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SYMMETRY GROUPS AND MOLECULAR STRUCTURE[1]

J. S. FRAME
Michigan Alpha, '44

Introduction. The theory of groups, which was used brilliantly by Abel in 1824 to prove the impossibility of expressing the roots of the general literal fifth degree equation in terms of radicals, remained for a century a theory of interest only to the pure mathematician, without so-called "practical" applications in other sciences. With the advent of the quantum theory in the 1920's, however, group theory became recognized as an important tool in the study of atomic and molecular structure. The purpose of this article is to explain at an introductory level one of the ways that the theory of symmetry groups is useful in a mathematical study of molecular structure.

1. ***Energy in a molecule.*** As a model of a molecule of n atoms, we consider a set of n particles of masses m_i, each oscillating with small amplitude about its own mean position in a suitable moving coordinate system. The mean positions form a so-called equilibrium configuration, that moves rigidly under translation and rotation in space. If the position of the i^{th} particle is described by 3 cartesian "displacement" coordinates ξ_i, η_i, ζ_i with origin at its equilibrium position P_i in the moving system, the internal kinetic energy T of the system (excluding translation and rotation of the configuration), is given by the formula

$$(1.1)\quad T = 1/2 \sum_{i=1}^{n} m_i(\dot\xi_i^2 + \dot\eta_i^2 + \dot\zeta_i^2) \text{ where } \dot\xi_i = d\xi_i/dt, \text{ etc.}$$

Mathematical simplicity is achieved by introducing "mass adjusted" displacement coordinates $x_1, x_2, \ldots x_{3n}$, of which the i^{th} triple is

$$(1.2)\quad (x_{3i-2}, x_{3i-1}, x_{3i}) = (\sqrt{m_i}\,\xi_i, \sqrt{m_i}\,\eta_i, \sqrt{m_i}\,\zeta_i)$$

In these coordinates the doubled internal kinetic energy has the simple form.

$$(1.3)\quad 2T = \sum_{i=1}^{3n} \dot x_i^2$$

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[1]Presented to the national meeting at Pennsylvania State University, August 27, 1957. Received by the editors March 23, 1958.

The internal potential energy of the system is an unknown function V of these adjusted displacement coordinates x_i. It has its minimum for the coordinates $x_i = 0$ that describe a state when each particle is at its own equilibrium position. We assume that 2V can be expanded in a convergent power series in the 3n variables x_i, with coefficients v_i, v_{ij}, . . .

(1.4)

$$2V = 2v_0 + \sum_{i=1}^{3n} v_i x_i + \sum_{i,j=1}^{3n} v_{ij} x_i x_j + \sum_{i,j,k=1}^{3n} v_{ijk} x_i x_j x_k + \cdots$$

Since the choice for 0-level of energy does not affect the study of energy changes, we may take $v_o = 0$. Since V has a minimum at equilibrium where $x_i = 0$,

$$(1.5) \quad 2\,\partial V/\partial x_i \Big|_o = v_i = 0.$$

Thus the constant and linear terms disappear. Next we assume that the cubic and higher order terms in the small quantities x_i can be neglected as compared with the quadratic terms, for a first approximation at least, and thus change (1.4) to

$$(1.6) \quad 2V = \sum_{i,j=1}^{3n} v_{ij} x_i x_j,$$

Still further simplification is desired, since even for a 5-atom molecule like methane (CH_4) this expression (1.6) involves

$3n(3n+1)/2 = 120$ coefficients a_{ij}. We seek a new set of coordinates, q_i called ***normal coordinates,*** that are linear functions of the x_i and are so chosen that the kinetic and potential energies are given by

$$(1.7) \quad 2T = \sum_{i=1}^{3n} \dot{q}_i^2 \quad , \quad 2V = \sum_{i=1}^{3n} \lambda_i q_i^2$$

Here only $3n(=15)$ coefficients λ_i appear, of which 6 can be shown to be 0.

Lagrange's equations of motion for the coordinates q_i take the form

$$(1.8) \quad \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}(T - V) = \frac{\partial}{\partial q_i}(T - V), \text{ or } \ddot{q}_i = -\lambda_i q_i .$$

These equations imply simply harmonic oscillations with frequencies $\lambda_i^{1/2}/2\pi$,

$$(1.9) \quad q_i = c_i \cos 2\pi f_i (t - t_i) \quad , \quad \lambda_i = 4\pi f_i^2 \quad ,$$

which do not quite represent the true motion, but give a close approximation provided that the cubic and higher order terms in (1.4) are relatively small.

Symmetry in an assumed equilibrium configuration for the molecule implies a certain symmetry in the formulas for kinetic and potential energy. Group theory can be used to show that certain of the eigenvalues λ_i must be equal, and thus to reduce the number of essential parameters. Group theory also assists powerfully in determining the normalcoordinates q_i as functions of the displacement coordinates x_i, and in determining which normal coordinates are associated with vibration energies observable as absorption lines in the infra-red spectrum of the molecule.

2. Symmetry groups. ***A*** symmetry operation defined on a rigid configuration of n points is a mapping that replaces each point either by itself or by some other "image" point in the configuration, in such a way that the distance $P_i P_j$ between any two of the points of the configuration is the same as the distance $P_i' P_j'$ between their image points under the given mapping. The result of following a mapping ***a*** by another mapping ***b*** is called the ***product ab*** of the two mappings. (It will be seen below that ***b a*** need not be the same as ***a b).*** The so-called identity mapping e (or 1 or I) maps each point into itself. If a mapping ***a*** maps each P_i into some suitable P_i', the mapping that takes each P_i' back to the original P_i is called the ***inverse*** of ***a*** and is written a^{-1}. We have $a^{-1}a = a^{-1}a = e$. The inverse of the product mapping a b is $b^{-1}a^{-1}$. For any three symmetry operations **a,b,c,** we have $(ab)c = a(bc)$. The set of all symmetry operations on a configuration is called its ***symmetry group.***

In general, a ***group*** G is defined in algebra to be a system of elements **a,b,c,...** connected by a ***law*** of ***combination*** $\circ$ (usually called multiplication or addition) such that if two of the three letters **a,b,c** represent elements of **G,** distinct or not, the equation $a \circ b = c$ can be ***solved uniquely in G*** for the third letter; and furthermore, such that the ***associative law*** $(a \circ b) \circ c = a \circ (b \circ c)$ is valid whenever **a,b,c** are elements of G. The symmetry operations on a configuration form a group under the associative product rule defined above.

The commutative law ***a b = b a*** may or may not hold for two group elements. For example, consider the symmetries of a regular **tetrahed-**

ron whose vertices are labeled 1, 2, 3, 4. Let *a* be a reflection in the plane 034 containing the center 0 and vertices 3, 4, and let *b* be a reflection in the plane 013 through the center 0 and vertices 1 and 3. Under *a b* the vertex originally at 1 moves to position 2 and then to 4, whereas under *b a* this vertex first stays at 1 and then moves to 2. The rotation *a b* takes 1 into 4, 4 into 2 and 2 into 1, leaving vertex 3 fixed, whereas the rotation *ba* takes 1 into 2, 2 into 4 and 4 into 1. Products of these reflections *a* and *b* satisfy

(2.1) $a \cdot a = e, \; b \cdot b = e, \; (ab)(ba) = a(bb)a = (ae)a = a \cdot a = e$

so that *a* is its own inverse, and *b* its own inverse, whereas *ba* is the inverse of *ab*.

3. ***Matrix representation of groups.*** Since multiplication of ordinary numbers is always commutative ($ab = ba$), it is necessary to employ quantities more complicated than single numbers to describe a non-commutative multiplication such as we have defined for the symmetry operations of a regular tetrahedron. Suppose coordinates (x, x_2, x_3) are assigned to each given vertex P_i in the configuration, and that after the mappinga the vertex P moves to a position with coordinates (y_1, y_2, y_3) and after *a b* to a position with coordinates (z_1, z_2, z_3). Then the mappings *a, b,* and *ab=c* can be described by expressing the coordinates x_i in terms of y_j, y_j in terms of z_k, and x_i in terms of z_k, thus:

(3.1) a: $x_i = \sum_j a_{ij} y_j$; b: $y_j = \sum_k b_{jk} z_k$; c = ab: $x_i = \sum_k c_{ik} z_k$

By substituting the values of y_j from the second equation into the first, and comparing coefficients of z_k with the third equation we find expressions for c_{ik} in terms of a_{ij} and b_{jk}.

(3.2) $$c_{ik} = \sum_j a_{ij} b_{jk} \qquad \text{or } C = AB$$

We denote by A the rectangular array of coefficients (a_{ij}) in which the entry a_{ij} appears in row i and column j, and call it a ***matrix.*** We denote by B the matrix (b_{jk}) with b_{jk} in row j, column k. We define the matrix product C=AB to be the matrix whose entry in row i column k is the sum c_{ik} of the products $a_{ij}b_{jk}$ of the j^{th} entries in row i of A and column k of B, summed for all j. With this definition of a product, the matrices that describe the symmetry operations of a configuration with respect to a chosen coordinate system form a group, and the mapping of symmetry operations onto the corresponding matrices is a ***single-valued mapping that preserves products.*** Any such mapping is called a representation of the group.

For example, let us choose coordinates x_i so that orginally the vertices of a given regular tetrahedron are at the points

(3.3) P_1: (1,-1,-I), P_2: (-1,1,-1), P_3: (-1,-1,1), P_4: (1,1,1)

Then the symmetry operations a, b, and a b above may be described by the coordinate transformations

(3.4) a: $\begin{aligned} x_1 &= y_2 \\ x_2 &= y_1 \\ x_3 &= y_3 \end{aligned}$ b: $\begin{aligned} y_1 &= -z_3 \\ y_2 &= z_2 \\ y_3 &= -z_1 \end{aligned}$ ab: $\begin{aligned} x_1 &= z_2 \\ x_2 &= -z_3 \\ x_3 &= -z_1 \end{aligned}$

or by the equivalent matrix equations

(3.5) a: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ b: $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0&0&-1 \\ 0&1&0 \\ -1&0&0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ ab: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0&1&0 \\ 0&0&-1 \\ -1&0&0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

written

(3.6) $X = AY \qquad Y = BZ \qquad X = CZ$

and we verify that the product **AB** of the matrices A and **B** is the matrix C.

(3.7) $X = AY = A(BZ) = (AB)Z = CZ$, where $C = AB$.

Let d be the symmetry operation that permutes the four vertices of the tetrahedron cyclically as follows: $P_1 \to P_2 \to P_3 \to P_4 \to P_1$.

Then the corresponding matrix D and its powers D^2, D^3, D^4 are

(3.8) $D = \begin{pmatrix} 0&0&-1 \\ 0&-1&0 \\ 1&0&0 \end{pmatrix} \quad D^2 = \begin{pmatrix} -1&0&0 \\ 0&1&0 \\ 0&0&-1 \end{pmatrix} \quad D^3 = \begin{pmatrix} 0&0&1 \\ 0&-1&0 \\ -1&0&0 \end{pmatrix} \quad D^4 = I = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$

The matrix D^4 leaves each point fixed and is known as the unit matrix, or identity matrix.

By taking all possible products of the two matrices A and D (allowing repetitions) we obtain 24 distinct matrices (including $B = AD^2AD$) that describe all the 24 symmetries of the regular tetrahedron.

These matrices all into five classes of similar matrices that are said to be ***conjugate*** in the symmetry group.

(3.9)

Class	*Type*	*No.*	*Elements*
I	Identity	1	I
II	180^0 rotations	3	D^2, DB, BD = $(AD^2)^2$
III	120^0 rotations	8	AB,BA,AD,D^3A, DA,AD^3, DAD^2,D^2AD^3.
IV	90^0 "rotary reflections"	6	D,D^3,ADA,AD^3 A, AD^2,D^2A
V	Plane reflections	6	A,ABD,B,BD^2, ABA, DAB

In general, two matrices A and B are called similar if a matrix S can be found such that $B = S^{-1}AS$, and they are called conjugate in a group G containing both, if such an S can be found in G. For the matrices A and B in class V of (3.9) defined by (3.5) and (3.6) the matrix ABA serves as S, since $B = BABAABA = S^{-1}AS$.

The trace of a matrix A is the sum of its principal diagonal entries a_{ii}. If $B = S^{-1}AS = TS$, then $T = S^{-1}A$ or $A = ST$, and it is found that the traces of A and B are equal since

$$(3.10)\quad \operatorname{tr} A = \sum_i a_{ii} = \sum_{ij} s_{ij}t_{ji} = \sum_{ji} t_{ji}s_{ij} = \sum_j b_{jj} = \operatorname{tr} B.$$

Thus two similar (or conjugate) matrices have the same trace. For any given representation of a group by matrices with real or complex entries we define the character to be the class function whose value in any class is the trace of a matrix in that class. Or we can think of the character as a vector whose components are these traces. For the representation by the matrices listed in (3.9) the character is seen from the matrices I, D^2, AB, D, A in (3.5) and (3.8) to be

$$(3.11)\quad \chi^{(4)} = (3, -1, 0, -1, 1)$$

We call this representation R_4 and its character $\chi^{(4)}$.

A different choice of coordinates for the vertices P_i in (3.3) would have yielded a representation similar to R_4 with the same character (3.11). Amazingly enough it can be shown that any two representations with the same character are similar to each other in this way. However, there exist other representations of our group that are not similar to R_4. Four of these are obtained by mapping a and d both on the number 1, or both on -1, or respectively onto the negatives of the matrices A and D, or onto A and the matrix D^+ that is obtained from D by changing its -1 entries to +1. Characters of these representations are as follows.

3.12) $\chi^{(1)} = (1, 1, 1, 1, 1)$. R_1 maps $a \to 1$, $d \to 1$.

3.13) $\chi^{(2)} = (1, 1, 1, -1, -1)$. R_2 maps $a \to -1$, $d \to -1$ (but $ad \to +1$).

