Do even perfect numbers have digital root 1?
Problem 1056

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TRANSLATION AND HOMOTHETY IN A PROBLEM

ALI R. AMIR-MOEZ*

Employing geometric transformations to construct geometrical figures with ruler and compass is quite interesting. In this note translation and homothety are used to solve a problem in ruler and compass construction. (Of course, after reading the problem, it is natural and expected for you to take out your ruler and compass, sharpen your pencil, and give the problem a try on your own).

PROBLEM. *Inscribe a regular pentagon in a given square such that the four vertices of the pentagon are on four sides of the square and the fifth vertex lies on a diagonal of the square.*

In order to make the note self-contained, a review of a construction of a regular pentagon will be presented first.

1. Geometric Solution of a Quadratic Equation. An outline of the construction is given. Let

$$ax^2 + bx + c = 0, a \neq 0$$

be a quadratic equation with real coefficients, and distinct real roots x_1 and x_2. It is known that the roots satisfy

$$x_1 + x_2 = \frac{-b}{a}, \qquad x_1 x_2 = \frac{c}{a}.$$

Consider a rectangular coordinate system, see Figure 1. Let A and B on the x-axis

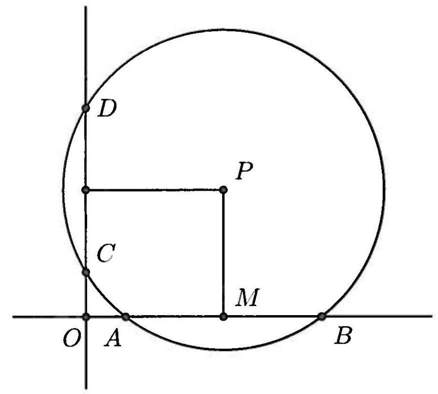

FIG. 1.

correspond to the values x_1 and x_2 respectively. We shall construct points C and D on the y-axis such that $OC = 1$ and $OD = c/a$. Observe that

$$(OC)(OD) = \frac{c}{a} = x_1 x_2 = (OA)(OB)$$

This implies that the four points A, B, C, and D all lie on a circle. Since point M, the midpoint of the line segment AB, corresponds to the x-value

$$\frac{x_1 + x_2}{2} = \frac{-b}{2b},$$

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one can construct the circle if the given quadratic coefficients a, b, and c are constructible numbers. First construct C, D and M. The center P of the circle is the point of intersection of the perpendicular bisector of CD with the line perpendicular to the x-axis as M. This constructs A and B, whose x-values are the solution of the given quadratic.

2. The Regular Pentagon. Consider the polynomial equation

$$z^5 = (z-1)(z^4 + z^3 + z^2 + z + 1) = 0 \tag{1}$$

over the complex field. The solutions of (1) are the fifth roots of unity, which form the vertices of a regular pentagon inscribed in the unit circle of the complex plane. One of the roots, 1, is real. Let the others, in clockwise order, be labelled $z_1 = x_1 + y_1 i$, $z_2 = z_1^2 = x_2 + y_2 i$, $z_3 = z_1^3 = x_2 - y_2 i$, and $z_4 = z_1^4 = x_1 - y_1 i$. To construct a pentagon it would be enough to construct any of the values x_i or y_i. Since z_1 is a root of $z^4 + z^3 + z^2 + z + 1$ we have

$$x_1 + x_2 = -\frac{1}{2}. \tag{2}$$

Since $z_2 = z_1^2$, we have $x_2 = x_1^2 - y_1^2 = 2x_1^2 - 1 = 2x_1(-x_2 - (1/2)) - 1$, so, using (2) we have

$$x_1 x_2 = \frac{-1}{4} \tag{3}$$

By (2) and (3) x_1 and x_2 are the roots of the quadratic

$$x^2 + \frac{1}{2}x - \frac{1}{4} \tag{4}$$

Now we can construct the pentagon using the method of the previous section, with

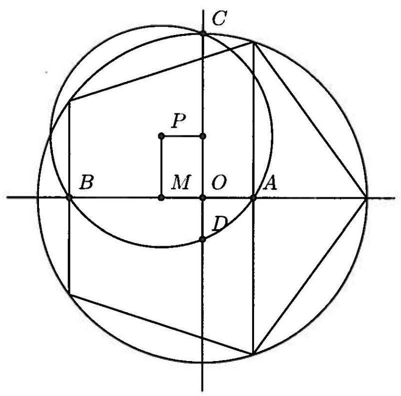

FIG. 2.

$C = (0,1)$, $D = (0, -1/4)$ and $P = (-1/4, 3/8)$. See Figure 2.

3. The Construction. The square $ABCD$ is given. Extend the diagonal BD and construct a regular pentagon such that the center of it is on the line BD and one side of it is perpendicular to BD. Let EF be the side of the pentagon perpendicular to the line BD. On the side EF construct the isosceles right triangle PDF and extend

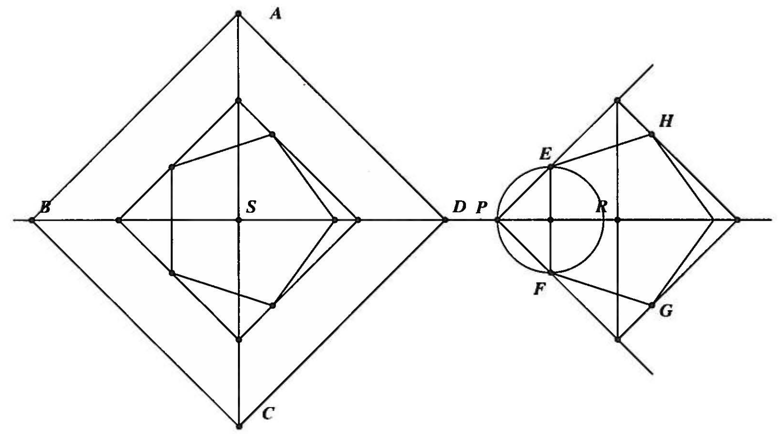

FIG. 3.

PE and PF. Perpendiculars form H to PF and from G to DF intersect at a point on the line BD. This way one obtains a square circumscribed about the pentagon. Let R be the center of this square. Translate this configuration through the vector $\overrightarrow{RS}$. Finally, by a homothety of center S and ratio SE/SM enlarge the pentagon, and the problem is solved.

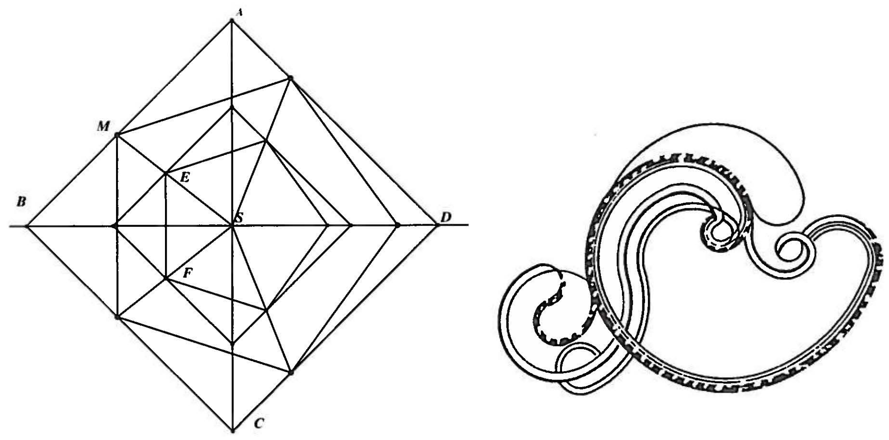

FIG. 4.

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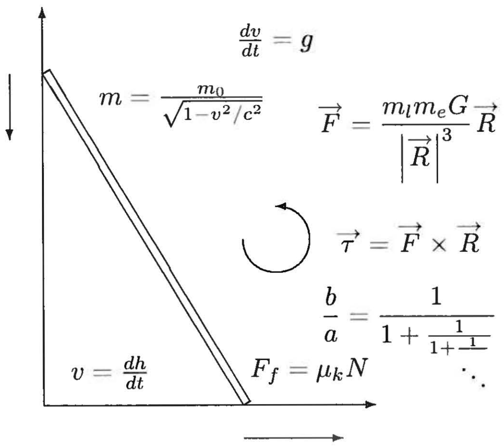
Problem. A painter is high atop a 40 foot ladder leaning against a 60 foot house at an angle such that the ratio of the lengths of the legs of the right triangle formed is the golden section. If the base of the ladder is moving away from the house at at the speed of light divided by Avogadro's number, what are the thoughts of the mathematician watching the ensuing disaster?



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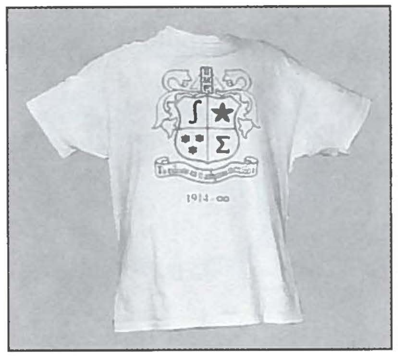

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A NOTE ON RANDOM LINEAR EQUATIONS WITH INTEGER SOLUTIONS

MICHAEL J. BOSSÉ* AND N. R. NANDAKUMAR†

1. Introduction. All too often, middle and secondary grades Algebra students lose sight of the significance of the properties of the real numbers and which of these properties are held within each number system. This lack of understanding is often exacerbated by countless textbook examples and problem sets that "work" or that produce answers within certain number systems. For instance, after countless examples in which linear equations with integer coefficients produce integer solutions, students may naturally induce the notion that the nature of the coefficients determines the nature of the solution.

This paper investigates the probability that a linear equation with randomly selected integer coefficients has an integer solution. This paper will demonstrate that this probability is zero. Students' experiences, however, will make the findings of this investigation seem dubious since so many of the problem that they initially experienced produced solution in the integers.

2. The Investigation. Given the linear equation, $ax + b = 0$, where $a, b, \in Z$ and a and b are randomly selected, what is the probability that the equation has an integer solution? Clearly, $ax + b = 0$ has an integer solution if a divides b.

Let us assume that a and b are randomly selected positive integers such that $1 \leq a, b \leq N$, where N is a positive natural number. (Although the preceding question allows for both positive and negative integers, without loss of generality, we will only consider positive values for N.) Then the number of possible ordered pairs (a, b) is equal to N^2. It is only necessary to determine an estimate of the number of pairs (a, b) such that $a|b$ (which is read a divides b).

We will begin with an example. Let $N = 10$. Then the set of all pairs (a, b) such that a divides b is a proper subset (denoted by bold faced letters) of the following:

(1,10)	(2,10)	(3,10)	(4,10)	**(5,10)**	(6,10)	(7,10)	(8,10)	(9,10)	**(10,10)**
(1,9)	(2,9)	**(3,9)**	(4,9)	(5,9)	(6,9)	(7,9)	(8,9)	**(9,9)**	
(1,8)	**(2,8)**	(3,8)	**(4,8)**	(5,8)	(6,8)	(7,8)	**(8,8)**		
(1,7)	(2,7)	(3,7)	(4,7)	(5,7)	(6,7)	**(7,7)**			
(1,6)	**(2,6)**	**(3,6)**	(4,6)	(5,6)	**(6,6)**				
(1,5)	(2,5)	(3,5)	(4,5)	**(5,5)**					
(1,4)	**(2,4)**	(3,4)	**(4,4)**						
(1,3)	(2,3)	**(3,3)**							
(1,2)	**(2,2)**								
(1,1)									

In each column, the number of pairs such that $a|b$ is equal to $\lfloor 10/i \rfloor$ where i represents the column number and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Thus, the total number of pairs such that $a|b$ is equal to

$$\sum_{i=1}^{10} \left\lfloor \frac{10}{i} \right\rfloor.$$

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Given two positive integers a and N, there exists two non-negative integers k and r with $r < k$ such that $(N/a) = k + (r/a)$. Thus the number of integers which are less than or equal to N and divisible by a are exactly equal to k. In other words, for given two positive integers a and N the number of ordered pairs (a, b) of integers such that $a|b$ and $b \leq N$ is equal to $\lfloor N/a \rfloor$. Thus we have for $1 \leq a, b \leq N$, the number of ordered pairs (a, b) such that $a|b$ is equal to

$$\sum_{i=1}^{N} \left\lfloor \frac{N}{i} \right\rfloor.$$

In what follows, $P_N(a, b)$ denotes the probability that an ordered pair of randomly selected integers such that $1 \leq a, b \leq N$ and $a|b$ and $P(a, b)$ denotes the probability that an ordered pair of randomly selected integers such that $1 \leq a, b \leq \infty$ and $a|b$. Next we show that $\lim_{N\to\infty} P_N(a, b) = 0$ which implies $P(a, b) = \lim_{N\to\infty} P_N(a, b) = 0$.

Since $\lfloor N/i \rfloor \leq (N/i)$, we obtain

$$P_N(a, b) = \frac{1}{N^2} \sum_{i=1}^{N} \left\lfloor \frac{N}{i} \right\rfloor \leq \frac{N}{N^2} \sum_{i=1}^{N} \frac{1}{i} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{i}.$$

To show that the right hand side approaches 0, first using integrals we get an upper bound for the harmonic series and then show that this upper bound goes to zero as N approaches infinity. In the Figure 1 the sum of the shaded rectangles is

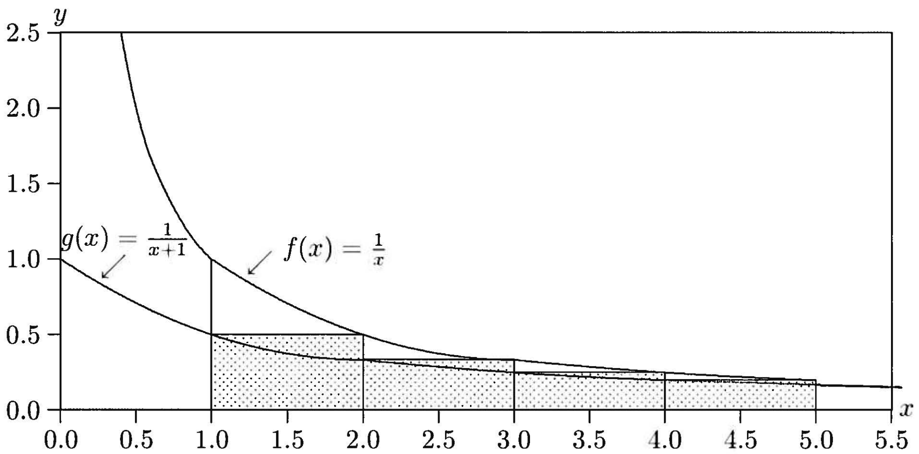

FIG. 1.

bounded by the area under the curve $f(x) = 1/x$. Since the area of each rectangle on each interval $[k-1, k]$ is $1 \cdot f(1/k) = 1/k$ we have

$$\sum_{i=2}^{N} \frac{1}{i} < \int_{1}^{N} \frac{1}{x}\,dx$$

and hence

$$\sum_{i=1}^{N} \frac{1}{i} < 1 + \ln N.$$

Using this upper bound we obtain

$$P(a, b) = \lim_{N\to\infty} P_N(a, b) \leq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{i} < \lim_{N\to\infty} \frac{1 + \ln N}{N} = 0.$$

Similarly, we obtain a lower bound for $P_N(a, b)$. As discussed earlier we can write $(N/a) = k + (r/a)$ where k and r are positive integers with $r < a$. It is obvious that if $a \leq \lfloor N/2 \rfloor$ then $k \geq 2$. Hence we have

$$\left\lfloor \frac{N}{a} \right\rfloor = k = \frac{k}{2} + \frac{r}{2a} + \left(\frac{k}{2} - \frac{r}{2a}\right) = \frac{ak + r}{2a} + \left(\frac{k}{2} - \frac{r}{2a}\right) = \frac{N}{2a} + \left(\frac{k}{2} - \frac{r}{2a}\right).$$

Since $k/2 \geq 1$ and $(r/2a) < 1$ we have $k/2 > r/2a$. Hence $\lfloor N/a \rfloor > (N/2a)$. Observing that $\lfloor N/a \rfloor = 1$ for $a > \lfloor (N+1)/2 \rfloor$ we obtain

$$\sum_{i=1}^{N} \left\lfloor \frac{N}{i} \right\rfloor = \sum_{i=1}^{\lfloor N/2 \rfloor} \left\lfloor \frac{N}{i} \right\rfloor + \left\lfloor \frac{N+1}{2} \right\rfloor > \sum_{i=1}^{\lfloor N/2 \rfloor} \frac{N}{2i} + \frac{N}{2} = \frac{N}{2} \sum_{i=1}^{\lfloor N/2 \rfloor} \frac{1}{i} + \frac{N}{2}.$$

Again using the figure above we see that the area under the curve of $g(x) = 1/(x+1)$ is less than the sum of the areas of rectangles from $x = 1$ to ∞. Thus, in particular,

$$\sum_{i=1}^{\lfloor N/2 \rfloor} \frac{1}{i} \geq 1 + \int_{1}^{\lfloor N/2 \rfloor} \frac{1}{x+1}\,dx = 1 + \ln(N/2).$$

Thus

$$\sum_{i=1}^{N} \left\lfloor \frac{N}{i} \right\rfloor > \frac{N}{2}(1 + \ln(N/2)) + \frac{N}{2} = \frac{N}{2}(2 + \ln(N/2)).$$

Combining upper bound and lower bound inequalities we obtain

$$\frac{N}{2}(2 + \ln(N/2)) < \sum_{i=1}^{N} \left\lfloor \frac{N}{i} \right\rfloor < N(1 + \ln N).$$

Dividing by N^2 we obtain

$$\frac{2 + \ln(N/2)}{2N} < P_N(a, b) < \frac{1 + \ln N}{N}.$$

The actual probabilities for various values of N comparing with the bounds are given in Table 1. The average of upper and lower bounds are close to the actual probabilities.
Remark. Although an upper bound for the harmonic series is obtained by simple calculus techniques, it has been shown that

$$\lim_{N\to\infty} \left(\sum_{i=1}^{N} \frac{1}{i} - \ln N \right)$$

exists and is denoted by γ. This constant γ, which is approximately equal to 0.577216, is called Euler's or Mascheroni's constant. It is unknown whether this constant γ is rational or not (see [1]). If this constant is rational it is known that the denominator has more than 244,000 decimal digits as of 2001.

