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SYMMETRY GROUPS AND MOLECULAR STRUCTURE

J. S. Frame
Michigan Alpha, '44

Introduction. The theory of groups, which was used brilliantly by Abel in 1824 to prove the impossibility of expressing the roots of the general literal fifth degree equation in terms of radicals, remained for a century a theory of interest only to the pure mathematician, without so-called "practical" applications in other sciences. With the advent of the quantum theory in the 1920's, however, group theory became recognized as an important tool in the study of atomic and molecular structure. The purpose of this article is to explain at an introductory level one of the ways that the theory of symmetry groups is useful in a mathematical study of molecular structure.

1. Energy in a molecule. As a model of a molecule of \( n \) atoms, we consider a set of \( n \) particles of masses \( m_i \), each oscillating with small amplitude about its own mean position in a suitable moving coordinate system. The mean positions form a so-called equilibrium configuration, that moves rigidly under translation and rotation in space. If the position of the \( i^{th} \) particle is described by 3 cartesian displacement coordinates \( \mathbf{x}_i \), with origin at its equilibrium position \( \mathbf{P}_i \) in the moving system, the internal kinetic energy \( T \) of the system (excluding translation and rotation of the configuration), is given by the formula

\[
T = \frac{1}{2} \sum_{i=1}^{n} m_i \left( \dot{x}_{i1}^2 + \dot{x}_{i2}^2 + \dot{x}_{i3}^2 \right)
\]

where \( \dot{x}_i = \frac{d\mathbf{x}_i}{dt} \), etc.

Mathematical simplicity is achieved by introducing "mass adjusted" displacement coordinates \( \mathbf{x}_1', \mathbf{x}_2', \ldots, \mathbf{x}_3n' \), of which the \( i^{th} \) triple is

\[
(\mathbf{x}_{3i-2}', \mathbf{x}_{3i-1}', \mathbf{x}_{3i}') = (\sqrt{m_i} \mathbf{e}_1', \sqrt{m_i} \mathbf{e}_2', \sqrt{m_i} \mathbf{e}_3')
\]

In these coordinates the doubled internal kinetic energy has the simple form

\[
2T = \sum_{i=1}^{3n} \mathbf{x}_i^2
\]

Presented to the national meeting at Pennsylvania State University, August 27, 1957. Received by the editors March 23, 1958.
The internal potential energy of the system is an unknown function $V$ of these adjusted displacement coordinates $x_i$. It has its minimum for the coordinates $x_i = 0$ that describe a state when each particle is at its own equilibrium position. We assume that $2V$ can be expanded in a convergent power series in the $3n$ variables $x_i$, with coefficients $v_\alpha$, $v_{ij}$, 

$$2V = 2v_0 + \sum_{i=1}^{3n} v_i x_i + \sum_{i,j=1}^{3n} v_{ij} x_i x_j + \sum_{i,j,k=1}^{3n} v_{ijk} x_i x_j x_k + \cdots$$

(1.4)

Since the choice for 0-level of energy does not affect the study of energy changes, we may take $v_0 = 0$. Since $V$ has a minimum at equilibrium where $x_1 = 0$,

$$2V = \frac{\partial V}{\partial x_i} \bigg|_o = v_i = 0.$$ 

(1.5) $2 \frac{\partial V}{\partial x_i} \bigg|_o = v_i = 0.$

Thus the constant and linear terms disappear. Next we assume that the cubic and higher order terms in the small quantities $x_i$ can be neglected as compared with the quadratic terms, for a first approximation at least, and thus change (1.4) to

$$2V = \sum_{i,j=1}^{3n} v_{ij} x_i x_j,$$ 

(1.6) $2V = \sum_{i,j=1}^{3n} v_{ij} x_i x_j.$

Still further simplification is desired, since even for a 5-atom molecule like methane (CH$_4$) this expression (1.6) involves

$$3n(3n+1)/2 = 120$$ coefficients $a_{ij}$. We seek a new set of coordinates, $q_i$ called normal coordinates, that are linear functions of the $x_i$ and are so chosen that the kinetic and potential energies are given by

$$3n(3n+1)/2 = 120$$ coefficients $a_{ij}$. We seek a new set of coordinates, $q_i$ called normal coordinates, that are linear functions of the $x_i$ and are so chosen that the kinetic and potential energies are given by

$$2T = \sum_{i=1}^{3n} q_i^2, \quad 2V = \sum_{i=1}^{3n} \lambda_i q_i^2$$

(1.7) $2T = \sum_{i=1}^{3n} q_i^2, \quad 2V = \sum_{i=1}^{3n} \lambda_i q_i^2$

Here only $3n(15)$ coefficients $\lambda_i$ appear, of which 6 can be shown to be 0.

Lagrange's equations of motion for the coordinates $q_i$ take the form

$$\frac{d}{dt} \frac{\partial}{\partial q_i} (T - V) = \frac{\partial}{\partial q_i} (T - V), \quad \dot{q}_i = -\lambda_i q_i.$$ 

(1.8) $\frac{d}{dt} \frac{\partial}{\partial q_i} (T - V) = \frac{\partial}{\partial q_i} (T - V), \quad \dot{q}_i = -\lambda_i q_i.$

These equations imply simply harmonic oscillations with frequencies $\lambda_i^{1/2}/2\pi$,

$$q_i = c_i \cos 2\pi f_i (t - t_i) \quad , \quad \lambda_i = 4\pi^2 f_i^2$$

(1.9) $q_i = c_i \cos 2\pi f_i (t - t_i) \quad , \quad \lambda_i = 4\pi^2 f_i^2$,

which do not quite represent the true motion, but give a close approximation provided that the cubic and higher order terms in (1.4) are relatively small.

