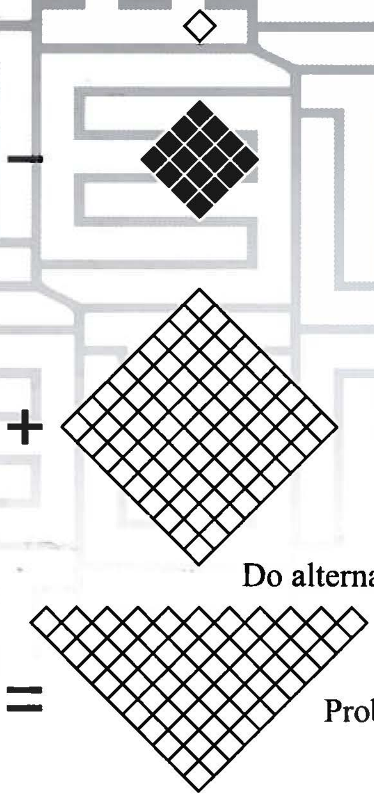

Do alternating squared squares yield a triangle?

Problem 1069

# The ΠME Journal

Volume 11, Number 9 Fall 2003

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# THE CURVATURE IN A FAMILY OF NESTED CONICS

AYOUB B. AYOUB*

The topics of conic sections and curvature are usually taught independently from each other. Here we will make a connection between them.

We will consider a family of conics, where all of them have the same vertex $A$ and the same focus $F$ as depicted in Figure 1. This figure suggests that the curvature of the conics at the vertex $A$ decreases when going from the ellipse to the parabola to the hyperbola. In this article, we will prove that this is indeed the case. Then we will show why this family of conics is considered nested.

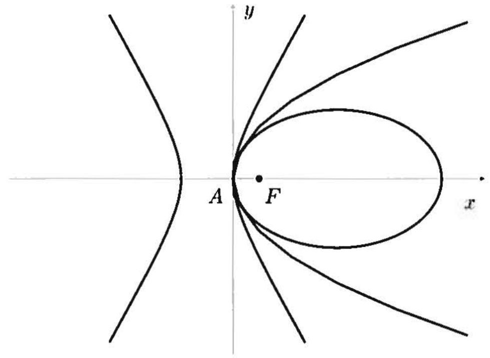


FIG. 1.

**1. The Conics Family's Equation.** In order to calculate the curvature, we need to derive the equation of the family of conics. Let $A$ be the origin, the line through $A$ and $F$ be the $x$-axis, and the perpendicular to $AF$ at $A$ be the $y$-axis. Also let the directrix corresponding to the focus $F$ meet the $x$-axis at $E$. If $P$ is a point on the conic, let its orthogonal projection on the directrix be $D$, see Figure 2.

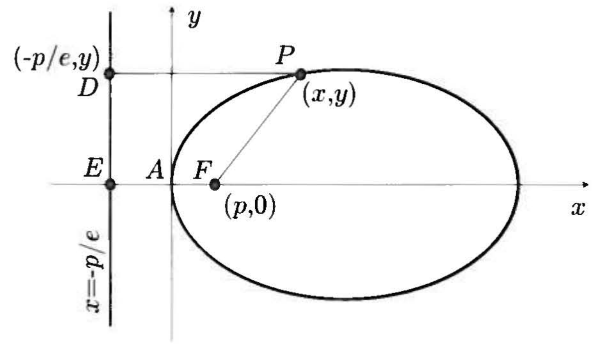


FIG. 2.

If we denote the eccentricity of the conic by $e$, then by the definition of the eccentricity, $e = PF/PD$. Since $A$ is also a point of the conic, then $e = AF/AE$. If

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$AF = p$, then $AE = p/e$ and consequently the equation of the directrix is $x = p/e$ .

Now, let $P$ be $(x, y)$, then $PD = x + p/e$. Since $PF = ePD$, and $F$ is $(p, 0)$, then $\sqrt{(x-p)^2 + y^2} = e\,(x + p/e)$.

After squaring and simplifying, we get:

$$y^2 = (e^2 - 1)x^2 + 2p(e+1)x \qquad (1)$$

This is the equation of the family of conics, where $e$ is playing the role of a parameter while $p$ is fixed.

**2. The Curvature.** The curvature at a point on a curve can be thought of as a measure of the rate of deviation of the curve from its tangent at that point. The greater the curvature, the sharper the curve will be bending away from the tangent.

If the equation of the curve is $y = f(x)$, then the curvature $K(x, y)$ at the point $(x, y)$ can be shown to be:

$$K(x, y) = \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$$

To use this formula for calculating the curvature of the conic $y^2 = (e^2 - 1)x^2 + 2p(e+1)x$ at the origin, we need to know the value of the derivative there. Since the conic has a vertical tangent at the origin, then $y'$ is undefined. To overcome this hurdle, we rotate the axes $90^\circ$ and the equation of the conic becomes $x^2 = (e^2-1)y^2 + 2p(e+1)y$. If we differentiate it twice, we get $2x = 2(e^2-1)yy' + 2p(e+1)y'$ and $2 = 2(e^2-1)(yy'' + (y')^2) + 2p(e+1)y''$. At $(0,0)$, the first equation gives $y' = 0$, then the second equation gives $y'' = 1/(p(e+1))$ . Substituting the values of $y'$ and $y''$ in the above curvature formula, we get

$$K(0,0) = \frac{1}{p(e+1)}$$

**3. Conclusion.** The relation $K(0,0) = 1/(p(e+1))$ implies that the curvature $K(0,0)$ decreases as the eccentricity $e$ increases. Since $e < 1$, $e = 1$, or $e > 1$ according to whether the conic is an ellipse, a parabola, or a hyperbola respectively, see [1], then the curvature at the vertex $A$ decreases starting with the ellipses whose $e < 1$, followed by the parabola whose $e = 1$ and finally the hyperbolas whose $e > 1$. The following table displays the eccentricities and curvatures of some of these conics;

| $e$ | Equation of Conic | $K(0,0)$ |
|---|---|---|
| $\frac{1}{2}$ | $\frac{(x-2p)^2}{4p^2} + \frac{y^2}{3p^2} = 1$ | $\frac{2}{3p}$ |
| $\frac{3}{4}$ | $\frac{(x-4p)^2}{16p^2} + \frac{y^2}{7p^2} = 1$ | $\frac{4}{7p}$ |
| $1$ | $y^2 = 4px$ | $\frac{1}{2p}$ |
| $\frac{3}{2}$ | $\frac{(x+2p)^2}{4p^2} - \frac{y^2}{5p^2} = 1$ | $\frac{2}{5p}$ |
| $2$ | $\frac{(x+p)^2}{p^2} - \frac{y^2}{3p^2} = 1$ | $\frac{1}{3p}$ |



There are two limiting cases for this family of conics. The first, when $e \to 0$, the ellipse becomes a circle of radius $p$ whose curvature is $1/p$. The second, when $e \to \infty$, the hyperbola degenerates to a double line $x^2 = 0$ along the $y$-axis. Of course, the curvature in this case is zero.

Figure 3 depicts the conics given in the table when $p = 1$. Also included in the figure, the two limiting cases.

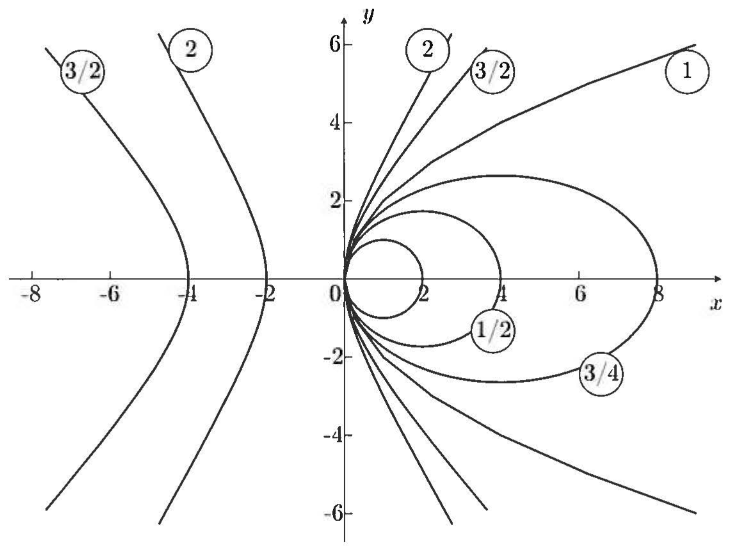


FIG. 3. *The circled values indicate the eccentricity.*

To justify the description of this family as nested, we consider two of its members whose equations are:

$$y^2 = (e_1^2 - 1)x^2 + 2p(e_1 + 1)x$$

and

$$y^2 = (e_2^2 - 1)x^2 + 2p(e_2 + 1)x\ .$$

If we subtract one from the other and simplify the difference, we get $(e_1+e_2)x^2 + 2px = 0$, the roots of which are $x = 0$ and $x = -2p/(e_1 + e_2)$. Since the corresponding $y$ coordinates are $y = 0$ and $y = \pm 2ip\sqrt{(1+e_1)(1+e_2)}/(e_1+e_2)$, then the conics of the family have only the point $(0,0)$ in common and that is where they touch each other.

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[1] H. ANTON. "Calculus, A New Horizon", 6th edition, John Wiley & Son, New York, 1999, pp. 743-744, p. 864, 1999.

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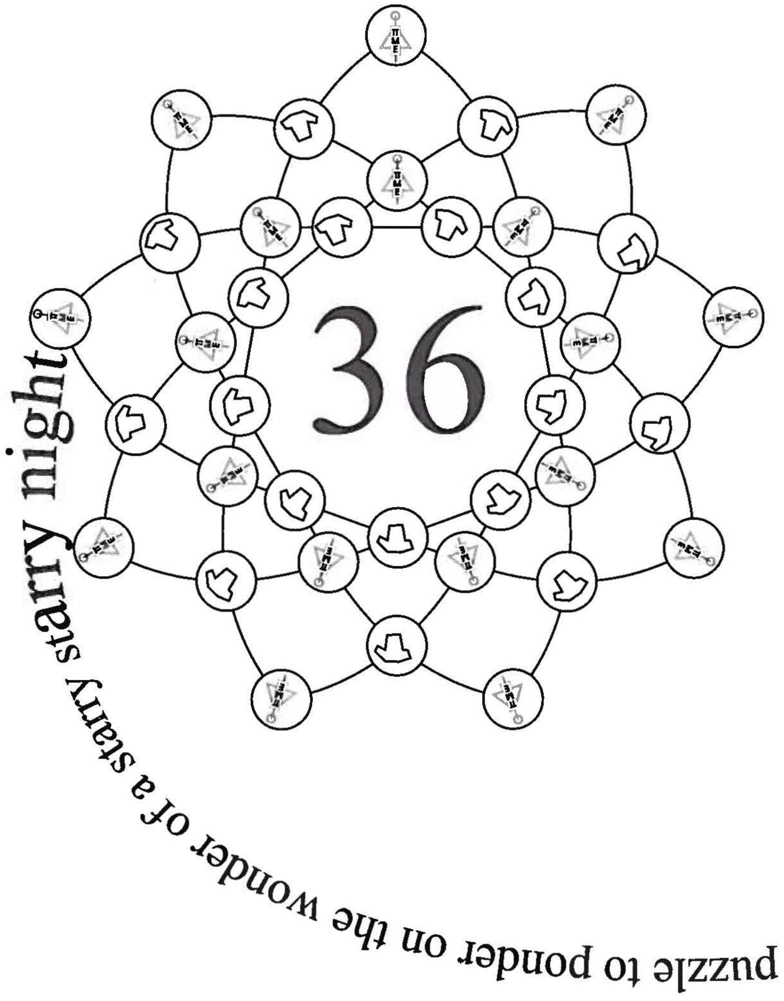


Can you place the numbers from 1 to 36 in the circles so that the sums along any of the 9 large arcs of the star are the same[1]?

While you are contemplating the beauty of this star, you can compare it to the beauty of the ΠME key pins and tee shirts, available as stocking stuffers. Check out `http://www.pme-math.org/membership/merchandise.html`

[1]The answer and more is to be found in the next issue!

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# THE STRONG SHADOWING PROPERTY ON THE UNIT INTERVAL: WHEN DO ORBITS STAY CLOSE TO THEIR SLOPPY COUSINS?

JOSEPH H. BROWN , TIMOTHY J. PENNINGS[*], AND JAMES O. WARREN[†]

**Abstract.** We study the dynamics of continuous maps on the unit interval. We determine necessary and sufficient conditions so that all pseudo-orbits can be approximated by orbits with the same initial point.

**1. Introduction.** When a calculator or computer is used to generate a sequence of numbers by iterating a function, how close is the computer-generated sequence to the intended one? Such a situation occurs, for example, whenever the Newton-Raphson Method is used to find roots of equations. The area of mathematics which deals with this question is called *dynamical systems*. It includes terms such as *orbit* – the true sequence of points generated by iterating a function, *pseudo-orbit* – the sequence of points generated by the computer as it attempts to form an orbit, and *shadowing property* – the property that every pseudo-orbit stays close to an actual orbit. In this paper we find necessary and sufficient conditions for the shadowing property to hold for continuous functions on the unit interval in the special case when the orbit and the pseudo-orbit begin at the same point which is often the case in practice. The general argument makes nice use of some standard undergraduate analysis ideas such as compactness, uniform continuity and uniform convergence.

In particular consider the functions

$$\begin{aligned} g(x) &= x + \frac{1}{4\pi}|\sin(2\pi x)|, \\ h(x) &= x + \frac{1}{4\pi}\sin(2\pi x), \text{ and} \\ k(x) &= \frac{3}{4}x + \frac{1}{8} + \frac{1}{4\pi}\sin(2\pi x) \end{aligned}$$

which are graphed in Figure 1 below. We will presently show that these functions all have distinct dynamics with regard to the shadowing property.

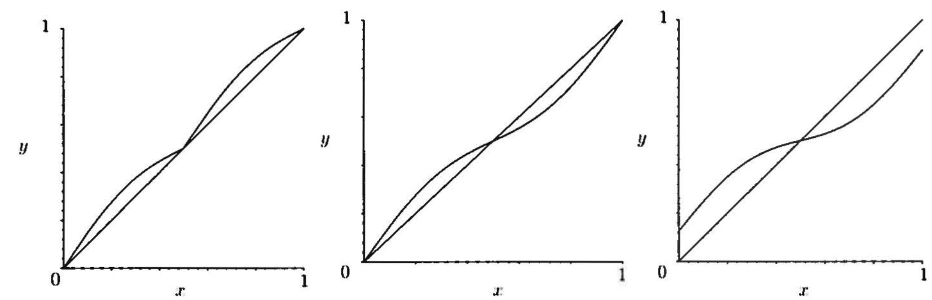


FIG. 1. *Graphs of g, h, and k respectively*

Let's begin by making the question precise. We start with any metric space - such as the unit interval with the distance between two points, $x$ and $y$, given by

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$d(x, y) = |x - y|$. For a metric space $X$, and a function $f : X \to X$, $(X, f, \mathbb{N})$ is a *dynamical system* where $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $x \in X$, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is called the *orbit* of $x$, where $f^0(x) = x$ and $f^n$ is the $n$-fold composition of $f$. In other words, each output of $f$ is fed back into the function to obtain the next point of the sequence. Orbits are easily pictured graphically by using the cobweb technique. Using the graph of $g$ above, we find the orbit of $x_0$ by alternately drawing a line vertically to the graph of $f$ which gives the next point of the sequence, and then horizontally to the line $y = x$ which in effect positions the point to be fed back into the function. See Figure 2 for an orbit of $g(x) = x + \frac{1}{4\pi}|\sin(2\pi x)|$. Notice that this dynamical system

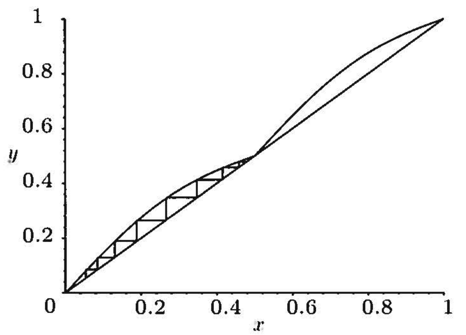


FIG. 2. *Graph of g and orbit*

has three fixed points - i.e., points where $f(x) = x$.

Computers simplify the task of calculating orbits, but round-off error will create a "sloppy orbit" called a pseudo-orbit. A pseudo-orbit may or may not stay close to an actual orbit. If every pseudo-orbit of $(X, f, \mathbb{N})$ stays close to an actual orbit, $f$ is said to have the shadowing property.

To be precise, for $\delta > 0$, a *$\delta$-pseudo-orbit* of $(X, f, \mathbb{N})$ is defined as a sequence $\{x_n\}_{n=0}^{\infty}$ such that $d(x_{n+1}, f(x_n)) \leq \delta$ for all $n \in \mathbb{N}$. Furthermore, a dynamical system $(X, f, \mathbb{N})$ has the *shadowing property* if for $\varepsilon > 0$, there exists $\delta > 0$ such that given a $\delta$-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ where $d(x_n, f^n(x)) < \varepsilon$ for all $n \in \mathbb{N}$.

Notice that this follows the format of a typical $\varepsilon - \delta$ definition. That is, first an $\varepsilon$ is given which sets the tolerance - i.e., how far things can be apart. Once that is given, a $\delta$ is determined which keeps the required points within $\varepsilon$ of each other. So a dynamical system has the shadowing property if given any $\varepsilon > 0$, there is a small enough $\delta > 0$, so that every $\delta$-pseudo-orbit will be followed $\varepsilon$ closely by an actual orbit. A $\delta$-pseudo-orbit being $\varepsilon$-shadowed by an actual orbit is shown in Figure 3 where $\varepsilon$ is the radius of the large circles and $\delta$ is the radius of the small circles - all of which are centered at the points of the pseudo-orbit.

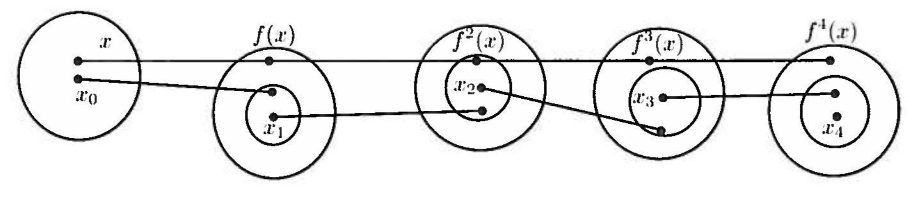


FIG. 3. *Diagram of orbit being $\varepsilon$-shadowed by a $\delta$-pseudo-orbit*



It may be the case that an actual orbit must start at a different point than a given pseudo-orbit in order to shadow it. Consider a pseudo-orbit in $g$ above which begins at $(0, 0)$. Although $g(0) = 0$, because of the sloppiness of the pseudo-orbit, the next term may be a positive number. From there the pseudo-orbit can climb away from $(0, 0)$. An orbit starting at $(0, 0)$ on the other hand, has no choice but to stay at $(0, 0)$ forever. Thus the only way to shadow such a pseudo-orbit is for the orbit to start at a positive value. Necessary and sufficient conditions for any non-decreasing continuous function on $[0, 1]$ to have the shadowing property are given in [6]. Essentially, each open interval around each fixed point between 0 and 1 must contain points where the graph of the function is both above and below the line $y = x$.

