begin{aligned} P_0 &= b_0, P_{-1} = 1; Q_0 = 1, Q_{-1} = 0; \\ P_n &= b_n P_{n-1} + a_n P_{n-2}, n \geq 1; \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2}, n \geq 1. \end{aligned} \tag{8}$$

These generalized formulas clearly resemble the formulas from Theorem 2 for the convergents of simple continued fractions. Again, if the cf in equation 7 converges, and the a_i and b_i are rational, then it is periodic if and only if it represents a quadratic irrational.

Notice that the a_n and the b_n are not necessarily integers, or even positive, or even rational (or, for that matter, even real). The great magical genius Ramanujan (1887-1920) had a particular liking for continued fractions with $b_n = 1$ and the a_n arbitrary real numbers, or even variables.

We can now solve the last puzzle, the one about the continued fraction

$$2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\frac{5}{5+\cdots}. \tag{9}$$

It is said to represent a real number θ. But what is θ?

Since $\theta = 2 + \dfrac{2}{2+p}$, where p is a positive number, we see that $2 < \theta < 3$. From the formulas in Equation 8, we see that $b_0 = 2$, and for $n \geq 1, b_n = a_n = n+1$. Thus,

$$\begin{aligned} P_0 &= 2, \\ P_1 &= 2 \cdot 2 + 2 \cdot 1 = 6, \\ P_2 &= 3 \cdot 6 + 3 \cdot 2 = 24, \\ P_3 &= 4 \cdot 24 + 4 \cdot 6 = 120, \\ P_4 &= 5 \cdot 120 + 5 \cdot 24 = 720, \end{aligned}$$

and we boldly guess that $P_n = (n+2)!$ for $n \geq 0$. Furthermore,

$$\begin{aligned} Q_1 &= 2 \cdot 1 + 2 \cdot 0 = 2, \\ Q_2 &= 3 \cdot 2 + 3 \cdot 1 = 9, \\ Q_3 &= 4 \cdot 9 + 4 \cdot 2 = 44, \\ Q_4 &= 5 \cdot 44 + 5 \cdot 9 = 265; \end{aligned}$$

the fourth convergent to θ is $P_4/Q_4 = 720/265 = 2.71698\ldots$, the fifth is $P_5/Q_5 = 5040/1854 = 2.71844\ldots$, and we guess that $\theta = e$.

In order to prove it, we need to make sense of the sequence $2, 9, 44, 265, \ldots$. Notice that

$$2 = 3 - 1 = \frac{3!}{2!} - \frac{3!}{3!},$$
$$9 = 4 \cdot 2 + 1 = 4\left(\frac{3!}{2!} - \frac{3!}{3!}\right) + 1 = \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!},$$
$$44 = 5 \cdot 9 - 1 = 5\left(\frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}\right) - 1 = \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!},$$

and in general, we guess that

$$Q_n = (n+2)!\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n+2}\frac{1}{(n+2)!}\right) = (n+2)!\sum_{k=2}^{n+2}\frac{(-1)^k}{k!} \qquad (10)$$

We'll leave it as an exercise to prove that the equation in Equation 10 holds for all $n \geq 1$.

Having done so, we conclude that the nth convergent C_n to the continued fraction in Equation 9 is given by

$$C_n = \frac{P_n}{Q_n} = \frac{(n+2)!}{(n+2)!\sum_{k=2}^{n+2}\frac{(-1)^k}{k!}} = \frac{1}{\sum_{k=2}^{n+2}\frac{(-1)^k}{k!}}.$$

Since $0! = 1! = 1$, we know that

$$\sum_{k=2}^{n+2}\frac{(-1)^k}{k!} = \frac{1}{1} - \frac{1}{1} + \sum_{k=0}^{n+2}\frac{(-1)^k}{k!} = \sum_{k=0}^{n+2}\frac{(-1)^k}{k!};$$

we also know that $\lim_{n\to\infty}\sum_{k=0}^{n+2}\frac{(-1)^k}{k!}$ exists by the Alternating Series Test. In fact, $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1}$ is what we get by substituting $x = -1$ in the usual Maclaurin series for e^x.

Hence, $\theta = \lim_{n\to\infty} C_n = 1/e^{-1} = e$, and our guess that

$$2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\frac{5}{5+\cdots} = e$$

was correct!

Questions. *Where can I find out more about continued fractions?* Most elementary number theory books have chapters devoted to continued fractions. See, for example, [6] (a classic), [7] (which also treats generalized continued fractions), [8] and [12]. Olds' book [10] is a very nice elementary introduction. Perron's book [11] will take you a long way into the subject—if you can read German. As for the history of continued fractions, both Volume 2 of Dickson's encyclopedic "History of the Theory of Numbers" [4] and Brezinski's work [2] contain tons of historical information; their styles are distinctly different. The above list is far from complete.

What else can you do with continued fractions? Among other things, you can use them to factor large integers. In the late 1960's, Brillhart and Morrison [3] developed an integer factoring algorithm called CFRAC, with which they factored the seventh Fermat number $F_7 = 2^{128} + 1$. Based on continued fractions, it was the world's

principle large-number cracker until being superseded by the Quadratic Sieve in the early 1980's.

What is the simple continued fraction expansion for e? The finite scf in Equation 3 for the rational number $\frac{1264}{465}$ is its beginning. (That's why I asked you to check the arithmetic.) In fact, $e = \langle 2, 1, 2, 1, 1, 4, 1, 1, \ldots, 2n, 1, 1, \ldots \rangle$; for a proof, see [7], Section 11.6.

What about continued fractions for π? The finite scf in Equation 2 for $\frac{355}{113}$ is how it begins; that six-decimal-place approximation to π was known to the Chinese mathematician Tsu Ch'ung Chi (430-501). It proceeds, with no apparent pattern, as follows: $\pi = \langle 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots \rangle$. As of 1999, it was known to 20,000,000 terms.

If Archimedes didn't make up the Cattle Problem, who did — and when? This problem surfaces in 1773 when Gotthold Ephraim Lessing published the Greek text of a 24-verse epigram translated from an Arabic manuscript. This epigram was the Cattle Problem as stated by Archimedes in a letter for the students at Alexandria which Archimedes sent to his friend Eratosthenes. A fair amount of ink has been spilled over the question, "What did Archimedes know, and when did he know it?" Most historians of mathematics agree that the problem is very likely due to Archimedes, although the epigram came later.

What's TEX? TEXis a mathematical typesetting program, but that is like saying that Hank Aaron was a home run hitter. For it has become a language that mathematicians use to communicate with each other in emails, handwritten notes, and other informal settings. It was developed by Donald Knuth, the legendary computer scientist, who did not cash in on his wonderful product, but *gave it away* to the American Mathematical Society. Such feats of invention and altruism are as admirable as they are rare.

What is so interesting about Ramanujan's continued fractions? Among other features, they look spectacular. One of my favorites is the following, which he included in a famous letter to the English mathematician G. H. Hardy:

$$\frac{1}{1+}\frac{e^{-2\pi\sqrt{5}}}{1+}\frac{e^{-4\pi\sqrt{5}}}{1+}\frac{e^{-6\pi\sqrt{5}}}{1+\ldots} = \left[\frac{\sqrt{5}}{1 + \left(5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2} - 1\right)^{1/5}} - \frac{\sqrt{5}+1}{2}\right] e^{2\pi/\sqrt{5}}$$

No, I don't know how to prove that these two expressions are equal. But there is some comfort: Hardy stated that when he first saw this equality, it defeated him completely.

Why is it called the Pell Equation if Pell had nothing to do with it? It was one of the few mistakes that the great Leonhard Euler (17078-1783) ever made. He wrongly attributed the equation to John Pell because it appeared in a book Pell wrote, but Pell had no connection with the equation. It would be appropriate to name it after either Brouncker, Fermat, or the Indian mathematician Bhaskara (1114-1185), all of whom studied the equation extensively. This is yet another example of Boyer's Law, which — what's Boyer's Law? That's another story.

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OPTIMAL INITIAL ANGLE TO FIRE A PROJECTILE

WILLIAM CHAU*

Assume a projectile is fired without air resistance and lands at a height y above its initial vertical position. What is the optimal initial angle of firing to maximize the horizontal distance travelled by the projectile? Most students learned that the optimal angle is $\pi/4$ when $y = 0$. Is the optimal angle still $\pi/4$ when $y \neq 0$?

In Physics, the formulae that describe the motion of a free falling body are

$$x = v_0 \cos(\theta) t \tag{1}$$

and

$$y = v_0 \sin(\theta) t - \frac{1}{2} g t^2, \tag{2}$$

where (x, y) is the position of the body at time t, projected at an angle θ, $0 \leq \theta \leq \pi/2$, with initial speed $v_0 > 0$, from the initial position $(0, 0)$, and subject to a gravitational pull of $-g$. Solve equation (1) for t and substitute that value into equation (2) to obtain

$$y = \tan(\theta) x - \frac{g x^2}{2 v_0 \cos^2(\theta)}. \tag{3}$$

Differentiate equation (3) with respect to θ to get

$$\frac{\partial y}{\partial \theta} = \sec^2(\theta) x + \tan(\theta) \frac{\partial x}{\partial \theta} - \frac{g}{v_0} \left(\frac{x}{\cos^2(\theta)} \frac{\partial x}{\partial \theta} + \frac{\sin(\theta) x^2}{\cos^3(\theta)} \right) \tag{4}$$

since x and y are functions of θ and t. We are searching for the maximum x for a given fixed y, so we take $\partial x/\partial \theta = \partial y/\partial \theta = 0$ in equation (4) and solve for x, obtaining $x = 0$ or $x = (v_0/g)\cot(\theta)$. Clearly, $x = 0$, which occurs when $t = 0$ or when $\theta = \pi/2$, does not produce a maximum, so we take

$$x = \frac{v_0}{g} \cot(\theta), \tag{5}$$

which, when put into equation (3) and solve for θ, yields

$$\theta = \arcsin\left(\frac{v_0}{\sqrt{2\left(v_0^2 - g y\right)}} \right). \tag{6}$$

Since θ lies in the first quadrant, the fraction in equation (6) lies between 0 and 1, and we must have

$$-\infty < y < \frac{v_0^2}{2g}. \tag{7}$$

Thus we see that $v_0^2/2g$ is the upper vertical limit to the height y at which the projectile can land.

Equation (6) shows us that the optimal angle θ is not always $\pi/4$. In fact, θ is a strictly increasing function of y in the interval described by (7). The following

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y	θ	x
$-\infty$	0	∞
$-v_0^2/g$	$\pi/6$	$v_0\sqrt{3}/g$
0	$\pi/4$	v_0/g
$v_0^2/3g$	$\pi/3$	$v_0/g\sqrt{3}$
$v_0^2/2g$	$\pi/2$	0

table displays some interesting values of y, θ and x and their limits, calculated from equations (5) and (6).

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EXPECTED AREAS OF RANDOMLY GENERATED TRIANGLES OF FIXED PERIMETER*

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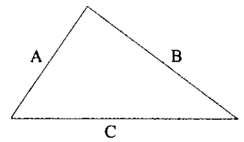
Abstract. We develop methods of randomly generating a triangle of fixed perimeter p. We then compute the mean and variance of the area of the triangle in terms of p.

1. Introduction. The project which led to this article began with the question: "If we were to generate triangles of fixed perimeter p by a random process, what would be the expected area?" In order to answer this question, we first needed to be more precise in its formulation.

We consider a triangle whose side lengths A, B and C are randomly chosen real numbers satisfying $A+B+C=p$.



In order for this to make sense, the triple (A,B,C) must be chosen randomly on the portion of the plane $A+B+C=p$ on which A, B and C are all positive and satisfy the triangle inequalities. It turns out that this leads us to require that $(A,B,C)\in S$, where S is the interior of the triangle in $\mathbb{R}^3$ with vertices $(p/2,p/2,0)$, $(p/2,0,p/2)$ and $(0,p/2,p/2)$ (see Section 3). In this way, choosing a point (A,B,C) in the interior of the triangular domain S corresponds to choosing a triangle with side lengths A, B and C. By "choosing (A,B,C) randomly on S," we mean that A, B and C are jointly distributed random variables with some joint probability density function $f(a,b,c)$, defined for all $(a,b,c)\in S$. In this article, we fix $m,n,l\in\mathbb{N}$, and we take f to be of the form $f(a,b,c)=ka^mb^nc^l$ for some positive constant k.

Let X be the area of the triangle whose sides have randomly chosen lengths $(A,B,C)\in S$, according to the joint probability density function f. In this paper, we determine the mean and variance of X.

Sections 2 through 4 present background material needed for the presentation of our results, including density functions, expectation, Heron's formula for computing the area of a triangle, the Gamma function and the Beta function.

In Section 5, we present the special case where $f(a,b,c)$ is a constant function. It follows that $f(a,b,c)=8\sqrt{3}/(3p^2)$. In this case, X has mean $\pi p^2/105$ and variance $p^4[(1/960)-(\pi/105)^2]$.

In Section 6, we state and prove the general result where $f(a,b,c)=ka^mb^nc^l$ for some $m,n,l\in\mathbb{N}$. We compute the value of k necessary to make f a joint probability density function on S. We then find a formula for the mean and variance of X in terms of m, n, l and p (Theorems 6.2 and 6.3). We also give a formula for the higher moments of X.

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2. Density Functions and Expectation. In this section, we will consider basic concepts in probability theory needed to solve the problem outlined in the introduction. For further information, the reader is referred to [2].

We consider a *continuous random variable* X which takes on values within a given interval $I \subseteq \mathbb{R}$. The probability distribution of X is determined by a *probability density function* (pdf). Recall that the pdf of a random variable X is a function $f : I \to \mathbb{R}$ with three main properties:

1. $f(x) \geq 0$ for all $x \in I$
2. $\int_I f(x)\,dx = 1$
3. for all $J \subseteq I$, the probability that $X \in J$ is $\int_J f(x)\,dx$.

Another important concept is that of the *expectation* (also called the *mean*) of a random variable X. Intuitively, the expectation of X is the 'average value' of X, and it is commonly denoted by $\mathrm{E}(X)$ or μ. For a random variable X with outcome space I and pdf $f(x)$, we have

$$\mathrm{E}(X) = \mu = \int_I x\, f(x)\,dx.$$

Similarly, for a function of X, say $g(X)$, we can determine the expected value as follows:

$$\mathrm{E}[g(X)] = \int_I g(x)f(x)\,dx.$$

Let S be a smooth surface in $\mathbb{R}^3$. Then the random variables A, B and C are said to be *jointly distributed on* S if $(A, B, C) \in S$ for all outcomes. The joint probability density function (jpdf) of A, B and C is a real-valued function f on S such that

1. $f(a, b, c) \geq 0$ for all $(a, b, c) \in S$
2. $\int_S f\,d\sigma = 1$
3. for any $S' \subseteq S$, we have $\mathrm{Prob}[(A, B, C) \in S'] = \int_{S'} f\,d\sigma$.

The notation $\int_{S'} f\,d\sigma$ is used to denote the surface integral over S' of a function f of three variables.