3.14) $\chi^{(5)} = (3, -1, 0, 1, -1)$. R_5 maps $a \to -A$, $d \to -D$.

3.15) $\chi''' = (3, 3, 0, 1, 1) = \chi^{(1)} + \chi^{(3)}$. R''' maps $a \to A$, $d \to D^+$.

Many apparently different groups of matrices can be used to describe a given symmetry group. Between any two, however, is a one-to-one correspondence of elements that preserves multiplication. For instance, if the k^{th} symmetry operation S_k is described in terms of linear transformations on the 3n mass adjusted coordinates x_i in (1.2) shown as a column matrix by the 3n x 3n matrix S_k, and if these coordinates are expressed in terms of some new normal coordinates q_i with corresponding column Q, by the matrix relation $X = UQ$, then the transformations

$$(3.16)\quad X = UQ \to S_k UQ = S_k X \text{ and } Q \to (U^{-1}S_k U)Q$$

are essentially the same, and are represented by similar matrices S_k and $U^{-1}S_k U$ which have the same trace for each choice of S_k.

By suitable choice of U it may be possible to partition the new variables q_i into two or more subsets each transformed within itself. The representation $s_k \to S_k$ is then called reducible, and $U^{-1}S_k U$ is called the direct sum of its component representations on the subsets. Its character is the sum of the characters of the components.

For example the representation of (3.15) is reduced by the following choice of U.

(3.17)

$$U = U^{-1} = \frac{1}{\sqrt{12}}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & 2 \\ -1-\sqrt{3} & -1+\sqrt{3} & 2 \\ 2 & 2 & 2 \end{pmatrix},\quad U^{-1}AU = A = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right),\quad U^{-1}D^+U = \left(\begin{array}{cc|c} \sqrt{3}/2 & -1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

In the lower right corners of $U^{-1}S_k U$ we find the 1- dimensional identity matrix, and in the upper left comers the generators of a two dimensional representation which is structurally the same as the group of symmetries of an equilateral triangle. Its character is

$$(3.18)\quad \chi^{(3)} = (2, 2, -1, 0, 0)$$

A representation which cannot be so reduced is called irreducible, and its character is called an irreducible character. The characters (3.11) (3.12) (3.13) (3.14) and (3.18) are all irreducible.

For any finite group the number of distinct irreducible characters (over the complex number field) is equal to the number of classes of conjugates, and these characters are rows of a square matrix whose columns are orthogonal.

In our example the character table is

Class:	I	II	III	IV	V
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	2	2	-1	0	0
$\chi^{(4)}$	3	-1	0	-1	1
$\chi^{(5)}$	3	-1	0	1	-1
Square sum	24	8	3	4	4
No. of els: g_i	1	3	8	6	6

The sum of squared absolute values in the j^{th} column is equal to the group order g (=24) divided by the number of elements g_j in the class C . Furthermore it can be shown that if χ_j and $\chi_j^{(i)}$ denote the values for the g elements in dass C_j of an arbitrary character χ, and of the irreducible character $\chi^{(i)}$ then

$$(3.20)\quad \chi = \sum_j m_i \chi^{(i)} \qquad \text{where } m_i = \sum_j g_j \chi_j \bar{\chi}_j^{(i)}/g$$

In particular, if

$$(3.21)\quad \sum_j g_j \chi_j \bar{\chi}_j = g \qquad \text{and } \chi_1 > 0$$

the character χ is irreducible. The general problem of finding all the irreducible representations of a given finite group is far from easy, and involves much more technique than can be presented here.

4. ***Applications*** to ***molecular structure.*** For a given molecule each of several possible symmetrical equilibrium configurations can be considered in turn. For each model the character of the symmetry group is first calculated in terms of the mass adjusted displacement coordinates. Then this character is decomposed into its component irreducible characters. The **degree (value** for unit element) of each irreducible character contained in it is important as the multiplicity (or degeneracy) of an eigenvalue λ_i of the potential energy matrix in (1.6). This follows from the facts that the same change of coordinates U that reduces the symmetry group to a direct sum of irreducible components, can be chosen that will diagonalize the potential energy matrix, and that only

a scalar multiple of the unit matrix is commutative with all matrices of an irreducible group. Thus the required normal coordinates are obtained by discovering the coordinate change that reduces the symmetry group.

The tetrahedral group will again be used as an illustration. All 15 coordinates for a methane (CH_4) molecule with hydrogen atoms near the vertices and a carbon atom near the center of an assumed tetrahedral equilibrium will be left fixed by the identity operation, so $\chi_1 = 15$. For a 180° rotation in class II that swaps pairs of hydrogen atoms, only the carbon atom is on the axis and its coordinates contribute $1 + 2\cos 180^0 = -1$ to the trace, so $\chi_2 = -1$. For the 120° rotations of class III of (3.9) the two atoms on the axis each contribute $1 + 2\cos 120^\circ = 0$ to the trace, so $\chi_3 = 0$. For the class IV type of 90° rotation combined with a reflection in a plane perpendicular to the axis of rotation, the central carbon atom contributes $-1 + 2\cos 90^0 = -1$, so $\chi_4 = -1$. For a plane reflection in class V, the three atoms in the plane each contribute $-1 + 2\cos 0^0 = 1$ so $\chi_5 = 3$. Hence

$$(4.1)\quad \chi = (15, -1, 0, -1, 3)$$

Each character value is then divided by the square sum at the bottom of table (3.19) to obtain the weighted vector

$$(4.2)\quad (\chi_i g_i/g) = (5/8, -1/8, 0, -1/4, 3/4)$$

Taking the scalar products of this weighted vector with each irreducible character vector in turn we obtain the multiplicities

$$(4.3)\quad m_1 = 5/8 - 1/8 - 1/4 + 3/4 = 1,\ m_2 = 0,\ m_3 = 10/8 - 2/8 = 1,$$
$$m_4 = 15/8 + 1/8 + 1/4 + 3/4 = 3,\ m_5 = 1.$$

Thus the normal coordinates divide into sets that are transformed as follows:

(4.4) 1 by R_1, 1 pair by R_3, 3 triples by R_4, 1 triple by R_5.

One triple of normalcoordinates that transform by R_4 contains the three coordinates of the centroid. For them equation (1.8) becomes $\ddot{q}_i = 0$, for uniform rectilinear motion. The corresponding eigenvalues λ_i are 0. The representation R_4 is called the translation representation, and the other two triples that transform by it undergo oscillations that can be excited by infra red radiation, whose eigenfrequencies show up as strong absorption lines in the infra-red spectrum. That two such strong lines should appear is determined from the simple calculation by group characters described above. A different assumption about the **equilib-**

rium configuration would lead to different multiplicities in (4.3). Another of these coordinate triples that transforms by R_4 and is infra-red active, describes the oscillation of the carbon centroid against the centroid of the four hydrogens. The third triple belonging to this representation R_4, describes an oscillation within the hydrogen configuration itself, measured by changing differences in lengths of opposite edges of the tetrahedron. The triple belonging to R describes the uniform rotation of the principal axes of inertia and has eigenvalues 0. Three coordinates measure changes in the sums of lengths of opposite sides of the tetrahedron and undergo a reducible representation like (3.15). The sum of these coordinates belongs to R_1 and describes a breather motion that cannot be activitated by infra-red bombardment. The other two dimensional component R_3 is also inactive in the infra red, but the corresponding energies can be observed as energy differences in the Raman effect.

If the coefficients in the irreducible representations of any symmetry group are known, they may be used to determine the transformation U that will reduce the symmetry group and define the normal coordinates q that simplify the expression for potential energy. Selection rules to determine which coordinates belong to infra-red active or Raman active frequencies depend on the interrelations among irreducible representations of the symmetry groups.

Finally, however, one must again look at the higher order terms neglected in (1.6) and the interaction term between rotation and inner vibrations, and replace the classical equations of motion by corresponding equations of quantum mechanics to match the observed spectra with patterns predictable from the assumed equilibrium configuration. Remarkable results have been achieved from this approach.

Michigan State University

NATURAL BOUNDARY OF A SERIES ASSOCIATED WITH THE LAMBERT SERIES

George R. Kuhn
Missouri Gamma, '55

Introduction. In 1913 Knopp (1)[1] gave a proof of the existence of a natural boundary of the Lambert series as well as of the generalized Lambert series. Since that time, methods similar to that used by Knopp have been used successfully (3, 4). We shall study in this paper the convergence and especially the natural boundary of a series associated with the Lambert series.

The series considered in this paper is the product of the derivative of the Lambert series and independent variable z; that is,

$$K(z) = z\,L'(z) = \sum nb_n z^n/(1-z^n)^2.^{2}$$

The first of the three parts of this paper will establish the region of convergence of the $K(z)$ series while the remaining two sections will prove the existence of its natural boundary when $b_n = 1/n$ and when b_n b_n is more broadly defined.

1. ***Convergence of*** $K(z)$. In this section we shall discuss the convergence and uniform convergence of $K(z)$.

Theorem 1. a). *If* $\sum b_n$ *is convergent, then the* $K(z)$ *series converges for all* $|z| \neq 1$.

b). *If* $\sum b_n$ *is not convergent, then the* $K(z)$ *series converges with power series* $\sum b_n z^n$.

Proof of (a). Since $\sum b_n$ is convergent, it follows that $\sum b_n z^n$ and $\sum nb_n z^n$ converges when $|z| < 1$. When $|z| < 1$, then for a sufficiently large m $|1 - z^m| > 1 - |z^m| > 1/2 > 0$. Hence

$$\left| \sum_{n=m}^{\infty} \frac{n\,b_n z^n}{(1-z^n)^2} \right| \leq 2^2 \sum_{n=m}^{\infty} \left| n b_n z^n \right|$$

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[1] The numbers appearing in ( ) refer to reference given in bibliograph! at end of paper.

[2] $\sum$ will indicate the sum from n or v = 1 to n or v = ∞ depending on the numbering index unless otherwise indicated.

from which follows that $K(z)$ converges for $|z| < 1$. Indeed, $K(z)$ converges absolutely and uniformly in $|z| \le \rho < 1$.

For $|z| > 1$, we have

$$\sum \frac{nb_n z^n}{(1-z^n)^2} = \sum \frac{nb_n(1/z)^n}{[1-(1/z)^n]^2} \tag{1.1}$$

and as $|1/z| < 1$, the right side of (1.1) is reduced to a form for which the preceding discussion applies and hence $K(z)$ converges for all $|z| \neq 1$.

Proof of (b). In order to establish this theorem the du Bois-Reymond Theorem[1] will be utilized. The series $K(z)$ may be written in the form

$$\sum nb_n z^n \cdot \frac{1}{(1-z^n)^2},$$

which is of the form $\sum a_n c_n$ and the du Bois-Reymond Theorem is applicable with $a_n = nb_n z^n$ and $c_n = 1/(1-z^n)^2$

Consider

$$\sum_{n=m}^{\infty} \left| \frac{1}{(1-z^n)^2} - \frac{1}{(1-z^{n+1})^2} \right| =$$

$$\sum_{n=m}^{\infty} \left| \frac{1-2z^{n+1}+z^{2n+2}-1+2z^n-z^{2n}}{(1-z^n)^2(1-z^{n+1})^2} \right| \tag{1.2}$$

for $m \ge 1$. For $|z| < 1$ and for sufficiently large m, we have $|1-z^m| > 1/2$.

As a consequence of this, it follows that for $|z| < 1$ and sufficiently large m that (1.2) is less than or equal to

$$64\,|1-z| \sum_{n=m}^{\infty} |z^n|$$

Hence, $K(z)$ converges with $\sum nb_n z^n$ and thus with $\sum b_n z^n$.

[1]The du Bois-Reymond Theorem can be stated as follows: $\sum a_n c_n$ is convergent if $\sum [c_n - c_{n-1}]$ is absolutely convergent and $\sum a_n$ is conditionally convergent (2, p, 315).

Corollary. The $K(z)$ series with hypotheses of either (a) or (b) of the above theorem is uniformly convergent in the regions which lie within the circles of convergence of the series.

In the proof of Theorem 1, Part (a), it was noted that $K(z)$ is absolutely and uniformly convergent for $|z| \le \rho < 1$ by the Weierstrass M-test.

When $|z| > 1$, and $\sum b_n$ is convergent, using (1.1) it is seen that $K(z)$ is uniformly convergent when $|z| \ge 1/\rho > 1$. On the other hand, when $\sum b_n$ is not convergent, $K(z)$ is uniformly convergent only if $|z| \le \rho < r < 1$, where r is the radius of convergence of $\sum b_n z^n$.

2. Natural Boundary Special Case. Let us consider the natural boundary for the $K(z)$ series when $b_n = 1/n$. Hence the series becomes

$$K(z) = \sum z^n/(1-z^n)^2.$$

Since this is derived from the Lambert Series which has the unit circle as a natural boundary (4) we would like the $K(z)$ series to also have the unit circle as a natural boundary. This is true.

Theorem 2. If in the $K(z)$ series, $b_n = 1/n$ and for the positive integers p and a relatively prime q $z_o = e^{2\pi i p/q}$ then for radial approach

$$\lim_{z \to z_o} (1 - z/z_o)^2 \sum z^n/(1-z^n)^2 = \sum_{v=1}^{\infty} \frac{1}{(qv)^2}$$

and the function $K(z)$ has the unit circle as a natural boundary. Showing the existence of the limit proves that the points z_o which are dense in themselves, are non-removable singularities. Any region completely enclosed by a curve composed of a set of singularities dense in themselves has this curve as a natural boundary.