Applying this result, the function $g$ above does not satisfy the shadowing property, while the function $h$ does. Imagine how a pseudo-orbit of $g$ might climb up to the point $(1/2, 1/2)$ and then leap over the fixed point and continue up to $(1, 1)$, while the actual orbit would be trapped at $(1/2, 1/2)$. Can you see why $h$, on the other hand, *does* have the shadowing property?

Even though a dynamical system may satisfy the shadowing property, sometimes a stronger condition is desired. Since a pseudo-orbit is typically generated when trying to generate an orbit, one may be interested in determining whether the pseudo-orbit will be $\varepsilon$-shadowed by an orbit *beginning at the same point.* This leads us to make the definition: $(X, f, \mathbb{N})$ has the *strong shadowing property* if for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\delta$-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$, $d(x_n, f^n(x_0)) < \varepsilon$ for all $n \in \mathbb{N}$. The purpose of this paper is to characterize all continuous functions on $[0, 1]$ which have the strong shadowing property.

Given a continuous function $f : [0, 1] \to [0, 1]$, let $\mathrm{gr}(f)$ denote the graph of $f$ and $\mathrm{gr}^{-1}(f)$ denote the reflection of $\mathrm{gr}(f)$ about the diagonal $y = x$. Also, $f$ has a *$k$-cycle* ($k \geq 2$) if there exists a set of $k$ distinct points, $\{x_0, x_1, \ldots, x_{k-1}\}$, such that $f(x_j) = x_{j+1 \pmod k}$. We show that the following three conditions are equivalent:

1. $f$ has the strong shadowing property
2. $f$ has no cycles and only one fixed point
3. $\mathrm{gr}(f)$ and $\mathrm{gr}^{-1}(f)$ have exactly one point in common.

The equivalence of (2) and (3) is easily seen. If $\mathrm{gr}(f)$ and $\mathrm{gr}^{-1}(f)$ have more than one point in common, then either $f$ has a 2-cycle or $f$ has more than one fixed point. Conversely, if $\mathrm{gr}(f)$ and $\mathrm{gr}^{-1}(f)$ have exactly one point in common, $f$ cannot have any 2-cycles. The Sarkovskii ordering of cycles then guarantees that $f$ has no cycles [3]. Furthermore, any fixed point of $f$ will be in the intersection of $\mathrm{gr}(f)$ and $\mathrm{gr}^{-1}(f)$, so $f$ has only one fixed point.

**2. Preliminaries.** Throughout this paper "$\subset$" denotes strict containment and all intervals are intersected with $[0, 1]$.

We begin with three lemmas first proved by Sarkovskii [7]. Drawing diagrams helps considerably in working through the details of the proofs. The culmination of the lemmas is that if $f : [0, 1] \to [0, 1]$ is continuous with no cycles, then the orbit of any point will converge (to a fixed point).

LEMMA 1. *Let $f : [0, 1] \to [0, 1]$ be a continuous function with no cycles. For $x \in [0, 1]$, if $f(x) > x$, then $f(y) > x$ for all $y \in [x, f(x)]$ (if $f(x) < x$, then $f(y) < x$ for all $y \in [f(x), x]$).*

*Proof.* We prove this lemma by contradiction. Assume $x \in [0, 1]$ with $f(x) > x$, and suppose there exists $z \in [x, f(x)]$ such that $f(z) \leq x$. Let $y = \max\{t \in [x, f(x)] : f(t) = x\}$. Since $f(x) > x$ and $f(y) = x < y$, $f$ must intersect the diagonal $i(t) = t$ in $[x, y]$. Thus, there exists at least one fixed point in $[x, y]$. Let $p_L = \max\{t \in [x, y] :$

$f(t) = t\}$. There are two cases to be considered.

Case 1: Suppose that $f$ has at least one fixed point $p \in [y,1]$, and let $p_R = \min\{t \in [y,1] : f(t) = t\}$. Since $f(y) = x < p_L$ and $f(p_R) = p_R > p_L$, there exists $s_1 \in (y, p_R)$ such that $f(s_1) = p_L$ by the Intermediate Value Theorem. Similarly, there exists $s_2 \in (y, p_R)$ such that $f(s_2) = s_1$. Since $f^2(y) = f(x) > y$ and $f^2(s_2) = p_L < s_2$ with $y, s_2 \in (p_L, p_R)$, $f^2$ must intersect the diagonal $i(t) = t$ in the interval $(p_L, p_R)$. Hence there exists $p \in (p_L, p_R)$ such that $f^2(p) = p$ and $f(p) \neq p$, so $f$ has at least one 2-cycle.

Case 2: Suppose that $f$ has no fixed points in $[y,1]$. The point $y$ was chosen so $f(y) = x$, $f^2(y) = f(x) > y$. Since the range of $f^2$ is contained in $[0,1]$, $f^2(1) < 1$. So $f^2$ must intersect the diagonal $i(t) = t$ in the interval $[y,1]$. Hence there exists $p \in [y,1]$ such that $f^2(p) = p$ and $f(p) \neq p$, so $f$ has at least one 2-cycle.

Each of these cases contradicted that $f$ has no cycles, so if $f(x) > x$, then $f(y) > x$ for all $y \in [x, f(x)]$. Similar arguments prove that if $f(x) < x$, then $f(y) < x$ for all $y \in [f(x), x]$. □

LEMMA 2. *Let $f\colon [0,1] \to [0,1]$ be a continuous function with no cycles. For all $n_1, n_2, n_3 \in \mathbb{N}$ $(n_1 < n_2 < n_3)$ and $x \in [0,1]$, if $f^{n_2}(x) \neq f^{n_3}(x)$, then $f^{n_1}(x)$ is not in the interval with endpoints $f^{n_2}(x)$ and $f^{n_3}(x)$.*

*Proof.* Choose $x \in [0,1]$ and $n_1, n_2, n_3 \in \mathbb{N}$ $(n_1 < n_2 < n_3)$ such that $f^{n_2}(x) \neq f^{n_3}(x)$. Since $f^{n_2}(x) \neq f^{n_3}(x)$, $f^{n_1}(x)$ cannot be a fixed point. Without loss of generality, assume $f^{n_1}(x) < f^{n_1+1}(x)$. We prove this lemma by using strong induction to show that $f^n(x) > f^{n_1}(x)$ for $n = n_1 + 1, \ldots, n_3$.

Assume $f^k(x) > f^{n_1}(x)$ for $k = n_1 + 1, \ldots, n$ where $n < n_3$. If $f^n(x) = f^{n-1}(x)$, then $f^{n+1}(x) = f^{n-1}(x) > f^{n_1}(x)$ by the induction hypothesis. If $f^n(x) > f^{n-1}(x)$, then $f(f^n(x)) > f^{n-1}(x)$ by Lemma 1. Thus, $f^{n+1}(x) > f^{n-1}(x) \geq f^{n_1}(x)$ by the induction hypothesis. If $f^n(x) < f^{n-1}(x)$, let

$$s = \max\{k \in \mathbb{N}, k < n : f^k(x) < f^{k+1}(x)\}.$$

We know $s$ exists since $f^{n_1}(x) < f^{n_1+1}(x)$, and so $s \geq n_1$ and $f^s(x) \geq f^{n_1}(x)$.

We claim that $f^s(x) < f^n(x) < f^{s+1}(x)$, for suppose $f^n(x) < f^s(x)$. By the definition of $s$, we have the inequalities $f^{s+1}(x) > f^{s+2}(x) > \cdots > f^n(x)$. Hence, there exists $p$, $s+1 \leq p \leq n-1$, such that $f^p(x) \in [f^s(x), f^{s+1}(x)]$ and $f^{p+1}(x) \notin [f^s(x), f^{s+1}(x)]$. This contradicts Lemma 1 since $f^s(x) \in [f^{p+1}(x), f^p(x)]$ while $f(f^s(x)) > f^p(x)$. Thus $f^s(x) < f^n(x)$. Furthermore, since $f^k(x) < f^{k-1}(x)$ for $k = s+2, \ldots, n$, $f^n(x) < f^{s+1}(x)$.

By Lemma 1, $f(f^n(x)) = f^{n+1}(x) > f^s(x) \geq f^{n_1}(x)$. Thus by induction, $f^n(x) > f^{n_1}(x)$ for $n = n_1 + 1, \ldots, n_3$. Therefore, $f^{n_1}(x)$ is not in the interval with endpoints $f^{n_2}(x)$ and $f^{n_3}(x)$. □

LEMMA 3. *Let $f\colon [0,1] \to [0,1]$ be a continuous function. If $f$ has no cycles, then for any $x \in [0,1]$, the orbit $\{f^n(x)\}_{n=0}^{\infty}$ will converge to a fixed point.*

*Proof.* First notice that $f^n(x) \to z$ implies $z$ is a fixed point of $f$. Let $x \in [0,1]$. Since $f$ has no cycles, $m \neq n$ implies $f^m(x) \neq f^n(x)$ or $f^m(x)$ is a fixed point. We can assume that $f^m(x) \neq f^n(x)$ for $m \neq n$. By the Bolzano-Weierstrass Theorem, $\{f^n(x)\}_{n=0}^{\infty}$ has a cluster point, $z$. We will show that $z$ is unique.

Suppose that $z'$ is another cluster point of $\{f^n(x)\}_{n=0}^{\infty}$. Without loss of generality, let $z < z'$. By Lemma 2, $(z, z')$ cannot contain any points from the orbit of $x$. Moreover, $\{f^n(x)\}_{n=0}^{\infty}$ is the union of two disjoint subsequences: $\{f^{n_k}(x)\} \uparrow z$ and $\{f^{m_k}(x)\} \downarrow z'$. Hence, there exists a subsequence $\{a_j\}$ of $\{f^{n_k}(x)\}_{n=0}^{\infty}$ such that $f(a_j) \to z'$. Thus, $f(z) = z'$ by continuity. Similarly, $f(z') = z$, which contradicts that $f$ has no cycles. Therefore, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ must converge to $z$, and $z$ must be a fixed point of $f$ by continuity. □



Our initial approach to proving that a function with one fixed point and no cycles has the strong shadowing property was to determine two arbitrarily small intervals containing the fixed point with one interval contained in the other. These intervals would have the property that the image of the larger interval would be contained within the smaller interval. Then we could determine $\delta$ such that all $\delta$-pseudo-orbits would eventually be contained in the larger interval, so $f$ would have the strong shadowing property.

However, we cannot guarantee such intervals for $f$. For example, a function with a fixed point which attracts on one side of the fixed point and repels on the other side cannot have these desired intervals. Nonetheless, we *can* prove the existence of these intervals for $f^2$:

LEMMA 4. *Let $f\colon [0,1] \to [0,1]$ be a continuous function with no cycles and one fixed point $p$. For $\varepsilon > 0$, there exists $\eta > 0$, $I = (a,b)$, and $I_\eta = (a - \eta, b + \eta)$ such that $I \subset I_\eta \subseteq B_\varepsilon(p)$ with $p \in I$ and $f^2(I_\eta) \subseteq I$.*

*Proof.* Let $\varepsilon > 0$ be given. By Lemma 3, the fixed point $p$ attracts the entire interval $[0,1]$. Since $p$ is the only fixed point of $f$, then $f(x) > x$ on $[0,p)$ and $f(x) < x$ on $(p,1]$. We begin by finding an interval $I_1 \subseteq B_\varepsilon(p)$ such that $f(I_1) \subseteq I_1$. There are two possible cases.

Case 1: If $f(B_\varepsilon(p)) \subseteq B_\varepsilon(p)$, then define $I_1 = B_\varepsilon(p)$ and $\varepsilon' = \varepsilon/2$.

Case 2: There exists $z \in B_\varepsilon(p)$ such that $f(z) \notin B_\varepsilon(p)$. Without loss of generality, assume $z \in (p, p+\varepsilon)$. Since $f(x) < x$ on $(p,1]$ and $f(z) \notin B_\varepsilon(p)$, we have that $f(z) < p - \varepsilon$. Let $m_1 = \min\{x \in [p,z] : f(x) = p - \varepsilon\}$, and define $I_1 = (p - \varepsilon, m_1)$, $\varepsilon' = m_1 - p$. We claim that $f(I_1) \subseteq I_1$. If $x \in (p, m_1)$, then $f(x) > p - \varepsilon$. Furthermore, $f(x) < x < m_1$, so $f(x) \in (p - \varepsilon, m_1)$. If $x \in (p - \varepsilon, p)$, then there exists $x_1 \in (p, m_1)$ such that $f(x_1) = x$ by the Intermediate Value Theorem. Since $f(x) > x$, Lemma 2 implies $f(x) < x_1 < m_1$. Thus, $f(x) \in (x, x_1) \subseteq (p - \varepsilon, m_1)$, so $f(I_1) \subseteq I_1$.

Construct $I_2$ for $B_{\varepsilon'}(p)$ as $I_1$ was constructed for $B_\varepsilon(p)$. Without loss of generality we have $I_2 = (a,b)$ such that $f(I_2) \subseteq I_2$. Let $J = (c,d)$ denote an open interval of $f^{-1}(I_2)$ such that $I_2 \subset J \subset I_1$. There are two cases to be considered.

Case 1: $c < a$, $b < d$. Taking $\eta = \min\{a - c, d - b\}$, we have $f(I_\eta) \subseteq I_2$, so $f^2(I_\eta) \subseteq I_2$.

Case 2: $c = a$, $b < d$ ($c < a$, $b = d$ done similarly). Then there exists $e < c$ such that $f((e,d)) \subseteq (c,d)$. We also know that $f((c,d)) \subseteq (a,b) = I_2$. By defining $\eta = \min\{a - e, d - b\}$, $f^2(I_\eta) \subseteq I_2$.

In each of these cases $I = I_2$ and $I_\eta$ are the required intervals. □

Using this lemma, we will prove that $f^2$ has the strong shadowing property when $f$ has no cycles and one fixed point $p$. To show that $f$ itself has the strong shadowing property, we first establish the uniform convergence of $\{f^k\}_{k=0}^{\infty}$ to $c(x) \equiv p$ on $[0,1]$. Given $\varepsilon > 0$, by the proof of Lemma 4, there exists an open interval $I \subseteq B_\varepsilon(p)$ such that $p \in I$ and $f(I) \subseteq I$. We have pointwise convergence of $\{f^k\}$ by Lemma 3, so for each $x \in [0,1]$ there exists $N_x \in \mathbb{N}$ such that $f^n(x) \in I$ if $n \geq N_x$. Therefore, $\bigcup_{n \in \mathbb{N}} f^{-n}(I)$ is an open cover of $[0,1]$. By the compactness of $[0,1]$, we can find $N \in \mathbb{N}$ such that $f^N(x) \in I$ for all $x \in [0,1]$. Thus, $|f^n(x) - p| < \varepsilon$ for all $x \in [0,1]$ and $n \geq N$, so $\{f^k\}_{k=0}^{\infty}$ must converge uniformly to $c(x) \equiv p$.

Finally we show that if $f^2$ has the strong shadowing property, then $f$ has the strong shadowing property.

LEMMA 5. *Let $f\colon [0,1] \to [0,1]$ be a continuous function with no cycles and one fixed* *point $p$. If $f^k$ has the strong shadowing property for some $k \in \mathbb{N}$, then $f^n$ has the strong shadowing property for all $n \in \mathbb{N}$.*

*Proof.* First we show that if $f^k$ has the strong shadowing property, then $f$ has the strong shadowing property. For arbitrary $\varepsilon > 0$, $f^k$ having the strong shadowing property implies there exists $\delta'$, $0 < \delta' < \varepsilon/2$, such that all $\delta'$-pseudo-orbits of $f^k$ are $\varepsilon/2$ strong shadowed. Moreover, we know from the uniform convergence of $\{f^n\}$ that there exists $N \in \mathbb{N}$ such that if $n \geq N$, $|f^n(x) - p| < \varepsilon/4$ for $x \in [0,1]$. Let $K = \max\{N, k\}$. From uniform continuity, we have the existence of $0 < \delta_1 < \delta_2 < \cdots < \delta_K < \delta'/K$ such that $d(x,t) < \delta_i \Rightarrow d(f(x), f(t)) < \delta_{i+1}$ for $i = 1, \ldots, K-1$.

We constructed this $\delta$-chain so that for any $\delta_1$-pseudo-orbit $\{x_m\}_{m=0}^{\infty}$ of $f$, the sequence $\{x_{j+nk}\}_{n=0}^{\infty}$ is a $\delta'$-pseudo-orbit of $f^k$, where $j \in \mathbb{N}$. To see this, take an arbitrary $j \in \mathbb{N}$. Then $d(x_{j+1}, f(x_j)) < \delta_1 \Rightarrow d(f(x_{j+1}), f^2(x_j)) < \delta_2 \Rightarrow \cdots \Rightarrow d(f^{k-1}(x_{j+1}), f^k(x_j)) < \delta_k$. Similarly, $d(f^{k-2}(x_{j+2}), f^{k-1}(x_{j+1})) < \delta_{k-1} < \delta_k$, $d(f^{k-3}(x_{j+3}), f^{k-2}(x_{j+2})) < \delta_{k-2} < \delta_k$, $\ldots$ $d(x_{j+k}, f(x_{j+k-1})) < \delta_1 < \delta_k$. Applying the triangle inequality,

$$d(x_{j+k}, f^k(x_j)) < k \cdot \delta_k < K \cdot \frac{\delta'}{K} = \delta'.$$

Therefore $\{x_{j+nk}\}_{n=0}^{\infty}$ is a $\delta'$-pseudo-orbit of $f^k$.

By the construction of the $\delta$-chain, $d(x_n, f^n(x_0)) < \delta_n < \varepsilon/2 < \varepsilon$ for $n = 1, \ldots, K$. If $n > K$, let $r \equiv n \bmod K$, $0 \leq r < K$. If $r = 0$, then $d(x_n, f^n(x_0)) = d(x_{mk}, f^{mk}(x_0)) < \delta' < \varepsilon$ since $f^k$ has the strong shadowing property. Otherwise, $n = mk + r$ where $r \neq 0$, so $d(x_{mk+r}, f^{mk}(x_r)) < \varepsilon/2$. Since $n > K$, then $mk \geq K$, which implies $mk \geq N$. Therefore, $d(f^{mk+r}(x_0), p) < \varepsilon/4$ and $d(f^{mk}(x_r), p) < \varepsilon/4$, so $d(f^{mk+r}(x_0), f^{mk}(x_r)) < \varepsilon/2$. Applying the triangle inequality again, we have

$$d(f^{mk+r}(x_0), x_{mk+r}) = d(f^n(x_0), x_n) < \varepsilon.$$

Thus, $d(x_0, f^n(x_0)) < \varepsilon$ for all $n \in \mathbb{N}$, so $f$ has the strong shadowing property.