The expected value of a function $g(A, B, C)$ of random variables A, B and C, which are jointly distributed on a surface S with jpdf f, is defined by

$$\mathrm{E}[g(A, B, C)] = \int_S gf\,d\sigma. \tag{1}$$

3. Preliminary Computations. We begin by considering Heron's formula, which states that the area of a triangle T of side lengths a, b, and c is

$$H(T) = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a + b + c)/2$. Writing this in terms of a, b and c, and letting $a + b + c = p$, gives

$$H(a, b, c) = \frac{\sqrt{p}}{4}\sqrt{(a + b - c)(a - b + c)(-a + b + c)}.$$

We generate random triangles of fixed perimeter p by considering the lengths of the sides as random variables A, B and C. The surface we work on is the plane determined by the perimeter: $A + B + C = p$.

It is necessary to calculate the set of possible outcomes (A, B, C) on the plane $A + B + C = p$, since this will determine our surface of integration. To do so, we use the triangle inequality. We have:

$$A + B \geq C \qquad A + C \geq B \qquad B + C \geq A$$

These three inequalities yield the triangle S in the abc-plane which has vertices $(p/2, p/2, 0)$, $(p/2, 0, p/2)$, and $(0, p/2, p/2)$. We assume that (A, B, C) is jointly distributed on the triangle S, and we denote the jpdf by $f(a, b, c)$.

For the remainder of the paper we assume X is the random variable defined by $X = H(A, B, C)$.

4. Special Functions. In order to describe our results, it is necessary to review the Gamma function, the classical Beta function, and a multivariate extension of the Beta function. The Gamma function extends the factorial to positive real arguments. For $t > 0$, we define

$$\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,dy$$

It is easily verified that $\Gamma(1) = 1$. When $t > 1$, one can use integration by parts to show that

$$\Gamma(t) = (t-1)\Gamma(t-1),$$

and so, by induction, $\Gamma(n) = (n-1)!$ for all positive integers n.

The Beta function is defined for positive real arguments a, b as

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

A transformation of variables argument (see [3, Section 7.6]) illustrates that

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Finally, we define a 3-parameter generalization of the Beta function. Given positive real numbers a, b, c, we define

$$N(a, b, c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}.$$

The following result is due to Dirichlet [1], and can be found in [3, Equations 7.7.4–5].

PROPOSITION 1. *For positive real numbers a, b, c,*

$$\int_0^1 \int_0^{1-x} x^{a-1}y^{b-1}(1-x-y)^{c-1}\,dy\,dx = N(a, b, c).$$

5. A Simple Example. Consider the case where $f(a,b,c) = k$, a constant. Then A, B and C have a *joint uniform distribution* on the triangle S in the abc-plane with vertices $(p/2,p/2,0)$, $(p/2,0,p/2)$, and $(0,p/2,p/2)$. It follows that

$$\int_S k\,d\sigma = k\int_S 1\,d\sigma = k\cdot \text{Area}(S) = 1.$$

The area of the equilateral triangle S is $p^2\sqrt{3}/8$, so we have

$$k = \frac{8\sqrt{3}}{3p^2}.$$

We now use (1) to compute the mean of X, the area of a random triangle. We obtain

$$\begin{aligned}\mathrm{E}(X) &= \int_S H(a,b,c)f(a,b,c)\,d\sigma\\ &= \int_S \frac{\sqrt{p}}{4}\sqrt{(a+b-c)(a-b+c)(-a+b+c)}\cdot\frac{8\sqrt{3}}{3p^2}\,d\sigma.\end{aligned}$$

By parameterizing S using $a = p(u+v)/2$, $b = p(1-v)/2$ and $c = p(1-u)/2$, the integral can be rewritten

$$\begin{aligned}\mathrm{E}(X) &= \int_0^1\int_0^{1-u}\frac{2\sqrt{3}}{3p^{3/2}}\sqrt{(pu)(pv)(p(1-u-v))}\left|\langle\tfrac{p}{2},0,-\tfrac{p}{2}\rangle\times\langle\tfrac{p}{2},-\tfrac{p}{2},0\rangle\right|dv\,du\\ &= \frac{p^2}{2}\int_0^1\int_0^{1-u}\sqrt{uv(1-u-v)}\,dv\,du\\ &= \frac{p^2}{2}N\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right) = \frac{p^2\pi}{105}.\end{aligned}$$

by Proposition 1.

A similar technique can be used to find the variance of X. First, recall that

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2.$$

To find $\mathrm{E}(X^2)$, we compute

$$\mathrm{E}(X^2) = \int_S H(a,b,c)^2 f(a,b,c)d\sigma = \int_S \frac{p}{16}(a+b-c)(a-b+c)(-a+b+c)\cdot\frac{8\sqrt{3}}{3p^2}\,d\sigma.$$

Parameterizing S as above, we find

$$\begin{aligned}\mathrm{E}(X^2) &= \frac{\sqrt{3}}{6p}\int_0^1\int_0^{1-u}(pu)(pv)(p(1-u-v))\left|\langle\tfrac{p}{2},0,-\tfrac{p}{2}\rangle\times\langle\tfrac{p}{2},-\tfrac{p}{2},0\rangle\right|dv\,du\\ &= \frac{p^4}{8}\int_0^1\int_0^{1-u}uv(1-u-v)\,dv\,du\\ &= \frac{p^4}{960}.\end{aligned}$$

Thus we have proven the following:

PROPOSITION 2. *Let X be the area of a triangle of perimeter p with side lengths A,B,C, where (A,B,C) is chosen uniformly on the triangle with vertices $(p/2,p/2,0)$, $(p/2,0,p/2)$, $(0,p/2,p/2)$. Then*

$$\mathrm{E}(X) = \frac{p^2\pi}{105}\quad\textit{and}\quad \mathrm{Var}(X) = p^4\left[\frac{1}{960}-\left(\frac{\pi}{105}\right)^2\right].$$

6. Main Theorem. In this section we solve the general problem of finding the mean and variance of the area of a randomly generated triangle of perimeter p. We assume that the side lengths (A,B,C) are distributed on the triangle S with vertices $(p/2,p/2,0)$, $(p/2,0,p/2)$, $(0,p/2,p/2)$ according to a jpdf of the form $f(a,b,c) = ka^mb^nc^l$.

It follows that

$$k = \left(\int_S a^mb^nc^l\,d\sigma\right)^{-1}. \tag{2}$$

LEMMA 3. *For nonnegative integers m, n and l, $\int_S a^mb^nc^l\,d\sigma$ is given by*

$$\frac{p^{m+n+l+2}\sqrt{3}}{2^{m+n+l+2}}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{h=0}^{l}\binom{m}{i}\binom{n}{j}\binom{l}{h}N(i+n-j+1,h+m-i+1,j+l-h+1).$$

Proof. We parameterize S as $a = p(u+v)/2$, $b = p(1-v)/2$, $c = p(1-u)/2$ as above. We find

$$\int_S a^mb^nc^l\,d\sigma = \frac{p^{m+n+l+2}\sqrt{3}}{2^{m+n+l+2}}\int_0^1\int_0^{1-u}(u+v)^m(1-v)^n(1-u)^l\,dv\,du.$$

We view this right hand side as

$$\frac{p^{m+n+l+2}\sqrt{3}}{2^{m+n+l+2}}\int_0^1\int_0^{1-u}(u+v)^m(u+(1-u-v))^n(v+(1-u-v))^l\,dv\,du$$

and expand via the binomial theorem to obtain

$$\frac{p^{m+n+l+2}\sqrt{3}}{2^{m+n+l+2}}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{h=0}^{l}\binom{m}{i}\binom{n}{j}\binom{l}{h}\int_0^1\int_0^{1-u}u^{i+n-j}v^{h+m-i}(1-u-v)^{j+l-h}\,dv\,du.$$

By Proposition 1, we see that the integral has the desired form. □

THEOREM 4. *Let S be the triangle with vertices $(p/2,p/2,0)$, $(p/2,0,p/2)$ and $(0,p/2,p/2)$. Let (A,B,C) have jpdf $f(a,b,c) = ka^mb^nc^l$ on S, where m, n and l are nonnegative integers and k is as in (2). Let X be the area of a triangle of side lengths (A,B,C). Then $\mathrm{E}(X)$ is given by*

$$\frac{p^2\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{h=0}^{l}\binom{m}{i}\binom{n}{j}\binom{l}{h}N(i+n-j+3/2,h+m-i+3/2,j+l-h+3/2)}{4\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{h=0}^{l}\binom{m}{i}\binom{n}{j}\binom{l}{h}N(i+n-j+1,h+m-i+1,j+l-h+1)}$$

Proof. We have

$$\mathrm{E}(X) = \frac{k\sqrt{p}}{4}\int_S a^mb^nc^l\sqrt{(a+b-c)(a-b+c)(-a+b+c)}\,d\sigma,$$

where k is given by Lemma 3.

As before, we parameterize S by $a = p(u+v)/2$, $b = p(1-v)/2$, $c = p(1-u)/2$. The integral becomes

$$\mathrm{E}(X) = \frac{kp^{m+n+l+4}\sqrt{3}}{2^{m+n+l+4}}\int_0^1\int_0^{1-u}(u+v)^m(1-v)^n(1-u)^l\sqrt{uv(1-u-v)}\,dv\,du.$$

Expanding via the binomial theorem as in Lemma 3, we see that the integral on the right hand side is equal to

$$\frac{kp^s\sqrt{3}}{2^s}\sum_{i,j,h=0}^{m,n,l}\binom{m}{i}\binom{n}{j}\binom{l}{h}\int_0^1\int_0^{1-u} u^{i+n-j+\frac{1}{2}}v^{h+m-i+\frac{1}{2}}(1-u-v)^{j+l-h+\frac{1}{2}}\,dv\,du,$$

where $s = m+n+l+4$. By Proposition 1, we see that

$$\mathrm{E}(X) = \frac{kp^s\sqrt{3}}{2^s}\sum_{i,j,h=0}^{m,n,l}\binom{m}{i}\binom{n}{j}\binom{l}{h}N\left(i+n-j+\frac{3}{2}, h+m-i+\frac{3}{2}, j+l-h+\frac{3}{2}\right)$$

and this proves the theorem. □

In order to compute $\mathrm{Var}(X)$, we need to find $\mathrm{E}(X^2)$. We actually compute the general moment $\mathrm{E}(X^z)$ for $z \geq 2$, since the added generality does not complicate things too much.

THEOREM 5. *Let S be the triangle with vertices $(p/2, p/2, 0)$, $(p/2, 0, p/2)$ and $(0, p/2, p/2)$. Let (A, B, C) have jpdf $f(a, b, c) = ka^m b^n c^l$ on S, where m, n and l are nonnegative integers and k is as in (2). Let X be the area of a triangle of side lengths (A, B, C). Then*

$$\mathrm{E}(X^z) = \left(\frac{p}{2}\right)^{2z}\frac{\sum_{i,j,h=0}^{m,n,l}\binom{m}{i}\binom{n}{j}\binom{l}{h}N(u+z/2, v+z/2, w+z/2)}{\sum_{i,j,h=0}^{m,n,l}\binom{m}{i}\binom{n}{j}\binom{l}{h}N(u,v,w)}$$

where

$$\begin{aligned} u &= i+n-j+1\\ v &= h+m-i+1\\ w &= j+l-h+1. \end{aligned}$$

This integration is similar to that in Theorem 4 and is left to the reader.

7. Closing Comments. This problem showed how an easily stated question in geometric probability can lead to some interesting results. The methods used to obtain the mean and variance of X also have the potential to answer some other questions. Of course, it is easy to extend our result to cover the case where the joint probability density function is any polynomial involving a, b and c. One could also explore whether it is possible to choose some well-behaved non-polynomial jpdf on the surface S and compute then mean and variance of X, or whether one can find an analogue of Theorem 4 in the case where m, n, l are arbitrary positive real numbers. We invite the interested reader to pursue these extensions.

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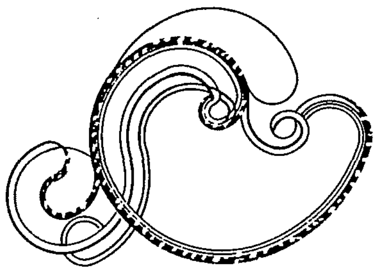
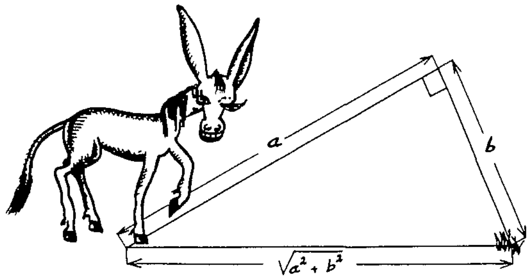
From the Right Side

Aladin, Jinni, and the Lamp. [sand] Ali R. Amir-Moéz

The artwork for this issue was contributed by Prof. Ali R. Amir-Moéz of Texas Tech University. A member of ΠΜΕ, he is also a painter, caricaturist, illustrator, playwright, dancer, actor and, as is fitting for a man born in Persia, a poet.

Besides his plays and literary works, he has written books on research mathematics as well as elementary topics, such as "Ruler, Compass, and Fun", (Ginn and Company, 1961) in which he describes how to draw spiral designs with the tools of classical geometry. The design at left below is his favorite and has become his trademark. The artist prefers the spiral to straight line despite the Theorem *Al Hamar* (Jackass) which states $\sqrt{a^2+b^2} \leq |a| + |b|$, since there is more to life than grass.





The ΠΜΕ *Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.*

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GROWTH RATES OF HYPERBOLIC GRAPHS – AN APPLICATION OF LINEAR ALGEBRA AND MATHEMATICA

BRIAN J. DUDO , ROBERT L. SCHEIB , AND C. CHRIS WU*

1. The Graphs. Imagine that you want to remodel your kitchen floor with ceramic tiles. Suppose you want the tiles to be identical size and identical shape, with the shape being an equal-angled equilateral polygon. Then you have only three choices of shapes to choose: the square, the triangle and the hexagon. For example, pentagonal tiles can not tile your kitchen floor. The three corresponding (infinite) graphs are called the square lattice, the triangular lattice and the hexagonal lattice respectively. Each of these graphs can be characterized by two integers, both ≥ 3: v, the number of neighbors of each vertex (two vertices are called neighbors if there is an edge connecting them); and p, the number of sides of each polygon. We denote such a graph by $G(v,p)$. For example, the triangular lattice is denoted by $G(6,3)$. Translating the above description that the kitchen floor can be covered by square, triangular and hexagonal tiles into mathematical terms, we say that the Euclidean plane R^2 can be tessellated (covered without gaps or overlaps) by $G(4,4)$,$G(6,3)$ and $G(3,6)$. More generally, consider the tessellation of the following three 2-dimensional spaces: the sphere S^2, the Euclidean plane R^2 and the hyperbolic plane H^2. For an intuitive tutorial of the hyperbolic plane H^2, we recommend readers check into an undergraduate web page at http://math.rice.edu/ joel/NonEuclid/, although it is not necessary to have any knowledge about H^2 in order to read this paper. Then $G(v,p)$

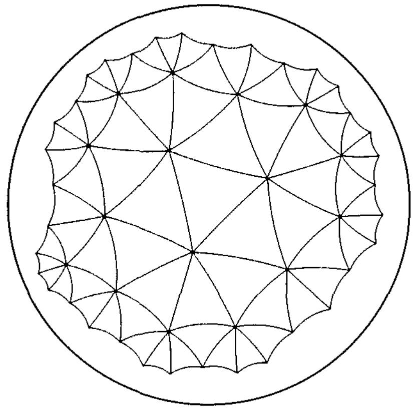

FIG. 1. *Part of*$H(7,3)$ *in the Poincaré disc.*

can tessellate S^2, R^2 or H^2 respectively if the quantity $(v-2)(p-2)$ is smaller than, equal to or larger than 4 respectively. Recalling that v and p are integers strictly greater than two, there are exactly five pairs of (v,p) satisfying $(v-2)(p-2) < 4$ and hence there are five graphs which tessellate S^2, which are $G(3,3)$, $G(3,4)$, $G(3,5)$, $G(4,3)$, and $G(5,3)$—all are finite graphs. For example, $G(5,3)$ is the logo of the Mathematical Association of America. There are exactly three pairs of (v,p) satisfying $(v-2)(p-2) = 4$. So, as we have seen previously, there are three graphs which tessellate R^2: $G(4,4)$, $G(6,3)$ and $G(3,6)$, all are infinite graphs.