Symmetry in an assumed equilibrium configuration for the molecule implies a certain symmetry in the formulas for kinetic and potential energy. Group theory can be used to show that certain of the eigenvalues $\lambda_i$ must be equal, and thus to reduce the number of essential parameters. Group theory also assists powerfully in determining the normal coordinates $q_i$ as functions of the displacement coordinates $x_i$, and in determining which normal coordinates are associated with vibration energies observable as absorption lines in the infra-red spectrum of the molecule.

2. Symmetry groups. A symmetry operation defined on a rigid configuration of $n$ points is a mapping that replaces each point either by itself or by some other "image" point in the configuration, in such a way that the distance $P_i P_j$ between any two of the points of the configuration is the same as the distance $P'_i P'_j$ between their image points under the given mapping. The result of following a mapping $a$ by another mapping $b$ is called the product $ab$ of the two mappings. (It will be seen below that $b a$ need not be the same as $a b$.) The so-called identity mapping $e$ (or $I$ or $I$) maps each point into itself. If a mapping $a$ maps each $P_i$ into some suitable $P'_i$, the mapping that takes each $P'_i$ back to the original $P_i$ is called the inverse of $a$ and is written $a^{-1}$. We have $a^{-1} = a^{-1} a = e$. The inverse of the product mapping $a b$ is $b^{-1} a^{-1}$. For any three symmetry operations $a, b, c$, we have $(ab)c = a(bc)$. The set of all symmetry operations on a configuration is called its symmetry group.

In general, a group $G$ is defined in algebra to be a system of elements $a, b, c, ...$ connected by a law of combination (usually called multiplication or addition) such that if two of the three letters $a, b, c$ represent elements of $G$, distinct or not, the equation $a b = c$ can be solved uniquely in $G$ for the third letter; and furthermore, such that the associative law $(a o b) o c = a o (b o c)$ is valid whenever $a, b, c$ are elements of $G$. The symmetry operations on a configuration form a group under the associative product rule defined above.

The commutative law $a b = b a$ may or may not hold for two group elements. For example, consider the symmetries of a regular tetrahedral...
ron whose vertices are labeled 1, 2, 3, 4. Let \( a \) be a reflection in the plane 034 containing the center 0 and vertices 3, 4, and let \( b \) be a reflection in the plane 013 through the center 0 and vertices 1 and 3. Under \( a b \) the vertex originally at 1 moves to position 2 and then to 4, whereas under \( b a \) this vertex first stays at 1 and then moves to 2. The rotation \( a b \) takes 1 into 4, 4 into 2 and 2 into 1, leaving vertex 3 fixed, whereas the rotation \( b a \) takes 1 into 2, 2 into 4 and 4 into 1. Products of these reflections \( a \) and \( b \) satisfy

\[
(2.1) \quad a \cdot a = e, \quad b \cdot b = e, \quad (ba) = (ab) = (ae) = a \cdot a = e
\]

so that \( a \) is its own inverse, and \( b \) its own inverse, whereas \( ba \) is the inverse of \( ab \).

3. Matrix representation of groups. Since multiplication of ordinary numbers is always commutative (\( ab = ba \)), it is necessary to employ quantities more complicated than single numbers to describe a non-commutative multiplication such as we have defined for the symmetry operations of a regular tetrahedron. Suppose coordinates \( (x, x_2, x_3) \) are assigned to each given vertex \( P_i \) in the configuration, and that after the mapping \( a \) the vertex \( P_i \) moves to a position with coordinates \( (y_1, y_2, y_3) \) and after \( b \) to a position with coordinates \( (z_1, z_2, z_3) \). Then the mappings \( a, b, c \) can be described by expressing the coordinates \( x_i \) in terms of \( y_j \), \( y_j \) in terms of \( z_k \), and \( x_i \) in terms of \( z_k \), thus:

\[
(3.1) \quad a: \quad x_i = \sum_j a_{ij} y_j; \quad b: \quad y_j = \sum_k b_{jk} z_k; \quad c = ab: \quad x_i = \sum_k c_{ik} z_k
\]

By substituting the values of \( y_j \) from the second equation into the first, and comparing coefficients of \( z_k \) with the third equation we find expressions for \( c_{ik} \) in terms of \( a_{ij} \) and \( b_{jk} \):

\[
(3.2) \quad c_{ik} = \sum_j a_{ij} b_{jk} \quad \text{or} \quad C = AB
\]

We denote by \( A \) the rectangular array of coefficients \( (a_{ij}) \) in which the entry \( a_{ij} \) appears in row \( i \) and column \( j \), and call it a matrix. We denote by \( B \) the matrix \( (b_{jk}) \) with \( b_{jk} \) in row \( j \), column \( k \). We define the matrix product \( C = AB \) to be the matrix whose entry in row \( i \) column \( k \) is the sum of \( C_{ik} \) of the products \( a_{ij} b_{jk} \) of the \( j \)th entries in row \( i \) of \( A \) and column \( k \) of \( B \), summed for all \( j \). With this definition of a product, the matrices that describe the symmetry operations of a configuration with respect to a chosen coordinate system form a group, and the mapping of symmetry operations onto the corresponding matrices is a single-valued mapping that preserves products. Any such mapping is called a representation of the group.