Finally, if $f$ has the strong shadowing property, then so does $f^n$ for all $n \in \mathbb{N}$. This follows because the $\delta$-pseudo-orbits for $f$ with $d(x_k, f(x_{k-1})) = 0$ for all $k \not\equiv 0 \bmod n$ are simply the $\delta$-pseudo-orbits for $f^n$. □

Notice that the last statement is independent of the condition that $f$ has no cycles and one fixed point. We use this fact in the proof of the main theorem.

### 3. Main Result - The Strong Shadowing Property.

THEOREM 6. *Given a continuous function $f : [0,1] \to [0,1]$, the following are equivalent:*

1. *$f$ has the strong shadowing property*
2. *$f$ has no cycles and one fixed point*
3. *$gr(f)$ and $gr^{-1}(f)$ have exactly one point in common.*

*Proof.* ($1 \Rightarrow 2$) Suppose $f$ has more than one fixed point. Notice that $f$ cannot have an interval of fixed points, otherwise the pseudo-orbit could move about the whole interval while the actual orbit remains fixed. Moreover, since $f$ is continuous, the fixed points of $f$ cannot even be dense in an interval by the same argument. Thus, there must exist fixed points $p_1 < p_2$ with no fixed points in the interval $(p_1, p_2)$.

Without loss of generality we can assume $f(x) > x$ on $(p_1, p_2)$. Given $\delta > 0$, consider the $\delta$-pseudo-orbit $x_0 = p_1$, $x_1 = x_0 + \delta/2$, $x_n = f(x_{n-1})$ for $n \geq 2$. Let $r = \min(f(x) - x)$ on the interval $[p_1 + \delta/2, p_2 - \delta/2]$; $r > 0$ since $f$ is a continuous function. Then the consecutive terms of the pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ must increase by at least $r$ until a term is at least as large as $p_2 - \delta/2$. Thus, $f$ does not have the strong shadowing property since the orbit remains at $p_1$.



We know by Lemma 5 that $f$ having the strong shadowing property implies $f^n$ has the strong shadowing property for all $n \in \mathbb{N}$. By the above, $f^n$ must have one fixed point for all $n \in \mathbb{N}$. Therefore, $f$ can have no cycles.

($2 \Rightarrow 1$) Let $g = f^2$, then $g$ has one fixed point $p$ and no cycles. For arbitrary $\varepsilon > 0$, Lemma 4 implies there exist $I = (a,b)$ and $I_\eta \subseteq (p-\varepsilon/2, p+\varepsilon/2)$ with $p \in I$ and $g(I_\eta) \subset I$. Furthermore, the uniform convergence of $\{g^n\}_{n=0}^{\infty}$ to $c(x) \equiv p$ guarantees the existence of $N \in \mathbb{N}$ such that $g^N(x) \in I$ for all $x \in [0,1]$.

Since $I_\eta \subseteq (p - \varepsilon/2, p + \varepsilon/2)$, $\eta < \varepsilon/2$. From the uniform continuity of $g$, we have the existence of $0 < \delta_1 < \delta_2 < \ldots < \delta_N < \eta/N$ such that $d(x,t) < \delta_i \Rightarrow d(g(x), g(t)) < \delta_{i+1}$ for $i = 1, \ldots, N$. Consider a $\delta_1$-pseudo-orbit of $g$, $\{x_n\}_{n=0}^{\infty}$. Thus,

$$d(x_1, g(x_0)) < \delta_1 \Rightarrow d(g(x_1), g^2(x_0)) < \delta_2 \Rightarrow \cdots \Rightarrow d(g^{N-1}(x_1), g^N(x_0)) < \delta_N.$$

Similarly,

$$d(g^{N-2}(x_2), g^{N-1}(x_1)) < \delta_{N-1} < \delta_N,$$

$$d(g^{N-3}(x_3), g^{N-2}(x_2)) < \delta_{N-2} < \delta_N, \ldots, d(x_N, g(x_{N-1})) < \delta_1 < \delta_N.$$

By applying the triangle inequality,

$$d(x_N, g^N(x_0)) < N \cdot \delta_N < N \cdot \frac{\eta}{N} \leq \eta < \varepsilon.$$

Repeating this procedure, we get $d(x_n, g^n(x_0)) < \varepsilon$ for $n = 1, \ldots, N$.

Since $N$ iterations have occurred, $g^k(x) \in I \subseteq I_\eta$ for all $k > N$. Furthermore, the $\delta_1$-pseudo-orbit $\{x_{N+k}\}_{k=1}^{\infty}$ cannot leave $I_\eta$ since $\delta < \eta$. Since diam $I_\eta < \varepsilon$, $d(x_k, g^k(x_0)) < \varepsilon$ for all $k > N$. Therefore, $g = f^2$ has the strong shadowing property, so $f$ has the strong shadowing property by 5. □

Returning to our original functions (Figure 1), although $h$ has the shadowing property, it does not have the strong shadowing property since it has more than one fixed point. On the other hand, $k$ does have the strong shadowing property since gr($k$) and gr$^{-1}$($k$) have exactly one point in common as shown in Figure 4a. In contrast,

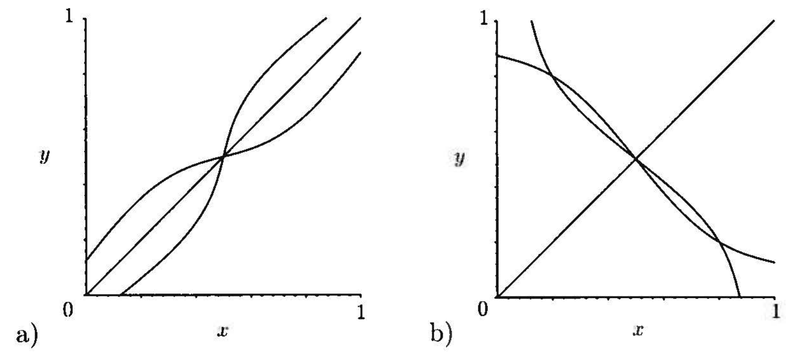


FIG. 4. *Graph of $k$ and $l$ and their reflections*

even though

$$l(x) = \frac{-3}{4}x + \frac{7}{8} + \frac{1}{4\pi}\sin(2\pi x)$$

has only one fixed point, its graph has three points in common with the graph of its reflection as seen in Figure 4b, thus it does not have the strong shadowing property.

Finally, a corollary to this theorem is that surjective functions on the unit interval do not have the strong shadowing property. By the theorem, we need only consider surjective functions with one fixed point. Let $f$ be such a function with fixed point $p$. Therefore, $f(x) > x$ on $[0, p)$ and $f(x) < x$ on $(p, 1]$. In particular, there exists $a \in (p, 1]$ so that $f(a) = 0$, and there exists $b \in [0, p)$ such that $f(b) = a$. Therefore, $f^2(x) < x$ on some interval of $[0, p)$, so $f$ has at least one 2-cycle. Then $f$ does not have the strong shadowing property.

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# ANOTHER PROOF OF THE STEINER-LEHMUS THEOREM

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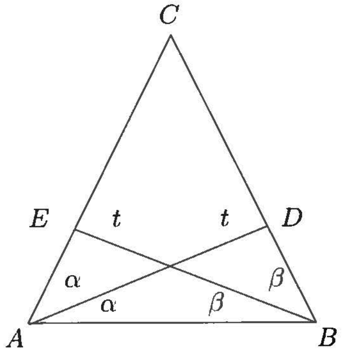
The Steiner-Lehmus theorem states that if a triangle has a pair of angle bisectors that are equal in length, then it is isosceles. The converse theorem, that an isosceles triangle has two equal angle bisectors, is quite elementary and readily proved by students of high school geometry. A direct proof of the Steiner-Lehmus theorem, on the other hand, is obscure, but many indirect proofs exist. In this note I add another elementary proof to that collection. For other proofs see [1, 2, 3, 4, 5].

Given $\triangle ABC$ with $BC = a$, $AC = b$ and $AB = c$. Draw the angle bisectors from $A$ and $B$ to intersect $BC$ and $AC$ at $D$ and $E$, respectively. Assume that $AD = BE = t$, $\angle BAD = \angle CAD = \alpha$ and $\angle ABE = \angle CBE = \beta$.




Let $A(XYZ)$ denotes the area of $\triangle XYZ$. We sum the area of triangles within $\triangle ABC$ in two different ways to get

$$A(BAD) + A(CAD) = A(ABE) + A(CBE),$$
$$\frac{1}{2}ct\sin(\alpha) + \frac{1}{2}bt\sin(\alpha) = \frac{1}{2}ct\sin(\beta) + \frac{1}{2}at\sin(\beta),$$
$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{a+c}{b+c}. \tag{1}$$

By the Law of Sines,

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b},$$
$$\frac{\sin(2\alpha)}{a} = \frac{\sin(2\beta)}{b},$$
$$\frac{2\sin(\alpha)\cos(\alpha)}{a} = \frac{2\sin(\beta)\cos(\beta)}{b},$$
$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{a\cos(\beta)}{b\cos(\alpha)}. \tag{2}$$

Equating (1) and (2), we get

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$$\frac{\cos(\beta)}{\cos(\alpha)} = \frac{b(a+c)}{a(b+c)}. \tag{3}$$

To prove that $\angle A = \angle B$, it is sufficient to show that both $\angle A < \angle B$ and $\angle A > \angle B$ lead to contradictions. We also need the fact that

$$\alpha, \beta < \pi/2, \tag{4}$$

for $2\alpha = \angle A < \pi$ and $2\beta = \angle B < \pi$. Assume $\angle A < \angle B$. It follows that $\alpha < \beta$ and $a < b$. By (4), the LHS of (3) is less than 1 while the RHS of (3) is greater than 1. Clearly we have an contradiction. Similarly we cannot have $\angle A > \angle B$ and it follows that $\angle A = \angle B$.

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# A DECOMPOSITION METHOD FOR SOLVING LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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**1. Introduction.** The study of linear systems of differential equations of the form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t), \quad \mathbf{x}(0) = \mathbf{c}, \tag{1.1}$$

where $\mathbf{x}(t) \in R^n$, $A$ an $n \times n$ matrix, $\mathbf{b}(t) \in R^n$ a given forcing function, and $\mathbf{x}(0) = \mathbf{c} \in R^n$ the initial condition, is one of the main topics that students study in the undergraduate curriculum. Indeed, students will frequently encounter linear systems in Linear Algebra, Differential Equations, Methods of Applied Mathematics courses and in other applied courses. Equation (1.1) is a mathematical model for many important applications that range over the spectrum from social and physical sciences to engineering. Variation of the parameter method is usually employed to solve the system in (1.1) (see, for examples, [2]-[4]). This method requires computing the exponential of the matrix $A$, $e^A$, and then expressing the solution as

$$\mathbf{x}(t) = e^{At}\mathbf{c} + \int_0^t e^{-A(t-\tau)}\mathbf{b}(\tau)d\tau. \tag{1.2}$$

The computation of the $e^{At}$ is not an easy task but may be achieved by finding the eigenvalues and eigenvectors of the matrix $A$. Another approach for solving the system (1.1) is to decouple it (if possible). Again this approach requires computing the eigenvalues and eigenvectors of the matrix $A$ and showing that $A$ is similar to a diagonal matrix $D$. The solution of the system (1.1) is then deduced from the solution of the decoupled system. We propose a decomposition method for solving (1.1). This method, modulo some theoretical background, is accessible to undergraduate students as it requires only basic knowledge of calculus and matrix algebra [1]. It provides in many instances a closed form solution and in others it provides an efficient way of computing a numerical solution. Thus the trade-off is the simplicity of the method and its suitability for numerical computation.

The decomposition method assumes a series solution; that is, we assume that the unknown vector $\mathbf{x}(t) \in R^n$ is a series of the form

$$\mathbf{x}(t) = \mathbf{x_1}(t) + \mathbf{x_2}(t) + \ldots = \sum_{i=0}^{\infty} \mathbf{x_i}(t) \tag{1.3}$$

and each iterate $\mathbf{x_i}(t) \in R^n$ is to be determined. In many instances few iterates are needed either to identify the closed form solution or to obtain an accurate numerical solution. We shall describe the decomposition method in Section 2 and illustrate several examples to clarify the algorithm in Section 3.

**2. Description of the Decomposition Method.** We will first give a brief description of the method that is normally employed to solve nonlinear problems of the form

$$x = Lx + f, \tag{2.1}$$

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where $L$ is a linear operator acting on an underlying function space $H$, and $f$ is a known function in $H$.

The method assumes a series solution

$$x = x_1 + x_2 + \ldots = \sum_{i=0}^{\infty} x_i, \tag{2.2}$$

where the iterates $x_i's$ are to be determined. Substituting (2.2) into (2.1) yield

$$\sum_{i=0}^{\infty} x_i = \sum_{i=0}^{\infty} Lx_i + f. \tag{2.3}$$

If the series converges, we can determine the iterates $x_i$ as in

$$x_0 = f \qquad x_n = Lx_{n-1}, n > 0. \tag{2.4}$$

Equation (2.4) is the decomposition algorithm that we use to solve (1.1). Indeed, we shall adapt the decomposition method to solve the linear system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t), \quad \mathbf{x}(0) = \mathbf{c}, \tag{2.5}$$

where $\mathbf{x}(t) \in R^n$, $\mathbf{x}(0) = \mathbf{c} \in R^n$, $A$ an $n \times n$ matrix, and $\mathbf{b}(t) \in R^n$ is a known forcing function.

Upon formally integrating (2.5), we get

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t A\mathbf{x}(\xi)d\xi + \int_0^t \mathbf{b}(\xi)d\xi. \tag{2.6}$$

Equation (2.6) is in the form (2.1) with $Lx = \int_0^t A\mathbf{x}(\xi)d\xi$ and $f = \mathbf{x_0} + \int_0^t \mathbf{b}(\xi)d\xi$. The solution vector $\mathbf{x}(t)$ is

$$\mathbf{x}(t) = \sum_{i=0}^{\infty} \mathbf{x_i}, \tag{2.7}$$

where each $\mathbf{x_i}$ is a vector in $R^n$. Substituting (2.7) into (2.6), we obtain

$$\sum_{i=0}^{\infty} \mathbf{x_i} = \mathbf{x}(0) + \sum_{i=0}^{\infty} \int_0^t A\mathbf{x_i}(\xi)d\xi + \int_0^t \mathbf{b}(\xi)d\xi. \tag{2.8}$$

For the series in (2.7) to converge, we set

$$\mathbf{x_0} = \mathbf{x}(0) + \int_0^t \mathbf{b}(\xi)d\xi, \qquad \mathbf{x_n} = \int_0^t A\mathbf{x_{n-1}}(\xi)d\xi, \; n > 0, \tag{2.9}$$

Equation (2.9) is now the basis of the decomposition algorithm for solving the linear system (2.5). Indeed, (2.9) determines all the iterates $\mathbf{x_i}$. We shall now demonstrate the method with some examples.



**3. Examples.** In this section we solve several examples of linear systems of differential equations to illustrate the decomposition method.

EXAMPLE 1. Consider

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2, & x_1(0) &= 1, \\ \dot{x}_2 &= 2x_2, & x_2(0) &= 1. \end{aligned} \tag{3.1}$$

The system can be written in matrix form as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.2}$$

Upon formally integrating the system in (3.2), we get

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t A\mathbf{x}(\xi)d\xi \tag{3.3}$$

Assuming the series solution to (3.2), we obtain

$$\mathbf{x_0} = \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x_n} = \int_0^t A\mathbf{x_{n-1}}(\xi)d\xi = \begin{bmatrix} (2t)^n/n! \\ (2t)^n/n! \end{bmatrix}. \tag{3.4}$$

Thus the solution is obtained by summing these iterates. It is clear that the sum of the components of these iterate $(1 + 2t + \frac{(2t)^2}{2!} + \ldots)$ add up the exponential function $e^{2t}$. Thus the solution to the system (3.2) is

$$\mathbf{x}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}. \tag{3.5}$$

EXAMPLE 2. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_3, & x_1(0) &= 0, \\ \dot{x}_2 &= x_1 + 2x_2, & x_2(0) &= 1, \\ \dot{x}_3 &= 8x_1 + 3x_3, & x_3(0) &= 0. \end{aligned} \tag{3.6}$$

We can write this system in matrix form as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 8 & 0 & 3 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.7}$$

Upon formally integrating the system in (3.7), we get

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t A\mathbf{x}(\xi)d\xi. \tag{3.8}$$

Assuming the series solution to (3.6), we obtain

$$\mathbf{x_0} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x_k} = \int_0^t A\mathbf{x_{k-1}}(\xi)d\xi = \begin{bmatrix} 0 \\ (2t)^k/k! \\ 0 \end{bmatrix}. \tag{3.9}$$

Thus the solution is obtained by summing the iterates. It is clear that the sum of the second component of these iterates $(1 + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \ldots)$ adds up to the exponential function $e^{2t}$. Thus the solution to the system (3.6) is

$$\mathbf{x}(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} \tag{3.10}$$

EXAMPLE 3. Consider the non-homogeneous system

$$\begin{aligned} \dot{x}_1 &= -x_2 + 1, & x_1(0) &= 0, \\ \dot{x}_2 &= x_1, & x_2(0) &= 0. \end{aligned} \tag{3.11}$$

We can write this system in matrix form as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.12}$$

Upon formally integrating the system in (3.12), we get

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t A\mathbf{x}(\xi)d\xi + \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\xi. \tag{3.13}$$

Assuming the series solution (3.12), we obtain $\mathbf{x_0} = [t, 0]$, and

$$\mathbf{x_n} = \int_0^t A\mathbf{x_{n-1}}(\xi)d\xi = \begin{cases} \begin{bmatrix} (-1)^{n(n+1)/2} t^{n+1}/(n+1)! \\ 0 \end{bmatrix}, & \text{for } n \text{ even}, \\ \begin{bmatrix} 0 \\ (-1)^{n(n+1)/2} t^{n+1}/(n+1)! \end{bmatrix}, & \text{for } n \text{ odd}. \end{cases}$$

Thus the solution is obtained by summing the iterates. It is clear that the sum of the first component $(t - \frac{t^3}{3!} + \ldots)$ and of the second component $(\frac{t^2}{2!} - \frac{t^4}{4!} + \ldots)$ of these iterates add up to $\sin t$ and $1 - \cos t$ respectively. Thus the solution to the system (3.12) is

$$\mathbf{x}(t) = \begin{bmatrix} \sin t \\ 1 - \cos t \end{bmatrix}. \tag{3.14}$$

Although the above examples are not that messy, yet they show the ease of obtaining the solution using the decomposition method. The last example that we present shows that, in the absence of a closed form solution, the decomposition method yields with relatively few iterates a "reasonable answer" when compared numerically to the answer obtained using the Computer Algebra System (Maple V).