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When tessellating the hyperbolic plane H^2, there are infinitely many choices of (v, p) satisfying $(v-2)(p-2) > 4$. In this case the graph is called a hyperbolic graph and will be denoted by $H(v,p)$ (in stead of $G(v,p)$). The hyperbolic graph $H(7,3)$ is shown in Figure 1. $H(4,5)$ is shown in Figure 3.

There is another type of infinite graphs which we will consider: the tree of degree v, which is defined to be the infinite connected graph without any loop such that each vertex has exactly v neighbors. We denote the tree of degree v by $T(v)$. $T(4)$ is shown in Figure 2. $T(v)$ can be regarded as the limit of the hyperbolic graph $H(v,p)$ as p approaches infinity.

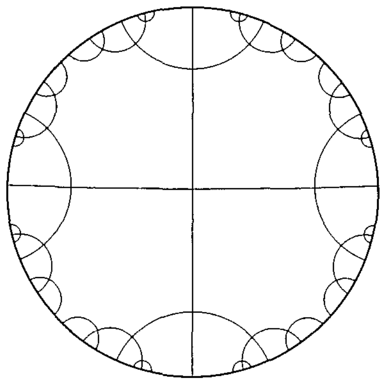

FIG. 2. *The tree* $T(4)$

2. The Growth Rate. There are several parameters associated with an infinite graph which describe how fast the infinite graph grows. In this note we define and calculate one such parameter, called the growth rate, for the square lattice, the trees, and the hyperbolic graph $H(v,p)$. When calculating the growth rate of $H(v,p)$, we will use Mathematica and the concept of diagonalizing a matrix. The growth rate defined in this paper actually equals Cheeger's constant defined in next section. The motivation for calculating such parameters comes from the study of some statistical physics models such as percolation and Ising models, which are currently among the most active research areas in probability theory and statistical mechanics [1,2,3].

We first introduce some notations. Let G be an infinite graph. Denote by o a distinguished vertex of G called the origin. Recall that two vertices x and y of G are called neighbors if there is an edge of G connecting them. Let A be a subgraph of G. Write ∂A for the exterior boundary of A, which is the set of vertices of G which are not in A but have a vertex of A as a neighbor, i. e.,

$$\partial A = \{x \in G : x \notin A, \text{ but } \exists y \in A \text{ such that } x \text{ and } y \text{ are neighbors } \}.$$

Write $|A|$ for the number of vertices in A.

For the square lattice $G(4,4)$, let B_n be the square of sidelength $2n$ centered at the origin. Define the growth rate by

$$R(G(4,4)) = \lim_{n\to\infty} \frac{|\partial B_n|}{|B_n|},$$

which describes the limit of the "perimeter" and "area" ratio of the box as the size of the box increases. It is easy to see that $|B_n| = (2n+1)^2$ and $|\partial B_n| = 4(2n+1)$, so $R(G(4,4)) = 0$.

For the tree $T(v)$, let l_k be the set of vertices of $T(v)$ which are which are exactly k steps away from origin (with l_0 = the origin). Let B_n be the set of vertices of $T(v)$ which are within k steps from the origin, so $|B_n| = \sum_{i=0}^{n} |l_i|$. As we will see next, it is not hard to show that

$$(1) \qquad |\partial B_n| = (v-1)^n v \text{ and } |B_n| = 1 + \frac{v}{v-2}((v-1)^n - 1).$$

So the growth rate of $T(v)$ is:

$$(2) \qquad R(T(v)) = \lim_{n\to\infty} \frac{|\partial B_n|}{|B_n|} = v - 2.$$

To get (1), first notice that $|l_0| = 1$, $|l_1| = v$, and

$$(3) \qquad |l_k| = (v-1)|l_{k-1}| \text{ for any } k \geq 2.$$

So $|\partial B_n| = |l_{n+1}| = (v-1)|l_n| = \cdots = (v-1)^n v$, and

$$|B_n| = \sum_{i=0}^{n} |l_i| = 1 + v\sum_{i=0}^{n-1}(v-1)^i = 1 + v\frac{(v-1)^n - 1}{(v-1)-1},$$

where in the last equality we use the formula

$$(4) \qquad \sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a-1}.$$

This justifies (1).

We now turn to the main purpose of this article—calculating the growth rate of the hyperbolic graph $H(v,p)$. We first partition the vertices into different layers. See Figure 3, where the vertices in each layer are connected by thick edges. The first layer consists of vertices on the central polygon. The second layer consists of those vertices which are not on the first layer but on a polygon which has a vertex in common with

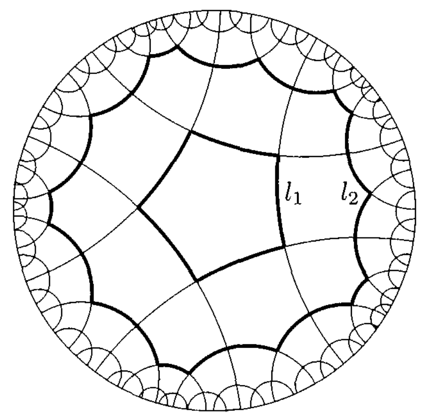

FIG. 3. $H(4,5)$

the first layer. The third, fourth, etc layers are formed in a similar way. Denote by l_k the set of vertices on the kth layer and write $B_n = \cup_{i=1}^n l_i$. The growth rate is then defined by

$$(5) \qquad R(H(v,p)) = \lim_{n\to\infty} \frac{|\partial B_n|}{|B_n|}.$$

We now calculate $|B_n|$ and $|\partial B_n|$. In order to get a recursive relation analogous to (3), let us for the moment assume $p \geq 4$. The vertices on the nth layer can be divided into two groups: I_n—the vertices that are connected by an edge to some vertex on the $(n-1)$th layer, and E_n—those that are not. Then

$$(6) \qquad |l_n| = |I_n| + |E_n| \text{ and } |\partial B_n| = |I_{n+1}|.$$

The following recursive relation is derived in [2]

$$(7) \qquad \begin{bmatrix} |I_n| \\ |E_n| \end{bmatrix} = \begin{bmatrix} v-3 & v-2 \\ pv-3p-3v+8 & pv-2p-3v+5 \end{bmatrix} \begin{bmatrix} |I_{n-1}| \\ |E_{n-1}| \end{bmatrix}$$

with initial values $|I_1| = 0$ and $|E_1| = p$. An outline of the proof of (7) is as follows. First, notice that the second row of the matrix in (7) is equal to $((v-3)(p-3)-1, (v-2)(p-3)-1)$. It is easy to see that $|I_n| = (v-3)|I_{n-1}| + (v-2)|E_{n-1}|$, since each vertex in I_{n-1} has $v-3$ of its neighbors in l_n and each vertex in E_{n-1} has $v-2$ of its neighbors in l_n. We next briefly explain why

$$|E_n| = ((v-3)(p-3)-1)|I_{n-1}| + ((v-2)(p-3)-1)|E_{n-1}|.$$

We start with the first term on the right-hand side of the equation. For each vertex in I_{n-1} there are v polygons sharing this vertex and two of these polygons have no vertices in l_n. Order the remaining $v-2$ polygons in the counter-clockwise direction. Then each of these polygons, except the first and the last, has $p-3$ vertices in E_n, and the first and the last polygon each has $(p-3)-1$ vertices in E_n. In order to handle the polygons in a systematical way, we only count the number of vertices of E_n on the first $v-3$ polygons, the vertices of E_n on the last (i.e., the (v-2)th) polygon will be counted into the second term of the right-hand side of the equation. The total number of vertices of E_n on the first $v-3$ polygons is $(v-3)(p-3)-1$, since the first polygon has $(p-3)-1$ vertices in E_n and each of the remaining $v-4$ polygons has $p-3$ vertices in E_n. So the total number of vertices of E_n contributed by the vertices in I_{n-1} is $((v-3)(p-3)-1)|I_{n-1}|$. In such a way of counting, we can similarly show that the total number of vertices of E_n contributed by the vertices in E_{n-1} is $((v-2)(p-3)-1)|E_{n-1}|$, which we leave as a exercise for interested readers. Hence (7) follows.

Write M for the matrix in (7) and let λ_1 and λ_2 be respectively the smaller and larger eigenvalue of M. Diagonalize M as

$$(8) \qquad M = S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^{-1},$$

where S is a 2×2 matrix. From (7) and (8)

$$(9) \qquad \begin{bmatrix} |I_n| \\ |E_n| \end{bmatrix} = M^{n-1} \begin{bmatrix} |I_1| \\ |E_1| \end{bmatrix} = S \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} S^{-1} \begin{bmatrix} |I_1| \\ |E_1| \end{bmatrix}.$$

We now calculate $|\partial B_n|$ and $|B_n|$ in order to get the growth rate in (5). By (6) and (9)

$$(10) \qquad |\partial B_n| = |I_{n+1}| = [1,0] \begin{bmatrix} |I_{n+1}| \\ |E_{n+1}| \end{bmatrix} = [1,0] S \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} S^{-1} \begin{bmatrix} |I_1| \\ |E_1| \end{bmatrix}.$$

From the definition of B_n (right before (5)), and by (6), (9) and (4)

$$(11) \qquad |B_n| = \sum_{i=1}^n |l_i| = \sum_{i=1}^n (|I_i| + |E_i|) = \sum_{i=1}^n [1,1] \begin{bmatrix} |I_i| \\ |E_i| \end{bmatrix} = [1,1] S \begin{bmatrix} \frac{\lambda_1^n - 1}{\lambda_1 - 1} & 0 \\ 0 & \frac{\lambda_2^n - 1}{\lambda_2 - 1} \end{bmatrix} S^{-1} \begin{bmatrix} |I_1| \\ |E_1| \end{bmatrix}.$$

Using Mathematica, it is very easy to calculate λ_1 and λ_2 and see that $\lambda_2 > \lambda_1$ and $\lambda_2 > 1$ for any $v \geq 3$ and $p \geq 3$. So

$$(12) \qquad \lim_{n\to\infty} (\lambda_1/\lambda_2)^n = 0 \text{ and } \lim_{n\to\infty} (1/\lambda_2)^n = 0$$

To calculate $\lim_{n\to\infty} |\partial B_n|/|B_n|$, factor out λ_2^n from the right hand sides of (10) and (11) respectively, cancel the factored out λ_2^n in the quotient $|\partial B_n|/|B_n|$, use the limits in (12), we then have (recall that $|I_1| = 0$ and $|E_1| = p$)

$$(13) \qquad R(H(v,p)) = \lim_{n\to\infty} |\partial B_n|/|B_n| = (\lambda_2 - 1)\frac{a}{b},$$

where

$$(14) \qquad a = [1,0] S \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \begin{bmatrix} 0 \\ p \end{bmatrix} \text{ and } b = [1,1] S \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \begin{bmatrix} 0 \\ p \end{bmatrix}.$$

We now use Mathematica, with the following commands, to evaluate $R(H(v,p))$ given by (13) and (14).

```
M = {{v - 3, v - 2}, {p v - 3p - 3v + 8, p v - 2p - 3v + 5}};
{S, W} = JordanDecomposition[M]; λ2= W[[2, 2]];
J = {{0, 0}, {0, 1}};
a = {1, 0}.S.J.Inverse[S].{0, p}; b = {1, 1}.S.J.Inverse[S].{0, p};
R[H[v,p]] = (λ2 - 1) a/b;
```

Here the first line is to assign the matrix in (7) to M. The second line is to diagonalize M and to find λ_2 and the 2×2 matrix S in (8). The fourth line is to evaluate a and b given in (14), and finally the last line is to evaluate $R(H(v,p))$ given in (13). Then we have the following formula for $R(H(v,p))$

$$(15) \qquad R(H(v,p)) = \frac{(r + p(v-2) - 2v)(-8 + r - p(v-2) + 4v)}{2(8 + r + p(v-4) - 2v)}$$

where $r = \sqrt{-4 + (2 + p(v-2) - 2v)^2}$.

To get the value of the growth rate for a specific graph, say $H(4,5)$, substitute 4 for v and 5 for p in $R(H(v,p))$ by entering the command `R[H[v,p]]/.{v→4.,p→5.}`. We then have $R(H(4,5)) = 1.1547$. Using Mathematica, we can also find the limit of

$R(H(v,p))$ as $p \to \infty$ by entering the command `Limit[R[H[v,p]], p→ ∞ ]`, which gives

$$\lim_{p\to\infty} R(H(v,p)) = v - 2.$$

That is, as p, the number of sides of the polygon increases, the growth rate of the hyperbolic graph $H(v,p)$ approaches that of the tree $T(v)$.

Let us not forget that what we have done above is only for the case $p \geq 4$. For $p = 3$, the recursive relation is slightly different from (7). In this case, for all layers but the first, the vertices fall into two classes: those that are connected to two vertices of the previous layer and those that are connected to only one vertex of the previous layer. Denoting the number of the vertices in the two groups by $|I_n|$ and $|E_n|$ respectively, we then have from [2] the following recursive relation:

$$\begin{bmatrix} |I_n| \\ |E_n| \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ v-6 & v-5 \end{bmatrix} \begin{bmatrix} |I_{n-1}| \\ |E_{n-1}| \end{bmatrix}$$

with initial values $|I_2| = 3$ and $|E_2| = 3v - 12$. Using this recursion and following the same argument as in the case $p \geq 4$, we have $R(H(v,3)) = \sqrt{12 - 8v + v^2}$, which is exactly the same as when substituting 3 for p in (15). So $R(H(v,p))$ is given by (15) for all $v \geq 3$ and $p \geq 3$.