### SYMMETRY GROUPS AND MOLECULAR STRUCTURE

For example, let us choose coordinates \( x_i \) so that originally the vertices of a given regular tetrahedron are at the points

\[
(3.3) \quad P_1: (1, -1, 1), \quad P_2: (-1, 1, 1), \quad P_3: (-1, -1, 1), \quad P_4: (1, 1, 1)
\]

Then the symmetry operations \( a, b, c \) above may be described by the coordinate transformations

\[
(3.4) \quad a: \quad x_1 = y_2, \quad y_1 = -z_3, \quad x_1 = z_2
\]

\[
(3.5) \quad a: \quad x_1 = (010) y_1, \quad y_1 = (001) z_1, \quad ab: \quad x_2 = (010) z_2
\]

or by the equivalent matrix equations

\[
(3.6) \quad X = AY, \quad Y = BZ, \quad X = C \quad \text{where} \quad C = AB
\]

Let \( d \) be the symmetry operation that permutes the four vertices of the tetrahedron cyclically as follows: \( P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \rightarrow P_1 \). Then the corresponding matrix \( D \) and its powers \( D^2, D^3, D^4 \) are

\[
(3.8) \quad D = \begin{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix}
\]

The matrix \( D^4 \) leaves each point fixed and is known as the unit matrix, or identity matrix.

By taking all possible products of the two matrices \( A \) and \( D \) (allowing repetitions) we obtain 24 distinct matrices \((\text{including} \quad B = AD^2, \quad AD)\) that describe all the 24 symmetries of the regular tetrahedron.

These matrices all into five classes of similar matrices that are said to be conjugate in the symmetry group.

### Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Type</th>
<th>No. Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Identity</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>180° rotations</td>
<td>3</td>
</tr>
<tr>
<td>III</td>
<td>120° rotations</td>
<td>8</td>
</tr>
<tr>
<td>IV</td>
<td>90° rotations</td>
<td>6</td>
</tr>
<tr>
<td>V</td>
<td>Plane reflections</td>
<td>6</td>
</tr>
</tbody>
</table>

These matrices describe the symmetry operations of a configuration with respect to a chosen coordinate system and the mapping of symmetry operations onto the corresponding matrices is a single-valued mapping that preserves products. Any such mapping is called a representation of the group.
In general, two matrices A and B are called similar if a matrix S can be found such that $B = S^{-1}AS$, and they are called conjugate in a group G containing both, if such an S can be found in G. For the matrices A and B in class V of (3.9) defined by (3.5) and (3.6) the matrix $ABA$ serves as $S$, since $B = BABABA = S^{-1}AS$.

The trace of a matrix A is the sum of its principal diagonal entries $a_{ii}$. If $B = S^{-1}AS = TS$, then $T = S^{-1}A$ or $A = ST$, and it is found that the traces of A and B are equal since

$$\text{tr} A = \sum_i a_{ii} = \sum_{ij} s_{ij} r_{ji} = \sum_{ji} r_{ji} s_{ij} = \sum_j b_{jj} = \text{tr} B.$$  

Thus two similar (or conjugate) matrices have the same trace. For any given representation of a group by matrices with real or complex entries we define the character to be the class function whose value in any class is the trace of a matrix in that class. Or we can think of the character as a vector whose components are these traces. For the representation by the matrices listed in (3.9) the character is seen from the matrices $I, D^2, AB, D, A$ in (3.5) and (3.8) to be

$$\chi^{(4)} = (3, -1, 0, -1, 1)$$

We call this representation $R_4$ and its character $\chi^{(4)}$.

A different choice of coordinates for the vertices $P_i$ in (3.3) would have yielded a representation similar to $R_4$ with the same character (3.11). Amazingly enough it can be shown that any two representations with the same character are similar to each other in this way. However, there exist other representations of our group that are not similar to $R_4$. Four of these are obtained by mapping a and d both on the number 1, or both on -1, or respectively onto the negatives of the matrices A and D, or onto A and the matrix $D^*$ that is obtained from D by changing its -1 entries to +1. Characters of these representations are as follows.

3.12) $\chi^{(1)} = (1, 1, 1, 1). R_1$ maps a $\rightarrow$ 1, d $\rightarrow$ 1.

3.13) $\chi^{(2)} = (1, 1, -1, -1). R_2$ maps a $\rightarrow$ -1, d $\rightarrow$ -1 (but ad $\rightarrow$ +1).

3.14) $\chi^{(3)} = (3, -1, 0, -1). R_3$ maps a $\rightarrow$ -A, d $\rightarrow$ -D.

3.15) $\chi'''' = (3, 3, 0, 1, 1) = \chi^{(1)} + \chi^{(3)}$. $R''''$ maps a $\rightarrow$ A, d $\rightarrow$ $D^*$.