EXAMPLE 4. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + 3x_2 + 2x_3, & x_1(0) &= 1, \\ \dot{x}_2 &= 2x_2 + 3x_3, & x_2(0) &= 0, \\ \dot{x}_3 &= x_1 + 2x_3, & x_3(0) &= 1. \end{aligned} \tag{3.15}$$

We can write this system in matrix form as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{3.16}$$



Upon formally integrating the system in (3.16), we get

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t A\mathbf{x}(\xi)d\xi. \tag{3.17}$$

Assuming the series solution of (3.16), we obtain

$$\begin{aligned} \mathbf{x_0} &= \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ \mathbf{x_1} &= \int_0^t A\mathbf{x_0}(\xi)d\xi = \begin{bmatrix} 3t \\ 3t \\ 3t \end{bmatrix} \\ \mathbf{x_2} &= \int_0^t A\mathbf{x_1}(\xi)d\xi = \begin{bmatrix} 9t^2 \\ 15t^2/2 \\ 9t^2/2 \end{bmatrix} \\ \mathbf{x_3} &= \int_0^t A\mathbf{x_2}(\xi)d\xi = \begin{bmatrix} 81t^3/6 \\ 19t^3/2 \\ 18t^3/3 \end{bmatrix}, \ldots \end{aligned} \tag{3.18}$$

Therefore we can get the approximate solution by choosing the several terms

$$\mathbf{x}(t) = \mathbf{x_0} + \mathbf{x_1} + \mathbf{x_2} + \ldots = \begin{bmatrix} 1 + 3t + 9t^2 + \frac{81}{6}t^3 + \frac{27}{2}t^4 \cdots \\ 3t + \frac{15}{2}t^2 + \frac{19}{2}t^3 + \frac{37}{4}t^4 + \cdots \\ 1 + 3t + \frac{9}{2}t^2 + \frac{18}{3}t^3 + \frac{51}{8}t^4 + \cdots \end{bmatrix}. \tag{3.19}$$

The error is less than $10^{-3}$ when we compare the solution of the decomposition method using only four terms with the solution obtained using Maple V (see Tables 1 - 3). The accuracy of the method can be improved by adding more iterates.

| t | Maple($x_1$) | Decomp($x_1$) |
|---|---|---|
| .02 | 1.063710 | 1.063710 |
| .04 | 1.135300 | 1.135299 |
| .06 | 1.215500 | 1.215491 |
| .08 | 1.305102 | 1.305065 |
| .10 | 1.404966 | 1.404850 |
| .12 | 1.516020 | 1.515727 |

TABLE 1

*The Comparison between the $1^{st}$ component of the solution obtained numerically using Maple and of the decomposition method solution in (3.19).*

Our goal was to introduce an alternate method for solving linear systems of differential equations that is accessible to undergraduate students with knowledge of basic calculus and matrix algebra.

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| t | Maple($x_2$) | Decomp($x_2$) |
|---|---|---|
| .02 | .630775 | .630775 |
| .04 | .132632 | .132632 |
| .06 | .209178 | .209172 |
| .08 | .293269 | .293243 |
| .10 | .385506 | .385425 |
| .12 | .486538 | .486334 |

TABLE 2
*The Comparison between the $2^{nd}$ component of the solution obtained numerically using Maple and of the decomposition method solution in (3.19).*

| t | Maple($x_3$) | Decomp($x_3$) |
|---|---|---|
| .02 | 1.061849 | 1.061849 |
| .04 | 1.127601 | 1.127600 |
| .06 | 1.197583 | 1.197579 |
| .08 | 1.272151 | 1.272133 |
| .10 | 1.351694 | 1.351638 |
| .12 | 1.436632 | 1.436490 |

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# ANALYSIS OF THE SUBTANGENT

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In volume five of Readings for Calculus [1], there is a section on Isaac Newton with an exercise that begins, "In the early days of Calculus the subtangent was considerably more important than it is now." The problem continues with questions about how to apply the subtangent operator to basic algebraic and trigonometric functions. This exercise prompted an exploration of the mathematical and historical significance of the subtangent.

The natural first approach to the subtangent is geometrical. The subtangent of a curve at a given point is defined as the line segment indicated in Figure 1. It also refers to the algebraic, or signed, length of this segment.

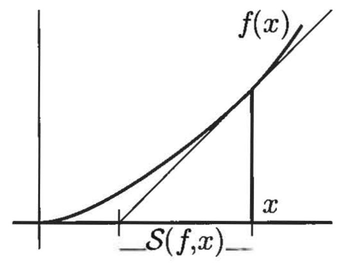


FIG. 1. *The subtangent, $\mathcal{S}(f,x)$, of $f$ at $x$.*

Before we can go into more detail about the subtangent, some notation needs to be defined. I will use $\mathcal{S}$ to denote the subtangent operator with either $\mathcal{S}f$ or $f^*$ representing the subtangent function of $f$ (similar to representing the derivative function of $f$ by either $\mathcal{D}f$ or $f'$).

For a case such as that in Figure 1, where $y = f(x)$, the subtangent can be calculated from the height of the vertical segment, $y$, and the slope of the tangent line, $y'$. From the definition of slope, $y' = \frac{y}{y^*}$, so $y^* = \frac{y}{y'}$. So for example, the length of the subtangent of the parabola $y = x^2$ is

$$y^* = \frac{x^2}{2x} = \frac{x}{2}.$$

For $y = x^3$, we have

$$y^* = \frac{x^3}{3x^2} = \frac{x}{3}.$$

If this formula works for all cases, then in general, $\mathcal{S}(x^n) = \frac{x}{n}$.

Like many problems in calculus, questions about the subtangent have both a geometrical and an algebraic side, and the two different conceptual approaches can yield different insights. One of the purposes of this paper is to exhibit this duality. A general argument that the formula for the subtangent given above is valid is easier with less geometry and more algebra. The argument runs as follows. Let $f(x)$ be a

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function of $x$, let $t(x)$ be the tangent line to $f(x)$ at $x_0$, and let $x_I$ be the $x$ intercept of $t(x)$, see Figure 2. Then $f^*(x_0)$ will be $x_0 - x_I$. The point-slope form of the tangent

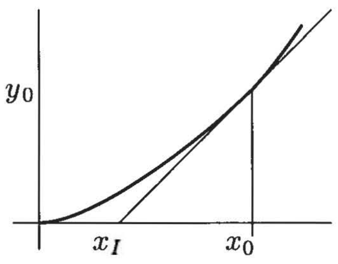


FIG. 2.

line gives

$$t(x) - t(x_0) = f'(x_0)(x - x_0).$$

Note that when $t(x) = 0$ in this formula, $x = x_I$, and also that, because $t(x)$ is tangent to $f(x)$ at $x_0$, $t(x_0) = f(x_0)$. Putting these two facts together gives us

$$x_0 - x_I = \frac{f(x_0)}{f'(x_0)} = f^*(x_o)$$

which says that the subtangent of a function $f(x)$ is given by $\mathcal{S}f(x) = f(x)/f'(x)$. Note that this implies that the subtangent of a function is undefined when the tangent line is horizontal (that is, when the derivative is 0). It follows that a constant function's subtangent is undefined everywhere.

The historical importance of the subtangent is its direct relationship to the derivative. Our perspective on the two has changed somewhat over time; we use the derivative exclusively and have forgotten about the subtangent. However, in the early development of calculus, the subtangent was primary and the derivative secondary. Pierre de Fermat's technique of drawing the tangent line was a precursor to the modern technique of differentiation. To draw a line tangent to a curve at a given point, he would first find the subtangent (through geometrical means other than the ones which we have employed). He would then have the two points $(x_I, 0)$ and $(x_0, f(x_0))$ through which to draw the tangent line [2], which is the reverse of what we just did. Similar techniques were used by Hudde, Sluze, and Barrow [3]. Even Leibniz, in his first paper on the differential calculus written in 1684, uses the defining proportion $dy : dx = y : y^*$.However, we will see that the derivative turns out to be the simpler and more useful of the two operators, which is probably why it eventually replaced the subtangent as the standard slope operator.

Now confident that our initial formula is correct, we can go on to compile a list of subtangent "rules." We have already established that $\mathcal{S}(x^n) = x/n$. Since the power rule for differentiation is valid for all real $n$ it follows that this rule is as well. The subtangents of other elementary functions can be calculated in the same manner. For instance, suppose we want to calculate the subtangent of $f(x) = \sin(x)$. Since $f'(x) = \cos(x)$, we have that $f^*(x) = \tan(x)$. The subtangent of $\cos(x)$ is the negative reciprocal of this, $-\cot(x)$.

A historical document illustrates what happens when we compute the subtangent of an exponential function. In the years 1676-1677, Newton and Leibniz corresponded



a number of times to discuss their respective methods of differentiation and integration. In a letter dated June 21, 1677, Leibniz gave a solution to the problem of finding a curve whose subtangent is constant [1]. This type of curve turns out to be an exponential, which can be seen easily with our modern advantages. For example, let $f(x) = e^x$. Since $f'(x) = f(x)$, $f^*(x) = 1$. It turns out that all exponential functions have a constant subtangent. This means that if you draw the subtangent at any point on an exponential curve, its length will always be the same, and is a result of the special characteristic of the exponential function that its growth rate, or slope, grows in proportion to its height, so that the two always have the same ratio.

While learning calculus, students are often asked to look at how a simple geometric transformation of a function changes its derivative. Similarly, one can ask how a transformation of a function changes its subtangent. We next examine various ways to transform a function, and the effects these transformations have on the subtangent.

First we consider vertically shifted functions of the form $f(x) = g(x) + a$, where $a$ is a constant. Since $f'(x) = g'(x)$, we have

$$f^*(x) = \frac{g(x) + a}{g'(x)} = g^*(x) + \frac{a}{g'(x)} = g^*(x) + \frac{ag^*(x)}{g(x)},$$

where we have made use of the fact that $f' = f/f^*$. Thus, the subtangent of a vertically shifted function is the subtangent of the original function plus a correction term. We can visualize this geometrically by noting that $x_I$ will move either to the left or right as the graph of the function moves up or down, but $x_0$ will remain unchanged.

Another simple transformation is a vertical scaling, which corresponds to multiplying a function by a constant. In this case, $f(x) = ag(x)$, and $f'(x) = ag'(x)$, so the constant cancels out and the subtangent is unchanged:

$$f^*(x) = \frac{ag(x)}{ag'(x)} = g^*(x).$$

This can be pictured geometrically as a function being scaled upward along the $y$-axis; its height and steepness will increase in the same ratio. It implies that a reflection about the $x$-axis, see Figure 3, doesn't change the subtangent, since this represents

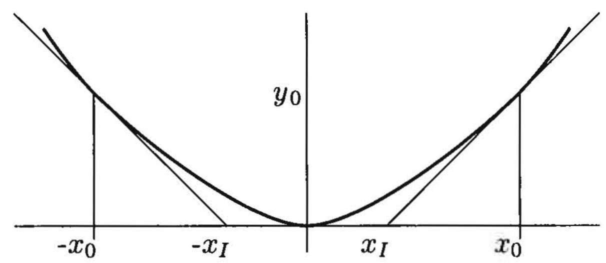


FIG. 3. *Reflection about the y axis.*

a multiplication of the original function by -1. Furthermore, it should be obvious that if a function is shifted along the $x$-axis the derivative function will be shifted by the same amount and the subtangent function will likewise. However, if a function is scaled along the $x$-axis the subtangent is changed. Suppose $f(x) = g(ax)$. Then $f'(x) = ag'(ax)$, so

$$f^*(x) = \frac{g(ax)}{ag'(ax)} = \frac{1}{a}g^*(ax)$$

This implies that a reflection about the $y$-axis, which corresponds to scaling along the $x$-axis by $-1$, will change the sign of the subtangent Figure 4. In all cases the sign of

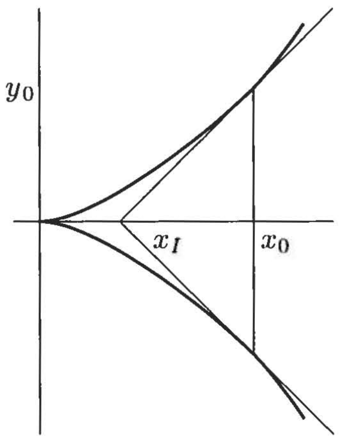


FIG. 4. *Reflection about the $x$ axis.*

the subtangent is an indication of the direction in which it is measured. If $x_0$ is to the left of $x_I$, the subtangent will be positive; if $x_I$ is to the left, the subtangent will be negative. Reflection about the $x$-axis doesn't change this; reflection about the $y$-axis does.

A list of subtangent rules must include product and quotient rules. If $f(x) = g(x)h(x)$, then $f'(x) = g(x)h'(x) + h(x)g'(x)$, so

$$f^*(x) = \frac{g(x)h(x)}{g(x)h'(x) + g'(x)h(x)} = \frac{g^*(x)h^*(x)}{g^*(x) + h^*(x)}.$$

And if $f(x) = \dfrac{g(x)}{h(x)}$, then $f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{h(x)^2}$, so

$$f^*(x) = \frac{g(x)h(x)}{h(x)g'(x) - g(x)h'(x)} = \frac{g^*(x)h^*(x)}{g^*(x) - h^*(x)}.$$

Two other scenarios we can look at are the sum of two functions and a composition of functions. In the first case, let $f(x) = g(x) + h(x)$, so that $f'(x) = g'(x) + h'(x)$. Then

$$f^*(x) = \frac{g(x) + h(x)}{g'(x) + h'(x)} = \frac{g^*(x)h^*(x)(g(x) + h(x))}{g(x)h^*(x) + g^*(x)h(x)}.$$

Note, therefore, that the subtangent is not a linear operator. To consider a composition, let $f(x) = g(h(x))$. Then $f'(x) = g'(h(x))h'(x)$ and

$$f^*(x) = \frac{g(h(x))}{g'(h(x))h'(x)} = \frac{g^*(h(x))}{h'(x)} = \frac{g^*(h(x))h^*(x)}{h(x)}.$$



| $f(x)$ | $f'(x)$ | $f^*(x)$ |
|---|---|---|
| $a$ | $0$ | undefined |
| $x^n$ | $nx^{n-1}$ | $x/n$ |
| $e^x$ | $e^x$ | $1$ |
| $a^x$ | $a^x \ln(a)$ | $1/\ln(a)$ |
| $\ln(x)$ | $1/x$ | $x\ln(x)$ |
| $\sin(x)$ | $\cos(x)$ | $\tan(x)$ |
| $\cos(x)$ | $-\sin(x)$ | $-\cot(x)$ |
| $\tan(x)$ | $\sec^2(x)$ | $\sin(x)\cos(x)$ |
| $g(x) + h(x)$ | $g'(x) + h'(x)$ | $\dfrac{g^*(x)h^*(x)(g(x)+h(x))}{g(x)h^*(x) + g^*(x)h(x)}$ |
| $g(x)h(x)$ | $g(x)h'(x) + h(x)g'(x)$ | $\dfrac{g^*(x)h^*(x)}{g^*(x) + h^*(x)}$ |
| $\dfrac{g(x)}{h(x)}$ | $\dfrac{h(x)g'(x) - g(x)h'(x)}{h(x)^2}$ | $\dfrac{g^*(x)h^*(x)}{g^*(x) - h^*(x)}$ |
| $g(h(x))$ | $g'(h(x))h'(x)$ | $\dfrac{g^*(h(x))h^*(x)}{h(x)}$ |
| $g(x) + a$ | $g'(x)$ | $g^*(x) + \dfrac{ag^*(x)}{g(x)}$ |
| $ag(x)$ | $ag'(x)$ | $g^*(x)$ |
| $g(x+a)$ | $g'(x+a)$ | $g * (x+a)$ |
| $g(ax)$ | $ag'(ax)$ | $g^*(ax)/a$ |

TABLE 1
*Rules for the subtangent*

Thus we have developed a chain rule for subtangents.

All of the foregoing results are summed up in Table 1.

Having developed the theory of the subtangent thus far, we will now take a step backwards conceptually to develop an analytical definition of the subtangent, similar to that of the derivative. We will develop this approach geometrically at first, closely following Barrow [3] (with the adaptation of limits, of course) and then establish it directly from the analytical definition of the derivative.

In Figure 5, as before, we have $f^*(x) = x - x_I$, and the ratio of the sides of the larger triangle will be equal to the ratio of the sides of the differential triangle as $h$ approaches 0. That is,

$$\frac{f^*(x)}{f(x)} = \lim_{h\to 0}\frac{h}{a} = \lim_{h\to 0}\frac{h}{f(x) - f(x-h)}.$$

Therefore,

$$f^*(x) = \lim_{h\to 0}\frac{f(x)h}{f(x) - f(x-h)}.$$

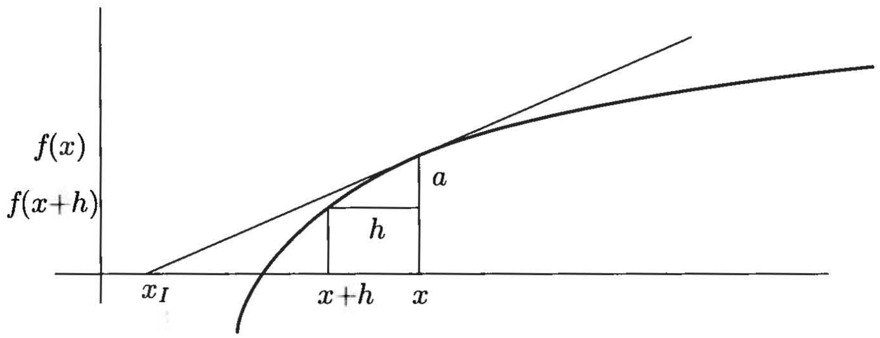


FIG. 5. *The differential triangle.*

It doesn't take much to see that this is just an alternate way of stating the definition of the derivative, for using $f^*(x) = f(x)/f'(x)$ gives us

$$\frac{f(x)}{f'(x)} = \lim_{h\to 0} \frac{f(x)h}{f(x) - f(x-h)}.$$

With the assumption that $f(x) \neq 0$, cancelling the $f(x)$ and inverting gives

$$f'(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h},$$

and we have successfully reinvented the wheel.