N	Lower Bound L	$P_N(a,b)$	Upper Bound U	Average $\frac{L+U}{2}$
10	0.1800	0.2700	0.3300	0.2550
20	0.1080	0.1650	0.2000	0.1540
50	0.0520	0.0800	0.0980	0.0750
100	0.0300	0.0487	0.0560	0.0430
500	0.0075	0.0128	0.0144	0.0109
1,000	0.0045	0.0071	0.0080	0.0063
5,000	0.0010	0.0017	0.0020	0.0015
10,000	0.0005	0.0009	0.0010	0.0008

TABLE 1

3. Student Consideration. Students may be somewhat confused by the findings above. They may question how the probability can be zero if they have so often found integer solutions to linear equations with integer coefficients. This misunderstanding is founded upon students' lack of understanding of the role infinity plays within this discussion. This discussion can lead to intuitive understandings of both infinity and limits.

4. Acknowledgements. The authors thank the referee for his valuable suggestions which have helped in the presentation of this paper aiming towards undergraduate students. The authors also acknowledge Michael Bossé, Jr., a high school student, for writing a computer program to generate actual probabilities given in the above table.

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ANALYZING THE AREA OF FRACTAL TILINGS*

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1. Introduction. The Koch island shown in Figure 1 is a classic example of a fractal curve that bounds a finite area. Discovered by Helge von Koch in 1904, the boundary of the Koch island is an example of a fractal curve with infinite length. The area of the enclosed island has been calculated precisely by exploiting the self-similarity of the region and summing a geometric series. (See [10],p.167.) In this note, we will examine other geometric regions with fractal boundaries and will determine the area of these regions.

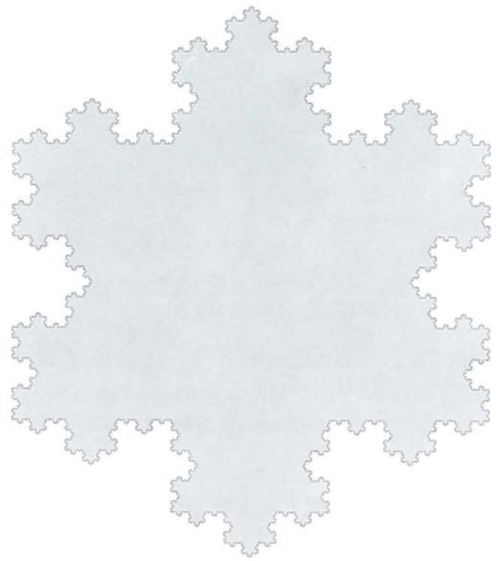

FIG. 1. *Koch island*

Specifically, we will examine geometric shapes with fractal boundaries that tile the plane. A tiling of the plane is a family of sets, called tiles, with disjoint interiors that cover the plane without gaps. For example, ordinary graph paper suggests that a square tiles the plane, since the plane can be covered with congruent copies of the square with no overlaps. First we will describe an iterative method to generate tilings of the plane with tiles (fractiles) whose boundaries are fractal curves. Then we will analyze the area enclosed by these irregular curves of infinite length. Such tiles arise in several contexts, including the study of self-similar tilings of the plane [1, 4, 11], the study of radix expansions [5, 6], the construction of fractals, and more recently the construction of families of orthonormal wavelets of compact support in the plane [9, 12]. An example of the fractal tiling of a portion of the plane is shown in Figure 2.

2. Generating the tilings. To generate the fractal tilings, we use an iterative process involving repeated compositions of two or more functions constructed from 2×2 integer matrices. The method is best introduced with an example.

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†Research funded by Faculty Research Grant 1518, University of Akron.

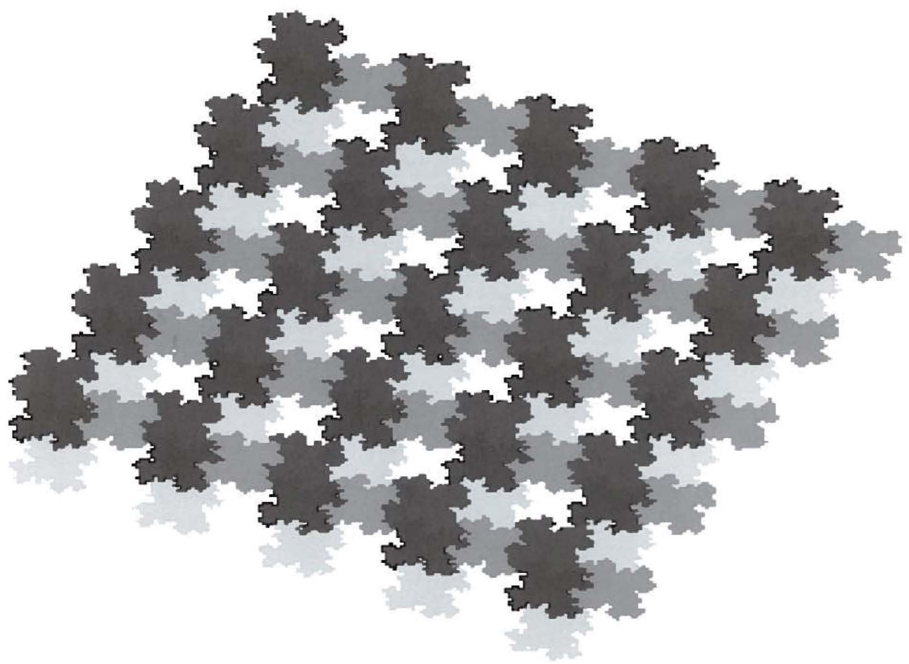


FIG. 2. *Tiling the plane*

EXAMPLE 1. *Let* $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ *and note* $m = \det A = 5$ *and* $A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$. *Choose*

$$D = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}.$$

Define the mappings $\varphi_i(z) = A^{-1}(z + d_i)$ *for* $d_i \in D$. *To initiate the iteration process we randomly choose any point* z_0 *in the plane and evaluate* $\varphi_i(z_0)$, $1 \le i \le 5$. *Then for* $n \ge 1$, *we choose recursively and randomly* $z_n \in \{\varphi_i(z_{n-1}),\ 1 \le i \le 5\}$. *After a few iterations the generated points lie near the tiling. The generated tiling is shown in Figure 3. The figures in this note show the results of several hundred thousands of iterations.*

The algorithm for generating the tiling is based on Barnsley's Random Iteration Algorithm [2, p.89]. Classical fractals such as the Cantor set (1883), the Koch curve (1904) and the Sierpinski triangle (1915) can be constructed using the same algorithm.

In order to use an integer 2×2 matrix A to generate a tiling of the plane, A must be an expanding matrix; that is, one with all eigenvalues $|\lambda_i| > 1$. If matrix A has determinant $|\det(A)| = m$ for some integer $m > 1$, then we choose $D = \{d_1, d_2, \ldots, d_m\}$ to be a finite set of m vectors in the plane that we will call a digit set. (A discussion of the careful selection of set D will follow later.) The linear maps

$$\varphi_i(z) = A^{-1}(z + d_i), 1 \le i \le m, \tag{1}$$

are all contractions. That is, if z is a vector in the plane, then $|A^{-1}z| < |z|$. A result of Hutchinson [7] states that there exists a unique nonempty compact set $T := T(A, D)$

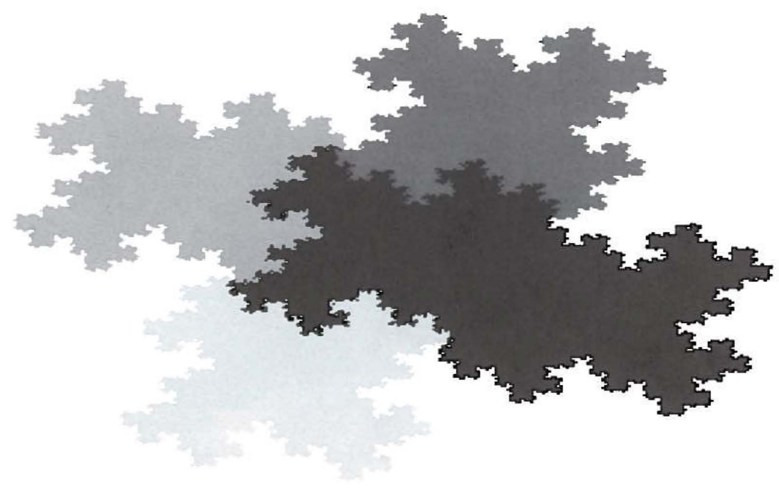


FIG. 3. *Fractal Cross*

satisfying the function equation

$$T = \bigcup_{i=1}^{m} \varphi_i(T). \tag{2}$$

Using the terminology of Barnsley [2], the collection $\{\varphi_i, 1 \le i \le m\}$ is an iterated function system and T is its attractor. We will denote by f_A the transformation induced by the matrix A. If we apply f_A to the set T,

$$f_A(T) = \bigcup_{i=1}^{m} (T + d_i). \tag{3}$$

Geometrically, the dilated set $f_A(T)$ is perfectly tiled by the m translates $T_j = T + d_j$ of T.

EXAMPLE 2. *Set* $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ *with* $m = \det A = 3$. *Choose*

$$D = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

The attractor $T(A, D)$ *is shown in Figure 4.*

For most pairs (A, D) the set $T = T(A, D)$ has area or Lebesgue measure $\mu(T) = 0$. This is the case for the Cantor set, the Koch curve and the Sierpinski triangle. If $T(A, D)$ has positive measure, we call $T(A, D)$ a self-affine tile. The fundamental question addressed in this paper is: Under what conditions can the area of the tile $T(A, D)$ be calculated?

3. The area of the tilings. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix as described above with all integer entries and $D = \{d_1, d_2, \ldots, d_m\}$ a collection of vectors with integer coordinates. Bandt [1] has shown that the attractor $T = T(A, D)$ is a tile when D is a complete set of coset representatives of $\mathbb{Z}^2/f_A(\mathbb{Z}^2)$. In this case such a set T tiles the plane by translations using some translation set ϑ of vectors with integer coordinates.

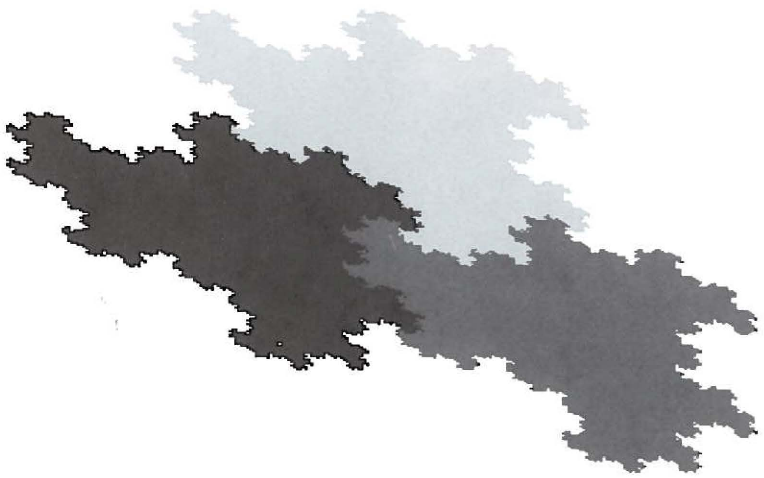


FIG. 4. *Slanted terdragon*

We will describe a geometric way to identify a complete set of coset representatives associated with a matrix A. Let S be the parallelogram formed by the columns of matrix A. Define E to be the set of vectors with integer coordinates that lie in or on S but not on the two outer edges that do not have the origin as a vertex. Then E contains exactly m vectors. The members of $E = \{e_j,\ j = 1, \ldots, m\}$ form a complete set of coset representatives of $\mathbb{Z}^2/f_A(\mathbb{Z}^2)$. The plane can be covered with parallelograms congruent to S, each containing m points with integer coordinates. Each point e in E is equivalent to a point e' (written $e \approx e'$) inside each of these congruent parallelograms. In our discussion we will always assume that $0 \in D$. Thus the digit set D can be any collection of points with $d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $d_j \approx e_j,\ j = 2, \ldots, m$. Figures 5 and 6 show the location of the set E and the congruent set D used in Examples 1 and 2.

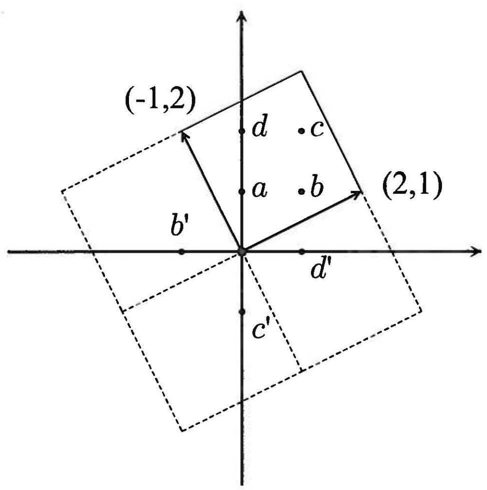


FIG. 5. *Location of congruent points*

Reference [4] has a complete discussion of the construction of tilings of the plane with

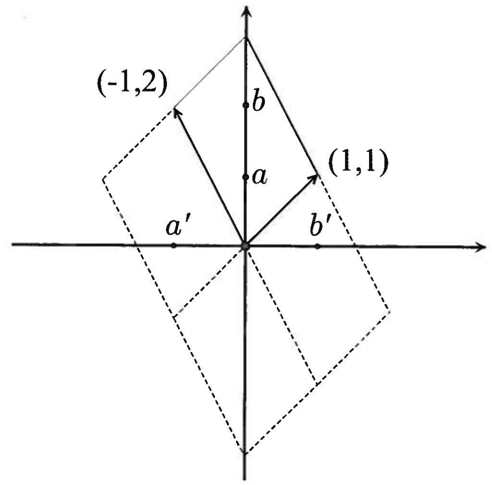


FIG. 6. *Congruent parallelograms*

fractal boundaries.

We will define the set $Z[A, D]$ to be the span, using coefficients in $\mathbb{Z}$, of the sets D and $f_A(D)$. A digit set D is *primitive* if $Z[A, D] = \mathbb{Z}^2$, and the associated tile $T(A, D)$ is called a *primitive tile.* For all expanding integer matrices A there exists some digit set D that is a complete set of coset representatives for $\mathbb{Z}^2/f_A(\mathbb{Z}^2)$ and for which $Z[A, D] = \mathbb{Z}^2$, [8, 9].

To begin the discussion of the area enclosed by tiles, we use a result of Wang [12, Theorem 5.3].

PROPOSITION: Let A be an expanding integer matrix that is irreducible over the rationals. Let D be a primitive set of complete coset representatives of $\mathbb{Z}^2/f_A(\mathbb{Z}^2)$. Then the measure of $T(A, D)$ is 1.

(We note that A is irreducible over the rationals means that the characteristic polynomial of A is irreducible over the rationals.)

In each of the two examples above, the set D is clearly a primitive set. Therefore, by Wang's result, the tilings shown in Figures 3 and 4 have area (measure) one.

EXAMPLE 3. *Set* $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ *with* $m = \det A = 2$. *Choose* $D = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. *Define the mappings* $\varphi_i(z) = A^{-1}(z + d_i)$ *for* $d_i \in D$. *Then* $f_A(D) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ *and* $Z[A, D] = \mathbb{Z}^2$. *The tile* $T(A, D)$ *is shown in Figure 7. If we choose* $S = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$, *then* $f_A(S) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$ *and we see that* S *is a complete set of coset representatives for* $\mathbb{Z}^2/f_A(\mathbb{Z}^2)$ *but* $Z[A, S] \neq \mathbb{Z}^2$. *The tile* $T(A, S)$ *is shown in Figure 8.*

A casual inspection of the tilings shown in Figures 7 and 8 reveals that the areas enclosed by the two tilings differ considerably. The shape and size of the tiles are clearly affected by the choice of digit sets.

4. Finding the primitive digit set. Suppose that for an integer matrix A, the set D is a complete set of coset representatives of $\mathbb{Z}^2/f_A(\mathbb{Z}^2)$, but $Z[A, D] \neq \mathbb{Z}^2$, as in

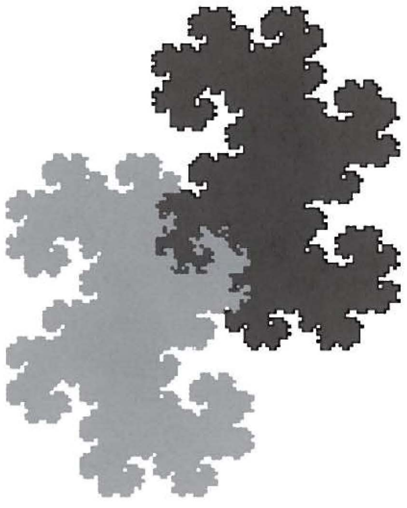


FIG. 7. *Small Twin Dragon*

the second part of Example 3. If we can find a matrix B such that the columns of B form a basis of $Z[A, D]$, that is $Z[A, D] = f_B(\mathbb{Z}^2)$, then there exists an integer matrix $\widetilde{A}$ and digit set $\widetilde{D} \subseteq \mathbb{Z}^2$ such that $Z[\widetilde{A}, \widetilde{D}] = \mathbb{Z}^2$ and $T(A, D) = f_B(T(\widetilde{A}, \widetilde{D}))$. In other words, it is always possible to find an integer matrix $\widetilde{A}$ and a digit set $\widetilde{D}$ such that $\mu(T(\widetilde{A}, \widetilde{D})) = 1$. For any tile T in the plane, we note that $\mu(f_B(T)) = |\det B|\, \mu(T)$. (This result is a consequence of the change-of-variables theorem for the Lebesgue integral. See [3].) Now since in our case, $T(A, D) = f_B(T(\widetilde{A}, \widetilde{D}))$,

$$\mu(T(A, D)) = |\det B|\, \mu(T(\widetilde{A}, \widetilde{D})) = |\det B|\,.$$

To illustrate, let us return to Example 3 where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $f_A(S) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$. Choose $B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ with $\det B = 5$. The columns of B form a basis for the set $Z[A, S]$ so that $Z[A, S] = f_B(\mathbb{Z}^2)$. We want to find $\widetilde{A}$ so that $AB = B\widetilde{A}$. So $\widetilde{A} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$. Set $S = B\widetilde{D}$ so that $\widetilde{D} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\widetilde{A}(\widetilde{D}) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Then $\widetilde{D}$ is a primitive complete digit set for $\widetilde{A}$. Since $A = B\widetilde{A}B^{-1}$, we conclude that $\mu T(A, S) = |\det B|\, \mu T(\widetilde{A}, \widetilde{D}) = 5$. This is the tile shown in Figure 8.