By suitable choice of U it may be possible to partition the new variables $q_i$ into two or more subsets each transformed within itself. The representation $s_k \rightarrow S_k$ is then called reducible, and $U^{-1}S_kU$ which have the same trace for each choice of $S_k$.

A representation which cannot be so reduced is called irreducible, and its character is called an irreducible character. The characters (3.11) (3.12) (3.13) (3.14) and (3.18) are all irreducible.

Many apparently different groups of matrices can be used to describe a given symmetry group. Between any two, however, is a one-to-one correspondence of elements that preserves multiplication. For instance, if the $k^{th}$ symmetry operation $S_k$ is described in terms of linear transformations on the 3n mass adjusted coordinates $x_i$ in (1.2) shown as a column matrix by the $3n \times 3n$ matrix $S_k$, and if these coordinates are expressed in terms of some new normal coordinates $q_i$ with corresponding column Q, by the matrix relation $X = UQ$, then the transformations

$$X = UQ \rightarrow S_k U Q = S_k X$$

are essentially the same, and are represented by similar matrices $S_k$ and $U^{-1}S_kU$ which have the same trace for each choice of $S_k$.

In the lower right corners of $U^{-1}S_k U$ we find the 1-dimensional identity matrix, and in the upper left corners the generators of a two dimensional representation which is structurally the same as the group of symmetries of an equilateral triangle. Its character is

$$\chi^{(3)} = (2, 2, -1, 0, 0)$$

A representation which cannot be so reduced is called irreducible, and its character is called an irreducible character. The characters (3.11) (3.12) (3.13) (3.14) and (3.18) are all irreducible.
For any finite group the number of distinct irreducible characters (over the complex number field) is equal to the number of classes of conjugates, and these characters are rows of a square matrix whose columns are orthogonal.

In our example the character table is

<table>
<thead>
<tr>
<th>Class:</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi^{(1)})</td>
<td>1 1</td>
<td>1 1</td>
<td>1 1</td>
<td>1 1</td>
<td></td>
</tr>
<tr>
<td>(\chi^{(2)})</td>
<td>2 2</td>
<td>-1 -1</td>
<td>0 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi^{(3)})</td>
<td>3 -1</td>
<td>0 -1</td>
<td>1 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi^{(4)})</td>
<td>3 -1</td>
<td>0 -1</td>
<td>1 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Square sum 24 8 3 4 4  
No. of els: \(g_i\) 1 3 8 6 6

The sum of squared absolute values in the \(j^{th}\) column is equal to the group order \(g\) (24) divided by the number of elements \(g_i\) in the class \(C\). Furthermore it can be shown that if \(\chi_j\) and \(\chi_j^{(1)}\) denote the values for the \(g\) elements in class \(C\) of an arbitrary character \(\chi\), and of the irreducible character \(\chi^{(1)}\) then

\[
(3.20) \ \chi = \sum_j m_j \chi_j^{(1)} \quad \text{where} \quad m_j = \sum_j g_j \chi_j \bar{\chi}_j^{(1)} / g
\]

In particular, if

(3.21) \(\sum_j g_j \chi_j \bar{\chi}_j = g \quad \text{and} \quad \chi_1 > 0\)

the character \(\chi\) is irreducible. The general problem of finding all the irreducible representations of a given finite group is far from easy, and involves much more technique than can be presented here.

4. Applications to molecular structure. For a given molecule each of several possible symmetrical equilibrium configurations can be considered in turn. For each model the character of the symmetry group is first calculated in terms of the mass adjusted displacement coordinates. Then this character is decomposed into its component irreducible characters. The degree (value) for unit element of each irreducible character contained in it is important as the multiplicity (or degeneracy) of an eigenvalue \(\chi_k\) of the potential energy matrix in (1.6). This follows from the facts that the same change of coordinates \(U\) that reduces the symmetry group to a direct sum of irreducible components, can be chosen that will diagonalize the potential energy matrix, and that only a scalar multiple of the unit matrix is commutative with all matrices of an irreducible group. Thus the required normal coordinates are obtained by discovering the coordinate change that reduces the symmetry group.

The tetrahedral group will again be used as an illustration. All 15 coordinates for a methane \(\text{C}_4\) molecule with hydrogen atoms near the vertices and a carbon atom near the center of an assumed tetrahedral equilibrium will be left fixed by the identity operation, so \(\chi_1^{(15)} = 15\). For a 180° rotation in class II that swaps pairs of hydrogen atoms, only the carbon atom is on the axis and its coordinates contribute \(1 + 2 \cos 180° = -1\) to the trace, so \(\chi_2 = -1\). For the 120° rotations of class III of (3.9) the two atoms on the axis each contribute \(1 + 2 \cos 120° = 0\) to the trace, so \(\chi_3 = 0\). For the class IV type of 90° rotation combined with a reflection in a plane perpendicular to the axis of rotation, the central carbon atom contributes \(-1 + 2 \cos 90° = -1\), so \(\chi_4 = -1\). For a plane reflection in class V, the three atoms in the plane each contribute \(-1 + 2 \cos 0° = 1\) so \(\chi_5 = 3\). Hence