The foregoing treatment of the subtangent shows that it is more complicated and less user-friendly than the derivative. The inclusion of $y$ and the inversion of $y'$ in the expression for the subtangent avoid what we now view as the heart of the matter, the slope of the tangent line itself, and makes the subtangent superfluous. However, the subtangent has managed to find its own useful niche. To finish up, here is a way that the subtangent has stuck with us. Although most calculus students never hear the term "subtangent," they may recognize the expression $y/y'$ from Newton's method for approximating the intercepts of a function. In Figure 2 it is obvious that $x_I$ is closer to a root of the function than $x_0$. So, having picked a value for $x_0$, you could compute the subtangent at that value by the formula $S(x) = y/y'$. Then you could use $S(x) = x_0 - x_I$ to find $x_I$. This value could be used for the new $x_0$, which would in turn produce a new subtangent and a new $x_I$ closer to the root of the function. Therefore, repeated applications of the formula $x_{n+1} = x_n - S(x_n)$ will produce numerical values that converge on the root of the function. This is Newton's method.

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# GEOMETRICAL ASPECTS OF AN OPTIMAL TRAJECTORY

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**1. Introduction.** The best-known fact about projectile motion, as familiar to high school track coaches as it is to calculus students, is that a launch angle of 45° produces the maximal horizontal range. Galileo Galilei [2] was the first to prove this result, which relies on the assumption that the only active agent is a constant vertically acting gravitational force. His analysis also showed that the projectile's trajectory is a parabola with vertical axis at mid-range. The symmetry of the parabolic trajectory implies that the triangle formed by the horizontal base of the trajectory, that is, the segment between the launch and impact points, and the tangents to the trajectory at the launch and impact points is isosceles. Further, in the case of a trajectory with maximal range, that is, one with a launch angle of 45°, this isosceles triangle is necessarily *right* and therefore the side of this triangle that is tangent to the trajectory at the launch point *bisects* the angle formed by the ray from the launch point through the impact point (i.e., the horizontal ray) and the vertical ray through the launch point.

In a recent note William Chau [1] gave a twist to the standard optimal trajectory problem. He treated the problem of maximizing the horizontal distance travelled when the projectile terminates on a given horizontal line $y = y_1$ (the case $y_1 = 0$ is then the *standard* problem). While in this modified problem the optimal launch angle is not necessarily 45°, we show nevertheless that the two geometrical features of the standard problem noted above continue to hold for the modified problem. Specifically, the tangents to the optimal trajectory at the launch and impact points, respectively, are *perpendicular*, and the launch vector *bisects* the angle formed by the ray from the launch point through the impact point and the vertical ray.

**2. Optimality Conditions.** We begin by recalling the parametric equations for the trajectory of a point particle of unit mass launched with initial speed $v$ from the origin at an angle $\theta$ with the positive horizontal axis:

$$\begin{aligned} x &= (v\cos\theta)t \\ y &= (v\sin\theta)t - (g/2)t^2 \end{aligned}$$

The first equation is a representation of the law of inertia and the second equation combines vertical inertial motion with Galileo's law of fall. As an alternative to Chau's derivation of the optimal launch angle, we consider the constrained maximization problem

$$\begin{aligned} \text{maximize: } & x(\theta,t) = (v\cos\theta)t \\ \text{subject to: } & y(\theta,t) = (v\sin\theta)t - (g/2)t^2 = y_1. \end{aligned}$$

A quick application of the Lagrange multiplier rule (see e.g., [4]) then gives the following optimality conditions

$$\begin{aligned} -(v\sin\theta)t &= \lambda(v\cos\theta)t \\ v\cos\theta &= \lambda(v\sin\theta - gt) \end{aligned}$$

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and hence, $\lambda = -\tan\theta$. Substituting this into the second condition gives the impact time:

$$t = \frac{v}{g}\csc\theta \tag{1}$$

Putting this into the constraint

$$(v\sin\theta)t - \frac{g}{2}t^2 = y_1 \tag{2}$$

then leads, after a little algebra, to Chau's characterization of the optimal launch angle:

$$\sin\theta = \frac{v}{\sqrt{2(v^2 - gy_1)}} \tag{3}$$

The Lagrange multiplier approach is very generous: it gives the impact time (1) as lagniappe! Alternatively, one could get the impact time (1) by solving for the larger root of the quadratic equation (2) but this is quite a bit more involved (Try it!).

In the next section (1) and (3) are put to use to derive the two promised geometrical characteristics of the optimal trajectory of the modified problem.

**3. Some Geometry.** The slope of the tangent line to the optimal trajectory at the launch point is $\tan\theta$, where $\theta$ is given by (3). At the impact time $t$ given by (1), we have

$$x'(t) = v\cos\theta$$

and

$$y'(t) = v\sin\theta - gt = v(\sin\theta - \csc\theta) = -v\cos\theta\cot\theta$$

and hence the slope of the tangent line to the optimal trajectory at the impact point is

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = -\cot\theta,$$

proving that the tangent lines to the optimal trajectory at the launch and impact points are perpendicular.

Now let $B$ be the intersection point of the tangents to the optimal trajectory at the launch point $O$ and impact point $A$, respectively, and let $OV$ and $OH$ be the vertical and horizontal rays, respectively, through the launch point (see Figure 1). Our second geometrical claim is that $\angle AOB = \angle BOV$. Since $\angle HOB = \theta$, the optimal launch angle, we have $\tan(\angle BOV) = \cot\theta$, and the required geometrical condition is equivalent to

$$\tan(\angle AOB) = \cot\theta.$$

The coordinates of the impact point $A$ are $((v\cos\theta)t, y_1) = \left(\frac{v^2}{g}\cot\theta, y_1\right)$ and the line through $B$ and $A$ therefore has equation

$$y - y_1 = -\cot\theta\left(x - \frac{v^2}{g}\cot\theta\right) = -(\cot\theta)x + \frac{v^2}{g}\cot^2\theta.$$



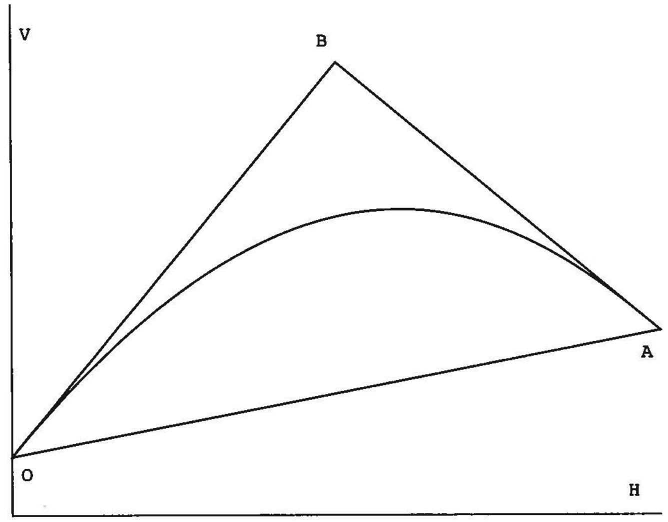


FIG. 1. *An Optimal Trajectory*

But by (3),

$$y_1 = \frac{v^2}{g}\left(1 - \frac{1}{2}\csc^2\theta\right)$$

and the line $OB$ has equation $y = (\tan\theta)x$. Making these substitutions we find the the first coordinate $x$ of the intersection point $B$ satisfies

$$(\tan\theta + \cot\theta)x = \frac{v^2}{g}(1 - \frac{1}{2}\csc^2\theta + \cot^2\theta) = \frac{v^2}{2g}\csc^2\theta$$

and hence

$$x = \frac{v^2}{2g}\cot\theta.$$

The coordinates of $B$ are therefore $\left(\frac{v^2}{2g}\cot\theta, \frac{v^2}{2g}\right)$ and hence

$$|OB| = \frac{v^2}{2g}\csc\theta.$$

Also,

$$|AB| = \sqrt{\left(\frac{v^2}{2g}\cot\theta\right)^2 + \left(y_1 - \frac{v^2}{2g}\right)^2} = \frac{v^2}{2g}\cot\theta csc\theta.$$

Finally, since $\angle ABO$ is right

$$\tan(\angle AOB) = \frac{|AB|}{|OB|} = \cot\theta = \tan(\angle BOV),$$

proving that the ray $OB$ bisects the angle formed by the rays $OA$ and $OV$, as promised.

**4. A Bow to Dr. Halley.** While the discussion above is a useful classroom demonstration of a number of topics from geometry, trigonometry and calculus, the geometrical characterization of the optimal launch angle also follows from an old result of Edmund Halley [3]. He treated the slightly different problem of firing up a fixed sloping battlefield. Halley showed that the optimal launch vector bisects the angle formed by the battlefield and the vertical (e.g., for a horizontal battlefield the launch angle would be $45^\circ$). If we think of the shot giving maximal horizontal range at height $y = y_1$ and consider the ray from the launch point through the impact point to be the sloping battlefield, then the given shot would also have to be optimal in Halley's sense. For otherwise a shot further up the sloping battlefield would be possible and this shot would impact on $y = y_1$ further down range than the original shot which was assumed to be optimal. So old dogs still do some pretty nice tricks!

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# FIBONACCI, LUCAS, AND EIGENVALUES

THOMAS KOSHY*

Fibonacci numbers $F_n$ and Lucas numbers $L_n$ are often defined recursively:

$$F_1 = 1,\ F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 3$$

and

$$L_1 = 1,\ L_2 = 3$$
$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 3$$

They are given explicitly by Binet's Formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ are solutions of the quadratic equation $t^2 - t - 1 = 0$.

In 1960, C. H. King studied a $2 \times 2$ matrix for his masters thesis at then San Jose State College in California, which he called the $Q$-matrix, [3, 4]. It is basically the same as the matrix

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Using induction, it can be shown that

$$Q^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$$

Since $|Q^n| = |Q|^n = (-1)^n$, this yields the Cassini formula, [2, 4], $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, where $|A|$ denotes the determinant of the square matrix $A$.

The eigenvalues of $Q$ are the solutions of the equation $|Q - \lambda I| = 0$, where $I$ denotes the $2 \times 2$ identity matirx, [1]. They are given by

$$\begin{vmatrix} \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = 0$$

that is, $\lambda^2 - \lambda - 1 = 0$; so $\lambda = \alpha, \beta$.

The eigenvector $X$ corresponding to $\lambda$ is given by $AX = \lambda X$, that is,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

*Framingham State College

This matrix equation yields $y = \lambda x$. Therefore, $(x, y) = (x, \lambda x) = (1, \lambda)x$. Thus the eigenvector corresponding to $\lambda$ is $(1, \lambda)x$, where $x$ is an arbitrary real number.

Geometrically, it represents the line $y = \lambda x$. Since $\alpha\beta = -1$, it follows that the lines $y = \alpha x$ and $y = \beta x$ are indeed perpendicular.

It is well-known that if $t$ is an eigenvalue of $A$ with the corresponding eigenvector $X$, then $t^n$ is an eigenvalue of $A^n$ with the corresponding eigenvector $X$, [1]. Consequently, $\lambda^n$ is an eigenvalue of $Q^n$, so $|Q^n - \lambda I| = 0$; that is,

$$\begin{vmatrix} F_{n-1} - \lambda & F_n \\ F_n & F_{n+1} - \lambda \end{vmatrix} = 0$$

Expanding this, we get

$$\lambda^{2n} - (F_{n-1} + F_{n+1})\lambda^n + F_{n-1}F_{n+1} - F_n^2 = 0$$

Since $F_{n-1} + F_{n+1} = L_n$, [2, 4], by Cassini's formula, this yields

$$\lambda^{2n} - L_n\lambda^n + (-1)^n = 0$$

Using the quadratic formula,

$$\lambda^n = \frac{L_n \pm \sqrt{L_n^2 - 4(-1)^n}}{2} \tag{1}$$

Since $\alpha > 0$ and $\beta < 0$, it follows that

$$\alpha^n = \frac{L_n + \sqrt{L_n^2 - 4(-1)^n}}{2} \tag{2}$$

and

$$\beta^n = \frac{L_n - \sqrt{L_n^2 - 4(-1)^n}}{2} \tag{3}$$

For example, $\alpha^5 = (L_5 + \sqrt{L_5^2 + 4}/2 = (11 + 5\sqrt{5})/2$ and $\beta^5 = (L_5 - \sqrt{L_5^2 + 4})/2 = (11 - 5\sqrt{5})/2$. Both may be verified using the actual values of $\alpha$ and $\beta$.

Formulas (2) and (3) yield an interesting formula for $F_n$ in terms of $L_n$:

$$\alpha^n - \beta^n = \sqrt{L_n^2 - 4(-1)^n}$$
$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \sqrt{\frac{L_n^2 - 4(-1)^n}{5}}$$

Using Binet's formula, this gives an explicit formula for $F_n$:

$$F_n = \sqrt{\frac{L_n^2 - 4(-1)^n}{5}} \tag{4}$$

For example,

$$F_{11} = \sqrt{\frac{L_{11}^2 + 4}{5}} = \sqrt{\frac{199^2 + 4}{5}} = 89$$

Notice that formula (4) can also be written, [2, 4], as

$$L_n^2 = 5F_n^2 + 4(-1)^n$$

Formula (4) yields three interesting byproducts:



- Since $F_n$ is an integer, $L_n^2 \equiv 4(-1)^n \pmod 5 \equiv (-1)^{n+1} \pmod 5$
- $L_{2n}^2 \equiv 4 \pmod 5$, so $L_{2n} \equiv \pm 2 \pmod 5$.
- $L_{2n+1}^2 \equiv -4 \pmod 5$, so $L_{2n+1} \equiv \pm 1 \pmod 5$.

For example, $L_6^2 = 324 \equiv (-1)^7 \pmod 5$, $L_{10} = 123 \equiv -2 \pmod 5$, and $L_{12} = 322 \equiv 2 \pmod 5$.

Properties (2) and (3) raise four interesting questions: Which Lucas numbers are congruent to $\pm 2$ modulo 5? Which are congruent to $\pm 1$ modulo 5?

Before we identify them, it is interesting to observe that $L_4, L_8, L_{12}, L_{16}$, and $L_{20}$ are congruent to 2 modulo 5, whereas $L_2, L_6, L_{10}, L_{14}$, and $L_{18}$ are congruent to -2 modulo 5. So we conjecture that $L_{4n} \equiv 2 \pmod 5$ and $L_{4n+2} \equiv -2 \pmod 5$.

Both can be established fairly easily. Since

$$L_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m$$

it follows by the binomial theorem that

$$\begin{aligned} 2^m L_m &= 2 + m(m-1)5\sqrt{5} + \cdots \\ &\equiv 2 \pmod 5 \end{aligned}$$

where $m \geq 1$.

In particular, $2^{4n}L_{4n} \equiv 2 \pmod 5$, so $L_{4n} \equiv 2 \pmod 5$. Also, $2^{4n+1}L_{4n+1} \equiv 2 \pmod 5$, so $L_{4n+1} \equiv 1 \pmod 5$; therefore, $L_{4n+2} = L_{4n+1} + L_{4n} \equiv 1 + 2 \equiv -2 \pmod 5$; and $L_{4n+3} = L_{4n+2} + L_{4n+1} \equiv -2 + 1 \equiv -1 \pmod 5$.

For example, $L_{40} = 228826127 \equiv 2 \pmod 5$, $L_{42} = 599074578 \equiv -2 \pmod 5$, $L_{41} = 370248451 \equiv 1 \pmod 5$, and $L_{43} = 969323029 \equiv -1 \pmod 5$.

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# From the Right Side

exp[mod[mix[sin[x[]],sin[mix[x[],mix[mod[y[],BW[0.672616]],exp[y[]],x[],mix[y[],BW[0.580041],
BW[0.578379],BW[-0.834412]]],mix[plus[y[],x[]],BW[-0.615139],plus[BW[-0.124235],y[]],mix[mod[y[],y[]],y[],
BW[-0.771782],div[BW[-0.241889],x[]]]],y[]]],div[mod[sin[div[if[x[],y[],RGB[-0.593078,-0.413547,0.00604378]],BW[0.0553808]]],
mod[sin[BW[0.63928]],sin[y[]]]],plus[y[],mod[sin[BW[0.608041]],div[y[],div[y[],BW[-0.663987]]]]]],mod[exp[RGB[-0.673357,-0.255538,0.34084]],
plus[sin[mod[mult[x[],x[]],x[]]],BW[-0.890933]]]],exp[exp[mix[mult[exp[BW[0.819036]],plus[x[],mod[y[],y[]]]],sin[plus[y[],y[]]],mix[mix[mix[exp[BW[-0.137998]],
[illegible]

Andrej Bauer, 2002.

In this work by Andrej Bauer we see a scene from the tale of Aladin, as the white geni hovers on the shoulder of the wizard and magics a lighting bolt. Or do we? The mathematical equation below the image is not just the unpronouncable title, it is in fact the mathematical formula for the image.

This example of what Andrej calls "Random Art" was created by having the computer randomly generate a function whose domain is a region of the complex plane, and whose range is an RGB vector. The images appear in a web gallery in which the visitors participate, rating the newest submissions 'good' or 'bad', with the bad images being discarded. So each work in the Random Art Gallery is a triple collaboration between Andrej, mathematics, and his fans on the web. This is a fruitful collaboration, with some images having been published as cover art for text books.

You can start your visit to the Random Art Gallery at

http://gs2.sp.cs.cmu.edu/art/random/archive/archive_0208/

where you can view or download a high quality color version of the image above.

*The ΠME Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.*

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## SOME PERFECT ORDER SUBSET GROUPS

STEPHANIE LIBERA* AND PAUL TLUCEK*

**1. Introduction.** We determine whether or not dihedral, quaternion, semi-dihedral, and quasi-dihedral groups are perfect order subset groups.

DEFINITION 1. *Let $G$ be a finite group and let $\Omega^r = \{g \in G \mid |g| = r\}$, where $|g|$ denotes the order of a group element. $G$ is a perfect order subset (POS) group if and only if $|\Omega^r|$ divides $|G|$ for all positive integers $r$ dividing the order of $G$.*

In [1], C. Finch and L. Jones investigate abelian POS groups. At the end of the paper they query whether there are any non-abelian POS groups other than the symmetric group $S_3$. In this paper, we look at a few examples of finite non-abelian groups with two generators – dihedral, quaternion, semi-dihedral, and quasi-dihedral groups – and determine that only the dihedral groups can be POS groups.

We begin with a useful calculation. The result is likely well known, but we provide the proof here for completeness.

LEMMA 2. *If $n > 1$, then $\varphi(n) \mid n$ if and only if $n = 2^k 3^l$, where $k \geq 1$ and $l \geq 0$.*

*Proof.* Suppose $\varphi(n) \mid n$. We must show that $n = 2^k 3^l$, where $k \geq 1$. Let $n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t}$ be the prime factorization of $n$ , where the $p_i$ are prime, $r_i \geq 1$ for all $i$, and $p_1 < p_2 < \cdots < p_t$. Then,

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right) \tag{1}$$

([2], Theorem 2.16), so

$$\begin{aligned}\varphi(n) &= n\left(\frac{p_1 - 1}{p_1}\right)\cdots\left(\frac{p_t - 1}{p_t}\right)\\ &= p_1^{r_1 - 1}\dots p_t^{r_t - 1}(p_1 - 1)\dots(p_t - 1).\end{aligned}$$

We see that $\varphi(n)$ divides $n$ if and only if $(p_1 - 1)(p_2 - 1)\dots(p_t - 1)$ divides $p_1 p_2 p_3 \dots p_t$.