Let the considered points within the unit circle be designated by

$$z = \rho e^{2\pi i p/q}, \text{ where } \rho < 1,$$

whereas points on the unit circle will be

$$z_o = e^{2\pi i p/q}$$

where p and q are relatively prime positive integers. Now divide the series into two sums such that $\sum_1$, is that part of $K(z)$ for which $n \equiv 0 \pmod q$; while $\sum_2$ is that part for which $n \not\equiv 0 \pmod q$.

Consider first the

$$\lim_{z \to z_o} (1 - z/z_o)^2 \sum z^{qv}/(1 - z^{qv})^2 =$$

$$\lim_{\rho \to 1} (1-\rho)^2 \sum \rho^{qv}/(1-\rho^{qv})^2 \qquad (2.1)$$

and multiplying both numerator and denominator by $(1-\rho^q)^2$, results in

$$\lim_{\rho \to 1} \frac{(1-\rho)^2}{(1-\rho q)^2} \cdot (1-\rho q)^2 \sum \rho^{qv}/(1-\rho^{qv})^2 =$$

$$\frac{1}{q^2} \lim_{\rho \to 1} (1-\rho^q)^2 \sum \rho^{qv}/(1-\rho^{qv})^2.$$

Substituting $y = \rho^q$ gives

$$1/q^2 \lim_{y \to 1} \sum y^v / \left[(1-y^v)/(1-y^2)\right]^2 =$$

$$\lim_{y \to 1} \sum \frac{v^2 y^v}{(qv)^2(1+y+y^2+\ldots+y^{v-1})^2}. \quad (2.2)$$

The final expression can be evaluated if the limit can be moved inside the summation. This can be done if the series is uniformly convergent in the interval $0 < y < 1$.[1]
This will now be shown.
To facilitate the proof, let

$$1/(qv)^2 = C_v, \qquad (2.3)$$
$$v^2y^v/(1+y+y^2+\ldots+y^{v-1})^2 = y_v,$$

and

$$r_v = \sum_{\lambda=v+1}^{\infty} C_\lambda.$$

Let us note that $\sum C_v$ is convergent, and that we must now consider uniform convergence of $\sum C_v y_v$. To show uniform convergence of this series, it must be shown that for every $e > 0$, it is possible to find an N_e such that for $m \geq N_e$ and $h > 1$, it is true that for $0 \leq y \leq 1$, that

$$\left| \sum_{v=m+1}^{m+h} C_v y_v \right| < e. \qquad (2.4)$$

The series $\sum_{v=m+1}^{m+h} C_v y_v$ may be expressed as

---

[1]As a matter of fact, the interval could be chosen as $0 < a \leq y \leq 1$.

$$\sum_{v=m+1}^{m+h} (r_{v-1} - r_v) y_v =$$

$$- \sum_{v=m+1}^{m+h} r_v (y_v - y_{v+1}) + r_m y_{m+1} - r_{m+h} y_{m+h+1}. \quad (2.5)$$

Each term of (2.5) must be examined for boundedness.

Since $\sum_{\lambda=m+1}^{\infty} C_\lambda = r_m$ is convergent, it is true that for a given e, an m can be found such thai $|r_m| < e$.

Showing that y_v and especially, $(y_v - y_{v+1})$ are bounded is a more complicated task. It will now be shown that $\{y_v\}$ is bounded and monotone in $0 \leq y \leq 1$.

Let us consider $(y_v)^{1/2} - (y_{v+1})^{1/2}$, where $0 < y < 1$. This difference may be shown to be

$$\frac{(1-y^{1/2})(1-y)y^{1/2 v}}{(1-y^v)(1-y^{v+1})} \left\{ v - \left[y^{1/2} + y + y^{3/2} + \ldots + y^v\right] + vy^{v+1/2} \right\}. \quad (2.6)$$

Let $f(y^{1/2})$ be the factor within the braces of (2.6). Hence, if $y = z^2$, we have

$$f(z) = v - \left[z + z^2 + z^3 + \ldots + z^{2v}\right] + vz^{2v+1}.$$

At $z = 0$, $f(z) = v$; while at $z = 1$, $f(z) = 0$.
Hence $f(z)$ passes through the points $(0,v)$ and $(1,0)$. The derivative of $f(z)$ is $f'(z) = -(1 + 2z + 3z^2 + \ldots + 2vz^{2v-1}) + v(2v+1)z^{2v}$ which for $0 < z < 1$ is negative. This indicates that $f(z)$ is monotone decreasing and positive for $0 < z < 1$. Since the fraction in (2.6) is positive for $0 < y < 1$, the sign of (2.6) is the same as that of $f(z)$. Consequently,

$$(y_v)^{1/2} - (y_{v+1})^{1/2} > 0, \text{ or } y_v > y_{v+1}$$

when $0 < y < 1$.
Furthermore, since at $y = 0$, $y_v = 0$ and at $y = 1$, $y_v = 1$, $\{y_v\}$ is bounded and is monotone non-increasing in the interval $0 \leq y \leq 1$.
Hence $|y_v| \leq 1$, since $|y_1| \leq 1$ for $0 \leq y \leq 1$, and $(y_v - y_{v+1})$ is positive for $0 < y < 1$ and zero when $y = 0$, or 1.

The absolute value of (2.5) now becomes

$$\left| \sum_{v=m+1}^{m+h} C_v y_v \right| \leq \sum_{v=m+1}^{m+h} |r_v| (y_v - y_{v+1}) + |r_m y_{m+1}| + |r_{m+h} y_{m+h+1}|$$

or

$$\le e \sum_{v=m+1}^{m+h} (y_v - y_{v+1}) + e + e = e(y_{m+1} - y_{m+h+1}) + e + e < 3e,$$

from which it follows that $\sum C_v y_v$ is uniformly convergent in $0 \le y \le 1$.

Taking the limit of (2.2), it follows that

$$\lim_{y\to 1} \sum \frac{v^2 y^v}{(qv)^2 (1+y+y^2+\dots+y^{v-1})^2} = \sum \frac{1}{(qv)^2} \qquad (2.7)$$

which is also the limit of (2.1)

To complete the proof of Theorem 2, we must show

$$\lim_{z\to z_0} (1-z/z_0)^2 \sum_2 = \lim_{z\to z_0} (1-z/z_0)^2 \sum z^{qv+s}/(1-z^{qv+s})^2 = 0. \qquad (2.8)$$

Firstly, let us show that for all n not divisible by q that

$$|1-z^n| \ge t, \qquad (2.9)$$

where $t = 1$ if $q = 2$ and $t = \sin 2\pi/q$ if $q > 2$. For if $q = 2$ and $0 \le \rho \le 1$, then $|1-z^n| = |1-\rho^n e^{2\pi inp/q}| = |1+\rho^n| \ge 1$.
If $q > 2$, then $2\pi inp/q = 2\pi i\alpha + 2\pi i\beta/q$ where α is an integer and $1 \le \beta \le q-1$. Thus $|1-z^n| = |1-\rho^n e^{2\pi inp/q}|$ which will be smallest when β is one or q-1. With either of these values, it is easy "to show that $|1-z^n| \ge \sin 2\pi/q$.
With the substitution $y = \rho^q$, the limit in (2.8) becomes

$$\lim_{y\to 1} \sum \frac{v^2 y^v z^s (1-y^v)^2}{(qv)^2 (1+y+y^2+\dots+y^{v-1})^2 (1-z^s y^v)^2} . \qquad (2.10)$$

Here again, the limit may be taken inside the summation symbol if the series is uniformly convergent for $0 \le y \le 1$. Uniform convergence will now be established for $0 \le y \le 1$.

Let

$$\sum_{v=m+1}^{\infty} \frac{v^2 y^2}{(qv)^2 (1+y+y^2+\dots+y^{v-1})^2} = \sum_{v=m+1}^{\infty} f_v = F_m(y),$$

which is the series (2.2 and is uniformly convergent for $0 \le y \le 1$. Hence, for $0 \le y \le 1$ and for m large enough $|F_m(y)| < e$. Also let

$$\frac{z^s(1-y^v)^2}{(1-y^v z^s)^2} = y_v .$$

With the aid of (2.9) it follows that $|y_v| < 1/t^2$ for $0 \le y \le 1$.

Consequently, from the series in (2.10), it is apparent that for $h > 1$

$$\left|\sum_{v=m+1}^{m+h} f_v y_v\right| = \left|\sum_{v=m+1}^{m+h} y_v (F_{v-1} - F_v)\right|$$

$$\le \sum_{v=m+1}^{m+h} |F_v| \cdot |y_v - y_{v+1}| + |F_m y_{m+1}| + |F_{m+h} y_{m+h+1}|$$

$$\le \frac{e}{t^4} \sum_{v=m+1}^{m+h} \left| -2y^v(1+z^s y^{2v+1})(1-z^s)(1-y) + y^{2v}(1-z^{2s})(1-y^2)\right| + \frac{2e}{t^2}$$

$$\le 10e/t^4 + 2e/t^2$$

for sufficiently large m.

Consequently, the series in (2.10) is uniformly convergent for $0 \le y \le 1$ and the limit of (2.10) may be moved within the summation symbol yielding the result of (2.8).

The combined facts as stated in (2.7) and (2.8) prove Theorem 2.

3. Natural Boundary General Case. We will now consider the $K(z)$ series in which the coefficients are less restrictive.

Theorem 3. *If the coefficients of $K(z) = \sum nb_n z^n/(1-z^n)^2$ are so restricted that for certain integers* q, *the series*

$$\sum \frac{b_{qv+s}}{(qv+s)} \qquad s = 0, 1, 2, \dots, q-1$$

is convergent and if for such a q *and a relatively prime* p, *we set* $z_0 = e^{2\pi ip/q}$ *then for radial approach*

$$\lim_{z\to z} (1-z/z_0)^2 \sum nb_n z^n/(1-z^n)^2 = \sum b_{qv}/qv.$$

Further, the $K(z)$ ***series has the unit circle as a natural boundary, if*** $\sum b_{qv}/qv \ne 0$ ***for a countable set*** q.

The proof of this theorem is similar to that of Theorem 3 and will not be given. It should be noted, however, that Theorem 2 is a corollary to Theorem 3 since for $b_n = 1/n$, the conditions of Theorem 3 become identical to the conditions of Theorem 2; that is

$$\sum \frac{b_{qv+s}}{(qv+s)} = \sum 1/(qv+s)^2$$

which converges for $s = 0, 1, \dots, q-1$. Similarly, the final limit $\sum b_{qv}/qv$ equals $\sum 1/(qv)^2$ of Theorem 2. Thus, Theorem 2 is but a special case of Theorem 3.

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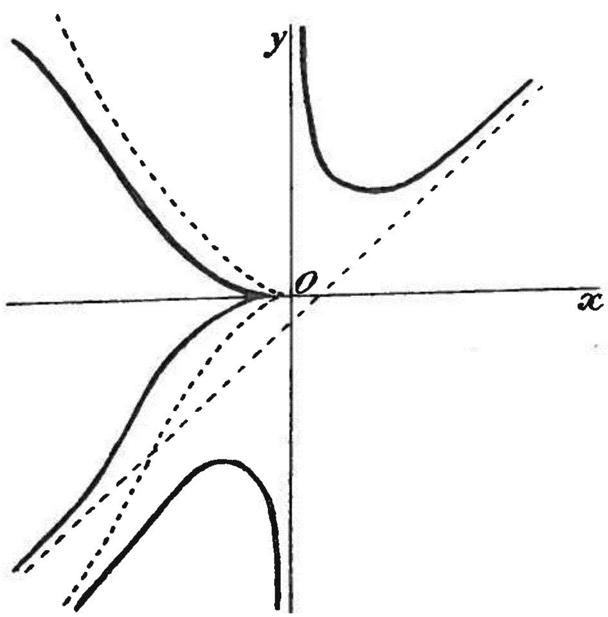
Battelle Memorial Institute
Columbus, Ohio



$$x^7 - x^4y^3 + a^3y^4 - axy^5 = 0$$

PROBLEM DEPARTMENT

Edited by
M. S. **Klamkin,**
Avco Research and Advanced Development Division

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity; but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate, signed sheets within four months after publication. Address all communications concerning problems to M. S. Klamkin, Avco Research and Advanced Development Division, **R-6, Wilmington,** Massachusetts.

107. Proposed by Michael J. Pascual, Burbank, California.

Professor Pushover being a sympathetic soul wishes to make it as easy as possible for his class on an analytic geometry exam, so he decides to set up second degree equations with integral co-efficients such that the rotation transformations necessary to eliminate the cross-product term involve onlv rational numbers. What set of formulas could the kind professor use?

108. Proposed by Daniel Block, Yeshiva University.

Evaluate the n^{th} order determinant D_n where

$$D_n = \begin{vmatrix} 1 & 1 & 0 & \ldots & & & & 0 \\ 1 & -3 & 1 & 0 & \ldots & & & 0 \\ 0 & 3 & -3 & 1 & 0 & \ldots & & 0 \\ 0 & -1 & 3 & -3 & 1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 3 & -3 & 1 & 0 \ldots & 0 \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 0 & -13 & -3 \end{vmatrix}$$

109. Proposed by Leo Moser, University of Alberta.

Fifteen professors attend regular committee meetings. At the end of each meeting every professor leaves with the hat brought by the man he most admires. Assume no professor is more popular than any other and that their relative admiration for one another does not change. After the first meeting, to which everyone brings his own hat, they agree to disband their committee as soon as everyone regains his own hat. They find that after 100 meetings they are still going strong. When will they disband?

*110.** Proposed by Lawrence **Shepp,** Princeton University.

If $P(z)$ is a polynomial of degree n, then $P(z)/z^{n+1}$ is real and positive for some z on the unit circle $|z| = 1$.

*111.** Proposed by M.S. **Klamkin,** Avco RAD, and D. J. **Newman,** M.I.T.