(4.1) \(\chi = (15, -1, 0, -1, 3)\)

Each character value is then divided by the square sum at the bottom of table (3.19) to obtain the weighted vector

(4.2) \(\chi_i g_i / g = (5/8, -1/8, 0, -1/4, 3/4)\)

Taking the scalar products of this weighted vector with each irreducible character vector in turn we obtain the multiplicities

(4.3) \(m_1 = 5/8 - 1/8 - 1/4 + 3/4 = 1, \ m_2 = 0, \ m_3 = 10/8 - 2/8 = 1, \ m_4 = 15/8 + 1/8 + 1/4 + 3/4 = 3, \ m_5 = 1, \)

Thus the normal coordinates divide into sets that are transformed as follows:

(4.4) 1 by \(R_1\), 1 pair by \(R_3\), 3 triples by \(R_4\), 1 triple by \(R_5\).

One triple of normal coordinates that transform by \(R_4\) contains the three coordinates of the centroid. For them equation (1.8) becomes \(\ddot{q}_1 = 0\), for uniform rectilinear motion. The corresponding eigenvalues \(\chi_1\) are \(0\). The representation \(R_4\) is called the translation representation, and the other two triples that transform by it undergo oscillations that can be excited by infra red radiation, whose eigenfrequencies show up as strong absorption lines in the infra-red spectrum. That two such strong lines should appear is determined from the simple calculation by group characters described above. A different assumption about the equilib-
rium configuration would lead to different multiplicities in (4.3).

Another of these coordinate triples that transforms by \( R_4 \) and is infra-red active, describes the oscillation of the carbon centroid against the centroid of the four hydrogens. The third triple belonging to this representation \( R_4 \), describes an oscillation within the hydrogen configuration itself, measured by changing differences in lengths of opposite edges of the tetrahedron. The triple belonging to \( R \) describes the uniform rotation of the principal axes of inertia and has eigenvalues 0. Three coordinates measure changes in the sums of lengths of opposite sides of the tetrahedron and undergo a reducible representation like (3.15). The sum of these coordinates belongs to \( R_1 \) and describes a breather motion that cannot be activated by infra-red bombardment. The other two dimensional component \( R_3 \) is also inactive in the infra red, but the corresponding energies can be observed as energy differences in the Raman effect.

If the coefficients in the irreducible representations of any symmetry group are known, they may be used to determine the transformation \( U \) that will reduce the symmetry group and define the normal coordinates \( q \) that simplify the expression for potential energy. Selection rules to determine which coordinates belong to infra-red active or Raman active frequencies depend on the interrelations among irreducible representations of the symmetry groups.

Finally, however, one must again look at the higher order terms neglected in (1.6) and the interaction term between rotation and inner vibrations, and replace the classical equations of motion by corresponding equations of quantum mechanics to match the observed spectra with patterns predictable from the assumed equilibrium configuration. Remarkable results have been achieved from this approach.

Michigan State University

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NATURAL BOUNDARY OF A SERIES ASSOCIATED WITH THE LAMBERT SERIES

George R. Kuhn
Missouri Gamma, '55

Introduction. In 1913 Knopp (1)\(^1\) gave a proof of the existence of a natural boundary of the Lambert series as well as of the generalized Lambert series. Since that time, methods similar to that used by Knopp have been used successfully (3, 4). We shall study in this paper the convergence and especially the natural boundary of a series associated with the Lambert series.

The series considered in this paper is the product of the derivative of the Lambert series and independent variable \( z \); that is,

\[
K(z) = z \sum_{n=1}^{\infty} b_n z^n / (1-z^n)^2.
\]

The first of the three parts of this paper will establish the region of convergence of the \( K(z) \) series while the remaining two sections will prove the existence of its natural boundary when \( b_n = 1/n \) and when \( b_n \) is more broadly defined.

1. Convergence of \( K(z) \). In this section we shall discuss the convergence and uniform convergence of \( K(z) \).

Theorem 1. a). If \( \sum b_n \) is convergent, then the \( K(z) \) series converges for all \( |z| \neq 1 \).

b). If \( \sum b_n \) is not convergent, then the \( K(z) \) series converges with power series \( \sum b_n z^n \).

Proof of (a). Since \( \sum b_n \) is convergent, it follows that \( \sum b_n z^n \) and \( \sum n b_n z^n \) converge when \( |z| < 1 \). When \( |z| < 1 \), then for a sufficiently large \( m \)

\[
|1 - z^m| > 1/2 > 0.
\]

Hence

\[
\left| \sum_{n=m}^{\infty} \frac{n b_n z^n}{(1-z^n)^2} \right| \leq 2^2 \sum_{n=m}^{\infty} |n b_n z^n|.
\]

\(^1\)The numbers appearing in ( ) refer to reference given in bibliography! at end of paper.

\(^2\) \( \sum \) will indicate the sum from \( n \) or \( v \) to \( n \) or \( v \) depending on the numbering index unless otherwise indicated.
For \(|z| > 1\), we have

\[
\sum \frac{nb_n z^n}{(1-z^n)^2} = \sum \frac{nb_n (1/z)^n}{(1 - (1/z)^n)^2}
\]  

(1.1)

and as \(1/|z| < 1\), the right side of (1.1) is reduced to a form for which the preceding discussion applies and hence \(K(z)\) converges for all \(|z| \neq 1\).