Since $p_1 p_2 \dots p_t$ is square-free, $\mathcal{A} = (p_1 - 1)(p_2 - 1)\dots(p_t - 1)$ must also be square-free. If $t \geq 3$ then $\mathcal{A}$ will be divisible by 4, so $t \leq 2$.

If $p_1 > 2$ then $p_1 - 1$ is even, and we must have $p_1 p_2 \dots p_t$ even too. Since $p_1 < p_2 < \cdots < p_t$, we have a contradiction. Therefore, $p_1 = 2$ and $k \geq 1$.

Finally, if $t > 1$ then $(p_2 - 1) \mid 2p_2$. Since $p_2 - 1$ and $p_2$ are relatively prime, we see that $(p_2 - 1) \mid 2$ and $p_2 = 3$. Putting all this together shows that if $n > 1$ and $n \mid \varphi(n)$, then $n$ is of the form $2^k 3^l$, with $k \geq 1$.

Now, we must show that if $n = 2^k 3^l$ with $k \geq 1$ and $l \geq 0$, then $\varphi(n) \mid n$.

If $l = 0$, then $\varphi(n) = \frac{n}{2}$ which certainly divides $n$.

If $l \geq 1$, then

$$\varphi(n) = n(1 - 1/2)(1 - 1/3) = n/3 = 2^k 3^l / 3 = 2^k 3^{l-1}.$$

Clearly $2^k 3^{l-1} \mid 2^k 3^l$, as desired. Thus, $\varphi(n) \mid n$ if and only if we have $n = 2^k 3^l$, where $k \geq 1$. ◻

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**2. Dihedral and Quaternion Groups.** Whether or not a dihedral group is a POS group depends on whether or not its subgroup of rotations is POS. Quaternion groups turn out not to be POS.

DEFINITION 3. *A dihedral group of order $2n$, where $n \geq 2$, is presented by generators and relations as follows:*

$$D_{2n} = \langle r, f \mid r^n = e, f^2 = e, fr = r^{-1}f \rangle.$$

THEOREM 4. *$D_{2n}$ is a POS group if and only if $n = 3^l$, for some $l \geq 1$.*

*Proof.* When we consider $D_{2n}$ as the group of symmetries of a regular $n$-gon, the dihedral group splits into $n$ rotations (by $360°k/n$ for $k = 1, 2, \ldots, n$) and $n$ reflections. Each reflection has order 2, while the rotations have order dividing $n$.

If $n$ is even, the only rotation of order 2 is the rotation by 180°. Since every reflection also has order 2, $|\Omega^2| = n + 1$. If $n$ is odd, only the reflections have order 2, so $|\Omega^2| = n$.

If $i \neq 2$, then $\Omega^i$ consists of rotations of order $i$. Since the subgroup of rotations is cyclic of order $n$, the order of a rotation $r^m$ is $n/\gcd(m, n)$, where $\gcd(m, n)$ denotes the greatest common divisor of $m$ and $n$. So the elements in $\Omega^i$ are all rotations of the form $r^s$ such that $0 < s < n$ and $\gcd(s, n) = n/i$. Let $n/i = d$, then $r^s \in \Omega^i$ if and only if $\gcd(s/d, n/d) = 1$ and $0 < s < n$. The number of such $s$ is $\varphi(n/d) = \varphi(i)$, where $\varphi$ is the Euler $\varphi$-function. Thus we have $|\Omega^i| = \varphi(i)$.

Suppose $n = 3^l$. Since $n$ is odd, $|\Omega^2| = n$ which clearly divides $2n$. When $i \neq 2$ divides $n$, we must have $i = 3^q$ for some $0 \leq q \leq l$. When $q > 0$, we have by equation (1), $\varphi(i) = \varphi(3^q) = 3^q(2/3) = 2 \cdot 3^{q-1}$. Clearly $\varphi(i)$ divides $2n$. When $q = 0$, $i = 1$ and $|\Omega^1| = 1$. We see that $D_{2n}$ is a POS group when $n = 3^l$.

Now suppose $D_{2n}$ is a POS group. If $n$ is even, then we need $|\Omega^2| = n + 1$ to divide $2n$. Since $n + 1$ is odd and greater than $n$, this is impossible. Hence $D_{2n}$ is not a POS group when $n$ is even.

Now assume $n$ is odd. In particular, we must have $|\Omega^n| \mid 2n$ and, hence, $\varphi(n) \mid 2n$. Since $\gcd(2, n) = 1$ and $\varphi(2) = 1$, we know $\varphi(n) = \varphi(n)\varphi(2) = \varphi(2n)$ [2]. From Lemma 2 we know $\varphi(2n) \mid 2n$ if and only if $2n = 2^k 3^l$ with $k \geq 1$. Since $n$ is odd we must have $n = 3^l$ for some $l \geq 1$. □

DEFINITION 5. *A quaternion group of order $n = 2^m$, $m \geq 3$, is presented as follows:*

$$Q_n = \langle x, y \mid x^{2^{m-1}} = e, y^2 = x^{2^{m-2}}, yx = x^{-1}y \rangle.$$

THEOREM 6. *$Q_n$ is not a POS group.*

*Proof.* We will show that $Q_n$ is not a POS group by showing $|\Omega^4| \nmid n$. First, note there are $n/2$ elements of the form $x^i y$, where $0 \leq i < n/2$.

CLAIM 7. *$|x^i y| = 4$ for all $0 \leq i < n/2$.*

*Proof.* We can see that $x^i y x^i y = x^i (x^i)^{-1} yy = y^2 = x^{2^{m-2}}$. So, the order of $x^i y$ is not two. Clearly, $|x^i y| \neq 3$ since 3 does not divide $n$. Finally, note that

$$(x^i y x^i y)^2 = (x^{2^{m-2}})^2 = x^{2^{m-1}} = e.$$

Thus, $|x^i y| = 4$ and we have at least $n/2$ elements of order 4. □

Now, let $j \in \mathbb{Z}^+$ and $0 < j < n/2$. There is at least one element of the form $x^j$ with order 4 since $|x^{n/8}| = 4$. Let $k \in \mathbb{Z}$ be the number of elements of the form $x^j$ that have order 4. For $Q_n$ to be a POS group, we must show that $(\frac{n}{2} + k) \mid n$.



There are no divisors of $n$ between $n/2$ and $n$, so $k = 0$ or $n/2$. We know that $k \neq 0$ since $\Omega^4$ is non-empty. So $k = n/2$, but there are only $(n/2) - 1$ elements of the form $x^j$ with $0 < j < n/2$, a contradiction, and $Q_n$ is not a POS group. □

**3. Semi-Dihedral and Quasi-Dihedral Groups.**

DEFINITION 8. *A semi-dihedral group of order $n = 2^m$, $m \geq 3$, is presented as follows:*

$$SD_n = \langle s, t \mid s^{2^{m-1}} = e, t^2 = e, ts = s^{2^{m-2}-1}t \rangle.$$

THEOREM 9. *$SD_n$ is not a POS group.*

*Proof.* We will show that $SD_n$ is not a POS group by showing $|\Omega^2| \nmid n$.

There are $n/2$ elements of the form $s^i t$, where $0 \leq i < n/2$. Furthermore, half of these elements have $i$ even, and half have $i$ odd. Thus, there are $n/4$ elements of the form $s^i t$ where $i$ is even, and $n/4$ elements of the form $s^i t$ where $i$ is odd.

CLAIM 10. *$|s^i t| = 2$ if and only if $i$ is even.*

*Proof.* Consider $(s^i t)^2$:

$$s^i t s^i t = s^i s^{i(n/4-1)} tt = s^i s^{i(n/4-1)} e = s^i s^{in/4-i} = s^{i+in/4-i} = s^{in/4}.$$

Since the order of $s$ is $n/2$, we see that $s^{in/4} = e$ if and only if $in/4 \equiv 0 \bmod n/2$. The latter equivalence is true if and only if $i \equiv 0 \bmod 2$. □

CLAIM 11. *Let $0 < j < n/2$. Then $|s^j| = 2$ if and only if $j = n/4$.*

*Proof.* If $j = n/4$ then $(s^j)^2 = s^{n/2} = e$. Now suppose $|s^j| = 2$ for some $0 < j < n/2$. Since $s^j s^j = s^{2j} = e$, we know that $2j \equiv 0 \bmod n/2$. This implies $j \equiv 0 \bmod n/4$. Since $0 < j < n/2$, we see that there is exactly one value for $j$, namely $j = n/4$, which satisfies this condition. □

Claims 10 and 11 imply that $|\Omega^2| = n/4 + 1$. Since $n = 2^m$, the divisors of $n$ that are larger than $n/4$ are only $n/2$ and $n$ itself. Now $n/4 + 1 = n$ does not have an integer solution, while $n/4 + 1 = n/2$ implies $n = 4$. Since $n \geq 8$ by assumption, we have a contradiction. Thus, $|\Omega^2|$ does not divide $n$, and $SD_n$ is not a POS group. □

DEFINITION 12. *A quasi-dihedral group of order $n = 2^m$, $m \geq 4$, is presented as follows:*

$$QD_n = \langle a, b \mid a^{2^{m-1}} = e, b^2 = e, ba = a^{2^{m-2}+1}b \rangle.$$

THEOREM 13. *$QD_n$ is not a POS group.*

*Proof.* We will show that $QD_n$ is not a POS group by showing that $|\Omega^2|$ does not divide $n$.

To begin, we will show that for $0 < i < n/2$, there is only one element of the form $a^i$ with order 2. If $|a^i| = 2$, then $2i \equiv 0 \bmod n/2$, so $i \equiv 0 \bmod n/4$. Since $0 < i < n/2$ we must have $i = n/4$.

Now, we will show that for $0 \leq j < n/2$, there are exactly 2 elements of the form $a^j b$ with order 2. We see that

$$a^j b a^j b = a^j a^{j(n/4+1)} bb = a^{j+jn/4+j} e = a^{j(2+n/4)}.$$

If $|a^j b| = 2$ then $j(2 + n/4) \equiv 0 \bmod n/2$. Now $n = 2^m$, so we can say $j(2 + 2^{m-2}) \equiv 0 \bmod 5$, which implies $j(1 + 2^{m-3}) \equiv 0 \bmod 2^{m-2}$.

Since $1 + 2^{m-3}$ is clearly relatively prime to $2^{m-2}$ when $m > 3$, it has a multiplicative inverse mod $2^{m-2}$, and we can say that $j \equiv 0 \bmod 2^{m-2}$.

We see that $j$ needs to be a multiple of $2^{m-2}$. Since $n = 2^m$, $j$ needs to be a multiple of $n/4$, and $0 \leq j < n/2$. Our possibilities are $j = 0$ and $j = n/4$. Thus, the number of elements of the form $a^j b$ with order 2 is 2.

Finally, $|\Omega^2| = 3$, and we know that 3 does not divide $n$ since $n = 2^m$. Thus, $QD_n$ is not a POS group. □

**4. Remarks.** In [1], the authors query whether a POS group $G$, whose order is divisible by an odd prime, must have 3 dividing $|G|$. Our dihedral group example provides further support for their conjecture.

The four types of groups we studied are examples of metacyclic groups (groups which have a cyclic normal subgroup and corresponding cyclic quotient group). A natural follow-up to this paper would be to study the family of all metacyclic groups, or at least the metacyclic $p$-groups.

One can use Theorems 1 and 3 of [1] as well as their Proposition 1 and Corollary 1 (which amount to our Lemma 2) to prove that $\mathbb{Z}_n$ is a POS group if and only if $n$ is of the form $2^k 3^l$, where $k \geq 1$ and $l \geq 0$. Now the dihedral group $D_{2n}$ is isomorphic to the semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$. The fact that $D_{2n}$ is a POS group if and only if $\mathbb{Z}_n$ is an odd order POS group suggests that there might be a semi-direct product version of the "Going-up" and "Going-down" theorems of [1] (Theorems 1 and 3 respectively).

**Acknowledgements.** We would like to thank the referees for their careful consideration of our paper. They provided useful comments and resources, and helped us simplify several of our proofs. Thanks also to Dr. Jill Dietz of St. Olaf College, who was our advisor on this project.

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# CONGRUENCES MODULO A PRODUCT OF PRIMES

JEREMY THIBODEAUX*

**Abstract.** In this paper, congruences of the form $a^Q \equiv a(\text{mod } m)$ are investigated. Where $a$ is a natural number, $Q$ is prime, and $m$ is a product of distinct primes, including $Q$. The result is a slight generalization of Fermat's Little Theorem.

It can be shown with little effort, by use of Fermat's Little Theorem, that the congruence $a^7 \equiv a(\text{mod } 42)$ holds for all natural numbers $a$. Note that 42 is a product of distinct primes, namely 2, 3, and 7. The same is also true for $a^{37} \equiv a(\text{mod } 1295)$. So one naturally asks what $Q$ and $m$ will allow the congruence $a^Q \equiv a(\text{mod } m)$ to hold for all natural numbers. Before this question is answered, the case $a^7 \equiv a(\text{mod } 42)$ will be shown as an example.

EXAMPLE: $a^7 \equiv a(\text{mod } 42)$ for all $a \in \mathbb{N}$.

*Proof.* If it can be shown that $a^7 \equiv a(\text{mod } 2)$, $a^7 \equiv a(\text{mod } 3)$ and $a^7 \equiv a(\text{mod } 7)$, then we have the result that $a^7 \equiv a(\text{mod } 42)$. By Fermat's Little Theorem,

$$a^2 \equiv a(\text{mod } 2)$$

multiplying by $a$ we have,

$$a^3 \equiv a^2 \equiv a$$

again multiplying by $a$,

$$a^4 \equiv a^3 \equiv a$$

therefore,

$$a^7 \equiv a^2 \equiv a(\text{mod } 2).$$

Also by Fermat's Little Theorem,

$$a^3 \equiv a(\text{mod } 3)$$

squaring both sides we get,

$$a^6 \equiv a^2$$

multiplying by $a$,

$$a^7 \equiv a^3 \equiv a(\text{mod } 3).$$

And of course,

$$a^7 \equiv a(\text{mod } 7).$$

So what we have found is that $2 \mid (a^7 - a)$, $3 \mid (a^7 - a)$, and $7 \mid (a^7 - a)$. Since 2, 3 and 7 are all prime, $2 \cdot 3 \cdot 7 \mid (a^7 - a)$, or $42 \mid (a^7 - a)$ and hence $a^7 \equiv a(\text{mod } 42)$ for all $a \in \mathbb{N}$. □

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The following lemma will be helpful in proving the general result.

LEMMA 1. *Let $n$ be a natural number and $p$ be a prime. Then $a^n \equiv a (\text{mod } p)$ for all $a \in \mathbb{N}$ if and only if $(p-1) \mid (n-1)$.*

*Proof.* Assume $(p-1) \mid (n-1)$ . Then,

$$\frac{n-1}{p-1} = k$$

for some $k \in \mathbb{N}$. Equivalently, $n - k(p-1) = 1$.

By Fermat's Little Theorem we have $a^p \equiv a(\text{mod } p)$ for all $a \in \mathbb{N}$. Note that the difference in the exponents of $a$ is $p-1$ and that this relationship is clearly preserved for any multiplication of the form $a^j$ where $j$ is a natural number. So for any natural number $m$ with $m \geq p$, we have $a^m \equiv a^{m-(p-1)}(\text{mod } p)$. This process can be continued until we reach $a^x$ where $x$ is a natural number such that $1 \leq x \leq p-1$. Therefore, we have

$$a^n \equiv a^{n-(p-1)} \equiv a^{n-2(p-1)} \equiv \cdots \equiv a^{n-k(p-1)} = a.$$

The contrapositive method is used for the converse. Assume $(p-1) \nmid (n-1)$. Then by the division algorithm, $n - k(p-1) = x+1$ for some $k \in \mathbb{N}$ and some natural number x such that $1 \leq x < p-1$. Therefore we have,

$$a^n \equiv a^{n-(p-1)} \equiv a^{n-2(p-1)} \equiv \cdots \equiv a^{n-k(p-1)} = a^{x+1}.$$

Since $x+1 < p$, it is not true in general that $a^{x+1} \equiv a(\text{mod } p)$. Thus $a^n \equiv a(\text{mod } p)$ does not hold for some $a \in \mathbb{N}$. Therefore if $a^n \equiv a(\text{mod } p)$ for all $a \in \mathbb{N}$ then $(p-1) \mid (n-1)$. □

THEOREM 2. *Let $S = \{1, 2, 3, \ldots, n\}$. Let $m = q_1 q_2 q_3 \cdots q_n Q$ , where each $q_i$ is a distinct prime and $Q$ is a prime larger than each $q_i$. Then $a^Q \equiv a(\text{mod } m)$ for all $a \in \mathbb{N}$ if and only if $(q_i - 1) \mid (Q-1)$ for all $i \in S$.*

*Proof.* Assume $(q_i - 1) \mid (Q-1)$ for all $i \in S$. Then by the lemma, $a^Q \equiv a(\text{mod } q_i)$ for each $i \in S$ and $a^Q \equiv a(\text{mod } Q)$ . Thus for each $i \in S$ we have that $q_i \mid (a^Q - a)$ and $Q \mid (a^Q - a)$. Since each $q_i$ and Q are prime, we have that $m \mid (a^Q - a)$. Therefore $a^Q \equiv a(\text{mod } m)$ for all $a \in \mathbb{N}$.

Now assume that for some $r \in S$, $(q_r - 1) \nmid (Q-1)$. Then by the lemma, $a^Q \equiv a(\text{mod } q_r)$ does not hold for some $a \in \mathbb{N}$. Therefore $a^Q \equiv a(\text{mod } m)$ does not hold for some $a \in \mathbb{N}$. Thus if $a^Q \equiv a(\text{mod } m)$ for all $a \in \mathbb{N}$ then $(q_i - 1) \mid (Q-1)$ for all $i \in S$. □

Now if we define an arithmetic function $P(n)$ to be the product of all primes $p$ less than or equal to $n$ such that $(p-1) \mid (n-1)$, then we can easily conclude that if Q is a prime then $a^Q \equiv a(\text{mod } P(Q))$ for all $a \in \mathbb{N}$.

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# PROBLEM DEPARTMENT

EDITED BY MICHAEL MCCONNELL AND JON A. BEAL

*This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

*All correspondence should be addressed to Michael McConnell, 840 Wood Street, Mathematics Department, Clarion University, Clarion, PA 16214, or sent by email to mmcconnell@clarion.edu. Electronic submissions using LaTeX are encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by* $\boxed{\textit{May 1, 2004}}$ *Solutions identified as by students are given preference.*

## Problems for Solution.

**1062.** *Proposed by M. Khoshnevisan, Gold Coast, Queensland, Australia*

A Generalized Smarandache Palindrome (GSP) is a concatenated number of the form: $a_1 a_2 \cdots a_n a_n \cdots a_2 a_1$ or $a_1 a_2 \cdots a_{n-1} a_n a_{n-1} \cdots a_2 a_1$, where all $a_1, a_2, \ldots a_n$ are positive integers of various numbers of digits. Find the number of GSP of four digits that are not palindromic numbers.