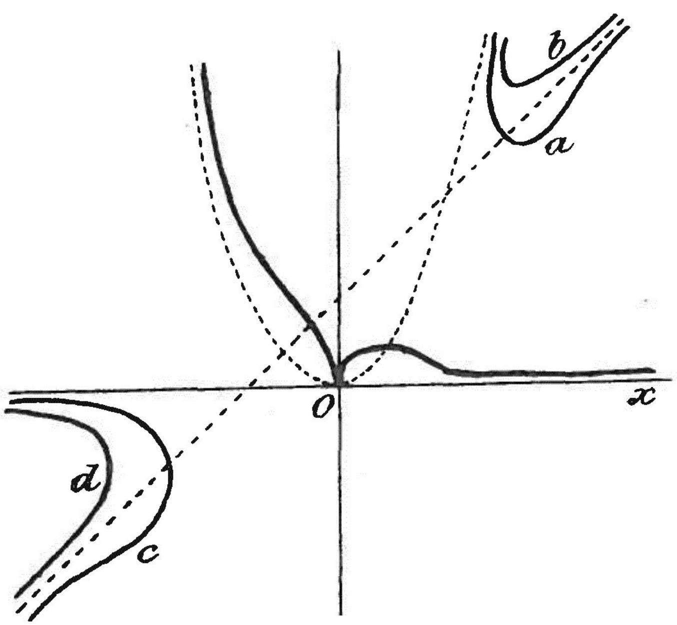
It is conjectured that at most $N-2$ super-queens can be placed on an $N \times N$ $(N > 2)$ chessboard so that none can take each other. A super-queen can move like an ordinary queen or a knight.

Late Solutions

97. George *E.* Andrews, Rockne *F.* Lambert

99-101. George *E.* Andrews

*No solution sent in by the proposer.



$$yx^2(y - x) - ay^3 - byx^2 + c^2x^2 = 0$$

BOOK REVIEWS

Edited by
Franz E. Hohn, University of Illinois

Modern Geometry, an Integrated Course. By Claire F. Adler. New York, McGraw-Hill, 1958, xiv + 216 pp., $6.00.

Modem Geometry, an Integrated First Course, by Claire Fisher Adler, is in many ways an admirable book. Its information as to the historical development of geometry shows how implicit assumptions have necessitated the recognition and clarifications of postulates and laws of reasoning. The subjects presented lead from interesting topics of Euclidean geometry (Menelaus' and Ceva's Theorems, harmonic elements, inversive geometry, etc.) to protective geometry. Coordinate projective geometry is introduced by Euclidean models; thus, the lines through the origin of Euclidean three-space serve to model the points of the projective plane. Transformations are given a central place in classifying geometries.

The theory is made interesting; the exercises are numerous and good. Especially to be commended is the inclusion of appendices A and B on Early Foundation of Euclidean Geometry and Hilbert's Axioms, respectively. The format of the book is excellent.

Due perhaps to the magnitude of the task set, there are inept and confusing statements, and even alarming mistakes. Some objections may be cited:

(1) Aristotle's Laws of Logic (p. 8, lines 3-5) are not too clearly or consistently stated: the "true-false" contrast is used for two of them, while the law of identity is given the misleadingly blunt form: "A thing is itself."

(2) It should be pointed out that the form of Pasch's axiom given on p. 26, lines 1-5, restricts the geometry to the plane.

(3) The necessity for giving a reference system of three rather than two elements when assigning a coordinate system to a pencil is ignored (p. 118, lines 1-10).

(4) The set of translations in the plane is given as

$$x' = x + a_i,$$
$$y' = y + b_i, \quad i = 1, 2, 3, \ldots, n,$$

though apparently a finite number is not intended (p. 139, lines 16-18).

(5) It is stated that the logarithmic function has period $2n\pi$ (p. 160, footnote; see also p. 161, line 20).

(6) It should be made clear that Part Three (Non-Euclidean and Metric Projective Geometries) is a suggestive description rather than a mathematical development. Even so, more care should be taken. Thus, the definition of parallel lines on p. 163-164, § 12.10, is not valid.

(7) In a book that seeks to present the unifying relations in mathematics (and so often succeeds), it is a pity that the implication of Pappus' Theorem as to commutative multiplication in the field of coordinates is not mentioned.

With some corrections and rewriting, this book can be made a very valuable text.

University of Illinois — Josephine H. Chanler

Introduction to Multivariate Statistical Analysis. By T. W. Anderson. New York, John Wiley, **1958. xiv + 374** pp., **$12.50.**

Some Aspects of Multivariate Analysis. By **S.** N. Roy. New York, John Wiley; Calcutta, Indian Statistical Institute, **1957.** xv + **214** pp., **$8.00.**

1. With the appearance of Anderson's book a real gap has been filled, since previously there was no modem textbook on multivariate normal analysis available. The book has a clear purpose and achieves this purpose very well. It is primarily intended as a textbook **for the** graduate student in mathematical statistics, who has a fair background in mathematics and statistics in general, but not necessarily any knowledge of multivariate analysis. There is even a review of the basic notions in continuous joint distributions, and a review of the necessary matrix algebra (in an appendix), so that the book is self-contained to a high degree. The book is well-organized, excellently written, contains many practical illustrations, and includes a host of interesting problems. This makes it an excellent textbook for the student of statistics, but there is no doubt that it will also be very useful as a reference work for the user of multivariate normal analysis.

The major topics covered are: the multivariate normal distribution, estimation and tests concerning the mean and covariance matrix of one population, correlation coefficients, classification of observations, linear hypotheses, independence of sets of variates, comparison between several populations, principal components, canonical correlations, and the distribution in the null case of the roots of a famous **determinantal** equation. There are, of course, many special topics not covered in the book, and of some of these a short account is given in the last chapter.

With so much to praise, it is only fair to point out some weakness. (a) To this reviewer's taste, too much emphasis is placed on the method of maximum likelihood In several cases the principle of invariance would have led to the same results in a more elegant way. As it is, in the main text little use has been made of invariance, and only very few problems touch on it. (b) It is unfortunate that the author does not mention a general theorem concerning the completeness of a family of exponential densities, contained in the **1955** Sankhya article of **Lehmann** and Scheffe. From this theorem the best unbiasedness of some of the usual estimates follows at once. Only in chapter 3 a special case of this theorem - the completeness of the sample mean vector for the population mean vector of the population covariance matrix is known - is delegated to a problem. (c) Chapters **4** and **5** should perhaps have been switched around, since chapter **4** seems to be the harder of the two. (d) In certain respects the book is not quite up to date. For instance, in chapter **13** the important methods introduced by A. T. James *(Ann. Math. Stat.* **1954),** using invariant measures and exterior differential forms, are not even mentioned.

The book has been printed with considerable care. The print is excellent and there are relatively few printing errors. Taking everything into account, the book can be warmly recommended to everyone learning, using, or teaching multivariate normal analysis.

2. Roy's book is essentially a rehash of papers by the author, and by the author and co-workers, which appeared in the *Annals* of *Mathematical Statistics* and in *Biometrika.* These papers deal with the author's heuristic method for test and confidence interval construction, and with the extension of multivariate ideas into multinomial situations. One soon becomes aware of the fact that the greater **part of** these papers have been taken over bodily into the book, and after that the book has been filled up with a little more discussion, with some classical material (chapters **3, 4,** and part of **12)** and with a set of appendices which occupy about one third of the book. It is inevitable that putting a book together in this way will lead to some incoherence and inconsistencies. At the top of p. **121,** for instance, the reader is referred to an earlier paper by the author instead of being referred to chapter **2.** A peculiar inconsistency appears in chapter **15.** After being instructed in section **15.1** that **'i'** stand for 'variate' and 'j' for 'classification', in subsection **15.2.1** both 'i' and 'j' are 'variates', and in subsection **15.2.2 'i'** is 'classification' and **'j'** is 'variate'. Subsequent subsections are similarly inconsistent.

One wonders what the aim of Roy's book is. It is clearly not directed toward the beginning student, but if it is intended for the more advanced worker in the field, then it is not clear why the classical material has been included at all. A reference to T. W. Anderson's book would have been appropriate. It is regrettable that apparently no attempt has been made to prevent overlapping of the books by Anderson and Roy, which is the stranger since both were published by Wiley.

The printing was done in Calcutta by the printers of *Sankhya.* The print is acceptable but far from excellent, and printing errors are numerous. Last and least, the book has the typical awkward Sankhya size, which won't fit on ordinary bookshelves. All in all, this reviewer sees little justification for Roy's book, since it is unsuitable for students, and the more advanced worker will feel happier reading the original papers.

University of Illinois — Robert A. Wijsman

Experimental Designs in Industry. Edited by Victor Chew. New York, John **Wiley, 1958.** xi + **268** pp., **$6.00.**

This is a collection of nine expository papers describing experimental designs, many of recent origin, particularly suited for industrial research and describing the experiences of industrial research workers with them. Most well known experimental designs were first used in agricultural research. However, industrial experimentation differs from agricultural experimentation in important ways. Industrial experiments are usually of shorter duration and therefore sequential methods may be more easily used with consequent savings in the number of observations needed, observations have less variability, and so forth. Accordingly, experimental designs invented for use in agricultural experimentation may not always be the best in industrial research. This book does not systematically develop the theory of experimental design. However, it does contain a great deal of material written at a fairly nontechnical level which should be of interest to industrial research workers and which might be of interest to anyone who is considering employment in industrial research. The book **contains** an extensive bibliography..

University of Illinois — D. Burkholder

Operational Mathematics. By **Ruel** V. Churchill. New York, **McGraw-Hill, 1958.** ix + **337** pp., **$7.00.**

This book is a revision of the well known "Modern Operational Mathematics in Engineering", A comparison of the table of contents in the first and second editions reveals the same topic headings and number of chapters. These comprise: The **Laplace** Transformation, Further Properties of the Transformation, Elementary Applications, Problems in Partial Differential Equations, Functions of a Complex Variable, The Inversion Integral, Problems in Heat Conduction, Problems in Mechanical Vibrations, Sturm-Liouville Systems, and Fourier Transforms.

It is within a number of these chapters, however, that differences between the first and second editions occur. In the first two chapters, for example, new proofs of some of the operational rules have been given and the step as well as the unit impulse functions are introduced. Chapter three on elementary applications is essentially the same as before, except for more problems involving electrical circuits and a section on servo-mechanisms. The next six chapters, except for minor modifications, are the same as in the first edition. The last chapter has been expanded **some-**

what and includes topics such as the "generalized Fourier transforms". The table of transforms is substantially the same as in the first edition.

The author has written an excellent revision in the same lucid manner characterized by its predecessor. It should serve as an important introduction to the growing field of operational mathematics.

University of Illinois E. J. Scott

Lectures on Ordinary Differential Equations. By Witold Hurewicz. Technology Press and John Wiley and Sons, New York, 1958. 122 pp., $5.00.

As indicated in the preface, this book is a printing of notes of some lectures given, in 1943, by the late Professor Hurewicz. The very fact that these notes have not only survived, but even found their way into print, is itself a testimonial.

This small volume will be of value to students with some background in the techniques of solution of ordinary differential equations who wish to go more deeply into certain of the mathematical aspects of this interesting field. To read it profitably they will need to have reached a level where they can read analytic proofs involving uniform continuity, uniform convergence, elementary estimates, etc., with some understanding. Also, a familiarity with the rudiments of matrix theory will be helpful.

An appealing feature of this book is its economy of style. Despite its small size, it treats (1) questions of existence and uniqueness of ordinary differential equations, (2) systems, both linear and non-linear, of differential equations, (3) the singularities of a second order autonomous system of equations, and (4) the global structure of solutions of such systems. While these topics are discussed in other works, it is certainly convenient to have this brief, fairly self-contained, and highly lucid treatment.

Of course, brevity is not without certain shortcomings. The shortcomings here are that there are no examples and no excercises. In our opinion this prevents its use as a text, but it will still be very valuable for supplementary reading.

University of Illinois Robert G. Bartle

Manual of Scientific Russian. By Thomas F. Magner. Minneapolis, Burgess Publishing Company, 1958. iii + 101 pp., $4.60.

This excellent manual provides, for the "mature person, used to complex ("linguistic"] structures," a brief and efficient discussion of the structure of the Russian language, a detailed reference outline to aid in the identification of the forms of nouns, adjectives, verbs, etc.; a vocabulary aid which groups in separate lists the prepositions, the connectives, the most frequently used adverbs, the most important phrases, etc.; and a guide to the technique of translation. The book concludes with a brief set of practice readings.

For the linguistically experienced and persistent do-it-yourself enthusiast, this book - together with a good dictionary - provides an effective means of learning to ***read*** scientific Russian without classroom instruction but not without effort. Pronunciation is ignored as much as possible here. Appropriately, conversation, colloquial and literary Russian, and the writing of Russian are not treated at all.

The person who wishes to learn to read Russian and who has the opportunity to take or audit a semester or moreof the language will find this a particularly useful reference volume when he begins the serious reading of scientific work.

The reviewer also recommends N. F. Potapova: ***Russian, Elementary Course,*** Volumes I and II (Moscow, Foreign Languages Publishing House, 1954, about $2.00 per volume) and A. I. Smirnitsky: ***Russian-English Dictionary*** (Moscow, State Publishing House of Foreign and National Dictionaries, 1958, 951 pp., about $4.50) as superior aids for the one who tackles the job by himself. These may be ordered through your local bookseller. For the mathematician, the pamphlet: ***Russian-English Vocabulary with a Grammatical Sketch*** (American Mathematical Society, 190 Hope Street, Providence 6, Rhode Island, $1.50) is excellent.

Russian is a difficult but by no means impossible language to learn. Thus the do-it-yourself student must expect only those miracles that are the result of hard work. However, the accomplishment is well worth the effort, for Russian is a language with a special beauty of its own. Moreover, it is decidedly sobering to observe at first hand, even if only to a limited extent, the maturity of the scientific work of a people whose intellectual efforts we so long have scorned because of political differences.