5. Noninteger tilings. In a slightly more general setting, we can define a tile to be a set $T(A, D)$ of positive measure, where A is an expanding real matrix such that $|\det A| = m \in \mathbb{Z}$, and $D = \{d_1, \ldots, d_m\}$ is a set of real-valued vectors. In this case, it is necessary that matrix A be similar to an integer matrix. Then the study of such tiles can be reduced to the study of integral self-affine tiles.

EXAMPLE 4. *Let* $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{15}}{2} \\ \frac{\sqrt{15}}{2} & \frac{1}{2} \end{bmatrix}$ *so that* $\det A = 4$. *With* $B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{\sqrt{15}}{2} \end{bmatrix}$ *we find that* $\widetilde{A} = \begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ *and* $A = B\widetilde{A}B^{-1}$. *The tiling* $T(A, f_B(D))$ *shown in*

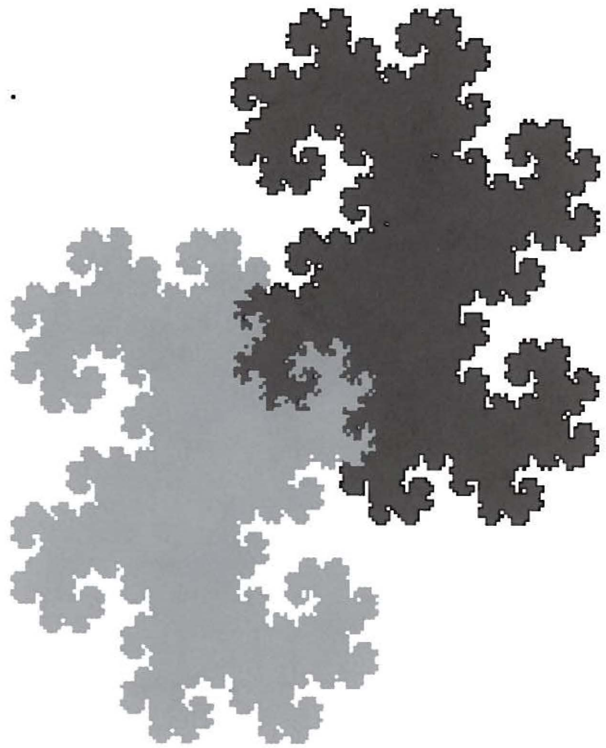


FIG. 8. *Big Twin Dragon*

Figure 9b is derived using the functions $f_j(z) = A^{-1}(z + Bd_j)$ *with*

$$D = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}.$$

The area of $T(\widetilde{A}, D)$, *shown in Figure 9a is one since* $Z[\widetilde{A}, D] = \mathbb{Z}^2$. *Note that* A *is similar to the integer matrix* $\widetilde{A}$. *So*

$$T(A, D) = f_B(T(\widetilde{A}, D))$$

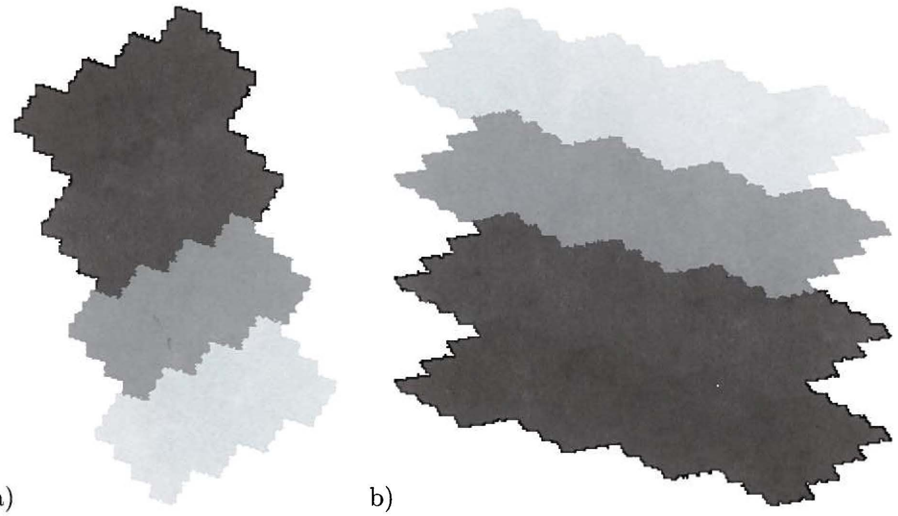
and

$$\mu(T(A, D) = |\det B|\, \mu(T(\widetilde{A}, D)) = \frac{\sqrt{15}}{2}.$$

This is the area of the tiling in Figure 9b.

6. Similarity Mappings. A similarity map g satisfies $|g(x) - g(y)| = r\,|x - y|$, for $r > 0$ and all x and y in the plane. If each mapping in our collection $\{\varphi_i, 1 \le i \le m\}$ is a similarity map with $0 < r < 1$, then the resulting tiling (attractor) $T(A, D)$ is self-similar. That is, $T(A, D)$ is the union of m smaller copies of itself. This phenomenon appears in Examples 1, 3 and 4, but not in Example 2. We can modify the tiles in Example 2 by introducing a change of basis matrix

$$B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}$$



a) b)

FIG. 9. *Four Tiling*

and defining a similarity mapping

$$h = BAB^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}.$$

We then iterate using the functions $f_j(z) = h^{-1}(z + Bd_j)$, $j = 1, 2, 3$. The attractor $T(A, f_B(D))$ is shown in Figure 10. However, the measure of the new tiling has

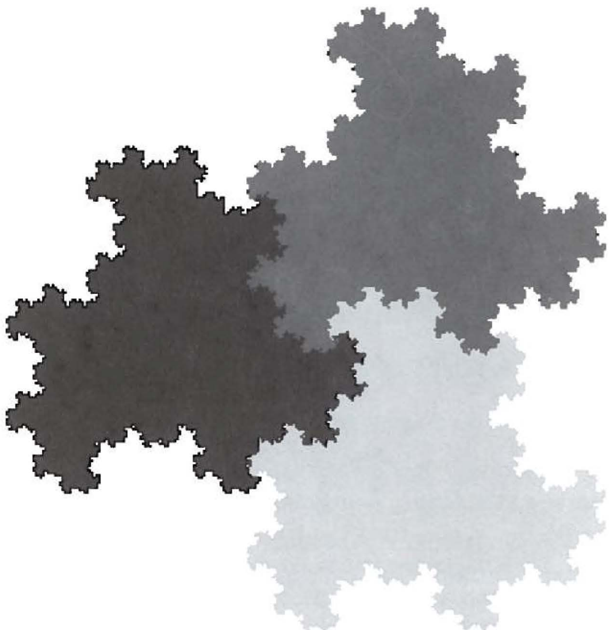


FIG. 10. *Terdragon from similarity map*

changed with this modification of the mappings. We now have

$$T(h, f_B(D)) = f_B(T(A, D)).$$

Therefore

$$\mu T(h, f_B(D)) = |\det B| \, \mu T(A, D)$$

So the tiling generated in Figure 10 has measure $\sqrt{3}/2$.

We can choose a matrix B such that $h = BAB^{-1}$ is a similarity map if the matrix A is diagonalizable over the complex numbers and its eigenvalues have equal modulus. (See [4] for details.)

A final example illustrates this technique.

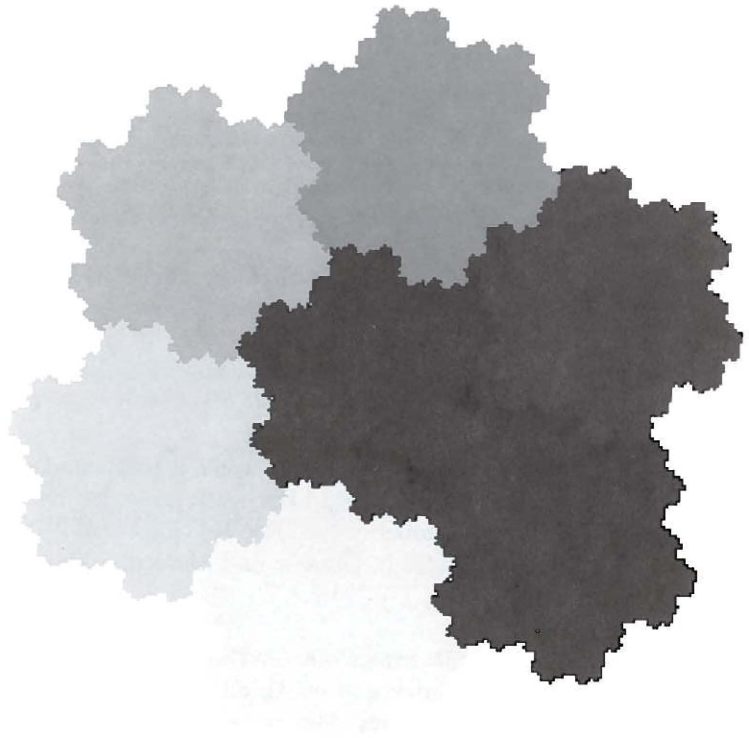


FIG. 11. *Seven tiling*

EXAMPLE 5. *Let* $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ *and*

$$D = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Since $Z[A, D] = \mathbb{Z}^2$, $\mu T(A, D) = 1$. *To generate the snowflake tiling shown in Figure 11 we must introduce the change of basis matrix* $B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$ *and define a similarity mapping* $h = BAB^{-1}$. *The tiling is generated using the iterated function* *system* $\{\varphi_i(z) = h^{-1}(z + Bd_i),\ d_i \in D, 1 \le i \le 7\}$. *Therefore* $\mu T(h, f_B(D)) = |\det B|\,\mu T(A, D) = \frac{\sqrt{3}}{2}$.

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EXPLORING FINITE-TIME BLOW-UP

DUFF CAMPBELL* AND JARED WILLIAMS*

1. Introduction. Given an initial value problem $dy/dt = f(t, y)$, $y(t_0) = y_0$, the Existence Theorem for Ordinary Differential Equations says that if $f(t, y)$ is continuous on an open rectangle R in the ty-plane containing the point (t_0, y_0), then there exists an $\epsilon > 0$ and a function $g(t)$ defined on the interval $(t_0 - \epsilon, t_0 + \epsilon)$ such that $g'(t) = f(t, g(t))$, and $g(t_0) = y_0$. While this theorem assures us that a solution exists, it does not assure us that it will exist for long. Most differential equations textbooks give examples where the conditions for the Existence Theorem are satisfied but where the solution "blows up" in finite time. This is generally demonstrated by analytically solving the differential equation. In this paper, we develop a method of determining whether the solution to an autonomous, first-order, scalar differential equation blows up in finite time *without* having to analytically solve the differential equation. In fact, our method expresses the blow-up time as an improper integral. Additionally, we use this result to develop a method for determining whether solutions to DEs go to zero in finite time.

We begin by defining the notion of finite time blow-up. Consider the following initial value problem (IVP):

$$\dot{P} = kP, \qquad P(0) = P_0 > 0, \qquad k > 0.$$

This can be solved using separation of variables to give $P(t) = P_0 e^{kt}$. Notice that this solution is defined for all $t \in \mathbf{R}$.

A variation of this is the IVP:

$$\dot{P} = kP^2, \qquad P(0) = P_0 > 0, \qquad k > 0.$$

Using separation of variables, we obtain:

$$P(t) = \frac{P_0}{1 - ktP_0}.$$

Observe that $P(t)$ has a vertical asymptote at $t = 1/kP_0$. This brings us to the following

Definition 1. *When a solution to an initial value problem "reaches infinity" in finite time, the solution is said to "blow up in finite time", or* **buift**. *That is, if $P(t)$ is a solution to $\dot{P} = f(t, P)$, $P(t_0) = P_0$, then*

$$P(t) \textbf{ buift} \iff \exists\, T > t_0 \text{ such that } \lim_{t \to T^-} P(t) = +\infty.$$

Generally, this is all that DE textbooks include in their discussions of **FTBU**. More specifically, the textbooks simply provide examples of DEs whose solutions blow up in finite time (**buift**), and they show that the solutions **buift** by solving the DE using separation of variables. This method of determining whether the solution to a DE **buift** requires solving the DE and expressing the solution as a function of t. An interesting question to ask is whether there exists a method of determining whether

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the solution to a DE **buift** *without* having to actually solve the DE; recall that many solutions of DEs cannot be expressed as elementary functions, and therefore another method would be necessary to determine whether such solutions **buift**. This paper examines the existence of such a method for the autonomous first order differential equation $\dot{P} = f(P)$.

2. Examples. First consider IVPs of the form:

$$\dot{P} = kP^n, \qquad P(0) = P_0 > 0, \qquad k > 0.$$

For these IVPs, we have:

$$\int P^{-n}\,dP = \int k\,dt.$$

If $0 < n < 1$ then let $u = -n + 1$ so $0 < u < 1$. Then, $P^u/u = kt + C$, so

$$P(t) = (ukt + P_0^u)^{\frac{1}{u}}.$$

Because $ukt + P_0^u > 0$ for $t > 0$, $P(t)$ is defined for all $t \geq 0$. Thus the solution does **not buift**.

If, on the other hand, $n > 1$, then let $u = n - 1$, so $u > 0$. Then $P^{-u} = -ukt + d$, where $d = P_0^{-u} > 0$, so

$$P(t) = \left(\frac{1}{-ukt + d}\right)^{1/u}.$$

This solution is only defined for $0 \leq t < d/uk$, and the solution **buift**.

Consider these results in dP/dt versus P space. For the case where $0 < n < 1$, the function $f(P) = kP^n$ is concave down, and thus there exists a linear function $g(P)$ (e.g. $g(P) = kP$) passing through the origin that "dominates" $f(P)$. (By this we mean that $g(P) > f(P)$ for all $P > P^*$ for some P^*. In this example, P^* could be any value greater than or equal to 1.)

For the case where $n > 1$, it can easily be shown that there does not exist a linear function $g(P)$ passing through the origin that "dominates" $f(P)$. Note that in the case where there exists a linear function through the origin which dominates kP^n, the solution to the IVP

$$\dot{P} = kP^n, \qquad P(0) = P_0 > 0$$

does **not buift**. Moreover, in the case where there does not exist a linear function through the origin which dominates kP^n, the solution to the DE does **buift**. Thus, it appears as though the solution to the IVP

$$\dot{P} = f(P), \qquad P(0) = P_0 > 0$$

might blow up in finite time if and only if the average rate of change of $f(P)$ is infinitely large. Based on the above discussion, it seems reasonable to formulate the following

CONJECTURE 1. *Given $\dot{P} = f(P)$, $P(0) = P_0$, $f(P) > 0$ for all $P \geq P_0$,*

$$P \textbf{ buift} \iff \lim_{x\to\infty} \frac{f(x) - f(P_0)}{x - P_0} = \infty.$$

If $f(P) = 0$ for some $P^* \geq P_0$, the solution to the DE cannot **buift**; rather, it will be bounded by the equilibrium solution P^*, and hence, it cannot even blow up in infinite time. Thus, the assumption that $f(P) > 0$ for all $P \geq P_0$ is included.

3. Testing the Conjecture. To test this conjecture, first consider the boundaries of **FTBU**, i.e., consider DEs whose solutions **buift** "slowly" or ones whose solutions increase rapidly without exhibiting **FTBU**. An example of the latter type of solution is $P(t) = e^{e^t}$.

An autonomous DE associated with this solution is: $\frac{dP}{dt} = P \ln P$. According to the conjecture, the solution will **buift** because

$$\lim_{x\to\infty} \frac{x\ln x - e\ln e}{x - e} = \lim_{x\to\infty} 1 + \ln x = \infty,$$

by l'Hopital's Rule. But it is clear that $P = e^{e^t}$ **does not buift**. Therefore, the conjecture is disproven. This seems to happen because the rate of change approaches infinity too "slowly" for **FTBU** to occur. But there does appear to be a connection between **FTBU** and convergence of reciprocals. Note that the solution to

$$\dot{P} = kP^n, \qquad P(0) = P_0 > 0, \qquad k > 0,$$

buift if and only if $n > 1$, and that, if $f(P) = P \ln P$, the solution does not **buift**. Further, note that the integral $\int_{P_0}^{\infty} (kP^n)^{-1} dP$ converges if and only if $n > 1$, and that $\int_{P_0}^{\infty} (P \ln P)^{-1} dP$ diverges. From this, there appears to be a relationship between **FTBU** of the solution to the IVP $\dot{P} = f(P)$, $P(0) = P_0 > 0$, and convergence of the integral $\int_{P_0}^{\infty} (f(P))^{-1} dP$. More specifically, it seems reasonable to formulate the following

THEOREM 2. *Given $\dot{P} = f(P)$, $P(0) = P_0$, $f(P) > 0$ for all $P \geq P_0$, then $P(t)$ "blows up" at time L if and only if $\int_{P_0}^{\infty} f(P)^{-1}\,dP$ converges to L.*

First, we have

LEMMA 3. *If $\dot{P} = f(P)$, $P(0) = P_0$, then*

$$\int_{P_0}^{P(T)} \frac{1}{f(u)}\,du = T$$

for all T for which the integral is defined.

Proof. Changing variables $u = P(t)$, $\dot{u} = \dot{P}$ in the integral

$$\int_0^T \frac{1}{f(P)} \frac{dP}{dt}\,dt = \int_0^T 1\,dt$$

gives

$$\int_{P_0}^{P(T)} \frac{1}{f(u)}\,du = \int_0^T 1\,dt = T. \quad \square$$

Proof. [of Theorem 2] Suppose $\dot{P} = f(P)$, $P(0) = P_0$. If P **buift**, then, by definition, there exists a finite T such that $\lim_{t\to T^-} P(t) = \infty$ and by the lemma

$$\int_{P_0}^{\infty} \frac{1}{f(P)}\,dP = \lim_{t\to T^-} \int_{P_0}^{P(t)} \frac{1}{f(P)}\,dP = \lim_{t\to T^-} t = T,$$

which is finite, so the integral converges.