**1063.** *Proposed by Monte J. Zerger, Adams State College, Alamosa, CO.*

Find all triples of consecutive integers (a,b,c) such that

$$\frac{a^3 + b^3 + c^3}{abc}$$

is integral.

**1064.** *Proposed by Karl David, Milwaukee School of Engineering, Milwaukee, WI*

Consider numbers formed by concatenating two or more successive powers of 2 [for example, 816 or 248163264]. Show that no such number is itself a power of 2. That is, show that for $n \geq 0$ and $k \geq 1$,

$$2^n 2^{n+1} \cdots 2^{n+k} \neq 2^m \text{ for any } m.$$

**1065.** *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*

For any triangle △ ABC, prove that

$$\frac{\sin^2 A + \sin^2 B}{\sin C} + \frac{\sin^2 B + \sin^2 C}{\sin A} + \frac{\sin^2 C + \sin^2 A}{\sin B} \geq 3\sqrt{3}$$

and determine when equality holds.

**1066.** *Proposed by Joe Howard, Portales, NM*

Let $a, b, c$ be sides of a triangle. Show that

$$\frac{b+c}{a} \leq \sqrt{\frac{bc}{(s-b)(s-c)}} \quad \text{where } s = \frac{a+b+c}{2}.$$

For which triangles does equality hold?

**1067.** *Proposed by Ayoub B. Ayoub, Penn State Abington, Abington, PA*
Suppose $p$ is an odd prime number. Show that

$$\sum_{r=0}^{p-1} \binom{2p-1}{r}^n$$

is divisible by $p$ if $n$ is even and not divisible by $p$ if $n$ is odd.

**1068.** *Proposed by William Chau, SoftTechies Corp., East Brunswick, NJ*
On pp. 35-39 of *On Prime Numbers and Perfect Numbers,* Scripta Math, Vol. 19, 1953, Jacques Touchard proved that any odd perfect number must be of the form $12k+1$ or $36k+9$. If an odd perfect number is of the form $36k+9$, prove that it can be further reduced to the form $108k+9$, $108k+45$, or $324k+81$.

**1069.** *Proposed by Monte J. Zerger, Adams State College, Alamosa, CO.*
Show that $1^4-2^4+3^4-4^4+\cdots+(-1)^{n+1}n^4=(-1)^n T_m$ where $T_m$ is a triangular number.

**1070.** *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI*
Determine the convergence of the series

$$\sum_{n=1}^{\infty}\left(1-\sqrt{\frac{2}{\pi}}\frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n+1}}\right).$$

**Corrections.** In the Spring 2003 issue, problem 1054 should read as:
Let $a_1, a_2, \ldots, a_n$ be integers such that $0 \le a_1 \le a_2 \le \ldots \le a_n$. If

$$\max_{k>j}\left\{\frac{a_k-a_j-1}{k-j}\right\} < \min_{k>j}\left\{\frac{a_k-a_j+1}{k-j}\right\}$$

then there exists $m$ and $b$ such that $a_i=[[mi+b]]$ for all $i$.

**Solutions.**

**1043.** [Fall 2002]*Proposed by Peter A. Lindstrom, Batavia, NY.*

The year 2002 is a four digit base ten palindrome as was the year 1991. (a.) Can 1991 be rewritten in a different base as a palindrome with four digits? (b.) Can 2002 be rewritten in a different base as a palindrome with four digits?

*Solution by* **William H. Peirce**, *Rangeley, ME*

Let $N>0$ be a base-ten palindrome which is expressed as a four-digit palindrome $xyyx$ in a base $b \neq 10$. $x$ and $y$ must satisfy $0 \le x, y \le (b-1)$ except that $x$ and $y$ cannot both be zero. $N = xb^3+yb^2+yb+x = (b+1)\{(b^2-b+1)x+by\}$ shows that the only allowable bases $b$, finite in number, are those for which $b+1$ divides $N$. Henceforth $b$ will be such a base.

By inspection, $x$ is the unique remainder $0 \le x \le (b-1)$ obtained when $N = xb^3+yb^2+yb+x$ is divided by $b$, and the integer $y$ is then found from

$$y=\frac{[N-x(b^3+1)]}{(b^2+b)}.$$

With $x$ and $y$ so determined, $N_b = xyyx$ is an arithmetic palindrome in base $b$, but $N_b$ may or may not be a valid palindrome since $y$ may or may not be in the range



$0 \le y \le (b-1)$. Let $N=1991$. The divisors of 1991 are 1,11,181,1991 and the choices of $b$, excluding 0 and 10, are 180 and 1990. For $b=180, x=11, y=-1969$, and $1991_{180}=11,-1969,-1969,11$. For $b=1990, x=1, y=-1989$, and $1991_{1990}=1,-1989,-1989,1$. Both palindromes are arithmetically correct, but neither is valid since $y$ in each is not in the proper ranger, $0 \le y \le (b-1)$. Therefore, 1991 cannot be written as a valid palindrome in a base $b \neq 10$. Let $N=2002$. The divisors of 2002 are

$$1, 2, 7, 11, 13, 14, 22, 26, 77, 91, 143, 154, 182, 286, 1001, 2002$$

and the choices for $b$, excluding 0 and 10, are 1,6,12,13,21,..., 2001. $2002_1 = 0, 1001, 1001, 0$ and $2002_6 = 4, 27, 27, 4$ are arithmetically correct palindromes but not valid since $y$ in each is greater that $b-1$. $2002_{12}, 2002_{21}, 2002_{25}, \ldots, 2002_{2001}$ are not valid palindromes since $y$ in each is negative. Therefore, $N_{13}=0, 11, 11, 0$ is the only valid palindrome for 2002 in a base other than 10.

Also solved by **The Cal Poly Pomona Problem Solving Group**, Pomona CA, **William Chau**, East Brunswick, NJ, **Richard I. Hess**, Rancho Palos Verdes, CA, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY **Yoshinobu Murayoshi** Okinawa, Japan, **Mike Pinter**, Belmont University, Nashville, TN, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

**1044.** [Fall 2002]*Proposed by Thomas J. Pfaff, University of Wisconsin-Superior, Superior, WI.*

Evaluate

$$\lim_{n\to\infty}\frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{ni-i^2}$$

*Solution by* **Justin Couchman, Mike Davis, and Whitney Kaczor**, *SUNY Fredonia, Fredonia, NY.*

Using partial fractions,

$$\frac{1}{i(n-i))}=\frac{1}{n}\left(\frac{1}{i}+\frac{1}{n-i}\right).$$

Thus we have

$$\lim_{n\to\infty}\frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{ni-i^2}=\lim_{n\to\infty}\frac{1}{\ln n}\sum_{i=1}^{n-1}\left(\frac{1}{i}+\frac{1}{n-i}\right).$$

Since

$$\sum_{i=1}^{n-1}\left(\frac{1}{i}+\frac{1}{n-i}\right)=\left(\frac{1}{1}+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\cdots+\left(\frac{1}{n-2}+\frac{1}{2}\right)+\left(\frac{1}{n-1}+\frac{1}{1}\right)=2\sum_{i=1}^{n-1}\frac{1}{i}$$

we obtain

$$\lim_{n\to\infty}\frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{ni-i^2}=\lim_{n\to\infty}\frac{2}{\ln n}\sum_{i=1}^{n-1}\frac{1}{i}.$$

Notice

$$\int_1^n \frac{1}{x}dx \le \sum_{i=1}^{n-1}\frac{1}{i} \le \int_2^n \frac{1}{x-1}dx+1$$

and therefore

$$\ln n \leq \sum_{i=1}^{n-1} \frac{1}{i} \leq \ln(n-1)+1.$$

Taking limits gives

$$\lim_{n\to\infty} \frac{\ln n}{\ln n} \leq \frac{1}{\ln n}\sum_{i=1}^{n-1}\frac{1}{i} \leq \frac{\ln(n-1)+1}{\ln n}.$$

By the sandwich theorem,

$$\lim_{n\to\infty} \frac{1}{\ln n}\sum_{i=1}^{n-1}\frac{1}{i} = 1.$$

Thus,

$$\lim_{n\to\infty} \frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{ni-i^2} = 2\lim_{n\to\infty}\frac{1}{\ln n}\sum_{i=1}^{n-1}\frac{1}{i} = 2.$$

Also solved by **The Cal Poly Pomona Problem Solving Group**, Pomona CA, **William Chau**, East Brunswick, NJ, **José Luis Díaz-Barrero**, Universitat Politècnica de Catalunya, Barcelona, Spain, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard,** Portales, NM, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY **Mike Pinter**, Belmont University, Nashville, TN, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

**1045.** [Fall 2002] *Proposed by Mohammad K. Azarian, Unversity of Evansville, Evansville, IN.*

Suppose that G is an abelian group with 2n elements, where n is odd. **Without using Sylow Theorems,** show that G has exactly one subgroup of order 2.

*Solution by* **Cal Poly Pomona Problem Solving Group,** *Cal Poly Pomona*

Let $G$ be an abelian group with $2n$ elements, where $n$ is odd. So $G$ has an even number of elements. Since $G$ is a group, there exists a unique identity $e$ in $G$, hence $G$ has an odd number of non-identity elements. We know that for every $x$ in $G$ there exists a unique $x^{-1}$ in $G$ such that $xx^{-1} = e$. So every element in $G$ can be "paired" up with its unique inverse. Since we know that $G$ has an odd number of non-identity elements, there exists at least one element in $G$, say $a$, which is its own inverse. Moreover, this element along with the identity make a subgroup of $G$, say $K$.

To show that there only exists one subgroup of order two, assume by contradiction that there are at least two. So let $K = \{e, a\}$ and $K' = \{e, b\}$, where $a \neq b$. If this is the case, we can construct the subgroup $M = \{e, a, b, ab\}$, where $M \leq G$. We know that $|G| = 2n$, where $n$ is odd and $|M| = 4$. Since $M \leq G$, $|M|$ divides $|G|$, which is a contradiction since 4 doesn't divide $2n$ when $n$ is odd. Therefore there is only one subgroup $K$ of $G$ where $|K| = 2$.

Also solved by **David E. Manes**, SUNY College at Oneonta, Oneonta NY **Rex H. Wu**, Brooklyn, NY and the **Proposer**.

**1046.** [Fall 2002] *Proposed by Paul S. Bruckman, Sacramento, CA.*

Let $S_1 = \{x_1, x_2, \ldots, x_n\}$, where the $x_j's$ are positive and not necesarily distinct. Let $S_k$ denote the set consisting of all $C_n^k$ possible products of the form $x_{j_1}x_{j_2}\cdots x_{j_k}$ where the $j_k's$ are distinct, $k = 1, 2, \ldots, n$. If $G_k$ represents the geometric mean of the elements of $S_k$, prove that $G_k = (G_1)^k$. *Solution by* **Kathleen E. Lewis,** *SUNY Oswego, Oswego NY*



First we note each element $x_{j_i}$ appears in $\binom{n-1}{k-1}$ of the products, as it will appear with every possible $k-1$-subset of the other elements. Since the geometric mean $G_k$ is the $\binom{n}{k}$'th root of the product of these subproducts, each $x_{j_i}$ will be raised to the power $\binom{n-1}{k-1}/\binom{n}{k}$. This simplifies to

$$\frac{(n-1)!}{(k-1)!(n-k)!}\cdot\frac{k!(n-k)!}{n!} = \frac{k}{n}.$$

Thus $G_k = (x_1x_2\cdots x_n)^{k/n}$. By substituting 1 for $k$, we see that $G_1 = (x_1x_2\cdots x_n)^{1/n}$, so $G_k = (G_1)^k$, as required.

Also solved by **William Chau**, East Brunswick, NJ, **William H. Peirce**, Rangeley Maine and the **Proposer**.

**1047.** [Fall 2002]*Proposed by Mohammad K. Azarian, Unversity of Evansville, Evansville, IN.*

Show that

$$\sec^2\alpha + \csc^2\alpha + \sec^2\alpha\csc^2\alpha \geq 8$$

for $0 < \alpha < \pi/2$. Determine when equality holds.

*Solution by* **The Cal Poly Pomona Problem Solving Group,** *Pomona, CA*

Consider the right triangle with sides $a, b, c$ where $c$ is the hypotenuse and $\alpha$ is the angle opposite $a$. Then we can rewrite the inequality as

$$\frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{c^2c^2}{a^2b^2} \geq 8.$$

Combining terms we get

$$\frac{a^2+b^2+c^2}{a^2b^2}c^2 \geq 8.$$

Using the Pythagorean Theorem, we arrive at

$$\frac{c^4}{a^2b^2} \geq 4.$$

Now we take the square root of both sides obtaining

$$\frac{c^2}{ab} \geq 2 \Leftrightarrow c^2 \geq 2ab \Leftrightarrow a^2+b^2 \geq 2ab \Leftrightarrow (a-b)^2 \geq 0.$$

Clearly this last inequality holds. Strict equality occurs when $a = b$. That is, when $\alpha = \pi/4$.

Also solved by **William Chau**, East Brunswick, NJ, **Kelly Chen**, 9th grade student, Wayne Hills H.S., Wayne, NJ, **Paul Dawkins**, student, Angelo State University, San Angelo, TX, **José Luis Díaz-Barrero**, Universitat Politècnica de Catalunya, Barcelona, Spain, **Richard I. Hess**, Rancho Palos Verdes, CA, **Joe Howard**, Portales, NM, **Kathleen E. Lewis**, SUNY Oswego, Oswego, NY, **Peter A. Lindstrom**,Batavia, NY, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **N.R. Nandakumar**, Delaware State University, Dover, DE, **Mike Pinter**, Belmont University, Nashville, TN, **Dale Wilger, Jennifer Wystup, Katie O'Hara**, SUNY Fredonia, Fredonia, NY, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

**1048.** [Fall 2002] *Proposed by Peter A. Lindstrom, Batavia, NY.*
Without using the Fundamental Theorem of Calculus, give a geometric argument to show that

$$\int_0^1 \frac{2}{t^2+1}\,dt = \frac{\pi}{2}.$$

I. *Solution by* **Peter A. Lindstrom**, *Batavia, NY.*

With the definite integral taken over [0,1] and the result being $\frac{\pi}{2}$ possibly suggests a unit circle is involved in some way or manner. Consider the set of parametric equations

$$x = \frac{t^2-1}{t^2+1} \text{ and } y = \frac{2t}{t^2+1}$$

Eliminating the parameter $t$, we see that

$$x^2+y^2 = (\frac{t^2-1}{t^2+1})^2 + (\frac{2t}{t^2+1})^2 = \frac{t^4-2t^2+1+4t^2}{(t^2+1)^2} = 1,$$

so that this set of parametric equations represents a unit circle centered at the origin and for $t \in [0,1]$, this represents that circle in the second quadrant. Since the area of the portion of the circle is *not* $\frac{\pi}{2}$, let's consider the arc length in the second quadrant which is $\frac{\pi}{2}$. Using the arc length formula for a curve given in parametric form, we obtain

$$\int_0^1 \sqrt{\left\{\frac{d}{dt}(\frac{t^2-1}{t^2+1})\right\}^2 + \left\{\frac{d}{dt}(\frac{2t}{t^2+1})\right\}^2}\,dt, \text{ which } = \frac{\pi}{2}.$$

Simplifying, we obtain

$$= \int_0^1 \sqrt{\left\{\frac{d}{dt}\left(\frac{t^2-1}{t^2+1}\right)\right\}^2 + \left\{\frac{d}{dt}\left(\frac{2t}{t^2+1}\right)\right\}^2}\,dt$$

$$= \int_0^1 \sqrt{\left\{\frac{4t}{(t^2+1)^2}\right\}^2 + \left\{\frac{2-2t^2}{(t^2+1)^2}\right\}^2}\,dt$$

$$= \int_0^1 \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}\,dt = \int_0^1 \frac{2}{t^2+1}\,dt.$$

II. *Solution by* **Mike Printer**, *Belmont Univeristy, Nashville, TN.*

Make the substitution $t = \tan\theta, dt = \sec^2\theta\ d\theta$, to obtain

$$\int_0^1 \frac{2}{t^2+1}\,dt = 2\int_{\arctan 0}^{\arctan 1} \frac{1}{\tan^2\theta+1}\sec^2\theta\ d\theta = 2\int_0^{\frac{\pi}{4}} 1\ d\theta$$

This last integral is equal to twice the area under $t=1$ from $\theta = 0$ to $\theta = \pi/4$. This equals twice the area of a 1 by $\pi/4$ rectangle which is $\pi/2$.

**1049.** [Fall 2002] *Andrew Cusumano, Great Neck, NY.* A Fibonacci-type sequence is defined by the rules $F_1 = A$, $F_2 = B$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$, where $A$ and $B$ are constants. Show that for each $n \geq 1$,

$$F_n^3 + F_{n+1}^3 + F_{n+2}^3 = F_{n+2}[2F_n + 2F_{n+1} + F_nF_{n+1}].$$



Editors' note: We incorrectly typed the problem, as shown below.

*Solution by* **Kelly Chen**, *student, Wayne Hills High School, Wayne NJ*

This is a wrong equation, here is a **counter-example**: Let $F_1 = 1, F_2 = 2$, then $F_3 = 3$. Substitute into the equation, we get that the left hand side is 36 while the right hand side is 24, therefore the equality does not hold in general.

Since I believe there is a typo in the journal, I tried to think about what a correct problem would be. There is a high likelihood that the correct equation is the following one

**Modification of the problem**: $F_n^3+F_{n+1}^3+F_{n+2}^3 = F_{n+2}[2F_n^2+2F_{n+1}^2+F_nF_{n+1}]$

*Proof:* Using $F_n^3 + F_{n+1}^3 + 3F_{n+1}^2F_n + 3F_{n+1}F_n^2 = [F_n + F_{n+1}]^3 = F_{n+2}^3$, we get

$$\begin{aligned} F_{n+2}[2F_n^2 + 2F_{n+1}^2 + F_nF_{n+1}] &= [F_n + F_{n+1}][2F_n^2 + 2F_{n+1}^2 + F_nF_{n+1}] \\ &= 2F_n^3 + 2F_{n+1}^3 + 3F_{n+1}^2F_n + 3F_{n+1}F_n^2 \\ &= F_n^3 + F_{n+1}^3 + [F_n + F_{n+1}]^3 \\ &= F_n^3 + F_{n+1}^3 + F_{n+2}^3. \qquad \square \end{aligned}$$

Also solved by **Scott H. Brown**, Auburn University, Montgomery AL, **William Chau**, Soft-Techies Corp., East Brunswick, NJ, **Richard Hess**, Rancho Palos Verdes, CA, **David E. Manes**, SUNY College at Oneonta, Oneonta NY, **Mike Pinter**, Belmont University, Nashville TN, **Rex H. Wu**, Brooklyn, NY and the **Proposer**.