University of Illinois Franz E. Hohn

Queues, Inventories and Maintenance. By Phillip M Morse. New York, John Wiley, 1958. ix + 202 pp., $6.50.

Finite Queuing Tables. By L. G. Peck and R. N. Hazelwood, New York, John Wiley, 1958. xvi + 210 pp., $8.50.

Imagine for a moment that you are driving from Long Island to Connecticut on a toll road one summer Sunday. At the toll booths are cars in queues waiting their turn to pay the toll and then move on. The mathematician in you might conjecture: At what rate do the queues grow? How long does it take each car to pay the toll? If there were more toll booths open, how long would a car have to wait? etc. The "toll booth" problem has its counterpart wherever people - or objects - wait for service.

Mr. Morse has written a timely book, ***Queues, Inventories and Maintenance,*** for "newcomers to the field of operations research". The newcomers are presumed to know mathematics on the first year graduate level, and to comprehend the scientific method. Mr. Morse explains that queues, inventories, and maintenance share a common mathematical formulation. He elaborates on the physical interpretation of the symbols in the equations. The interpreting is the substance of the book. The mathematics of matrix algebra and probability which underly the work are scarcely mentioned. Yet the book can be appreciated only superficially without the reader's having read, say, Feller's ***Probability Theory and its Applications.***

The book will be most interesting and helpful to a person considering operations research as a profession. Mr. Morse has captured the delight of formulating very common real world situations mathematically.

In Finite Queuing Tables there are good derivations and explanations of the equations used, as well as of the method of calculation. These tables should be of value to those engaged in queuing theory.

Jane I. Robertson

An Introduction to Combinotorial Analysis. By John Riordan. New York, John Wiley, 1958. x + 244 pp., $8.50.

This is a book written by an expert andit reflects this fact in every page. It is very readable and deals with thesubject with the thoroughness that is characteristic of the author.

The author has taken Combinatorial Analysis in its broad sense of "enumeration". Thus problems arising in numerous applications such as Statistics and Network Theory are treated alongside the classical combinatorial problems. However, as the title implies, only the most important tool – the enumerating generating function – is introduced and only such problems as lend themselves easily to this technique are considered. These are mainly problems that are describable by linear difference equations or difference equations of the convolution type. The contributions of Slepian (Jour. Math. Phys., 1953, pp 185-193) and Davis (Proc. Amer. Math. Soc., 1953, pp 486-495) are examples of problems that are not considered.

The concept of a generating function is introduced in chapter 1, after an elementary discussion of permutations and combinations. Chapter 2 is devoted entirely to generating functions, their elementary algebraic properties and their application to linear difference equations and related problems in statistics. The principle of inclusion and exclusion is discussed briefly in chapter 3. The next chapter contains a brief treatment of the enumeration of permutations by cycles. The important problem of distributions and the related problem of occupancy are given a thorough treatment in chapter 5. The following chapter is one of the best in the book, dealing mainly with the problem of enumerating the number of linear graphs of various types - trees, rooted trees, series-parallel two terminal graphs, labelled graphs, etc. Since the author has made many original contributions to these problems it is natural that this chapter should be excellent. The last two chapters concern the enumeration of permutations in in which certain elements are restricted to occupy or not to occupy certain positions. The traditional combinatorial problems (probleme des rencontres, probleme des menages, the rook problem, card matching, etc.) are discussed in detail and in a unified manner.

Within the scope considered in the book, the coverage is excellent. Virtually all the major contributions are included, with special emphasis on those of Euler, Touchard, Bell, MacMahon and, of course, Riordan. There is a short but well-chosen bibliography in each chapter, with many references as recent as 1956. Each chapter is followed by a long list of well-chosen problems, many of which are far from trivial.

The faults that can be found with the book are mainly those ascribable to the subject itself and to the relatively small number of pages in the present text. Many non-specialists will find this book difficult to refer to, due to the special notations of combinatoric (which are necessary to keep the formulas brief). A table of symbols would have been a useful addition to the book. An abstract study such as this always suffers from lack of motivation. The author has tried to compensate for it by giving the origins of the various problems but space considerations have necessarily left these discussions very brief.

There are very few typographical errors in the book.

University of Toronto — Sundaram Seshu

Introduction to Functional Analysis. By Angus E. Taylor. New York. John Wiley, 1958, 423 pp., $12.50.

This book is a very attractive introduction to the theory of linear operators and topological linear spaces that has been developed in this century. This field has grown from studies in linear differential and integral equations of the sort arising in potential theory and quantum physics, and is in turn applicable to these subjects. Taylor's book is written primarily for graduate students of mathematics and "is an introduction, not a treatise" which is "meant to open doors for the student." In our opinion it should be highly successful in this goal.

In the past five years at least a dozen books on this general area have appeared in this country or abroad. In a review of this nature it is not appropriate to contrast and analyze the varying features of these books. We shall content ourselves with expressing the opinion that Taylor is the most suitable one for use as a textbook in a first course in functional analysis, or for independent reading.

Each new circle of ideas is introduced carefully with considerable motivational discussion. Many examples and applications are given, so the importance and relevance of the ideas are demonstrated. The actual development is rigorously done but flows smoothly. Finally, there are numerous well-chosen exercises on almost every topic that is discussed.

In summary, we recommend this text most highly as an introduction to abstract analysis.

University of Illinois — Robert G. Bartle

Elementary Matrix Algebra. By Franz E. Hohn, The McMillan Company, New York, 1958. xi + 305 pp., $7.50.

Professor Hohn's book is designed to serve many purposes. It presupposes little background on the part of the student and could perhaps be used profitably in a course for students with no more formal background than a course in college algebra. On the other hand it contains material that is vital to the development of anyone seriously interested in mathematics or in the application of mathematics. Thus it could be used properly for a beginning graduate course.

In the preface the author states that the book makes no pretense of being complete. However, in the opinion of the reviewer, it covers a surprising number of details in matrix theory. It covers well the usual concepts in matrix algebra and determinant theory. Vector spaces and linear transformations are developed quite thoroughly. The last two chapters deal with characteristic-equations of matrices, and bilinear, quadratic and Hermitian forms. Three appendices are added. They treat the summation and product symbols, complex numbers and the general concept of isomorphism.

The best feature of the book is the obvious care taken to introduce each topic in as natural a fashion as possible. The book is quite readable. Since the author considers the person with a need for matrix algebra as a tool, emphasis is placed on techniques for actually calculating such things as inverses of matrices and evaluating determinants. On the other hand, for the sake of the person interested in the theoretical aspects of the subject, the concepts of field and group are introduced. The chapter on linear transformations is well done and the isomorphism between the set of linear transformations on an n-dimentional vector space over a field and the set of n x n matrices with elements in the field is established very nicely.

The numbering system and notation is consistent and good throughout the book and makes for easy reference. The book is well indexed, listing symbols as well as terms. The exercises vary in difficulty and supplement well the material in the text. They are often used to point out areas of application for the topics treated.

In the opinion of this reviewer, the book could be used by the person who wants the material in the text as a tool or it could be used by the person wanting a good introduction to linear algebra. It treats both extremes well. It is a very readable book and this readability is effected by the care taken by the author ro introduce new concepts naturally rather than formally. It should be well received.

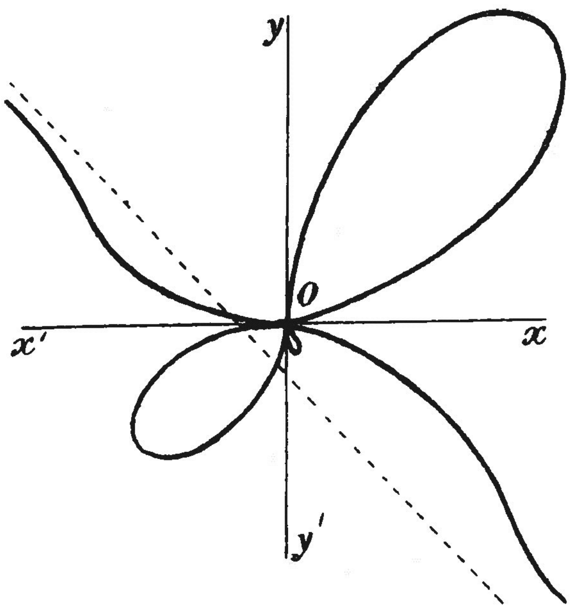
St. Louis University — John W. Riner

BOOKS RECEIVED FOR REVIEW

*V. Chew (Editor): Experimental Designs in Industry, New York, John Wiley, **1958, $6.00.**
*R. V. Churchill: Operational Mathematics, New York, **McGraw** Hill, **1958, $7.00.**
R. E. Gaskell: Engineering Mathematics, New York, Henry Holt, **1958, $7.25.**
S. E. Gass: Linear Programming: Methods and Applications, New York, McGraw-Hill, **1958, $6.50.**
W. Kaplan: Ordinary Differential Equations, Reading, Mass., **Addison-**Wesley, **1958, $8.50.**
J. L. Marks, C. R. Purdy, and L. B. Kinney: Teaching Arithmetic for Understanding, New York, **McGraw-Hill, 1959, $6.00.**
I. S. Sokolnikoff and R. M Redheffer: Mathematics of Physics and Modern Engineering, New York, **McGraw-Hill, 1958, $9.50.**
*A. E. Taylor: Introduction to Functional Analysis, New York, John Wiley, **1958, $12.50.**

*See review, this issue.

NOTE: All correspondence regarding book reviews should be addressed to FRANZ E. HOHN, **374** ALTGELD HALL, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS.



$$x^3y^2 + xy^5 - y^7 - x^7 = 0$$

OPERATIONS

Science needs you

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UNLIMITED

This section of the Journal is devoted to encouraging advanced study in mathematics and the sciences. Never has the need for advanced study been as essential as today.

Your election as members of Pi Mu Epsilon Fraternity is an indication of scientific potential. Can you pursue advanced study in your field of specialization?

To point out the need of advanced study, the self-satisfaction of scientific achievement, the rewards for advanced preparation, the assistance available for qualified students, etc., it is planned to publish editorials, prepared by our country's leading scientific institutions, to show their interest in advanced study and in you.

Through these and future editorials it is planned to show the need of America's scientific industries for more highly trained personnel and their interest in scholars with advanced training.

The following lists contributing corporations with the issue in which their editorials appeared.

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AVCO, Research and Advanced Development	Vol. 2, No. 10
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We are grateful to these scientific institutions for their editorials and their material support of this publication.

May the mind that holds the knowledge have the heart to use it well.

ARMY BALLISTIC MISSILE AGENCY

THE SPACE AGE - AN EDUCATIONAL CHALLENGE

By WERNHER VON BRAUN
Director, Development Operations Division
Redstone Arsenal, Alabama

Wernher von Braun

It is of interest to notice that of the many men who have made great contributions to the field of science a large percentage were men of wealth and position, and therefore had no monetary incentive to spur them forward in their great tasks. It is also interesting that many of the early researchers actually jeopardized their social and financial positions by pursuing their interests and presenting their theories with such vigor. In view of these observations, we are moved to look for the motivating force which drove these men to immortality.

This motivating force is not characteristic of scientists alone, nor of humans alone, but of all living creatures. It is known as curiosity. The small child is consumed by an overwhelming curiosity and thirst for knowledge, (much to the distress of some parents). As the child grows older, the curiosity gradually subsides until the pedestal of adulthood is surmounted from which vantage point we survey the world in blissful ignorance, entirely too sophisticated to trouble our minds over things we cannot understand.

Fortunately there are those in whom the spark of curiosity is not smothered, but rather grows into a bright flame to light the way for others who follow. Hence we have our Galileos, our Newtons, and our Einsteins.

Man now finds himself faced with history's greatest challenge - the conquest of Space. Never before has the nurturing of the spark of curiosity been so crucial to the welfare of mankind, for this age bears with it not only a technological challenge but also a tremendous philosophical challenge. There are forces in the world today which would thwart the very purpose of man's existence. These forces which are gathering momentum at a staggering rate must be met and subdued. This can only be accomplished by cultivating that with which all men are born - the thirst for knowledge.

The need for sound basic research is self evident. In many fields of scientific endeavor, the lack of basic research has seriously retarded progress to the extent that grave errors, which should have been easily avoided, have been made.

A sound basic research program can only be an outgrowth of a sound educational program. The goal of this program must be not only to encourage more young people to study the sciences but also to encourage advanced study in these areas, for it is only through advanced study that a genuine understanding is achieved - the kind of understanding that is so essential to creative work in the field of scientific research.

It is at this level of understanding that the beauty of science unfolds, and it is at this moment that the spark of curiosity bursts into a brilliant flame. The path of scientific research can be one of the most gratifying paths which men may choose to follow. No artist nor musician has ever revealed a stronger devotion to his work than did Albert Einstein when he said:

> "The most beautiful and most profound emotion we can experience is the sensation of the mystical. It is the sower of all true science. He to whom this emotion is a stranger, who can no longer wonder and stand rapt in awe, is as good as dead. To know that what is impenetrable to us really exists, manifesting itself as the highest wisdom and the most radiant beauty which our dull faculties can comprehend only in their most primitive forms - this knowledge, this feeling is at the center of all true religiousness."

The technological problems associated with the Space Age present an even greater challenge to scientific research than in the past. This challenge must be met by the students of today and tomorrow, and if they are to meet this challenge they must be equipped with a sound knowledge of the basic sciences such as mathematics and physics. Interest must be cultivated in these subjects early in life in order that a student will have the proper background when he enters the university.

A proper combination of educational support and guidance along with the cultivation of a high sense of values will equip both present and future generations to meet any challenge which may be presented.

BELL TELEPHONE LABORATORIES

MATHEMATICAL RESEARCH IN THE COMMUNICATIONS INDUSTRY

Henry O. Pollak

By **HENRY O. POLLAK**
Supervisory Member of Technical Staff
Murray Hill, N.J.