For the converse, assume that $\int_{P_0}^{\infty} \frac{1}{f(P)}\,dP$ converges to L. Let A be a value of t such that $(A, P(A))$ exists. Then, from the lemma,

$$A = \int_{P_0}^{P(A)} \frac{1}{f(P)}\,dP < \int_{P_0}^{\infty} \frac{1}{f(P)}\,dP = L,$$

because $f(P) > 0$ for all $P \geq P_0$. So A is bounded by L. Notice also that $P(A) \to +\infty$ as $A \to L^-$. Hence P **buift**. Moreover, L represents the time at which P **buift**. ☐

Now consider the following example: Suppose $\dot{P} = P^2 - P$, $P(0) = P_0 > 1$. We want to know if the solution to this IVP blows up in finite time. We check $\int_{P_0}^{\infty}(P^2-P)^{-1}\,dP$ for convergence.

Clearly $(P^2-P)^{-1} < (P^2 - \frac{P}{P_0}P)^{-1} = P^{-2}P_0/(P_0-1)$ when $P > P_0$. Thus

$$\int_{P_0}^{\infty} \frac{1}{P^2-P}\,dP < \int_{P_0}^{\infty} \frac{P_0}{P_0-1}P^{-2}\,dP = \frac{1}{P_0-1}$$

so we know that $\int_{P_0}^{\infty}(P^2-P)^{-1}\,dP$ converges to a value less than $1/(P_0-1)$. Thus, by the theorem, we know $P(t)$ blows up before $t = 1/(P_0-1)$. To verify this result, we solve the DE using separation of variables:

$$\int \frac{dP}{P^2-P} = \int 1\,dt \implies P = \frac{1}{1-Ae^t}$$

From this, it is clear that $P(t)$ blows up when $1 - Ae^t = 0$, i.e., when

$$e^t = \frac{1}{A} = \frac{P_0}{P_0-1}, \qquad \text{or} \qquad t = \ln\left(\frac{P_0}{P_0-1}\right).$$

Thus, the theorem was correct in predicting that the solution will **buift**. To check whether or not the theorem gave a valid upper bound for the blow-up time of P, we compare $\ln(P_0/(P_0-1))$ with $1/(P_0-1)$. By using the inequality $1+x < e^x$ for all $x > 0$, it is easily shown that $\ln(P_0/(P_0-1)) < 1/(P_0-1)$ for all $P_0 > 1$ (which we assumed). Thus, the value obtained from the theorem, $1/(P_0-1)$, is a valid upper bound for the time of blow-up.

4. Consequences and an Application. This theorem is much more powerful for DEs that cannot be solved analytically; it allows us to utilize all the calculus integral comparison tests to determine whether the solution of an autonomous DE **buift**, and if the solution does **buift**, the theorem often allows us to find a time by which the solution will blow up.

Consider the IVP $\dot{y} = y^2(2+\cos(y))$, $y(0) = 1$. Though the IVP is separable, the solution cannot be written in terms of elementary functions. However, using Theorem 1, we see that the solution **buift** between $t = 1/3$ and $t = 1$:

$$\frac{1}{3} = \int_1^{\infty} \frac{1}{3y^2}\,dy < \int_1^{\infty} \frac{1}{y^2(2+\cos(y))}\,dy < \int_1^{\infty} \frac{1}{y^2}\,dy = 1.$$

A more striking example is the IVP $\dot{y} = e^{y^2}$, $y(0) = 1$. Comparison with $\dot{y} = 1 + y^2$ shows that the solution **buift**, but we can use Theorem 1 to assert that the solution will blow up at time

$$T = \int_1^{\infty} e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}(1 - \operatorname{Erf}(1)) \approx 0.139403.$$

Moreover, this theorem can be used to determine if and when solutions to autonomous DEs go to zero. Consider the autonomous DE $\dot{P} = f(P)$, $P(0) = P_0$. To determine whether the solution goes to zero in finite time, we should check its reciprocal for **FTBU**. More specifically, P will go to zero in finite time if and only if P^{-1} **buift**.

To find an efficient method for checking whether P^{-1} **buift**, first let $u = P^{-1}$. Then

$$\dot{u} = -\frac{1}{P^2}\dot{P} = -\frac{1}{P^2}f(P) = -u^2 f(u^{-1}).$$

We check whether u **buift**. From the theorem, u will **buift** if and only if

$$\int_{u_0}^{\infty} \frac{1}{-u^2 f(u^{-1})}\,du = \int_{\frac{1}{P_0}}^{\infty} \frac{1}{-u^2 f(u^{-1})}\,du$$

converges.

For an example, consider $P(t) = 2e^{-t} - 1$. An IVP to which this is the solution is $\dot{P} = -P - 1$, $P_0 = 1$. To determine if and when P goes to zero, we check the integral $\int_1^{\infty}(u+u^2)^{-1}\,du$ for convergence. Using partial fractions, this integral is found to converge to $\ln 2$. To test whether the corollary actually worked, we evaluate P at $t = \ln 2$; $P(\ln 2) = 2e^{-\ln 2} - 1 = 0$. Thus, the corollary accurately predicted when the solution to the autonomous DE would go to zero.

As a further example, this corollary can be applied to autonomous DEs that model logistic growth with harvesting. Consider the differential equation $\dot{P} = kP(1 - \frac{P}{N}) - H$, where H represents the harvesting rate and N represents the carrying capacity. Scaling P and t appropriately, one may assume that $k = N = 1$, so $f(P) = P(1-P) - H$. It is then easily verified that, if $H > 1/4$, then all solutions with $P(0) = P_0 > 0$ will eventually go to zero. With our corollary, though, we can do better: we can obtain a (rough) upper bound for the time at which the population will go to zero. It is given by

$$\int_{\frac{1}{P_0}}^{\infty} \frac{1}{-u^2 f(\frac{1}{u})}\,du = \int_{\frac{1}{P_0}}^{\infty} \frac{1}{Hu^2 - u + 1}\,du$$
$$< \int_{\frac{1}{P_0}}^{\infty} \frac{1}{Hu^2 - u + \frac{1}{4H}}\,du = \frac{1}{\frac{H}{P_0} - \frac{1}{2}}.$$

This upper bound is of course only good if $2H > P_0$, but it provides a good estimate when P_0 is small, or when H is close to $1/4$; furthermore, obtaining this estimate was fairly easy. A lower bound may be obtained similarly from the integral

$$\int_{\frac{1}{P_0}}^{\infty} \frac{1}{Hu^2 - u + 1}\,du > \int_{\frac{1}{P_0}}^{\infty} \frac{1}{Hu^2 + 1}\,du$$
$$= \frac{1}{\sqrt{H}}\left(\frac{\pi}{2} - \arctan\left(\frac{\sqrt{H}}{P_0}\right)\right).$$

The effort involved may be compared with that expended via the method of explicit solutions: using separation of variables and completing the square, we can see that the general solution to the DE above is

$$P(t) = \frac{1}{2} + \sqrt{H - \frac{1}{4}}\tan\left(C - t\sqrt{H - \frac{1}{4}}\right).$$

The initial condition shows that

$$C = \arctan\left(\frac{P_0 - \frac{1}{2}}{\sqrt{H - \frac{1}{4}}}\right),$$

so the particular solution to our IVP is

$$P(t) = \frac{1}{2} + \sqrt{H - \frac{1}{4}} \tan\left(\arctan\left(\frac{P_0 - \frac{1}{2}}{\sqrt{H - \frac{1}{4}}}\right) - t\sqrt{H - \frac{1}{4}}\right).$$

Finally, this solution equals zero when

$$t = \frac{1}{\sqrt{H - \frac{1}{4}}}\left(\arctan\left(\frac{P_0 - \frac{1}{2}}{\sqrt{H - \frac{1}{4}}}\right) + \arctan\left(\frac{1}{2\sqrt{H - \frac{1}{4}}}\right)\right).$$

5. Conclusion. Our method for determining whether finite time blowup occurs in solutions of first order, autonomous, scalar differential equations is more efficient than the method included in DE textbooks: it allows one to use the convergence of an improper integral, rather than the evaluation of an indefinite integral, to determine whether a solution blows up in finite time. If the solution does blow up in finite time, the integral comparison tests often allow us to obtain upper and lower bounds for the blowup time. Also, our method is applicable even for DEs whose solutions cannot be expressed as an elementary function of the independent variable, whereas the method included in DE textbooks is useless for such DEs. Finally, by considering finite time blowup of the reciprocal of solutions to DEs, we developed a method for determining whether solutions to DEs go to zero in finite time.

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SOLID CONSTRUCTIONS USING ELLIPSES

PATRICK HUMMEL*

Over two thousand years ago, the Greeks wondered whether certain geometric figures, lengths, and angles could be constructed using only a compass and a straightedge. The Greeks even developed nomenclature to describe how easily a construction problem could be solved. A problem was called 'plane' if it could be solved using only a compass and a straightedge. If a problem required one or more conic sections in addition to the compass and straightedge, it was called 'solid'. The Greeks suspected that many problems solvable using solid tools were non-planar but were unable to prove their suspicions.

More recently, an algebraic characterization of the set of points constructible from a compass and a straightedge has been found. A point (x, y) is constructible if and only if $x + iy$ lies in a subfield K of $\mathbb{C}$ such that there exists a finite sequence of fields, $\mathbb{Q} = K_0 \subset K_1 \subset \ldots \subset K_n = K$ such that the index $[K_{j+1} : K_j]$ at each step is 2 for all j. From this, it follows that problems such as constructing the cube root of an arbitrary length and trisecting an arbitrary angle are not planar in general. However, these problems are solid because they can be solved using conics in addition to a compass and a straightedge.

In [3], a characterization of the set of points constructible from conic sections is demonstrated. In particular, a complex number z is found to be conic-constructible if and only if z lies in a subfield K of $\mathbb{C}$ such that there exists a finite sequence of fields, $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that the index $[K_{j+1} : K_j]$ at each step is 2 or 3 for all j. In the same paper, Videla wonders if all points constructible from the set of conics are also constructible using only particular conic sections. He points out that all points constructible from parabolas, hyperbolas, and ellipses can also be constructed using only hyperbolas and parabolas and leaves whether such points are also constructible using only hyperbolas as an open question. In [2], it is shown that a single parabola is an equally powerful tool as the set of all conic sections. However, it is not possible to draw a parabola freehand. By contrast, ellipses can easily be sketched by simply placing two pins at the foci. This paper demonstrates that all points constructible from conics can also be constructed using only ellipses.

In using ellipses to solve construction problems, we are bound by certain restrictions governing what kind of ellipses can actually be drawn. For an ellipse to be constructible it must have constructible foci, and the sum of the distances between any point on the ellipse and the two foci must be constructible as well.

THEOREM 1. *Given an arbitrary constructible length, it is always possible to construct the cube root of the length using a single ellipse in addition to a compass and a straightedge.*

Proof. Suppose we draw an ellipse with center $(a, 0)$ and vertical major axis such that the ellipse goes through the points $(0, c)$ and (a, b), where a, b, and c are all constructible lengths such that $a^2 < b^2 - c^2$. Such an ellipse can be constructed because the foci lie at $(a, b\sqrt{\frac{b^2-c^2-a^2}{b^2-c^2}})$ and $(a, -b\sqrt{\frac{b^2-c^2-a^2}{b^2-c^2}})$, both of which are constructible, and the sum of the distances between a point on the ellipse and the two foci is $2b$, facts which are easy enough to verify.

We can use the compass to draw a circle centered at constructible point, (h, k)

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that intersects $(0,c)$. To show that the cube root of an arbitrary length, say r, can always be constructed using these tools, it suffices to show that there are always values that can be substituted for the constants such that the x-coordinate of one of the points of intersection of the circle and the ellipse is equal to $\sqrt[3]{r}$.

To demonstrate this, consider the equations for the ellipse and the circle, and find the x-coordinates of the points of intersection. Note that the ellipse can be represented as

$$\frac{(b^2-c^2)(x-a)^2}{b^2a^2}+\frac{y^2}{b^2}=1.$$

The equation for the circle is given by

$$(x-h)^2+(y-k)^2=h^2+(c-k)^2,$$

which we can rewrite as

$$x^2-2xh+y^2-2yk=c^2-2ck. \tag{1}$$

Rearranging the equation for the ellipse, we get

$$y=\pm\sqrt{c^2-\frac{(b^2-c^2)(x^2-2xa)}{a^2}},$$

and substituting this in for (1) gives

$$x^2-2xh+c^2-\frac{(b^2-c^2)(x^2-2xa)}{a^2}\mp 2k\sqrt{c^2-\frac{(b^2-c^2)(x^2-2xa)}{a^2}}=c^2-2ck,$$

$$\left(1-\frac{(b^2-c^2)}{a^2}\right)x^2+2\left(\frac{(b^2-c^2)}{a}-h\right)x+2ck=\pm 2k\sqrt{c^2-\frac{(b^2-c^2)(x^2-2xa)}{a^2}},$$

which simplifies to

$$x^3+4\frac{\left(\frac{b^2-c^2}{a}-h\right)}{1-\left(\frac{b^2-c^2}{a^2}\right)}x^2+\frac{4ck}{1-\left(\frac{b^2-c^2}{a^2}\right)}x+4\frac{\left(\frac{b^2-c^2}{a}-h\right)^2}{\left(1-\left(\frac{b^2-c^2}{a^2}\right)\right)^2}x+\frac{4k^2\left(\frac{b^2-c^2}{a^2}\right)}{\left(1-\left(\frac{b^2-c^2}{a^2}\right)\right)^2}x$$

$$=\frac{8k^2\left(\frac{b^2-c^2}{a^2}\right)-8ck\left(\frac{(b^2-c^2)}{a}-h\right)}{\left(1-\left(\frac{b^2-c^2}{a^2}\right)\right)^2}. \tag{2}$$

after squaring both sides and doing a little algebra.

To prove that there are always values for the constants that one can substitute in such that $\sqrt[3]{r}$ is a solution to the equation for any constructible length r, it suffices to show that there are constants for which (2) is of the form $x^3=r$. This is satisfied if the x^2 and the x terms on the left-hand side are zero, and the term on the right-hand side is equal to r, or if the following system of three equations is satisfied:

$$4\frac{\left(\frac{b^2-c^2}{a}-h\right)}{1-\frac{b^2-c^2}{a^2}}=0, \tag{3}$$

$$\frac{4ck}{1-\frac{b^2-c^2}{a^2}}+4\frac{\left(\frac{b^2-c^2}{a}-h\right)^2}{\left(1-\frac{b^2-c^2}{a^2}\right)^2}+\frac{4k^2\left(\frac{b^2-c^2}{a^2}\right)}{\left(1-\frac{b^2-c^2}{a^2}\right)^2}=0, \tag{4}$$

$$\frac{8k^2\left(\frac{b^2-c^2}{a}\right)-8ck\left(\frac{b^2-c^2}{a}-h\right)}{\left(1-\frac{b^2-c^2}{a^2}\right)^2}=r. \tag{5}$$

Equation (3) simplifies to

$$h=\frac{b^2-c^2}{a}. \tag{6}$$

Using this, we find equation (4) simplifies to

$$ck\left(1-\frac{b^2-c^2}{a^2}\right)+k^2\frac{(b^2-c^2)}{a^2}=0,$$

$$k\frac{(b^2-c^2)}{a^2}=c\left(\frac{b^2-c^2}{a^2}-1\right),$$

$$k=c\left(1-\frac{a^2}{b^2-c^2}\right). \tag{7}$$

With equations (6) and (7) in mind, we find (5) simplifies to

$$\frac{8c^2\left(1-\frac{a^2}{b^2-c^2}\right)^2\left(\frac{b^2-c^2}{a}\right)}{\left(1-\frac{b^2-c^2}{a^2}\right)^2}=r. \tag{8}$$

To demonstrate that there are always constants that satisfy this equation, introduce a temporary variable d such that

$$d=b^2-c^2. \tag{9}$$

Then, we can rewrite (8) as

$$\frac{8c^2\left(1-\frac{a^2}{d}\right)^2\left(\frac{d}{a}\right)}{\left(1-\frac{d}{a^2}\right)^2}=r,$$

$$c=\sqrt{\frac{ra\left(1-\frac{d}{a^2}\right)^2}{8d\left(1-\frac{a^2}{d}\right)^2}}. \tag{10}$$

Thus if we choose constructible lengths for a and d such that $a^2<d$ and $d>0$, we can use (10), (9), (6), and (7) to determine constructible lengths, c, b, k, and h such that the x-coordinate of the only other point of intersection is equal to $\sqrt[3]{r}$. Therefore, the cube root of any constructible length can always be constructed using an ellipse in addition to a compass and a straightedge. □

EXAMPLE 1. *One problem the Greeks were highly concerned with solving was the problem of doubling a cube, or constructing a cube with twice the volume of an already given cube. Though unsolvable using only a compass and straightedge, this problem can be solved using a single ellipse as an additional tool.*

Solving this problem is contingent on the ability to construct the cube root of 2. *Selecting* $a = 1$ *and* $d = 2$ *and using the equations mentioned in the previous proof, we find that* $\sqrt[3]{2}$ *can be constructed by drawing an ellipse with center* $(1, 0)$ *that passes through* $(0, \frac{\sqrt{2}}{2})$ *and* $(1, \sqrt{\frac{5}{2}})$, *and then using the compass to construct a circle centered at* $(2, \frac{\sqrt{2}}{4})$ *that intersects the ellipse at* $(0, \frac{\sqrt{2}}{2})$. *It is easy to verify that the* x*-coordinate of the other point of intersection is* $\sqrt[3]{2}$. *(See Figure 1.)*

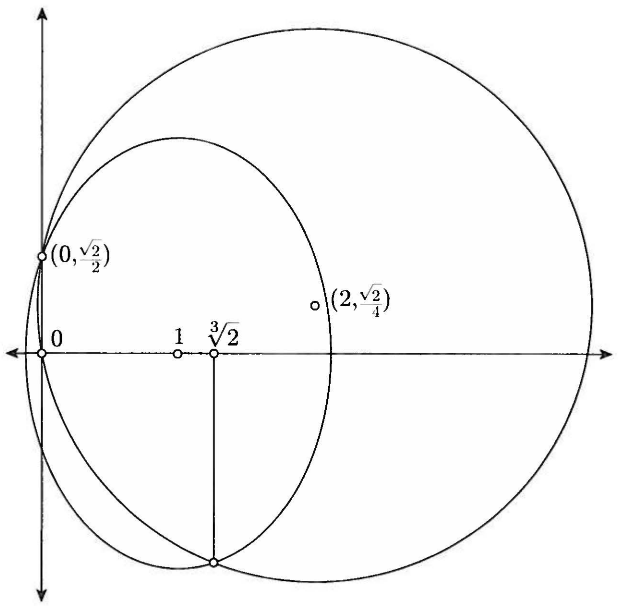

FIG. 1. *A solid construction of* $\sqrt[3]{2}$.