3. The Growth Rate Equals Cheeger's Constant. Let G be the square lattice, the trees, or the hyperbolic graphs. Cheeger's constant of G is defined to be

$$\rho(G) = \inf\left\{\frac{|\partial A|}{|A|} : A \text{ is a finite subgraph of } G\right\},$$

where ∂A is the exterior boundary of A defined in last section. It is not hard to see that $\rho(G) \leq R(G)$, the growth rate of G. It can be argued that

$$\rho(G) \geq R(G). \tag{16}$$

In fact, it can be argued that for any finite subgraph $A \subset G$, let B_{n_0} be the smallest B_n (defined in last section) which contains A, then $|\partial A|/|A| \geq |\partial B_{n_0}|/|B_{n_0}|$, which implies (16). Therefore

$$\rho(G) = R(G). \tag{17}$$

In [1], (17) is rigorously proved. In fact, a theorem in [1], in a simplified version, states that if B_n is defined as above, then the limit $\lim_{n\to\infty} |\partial B_n|/|B_n|$ exists and equals $\rho(G)$. It is also calculated in [1], independently from the present work, that $\rho(H(v,p)) = (v-2)\sqrt{1 - 4/((v-2)(p-2))}$. The argument there is more advanced (and rigorous) but beyond the scope of an undergraduate research project, and the present argument is more elementary.

We end this article by a problem in College Algebra which we can not solve. From (17), we have that $\rho(H(v,p)) = R(H(v,p))$. We also know that $\rho(H(v,p)) = (v-2)\sqrt{1 - 4/((v-2)(p-2))}$ and that $R(H(v,p))$ is given by (15). Therefore we have that the expression in (15) is

$$(v-2)\sqrt{1 - 4/((v-2)(p-2))}. \tag{18}$$

But we can not do the algebra to show (18). We leave it as a problem for the reader.

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Dear Ghost of Gauss,

You are the master of all mathematical wisdom and grow the sideburns we are not worthy to brush. Can you use your genius to answer this riddle:

"Which weighs more, a pound of feathers or a pound of gold?"

Worried but Curious.

Dear WBC,

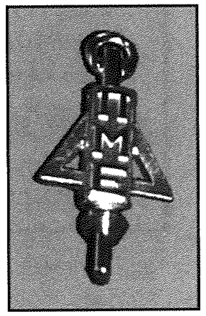
A pound of feathers will weigh you down and make you look rather odd, but with just the gold to be found in a IIME lapel pin you will rise in the esteem of your colleagues. Go for the gold.

I have spoken.

P.S. The gold clad keypins are available at the national office at the price of $12 each. To purchase a keypin go to the IIME website

`http://www.pme-math.org/Merchandise/index.html`

or write to the secretary-treasurer:



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A NOTE ON THE VARIATION OF THE TAXICAB LENGTHS UNDER ROTATIONS

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1. Introduction. The taxicab metric d_T is defined by $d_T(A,B) = |x_1 - x_2| + |y_1 - y_2|$ instead of the Euclidean metric $d_E(A,B) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ in the analytical plane, where $A = (x_1, y_1)$, $B = (x_2, y_2)$.

For a line segment l which is not parallel to the coordinate axes, $d_T(l) \neq d_E(l)$. It is known that rotations and translations preserve the Euclidean distance. In the taxicab plane all translations and the rotations with Euclidean angles $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (or taxicab angles $\theta_T = \frac{\pi_T}{2}, \pi_T, \frac{3\pi_T}{2}, 2\pi_T$, where $\pi_T = 4$) preserve the taxicab distance, [3]. However, the rotations with the other angles change the taxicab distance. In this work we study variations of taxicab lengths under such rotations.

2. Taxicab Angles and Trigonometry. In [1, 2, 4], the taxicab measures of angles in the first quadrant have been determined by using the taxicab unit circle as follows: As shown in Figure 1b the reference angle α of an angle θ is the angle between

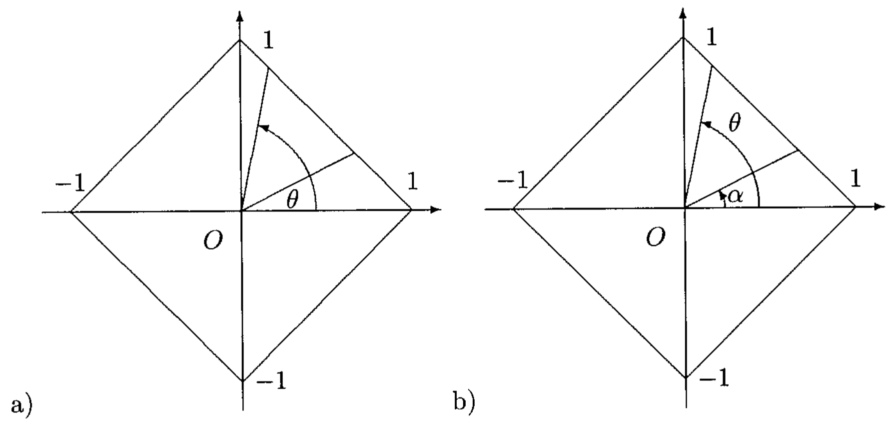

FIG. 1. *θ in standard position and with reference angle α.*

θ and the positive direction of the x-axis. For an acute angle θ with reference angle α contained entirely in a quadrant, the taxicab measure of angle θ is

$$\theta_T = \frac{2}{1+\tan\alpha} - \frac{2}{1+\tan(\theta+\alpha)}$$

$$(1) \qquad = \frac{2\sin\theta}{(\cos(\theta+\alpha)+\sin(\theta+\alpha))(\cos\alpha+\sin\alpha)}.$$

Let θ be in the standard position (Figure 1a), that is $\alpha = 0$, then

$$(2) \qquad \theta_T = 2 - \frac{2}{1+\tan\theta} = \frac{2\sin\theta}{\sin\theta+\cos\theta}.$$

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Here we determine the taxicab measure θ_T of the angle θ in standard position according to the quadrants containing θ.

$$(3)\qquad \theta_T = \begin{cases} \frac{2\sin\theta}{\sin\theta+\cos\theta}, & \text{if } 0<\theta\le\frac{\pi}{2} \\ \pi_T - \frac{2\sin\theta}{\sin\theta-\cos\theta}, & \text{if } \frac{\pi}{2}<\theta\le\pi \\ \pi_T + \frac{2\sin\theta}{\sin\theta+\cos\theta}, & \text{if } \pi<\theta\le\frac{3\pi}{2} \\ 2\pi_T + \frac{2\sin\theta}{-\sin\theta+\cos\theta}, & \text{if } \frac{3\pi}{2}<\theta\le 2\pi \end{cases}$$

Considering the known definitions of the taxicab trigonometric functions $\sin_T$ and $\cos_T$

$$\sin_T\theta_T = \begin{cases} \frac{1}{2}\theta_T, & 0<\theta_T\le 2 \\ 2-\frac{1}{2}\theta_T, & 2<\theta_T\le 6 \\ -4+\frac{1}{2}\theta_T, & 6<\theta_T\le 8 \end{cases}$$

$$\cos_T\theta_T = \begin{cases} 1-\frac{1}{2}\theta_T, & 0<\theta_T\le 4 \\ -3+\frac{1}{2}\theta_T, & 4<\theta_T\le 8 \end{cases}$$

and using Equation 1 for the first quadrant one gets

$$(4)\qquad \sin_T\theta_T = \frac{\sin\theta}{\cos\theta+\sin(\theta+2\alpha)}$$

$$(5)\qquad \cos_T\theta_T = \frac{\cos\theta-\sin\theta+\sin(\theta+2\alpha)}{\cos\theta+\sin(\theta+2\alpha)}.$$

It can be computed with Equations 4 and 5

$$\sin_T\alpha_T = \frac{\sin\alpha}{\cos\alpha+\sin\alpha},\qquad \cos_T\alpha_T = \frac{\cos\alpha}{\cos\alpha+\sin\alpha}$$

setting α to zero and θ to α.

Using Equations 4 and 5 one gets the Euclidean trigonometric functions sin and cos in terms of $\sin_T$ and $\cos_T$ as follows:

$$(6)\qquad \sin\theta = \frac{\sin_T\theta_T}{\sqrt{[(1-2\sin_T\alpha_T)\cos_T\theta_T+2\sin_T^2\alpha_T]^2+\sin_T^2\theta_T}} = \frac{\theta_T}{\sqrt{[(1-\alpha_T)(2-\theta_T)+\alpha_T^2]^2+\theta_T^2}}$$

$$(7)\qquad \cos\theta = \frac{(1-2\sin_T\alpha_T)\cos_T\theta_T+2\sin_T^2\alpha_T}{\sqrt{[(1-2\sin_T\alpha_T)\cos_T\theta_T+2\sin_T^2\alpha_T]^2+\sin_T^2\theta_T}} = \frac{(1-\alpha_T)(2-\theta_T)+\alpha_T^2}{\sqrt{[(1-\alpha_T)(2-\theta_T)+\alpha_T^2]^2+\theta_T^2}}.$$

If any angle θ is not in standard position, there is a reference angle α and $\theta_T = (\alpha+\theta)_T-\alpha_T$.Therefore θ_T, $\sin_T\theta_T$, $\cos_T\theta_T$, $\sin\theta$ and $\cos\theta$ can be calculated in a similar way for the other quadrants.

3. Change of Taxicab Length under Rotations. Now let us find the change of the taxicab length of a line segment after rotations.

THEOREM 1. *Let OA be a line segment, not on the x-axis with reference angle α and $d_T(O,A)=k$. If OA' is the image of OA under the rotation with an angle θ_T (or θ) then*

$$d_T(O,A') = k\cdot\sqrt{\frac{\sin_T^2\alpha_T+\cos_T^2\alpha_T}{\sin_T^2(\theta+\alpha)_T+\cos_T^2(\theta+\alpha)_T}} = k\cdot\sqrt{\frac{1+|\sin 2(\theta+\alpha)|}{1+|\sin 2\alpha|}}.$$

Proof. Let $d_T(O,A)=k$ be the taxicab length of the line segment OA. Rotating OA through an angle θ we get the line segment OA'. If α is the reference angle of θ then $A=(k\cos_T\alpha_T,\ k\sin_T\alpha_T)$.

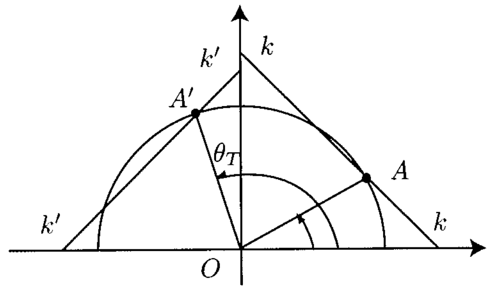

FIG. 2. *Rotation of the line segment OA with angle θ.*

Let us calculate $d_T(O,A')=k'$, where $A'=(k'\cos_T(\theta+\alpha)_T,\ k'\sin_T(\theta+\alpha)_T)$. Because of the equality of Euclidean lengths of the line segments OA and OA' we get

$$d_E(O,A)=d_E(O,A')$$

and therefore

$$(k\cos_T\alpha_T)^2+(k\sin_T\alpha_T)^2 = (k'\cos_T(\theta+\alpha)_T)^2+(k'\sin_T(\theta+\alpha)_T)^2$$

$$k^2(\cos_T^2\alpha_T+\sin_T^2\alpha_T) = (k')^2(\cos_T^2(\theta+\alpha)_T+\sin_T^2(\theta+\alpha)_T)$$

$$(8)\qquad k' = k\cdot\sqrt{\frac{\cos_T^2\alpha_T+\sin_T^2\alpha_T}{\cos_T^2(\theta+\alpha)_T+\sin_T^2(\theta+\alpha)_T}}.$$

Using $d_E(O,A)=d_E(O,A')$ one obtains

$$\frac{k}{|\cos\alpha|+|\sin\alpha|} = \frac{k'}{|\cos(\alpha+\theta)|+|\sin(\alpha+\theta)|}$$

and

$$(9)\qquad d_T(O,A') = k' = k\cdot\sqrt{\frac{1+|\sin 2(\theta+\alpha)|}{1+|\sin 2\alpha|}}$$

in terms of the Euclidean sine and cosine functions. □

The following corollary shows how one can find the taxicab length, after a rotation of a line segment with an angle θ in standard form.

COROLLARY 2. *Let OA be a line segment on the x-axis. If OA' is the image of OA under the rotation with an angle θ_T (or θ in the standard form) then*

$$d_T(O,A') = \frac{k}{\sqrt{\cos_T^2 \theta_T + \sin_T^2 \theta_T}} = k \cdot \sqrt{1 + |\sin 2\theta|} \ .$$

Proof. Using the value $\alpha = 0$ in Equations 8 and 9, one get the equations in the corollary.

If there is a line not passing through the origin, the above discussion is true. Because in the taxicab plane translations preserve the taxicab length. □

Acknowledgement. The authors would like to thank Prof. Dr. Rüstem Kaya for his kind help and valuable suggestions during the preparation of this note.

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THE LAH IDENTITY AND THE ARGONAUTS

MARKO PETKOVŠEK AND TOMAŽ PISANSKI*

According to a legend, *Ljubljana*, the capital of modern Slovenia, known as *Emona* in Roman times, was founded by *Jason and the Argonauts* when they were fleeing on the ship *Argo* with the *Golden Fleece* from the *Black See* via the *Danube River*, upstream the *Sava River* and finally reaching a dead-end at the source of the *Ljubljanica River*. The dragon that Jason supposedly slaughtered when camping on the bank of Ljubljanica is still the dominant symbol in the city coat-of-arms.

The legend tells us that the Argonauts took their ship Argo apart and carried it over the *Slovenian Carst* to the *Adriatic Sea.* Since there are numerous trails leading from Ljubljana to the coast we may assume that each part of Argo was carried for security reasons along a separate trail thereby minimizing the possibility of losing the Golden Fleece to the pursuing party.

Much later, in 1955, the Slovenian mathematician *Ivo Lah* (Štrukljeva vas, 1896 - Ljubljana, 1979) proved in [1] for any positive integer n the following identity between the two n-degree polynomials in x:

$$\prod_{i=0}^{n-1}(x+i) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}\prod_{j=0}^{k-1}(x-j) \qquad \text{(Lah Identity)}$$

The coefficients

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1} \qquad \text{(Unsigned Lah Numbers)}$$

are called the unsigned *Lah numbers* [2]. Here we give a combinatorial proof of the Lah identity. Note that it suffices to prove it for sufficiently many values of x. In fact we will prove it for all integers $x \geq n$. The story with the Argonauts will help us visualize the proof. Let n be the number of Argonauts. Say, each Argonaut has a number assigned. The numbers are: $1, 2, \ldots, n$. Let $x \geq n$ be the number of trails from Ljubljana to the coast of the Adriatic Sea. Finally, let $k \leq n$ be the number of indistinguishable parts that the ship Argo is divided into for transportation. We further assume that each part is carried by some positive number of linearly ordered Argonauts along a separate trail. Let us take a special example. Let $x = 10, n = 5$ and $k = 3$. The parts are carried as follows: (3,1) take route 1. (2) takes route 2.(4,5) take route 10. This can be encoded as a vector:

$$[31 - 2 - - - - - - - - 45].$$

There are $(x-1) = 9$ dashes that separate trails from each other. One can choose the positions of the dashes in $\binom{n+x-1}{n}$ ways. Combined with $n!$ independent ways to permute the n Argonauts, we obtain the left hand side of the Lah identity.