When I am asked for a definition of applied mathematics, I usually reply - without any expectation of ending the discussion - that applied mathematics is what I do. For our present purposes, let me broaden this to include the activities of research mathematicians on the Ph.D. level at the Bell Telephone Laboratories; there are, depending on exactly what areas of specialization you count, between 50 and 100 of us. There is, of course, much exciting work for those with less advanced training, but it will not be considered here.

What we do is, first and foremost, solid mathematical research, new mathematics which is of interest to the scientific community and publishable in an appropriate research journal. Industrial mathematical research takes no back seat in either number or content. The recent Survey of Research Potential in the Mathematical Sciences showed that industrial mathematicians on the Ph.D. level published at least as much as their academic colleagues - somewhat more, in fact, among recent graduates. Nor is there a difference in quality, as (to name just two out of many) Shannon's Information Theory and Rice's studies of noise will attest. The real distinctions, as we shall see, lie in the origin of the research, and in the nature of its successful conclusion.

Many people would, without further thought, claim that the primary function of a mathematician in industry is problem solving. This is rather far from the truth; the solution of a specific mathematical problem is only a small part of the picture. As a matter of fact, there seem to be five stages in the evolution and dispatch of a problem, and while there are no sharp boundaries between these, they do catch the mathematician in different attitudes. They are, to use a one-word title for each, recognition, formulation, solution, computation, explanation.

An industrial laboratory parades before the open mathematical mind a host of simultaneous and exciting activities. Every so often, you notice that there is in one of these activities something fundamental which is a stumbling block, which requires exploration, which is not understood. You don't as yet know what to try to do, but you

have discovered a situation that needs thought; you have recognized that there is a problem.

Well then, see if you can formulate something precise to work on. You need to make a mathematical model which is simple enough to permit of mathematical analysis, and yet sufficiently close to the real situation to be relevant. This model building is probably the most difficult, and the most valuable, task of the industrial mathematician. To quote T. C. Fry, who has done this as successfully as anyone, "It is because the mathematician is expert in analyzing relations, in distinguishing what is essential from what is superficial in the statement of these relations, and in formulating broad and meaningful problems, that he has come to be an important figure in industrial research teams." To suppress the irrelevant details, and get to the heart of the matter, that is the formulation of the problem. And do not imagine that because someone else has brought the problem area to your attention, that he can therefore formulate the problem for you. Many times the original mathematical question which someone else has asked turns out to have been the wrong question. To avoid the irrelevant and misleading, a mathematician must usually do his own model-building.

Mathematical exploration and solution is the next step. The branches of mathematics which come into play here may be quite unexpected; number theory and algebraic topology have on occasion turned out to be as important as classical and abstract analysis. The key points are two: The relevant mathematics is unpredictable, and so is the difficulty of the problem. While everyone appreciates a neat and beautiful theory when it can be found, there are times when you want the answer and have to slug to get it. The neatness of textbook problems cannot be guaranteed by the real world; and yet you must not give up too soon, for neatness is frequently a symptom of real understanding.

The computation of interesting cases is to be recommended to any mathematician, industrial or otherwise, whenever it is meaningful, for planning a numerical program and understanding its output frequently suggests further directions for research. But when you hope that your mathematics says something significant about a real situation, the computing is almost essential, and a source of genuine satisfaction when all goes well.

At this stage, you are certainly tempted to call it a day and a job well done. And yet it is not so, for a beautiful theory in either head or notebook is still of no value to anyone else. The industrial mathematician must make his knowledge available to those who can use it, and this is a non-trivial process. The typical mathematical paper, written in a typical mathematical exposition, is not enough, for this kind of presentation is too compact, and **probably** in the wrong language, for the customer. The significance of the result must be explained without assuming the specialized terminology of the mathematical field which was used to obtain it. What is needed is a leisurely exposition, full of motivation, in the customer's own terms, with detailed proofs relegated to the appendix unless it is the proof that provides the real understanding.

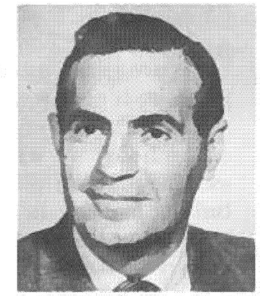
After this description of the nature of mathematical research in industry as it appears from the Bell Labs, the next step is to inquire into the character and training of the man for the job. When you look back over the description, three features stand out: Mathematics, physical world, exposition. First of all, it is clear that industry is looking for a man who knows his mathematics and is really good at it, one who has an interest in, and knows the fundamentals of, a great many fields. For you never know what you are going to get into; if your only interest is in topological groups, or even in hyperbolic partial differential equations, your path will not be easy. Thus the work is certainly mathematics on the Ph.D. level, but with this difference: The best man is a mathematical ***generalist,*** rather than a narrow specialist. And such a mathematician will have as broad an outlook as anyone in the company. Secondly, you ought to know the language, and the basic facts, of a number of other fields of science, for these are the source of many of your problems, and other scientists should be able to understand what you have to say. Much of this **you** can learn on the job, but a good background certainly helps. Thirdly, you should know how to speak and to write, and be willing to work at a clear presentation. A man likes to have his labors appreciated by those around him, and unhappy is he who cannot be understood. When you have spent months in cracking a beautiful problem, isn't it sensible to spend a few extra days striving for a beautiful write-up?

The most important question of all has been left for last. Is this sort of work a satisfying experience, and is it fun? There is no need to dwell on the question of money; industrial mathematical research is certainly well paid, but no scientist will be happy for any length of time if the intellectual rewards are not there. And what are these intangibles? You have, first of all, the thrill of discovery, of seeing pieces fall into place and form a theory. You are not without the satisfaction of teaching, for you have the opportunity to consult for other scientists and engineers, and to teach them what they want to know. Finally, every once in a while, you can actually see greater understanding, and faster technical progress, because of what you have been able to do. Each of these contributes to the satisfaction derived from industrial mathematical research.

AVCO RESEARCH AND ADVANCED DEVELOPMENT DIVISION

ON SOME MATHEMATICAL PROBLEMS ARISING IN THE MISSILE INDUSTRY

By M. S. KLAMKIN
Senior Staff Mathematician
Physics Section

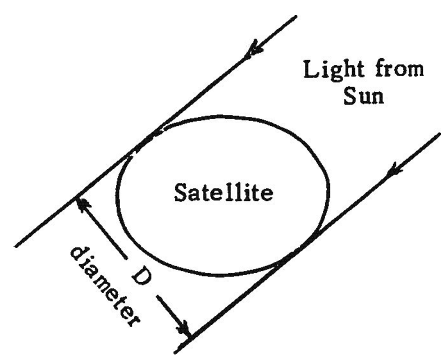


Murray S. Klamkin

In previous articles in this series of "Operations Unlimited", a general picture has been given of the need, the importance, and the application of mathematics to a broad spectrum of industry. In this article, I will take an opposite tack and give some specific problems which have arisen directly in the course of my work plus several interesting ones which have arisen as by-products. It should be kept in mind that this is only a small sample of only one person's work and consequently will probably not be truly representative. However, it should provide, in a specific manner, indications of the mathematical work being done in the missile industry.

Problem 1. The Satellite Solar Battery

A satellite of given shape is rotating in some random way in its orbit. We wish to determine the average flux of light from the sun which intercepts a specified portion of the satellite's surface. To simplify the discussion here, we will consider the two-dimensional analog. Corresponding results will hold in three dimensions. If the body is a smooth convex one and we consider that the entire length is light absorbing, then it follows from a known theorem in integral geometry that the average diameter (which is proportional to the light flux) is given by the diameter $\overline{D}$ of a circle with the same perimeter. If only half the length is light absorbing, it will follow by symmetry that, if the body is **centro-**symmetric, the average diameter will be $\overline{D}/2$. An interesting problem is the converse one; if the average diameter is



$\bar{D}/2$, must the body be *centro-symmetric.* Also, for what bodies is the average diameter $r\bar{D}$ where r is the fraction of the length which is light absorbing.

Problem 2. The Dynamic and Static Balancing of a Rigid Body

We are given an axial-symmetric body (on the outside) but inhomogeneous inside which is almost balanced **i.e.,** one of the principal centroidal axes of inertia almost coincides with the geometric axis of the body. We wish to add masses in such a way that the geometric axis becomes a principal centroidal one and that the sum of the perturbing masses is a minimum. It has been shown that[1] if the body is connected then the addition of four mass points will suffice to achieve balance and also a method is given for the location of the mass points. However, there still remains a problem if the mass point locations do not occur in the voids of the body. This can be taken care of in an empirical manner using five mass points.

Problem 3. Reducibility of Some Linear Differential Operators

The differential equation, $\{x^2D^4 - \lambda\}\, y = 0$, arose in the solution of a certain stress problem. This equation and extensions are easily solved by establishing, among others, the identities[2]

$$x^n D^{2n} \equiv \{xD^2 - (n-1)D\}^n,$$

$$x^{2n} D^n \equiv \{x^2D - (n-1)x\}^n.$$

Problem 4. An N-Dimensional Volume

Determine the volume in N-space bounded by the region

$$0 \le a_1x_1 + a_2x_2 + \cdots\cdots a_nx_n \le 1, \quad (a_r \ge 0)$$

$$b_r \ge x_r \ge c_r \qquad (r = 1, 2, \ldots n).$$

This problem[3] originated in determining the distribution function of a series parallel circuit of resistances where each resistance is distributed uniformly over given ranges. This problem has been solved by L. Shepp. A special case of this problem is to determine the probability that N points picked at random in a plane form a convex polygon. This is a generalization of No. 95 in the problem section of this journal. (To make the problem unique, one would have to specify the random method of selecting the points.)

Problem 5. On a Maximum Perturbation of a Center of Gravity

The physical problem here is to determine the largest displacement

1. J. Warga, On Certain Optimal Mass Distributions, AVCO **RAD-U-58-15.**
2. M. S. Klamkin, D. J. **Newman,** On the Reducibility of Some Linear Differential Operators, AVCO RAD-U-58-10 (to be published shortly in "The American Mathematical Monthly").
3. For a more complete discussion of this problem and other applied ones, see the Applied Problem Section in the new journal, "The SIAM Review", January 1958.

one can have, in a plane perpendicular to the axis of an axial-symmetric shell, of the C.G. of the component parts (lying inside) of the shell due to possible variations in weight of $\pm$ p% of the component parts. This problem would be a simple one but for the fact that there are in the order of fifty component parts to be accounted for. Mathematically, the problem reduces to determining a vector **E** lying on the surface of the N-dimensional hypercube $(\pm 1, \pm 1, \ldots\ldots\ldots \pm 1)$ $(N \cong 50)$ such that $\{A\cdot E\}^2 + \{B\cdot E\}^2$ is a maximum (here A and B are given vectors).

Problem 6. A Non-Linear Integral Equation

In a study of the distribution of particle sizes in a fog, the following non-linear integral equation has come up:

$$\int_0^\infty r^3 \Phi(\xi - r)\,dr = a\left\{\int_0^\infty r^2 \Phi(\xi - r)\,dr\right\}^b,$$

where a and b are independent of ξ. Extraordinarily, this non-linear equation has an elegant solution (assuming differentiability of Φ). Unfortunately, the solution did not correspond to experimental data.

Problem 7. On the Propagation of a Wave

Determine the equation of an axial-symmetric wave which propagates normally to itself with a velocity varying with distance from the axis. This leads to the P.D.E.

$$\left(\frac{\partial F}{\partial t}\right)^2 = V(r)^2\left\{1 + \left(\frac{\partial F}{\partial r}\right)^2\right\},$$

subject to the initial condition $F(r,0) = G(r)$ (the initial wave front). This is solved by finding the envelope of the complete integral of the equation. If $V(r)$ is constant, we obtain parametric equations for the class of axial-symmetric parallel surfaces.

Problem 8. Steady-State Temperature Solution in a Half-Space

Consider a half-space which is subject to a constant heat flux Q over a circle on the bounding plane and insulated elsewhere. It can be shown[4] that the steady-state solution is given by

4. **Carslaw** and Jaeger, Conduction of Heat in Solids; for the transient case in the region bounded by two parallel planes, M. S. Klamkin, AVCO RAD-2-TM-58-119.

$$T_s(r,z) = \frac{Qa}{k}\int_0^\infty e^{-\lambda z}J_0(\lambda r)J_1(\lambda a)\,d\lambda/\lambda \,.$$

The problem here is to obtain numerical results. It can be shown that

$$T_s(0,z) = \frac{Qa}{k}\left\{ z/a + \sqrt{(z/a)^2+1} \right\}^{-1},$$

$$T_s(a,0) = \frac{Qa}{k}\,\frac{2}{\pi}$$

$$T_s(a\sqrt{2},0) = \frac{Qa}{k}\,\frac{\sqrt{\pi}}{4\sqrt{2}\,\Gamma(5/4)^2}$$

For other r and z, rapidly convergent series expansions are obtained. As a side result, we obtain the two interesting sums

$$\frac{2}{\pi} = \sum_{m=0}^{\infty} \frac{(2m)!^2}{2^{4m}(2m+2)\,m!^4} = 1 - \sum_{m=0}^{\infty} \frac{(2m)!^2}{2^{4m}(2m-1)\,m!^4}$$

Problem 9. Unsteady-State Heat Transfer Due to A Source Between Two Parallel Insulated Planes

The physical problem here is to determine the temperature distribution due to a continuous source lying between two parallel insulated planes. Since the solution for a source in all-space is well known, the problem is solved very simply by the method of images. This leads to a solution which is very convenient for small time numerical calculations. However, for large times, the solution is not at all suitable since it is extremely slow in convergence. For this case, suitable asymptotic expansions are derived. The following are two interesting problems which came up in determining the asymptotic expansions:

a.

$$\lim_{r\to 0^+}\left\{-\log r + 2\sum_{n=1}^{\infty} K_0\left(\frac{n\pi r}{H}\right)\cos\frac{n\pi z}{H}\cos\frac{n\pi h}{H}\right\},$$

b. is the function $F(x) = x^{-1}-(x^2 + a^2)^{-1/2}$ completely monotone (i.e., $(-1)^n F^n(x) \geq 0$) for $a,x > 0$?