THEOREM 2. *Given an arbitrary constructible angle, it is always possible to trisect the angle using a single ellipse in addition to a compass and a straightedge.*

Proof. First consider the case where the angle is acute. We can trisect a given acute angle α if we can construct $x = e^{i\alpha/3}$. Note that

$$x^3 = e^{i\alpha},$$

$$x^{-3} = e^{-i\alpha},$$

(11) $$x^3 + x^{-3} = 2\cos\alpha.$$

Now let $\omega = x + x^{-1} = 2\cos(\alpha/3)$. Then

$$\omega^3 = x^3 + 3x + 3x^{-1} + x^{-3}.$$

This means that (11) can be rewritten in terms of ω as follows:

$$\omega^3 - 3\omega = 2\cos\alpha.$$

If we can always construct the solutions to this equation, we can always trisect a constructible acute angle. To demonstrate this, recall equation (2) for the x-coordinates of the points of intersection of a circle with an ellipse. If we use the same equation for this problem, we must demonstrate that there are always solutions to the following simultaneous set of equations:

(12) $$4\frac{(\frac{b^2-c^2}{a} - h)}{1 - (\frac{b^2-c^2}{a^2})} = 0,$$

(13) $$\frac{4ck}{1 - (\frac{b^2-c^2}{a^2})} + 4\frac{\left(\frac{b^2-c^2}{a} - h\right)^2}{\left(1 - (\frac{b^2-c^2}{a^2})\right)^2} + \frac{4k^2\left(\frac{b^2-c^2}{a^2}\right)}{\left(1 - (\frac{b^2-c^2}{a^2})\right)^2} = -3,$$

(14) $$\frac{8k^2\left(\frac{b^2-c^2}{a^2}\right) - 8ck\left(\frac{b^2-c^2}{a} - h\right)}{\left(1 - (\frac{b^2-c^2}{a^2})\right)^2} = 2\cos\alpha.$$

Equation (12) simplifies to

(15) $$h = \frac{b^2 - c^2}{a}.$$

Using this, we find equation (13) simplifies to

$$\frac{4ck}{1 - (\frac{b^2-c^2}{a^2})} + \frac{4k^2\left(\frac{b^2-c^2}{a^2}\right)}{\left(1 - (\frac{b^2-c^2}{a^2})\right)^2} = -3,$$

$$c = -\frac{3\left(1 - (\frac{b^2-c^2}{a^2})\right)}{4k} - \frac{k\left(\frac{b^2-c^2}{a^2}\right)}{1 - (\frac{b^2-c^2}{a^2})},$$

(16) $$c = -\frac{3(1 - \frac{d}{a^2})}{4k} - \frac{kd}{a^2 - d},$$

meaning c is a constructible length whenever $k > 0$, and $a^2 < d$. Also, equation (14) yields

$$\cos\alpha = \frac{4k^2\left(\frac{b^2-c^2}{a}\right)}{\left(1 - (\frac{b^2-c^2}{a^2})\right)^2},$$
$$= \frac{4k^2 d}{a\left(1 - \frac{d}{a^2}\right)^2},$$

$$k = \sqrt{\frac{a\cos\alpha\left(1 - \frac{d}{a^2}\right)^2}{4d}}.$$

Since α is acute, $\cos\alpha$ is positive, and we can choose constructible lengths, a and d such that k is a constructible length. Then, along with equations (16), (9), and (15), we can find lengths for the constants, c, b, and h that are constructible. This means that we can use the ellipse to trisect α.

It is quite simple to prove that α can be trisected if it is obtuse once we know it can be trisected if it is acute. Simply divide α up into a right angle and an acute angle. A right angle can be trisected using a compass and straightedge, and an acute angle can be trisected using a single ellipse, so if α is obtuse, it can also be trisected. □

EXAMPLE 2. *We can use the above result to construct a regular 9-gon, a figure not constructible using only a compass and straightedge. Constructing the regular 9-gon amounts to nothing more than constructing $\frac{2\pi}{9}$ and then using this to construct all the rest of the vertices. To construct $\frac{2\pi}{9}$, it suffices to construct $\frac{\pi}{9}$, or trisect $\frac{\pi}{3}$, and use $\frac{\pi}{9}$ to construct $\frac{2\pi}{9}$.*

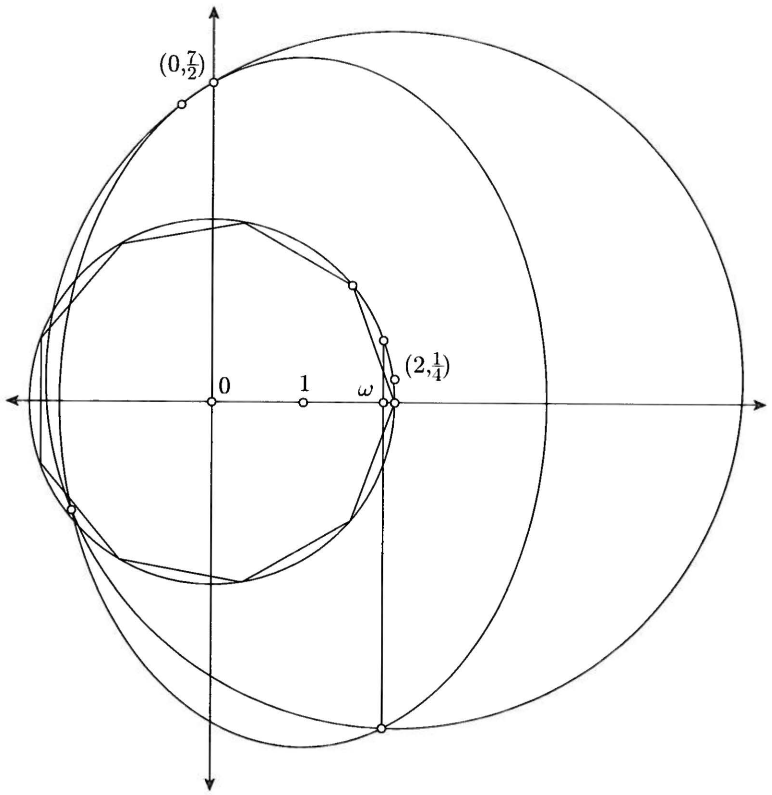

FIG. 2. *A solid construction of the regular 9-gon.*

To construct $\frac{\pi}{9}$, select $\alpha = \frac{\pi}{3}$, $a = 1$, and $d = 2$, and use the equations mentioned in the previous proof. First draw an ellipse centered at $(1,0)$ that goes through $(0,\frac{7}{2})$ and $(1,\frac{\sqrt{57}}{2})$. Then use the compass to draw a circle centered at $(2,\frac{1}{4})$ that intersects the ellipse at $(0,\frac{7}{2})$. Note that one of the points of intersection is $\omega = 2\cos(\frac{\pi}{9})$. By dropping a perpendicular from this point of intersection to the circle, $x^2+y^2=4$, we can then construct an angle of measure $\frac{\pi}{9}$, which we can in turn use to construct an angle of measure $\frac{2\pi}{9}$, the angle we needed to solve the problem. (See Figure 2.)

Recalling the first two theorems presented in this paper, we realize that we can construct the cube roots of arbitrary lengths, and we can trisect arbitrary angles. This implies that we can find the cube root of any constructible complex number. In turn, we know that a point (x, y) can be constructed from ellipses if and only if, when represented as a complex number, $x + iy$, it lies in a subfield K of $\mathbb{C}$ for which there exists a finite sequence of subfields, $K_0, K_1, \ldots, K_n$ satisfying $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n = K$, where $[K_{j+1} : K_j]$ is 2 or 3 for all j. (This is demonstrated in Videla's paper[3].) In light of Videla's result for the characterization of all points constructible from conics, we can use the above results to obtain the following theorem:

THEOREM 3. *All points constructible from conics, a compass, and a straightedge are also constructible using only ellipses in addition to a compass and a straightedge.*

Remark. The above results are contingent on the ability to use different ellipses for different constructions. Can all points constructible from conics also be constructed using the same ellipse in addition to a compass and straightedge? Or, perhaps a less restrictive question: We can draw ellipses using a piece of string strung between two foci. Is it possible to solve all these construction problems using the same string length for all ellipses?

I am deeply grateful and indebted to Dr. Arthur Baragar for his guidance in many aspects of this project.

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Dear Ghost of Gauss,

You are the master of all mathematical wisdom. I have studied and computed and cannot determine the answer to this question. Please, with your unsurpassed mathematical perspective, settle this question for me:

As I was going to St. Ives
I met a man with seven wives
The seven wives had seven sacks
The seven sacks had seven cats
The seven cats had seven kits
Kits, cats, sacks and wives
How many going to St. Ives?

Bothered and Bewildered.

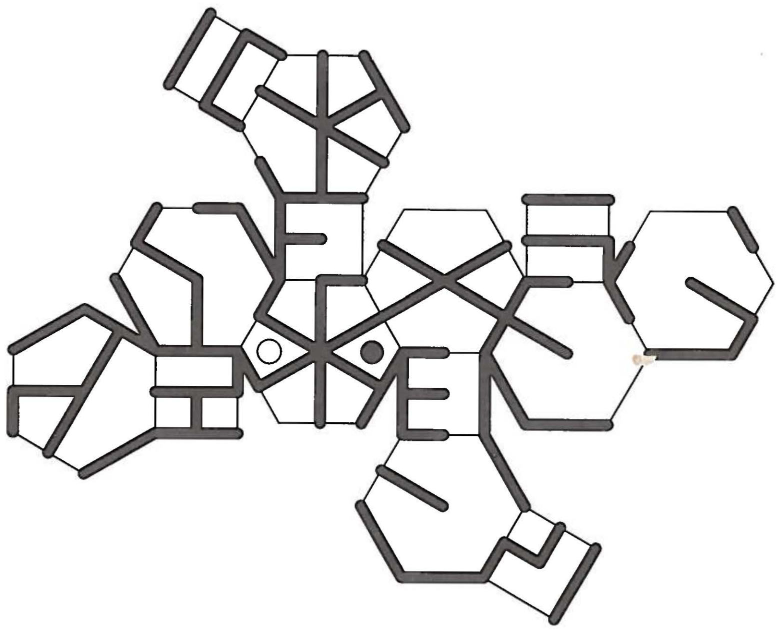
Dear B & B,

Those who have learned the formula for the Geometric Series

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

know the difference between going to St. Ives and going to the riverbank.

A good way to show that you know where you are going is to wear a ΠΜΕ keypin. Gold clad keypins are available at the national office at the price of $20 each. To purchase a keypin visit http://www.pme-math.org/Merchandise/pin.html G.G.



Polyhedral Maze contributed by Prof. Izidor Hafner from the University of Ljubljana (izidor.hafner@fe.uni-lj.si).

PASCAL MATRICES AND PARTICULAR SOLUTIONS TO DIFFERENTIAL EQUATIONS

JOHN M. ZOBITZ*

Abstract. In this paper any non-homogeneous differential equation with constant coefficients is reduced to a matrix equation $\vec{q} = \vec{c}P$. For the discussion, $\vec{q}$ represents a matrix of constant coefficients to the differential equation, $\vec{c}$ a matrix of arbitrary constants to the solution, and P is a lower triangular matrix with entries that are derivatives of the characteristic polynomial of the differential equation. After careful development, the task becomes finding a inverse to the matrix P. Interestingly enough, P is a generalized form of what is termed a Pascal Matrix, [1]. An inverse for certain conditions to such a matrix is proven to exist by the theorem given in the paper.

This approach was developed in earlier research, [2]. The advantage is that it uses fundamental concepts such as the linearity of the derivative, matrix multiplication, and product rule for derivatives. Furthermore a precise algorithm to solve a wide variety of differential equations is given with this approach.

1. Demonstration of the Method. How would one find a particular solution to the following differential equation?

$$y''' - y' + 3y = (1+5t)e^{4t} \tag{1}$$

We begin by defining an operator L so that $L = D^3 - D + 3$. In this instance D^k is the k^{th} derivative of Y with respect to t. Note that:

$$L(e^{4t}) = 63e^{4t} = p(4)e^{4t} \text{ where } p(a) = a^3 - a + 3.$$

Let us assume a particular solution of $y^* = (c_0 + c_1 t)e^{4t}$. Our the strategy will be to differentiate the particular solution and compare it to the right hand side of Equation 1.

We can also rewrite the differential equation in matrix format:

$$L(y) = \begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}1\\ t\end{bmatrix}e^{4t}.$$

By applying L to y^* we obtain the following:

$$L(y^*) = L\left(\begin{bmatrix}c_0 & c_1\end{bmatrix}\begin{bmatrix}1\\ t\end{bmatrix}e^{4t}\right) = \begin{bmatrix}c_0 & c_1\end{bmatrix} L\left(\begin{bmatrix}1\\ t\end{bmatrix}e^{4t}\right).$$

By the linearity of L, one can apply L to each entry in the column vector. On the other hand, by direct calculation, $L(te^{4t}) = (63t+47)e^{4t}$. These results can be subsequently written in matrix format:

$$\begin{bmatrix}c_0 & c_1\end{bmatrix}\begin{bmatrix}L(e^{4t})\\ L(te^{4t})\end{bmatrix} = \begin{bmatrix}c_0 & c_1\end{bmatrix}\begin{bmatrix}63 & 0\\ 47 & 63\end{bmatrix}\begin{bmatrix}1\\ t\end{bmatrix}e^{4t} = \begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}1\\ t\end{bmatrix}e^{4t}.$$

As a result, this problem easily reduces to a simple matrix equation:

$$\begin{bmatrix}c_0 & c_1\end{bmatrix}\begin{bmatrix}63 & 0\\ 47 & 63\end{bmatrix} = \begin{bmatrix}1 & 5\end{bmatrix}$$

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The inverse of the 2 x 2 matrix is easily calculated to find c_0 and c_1. So the particular solution to the differential equation is:

$$y^* = \left(\frac{-172}{3969} + \frac{5}{63}t\right) e^{4t}.$$

Certain questions naturally arise from this example.

1. How was the form of the particular solution determined?
2. If k and a are any real numbers, are there shortcuts to apply L to $t^k e^{at}$?,
3. Is there an easy way other than the usual techniques to construct the 2 x 2 matrix that we inverted?
4. Is that matrix always invertible? What is its inverse?

Such questions lie at the heart of the method developed in this paper. These questions and others will be answered if we consider the problem more generally.

2. Theory.

2.1. Transforming the Problem. Consider a n'th order differential equation with constant coefficients of the form:

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = (q_0 + q_1 t + \cdots + q_m t^m)e^{at}. \quad (2)$$

where $n,m \geq 0$ and a_k and q_k represent constant coefficients.

Define a linear operator $L(y)$ such that:

$$L(y) = \sum_{k=0}^{n} a_k D^k(y),$$

where D^k is the k^{th} derivative with respect to t. Thus the left hand side of Equation 2 is represented by $L(y)$. Furthermore, the right hand side of Equation 2 can be represented in matrix form:

$$(q_0 + q_1 t + \cdots + q_m t^m)e^{at} = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots & q_m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}.$$

So Equation 2 is transformed into the following equation:

$$L(y) = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots & q_m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}. \quad (3)$$

2.2. Finding a particular solution. We assume a particular solution y^* of the form $y^* = (c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m)e^{at}$. We do this by selecting the order of the polynomial $c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m$ to be of the same order as the polynomial on the right hand side of Equation 3. The plan is to compute $L(y^*)$ and then compare it

to the right hand side of Equation 3. This will allow us to determine the coefficients c_i.

Writing y^* in matrix format we can see that:

$$L\left(\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}\right) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix} L\left(\begin{bmatrix} e^{at} \\ te^{at} \\ t^2 e^{at} \\ \vdots \\ t^m e^{at} \end{bmatrix}\right)$$

$$= \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix} \begin{bmatrix} L(e^{at}) \\ L(te^{at}) \\ L(t^2 e^{at}) \\ \vdots \\ L(t^m e^{at}) \end{bmatrix}. \quad (4)$$

Now we develop a formula for $L(t^k e^{at})$ where $0 \leq k \leq m$. Observe that:

$$L(e^{at}) = \sum_{k=0}^{n} a_k D^k(e^{at}) = \sum_{k=0}^{n} a_k a^k e^{at} = e^{at} \sum_{k=0}^{n} a_k a^k = p(a)e^{at}$$

where $p(a) = \sum_{k=0}^{n} a_k a^k$.

Note that $p(a)$ is the characteristic polynomial to the differential equation. To find higher derivatives of $L(e^{at})$ we note two things. First, by considering $L(e^{at})$ as a function of a and t, we can utilize the fact that in this case the order of differentiation of mixed partial derivatives can be interchanged. Using this we see:

$$L(t^k e^{at}) = L\left(\frac{\partial^k (e^{at})}{\partial a^k}\right) = \frac{\partial^k (L(e^{at}))}{\partial a^k} = \frac{\partial^k (p(a)e^{at})}{\partial a^k}.$$

Second, from [3] we can invoke Leibnitz's Rule for higher derivatives of the product of two functions u and v:

$$(uv)^{(n)} = uv^{(n)} + \binom{n}{1} u' v^{(n-1)} + \cdots + vu^{(n)} = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}.$$

Thus, to calculate $L(t^k e^{at})$ we apply this rule to $\frac{\partial^k (p(a)e^{at})}{\partial a^k}$ and simplify:

$$\begin{aligned} L(t^k e^{at}) &= \frac{\partial^k (p(a)e^{at})}{\partial a^k} = \sum_{l=0}^{k} \binom{k}{l} [p(a)]^{(k-l)} [e^{at}]^{(l)} \\ &= \sum_{l=0}^{k} \binom{k}{l} p^{(k-l)} t^l e^{at} = e^{at} \sum_{l=0}^{k} \binom{k}{l} p^{(k-l)} t^l, \end{aligned} \quad (5)$$

where $p^{(k-l)}$ signifies the $k - l^{th}$ derivative of P with respect to a.