Proof of (b). In order to establish this theorem the du Bois-Reymond Theorem \(^1\) will be utilized. The series \(K(z)\) may be written in the form

\[
\sum nb_n z^n \cdot \frac{1}{(1-z^n)^2}
\]

which is of the form \(\sum a_n c_n\) and the du Bois-Reymond Theorem is applicable with \(a_n = nb_n z^n\) and \(c_n = 1/(1-z^n)^2\).

Consider

\[
\sum_{n=m}^{\infty} \frac{1}{(1-z^n)^2} - \frac{1}{(1-z^{n+1})^2}
\]

for \(m \geq 1\). For \(|z| < 1\) and for sufficiently large \(m\), we have \(|1 - z^m| > 1/2\).

\[|1 - z| \sum_{n=m}^{\infty} |z^n| \]

Hence, \(K(z)\) converges with \(\sum nb_n z^n\) and thus with \(\sum b_n z^n\).

\(^1\)The du Bois-Reymond Theorem can be stated as follows: \(\sum a_n c_n\) is convergent if \(\sum \left[ c_n - c_{n-1} \right]\) is absolutely convergent and \(\sum a_n\) is conditionally convergent (2, p. 315).

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Corollary. The \(K(z)\) series with hypotheses of either (a) or (b) of the above theorem is uniformly convergent in the regions which lie within the circles of convergence of the series.

In the proof of Theorem 1, Part (a), it was noted that \(K(z)\) is absolutely and uniformly convergent for \(|z| < \rho < 1\) by the Weierstrass M-test.

When \(|z| > 1\), and \(\sum b_n\) is convergent, using (1.1) it is seen that \(K(z)\) is uniformly convergent when \(|z| \geq 1/\rho > 1\). On the other hand, when \(\sum b_n\) is not convergent, \(K(z)\) is uniformly convergent only if \(|z| < \rho < r < 1\), where \(r\) is the radius of convergence of \(\sum b_n z^n\).

2. Natural Boundary Special Case. Let us consider the natural boundary for the \(K(z)\) series when \(b_n = 1/n\). Hence the series becomes

\[K(z) = \sum z^n/(1-z^n)^2\]

Since this is derived from the Lambert Series which has the unit circle as a natural boundary (4) we would like the \(K(z)\) series to also have the unit circle as a natural boundary. This is true.

Theorem 2. If in the \(K(z)\) series, \(b_n = 1/n\) and for the positive integers \(p\) and a relatively prime \(q\); \(z_0 = e^{2\pi ip/q}\), then for radial approach

\[
\lim_{z \to z_0} (1 - z/z_0)^2 \sum z^n/(1-z^n)^2 = \sum_{v=1}^{\infty} \frac{1}{(qv)^2}
\]

and the function \(K(z)\) has the unit circle as a natural boundary.

Showing the existence of the limit proves that the points \(z_0\) which are dense in themselves, are non-removable singularities. Any region completely enclosed by a curve composed of a set of singularities dense in themselves has this curve as a natural boundary.

Let the considered points within the unit circle be designated by \(z = \rho e^{2\pi ip/q}\), where \(\rho < 1\), whereas points on the unit circle will be

\[z_0 = e^{2\pi ip/q}\]

where \(p\) and \(q\) are relatively prime positive integers. Now divide the series into two sums such that \(\sum_1\) is that part of \(K(z)\) for which \(n \equiv 0 \pmod{q}\); while \(\sum_2\) is that part for which \(n \not\equiv 0 \pmod{q}\).

Consider first the
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\[ \sum_{v=m+1}^{m+h} (r_{v-1} - r_v) y_v = \]
\[ \sum_{v=m+1}^{m+h} r_v (y_v - y_{v+1}) + r_m y_{m+1} - r_{m+h} y_{m+h+1}. \]  

Each term of (2.5) must be examined for boundedness.

Since \( \sum_{\lambda=m+1}^{\infty} C_{\lambda} r_{\lambda} \)

is convergent, it is true that for a given \( \epsilon \), an \( m \) can be found such that

\[ |r_m| < \epsilon. \]

Showing that \( y_v \) and especially, \((y_v - y_{v+1})\) are bounded is a more complicated task. It will now be shown that \( \{ y_v \}_{v=1}^{\infty} \) is bounded and monotone in \( 0 < y < 1 \).

Let us consider \( (y_v)^{1/h} - (y_{v+1})^{1/h} \), where \( 0 < y < 1 \). This difference may be shown to be

\[ \frac{(1-y^{1/h})(1-y^{1/h})^{2}}{(1-y^{1/h})(1-y^{1/h}+1)} \left\{ v - \left[ y^{1/h} t y + y^{3/2} + ... (y^{1/h}) + vy^{1+h} \right] \right\}. \]

Let \( f(y^{1/h}) \) be the factor within the braces of (2.6). Hence, if \( y = z^2 \), we have

\[ f(z) = v - \left[ z + z^2 + z^3 + ... + z^{1/h} \right] + vz^{2+h} + 1. \]

At \( z = 0, f(z) = v; \) while at \( z = 1, f(z) = 0. \)

Hence \( f(z) \) passes through the points \((0,v)\) and \((1,0)\). The derivative of \( f(z) \) is \( f'(z) = -(1 + 2z + 3z^2 + ... + 2z^{1/h}) \) for \( O < z < 1 \) is negative. This indicates that \( f(z) \) is monotone decreasing and positive for \( 0 < z < 1 \). Since the fraction in (2.6) is positive for \( O < y < 1 \), the sign of (2.6) is the same as that of \( f(z) \). Consequently,

\[ (y_v)^{1/h} - (y_{v+1})^{1/h} > O, or y_v > y_{v+1} \]

when \( 0 < y < 1 \).