Problem 10. Evaluation of Some Integrals Arising in Corona Light Scattering

$$\int_0^\infty \left\{ J_0(x)^2 - J_1(x)^2 \right\} dx = 2/\pi \quad ,$$

$$\int_0^\infty \left\{ J_0(x)^2 - J_2(x)^2 \right\} dx = 8/3\,\pi \;.$$

These results can be generalized in the one integral

$$\int_0^\infty \left\{ J_m(x)^2 - J_n(x)^2 \right\} dx$$

(m, n rational) which can also be evaluated in closed form.

Problem 11. Solution of a Cubic with Complex Coefficients

This problem arises in stability considerations of a liquid being heated from below. The trigonometric solution for the cubic with real coefficients is extended to take care of this case.

Problem 12. An Extension of the Mean Value Theorem for Harmonic Functions

Consider the heat flow equation.

$$\frac{\partial T}{\partial t} = \alpha\left\{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right\}$$

for a circular region subject to the initial condition* $T(r, 9, 0) = 0$, and boundary condition $\oint T(R, \theta, t)\,d\theta = 0$. It is easily shown that $T(O, 9, t) = 0$ (this is also true for an N-sphere). If $T(R, \theta, t)$ is independent of t, we obtain a steady-state solution which gives the mean value theorem for harmonic functions. One simple intuitive method of solution is to consider the problem for a regular polygon where the result is easily obtained by superposition. Then take the limit as the number of sides increases without bounds. The same results also hold for the Neumann problem.

* Professor L. Nirenberg has pointed out that this can be extended to $\oint T(r,\theta,t)\,d\theta = 0$.

Problem 13. On Sets Preserving an Axis of Symmetry
Under the Group of Affine Transformations

This problem arose in an unsuccessful attempt at solving a certain heat flow problem. It is well known that conics transform into conics under affine transformations. Consequently, an axis of symmetry (actually two) is preserved. Are there any other sets having an axis of symmetry which is preserved under the group of affine transformations? The following partial result has been obtained by D. J. **Newman:**

Consider a set, symmetric to both x and y axes, passing thru $(\pm a, 0)$, $(0, \pm b)$ and, except for these latter points, lying wholly inside the ellipse $x^2/a^2 + y^2/b^2 = 1$. Then, in general, there is no axis of symmetry in the transformed set.

Problem 14. A Numerical Solution of a Boundary Value Problem

This last problem has been solved "experimentally" using an I.B.M. 704 computer. This is typical of a wide class of problems in which the mathematical theory available is not sufficient to resolve the problem.

Consider the one-dimensional heat flow equation

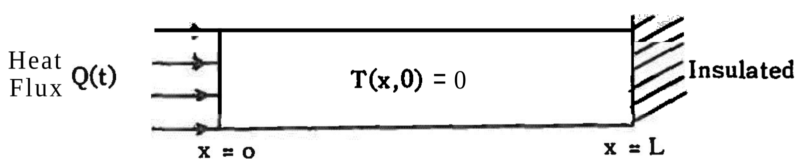
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

subject to following initial and boundary conditions:

$$\text{I.C.} \quad T(x,0) = 0$$

$$\text{B.C. 1} \quad -k \left.\frac{\partial T}{\partial x}\right]_{x=0} = Q(t)$$

$$\text{B.C. 2} \quad \left.\frac{\partial T}{\partial x}\right]_{x=L} = 0$$



The usual problem is to determine the temperature function $T(x,t)$. This can be done by using the difference equations

$$T(x,t+\Delta t) = r\left\{T(x+\Delta x,t) + T(x-\Delta x,t)\right\} + (1-2r)\,T(x,t)$$

where $r = \alpha\,\Delta t/\Delta x^2$. In order that the numerical process be stable $r < 1/2$. To increase the accuracy, we decrease the size of the increment Δx.

In the problem here, $Q(t)$ cannot (at present) be determined accurately. Consequently, we wish to determine $Q(t)$ from temperature measurements at an interior point x_1, which can be determined by means of a thermocouple. From a knowledge of $T(x_1,t)$ and $\left.\frac{\partial T}{\partial x}\right]_{x=L} = 0$, we can calculate $\left.\frac{\partial T}{\partial x}\right]_{x=x_1}$. We now have a Cauchy problem for determining $Q(t)$. Since the Cauchy data is given numerically, there is no solution in the sense of **Hadamard.** If one attempts to compute $Q(t)$ by means of the difference equation, it will turn out the process is unstable; the results getting worse as we decrease Δx. If we take Δx too large, we will not have a good approximation to the heat flow equation. Consequently, the problem here is to determine the optimum size of Δx and how good a result we can get using this Δx. For the problem actually considered, an accuracy of 6% was achieved.

DO YOU KNOW

Not All The Good Die Young

Evariste Galois	Died at the age of	21
Niels H. Abel	Died at the age of	27
Blaise Pascal	Died at the age of	39
George F. B. Riemann	Died at the age of	40
Carl G. J. Jacobi	Died at the age of	47
Johann F. C. Gauss	Died at the age of	78
Jean-Victor Poncelet	Died at the age of	79
Richard Dedekind	Died at the age of	85
Isaac Newton	Died at the age of	85
Albertus Magnus	Died at the age of	87

NEWS AND NOTICES

Edited By
Mary L. Cummings, University of Missouri

Pi Mu Epsilon members are invited to a Dutch-treat luncheon or breakfast to be held during the annual summer meetings of the M.A.A. and the Society, at the University of Utah, Salt Lake City, August 31 – September 3, 1959.

The next delegate meeting of Pi Mu Epsilon will be at Michigan State University, East Lansing, Michigan, in the summer of 1960.

Dr. Bernard Derwort, Missouri Gamma, '48, formerly of North American Aviation, Columbus, Ohio is now head of the Mathematics department at St. Thomas College, St. Paul.

Mr. Thomas K. Boehme, Oklahoma Alpha, '52, who received his Master's in mathematics from Oklahoma State University is now working on his Doctor's at California Institute of Technology.

News items should be sent to Mary L. Cummings, Mathematics Department, University of Missouri, Columbia, Missouri.

NOTICE TO INITIATES

On initiation into Pi Mu Epsilon Fraternity, you are entitled to two copies of the Journal. It is your **responsibility** to keep the business office informed of your correct address, at which delivery will be assured. When you change address, please advise the business office of the Journal.

DEPARTMENT DEVOTED TO CHAPTER ACTIVITIES

Edited by
Houston T. Karnes, Louisiana State University

EDITORS NOTE. According to Article VI, Section 3 of the Constitution: "The Secretary shall keep account of all meetings and transactions of the chapter and, before the close of the academic year, shall send to the Secretary General and to the Director General, an annual report of the chapter activities including programs, results of elections, etc." The Secretary General now suggests that an additional copy of the annual report of each chapter be sent to the editor of this department of the Pi Mu Epsilon Journal'. Besides the information listed above we are especially interested in learning what the chapters are doing by way of competitive examinations, medals, prizes and scholarships, news and notices concerning members, active and alumni. Please send reports to Chapter Activities Editor Houston T. Kames, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana. These reports will be published in the chronological order in which they are received.

REPORTS OF THE CHAPTERS

ALPHA OF MONTANA, University of Montana

The Montana Alpha Chapter began the 1957-58 year by initiating ten new members. There were six program meetings and the year was closed with the annual Spring picnic. The following papers were presented during the year:

"Report on the Michigan Meeting on Differential Equations" by Dr. Louis J. Schmittroth
"Random Walks and the Gambler's Ruin" by Mr. Howard Reinhardt
"The Stone-Weierstrass Theorem" by Dr. William R. Ballard
"Veblem-Wedderburn Planes" by Dr. Wayne Cowell
"Boolean Functions" by Dr. Frederick Young
"Coordinatization of the Plane" by Mr. Donald Sward

Due to a change in course offerings, the traditional Pi Mu Epsilon prizes, formerly awarded to the three freshmen placing highest in entrance examinations, were combined in a single prize of $50.00 given to the high school student getting the top award in the Mathematics Section of the Montana Science Fair, which is held on the University Campus each Spring. This year the prize went to Lindsey Hess, Gallatin County High School of Bozeman.

The election of 1958-59 officers was postponed until the first meeting in the Fall.

ALPHA OF KANSAS, University of Kansas

The Kansas Alpha Chapter held seven meetings during 1957-58. Six of these meetings were business meetings. The seventh meeting was the annual initiation and the election of officers for 1958-59. At this session Professor Stifel, of Zurich, Switzerland, presented a paper on: "The Calculation of π".

DELTA OF ILLINOIS, Southern Illinois University

The Illinois Delta Chapter held three regular program meetings during 1957-58. The following papers were presented:

"Mathematics in the Social Sciences" by Professor Mark
"Bendixson Theory of Non-linear Differential Equations" by Professor Wilson
"Normal Numbers" by Paul Phillips, chapter president

In April, the chapter sponsored a Mathematics Field Day, which was attended by about 600 students in the high schools of Southern Illinois. The high school students were given a competitive examination in the morning and in the afternoon attended a program prepared by chapter members. The talks, on the general topic of information, presented by the chapter members were:

"Bits" by David Phillips
"Redundancy" by Miss Janet Messerli
"Noise" by Lawrence Larson
"How Information is Limited by Noise" by Dennis Bechtlofft.

The annual award presented by the Department of Mathematics to the outstanding senior of the chapter was given to Pauline Brigham.

Winners of the annual problem contest sponsored by the chapter were:
First Place - Lawrence Larson
Second Place - Richard Allen

Officers for 1958-59 are: President, Donald Parker; Secretary, Barbara Craig; Treasurer, Joyce Perkins.

BETA OF NEW YORK, Hunter College

The New York Beta Chapter held four program meetings and two initiation dinners during the year 1957-58. Twenty new members were initiated during the year.

The following papers were presented at the program meetings:
"The Theory of Strictly Determined Games" by Barbara Saul, Annabelle Siegel, and Frances Testa under the direction of Professor Marguerite Darkow.
"The Theory of Non-Strictly Determined Games" by Lillian Scott, Natalie Horowitz, and Ruby Freed under the direction of Professor Marguerite Darkow.
"Calendar Problems" by Marianne Sinapi, Stephanie White, and Helen Kaufman under the direction of Professor A. Day Bradley.
"Calculus of Variations" by Paul Freund, Linda Rosen, and Hilda Berglas under the direction of Professor Mary K. Landers.

Officers of the Fall term of 1957 were: Director, Professor Laura Guggenbuhl; Permanent Secretary, Professor Isabel McLaughlin; President, Barbara Saul; Recording Secretary, Natalie Horowitz; Corresponding Secretary, Lillian Scott; Treasurer, Frances Testa; Librarian, Annabelle Siegel.

Officers of the Spring term of 1958 were: Director, Professor Laura Guggenbuhl; Permanent Secretary, Professor Isabel McLaughlin; President, Ruby Freed; Recording Secretary, Stephanie White; Corresponding Secretary, Helen Kaufman; Treasurer, Annabelle Siegel; Librarian. Kathryn Miranda.

ALPHA OF PENNSYLVANIA, University of Pennsylvania

The Pennsylvania Alpha Chapter held five program meetings during 1957-58. The following papers were presented:
"Computers" by Dr. Robert McNaughton of the Moore School of Electrical Engineering, University of Pennsylvania
"Quaternions" by Professor Margaret Lehr, Bryn Mawr College
"Mathematics in Biophysics" by Dr. Joseph Higgins, Johnson Foundation for Medical Physics, University of Pennsylvania
"Euler's Life and Works" by Professors Hans Rademacher and Pincus Schub. This was an Euler Memorial Meeting on the 250th anniversary of his birth.
"Partitions" by Professor Nathan J. Fine.

ALPHA OF OREGON, University of Oregon

The Oregon Alpha Chapter held four program meetings, which included business sessions and social periods, during 1957-58. In addition the annual initiation was held and concluded with a picnic at which 175 members, families and friends were in attendance. Fifty-six new members were inducted at the initiation ceremonies. The chapter presented the library with a new book for the browsing room. The following papers were presented during the program meetings:
"Mathematics in Economics" by Professor Paul Simpson, Department of Economics
"Farey Sequences" by Professor Ivan Niven
"A Problem of Information Transfer in Biology" by Professor T. L. Hill. Department of Chemistry
"Some Famous Problems in Set Theory" by Professor Herman Rubin.

ALPHA OF ALABAMA, University of Alabama.

The Alabama Alpha Chapter held four program meetings during 1957-58. In addition there was a Fall banquet and a Spring picnic. The following papers were presented during the program meetings:
"Alabama Paradox" by Dr. J. H. Hornback
"Discussion of the Properties of the Solution of the Differential Equation $y''(x) + y(x) = 0$" by Mr. Harold Sellars
"Circulants and Groups" by Dr. F. A. Lewis
"Use and Abuse of Proofs by Finite Induction" by Dr. J. L. Howell

Dwight Eddins, a freshman in Arts and Sciences, was chosen to receive a book of mathematical tables for his outstanding achievements in mathematics and his overall scholastic qualifications.

The officers for 1957-58 were: Director, Ann Richardson; Vice-Director, Kelly Grider; Secretary, Beverly Ryan; Treasurer, Rodger Comer; Social Chairman, Bunny Crawford; Scholarship Advisor, Mrs. Edith Ainsworth; Faculty Advisor, Dr. Holland Filgo.