Note that $\sum_{l=0}^{k} \binom{k}{l} p^{(k-l)} t^l$ is just a polynomial with coefficients involving derivatives of P. By writing out the terms, we can use a $(m+1) \times (m+1)$ matrix P to represent Equation 5.

Let P_{k+1} represent the $k+1^{th}$ row of P. Naturally we would like to organize P so that $L(e^{at}) = L(t^0 e^{at}) = p(a)e^{at}$ corresponds to entry P_{11} of the matrix. Since k ranges from 0 to m, $L(t^k e^{at})$ will correspond to row $k+1$. Similarly, we order the columns by the power of t in Equation (5) which is L.

Note then that P is a lower triangular matrix, so $P_{kl} = 0$ for $k < l$, Thus,

$$P_{k+1,l+1} = \begin{cases} \binom{k}{l} p^{(k-l)} & k \geq l \\ 0 & k < l \end{cases} \text{ where } 0 \leq k, l \leq m \tag{6}$$

By writing the powers of t as a column vector we have the following:

$$L\left(\begin{bmatrix} e^{at} \\ te^{at} \\ t^2 e^{at} \\ \vdots \\ t^m e^{at} \end{bmatrix}\right) = \begin{bmatrix} p & 0 & 0 & \cdots & 0 \\ p' & p & 0 & \cdots & 0 \\ p'' & 2p' & p & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p^{(m)} & \binom{m}{1} p^{(m-1)} & \cdots & \binom{m}{m-1} p' & p \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at} \tag{7}$$

Set $\vec{c} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots & q_m \end{bmatrix}$.

Comparing Equations 3 and 7 we obtain:

$$L(y_p) = \vec{c}P \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at} = \vec{q} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}. \tag{8}$$

Thus Equation 8 reduces to the matrix equation:

$$\vec{q} = \vec{c}P. \tag{9}$$

P is a general from of a Pascal Matrix, a lower triangular matrix with entries that correspond to Pascal's Triangle. Reference [1] only discusses Pascal Matrices with integer entries. In this situation the entries of P are higher order derivatives of a function. With [1] as a guide, we can generalize an inverse to such matrices to find an inverse for P, which is what we need to solve the differential equation.

2.3. Finding a Solution. In order to solve for $\vec{c}$, we need to find P^{-1}. In order for P^{-1} to exist, $\det(P)$ must be nonzero. Since P is a lower triangular matrix, $\det(P) = [p(a)]^m$. For the moment, assume that $p(a) \neq 0$. (The case $p(a) = 0$ will be considered later.)

THEOREM 1. *Let P be as in Equation 6 and assume $p(a) \neq 0$. Let*

$$Q_{k+1,l+1} = \begin{cases} \binom{k}{l}\left(\frac{1}{p}\right)^{(k-l)} & k \geq l \\ 0 & k < l \end{cases} \tag{10}$$

Then $P^{-1} = Q$.

Proof. It is clear from properties of lower triangular matrices that:

$$(PQ)_{k+1,l+1} = \begin{cases} 0 & k < l \\ 1 & k = l \end{cases}$$

Because when $k < l$ the matrix entry will be zero in any case. If $k = l$, then $k - l = 0$ and $p^{(k-l)} = p$. Similarly, $(1/p)^{(k-l)} = 1/p$, and the product of the two is 1.

Now we need to show that $(PQ)_{k+1,l+1} = 0$ if $k > l$. Suppose $k > l$. Then $k = l + q$ for some $q > 0$. We see that:

$$(PQ)_{k+1,l+1} = \sum_{r=0}^{q} P_{l+q+1,l+r+1} Q_{l+r+1,l+1}$$

$$= \sum_{r=0}^{q} \binom{l+q}{l+r}\binom{l+q}{l} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}. \tag{11}$$

Expanding the binomial terms, Equation 11 is equivalent to:

$$(PQ)_{k+1,l+1} = \sum_{r=0}^{q} \frac{(l+q)!(l+r)!}{(l+r)!(q-r)!r!l!} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}.$$

If we multiply both the numerator and denominator by $q!$ and factor out the terms not dependent on r we obtain:

$$(PQ)_{k+1,l+1} = \frac{(l+q)!}{l!q!} \sum_{r=0}^{q} \frac{q!}{(q-r)!r!} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}. \tag{12}$$

or:

$$(PQ)_{k+1,l+1} = \binom{l+q}{q} \sum_{r=0}^{q} \binom{q}{r} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}.$$

Since $k = l + q$ we have,

$$(PQ)_{k+1,l+1} = \binom{k}{q} \sum_{r=0}^{q} \binom{q}{r} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}. \tag{13}$$

Using Leibnitz's Rule for Higher Derivatives of Products once more, Equation 13 becomes:

$$(PQ)_{k+1,l+1} = \binom{k}{q} \sum_{r=0}^{q} \binom{q}{r} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)} = \binom{k}{q}\left(\frac{1}{p}p\right)^{(q)}.$$

Since $q > 0$, $\binom{k}{q}\left(\frac{1}{p}\cdot p\right)^{(q)} = \binom{k}{q}(1)^{(q)} = 0$. Thus, when $k > l$, $(PQ)_{k+1,l+1} = 0$.

So, $(PQ)_{k+1,l+1} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$ Showing that $P^{-1} = Q$. □

For our purposes p is specified as the characteristic polynomial of the differential equation, yet this theorem generalizes the results of [2] to include functions, instead of integers, as entries in the Pascal Matrix.

Now that P^{-1} is found, Equation 9 can be solved and the coefficients of the vector $\vec{c}$ can be determined. Thus, the particular solution y^* to the differential equation is obtained, that is:

$$\vec{c} = \vec{q}P^{-1} \text{ and } y^* = \vec{c}\vec{t}e^{at}. \tag{14}$$

A good exercise would be to use the methods described and apply them to the initial example. As one can see, we used the same method outlined thus far to solve the first example.

2.4. Adjusting the Method. The restriction that $p(a) \neq 0$ is a limitation. Note that finding P^{-1} hinged on the assumption that P has a nonzero determinant. Consider the following differential equation:

$$y'' - 16y = (1+t^2)e^{4t}. \tag{15}$$

Applying our method to this differential equation reveals that $p(4) = 0$. This gives a matrix P with 0's along the main diagonal, which makes its determinant zero. Thus the technique seems to fail.

If $p^{(j)}(a) = 0$ for all $0 \leq j < q < m$, this suggests that y_h, the homogenous solution to Equation 2 is:

$$\sum_{j=0}^{q-1} r_j t^j e^{at} \text{ where } r_j \text{is a constant.}$$

A lower triangular matrix can still be constructed to determine the particular solution. In this case we assume a particular solution of the form:

$$(c_q t^q + c_{q+1}t^{q+1} + \cdots + c_m t^m + \cdots + c_{q+m}t^{q+m})e^{at}.$$

To find the value of each c_i we can follow a similar process outlined previously. Since $p^{(j)}(a) = 0$ for $j < q$, the corresponding entries in Equation 7 will be zero. As a result, Equation 7 reduces to a $(m+1) \times (m+1)$ lower triangular matrix:

$$\begin{bmatrix} p^{(q)} & 0 & 0 & \cdots & 0 \\ p^{(q+1)} & \binom{q+1}{1}p^{(q)} & 0 & \cdots & 0 \\ p^{(q+2)} & \binom{q+2}{1}p^{(q+1)} & \binom{q+2}{2}p^{(q)} & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p^{(q+m)} & \binom{q+m}{1}p^{(q+m-1)} & \cdots & \binom{q+m}{m-1}p^{(q+1)} & \binom{q+m}{m}p^{(q)} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}.$$

Thus, we can extend our previous formulation of matrix P to the following:

$$P_{k+1,l+1} = \begin{cases} \binom{q+k}{l}\left(p^{(q)}\right)^{(k-l)} & k \geq l \\ 0 & k < l \end{cases}, \tag{16}$$

where $0 \leq k, l \leq m$ and $p^{(j)} = 0$ for $j < q$. Since $p^{(q)} \neq 0$, $\det(P) \neq 0$ and P^{-1} exists.

Returning to Equation 15, let's assume a particular solution of the form:

$$y^* = (c_1 t + c_2 t^2 + c_3 t^3)e^{4t}$$

Since $p'(4) = 8$, by emphasizing the binomial coefficients, we see that this leads to a matrix P:

$$\begin{bmatrix} 1\cdot 8 & 0 & 0 \\ 1\cdot 2 & 2\cdot 8 & 0 \\ 1\cdot 0 & 3\cdot 2 & 3\cdot 8 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 16 & 0 \\ 0 & 6 & 24 \end{bmatrix}.$$

One may innocently think that the form of P^{-1} would be similar to (10). Unfortunately, this is not correct. The key to finding an P^{-1} rested in the fact that P had the form of a Pascal Matrix. Equation 16 certainly has entries from Pascal's Triangle

yet that is not a strong enough condition to guarantee that in this case P^{-1} can be generalized. As a result, we need to find P^{-1} from standard techniques.

For completion, P^{-1} for Equation 15 is given by:

$$\begin{bmatrix} \frac{1}{8} & 0 & 0 \\ -\frac{1}{64} & \frac{1}{16} & 0 \\ \frac{1}{256} & -\frac{1}{64} & \frac{1}{24} \end{bmatrix}.$$

So the particular solution to Equation 15 is:

$$y^* = \left(\frac{33}{256}t - \frac{1}{64}t^2 + \frac{1}{24}t^3\right)e^{4t}.$$

2.5. Comments. If the order n of the differential equation is less than the order m of the row vector $\vec{q}$, then $p^{(j)}(a) = 0$ for $n \leq j \leq m$. Due to the restriction that $p(a) \neq 0$, it will never be the case that P will be a zero matrix. In fact, if $m = 0$, the particular solution is quite simple:

$$y^* = \frac{q_0}{p(a)e^{at}} \text{when } m = 0.$$

As Gollwitzer remarks, this method can be used to solve a wide variety of non-homogeneous equations. If faced with a trigonometric equation on the right-hand side, one can use Euler's identity and set $a = i\omega$. The final solution will either be the real or the imaginary part of the particular solution. Table 1 summarizes these adjustments to Equation 14:

Right hand side of Equation 2	Adjustment to Equation 14
$q_0 + q_1 t + \cdots + q_m t^m$	$a = 0$
$(q_0 + q_1 t + \cdots + q_m t^m)\sin(\omega t)$	$a = i\omega$, Im(Equation 14).
$(q_0 + q_1 t + \cdots + q_m t^m)\cos(\omega t)$	$a = i\omega$, Re(Equation 14).

TABLE 2.1
Adjustments to Equation 14.

3. Conclusions. Non-homogeneous differential equations with constant coefficients arise frequently in physics, chemistry, and engineering. For example, forced motion of a pendulum and LRC circuits generate such differential equations. In practice, this method could be applied to many cases encountered by a physicist, chemist, biologist, as well as a mathematician.

The following procedure can be applied to solve most differential equations of the form given in Equation 2:

1. Identify $p(a)$, the characteristic polynomial and m, the number that determines the size of P.
2. Construct $Q = P^{-1}$ from (10).
3. Multiply $\vec{q}$ and P^{-1} to find $\vec{c}$, the coefficients to the particular solution.
4. Make necessary adjustments to Table 1 if needed.

As it can be seen, this procedure could be implemented by a computer program. In general, computers could solve Equation 2 with less time using this method than with a method such as undetermined coefficients because this method utilizes matrices, which generally take less computational time.

4. Acknowledgements. Many thanks to my advisor, Dr. Shobha Deshmukh, for bringing this project to my attention and to her gentle guidance and patience while working through it.

Thanks to Dr. Jennifer Galovich for her careful revisions of this work. Her editing skills made this more readable.

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[1] Call G. S. and Vellman D. J., *Pascal's Matrices,* American Mathematical Monthly, April 1993, pp 372–376.

[2] Gollwitzer H., *Matrix Patterns and Undetermined Coefficients,* College Mathematics Journal, Vol. 25, No. 5, pp 445–448, 1994.

[3] Speigel M. R. and Liu J., "Mathematical Handbook of Formulas and Tables," 2nd ed, McGraw Hill, 1999.

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The Richard V. Andree Awards. The Richard V. Andree Awards are given annually to the authors of the papers, written by undergraduate students, that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the Pi Mu Epsilon Journal in the past year.

Until his death in 1987, Richard V. Andree was Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon for many years and in a variety of capacities: as President, as Secretary-Treasurer, and as Editor of this Journal.

The awards for papers appearing in 2002 are announced on the next page. The officers and councilors of the Society congratulate the winners on their achievements and wish them well for their futures.

2002 2002

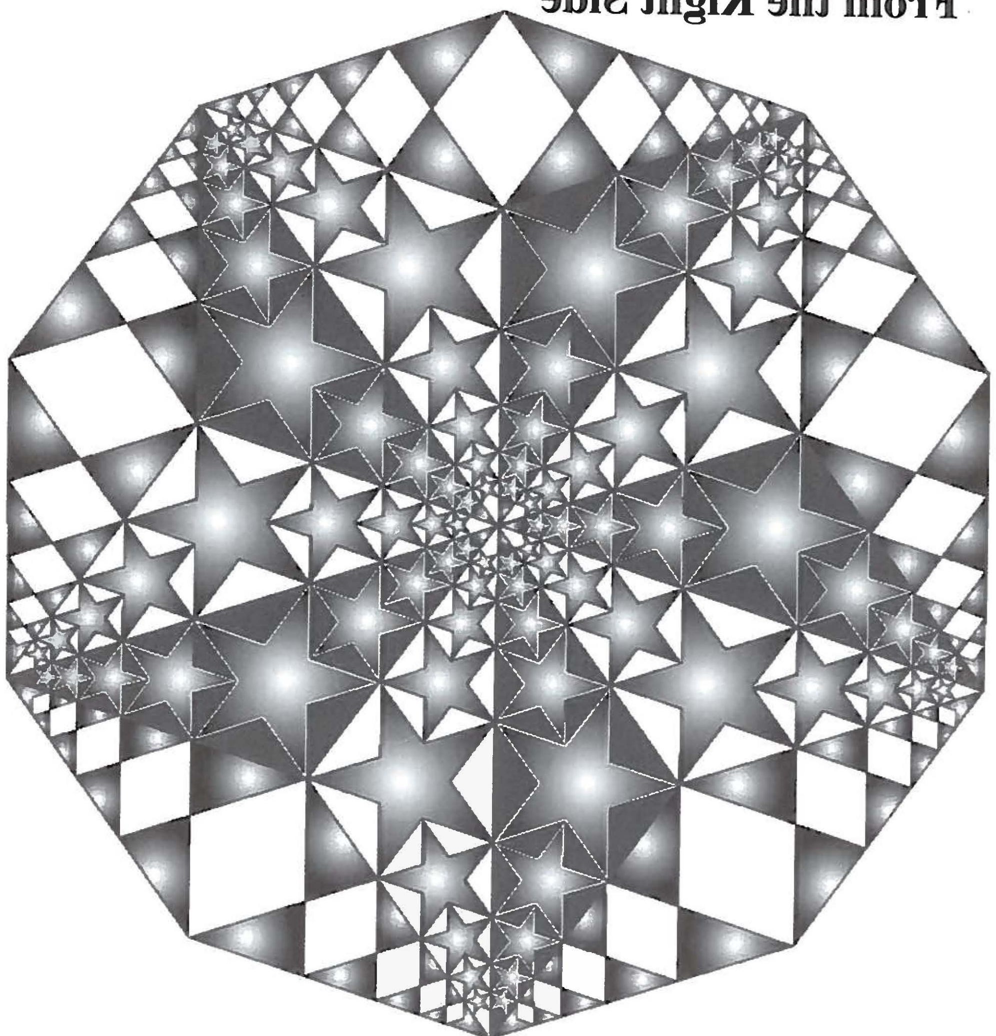
The Richard V. Andree Awards

A. Douglass, C. Fitzgerald and S. Mihalik,
"Expected Areas of Randomly Generated Triangles",
Pi Mu Epsilon Journal, Vol. 11, No. 7, Fall 2002.

Kimberly L. Patti,
"Randomly Generating a Finite Group",
Pi Mu Epsilon Journal, Vol. 11, No. 6, Spring 2002.

2002 2002

From the Right Side



Stardust, M. T. Krašek, 1999.

In *Stardust* we see harmony between light and dark, large and small, near and far, motion and rest. Matjuska Teja Krašek holds a B.A. degree in painting from Arthouse-College for Visual Arts, Ljubljana, and is a freelance artist who lives and works in Ljubljana, Slovenia. Her works in acrylic on canvas are seen in exhibitions around the world and several are permanently displayed in the mathematics department of the University of Ljubljana, where Teja regularly attends seminars and colloquia, since she sees symmetry in all its various forms as a linking concept between art and science. Some of her computer graphics can be viewed at

http://mitpress.mit.edu/e-journals/Leonardo/gallery/gallery331/homageescher.htm

The ΠME *Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.*

ΠME Journal, Vol. 11, No. 8, pp 447–456, 2003.

PROBLEM DEPARTMENT

EDITED BY MICHAEL MCCONNELL, AND JON A. BEAL

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Michael McConnell, 840 Wood Street, Mathematics Department, Clarion University, Clarion, PA 16214, or sent by email to mmcconnell@clarion.edu. Electronic submissions using LaTeX are encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by December 1, 2003. Solutions identified as by students are given preference.

Problems for Solution.

1052. *Proposed by Peter A. Lindstrom, Batavia, NY.*

The previous Problem Editor, Clayton DODGE, was a GREAT EDITOR. Solve the following addition alphametic in base ten:

$$\begin{array}{r} D\,O\,D\,G\,E \\ \underline{G\,R\,E\,A\,T} \\ E\,D\,I\,T\,O\,R \end{array}$$

1053. *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*

Find exactly

$$\int_0^\infty \frac{u}{e^u+1}\,du \quad \text{and} \quad \int_0^\infty \frac{u^3}{e^u+1}\,du.$$

1054. *Proposed by Ronald Kopas, Clarion, PA.*

Let $a_1, a_2, \ldots, a_n$ be integers such that $0 \le a_1 \le a_2 \le \ldots \le a_n$. If

$$\max_{k>j}\left\{\frac{a_k - a_{j-1}}{k-j}\right\} < \min_{k>j}\left\{\frac{a_k - a_{j+1}}{k-j}\right\}$$

then there exists m and b such that $a_i = [\![mi + b]\!]$ for all i.