Furthermore, since at \( y = 0, y_v = 0 \) and at \( y = 1, y_v = 1, \) \( \{ y_v \}_{v=1}^{\infty} \) is bounded and is monotone non-increasing in the interval \( 0 < y < 1 \).

Hence \( \{ y_v \}_{v=1}^{\infty} \) is positive for \( 0 < y < 1 \) and zero when \( y = 0, \) or 1.

The absolute value of (2.5) now becomes

\[ \left| \sum_{v=m+1}^{m+h} C_v y_v \right| < \left| \sum_{v=m+1}^{m+h} \right| r_v \left| y_v - y_{v+1} \right| + |r_m y_{m+1} + |r_{m+h} y_{m+h+1} | \]

or

\[ \text{As a matter of fact, the interval could be chosen as } 0 < a \leq y \leq 1. \]
from which it follows that \( \sum c_v y_v \) is uniformly convergent in \( 0 \leq y \leq 1 \).

Taking the limit of (2.2), it follows that

\[
\lim_{y \to 1} \sum_{v=m+1}^{m+h} \frac{v^2 y^v}{(qv)^2 (1 + y + y^2 + \ldots + y^{qv-1})^2} = \sum_{v=m+1}^{m+h} \frac{1}{(qv)^2} \quad (2.7)
\]

which is also the limit of (2.1).

To complete the proof of Theorem 2, we must show

\[
\lim_{z \to z_0} \left( 1 - \frac{z}{z_0} \right)^2 \sum_{q=1}^{q-1} z^{qv+s} / (1 - z^{qv+s})^2 = 0. \quad (2.8)
\]

Firstly, let us show that for all \( n \) not divisible by \( q \) that

\[ |1 - z^n| \geq t, \quad (2.9) \]

where \( t = 1 \) if \( q = 2 \) and \( t = \sin 2 \pi/q \) if \( q > 2 \). If \( q = 2 \) and \( 0 \leq q \leq 1 \),

\[ |1 - z^n| = \left| 1 - q^n \right| = 1 + q^n \geq 1. \]

If \( q > 2 \), then \( 2 \pi q/\pi q = 2/\pi q \) is an integer and \( 1 \leq q \leq q - 1 \).

Thus, \( |1 - z^n| = \left| 1 - q^n \right| = 1 + q^n \) which will be smallest when \( n \) is one or \( q - 1 \). With either of these values, it is easy to show that \( |1 - z^n| \geq \sin 2 \pi q/\pi q. \)

With the substitution \( y = q^n \), the limit in (2.8) becomes

\[
\lim_{y \to 1} \sum_{v=m+1}^{m+h} \frac{v^2 y^v}{(qv)^2 (1 + y + y^2 + \ldots + y^{qv-1})^2} \quad (2.10)
\]

Here again, the limit may be taken inside the summation symbol if the series is uniformly convergent for \( 0 \leq y \leq 1 \). Uniform convergence will now be established for \( 0 \leq y \leq 1 \).

Let

\[
\sum_{v=m+1}^{m+h} \frac{v^2 y^v}{(qv)^2 (1 + y + y^2 + \ldots + y^{qv-1})^2} = \sum_{v=m+1}^{m+h} F_v = F_m(y),
\]

which is the series (2.2) and is uniformly convergent for \( 0 \leq y \leq 1 \). Hence, for \( O \leq y \leq 1 \) and for \( m \) large enough \( |F_m(y)| < e \).

Also let

\[
\frac{y^n(1-y^n)^2}{(1-yn^2)(1-yn^3)^2} = y_v.
\]

With the aid of (2.9) it follows that \( |y_v| < 1/t^2 \) for \( O \leq y \leq 1 \). Consequently, from the series in (2.10), it is apparent that for \( h > 1 \)

\[
|\sum_{v=m+1}^{m+h} f_v y_v| = |\sum_{v=m+1}^{m+h} y_v (F_{v-1} - F_v)|
\]

\[
\leq \sum_{v=m+1}^{m+h} |F_v| \cdot \left| y_v - y_{v+1} \right| + \left| F_m y_{m+1} \right| + \left| F_{m+h} y_{m+h+1} \right|
\]

\[
\leq \frac{e}{t^2} \sum_{v=m+1}^{m+h} \left| -2y^2(1 + y^n)^2 (1-y^n)^2 \right| + \frac{2e}{t^2}
\]

\[ \leq 10e/t^4 + 2e/t^2 \]

for sufficiently large \( m \).

Consequently, the series in (2.10) is uniformly convergent for \( 0 \leq y \leq 1 \) and the limit of (2.10) may be moved within the summation symbol yielding the result of (2.8).