The officers for 1958-59 are: Director, James Herod; Vice-Director, Rayburn Homer; Secretary. Samuel Dickson; Treasurer, Kenneth Harwell; Social Chairman, Joyce Smelley; Scholarship Advisor, Mr. H. C. Miller; Faculty Advisor, Dr. J. L. Howell.

ALPHA OF NEW HAMPSHIRE, University of New Hampshire

The New Hampshire Alpha Chapter held two initiations during 1957-58. At the winter initiation nine students were inducted. The ceremony was followed by a banquet with Dr. Cecil J. Schneer of the Department of Geology as the main speaker. At the Spring initiation five students were inducted.

A major project of the chapter was the complete revision of the by-laws with particular attention to the requirements for membership which the local organization has established in addition to the national requirements. To become a member of the New Hampshire Alpha Chapter one must be an upperclassman, graduate student or member of the faculty; must have completed at least three semesters of calculus or the equivalent with a 3.5 grade point average (based on the 4.0 system) in all mathematics courses; and must be a Dean's List student (must have a 3.0 general average based on the 4.0 system).

An award was presented to the student in the class of 1960 who had the highest scholastic average in mathematics during his freshman year. Elizabeth Tuttle and Verne Brown tied for this award and received duplicate prizes.

Throughout the year members of Pi Mu Epsilon conducted weekly classes for students in freshman and sophomore mathematics courses who wished to receive assistance in their studies.

The Officers for 1957-58 were: President, Mary P. Todt; Vice President. J. Douglas Cowie; Secretary, Margaret Ann Shea; and Treasurer, Richard S. Gaudette. The advisor was Dr. Robert H. Owens.

ALPHA OF MICHIGAN,, Michigan State University.

The Michigan Alpha Chapter held meetings every few weeks during 1957-58. On these occasions papers were presented by both faculty and students. A special event was a tour to observe the new electronic computer built at Michigan State University.

Other activities of the year included three initiations, a winter banquet and a spring picnic. The guest speaker for the banquet was Professor Marshall Hall of Ohio State University.

Officers for 1957-58 were President, Robert Powell; Vice-president, Pat Hertzler; Secretary, Margo Harrison; Treasurer, Sue Marzke; Faculty Advisor, Dr. R. H. Oehmke.

ALABAMA ALPHA, University of Alabama (November 26, 1958)

Max L. Allen
Joseph H. Campbell
Martha S. Duke
Mary Ann Higgs
Fred A. Hollub
Ellen L. Manner
Joseph D. Naughton
Claude C. Priest, Jt.
Thomas D. Roberts
Max E. Rosenthal
Ewell M Scott, Jr.
Dale R. Summers
Billy D. Weaver
Robert E. Yotk

ALABAMA BETA, Alabama Polytechnic Institute (December 4, 1958)

Richard Alexander
Joseph A. Cares
Ronald L. Coleman
Jack Cumbee
John T. Cutchen
George J. Dezenberg
Thomas D. Floyd
Paul J. Hayes
James E. Horn
Albert A. Howard, Jr.
Johnny M Humphrey
Edgar G. Johnson, Jr.
Paul M Julich
Robert G. Lackey
Stewart Langdon
William F. Lineberger
Joe M McGuire
John G. Meadors
Cecil W. Messer
Rochelle Morriss
Wilbur C. Mosby, Jr.
Chandler R. Murton, Jr.
James M. O'Neil
William D. Parker
Thomas F. Rogan
Izzydor Shever
Barney D. Thornton
Gary D. Thornton
Harry Weaver
Richard E. Whitt
Patsy Woodham

DISTRICT OF COLUMBIA ALPHA, Howard University (May 31, 1958)

Jocelyn M. Andrews
Edna-Marie Boiling
James W. Breedlove
Paul Brown
Anna J. Coble
Rosa L. Hill
Almeta R. Kimber
W. V. Udipi-Krishna
Ivary M. Langley
Ramsel A. Mongol
John E. Thomas
Lois J. Waters
Charles R. White

FLORIDA BETA, Florida State University (December 10, 1958)

Patricia J. Cale
Valma Y. Edwards
Gordon M Feathers
John D. Grow
Edwin L. Lindman
Dorothy A. Linley
George L. Markey
James R. Neetgaard
Stephen Peyton
Michael S. Saliba, Jr.
Richard L. Schulz
Robert E. Smith
John C. Steadman
Sandra D. Stewart
Marion F. Tinsley
William O. Williford
Barbara J. Willits

GEORGIA ALPHA, University of Georgia (November 12, 1958)

Curtis P. Bell
Walter G. Blanchard
Peter N. Henriksen
Carol R. Wallace
Florence E. Whitfield

ILLINOIS ALPHA, University of Illinois (1941)

Geraldine A. Galloway

KANSAS ALPHA, University of Kansas (December 10, 1958)

Barbara L. Blake
Charles O. Christenson
Gail A. Cotdes
Spencer E. Dickson
Hugo F. Ftanzen
Joyce E. Isaacson
Walter Kintsch
G. Lawrence Lane
Guy K. Magnuson
Harold McBeth
Lucille M. Parks
David E. Pellett
Max L. Slankard
Donald B. Small
Harley R. Stafford

KANSAS GAMMA, University of Wichita (December 15, 1958)

Richard E. Brown
Bruce W DuVall
Richard T. Feeney
William M. Gertsen
Janice E. Hart
Edward W. Johnson
Ellis R. McDaniels
Mary C. McMullen
Myron Dale Walter

MISSOURI ALPHA, University of Missouri (December 11, 1958)

Robert E. Berry
Orville L. Brill
Alfred E. Brims
Marvin Cohen
James E. Delmore, Jr.
Ronald S. Dingus
Glen A. Edwards
Larry B. Feldcamp
Thomas A. Janes, III
Jack L. Kampman
William D. Logan
Gerald T. Magee
John E. Magee
Charles R. Norris
Bill R. Premer
Roy Dean D. Reed
Lanny W Shockley
Donald F. Steinbrueck
Paul M. Toler
Charles N. Wright

MISSOURI GAMMA, St. Louis University (December 6, 1958)

Dr. Wernher Von Btaun

NEBRASKA ALPHA, University of Nebraska (November 18. 1958)

Hugo T. Alarcao
Dennis R. Bonge
Paul B. Bower
Ernest B. Cobb
Troy D. Fuchser
James O. Jirsa
Earl A. Leonhardt
Ned A. Lindsay
Loren D. Lutes
Ronald H. McKnight
Wellington R. Meier, Jr.
Jack K. NyQuist
Lynn H. Peterson
Theodore G. Reiss
Robert J. Schwabauer
Sid L. R. Snyder
Richard T. Sokol
William E. Spencer
Norman V. Stones
Harry R. Tolly
Alfred H. Wine, Jr.
Allan J. Worrest

(December 9, 1958)

Diana Yun-dee Fan
Bernard Harris
John E. Kimber

NEW HAMPSHIRE ALPHA, University of New Hampshire (December 11, 1958)

Adrienne R. Beaudoin
Charles Boghosian
Verne R. Brown
Archer G. Buck
Robert B. Hardy
Richard M. Kimball
Nelson C. Maynard
Jack C. Norman
Richard L. O'Malley
Robert E. O'Malley
Elizabeth R. Tuttle
Roland H. Therrien
Robert E. Smith
John T. Viola

NEW JERSEY BETA, Douglass College (November 14, 1958)

Jean M. Liska
Joan A. Shrout

NEW YORK ALPHA, Syracuse University (December 16, 1958)

John W. Allis
Carol Block
Laurence E. Broniwitz
Larry H. Buss
John E. Campbell
Carol A. Donovan
William F. Heagerty
Ivars Irbe
Leon S. Lasdon
John W. Lewis
Robert J. Lundegard
Joseph A. Mercurio
Richard C. Morey
Barbara A. Nelson
Susan E. Romes
Ethel M. Stevens
Albert C. Vosburg

NEW YORK GAMMA, Brooklyn College (November 25, 1958)

Harry Allen
Susan R. Balsam
Norman Bleistein
Robert Ehrlich
Lawrence Freundlich
Rochelle M. Friedlieb
Judith M Gillman
Paul S. Kitshenbaum
Linda Lerman
Bernard J. Sackaroff
Edward A. Spargo
Hanna L. Wolfson

NEW YORK ETA, University of Buffalo (December 17, 1958)

Richard W Feldmann, Jr.
Donald R. Getzin
Stephen R. Graczyk
Alburt E. Pifer
Joe H. Pifer
Elizabeth M. Schuler

NORTH CAROLINA ALPHA, Duke University (December 3, 1958)

Linda L. Blacketby
Lena M Bradley
Gail E. Foster
Gail K. Lundberg
Sarah A. Myers
Lawrence I. Peterson
Thomas M. Prather
Jack E. Rathmell
Seldon L. Stewart, III

OHIO DELTA, Miami University (October 30, 1958)

Carol J. Argus
Charlotte M. Ashdown
Virginia D. Bittel
Claudette M Cook
William G. Corkins
Paul E. Dahnke
Richard A. Grosse, Jr.
Susan L. Hamilton
Kenneth R. Hammond
Gerald D. Hartsel
David L. Hirst, Jr.
Ramsey D. Johnson
John F. Kramer
Joyce A. Longley
Allan J. MacKinnon
C. Maxine Mikels
Allen A. Montgomery
Mary J. Oring
James E. Peirce
Mary A. Pound
Beverly D. Quanstrom
hlarcina J. Quimby
Herbert F. Rowan
Mary E. Smith
Janice J. Thompson

OKLAHOMA ALPHA, University of Oklahoma (November 11, 1958)

Phillip Briggs
Gene M. Cunningham
Bob R. Evans
Janice K. Hann
Paul R. Hinrichs
Harry McLeod
Richard G. Martin
An Mihram
Marquis Murdock
James E. Palmer
Robert C. Poe
Harry Otis Sims
Hugh A. Walls
Joni S. Williams

OKLAHOMA BETA, Oklahoma State University (May 12, 1957)

Charles Bensinger
Thomas Cairns
Roland V. Decker
Charles W. Friend
Dick George
Aaron Goldman
Charles MacDonald
James M. Means
Gene Pulley
Charles A. Ranck
Albert Reynolds
Walter B. Sturek
Attila Taluy
Jack Warner
Lyman Yarborough

(March 30, 1958)

Merlin R. Ahrens
Harvey Beets
William F. Blose
Donald W. Dareing
Richard A Dupree
Earl. L. Dowty
George C. Dysinger
Glen D. Farquharson
Dale Kirmse
Alfonzo Ledezma
Mahmoud H. Sabet
Myron D. Simons
James G. Thomas
Janice Van Deventer
Thomas D. Wallace
Fred M. Welsh
Fred W. Wenninger, Jr.

PENNSYLVANIA ALPHA, University of Pennsylvania (1957-1958)

Harold Byer
William Eaton
Murray Eisenberg
Stanley Foster
Donald Goldsmith
Eugene Gruber, Jr.
Barry Halpern
Richard Josephs
Jonathan Kurland
Philip Leifer
Arthur Schless
Joel Sinsky
Diane Splaver
Barry Stein
Betel Sternthal
Joel Stiman
David Wolfe

PENNSYLVANIA BETA, Bucknell University (December 11, 1958)

Dawn M. Beatty
Oliver W. Clowe
Robert A. Crovelli
Vadim G. Drozixn
Barbara E. Dunning
Robert E. Ebersole
Norman S. Edgett
Marilyn J. Ekiss
Kenneth E. Farr
Leroy R. Hettche
Robert A. Howell
Joyce S. Mehring
Bertram Mond
Herman T. Palmer
Joan M. Piersol
Sherry L. Rhone
George M. Shannon
Patricia M. Shapoe
Janet W. Stockham
Robert A. Thompson
Joanna R. Ziegler

VIRGINIA ALPHA, University of Richmond (October 27, 1958)

Dr. H. P. Atkins
William J. Bugg, Jr.
Mary R. Burruss
Donald F. Clement
Patricia A. Crawford
Howard P. Estes
Lois M. Grasty
Edith M. Jones
Kay C. Lacy
Anne B. Loving
Mary J. McWilliams
Paula Williams

WISCONSIN ALPHA, Marquette University (May 1, 1956)

John H. Barton
Lawrence G. Deysach
Robert A. Gehrke
James M. Geyer
Thomas Krobikowski
Peter A. Mercola
Robert E. hlierendorf
Tom P. Miotke
Mary M. Mistele
Marvin J. Mundshaw
Marleen A. Pytlik
Raymond G. Schliesmann
George R. Sell
Thomas N. Tischer
John R. Van Ryzin

(October 26, 1958)

Kenneth Batinovich
Ernest E. Gloyd
Atilano Gomez
Ralph R. Hoepfner
Leo R. Kasun
Roger L. Kroes
Mildred Sue Leslie
Mrs. Cyril O'Brien
Rose Maire O'Connor
Glenn Patton
Lee G. Radosevich
Robert Rampetsreiter
Charles Russ
Timothy Schlax
Elroy Slawinski
Leon Winslow

(March 27, 1958)

Ronald Boyle
Kathryn R. Buttice
Ernest K. Lee
Donald J. Kopydlowski
Mary L. Kowalski
James E. Maher, Jr.
Norbert Reis

WISCONSIN BETA, University of Wisconsin (October 3, 1958)

Howard E. Bell
David G. Beverage
David A. Blaeuer
Lawrence O. Cannon
Douglas A. Clarke
William F. Furr
Wilbur F. Hoppe
Donald R. Jacobs
A. Alex Khoury
Nora Lazier
William B. Morgan
J. Marshall Osborn
Dean Phelps
Howard P. Rawlings
Enders A. Robinson
Joan H. Rohrer
Maria E. Sotelo de Garcia
Charles B. Stafford
D. V. Thampuran
Maynard DeWayne Thompson