1055. *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*

Find a function $f(x) \in \mathbf{C}^\infty(-\infty, \infty)$, such that there are exactly three different solutions for $f(x) = k$ for all $k \in \mathbf{R}$. In other words, any horizontal line in the XY-plane will intersect the plot of $f(x)$ at exactly three places.

1056. *Proposed by William Chau, SoftTechies Corp., East Brunswich, NJ.*

Given a positive integer n, take the sum of its digits to obtain a different number, then take the sum of the digits of the new number to obtain yet another number, and so on until the remaining number has only one digit. We call the one digit number the digital root of n. Taking the digital roots of the first five even perfect numbers 6, 28, 496, 8128, and 33550336, we found that they are 6, 1, 1, 1, and 1, respectively. Is it true that all even perfect numbers except 6 have digital root 1?

1057. *Proposed by Mark Snavely, Mathematics Department, Carthage College, Kenosha, Wisconsin.*

Many book include exercises similar to the following example.

Prove using induction that 4 divides $5^n - 1$ for all $n \in \mathbb{N}$.

For natural numbers p, q and r, show that r divides $p^n - q$ for all $n \in \mathbb{N}$ if and only if r divides $q^n - p$ for all $n \in \mathbb{N}$. [Hint: As a first step, characterize all natural numbers p, q and r such that r divides $p^n - q$ for all $n \in \mathbb{N}$.]

1058. *Proposed by Peter A. Lindstrom, Batavia, NY.*

Suppose that $\triangle ABC$ has an interior point P. Let D, E, and F be points on sides AB, BC, and CA, respectively, so that $PD \perp AB$, $PE \perp BC$, and $PF \perp CA$. Let $|AB| = x$, $|BC| = y$, $|CA| = z$, $|AD| = a$, $|BE| = b$ and $|CF| = c$.

1. Show that $(x-a)^2 + (y-b)^2 + (z-c)^2 = a^2 + b^2 + c^2$.
2. Show that if $\triangle ABC$ is an equilateral triangle, then $a + b + c = \frac{1}{2}$(perimeter of $\triangle ABC$).

1059. *Proposed by Peter A. Lindstrom, Batavia, NY.*

Student solutions especially solicited

Every even perfect number is of the form $2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are primes. If $X = 2^{p-1}(2^p - 1)$ is a perfect number, show that

$$\prod_{d|X} d = X^p.$$

1060. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abingtion PA.*

Suppose $\triangle ABC$ is an equilateral triangle. The points D, E, and F are on AB, BC and CA respectively such that $|AD| = |BE| = |CF|$. Show that the circumcircles of $\triangle ABC$ and $\triangle DEF$ are concentric.

1061. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abingtion PA.*

Let

$$x = \sum_{k=0}^{n} 2^k \binom{2n+1}{2k+1}.$$

Then $\dfrac{x^2-1}{2}$ is the product of two consecutive whole numbers.

Solutions.

1034. [Spring 2003]*Proposed by Norman Schauiuberger, Douglaston, New York.*

Let $a, b, c \in \mathbf{Z}^+$. Show that

$$\left(\frac{a+b+c}{3}\right)^{a+b+c} \geq \left(\frac{a+b}{2}\right)^{c} \left(\frac{b+c}{2}\right)^{a} \left(\frac{c+a}{2}\right)^{b}$$

Solution by **Murray S. Klamkin**, *University of Alberta, Edmonton, Alberta, Canada.*

More generally we will show that for positive numbers $a_1, a_2, \ldots, a_n$ with sum S that

$$\left(\frac{S}{n}\right)^S \geq \left(\frac{S-a_n}{n-1}\right)^{a_n} \left(\frac{S-a_{n-1}}{n-1}\right)^{a_{n-1}} \cdots \left(\frac{S-a_1}{n-1}\right)^{a_1}.$$

First we take the S^{th} root of both sides. Then by the weighted AM-GM inequality, it suffices to show that

$$\left(\frac{S}{n}\right) \geq \frac{a_n(S-a_n) + a_{n-1}(S-a_{n-1}) + \cdots + a_1(S-a_1)}{(n-1)S}$$

which reduces after some algebra to

$$\sum_{i,j=1}^{n} (a_i - a_j)^2 \geq 0.$$

Also solved by **Ovidiu Furdui**, Kalamazoo, MI, **Rex H. Wu**, Brooklyn, NY and by the **Proposer**

1035. [Spring 2002] *From a problem proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College.*

It occasionally happens that mistake can have a serendipitious outcome, as in the case of this problem. The problem, as originally proposed, is now problem 1061 in the problems for solution this issue. We incorrectly printed x as

$$x = \sum_{k=1}^{2n+1} 2^{k-1} \binom{2n+1}{k}.$$

A number of astute readers noticed that, while, for this x, $\dfrac{x^2-1}{2}$ doesn't need to be the product of consecutive whole numbers, $\dfrac{x^2-1}{4}$ does. We offer our apologies to the proposer for the mistake. However, it did provide a second problem for our readers to tackle.

The solutions below use the x defined above and show that $\dfrac{x^2-1}{4}$ is a product of two consecutive whole numbers.

Solution (1) by **Ellen M. Ellis** *and* **Tracey M. Hagedorn**, *students, Angelo State University, San Angelo, TX.*

By the Binomial Theorem we have that

$$\begin{aligned} x &= \sum_{k=1}^{2n+1} 2^{k-1} \binom{2n+1}{k} \\ &= \frac{1}{2} \sum_{k=0}^{2n+1} 2^k \binom{2n+1}{k} - \frac{1}{2} \\ &= \frac{1}{2}(1+2)^{2n+1} - \frac{1}{2} \\ &= \frac{3^{2n+1}-1}{2}. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{4}(x^2-1) &= \frac{1}{4}(x-1)(x+1) \\ &= \frac{1}{4}\left(\frac{3^{2n+1}-1}{2} - 1\right)\left(\frac{3^{2n+1}-1}{2} + 1\right) \\ &= \left(\frac{3^{2n+1}-3}{4}\right)\left(\frac{3^{2n+1}+1}{4}\right) \end{aligned}$$

and we must show that $\left(\frac{3^{2n+1}-3}{4}\right)$ and $\left(\frac{3^{2n+1}+1}{4}\right)$ are both whole numbers and consecutive.

By modular arithmetic, we have that

$$\begin{aligned} 3^{2n+1} &\equiv (-1)^{2n+1} \\ &\equiv -1 \pmod 4. \end{aligned}$$

So $\left(\frac{3^{2n+1}-3}{4}\right)$ is always a whole number.
Further, since

$$\left(\frac{3^{2n+1}+1}{4}\right) - 1 = \left(\frac{3^{2n+1}-3}{4}\right),$$

the two numbers are consecutive.

Solution (2) by **Harry Sedinger**, *St. Bonaventure University, St. Bonaventure, NY.*

We show that $\frac{1}{4}(x^2-1)$ is the product of two consectutive whole numbers. The first term for the sum for x is odd, while the rest of the terms are even, so x is odd. It follows that $x+1$ and $x-1$ are both even, and hence $\frac{1}{2}(x+1)$ and $\frac{1}{2}(x-1)$ are consecutive whole numbers. Their product is $\frac{1}{4}(x^2-1)$.

Also solved by **Ovidiu Furdui**, student, Western Michigan Unversity, **Steven Gendler**, Clarion University, **Murray S. Klamkin**, University of Alberta, **William Peirce**, Rangeley, Maine and **Rex H. Wu**, Brooklyn, NY.

1036. [Spring 2003] *Proposed by Shiva Saksena, Univ. of North Carolina at Wilmington, Wilmington, North Carolina.*

Student solutions solicited

Let $f(x) = \prod_{i=0}^{\infty}(1+x^{2^i})$, find c such that $\int_0^c f(x)\,dx = \pi$.

Solution by **Mike Pinter**, *Belmont University, Nashville, TN.*

Since

$$\begin{aligned} f(x) &= (1+x)(1+x^2)(1+x^4)\cdots \\ &= 1+x+x^2+x^3+x^4+\cdots = \frac{1}{1-x}, \text{ for } |x|<1, \end{aligned}$$

we have

$$\pi = \int_0^c f(x)\,dx = \int_0^c \frac{1}{1-x}\,dx = \ln(\frac{1}{1-c}).$$

It follows that $c = 1 - e^{-\pi}$.

Also solved by **Ellen Ellis** and **Jason Davis**, Angelo State University, San Angelo, TX, **Ovidiu Furdui**, Kalamazoo, MI, **Richard Hess**, Rancho Palos Verdes CA, **Murray S. Klamkin**, University of Alberta, Edmonton, Alberta, Canada, **Skidmore College Problem Group**, Saratoga Springs, NY, **Rex H. Wu**, Brooklyn, NY and by the **Proposer**

1037. [Spring 2003] *Proposed by Jim Vandergriff, Austin Peay State University, Clarksville, TN.*

Evaluate

$$\lim_{n\to\infty}\int_0^1 \left(\frac{\lfloor nx\rfloor}{n}\right)^2 dx$$

Solution by **Rex H. Wu**, *Brooklyn, NY.*

Observe that $\lfloor nx\rfloor = i$ for $x \in [\frac{i}{n}, \frac{i+1}{n})$ It follows that

$$\lim_{n\to\infty}\int_0^1 \left(\frac{\lfloor nx\rfloor}{n}\right)^2 dx = \lim_{n\to\infty}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{\lfloor nx\rfloor}{n}\right)^2 dx =$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{i}{n}\right)^2 dx = \lim_{n\to\infty}\left(\frac{1}{n^3}\right)\sum_{i=0}^{n-1} i^2 = \frac{1}{3}.$$

Also solved by **Frank P. Battles**, Massachusetts Maritime Academy, Buzzards Bay, MA, **Rob Downes**, Rockaway, NJ, **Mark D. Evans**, Louisville, KY, **Ovidiu Furdui**, Kalamazoo, MI, **Steve Gendler**, Clarion University, Clarion, PA, **Richard Hess**, Rancho Palos Verdes CA, and by the **Proposer**

1038. [Spring 2003] *Proposed by Dr. Shiva K. Saksena, University of North Carolina at Wilmington, Wilmington, North Carolina.*

Find all solutions of the equation

$$\ln(\log x)) = \log(\ln x)).$$

Solution by **Peter A. Lindstrom**, *Batavia, NY.*

If

$$\ln(\log x)) = \log(\ln x)),$$

then

$$\ln((\ln x)(\log e)) = (\ln(\ln x))(\log e),$$

since $\log(A) = (\ln A)(\log e)$. It follows that

$$\ln(\ln x) + \ln(\log e) = \ln(\ln x)(\log e)$$

Therefore,

$$\ln(\ln x)[\log e \;-1] = \ln(\log e)$$

and so

$$\ln(\ln x) = \frac{\ln(\log e)}{\log e \;-1},$$

and

$$x = e^{\left(e^{\left(\frac{\ln(\log e)}{\log e \;-1}\right)}\right)}$$

But we are not done yet, as all solutions to any equation must be checked in the original equation. The left-hand side of the equation becomes

$$\begin{aligned}\ln(\log x)) &= \ln\log e^{\left(e^{\frac{\ln(\log e)}{\log e\ -1}}\right)}\\ &= \ln\left(e^{\frac{\ln(\log e)}{\log e\ -1}}\log e\right)\\ &= \frac{\ln(\log e)}{\log e\ -1} + \ln(\log e)\\ &= \frac{\ln(\log e)}{\log e\ -1}\log e\\ &= \log\left(e^{\left(\frac{\ln(\log e)}{\log e\ -1}\right)}\right)\\ &= \log\ln e^{\left(e^{\frac{\ln(\log e)}{\log e\ -1}}\right)},\end{aligned}$$

which is the right-hand side of the equation.

Also solved by **Ayoub B. Ayoub**, Penn State Abington College, Abington, PA, **Maureen P. Cox** and **Albert White**, St. Bonaventure University, St. Bonaventure, NY, **George P. Evanovich**, Saint Peters College, Jersey City, NJ, **Mark D. Evans**, Louisville, KY, **Ovidiu Furdui**, Kalamazoo, MI, **Steve Gendler**, Clarion University, Clarion, PA, **Richard Hess**, Rancho Palos Verdes CA, **Murray S. Klamkin**, University of Alberta, Edmonton, Alberta, Canada, **George W. Rainey**, Los Angeles, CA **Skidmore College Problem Group**, Saratoga Springs, NY, **Rex H. Wu**, Brooklyn, NY and by the **Proposer**

1039. [Spring 2003] *Proposed by Cecil Rousseau, The University of Memphis.*

(Erdös) Let n be a natural number. The number of odd divisors of n equals the number of representations of n as the sum of consecutive natural numbers. *Note:* Sums with one term are counted.

Solution by **Harry Sedinger**, *St. Bonaventure University, St. Bonaventure, NY.*

Consider the sequence of consecutive natural numbers starting with a and ending with b. Let $s(a,b)$ denote its sum and $t(a,b)$ its length. Note that $t(a,b) = b - a + 1$.

Let n be a given natural number. If n has an odd divisor $2k+1$, then $n = m(2k+1)$ for some natural number m. It is easily seen that if $m > k$, then $n = s(m-k, m+k)$ with the length of the sequence being $2k+1$. If $m \leq k$, then $n = s(k-m+1, k+m)$ with the length of the sequence being $2m$. It follows that each odd divisor of n determines a unique consecutive sequence whose sum is n.

Conversely, let $n = s(a,b)$. If $t(a,b)$ is odd, then $a+b$ is even and $n = s(a,b) = t(a,b)\left[\frac{(a+b)}{2}\right]$. If $t(a,b)$ is even, then $a+b$ is odd and $n = s(a,b) = \left[\frac{t(a,b)}{2}\right](a+b)$. In either case, n has a uniquely determined odd divisor.

The desired result follows.

Also solved by **Richard Hess**, Rancho Palos Verdes CA, **Murray S. Klamkin**, University of Alberta, Edmonton, Alberta, Canada, **Kee-Wai Lau**, Hong Kong, China, **Mike Pinter**, Belmont University, Nashville, TN, **Skidmore College Problem Group**, Saratoga Springs, NY, **Rex H. Wu**, Brooklyn, NY and by the **Proposer**

1040. [Spring 2003] *Proposed by Andrew Cusumano, Great Neck, New York.*

Define $a_n = \sum_{i=0}^{n} \frac{1}{n+2i}$. Show that $\{a_n\}$ is a decreasing sequence and $\lim_{n\to\infty} a_n > \frac{1}{2}$.

Solution by **Kee-Wai Lau**, *Hong Kong, China.*

For $n \geq 1$, we have

$$\begin{aligned}a_n - a_{n+1} &= \sum_{i=0}^{n}\frac{1}{n+2i} - \sum_{i=0}^{n+1}\frac{1}{n+1+2i}\\ &= \frac{1}{2n} + \frac{1}{2(n+2n)} - \frac{1}{n+1+2n} - \frac{1}{n+1+2(n+1)} +\\ &\quad \frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{1}{n+2i} - \frac{2}{n+1+2i} + \frac{1}{n+2+2i}\right)\\ &= \frac{1}{2n} + \frac{1}{6n} - \frac{1}{3n+1} - \frac{1}{3n+3} +\\ &\quad \sum_{i=0}^{n-1}\frac{1}{(n+2i)(n+1+2i)(n+2+2i)}\\ &> \frac{1}{2n} + \frac{1}{6n} - \frac{1}{3n} - \frac{1}{3n} = 0,\end{aligned}$$

so $\{a_n\}$ is decreasing. Also,

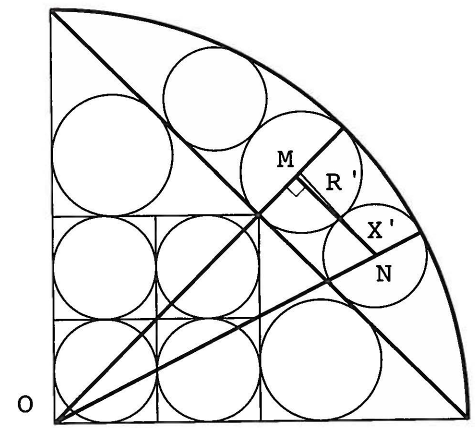
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{n}\frac{1}{\left(1+\frac{2i}{n}\right)} = \int_0^1 \frac{dx}{1+2x} = \frac{1}{2}\ln 3 > \frac{1}{2}$$

Also solved by **Mark D. Evans**, Louisville, KY, **Ovidiu Furdui**, Kalamazoo, MI, **Steve Gendler**, Clarion University, Clarion, PA, **Richard Hess**, Rancho Palos Verdes CA, **Shiva K. Saksena**, University of North Carolina at Wilmington, NC, **Rex H. Wu**, Brooklyn, NY and by the **Proposer**

1041. [Spring 2002] *Proposed by Leon Bankoff, Los Angeles, California.*

The figure below shows a quarter circle with smaller circles inside.

1. Prove that the three larger circles have radii of equal length.
2. Prove the the remaining six smaller circles also have radii of equal length.



Solution by **Ayoub B. Ayoub**, *Pennsylvania State University, Abington College*

For the larger circles inscribed in right angle triangles, the area of the triangle is

$$A = \frac{1}{2}R(\text{perimeter})$$

where R is the radius of the circle. If r is the radius of the original circle, then

$$\frac{1}{2}\cdot\frac{r}{2}\cdot\frac{r}{2}=\frac{1}{2}R\left(\frac{r}{2}+\frac{r}{2}+\frac{r}{2}\sqrt{2}\right),$$

hence

$$R=\frac{r}{2(2+\sqrt{2})}=\frac{r}{4}(2-\sqrt{2}).$$

For the larger circle inscribed in the circular segment, we will denote it's radius by R', then $2R' = r -$ (diagonal of the larger square). (See the figure above.) Hence $2R' = r - \frac{r}{2}\sqrt{2}$ which implies $R' = \frac{r}{4}(2-\sqrt{2}) = R$. This proves part 1.