The same possibilities can be counted in a different way, namely depending on k, the number of indistinguishable parts that the Argo is divided into. Let us take the same permutation of $n = 5$ Argonauts $< 31245 >$ and split it with two vertical lines into three nonempty parts: $[31|2|45]$. Since each part is non-empty we may first take a permutation of the form $\{= || =\}$ and extend it by introducing in each part an

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extra symbol (without increasing the number of possibilities) $\{==\mid=\mid==\}$. There are obviously

$$\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$$

possibilities. Every one of them has to be multiplied by $n!$ since there are $n!$ permutations of Argonauts and divided by $k!$ since the order of the parts is irrelevant. the result is actually the Lah number $L(n,k)$. This number is to be multiplied by $x(x-1)\cdots(x-k+1)$ since this the number of ways to assign trails to the parts, and the Lah identity follows when we sum over all possible values of k.

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ON THE NUMBER 495
A CONSTANT SIMILAR TO KAPREKAR'S CONSTANT

PRATED KUMARA RAY*

Introduction. According to Gauss, mathematics is the queen of sciences and the theory of numbers is the queen of mathematics [1]. Number theory created interest among noted mathematicians as well as numerous amateurs from the time of Pythagoras.

Nannei discussed a problem by Barisien on one characteristic of 6-digit numbers [2]. Kaprekar dealt with four digit numbers and the number 6174 is known as Kaprekar's constant [3,4] (discussed later in this paper). This constant is obtained by a set of recurrent mathematical operations on any 4-digit number with at least two distinct digits. The present paper deals with a similar characteristic of 3-digit numbers.

An algorithm. The input is a 3-digit natural number N_0 such that all its digits are not identical.

Set $i = 0$
while $N_i \neq 495$ **do**
begin
1. Arrange the digits of N_i in descending order to get the number A_i.
2. Arrange the digits of N_i in ascending order to get the number B_i.
3. **Set** $N_{i+1} := A_i - B_i$.
4. **Set** $i := i + 1$

end.

To illustrate the above, a numerical example is considered below.

Example: Let $N_0 = 585$.
Output 1: $A_0 = 855$, $B_0 = 558$, and $N_1 = 297$.
Output 2: $A_1 = 972$, $B_1 = 279$, and $N_2 = 693$.
Output 3: $A_2 = 963$, $B_2 = 369$, and $N_3 = 594$.
Output 4: $A_3 = 954$, $B_3 = 459$, and $N_4 = 495$.

Observe that the while loop is executed four times before the algorithm terminates.

Theorem 1. *The above algorithm terminates with the number 495 after at most six executions of the while loop.*

Proof. Let $a_1 \geq a_2 \geq a_3$ be the digits of N_0. Since not all three digits are identical, we know that $a_1 > a_3$. The digits of N_1 are $10 + a_3 - a_1$, 9, and $a_1 - a_3 - 1$. The middle digit must be 9 and the sum of the first and last digit is 9 as well. Clearly the first digit is at most 8 and the first and last digit can not be equal since this would imply $2a_3 - 2a_1 + 11 = 0$ which is impossible. So N_1 is a proper input. Let us denote the digits of N_1 by 9, b_1 and b_2 with $b_1 \geq b_2$. Computing N_2 we get the digits $10 + b_2 - 9 = b_2 + 1$, 9 and $9 - b_2 - 1 = 8 - b_2$. The middle digit is again 9, but the smaller digit of N_1 was increased by 1, while the larger was decreased by 1. Since after the first execution of the while loop the largest (non middle) digit is 9, we reach 495 after at most 6 executions of the loop. □

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Executing the while loop on the number 495 returns the number 495 again, so 495 is a fixed point of our algorithm. If we lift the restriction that not all digits are the same, 000 is another fixed point.

If we change the input to our algorithm from 3 to 4-digit numbers, the fixed point 6174 is reached after at most 8 while-loop executions. 6174 is known as Kaprekar's constant [3,4]. For 5-digit numbers we do not obtain a fixed point, so the corresponding algorithm would not terminate, however, depending on the input, the output sequence becomes stationary repeating a set of two or four numbers. We observe that the sum of the digits of any output number is a multiple of 9. The interested readers are encouraged to prove that executing the while loop on an n-digit number yields a number whose digit sum is a multiple of 9.

Another spin-off is as follows. Instead of subtracting the two numbers after ordering the digits, we could add them. In [5] it is mentioned as an open problem(on page 82) whether this leads to a palindromic number after a finite number of steps. Note that the problem there is a bit different in that the digits of the numbers are merely reversed, but not ordered. We propose the following problem: Is it true that , given a 3-digit number, adding the two numbers obtained from it by ordering the digits in increasing and decreasing order and repeating this process, yields a palindromic number in a finite number of steps?

For example, performing this process on the number 196 yields 961+169=1130. Repeating, we get 3110+0113=3223, which is a palindrome. However, without ordering the digits but merely reversing them, according to [5], might never yield a palindrome.

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AN IDENTITY IN $\mathbb{R}^3$

EUGENE SPIEGEL*

When one is a student in Calculus III studying 3-dimensional Euclidean space, $\mathbb{R}^3$, there is the possibility for confusion with three products, the dot product, $(\cdot)$, and the cross product, $(\times)$, in $\mathbb{R}^3$, and multiplication in R, all playing essential roles. One way to distinguish between these products is to experience identities that simultaneously exhibit all of them. This note presents one such identity, and suggests different methods of verifying it. The identity is the following:

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are elements of $\mathbb{R}^3$ then

$$(\star) \qquad (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) + (\mathbf{A} \cdot \mathbf{B})(\mathbf{A} \cdot \mathbf{C}) = |\mathbf{A}|^2(\mathbf{B} \cdot \mathbf{C}).$$

Here $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$ is the square of the length of the vector $\mathbf{A}$. If $\mathbf{A}$ has unit length, then $(\star)$ can be thought of as an expression for the dot product of two vectors $\mathbf{B}$ and $\mathbf{C}$, which involves an arbitrary unit vector $\mathbf{A}$. Notice, too, that the left hand side of $(\star)$ is the sum of two terms, the first the dot product of vectors and the second the product of real numbers.

The reader might want to verify that $(\star)$ holds in a specific case, for example when $\mathbf{A} = (1, 2, 3)$, $\mathbf{B} = (4 - 1, 0)$ and $\mathbf{C} = (2, 5, -3)$. In this case $\mathbf{A} \times \mathbf{B} = (3, 12, -9)$, and $\mathbf{A} \times \mathbf{C} = (-21, 9, 1)$, so $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = 36$. Now $\mathbf{A} \cdot \mathbf{B} = 2$, $\mathbf{A} \cdot \mathbf{C} = 3$ and then $(\mathbf{A} \cdot \mathbf{B})(\mathbf{A} \cdot \mathbf{C}) = 6$. The left hand side of the identity is then 42. Now check that $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A} = 14$, and $\mathbf{B} \cdot \mathbf{C} = 3$ so that the right hand side of $(\star)$ is also 42.

We now suggest three methods of verifying $(\star)$ in general.

1) Let $\mathbf{A} = (a_1, a_2, a_3), \mathbf{B} = (b_1, b_2, b_3)$ and $\mathbf{C} = (c_1, c_2, c_3)$. As in the previous special case, compute the left and right hand sides of the identity algebraically and see that they are the same.

2) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are each standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we can check that $(\star)$ holds in all possible situations. While this involves 27 cases, the dot product of perpendicular vectors being zero simplifies our work. For example, if $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, then $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{k}) \cdot (-\mathbf{j}) = 0$ while $(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \mathbf{C}) = 0$.

We leave it to the reader to use the linearity of each of the three products involved, that is let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, etc., to see that the general identity follows from the cases already verified.

3) This method suggests a geometric reason why $(\star)$ holds. The identity holds when any of $\mathbf{A}$, $\mathbf{B}$ or $\mathbf{C}$ is zero. So suppose that none of these vectors is zero. Now, if $\mathbf{A}$ is a non-zero multiple of the vector $\mathbf{B}$, say $\mathbf{A} = \alpha\mathbf{B}$, with $\alpha \neq 0$, then $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ and

$$(\mathbf{A} \cdot \mathbf{B})(\mathbf{A} \cdot \mathbf{C}) = (\mathbf{A} \cdot \frac{1}{\alpha}\mathbf{A})(\alpha\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{C}) = |\mathbf{A}|^2(\mathbf{B} \cdot \mathbf{C}).$$

The identity then holds in this case. A similar argument also shows that the identity is true when $\mathbf{A}$ is a multiple of $\mathbf{C}$. Our next simplification is to assume that $\mathbf{A}$ is a unit vector. (Why can we do this?) So we must prove $(\star)$ when $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are non-zero vectors, with $\mathbf{A}$ neither a multiple of $\mathbf{B}$ nor $\mathbf{C}$, and $\mathbf{A}$ of unit length.

Let Π_1 be the plane determined by the three points $0, A, B$, and Π_2 the plane determined by $0, A, C$. The planes Π_1 and Π_2 intersect in the line through 0 and A.

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Then $\mathbf{A}\times\mathbf{B}$ and $\mathbf{A}\times\mathbf{C}$ are normals to the planes Π_1 and Π_2 respectively, and the first term in $(\star)$ is the product of the lengths of $\mathbf{A}\times\mathbf{B}$ and $\mathbf{A}\times\mathbf{C}$ and the cosine of the angle, θ, between the planes Π_1 and Π_2. If ϕ is the angle between $\mathbf{A}$ and $\mathbf{B}$, then $|\mathbf{A}\times\mathbf{B}| = |\mathbf{B}|\sin\phi$. This quantity is equal to the length of the component of $\mathbf{B}$ which is perpendicular to $\mathbf{A}$. Similarly, $\mathbf{A}\times\mathbf{C}$ is the component of $\mathbf{C}$ in the direction of $\mathbf{B}$. Hence $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{C})$ is equal to the product of the lengths of the components of $\mathbf{B}$ and $\mathbf{C}$ which are perpendicular to $\mathbf{A}$, and $\cos\theta$.

We now recompute this same quantity another way. The angle between the planes Π_1 and Π_2 is the angle between a vector in Π_1 and one in Π_2, each of which is perpendicular to $\mathbf{A}$. (Draw a picture.) But the component of $\mathbf{B}$ perpendicular to $\mathbf{A}$ is $\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{A}$ and so the dot product of the vectors $\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{A}$ and $\mathbf{C}-(\mathbf{A}\cdot\mathbf{C})\mathbf{A}$ is the length of these vectors times $\cos\theta$. Hence

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{C}) &= (\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{A})\cdot(\mathbf{C}-(\mathbf{A}\cdot\mathbf{C})\mathbf{A})\\ &= \mathbf{B}\cdot\mathbf{C}-(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{A})+(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})|\mathbf{A}|^2\\ &= (\mathbf{B}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})+(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})\\ &= (\mathbf{B}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C})\end{aligned}$$

This verifies $(\star)$.

The author thanks the referee for suggesting the following application of the identity.

There is a standard example from plasma physics or electricity and magnetism where a charged particle is moving in a magnetic field, $\mathbf{B}$, and traces out helical patterns around the magnetic field lines. One would like to know the torque, $\tau = \mathbf{r}\times\mathbf{F}$, the force that makes the particle rotate, and specifically the magnitude of the torque component, $\tau_{\mathbf{B}}$, in the direction of the magnetic field. The Lorentz force is $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$, where $\mathbf{r}$, $\mathbf{v}$ and q are, respectively, the position, velocity, and charge of the particle. Further, let $\mathbf{r}_\perp$ and $\mathbf{v}_\perp$ be the components of $\mathbf{r}$ and $\mathbf{v}$ perpendicular to $\mathbf{B}$. Then

$$\begin{aligned}\tau_{\mathbf{B}} &= \frac{\mathbf{B}}{|\mathbf{B}|}\cdot\tau\\ &= \frac{\mathbf{B}}{|\mathbf{B}|}\cdot(\mathbf{r}\times(q\mathbf{v}\times\mathbf{B}))\\ &= \frac{q}{|\mathbf{B}|}(\mathbf{B}\cdot(\mathbf{r}\times(\mathbf{v}\times\mathbf{B})))\\ &= \frac{q}{|\mathbf{B}|}((\mathbf{B}\times\mathbf{r})\cdot(\mathbf{v}\times\mathbf{B}))\\ &= -q|\mathbf{B}|((\frac{\mathbf{B}}{|\mathbf{B}|})\times\mathbf{r})\cdot(\frac{\mathbf{B}}{|\mathbf{B}|}\times\mathbf{v}))\\ &= -q|\mathbf{B}|((\mathbf{r}\cdot\mathbf{v})-(\frac{\mathbf{B}}{|\mathbf{B}|}\cdot r)(\frac{\mathbf{B}}{|\mathbf{B}|}\cdot\mathbf{v})) \qquad \text{(by the identity)}\\ &= -q|\mathbf{B}|\mathbf{r}_\perp\cdot\mathbf{v}_\perp \qquad \text{(as in (3)).}\end{aligned}$$

We arrive at $\tau_{\mathbf{B}} = -q|\mathbf{B}|\mathbf{r}_\perp\cdot\mathbf{v}_\perp$, which is the classical expression for the magnitude of the torque component.

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PROBLEM DEPARTMENT

EDITED BY MICHAEL MCCONNELL AND JON A. BEAL

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Michael McConnell, 840 Wood Street, Mathematics Department, Clarion University, Clarion, PA 16214, or sent by email to mmcconnell@clarion.edu. Electronic submissions using LaTeX are encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by December 13, 2003. Solutions identified as by students are given preference.

Problems for Solution.

1043. *Proposed by Peter A. Lindstrom, Batavia, NY.*
The year 2002 is a four digit base ten palindrome as was the year 1991. (a.) Can 1991 be rewritten in a different base as a palindrome with four digits? (b.) Can 2002 be rewritten in a different base as a palindrome with four digits?

1044. *Proposed by Thomas J. Pfaff, University of Wisconsin-Superior, Superior, WI.*

Evaluate

$$\lim_{n\to\infty}\frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{ni-i^2}$$

1045. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN.*
Student solutions solicited
Suppose that G is an abelian group with 2n elements, where n is odd. *Without using Sylow Theorems,* show that G has exactly one subgroup of order 2.

1046. *Proposed by Paul S. Bruckman, Sacramento, CA.*
Let $S_1 = \{x_1, x_2, \ldots, x_n\}$, where the $x_j's$ are positive and not necessarily distinct. Let S_k denote the set consisting of all C_n^k possible products of the form $x_{j_1}x_{j_2}\cdots x_{j_k}$ where the $j_k's$ are distinct, $k = 1, 2, \ldots, n$. If G_k represents the geometric mean of the elements of S_k, prove that $G_k = (G_1)^k$.