**1050.** [Fall 2002] *Ronald Kopas, Clarion University, Clarion, PA.*

A lottery uses 31 balls, numbered 1 through 31. Six of these balls are selected in the drawing, so each lottery ticket contains six numbers from 1 through 31. Show that it is possible to buy exactly 31 tickets so that each pair of numbers appears on exactly one of the tickets.

*Solution by* **The Skidmore College Problem Group**, *Saratoga Springs, NY*

Organize the 31 numbers as follows: $n_0, n_1, \ldots, n_5, n_{1,1}, n_{1,2}, \ldots, n_{5,5}$. Now let ticket $T_0 = \{n_0, \ldots, n_5\}$. For $j = 1, \ldots, 5$, let ticket $T_{0,j} = \{n_0\} \cup \{n_{j,i+1} : i = 0, \ldots, 4\}$. Now, for $j, k = 1, \ldots 5$, define ticket $T_{k,j} = \{n_k\} \cup \{n_{i+1,\underline{j+ki}} : i = 0, \ldots, 4\}$, where $\underline{m} \equiv m \pmod 5), \underline{m} \in \{1, \ldots, 5\}$. Notice that this defines 31 tickets, each containing $\binom{6}{2} = 15$ different pairs of numbers. Since there are $\binom{31}{2} = 495 = 31 \times 15$ different possible pairs among the 31 numbers, it will suffice to show that no two of our tickets can have more than one number in common.

PROPOSITION 1. *No two tickets have more than one number in common.*

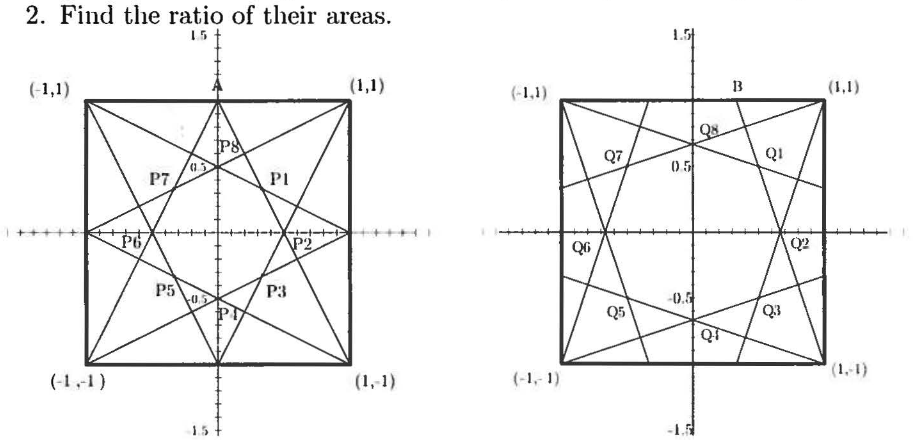
*Proof.* First, note that by design, $T_0$ shares exactly one number with every other ticket. So suppose $T_{0,j_1} \cap T_{0,j_2} \supseteq \{n_0, n_{r,s}\}$ for some $r, s$. Then $n_{r,s} \in T_{0,j_1}$ implies $r = j_1$ and $n_{r,s} \in T_{0,j_2}$ implies $r = j_2$. Therefore, $j_1 = j_2$. Next, suppose that $T_{0,j_1} \cap T_{k,j_2} \supseteq \{n_{r,s}, n_{t,u}\}$. As above, $r = j_1$ and $t = j_1$ gives $r = t$. But now $n_{r,s} \in T_{k,j_2}$ and $n_{r,u} \in T_{k,j_2}$ gives $s = u$. Finally, suppose that $T_{k_1,j_1} \cap T_{k_2,j_2} \supseteq \{n_{r,s}, n_{t,u}\}$. Then $r = i_1 + 1$ for some $i_1 = 0, \ldots 4$. So $s = \underline{j_1 + k_1 i_1} = \underline{j_2 + k_2 i_1}$. If $t = i_2 + 1$ for some $i_2 = 0, \ldots 4$, then $u = \underline{j_1 + k_1 i_2} = \underline{j_2 + k_2 i_2}$. These give $\underline{j_1 - j_2} = \underline{i_1(k_2 - k_1)} = \underline{i_2(k_2 - k_1)}$. So either $i_1 = i_2$ or $k_1 = k_2$. In either case we get $r = t$ and $s = u$. $\square$

Also solved by **Paul Dawkins**, student, Angelo State University, San Angelo, TX, **Richard I. Hess**, Rancho Palos Verdes, CA, **Kathleen E. Lewis**, SUNY Oswego, Oswego, NY, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **Mike Pinter**, Belmont University, Nashville, TN, **Rex H. Wu**, Brooklyn, NY, and the **Proposer**.

**1051.** [Fall 2002] *Monte J. Zerger, Adams State College, Alamosa, CO.*

The two squares in the figure below are congruent. In the figure on the left, the octagon is formed by joining the bisection points of the sides of the square to vertices as shown. In the second figure on the right, the trisection points of the sides are used instead.

1. Show that the octagons are similar, equilateral, but not equiangular.
2. Find the ratio of their areas.




*Solution by* **William Peirce**, *Rangeley, ME*

Without loss of generality, let the square in which each octagon is placed have corners (1,1), (1,-1), (-1,-1), (-1,1) and let $A=(a,1)$ and $B=(b,1)$, $-1<a<b<1$, be two points on the top side of the square. These two points define the (convex) octagons. In this problem, $a=0$ and $b=1/3$. The octagon vertices and other points are marked on the problem statement.

By routine analytic geometry using $a=0$ and $b=\frac{1}{3}$, we have $P_1=(\frac{1}{3},\frac{1}{3})$, $P_2=(\frac{1}{2},0)$, $P_4=(\frac{1}{3},-\frac{1}{3})$, $P_2=(0,-\frac{1}{2})$, $P_5=-P_1$, etc., and $Q_1=(\frac{1}{2},\frac{1}{2})$, $Q_2=(\frac{2}{3},0)$, $Q_3=(\frac{1}{2},-\frac{1}{2})$, $Q_4=(0,-\frac{2}{3})$, $Q_5=-Q_1$, etc. Each side of the left octagon in $\sqrt{5}/6$ units long and side of the right octagon is $\sqrt{5/18}$ units long, so each octagon is equilateral.

The segments $P_1P_3$ and $P_2P_4$ are 2/3 and $\sqrt{1/2}$ units long, respectively, so the interior angles at $P_2$ and $P_3$ are not equal, and the left octagon is not equiangular. Likewise, the segments $Q_1Q_3$ and $Q_2Q_4$ are 1 and $\sqrt{8/9}$ units long, respectively, so that the interior angles at $Q_2$ and $Q_3$ are not equal and the right octagon is not equiangular.

Now consider triangle $P_1P_2P_3$, whose sides squared are $(\frac{5}{36},\frac{4}{9},\frac{5}{36})=(5,16,5)/36$, ad triangle $Q_2Q_3Q_4$ whose sides squared are $(\frac{5}{18},\frac{8}{9},\frac{5}{18})=(5,16,5)/18$. These triangles with their sides squared are similar as are these triangles without their sides squared. Thus $\angle P_2=\angle Q_3$. Similarly, $\angle P_3=\angle Q_2$. Since the alternate angles of the two octagons are equal, the octagons are similar.

The area ratio of two similar polygons is equal to the ratio of the squares of corresponding sides, in this case $\frac{5/18}{5/36}=\frac{2}{1}$, larger octagon to smaller.

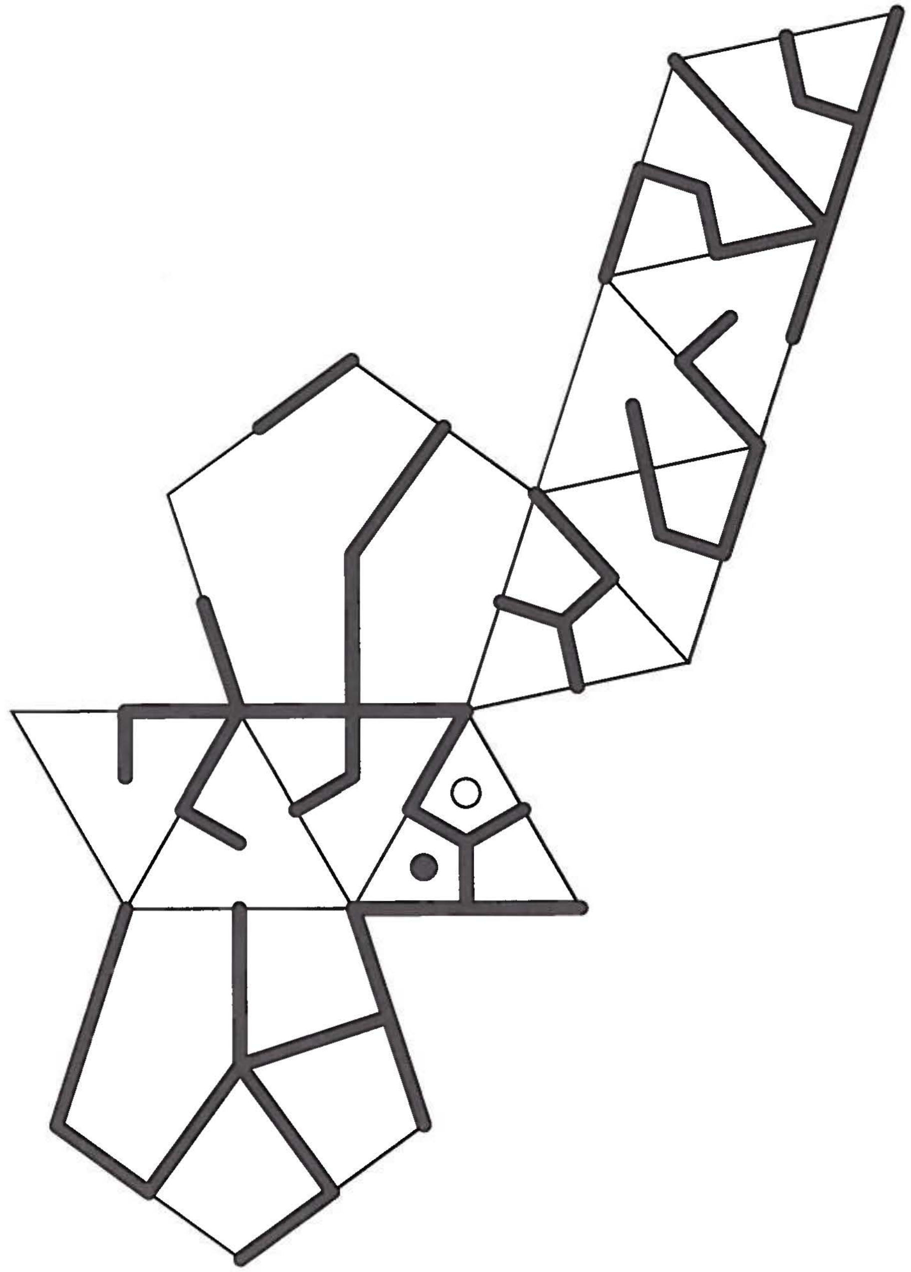
For general $a$ and $b$ in the range $-1<a<b<1$, the two octagons are equilateral but not equiangular. they are similar if and only is $(3-a)(3-b)=8$, which includes the case $a=0$ and $b=1/3$. The area ratio for similar polygons is

$$\frac{2(1-a)^2}{(1+a)^2}=\frac{(1+b)^2}{2(1-b)^2}$$



larger octagon to smaller.

Also solved by **Richard Hess**, Rancho Palos Verdes, CA, **Gus Mavrigian**, Youngstown OH, **Rex H. Wu**, Brooklyn, NY and the **Proposer**.



Polyhedral Maze contributed by Prof. Izidor Hafner from the University of Ljubljana (izidor.hafner@fe.uni-lj.si).

# Whom Isabelle could not disturb

Sustained winds near 140 mph, large ocean swells and dangerous surf disrupted mail service, caused power outages, frightened people, and damaged property but could not stop our referees from doing their valuable duty. After all, as one of them put it, in times of great distress, mathematics, as it occupies the mind and does not require electricity, reveals strengthening and soothing qualities.

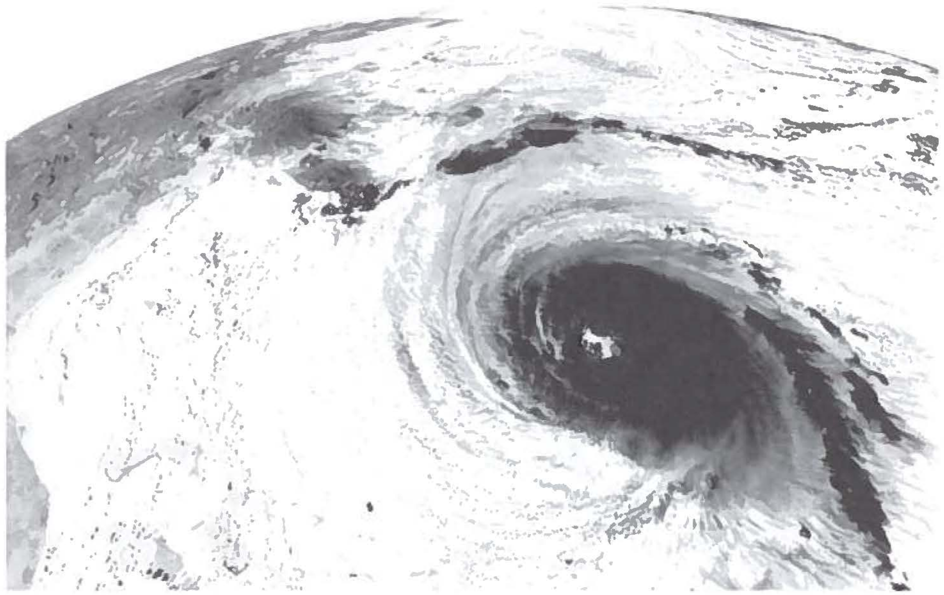


Photo courtesy of NOAA

The editor wishes to acknowledge the substantial contributions made by the following mathematicians who refereed manuscripts for the Pi Mu Epsilon Journal during the past year.



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# MATH ACROSTIC

Dan Hurwitz, Skidmore College

| | |
|---|---|
| a. The ___ Principle relates genotypes in succeeding generations | 037 073 156 112 196 043 152 144 055 013 172 123 090 079 097 117 135 188 004 162 062 025 |
| b. ___ stroke, meaning the negation of a conjunction | 058 093 077 048 113 183 016 147 |
| c. Calculus in which partial derivatives are used. | 134 010 195 092 072 030 168 064 114 158 105 047 |
| d. ___ ratio test | 082 006 027 053 154 036 124 |
| e. Frisco to LA (2 wds) | 045 061 108 008 089 017 184 121 |
| f. Greek mathematician, long time head of the Academy | 081 065 042 099 145 166 192 019 131 189 |
| g. Famous graphic artist | 051 078 174 020 157 104 |
| h. They follow from the axioms | 012 126 199 034 159 146 100 056 |
| i. Battles around Moscow didn't stop publication of Laplace's seminal work on probability this year. | 150 057 170 028 125 141 202 111 070 005 160 035 102 136 |
| j. The greatest or least on an interval | 142 155 171 049 022 063 115 101 127 191 180 187 038 084 133 009 |
| k. Used in Gaussian elimination | 033 177 163 138 015 |
| l. Having excess mass | 175 201 031 040 148 103 001 075 130 088 |
| m. False use of probability to predict frequency (3wds.) | 076 096 198 119 161 029 140 186 151 110 087 190 046 |
| n. One half when used to make a continuity correction on a chi-square test (3 wds.) | 067 032 143 122 011 050 054 137 071 116 107 153 080 041 178 193 181 086 098 197 021 128 165 |
| o. Linear transformation not including a rotation | 185 068 176 026 120 149 091 059 007 |
| p. Optional for conditional | 018 200 173 094 |
| q. Not in agreement | 039 023 052 109 164 074 002 085 132 182 |
| r. How Germans read numerals from 21 through 29 (3 wds.) | 095 167 179 044 129 194 083 066 014 139 118 |
| s. Preserving collinearity | 003 024 069 106 169 060 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 001l | 002q | | 003s | 004a | 005i | 006d | 007o | 008e | | 009j | 010c | 011n | 012h | | 013a | 014r |
| | 015k | 016b | 017e | 018p | | 019f | 020g | 021n | 022j | | 023q | 024s | | 025a | 026o | 027d |
| | 028i | 029m | 030c | 031l | | 032n | | 033k | 034h | 035i | 036d | 037a | 038j | 039q | 040l | 041n |
| 042f | | 043a | 044r | 045c | 046m | 047c | | 048b | 049j | 050n | 051g | 052q | | 053d | 054n | 055a |
| 056h | 057i | 058b | 059o | | 060s | 061e | 062a | 063j | 064c | 065f | 066r | 067n | | 068o | 069s | |
| 070i | 071n | 072c | 073a | 074q | 075l | 076m | 077b | 078g | | 079a | 080n | 081f | | 082d | 083r | 084j |
| 085q | 086n | 087m | | 088l | 089e | 090a | 091o | 092c | 093b | 094p | 095r | | 096m | 097a | | 098n |
| 099f | 100h | 101j | | 102i | 103l | 104g | 105c | 106s | 107n | 108e | 109q | | 110m | 111i | 112a | |
| 113b | 114c | 115j | 116n | | 117a | 118r | | 119m | 120o | 121e | 122n | 123a | 124d | | 125i | 126h |
| 127j | 128n | | 129r | 130l | 131f | | 132q | 133j | 134c | 135a | 136i | 137n | | 138k | 139r | |
| 140m | 141i | 142j | 143n | 144a | 145f | 146h | 147b | | 148l | 149o | 150i | 151m | 152a | | 153n | 154d |
| 155j | 156a | 157g | | 158c | 159h | 160i | | 161m | 162a | 163k | 164q | | 165n | 166f | 167r | 168c |
| 169s | 170i | 171j | 172a | 173p | | 174g | 175l | 176o | 177k | 178n | 179r | | 180j | 181n | 182q | 183b |
| 184e | 185o | 186m | 187j | | 188a | 189f | | 190m | 191j | 192f | 193n | 194r | 195c | 196a | | 197n |
| 198m | 199h | 200p | 201l | 202i | | | | | | | | | | | | |

Last issue's mathacrostic was taken from "How to Solve it", by George Polya.

Setting up equations is like translation from one language into another. This comparison, used by Newton in his Arithmetica Universalis, may help to clarify the nature of certain difficulties often felt both by students and by teachers.

## Contents



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