For each of the smaller circles inscribed in the smaller squares, the radius is $x = \frac{1}{4}(\frac{r}{2}) = \frac{1}{8}r$.

For each of the smaller circles inscribed in the circular segment, we will denote its radius by x'. Then the sides of triangle ΔOMN have lengths $r-R$, $r-x'$, and $R-x'$. This implies that its half perimeter is r. Using Heron's Formula, the area of ΔOMN is $\sqrt{r\cdot R\cdot x'(r-R-x')}$. Hence the height of ΔOMN from N is $\frac{2\sqrt{r\cdot R\cdot x'(r-R-x')}}{r-R}$. Applying the Pythagorean Theorem to the triangle whose hypotenuse is MN we get $(R+x')^2 = \frac{4rRx'(r-R-x')}{(r-R)^2} + (R-x')^2$ which implies $x' = \frac{rR-R^2}{r}$. Thus $x' = \frac{r}{4}(2-\sqrt{2}) - \frac{r}{16}(2-\sqrt{2})^2$, which implies $x' = \frac{r}{8} = x$. This proves part 2.

Also solved by **Mark Evans**, Louisville, KY, **Richard Hess**, Ranch Palos Verdes, CA, **Gus Mavrigian**, Youngstown, OH, **Yoshinobu Murayoshi**, Okinawa, Japan, **Rex H. Wu**, Brooklyn, NY and the **Proposer**.

1042. [Spring 2002] *Proposed by Robert C. Gebhardt.*

In a simple roulette game, there are thirty-six numbers, a predetermined half of the numbers are black and the the other half are red.

1. In how many ways cna the numbers be arranged in slots around the wheel if no two adjacent slits can have the same-colored number?
2. European roulette wheels also have a green "0". Repeat the question from part (1) for this situation.
3. American roulette wheels have a green "0" and a green "00". Repeat the question for this situation.

Note: This problem shows a subtlety that evaded many, including the poser and the editors. This deals with the solution to the third part of the problem. While we received correct solutions to parts 1 and 2, the only complete solution is presented below.

Solution by **Steve Gendler**, *Clarion University, Clarion PA*

1. If one of the numbers, say #1, is fixed then 17 numbers of its color and 18 of the opposite color are permuted to obtain

$$P_1 = 2(17!)(18!)$$

possible arrangements.

2. If there is a green slot, fix it and allow a red to it's right. Then there are 18 of each type to permute, giving $(18!)(18!)$ arrangements. But this occurs also if there is a black to its right. So there are

$$P_2 = 2(18!)(18!)$$

possible arrangements.

3. When we add a second green, it cannot be adjacent to the first. If it appears an even number of spaces to the right of the first green, then the segment to the left of the second green may be reversed (if not for the second green, this reversal would have adjacent slots of the same color and so could not have been counted). If it appears an odd number of spaces to the right of the first green, then a reversal will not create a new arrangement since the segment will begin and end with the same color. Since the second green can fall in 35 locations and 17 of them allow a reversal, we obtain

$$P_3 = 2(18!)(18!)35 + 2(18!)(18!)17 = 2(18!)(18!)52$$

different arrangements.

Parts 1 and 2 also solved by **Mark D. Evans**, Louisville, KY, **Mike Pinter**, Belmont University, Nashville, TN, and the **Proposer**.

1043. [Spring 2002] *Proposed by Mohd Nadeem Khan, New Abadi, Aligarh, INDIA.*

Find all quadruples of distinct integers x, y, u, and v such that

$$\begin{aligned} xy &= uv \\ x-y &= u+v \\ \gcd(x,y) &= 1 \\ \gcd(u,v) &= 1 \\ x &> y \qquad \text{and} \\ u &> v. \end{aligned}$$

Solution by **Rex H. Wu**, *Brooklyn, NY*

Suppose $\gcd(x,v) = A$, $\gcd(y,u) = D$, $x = AB$, $y = DE$, $u = EF$, and $v = AC$ for some integers A, B, C, D and E

Then $\gcd(x,y) = 1 \Rightarrow \gcd(A,D) = \gcd(A,E) = \gcd(B,D) = \gcd(B,E) = 1$. Also $\gcd(u,v) = 1 \Rightarrow \gcd(C,D) = \gcd(C,F) = \gcd(A,F) = 1$. And finally $\gcd(y,u) = D \Rightarrow \gcd(E,F) = 1$.

From $xy = uv$, we have $(AB)(DE) = (AC)(DF)$, or $BE = CF$. Since $\gcd(B,C) = 1$ and $\gcd(E,F) = 1$, we can only have $B = F$ and $C = E$.

$$\begin{aligned} x-y &= u+v \\ x-v &= y+u \\ A(B-C) &= D(E+F) \\ A(B-C) &= D(C+B). \end{aligned}$$

Since $\gcd(A,D) = 1$ we can only have A divides $C+B$ and D divides $B-C$. In other words if $\alpha A = (C+B)$, then $(B-C) = \alpha D$.

By adding and subtracting the two equations, we get $\alpha(A+D) = 2B$ and $\alpha(A-D) = 2C$. Then $\alpha = 1$ or $\alpha = 2$. If $\alpha \geq 3$, then $\gcd(B,C) \neq 1$, a contradiction.

Case (i) $\alpha = 1$

We have $A + D = 2B$ and $A - D = 2C$. Or $A = D + 2C$. Here C can be any positive integer, $C = 1, 2, 3, 4, \ldots$. Since $\gcd(A,D) = 1$, D cannot be even. We also know that $\gcd(D,C) = 1$. Therefore, D can be any integer with the restriction that $\gcd(D,2C) = 1$. Let $A = D + 2C$ and $B = (A+D)/2 = C + D$.

Then $x = AB = (D+2C)(C+D) = D^2 + 2C^2 + 3CD$, $y = CD$, $u = \max\{AC, BD\} = \max\{(D+2C)C, (C+D)D\} = \max\{CD+2C^2, CD+D^2\}$, and $v = \min\{AC, BD\} = \min\{CD+2C^2, CD+D^2\}$. (The max and min functions are used to satisfy the condition $u > v$.) We have our first class of solutions:

$$
\begin{aligned}
x &= D^2 + 2C^2 + 3CD \\
y &= CD \\
u &= \max\{CD+2C^2, CD+D^2\} \\
v &= \min\{CD+2C^2, CD+D^2\}
\end{aligned}
$$

where $C = 1, 2, 3, 4, \ldots$ and $\gcd(D, 2C) = 1$.

Case (ii) $\alpha = 2$

We have $2(A+D) = 2B$ and $2(A-D) = 2C$, or $A = D + C$. Not that A and D are of opposite parity, otherwise $\gcd(B, C) \neq 1$. This implies that C is odd, $C = 2i-1$ for $i = 1, 2, 3, 4, \ldots$. Then D can be any integer with $\gcd(C, D) = 1$, $A = D + C$, and $B = A + D = C + 2D$.

Now $x = AB = C^2 + 2D^2 + 3CD$, $y = CD$, $u = \max\{AC, BD\} = \max\{CD + C^2, CD + 2D^2\}$, and $v = \min\{AC, BD\} = \min\{CD + C^2, CD + 2D^2\}$ with $C = 1, 3, 5, 7, \ldots$ and $\gcd(D, C) = 1$.

But this is the exact expression as in (i) with C and D switched.

In conculusion,

$$
\begin{aligned}
x &= D^2 + 2C^2 + 3CD \\
y &= CD \\
u &= \max\{CD+2C^2, CD+D^2\} \\
v &= \min\{CD+2C^2, CD+D^2\}
\end{aligned}
$$

where $C = 1, 2, 3, 4, \ldots$ and $\gcd(D, 2C) = 1$.

Note: The solver hasn't included it, but it does follow that in these cases, $\gcd(x, y) = 1$, $\gcd(u, v) = 1$, $x > y$, and $x - y = u + v$.

Also solved by **Richard Hess**, Ranch Palos Verdes, CA, partial solutions by **Mike Pintner**, Belmont University, Nashville TN and the **Proposer**.

The 2002 National Pi Mu Epsilon Meeting

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held in Burlington, VT from August 1–2, 2002. As in the past, the meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections.

The J. Sutherland Frame Lecturer was **Frank Morgan** from Williams College. His presentation was entitled "Soap Bubbles: Open Problems".

Student Presentations. The following student papers were presented at the meeting. An asterisk(*) after the name of the presenter indicates that the speaker received a best paper award.

*Tom Wakefield**, Youngstown State University – Ohio Xi
Factorization and $PSL_2(13)$

Lorne Fairbairn, SUNY Potsdam – New York Phi
Carry Groups

Nicole Miller, Salisbury University – Maryland Zeta
The Evolution Homomorphisms and Classification of Cellular Automata

*Ed Kenney**, University of Richmond – Virginia Alpha
Search for Constructions of Partial Difference Sets

Eric C. Polley, St. Johns University – Minnesota Delta
How to Color a Graph

Christopher Jones, Youngstown State University – Ohio Xi
Analysis of the Closure and Interior of Topological spaces

Conrad Miller, Southwestern University – Texas Pi
Implementation of Error Correcting Codes

James Sloan, Southwestern University – Texas Pi
Check, Please!

Michael B. Henry, Augustana College – Illinois Eta
The Illustrated Analyst

Elizabeth Fite, Hendrix College – Arkansas Beta
Binarizing Text Images

Igor Crk, Carthage College – Wisconsin Epsilon
Mathematics of High Performance Computer Graphics

Noorie Hanum, Angelo State University – Texas Zeta
Mathematical Analysis of Computing Algorithms

Jonathan Moussa, Worcester Polytechnic Institute – Massachusetts Alpha
Recursive Method for Solving the Many-Body Quantum Problem

Ben Blaiszik, Elmhurst College – Illinois Iota
Kicking the System: The Effect of 4:1 Forcing on Stable Pulse Length

Valerie Kunde, Aquinas College – Michigan Lambda
Delayed Resonance

Joel Lepak, Youngstown State University – Ohio Xi
Dynamics of Population Modeling

*Robert Shuttleworth**, Youngstown State University – Ohio Xi
Numerical Solutions of PDEs

Amanda Milby, Southwestern University – Texas Pi
Driving Mis-Coding

Philip Busse, St. Norbert College – Wisconsin Delta
Error-Correcting Codes and Abstract Algebra

Casey Douglas, Southwestern University – Texas Pi
A Hamming Code by Any Other Name...

Christian Jason Maier, Alfred University
New Primality Testing

Brian Street, University of Virginia – Virginia Kappa
Uniformly Sweeping Out for Measure Preserving Group Actions

Catharine Wright, University of Maine – Maine Alpha
Topological Graph Theory

*Elizabeth Donovan**, Worcester Polytechnic Institute – Massachusetts Alpha
Maximum Chromatic Status of a Graph

Kelly Wroblewski, University of Houston – Downtown – Texas Nu
A Look at Triangles with Graffiti.pc

*Anupam Bhatnagar and Borislav Mezhericher**, Queens College – New York Alpha
Graphs that Count: Generalized Catalan Numbers

Yana Malysheva, University of Illinois – Illinois Alpha
illi-Tantrix: New Ways of Looking at Knots

Eman Kunz and Quincy Loney, SUNY Potsdam – New York Phi
Intrinsically Chiral Graphs

David Gohlke, Youngstown State University – Ohio Xi
The Mathematics of Soccer

John Angelis, Youngstown State University – Ohio Xi
God Knows Markov

*Teresa Selee**, Youngstown State University – Ohio Xi
The Assumptions and Strategies of Repeated Games

*Brian Wyman**, University of Richmond – Virginia Alpha
Game Strategy Development

Tricia Hemmesch, College of St. Benedict – Minnesota Delta
Escalating Behavior in the Dollar Auction?

Nathan A. Lewallen, North Carolina State University – North Carolina Gamma
Analysis of Shocks in Granular Material Flows

*F. Ronald Ogborne**, SUNY Fredonia – New York Pi
Reciprocity Gap and General Linear "Crack" Identification

Lara Stroud, Meredith College – North Carolina Mu
Modeling Tricholorethylene

Carrie Diaz Eaton, University of Maine – Maine Alpha
Fast-Spiking Cell and Networked Cell Models

Joseph Boley, University of Houston – Downtown – Texas Nu
A Numerical Study of the Beta Insulin Glucose Model of Diabetes

Fei Sun, Moravian College – Pennsylvania Omicron
A Smart Measurebot

Amanda Szymanski, Aquinas College – Michigan Lambda
Centroids are Central

Hai He, Hunter College – New York Beta
The Indeterminate Case [0/0] – A Closer Look

Mehrdad Khosravi, University of Central Florida – Florida Alpha Mu
Law of cosines in n Dimensions

Nicole J. Munden, Southern Illinois University at Edwardsville – Illinois Zeta
Monte Carlo Integration

Mathfest 2003

Call For Papers

The next ΠΜΕ meeting will take place at Mathfest 2003 in Boulder, Colorado, July 31 – August 2. See the ΠΜΕ webpage (http://www.pme-math.org/) for application deadlines and forms. There will be mathematics talks and social events and don't forget, the ΠΜΕ banquet. See also the MAA webpage for details as well as for other activities in the Mile-High State.

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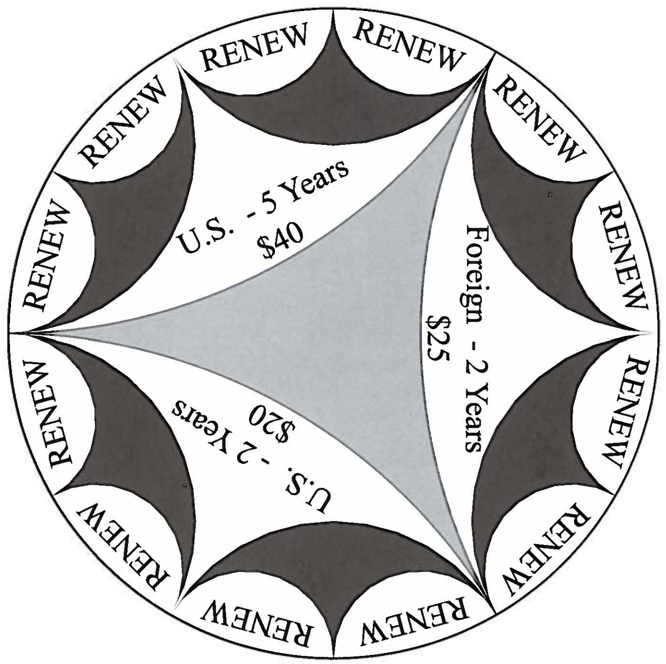

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Joan Weiss, Dept. of Math., Fairfield University, Fairfield, CT 06824-5195.

MATH ACROSTIC

Dan Hurwitz, Skidmore College

a. Numerology — 049 139 115 184 100 071 088 126

b. Contraversial postulate (2 wds.) — 050 008 066 112 128 149 029 162 022 141 096 132

c. Non-perpendicularity often refers back to this(2 wds.) — 074 174 125 033 011 155 024 105 190 65 183

d. Participant in Fibonacci example — 037 111 187 080 147 131

e. Application area for probability theory — 046 194 082 119 058 153 102 092

f. Has good experimental design and predicted values — 060 166 151 072 142 095 002 041 025

g. Another interest of Russell and Whitehead — 009 173 106 030 057 160 040 069 097 188

h. He found a geometric solution to a cubic (full name) — 171 068 013 127 023 193 055 081 117 093 039

i. – rectum, runs through a focus — 168 048 182 137 110

j. An authority on repunits — 130 116 169 143 018

k. Distinguished journal (nickname) — 031 006 185 103 021 159 052

l. Space where points can be separated — 090 070 104 176 079 016 061 152 036

m. Usually indicated by a circular arrow — 086 094 113 099 075 032 146 157 019 161 056

n. One joule per second — 084 135 145 053

o. Sums of partialities — 122 038 163 027 043 150 003 101 180 073

p. Slide rule inventor — 123 178 007 063 014 144 042 186

q. Function required for Peano axioms — 077 012 154 192 164 020 001 140 109

r. Rotated about minor axis — 054 170 120 044 189 083

s. Raise to covering space — 156 091 129 177

t. He axiomatized quantum mechanics — 107 034 181 134 010 076 098 191 045 148

u. German number theorist, Gauss' contemporary(1823-1852) — 133 064 114 078 035 196 172 167 005 087

v. Gives scalars from vectors (2 wds.) — 051 165 017 108 138 121 026 067 179 047 124 062

w. Perpetual Jack Benny birthday ordinal (hyph.) — 136 059 015 195 085 175 028 158 089 004 118

001q	002f	003o	004w	005u	006k	007p		008b	009g		010t	011c	012q	013h	014p	015w
016l	017v	018j		019m	020q		021k	022b	023h	024c		025f	026v	027o	028w	029b
030g	031k	032m	033c	034t	035u		036l	037d	038o	039h		040g	041f	042p		043o
044r	045t	046e	047v	048i	049a	050b		051v	052k	053n	054r		055h	056m	057g	058e
059w	060f	061l		062v	063p	064u	065c		066b	067v	068h	069g	070l	071a	072f	073o
074c	075m		076t	077q	078u	079l		080d	081h		082e	083r	084n	085w	086m	087u
	088a	089w		090l	091s	092e		093h	094m	095f	096b	097g	098t	099m	100a	101o
102e	103k		104l	105c	106g	107t	108v	109q	110i	111d	112b	113m	114u		115a	116j
117h		118w	119e	120r	121v		122o	123p		124v	125c	126a	127h	128b	129s	130j
	131d	132b	133u		134t	135n	136w	137i	138v	139a		140q	141b		142f	143j
144p	145n	146m	147d	148t		149b	150o	151f	152l	153e	154q	155c	156s	157m	158w	159k
160g		161m	162b	163o	164q	165v		166f	167u	168i	169j		170r	171h	172u	173g
	174c	175w		176l	177s	178p	179v	180o	181t	182i	183c		184a	185k	186p	
187d	188g		189r	190c	191t	192q	193h	194e	195w	196u						

Last month's mathacrostic was taken from "Archimedes' Revenge" by Hoffman.

"Pappas observed that, besides the hexagon, the square and the equilateral triangle are the only other regular polygons that can tile the plane. But, for the bee, the hexagon is superior because it encloses the most area for a given perimeter."

Contents

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