1047. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN.*
Show that

$$\sec^2\alpha + \csc^2\alpha + \sec^2\alpha\csc^2\alpha \geq 8$$

for $0 < \alpha < \pi/2$. Determine when equality holds.

1048. *Proposed by Peter A. Lindstrom, Batavia, NY.*
Without using the Fundamental Theorem of Calculus, give a geometric argument to show that

$$\int_0^1 \frac{2}{t^2+1}\,dt = \frac{\pi}{2}.$$

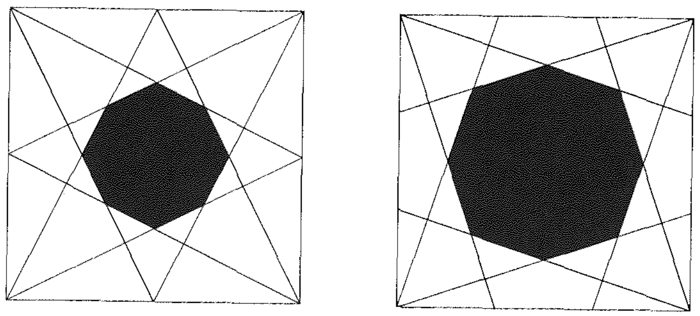
1049. *Andrew Cusumono, Great Neck, NY.*
A Fibonacci-type sequence is defined by the rules $F_1 = A$, $F_2 = B$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$, where A and B are constants. Show that for each $n \geq 1$,

$$F_n^3 + F_{n+1}^3 + F_{n+2}^3 = F_{n+2}[2F_n + 2F_{n+1} + F_nF_{n+1}].$$

1050. *Ronald Kopas, Clarion University, Clarion, PA.*
A lottery uses 31 balls, numbered 1 through 31. Six of these balls are selected in the drawing, so each lottery ticket contains six numbers from 1 through 31. Show that it is possible to buy exactly 31 tickets so that each pair of numbers appears on exactly one of the tickets.

1051. *Monte J. Zerger, Adams State College, Alamosa, CO.*
The two squares in the figure below are congruent. In the figure on the left, the octagon is formed by joining the bisection points of the sides of the square to vertices as shown. In the second figure on the right, the trisection points of the sides are used instead.

1. Show that the octagons are similar, equilateral, but not equiangular.
2. Find the ratio of their areas.



Corrections. We would like to note two typos in the Spring 2002 problem proposals:

In Problem 1035 the proof should be that $\frac{1}{4}(x^2-1)$ is the product of two consecutive integers, rather than the incorrect $\frac{1}{2}(x^2-1)$.

In Problem 1037, the limit should be $\lim_{n\to\infty}$ rather than the incorrect $\lim_{x\to\infty}$.

Solutions.

1021. [Fall 2001] *Proposed by Tom Moore, Bridgewater State College, Bridgewater, MA.*

Let $D(n)$ be the sum of the (base 10) digits of the positive integer n. Are there twin primes p and $p+2$ such that $D(p) = D(p+2)$?

I. *Solution by* **Scott Parker**, *student at Messiah College, Grantham, PA.*

We will solve the stated problem by proving the following more general result:

THEOREM: *Let $D(n)$ be the sum of the (base 10) digits of the positive integer n. Then for any positive integer m, $D(m+2) \neq D(m)$.*

Proof. We will consider a proof by cases.

Case 1: Suppose that the addition of 2 to m involves no carrying, i.e. the ones digit of m is neither 8 or 9. then the ones digit of the integer $m+2$ is two greater than that of m, and all other digits (if any) in the two numbers are identical. So we have that $D(m+2) = D(m) + 2 \neq D(m)$.

Case 2: Suppose that the addition of 2 to m involves exactly one carry, i.e. the ones digit of m is either 8 or 9 and the tens digit of m (taken to be zero if $m < 10$) does not equal 9. Then the ones digit of the number $m+2$ is eight less than that of m and the tens digit of $m+2$ is one greater than that of m. All other digits (if any) in the two numbers are identical. So we have that $D(m+2) = D(m)-8+1 = D(m)-7 \neq D(m)$.

Case 3: Suppose that the addition of 2 to m involves exactly n carries, where $n \geq 2$, i.e. $m = a\cdot 10^{n+1} + b\cdot 10^n + 9(10^{n-1} + 10^{n-2} + \cdots + 10^2 + 10) + c$, where a is a nonnegative integer, $b \in \{0,1,2,\ldots,7,8\}$, and $c \in \{8,9\}$. then the ones digit of the integer $m+2$ is eight less than that of m; the 10^ks digit of $m+2$, for $1 \leq k \leq n-1$, becomes 0; and the 10^ns digit of $m+2$ is one greater than that of m. All other digits (if any) in the two numbers are identical. So we have $D(m+2) = D(m)-8-9(n-1)+1 = D(m)+2-9n \neq D(m)$.

In all cases, given any positive integer m, $D(m) \neq D(m+2)$. □

Note: If m is limited to prime values such that m and $m+2$ are twin primes, as in the stated problem, then in the proof given above we can delete any mention of the unit digit of m being 8.

II. *Solution and comment by* **J. Ernest Wilkins, Jr.**, *Clark Atlanta University, Atlanta, GA.*

Let n be any integer. It is known that $D(n) \equiv n \pmod 9$; hence $D(n+2) - D(n) \equiv 2 \pmod 9$, so that there is no integer n such that $D(n) = D(n+2)$. Neither the sign nor the primality of n is relevant. The same analysis also shows that, if a is any integer such that $a \not\equiv 0 \pmod 9$, there is no integer n such that $D(n) = D(n+a)$. When $a = 9$, however, any positive integer n which is not a multiple of 10 and whose last digits do not like in the closed interval [91, 99] will satisfy the condition that $D(n) = D(n+9)$. The only such n such that n and $n+9$ are prime is 2.

Also solved by **Paul S. Bruckman**, Sacremento CA, **William Chau**, New Brunswick, NJ, **Mark D. Evans**, Louisville, KY, **Richard I. Hess**, Rancho Palos Verdes, CA, **Justin Jordan**, Belmont University, Nashville, TN, **Adam Mark**, Belmont University, Nashville, TN, **Joanna Oakland**, Belmont University, Nashville, TN, **Mike Pinter**, Belmont University, Nashville, TN, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

1022. [Fall 2001] *Proposed by William Chau, Middletown, NJ.*
Find an ordered pair (n, m) where n and m are composite numbers such that $n! = m^2$, or prove that there is none.

Solution by **Nicholas Zoller**, *student at Messiah College, Grantham, PA.*

The problem is solved by proving a more general result, using a theorem from number theory known as Bertrand's Postulate: For each integer $n > 3$ there exists a prime p such that $n < p < 2n$ (Anderson and Bell, 394).

THEOREM: *For all $n > 3$, $n!$ is not a perfect square.*

Proof. If n is even, then there exists a prime p such that $n/2 < p < n$. If n is odd, then there exists a prime p such that $(n+1)/2 < p < n+1$. In either case, we have that $n < 2p$. Therefore $p|n!$, but $p^2 \not| n!$, and n is not a perfect square. □

Reference: ANDERSON, JAMES A., and BELL, JAMES M. "Number Theory with Applications", Prentice Hall, NJ, 1997.

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, **Paul S. Bruckman**, Sacremento CA, **Mark Evans**, Louisville, KY, **Ovidiu Furdui**, Western Michigan University, Kalamzoo, **Joe Howard**, Portales, NM, **Murray S. Klamkin**, University of Alberta, **Mike Pinter**, Belmont University, Nashville, TN, **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY **H.-J. Seiffert**, Berlin, Deutschland, **J. Ernest Wilkins, Jr.**, Clark Atlanta University, Atlanta GA, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

1023. [Fall 2001] *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, NY.*
If $U_1 = 16$ and $U_{n+1} = U_n + 8n + 12$, find $\sum_{n=0}^{\infty}(U_{n+1})^{-1}$.

Solution by **Eric Heinzman,** *student at Ithaca College, Ithaca, NY.*

First we will show that $U_n = (2n+2)^2$, for $n \geq 1$, by induction. Note that $U_1 = 16 = 4^2 = (2(1)+2)^2$. Assume that $U_j = (2j+2)^2$. So

$$U_{j+1} = U_j + 8j + 12 = (2j+2)^2 + 8j + 12 = 4j^2 + 16j + 16 = (2(j+1)+2)^2.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right).$$

we have therefore

$$\sum_{n=1}^{\infty}\frac{1}{U_n} = \sum_{n=1}^{\infty}\frac{1}{(2n+2)^2} = \sum_{n=2}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right).$$

Also solved by **Jean-Claude Andrieux**, Beaune, France, **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA, **Paul S. Bruckman**, Sacramento, CA, **Kenneth B. Davenport**, Frackville, PA, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **George P. Evanovich**, Saint Peters College, Jersey City, NJ, **Ovidiu Furdui**, Kalamazoo, MI, **Robert C. Gebhardt**, Hopatcong, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **Murray S. Klamkin**, University of Alberta, Edmonton, Alberta, Canada, **Yoshinobu Murayoshi** Okinawa, Japan, **Shiva K. Saksena**, University of North Carolina at Wilmington, Wilmington, NC, **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY, **H.-J. Seiffert** Berlin, Germany, **Ernest Wilkins, Jr.**, Clark Atlanta Unversity, Atlanta, GA, **Monte Zerger**, Adams State College, Alamosa, CO, and by the **Proposer**.

1024. [Fall 2001] *Proposed by Clayton W. Dodge, University of Maine, Orono, ME.*
Find the largest positive integer b and an integer c such that

$$\sqrt{2002 + b\sqrt{c}} + \sqrt{2002 - b\sqrt{c}} = 64.$$

Solution by **Yu Gan,** *student, Loch Raven High School, Baltimore, MD.*

Squaring both sides of the equation, we have

$$2002 + b\sqrt{c} + 2\sqrt{(2002 + b\sqrt{c})(2002 - b\sqrt{c})} + 2002 - b\sqrt{c} = (64)^2.$$

Then $\sqrt{2002^2 - b^2c} = 46$. Then $b^2c = 2002^2 - 46^2 = 2^{12} \times 2 \times 3 \times 163$. Thus, the largest positive integer b is $b = 2^6 = 64$ and $c = 978$. Checking the solutions, we confirm that they are solutions of the original equation.

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA, **Paul S. Bruckman**, Sacramento, CA, **William Chau**, SoftTechies Corp., East Brunswick, NJ, **Kenneth B. Davenport**, Frackville, PA, **Jose Luis Diaz-Barrero**, Universitat Politecnica de Catalunya, Barcelona, Spain, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **George P. Evanovich**, Saint Peters College, Jersey City, NJ, **Mark D. Evans**, Louisville, KY, **Ovidiu Furdui**, Kalamazoo, MI, **Robert C. Gebhardt**, Hopatcong, NJ, **Tracey M. Hagedorn**, Angelo State University, San Angelo, TX, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **Murray S. Klamkin**, University of Alberta, Edmonton, Alberta, Canada, **Yoshinobu Murayoshi** Okinawa, Japan, **Scott Parker**, Messiah College, Shrewsbury, NJ, **Shiva K. Saksena**, University of North Carolina at Wilmington, Wilmington, NC, **H.-J. Seiffert** Berlin, Germany, **Ernest Wilkins, Jr.**, Clark Atlanta Unversity, Atlanta, GA, **Monte Zerger**, Adams State College, Alamosa, CO, and by the **Proposer**.

1025. [Fall 2001] *Proposed by Ayoub B. Ayoub, Pennsylvania State University-Abington College, Abington, PA.*
Find the sum of the following in simplest form;

$$\frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \frac{\binom{n}{3}}{x+3} + \cdots + (-1)^n\frac{\binom{n}{n}}{x+n}$$

or

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k}$$

Solution by **Lamarr Widmer**, *Messiah College, Grantham, PA.*

We will consider $P(x) = x(x+1)(x+2)\dots(x+n)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k}$ Distributing and cancelling, we have

$$P(x) = \sum_{k=0}^{n}\left((-1)^k\binom{n}{k}\prod_{\substack{i=0\\i\neq k}}^{n}(x+i)\right).$$

We note that $P(x)$ is a polynomial of degree at most n. Now we want to evaluate $P(x)$ when $x = -m$ where $m = 0, 1, \dots, n$. When $k \neq m$, we have

$$\prod_{\substack{i=0\\i\neq k}}^{n}(x+i) = \prod_{\substack{i=0\\i\neq k}}^{n}(-m+i) = 0$$

since there is a zero factor corresponding to $i = m$. When $k = m$, the product becomes

$$\prod_{\substack{i=0\\i\neq k}}^{n}(i-m) = \left[\prod_{i=0}^{m-1}(i-m)\right]\left[\prod_{i=m+1}^{n}(i-m)\right] = (-1)^m m!(n-m)!$$

So for $P(-m)$ we have only one nonzero term in the sum, occurring when $k = m$. We compute

$$P(-m) = (-1)^m\binom{n}{m}\prod_{\substack{i=0\\i\neq m}}^{n}(-m+i) = (-1)^m\frac{n!}{m!(n-m)!}(-1)^m m!(n-m)! = n!.$$

Now $P(x)$ is a polynomial of degree at most n which assumes the value $n!$ for $n+1$ different values of x. So $P(x) \equiv n!$. So we have

$$\sum_{k=0}^{n}(-1)^k \frac{\binom{n}{k}}{x+k} = \frac{n!}{x(x+1)(x+2)\dots(x+n)}.$$

[A number of references were noted by the problem solvers. William Chau and H.-J. Seiffert noted "Concrete Mathematics", by R. L. Graham, D. E. Knuth, and O. Patashnik, 2nd ed., Addison-Wesley, 1994, p. 188; Joe Howard noted *Mathematics Gazette*, October 1982, p. 221; and Murray S. Klamkin noted "Combinatorial Identities" by H.W. Gould, Morgantown Printing and Binding Co.,1972, p.6.]

Also solved by **Paul S. Bruckman**, Sacramento, CA, **William Chau**, SoftTechies Corp., East Brunswick, NJ, **Kenneth B. Davenport**, Frackville, PA, **Ovidiu Furdui**, Kalamazoo, MI, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **Murray S. Klamkin**, University of Alberta, Edmonton, Alberta, Canada, **Yoshinobu Murayoshi** Okinawa, Japan, **H.-J. Seiffert** Berlin, Germany, and by the **Proposer**.

1026. [Fall 2001] *Proposed by Ayoub B. Ayoub, Pennsylvania State University-Abington College, Abington, PA.*

Express the value of the following $(n+1)\times(n+1)$ as a product involving linear factors of $x_i's$ and $\alpha_i's$.

$$\begin{vmatrix} (x_1+\alpha_1)^n & (x_1+\alpha_2)^n & \dots & (x_1+\alpha_{n+1})^n \\ (x_2+\alpha_1)^n & (x_2+\alpha_2)^n & \dots & (x_2+\alpha_{n+1})^n \\ \vdots & \vdots & & \vdots \\ (x_{n+1}+\alpha_1)^n & (x_{n+1}+\alpha_2)^n & \dots & (x_{n+1}+\alpha_{n+1})^n \end{vmatrix}.$$

Solution by **Murray S. Klamkin**, *University of Alberta, Edmonton, Alberta, Canada.*

On setting $x_i = x_j$, we get two identical rows so that $(x_i - x_j)$ is a factor for all $i \neq j$. Similarly, $(\alpha_i - \alpha_j)$ is a factor for all $i \neq j$. We now show that the remaining factor is the constant $\pm\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}$. The given alternant determinant can be expressed as the product of the following two $(n+1)\times(n+1)$ alternant determinants:

$$\begin{vmatrix} 1 & \binom{n}{1}\alpha_1 & \dots & \binom{n}{n}\alpha_1^n \\ 1 & \binom{n}{1}\alpha_2 & \dots & \binom{n}{n}\alpha_2^n \\ \vdots & \vdots & & \vdots \\ 1 & \binom{n}{1}\alpha_{n+1} & \dots & \binom{n}{n}\alpha_{n+1}^n \end{vmatrix} \times \begin{vmatrix} x_1^n & x_2^n & \dots & x_{n+1}^n \\ x_1^{n-1} & x_2^{n-1} & \dots & x_{n+1}^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

See T. Muir, and H. Metzler, "A Treatise on the Theory of Determinants", Dover, NY, 1960, p. 341. The rest now follows.

[An alternant determinant (of order n) is a determinant where the entries of the first row are functions $g_1, g_2, \dots, g_n$ of one variable, say x_1, the entries of the second row are the same functions $g_1, g_2, \dots, g_n$ of a second variable, say x_2, and so on. Eds.]

[Comment by H.-J. Seiffert. G. Polya and G. Szego in "Problems and Theorems in Analysis II", Springer, 1976, p. 92, exercise 5, asks for the value of this determinant and gives the solution on p. 280.]

Also solved by **Paul S. Bruckman**, Sacramento, CA, **Ovidiu Furdui**, Kalamazoo, MI, **William H. Pierce**, Rangeley, ME, **Shiva K. Saksena**, University of North Carolina at Wilmington, Wilmington, NC, **H.-J. Seiffert** Berlin, Germany, and the **Proposer**.

1027. [Fall 2001] *Proposed by James Chew, North Carolina Agricultural and Technical State University, Greensboro, NC.*

Let a jar contain 1 green marble and 9 red marbles, thoroughly mixed. One marble is randomly drawn, and its color is noted. A second jar contains 2 green marbles and 8 red marbles. One marble is drawn from the second jar and again the color is noted. The next jar contains 3 green marbles and 7 red marbles. One marble is drawn from the third jar and again the color is noted. Repeat this process until a fifth marble has been drawn from the jar containing 5 green and 5 red marbles. Let X be the number of green marbles drawn. Calculate $P(X=i)$, $i = 0, 1, 2, \dots, 5$. A local newspaper gives probabilities of rain for the next 5 days as: 10% , 20% , 30% , 40% ,50% . Use the marbles-in-the-jar model to determine the probability of getting a) exactly two days of rain, b) at least two days of rain.

I. *Solution by* **Joy Schneider**, *student at Belmont University, Nashville, TN.*

The probabilities of red and green in each jar are given below.

red	$\frac{9}{10}$	$\frac{8}{10}$	$\frac{7}{10}$	$\frac{6}{10}$	$\frac{5}{10}$
green	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$

Since the events are independent, we must multiply the probabilities for each jar in all the relevant combinations.

$$\begin{aligned}
P(X=0) &= (9\cdot 8\cdot 7\cdot 6\cdot 5)\div 10^5 = .1512 \\
P(X=1) &= (8\cdot 7\cdot 6\cdot 5 + 9\cdot 2\cdot 7\cdot 6\cdot 5 + 9\cdot 8\cdot 3\cdot 6\cdot 5 + 9\cdot 8\cdot 7\cdot 4\cdot 5 \\
&\quad + 9\cdot 8\cdot 7\cdot 6\cdot 5)\div 10^5 = .3714 \\
P(X=2) &= (1\cdot 2\cdot 7\cdot 6\cdot 5 + 1\cdot 8\cdot 3\cdot 6\cdot 5 + 1\cdot 8\cdot 7\cdot 4\cdot 5 + 1\cdot 8\cdot 7\cdot 6\cdot 5)\div 10^5 \\
&\quad + (9\cdot 2\cdot 3\cdot 6\cdot 5 + 9\cdot 2\cdot 7\cdot 4\cdot 5 + 9\cdot 8\cdot 7\cdot 6\cdot 5)\div 10^5 \\
&\quad + (9\cdot 8\cdot 3\cdot 4\cdot 5 + 9\cdot 8\cdot 3\cdot 6\cdot 5)\div 10^5 \\
&\quad + (9\cdot 8\cdot 7\cdot 4\cdot 5)\div 10^5 \\
&= \frac{3940}{10^5} + \frac{7920}{10^5} + \frac{10800}{10^5} + \frac{10080}{10^5} = \frac{32740}{100,000} = .3274 \\
P(X=3) &= (1\cdot 2\cdot 3\cdot 6\cdot 5 + 1\cdot 2\cdot 7\cdot 4\cdot 5 + 1\cdot 2\cdot 7\cdot 6\cdot 5)\div 10^5 \\
&\quad + (1\cdot 8\cdot 3\cdot 4\cdot 5 + 1\cdot 8\cdot 3\cdot 6\cdot 5)\div 10^5 + (1\cdot 8\cdot 7\cdot 4\cdot 5)\div 10^5 \\
&\quad + (9\cdot 2\cdot 3\cdot 4\cdot 5 + 9\cdot 2\cdot 3\cdot 6\cdot 5)\div 10^5 + (9\cdot 2\cdot 7\cdot 4\cdot 5)\div 10^5 \\
&\quad + (9\cdot 8\cdot 3\cdot 4\cdot 5)\div 10^5 \\
&= \frac{880}{10^5} + \frac{1200}{10^5} + \frac{1120}{10^5} + \frac{2700}{10^5} + \frac{2520}{10^5} + \frac{4320}{10^5} + = \frac{12740}{100,000} = .1274 \\
P(X=4) &= (1\cdot 2\cdot 3\cdot 4\cdot 5 + 1\cdot 2\cdot 3\cdot 6\cdot 5 \\
&\quad + 1\cdot 2\cdot 7\cdot 4\cdot 5 + 1\cdot 8\cdot 3\cdot 4\cdot 5 + 9\cdot 2\cdot 3\cdot 4\cdot 5)\div 10^5 = .0214 \\
P(X=5) &= (1\cdot 2\cdot 3\cdot 4\cdot 5)\div 10^5 = .0012
\end{aligned}$$

So the probability of rain for exactly two days is $P(X=2) = .3274$, and the probability of rain for at least two days is

$$P(X=2)+P(X=3)+P(X=4)+P(X=5) = .3274+.1274+.0214+.0012 = .4774.$$

II. *Solution by* **Richard Hess**, *Rancho Palos Verdes, CA.*

Since

$$\begin{aligned}
&\frac{9+x}{10}\cdot\frac{8+2x}{10}\cdot\frac{7+3x}{10}\cdot\frac{6+4x}{10}\cdot\frac{5+5x}{10}\\
&=10^{-5}(72+26x+2x^2)(42+46x+12x^2)(5+5x)\\
&=10^{-5}(72+26x+2x^2)(210+440x+290x^2+60x^3)\\
&=10^{-5}(15120+37140x+32740x^2+12740x^3+2140x^4+120x^5)
\end{aligned}$$

we have

$$\begin{array}{lll}
P(X=0)=.1512 & P(X=1)=.3714 & P(X=2)=.3274\\
P(X=3)=.1274 & P(X=4)=.0214 & P(X=5)=.0012.
\end{array}$$

So the probability of rain for exactly 2 days is .3274 while the probability of rain for at least 2 days is .4774.

[Although it doesn't provide explanation, this solution to the problem is both efficient and elegant. The solver uses a fifth degree polynomial as a "bookkeeping" device to keep track of all of the probabilities. It generalizes the appearance of $\binom{n}{k}$ as the coefficient of x^k in the expansion of $(x+1)^n$. Eds.]

Also solved by **Paul S. Bruckman**, Sacremento CA, **Mark D. Evans**, Louisville, KY, **Robert C. Gebhhardt**, Hopatcong, NJ **Mike Pinter**, Belmont University, Nashville, TN, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

1028. [Fall 2001] *Proposed by Editors.*
As a modification of #1027, explain how to modify the model in problem 1027 so that the assumption of independence is removed. Based on your new model, determine the probability of getting a) exactly two days of rain, b) at least two days of rain.

I. *Solution by* Paul S. Bruckman, *Sacramento, CA.*

The modified model is obtained by combining the contents of all the jars into one jar, and drawing five marbles *without replacement.* Our jar now contains 15 green marbles and 35 red ones. Again, let $P(i)$ denote $Pr(X=i), i=0,1,2,\ldots,5$, where X is the total number of green marbles drawn in our modified experiment. This time, we obtain the following:

$$P(i)=\frac{\binom{15}{i}\binom{35}{5-i}}{\binom{50}{5}}, i=0,1,\ldots,5.$$

Upon calculation, this yields the following estimated probabilities:

$$\begin{array}{lll}
P(0)=0.153218, & P(1)=0.370689, & P(2)=0.324352,\\
P(3)=0.127775, & P(4)=0.022549, & P(5)=0.001417;
\end{array}$$

these are not exact probabilities.

Using these probabilities as our weather model, the probability of exactly two days of rain is now $P(2)=0.324351$, and the probability of at least two days of rain is $1-P(0)-P(1)=0.476093$ (approximately).

II. *Solution by* Mark D. Evans, *Louisville, KY.*

The classical probability problem involves drawing marbles from a single jar with or without replacement where the former represents independence and the later dependence. The analogous approach here is to remove marbles of the same color as the marble drawn from the remaining jars. For example, if the first draw is red, then

Draw	$P(0)$	$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$
1	$\frac{9}{10}$	$\frac{1}{10}$				
2	$\frac{63}{90}$	$\frac{26}{90}$	$\frac{1}{90}$			
3	$\frac{315}{720}$	$\frac{345}{720}$	$\frac{59}{720}$	$\frac{1}{720}$		
4	$\frac{1330}{5040}$	$\frac{2640}{5040}$	$\frac{1330}{5040}$	$\frac{124}{5040}$	$\frac{1}{5040}$	
5	$\frac{945}{30240}$	$\frac{10005}{30240}$	$\frac{14550}{30240}$	$\frac{4486}{30240}$	$\frac{253}{30240}$	$\frac{1}{30240}$

one red marble would be removed from each of the remaining jars. The values of $P(x)$ can be found via a binary tree approach. Here $P(x)$ will be based on the draws performed so far as we progress through the binary tree.

It follows that $P(2)=\frac{14550}{30240}\approx 0.4812$ and that the probability of at least two days of rain is $1-P(0)-P(1)=\frac{19290}{30240}\approx 0.6379$.

[As in the case with all models, various assumptions must be made during the formulation of the model. One assumption might be based on whether it is more likely to rain on a given day if it rains the previous day or whether it is less likely to rain. Hence another model would be to add the ball drawn from the jar to the next jar before drawing. Ed.]

***1029.** [Fall 2001] *Proposed by Ice B. Risteski, Skopje, Macedonia.*
If P and Q denote the linear differential operators

$$P=\sum_{i=0}^{m}p_i(x)D^i, \qquad Q=\sum_{j=0}^{n}q_j(x)D^j, \qquad \left(D=\frac{d}{dx}\right)$$

show that

$$QP=\sum_{s=0}^{m+n}r_s(x)D^s,$$

where

$$r_s(x)=\sum_{j=\max(0,s-m)}^{n}\left\{\left[\sum_{i=\max(0,s-j)}^{\min(s,m)}\left(\binom{j}{s-i}\right)p_i^{(i+j-s)}(x)\right]q_j(x)\right\}$$

Solution by **H.-J. Seiffert**, *Berlin, Germany.*
Using Leibniz' rule, we find

$$\begin{aligned}
QP&=\sum_{j=0}^{n}\sum_{i=0}^{m}q_j(x)D^j\left(p_i(x)D^i\right)\\
&=\sum_{j=0}^{n}\sum_{i=0}^{m}q_j(x)\sum_{k=0}^{j}\binom{j}{k}p_i^{(j-k)}(x)D^{i+k}\\
&=\sum_{j=0}^{n}\sum_{i=0}^{m}q_j(x)\sum_{i}^{i+j}\binom{j}{s-i}p_i^{(i+j-s)}(x)D^s
\end{aligned}$$

Changing the summations gives

$$QP=\sum_{s=0}^{m+n}\sum_{j=0}^{n}\left[\sum_{i}^{m}\binom{j}{s-i}p_i^{(i+j-s)}(x)\right]q_j(x)D^s$$

where $\binom{u}{v} = 0$ if $v < 0$ or $v > u$. Since $\binom{j}{s-i} = 0$ if $i > s$ or $i < s - j$, we have

$$QP = \sum_{s=0}^{m+n} \sum_{j=0}^{n} \left[\sum_{i=\max(0,s-j)}^{\min(s,m)} \binom{j}{s-i} p_i^{(i+j-s)}(x) \right] q_j(x) D^s$$

If $j < s - m$, then $\min(s, m) \le m < s - j \le \max(0, s - j)$ and so the sum in the parenthesis is empty. Hence

$$QP = \sum_{s=0}^{m+n} \sum_{j=\max(0,s-m)}^{n} \left[\sum_{i=\max(0,s-j)}^{\min(s,m)} \binom{j}{s-i} p_i^{(i+j-s)}(x) \right] q_j(x) D^s$$

which is the desired equation.

Also solved by **Paul S. Bruckman**, Sacramento, CA.

1030. [Fall 2001] *Proposed by Ayoub B. Ayoub, Pennsylvania University–Abington College, Abington, PA.*

On the sides of an arbitrary triangle ABC, three equilateral triangles A_1BC, AB_1C, and ABC_1 are drawn outward. The on the sides of the triangle $A_1B_1C_1$ another three equilateral triangles $A_2B_1C_1$, $A_1B_2C_1$, and $A_1B_1C_2$ are drawn outward relative to the triangle $A_1B_1C_1$. Show that each set of points $\{A_2, A, A_1$, $\{B_2, B, B_1$, and $\{C_2, C, C_1$ lie on a straight line and that the three lines meet in one point.

I. *Solution by* **Paul S. Bruckman**, *Sacramento, CA.*

Embed the figure in the complex plane, with points represented by their affines. Let $\theta = e^{i\pi/3}$. Note that $\theta\overline{\theta} = \theta + \overline{\theta} = 1$, $\theta^2 = -\overline{\theta}$, and $(\overline{\theta})^2 = -\theta$. Then $A_1 = C + \theta(B - C) = \overline{\theta}C + \theta B$. Likewise, $B_1 = \overline{\theta}A + \theta C$ and $C_1 = \overline{\theta}B + \theta A$. Similarly

$$\begin{aligned} A_2 &= \overline{\theta}C_1 + \theta B_1 \\ &= \overline{\theta}(\overline{\theta}B + \theta A) + \theta(\overline{\theta}A + \theta C) \\ &= 2A - \theta B - \overline{\theta}C; \end{aligned}$$

likewise, $B_2 = -\overline{\theta}A + 2B - \theta C$ and $C_2 = -\theta A - \overline{\theta}B + 2C$.

We now show the following:

$$(1) \qquad A = \frac{A_1 + A_2}{2},\ B = \frac{B_1 + B_2}{2},\ C = \frac{C_1 + C_2}{2}.$$

Note that $A_1 + A_2 = \overline{\theta}C + \theta B + 2A - \theta B - \overline{\theta}C = 2A$, hence $A = (A_1 + A + 2)/2$; The other relations in (1) follow similarly. This shows that A is the midpoint of the segment connecting A_1 and A_2, which implies that A, A_1, and A_2 are collinear. (Actually, we have shown a stronger result.) Similar conclusions apply for the other points.

The concurrence of the three lines A_2AA_1, B_2BB_1, and C_2CC_1 is a well-know result, attributable to the joint efforts of Fermat and Torricelli. The lines AA_1, BB_2, and CC_1 meet at a single point, known as the *Fermat Point* (or the *Fermat-Torricelli Point*); we denote such a point as F in the figure. It also known that the angles B_1FA_1, A_1FC_1, and C_1FB_1 are equal (and are therefore 120°). In the figure, the largest angle in triangle ABC is less than 120°. If the largest angle is 120°, then F coincides with the vertex at which the angle is 120°; if the largest angle is greater than 120°, then F lies outside of triangle ABC.

Also solved by **Mark Evans**, Louisville, KY, **Ovidiu Furdui**, Western Michigan University, Kalamzoo, **Murray S. Klamkin**, University of Alberta, **William H. Pierce**, Rangeley, ME, and the **Proposer**.

1031. [Fall 2001] *Proposed by Andrew Cusumano, Great Neck, NY.*

Notice that

$$\sqrt[3]{9 + \sqrt{80}} + \sqrt[3]{9 - \sqrt{80}} = 3$$

and

$$\sqrt[3]{161 + \sqrt{25920}} + \sqrt[3]{161 - \sqrt{25920}} = 7.$$

Generalize this by showing that

$$\sqrt[3]{\frac{x^3 - 3x}{2} + \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}} + \sqrt[3]{\frac{x^3 - 3x}{2} - \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}} = x.$$

Counter-example by **Rex H. Wu**, *Brooklyn, NY.*

This counter-example shows that it is important to place restrictions on x. When $x = 1$,

$$\begin{aligned} \sqrt[3]{\frac{1^3 - 3}{2} + \frac{(1^2 - 1)\sqrt{1^2 - 4}}{2}} + \sqrt[3]{\frac{1^3 - 3}{2} - \frac{(1^2 - 1)\sqrt{1^2 - 4}}{2}} &= \sqrt[3]{-1} + \sqrt[3]{-1} \\ &= -2 \\ &\neq 1. \end{aligned}$$

Another counter-example is when $x = -1$.

Solution by **Ovidiu Furdui**, *student at Western Michigan University, Kalamazoo, MI.*

Notice that we have to have $x^2 - 4 \ge 0$, that is, $x^2 \ge 4$. Let us denote

$$S = \sqrt[3]{\frac{x^3 - x}{2} + \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}} + \sqrt[3]{\frac{x^3 - x}{2} - \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}}.$$

I'll use the formula $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$. Raising to the third power the above equality we get

$$S^3 = x^3 - 3x + 3 \cdot S \cdot \sqrt[3]{\frac{x^3 - x}{2} + \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}} \cdot \sqrt[3]{\frac{x^3 - x}{2} - \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}}.$$

Notice that

$$\sqrt[3]{\frac{x^3 - x}{2} + \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}} \sqrt[3]{\frac{x^3 - x}{2} - \frac{(x^2 - 1)\sqrt{x^2 - 4}}{2}} = 1.$$

So we obtain

$$\begin{aligned} S^3 &= x^3 - 3x + 3S \\ S^3 - x^3 + 3x - 3S &= 0 \\ (s - x)(s^2 + sx + x^2 - 3) &= 0. \qquad (*) \end{aligned}$$

But $s^2 + sx + x^2 - 3 = (s + \frac{x}{2})^2 + \frac{3x^2}{4} - 3) = (s + \frac{x}{2})^2 + \frac{3}{4}(x^2 - 4)$.

I distinguish here 2 cases:

a) If $x^2 > 4$ then $(s + \frac{x}{2})^2 + \frac{3}{4}(x^2 - 4) > 0$, so from (*) we get $s - x = 0$. Thus $S - x = 0$ and $S = x$.

b) $x^2 = 4$. In this case, the initial identity becomes

$$\sqrt[3]{\frac{8-6}{2}} + \sqrt[3]{\frac{8-6}{2}} = 1 + 1 = 2 = x \qquad \text{for } x = 2, \text{ and}$$
$$\sqrt[3]{\frac{-8+6}{2}} + \sqrt[3]{\frac{-8+6}{2}} = -1 - 1 = -2 = x \qquad \text{for } x = -2.$$

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA, **Scott H. Brown**, Auburn University at Montgomery, AL, **Paul S. Bruckman**, Sacremento CA, **William Chau**, New Brunswick, NJ, **Kenneth B. Davenport**, Frackville, PA, **José Luis Díaz-Barrero**, Universitat Politècnica de Catalunya, Barcelona, Spain, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **Mark D. Evans**, Louisville, KY, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **Murray S. Klamkin**, University of Alberta, **Yoshimobu Murayoshi**, Okinawa, Japan, **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY **J. Ernest Wilkins, Jr.**, Clark Atlanta University, Atlanta, GA, **Jason T. Woodson**, Messiah College, Grantham, Pa, **Monte J. Zerger**, Adams State College, Alamosa, CO, and the **Proposer**.

1032. [Fall 2001] *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*

Consider an equilateral triangle with sides of length 1 unit. From an arbitrary interior point P, draw perpendiculars PQ, PR, and PS to the sides of the triangle. Find the sum of the lengths of PQ, PR, and PS.

Solution by **Tracey M. Hagedorn**, *student at Angelo State University, San Angelo TX.*

Let A, B, C be the vertices of the triangles and let $AB = AC = BC = a$. When lines are drawn from each of the vertices to the point P, we get

$$\text{Area}(\Delta BPC) = \frac{a(PQ)}{2},$$
$$\text{Area}(\Delta APB) = \frac{a(PS)}{2},$$
$$\text{Area}(\Delta APC) = \frac{a(PR)}{2}.$$

Since P is an interior point of ΔABC,

$$\text{Area}(\Delta ABC) = a\frac{PQ + PS + PR}{2}.$$

Substituting $\text{Area}(\Delta ABC) = \sqrt{3}a^2/4$ we get $PQ + PR + PS = \sqrt{3}/2$ which is the height of ΔABC.

Also solved by **Ayoub B. Ayoub**, Penn State Abington College, PA, **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA, **Dipendra Bhattacharya and Stephen Gendler**, Clarion University, Clarion, PA, **Brian Bradie**, Chrisopher Newport University, Newport News, VA, **Scott H. Brown**, Auburn University at Montgomery, AL, **Paul S. Bruckman**, Sacremento CA, **William Chau**, New Brunswick, NJ, **Brian Clester**, Perry, GA, **Kenneth B. Davenport**, Frackville, PA, **Mark D. Evans**, Louisville, KY, **George P. Evanovich**, Saint Peter's College, Jersey City, NJ, **Ovidiu Furdui**, Western Michigan University, Kalamzoo, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **R. Daniel Hurwitz**, Skidmore College, Saratoga Springs, NY, **Murray S. Klamkin**, University of Alberta, **Jim Messinger**, University of Texas at Arlington, TX **Yoshimobu Murayoshi**, Okinawa, Japan, **Roger B. Nelsen**, Lewis & Clark College, Portland, OR, **Melanie Parker**, Clarion University, Clarion, PA, **Supun Pathirana**, Montclair State University, Upper Montclair, NJ, **Shiva K. Saksena**, University of North Carolina at Wilmington, NC, **Harry Sedinger**, St. Bonaventure University, St. Bonaventure, NY **H.-J. Seiffert**, Berlin, Deutschland, **Rex H. Wu**, Brooklyn, NY, **Monte J. Zerger**, Adams State College, Alamosa, CO, and the **Proposer**.

1033. [Fall 2001] *Proposed by Kenneth B. Davenport Frackville, PA.*

Student solutions solicited

Show that

$$\frac{2\sin(2\theta) - 3\sin(\theta)}{1 - \cos(\theta) - 2\cos(2\theta)} = -\tan\left(\frac{\theta}{2}\right)$$

for all values of θ where both sides are defined.

Solution by **Alex Rand**, *student, New Mexico Tech, Socorro, NM.*

First, note applying the double angle formula for sine on the numerator of the left-hand side yields:

$$2\sin(2\theta) - 3\sin(\theta) = 2(2\sin(\theta)\cos(\theta) - 3\sin(\theta) = \sin(\theta)(4\cos(\theta) - 3)$$

Likewise, applying the double angle formula for cosine to the denominator gives:

$$1 - \cos(\theta) - 2\cos(2\theta) = 3 - \cos(\theta) - 4\cos^2(\theta) = -1(4\cos(\theta) - 3)(1 + \cos(\theta)).$$

Altogether this means that

$$\frac{2\sin(2\theta) - 3\sin(\theta)}{1 - \cos(\theta) - 2\cos(2\theta)} = \frac{\sin(\theta)(4\cos(\theta) - 3)}{-1(4\cos(\theta) - 3)(1 + \cos(\theta))} = -\frac{\sin(\theta)}{1 + \cos(\theta)}.$$

Now, this is simply the half angle formula for tangent.

$$-\frac{\sin(\theta)}{1 + \cos(\theta)} = -\tan(\theta/2).$$

So it has been shown that

$$\frac{2\sin(2\theta) - 3\sin(\theta)}{1 - \cos(\theta) - 2\cos(2\theta)} = -\tan(\theta/2)$$

for all θ for which both sides of the equation are defined.

Also solved by **Jean-Claude Andrieux**, Beaune, France, **Ayoub B. Ayoub**, Penn State Abington College, Abington, PA, **Paul S. Bruckman**, Sacramento, CA, **George P. Evanovich**, Saint Peters College, Jersey City, NJ, **Mark D. Evans**, Louisville, KY, **Ovidiu Furdui**, Kalamazoo, MI, **Robert C. Gebhardt**, Hopatcong, NJ, **Tracey M. Hagedorn** Angelo State University, San Angelo, TX, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **Jim Messinger**, University of Texas at Arlington, Graford **Yoshinobu Murayoshi**, Okinawa, Japan, **Skidmore College Problem Group**, Saratoga Springs, NY, **H.-J. Seiffert** Berlin, Germany, **Jason T. Woodson**, Messiah College, Grantham, PA, and by the **Proposer**.

MATH ACROSTIC

Dan Hurwitz, Skidmore College

a. Odd prime sided (one case) — 049 146 114 018 182 067 123 161 154 089

b. Epic — 093 140 057 172

c. He did foundation work in statistics — 179 108 142 029 075 063

d. What Peano's curve does (3 wds.) — 122 022 163 094 009 194 134 051 173 001 042 104 083

e. They are included in $n\mathbb{Z}$ — 192 088 109 036 148 003 055 024 165

f. Sets on which concatenation can be based — 116 061 092 101 032 019 189 073 143

g. Ball around a point in a metric space — 066 010 191 096 074 119 180 043 169 034 139 023

h. Indian mathematician (c.476-550), calculated chord lengths — 178 196 079 105 151 126 002 168 045

i. His *Grounde of Artes* was a classic arithmetic text (full name) — 190 076 128 020 150 062 011 110 162 097 084 047 177

j. Number code — 153 184 004 082 130 071

k. Friction makes this out of kinetic energy — 112 157 060 121

l. Linear programming constraint — 064 106 160 052 041 017 115 141 131 095

m. Primes used to generate perfect numbers — 171 129 059 025 118 187 035 152

n. Gödel showed that proofs of true statements don't always do this — 028 136 158 167 048

o. Halley used tables to estimate its probability — 014 133 070 103 037

p. Russian mathematician, studied measurable functions — 113 033 149 176 007 185

q. Part of a covering space — 021 132 186 044 111

r. He used the modern division symbol in *Teutsche Algebra* — 181 090 027 046

s. Consider interchangeable — 135 040 155 005 015 050

t. Difference in electrical potential — 012 164 078 100 065 087 038

u. Reverse a logarithm — 193 031 145 080 117 072 098 159 054 175 125 013

v. Expert on the mathematics of antiquity — 077 170 144 183 030 008 102 053 127 085

w. This causes maximal deformation (2 wds.) — 138 147 086 056 026 195 099 174 006 107 124 069 156 039

x. No jumps and no retracings (2 wds.) — 166 120 068 058 091 188 137 081 016

001d	002h	003e	004j	005s	006w	■	007p	008v	009d	010g	011i	012t	013u	014o	■	015s
016x	017l	018a	■	019f	020i	021q	022d	023g	024e	025m	■	026w	027r	028n	■	029c
030v	031u	032f	033p	034g	035m	■	036e	037o	038t	■	039w	040s	041l	042d	043g	044q
■	045h	046r	047i	■	048n	049a	050s	■	051d	052l	053v	054u	055e	056w	057b	058x
059m	060k	061f	■	062i	063c	064l	065t	066g	067a	068x	069w	■	070o	071j	072u	■
073f	074g	075c	■	076i	077v	078t	079h	■	080u	081x	082j	083d	084i	■	085v	086w
087t	088e	089a	090r	091x	■	092f	093b	094d	095l	096g	097i	098u	099w	■	100t	101f
102v	103o	■	104d	105h	106l	■	107w	108c	109e	110i	■	111q	112k	113p	■	114a
115l	116f	117u	118n	■	119g	120x	121k	■	122d	123a	124w	■	125u	126h	127v	■
128i	129m	130j	■	131l	132q	133o	■	134d	135s	136n	137x	138w	139g	140b	■	141l
142c	■	143f	144v	145u	146a	147w	148e	149p	150i	■	151h	152m	153j	154a	155s	156w
157k	■	158n	159u	■	160l	161a	162i	163d	164t	165e	166x	167n	■	168h	169g	170v
■	171m	172b	173d	174w	■	175u	176p	177i	178h	■	179c	180g	181r	■	182a	■
183v	184j	185p	186q	187m	■	188x	189f	190i	191g	192e	193u	194d	195w	196h	■	■

Last issue's mathacrostic was taken from "Recursive Functions" by R. Peter.

> "The problem I dealt with arose as a consequence of inevitable inner developments in mathematics. ... It should be a warning example to all those who want to discourage research in pure mathematics that they are preventing the cause of the applications of mathematics as well. "

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