The combined facts as stated in (2.7) and (2.8) prove Theorem 2.

3. Natural Boundary General Case. We will now consider the \( K(z) \) series in which the coefficients are less restrictive.

Theorem 3. If the coefficients of \( K(z) = \sum b_n z^n / (1-z^n)^2 \) are so restricted that for certain integers \( q \), the series

\[
\sum_{q=0}^{q-1} b_{qv+s} / (qv+s) \quad s = 0, 1, 2, \ldots, q-1
\]

is convergent and if for such a \( q \) and a relatively prime \( p \), we set

\[
z_0 = e^{i2\pi p/q}
\]

then for radial approach

\[
\lim_{z \to z_0} \sum_{q=0}^{q-1} b_n z^n / (1-z^n)^2 = \sum b_q / (qv)
\]

Further, the \( K(z) \) series has the unit circle as a natural boundary, if \( \sum b_q / (qv) \notin O \) for a countable set \( q \).

The proof of this theorem is similar to that of Theorem 3 and will not be given. It should be noted, however, that Theorem 2 is a corollary to Theorem 3 since for \( b_n = 1/n \), the conditions of Theorem 3 become identical to the conditions of Theorem 2; that is

\[
\sum_{q=0}^{q-1} b_{qv+s} / (qv+s) = \sum 1/(qv+s)^2
\]

which converges for \( s = 0, 1, \ldots, q-1 \). Similarly, the final limit

\[
\sum b_q / (qv)
\]

equals \( \sum 1/(qv)^2 \) of Theorem 2. Thus, Theorem 2 is but a special case of Theorem 3.
107. Proposed by Michael J. Pascual, Burbank, California.
Professor Pushover being a sympathetic soul wishes to make it as easy as possible for his class on an analytic geometry exam, so he decides to set up second degree equations with integral coefficients such that the rotation transformations necessary to eliminate the cross-product term involve only rational numbers. What set of formulas could the kind professor use?

108. Proposed by Daniel Block, Yeshiva University.
Evaluate the $n^{th}$ order determinant $D_n$ where

\[
D_n = \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 \\
1 & -3 & 1 & \cdots & 0 \\
0 & 3 & -3 & 1 & 0 \\
0 & -1 & 3 & -3 & 1 & 0 \\
0 & 0 & -1 & 3 & -3 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -13 & -3 \\
\end{vmatrix}
\]

109. Proposed by Leo Moser, University of Alberta.
Fifteen professors attend regular committee meetings. At the end of each meeting every professor leaves with the hat brought by the
man he most admires. Assume no professor is more popular than any other and that their relative admiration for one another does not change. After the first meeting, to which everyone brings his own hat, they agree to disband their committee as soon as everyone regains his own hat. They find that after 100 meetings they are still going strong. When will they disband?

110.* Proposed by Lawrence Shepp, Princeton University.
If \( P(z) \) is a polynomial of degree \( n \), then \( \frac{P(z)}{z^{n+1}} \) is real and positive for some \( z \) on the unit circle \( |z| = 1 \).

It is conjectured that at most \( N-2 \) super-queens can be placed on an \( N \times N \) (\( N \geq 2 \)) chessboard so that none can take each other. A super-queen can move like an ordinary queen or a knight.

Late Solutions
97. George E. Andrews, Rockne F. Lambert

*No solution sent in by the proposer.

\[ yx^2(y - x) - ay^3 - byx^2 + c^2x^2 = 0 \]


1. With the appearance of Anderson's book a real gap has been filled, since previously there was no modem textbook on multivariate normal analysis available. The book has a clear purpose and achieves this purpose very well. It is primarily intended as a textbook for the graduate student in mathematical statistics, who has a fair background in mathematics and statistics in general, but not necessarily any knowledge of multivariate analysis. There is even a review of the basic notions in continuous joint distributions, and a review of the necessary matrix algebra (in an appendix), so that the book is self-contained to a high degree. The book is well-organized, excellently written, contains many practical illustrations, and includes a host of interesting problems. This makes it an excellent textbook for the student of statistics, but there is no doubt that it will also be very useful as a reference work for the user of multivariate normal analysis.

The major topics covered are: the multivariate normal distribution, estimation and tests concerning the mean and covariance matrix of one population, correlation coefficients, classification of observations, linear hypotheses, independence of sets of variates, comparison between several populations, principal components, canonical correlations, and the distribution in the null case of the roots of a famous determinantial equation. There are, of course, many special topics not covered in the book, and of some of these a short account is given in the last chapter.

With so much to praise, it is only fair to point out some weakness. (a) To this reviewer's taste, too much emphasis is placed on the method of maximum likelihood. In several cases the principle of invariance would have resulted in a more elegant theory. As it is, in many cases little use has been made of invariance, and only very few problems touch on it. (b) It is unfortunate that the author does not mention a general theorem concerning the completeness of a family of exponential densities, contained in the 1955 Sankhya article of Lehmann and Scheffe. From this theorem the best unbiasedness of some of the usual estimates follows at once. Only in chapter 3 a special case of this theorem—the completeness of the sample mean vector for the population mean vector of the population covariance matrix—is known—is delegated to a problem. (c) Chapters 4 and 5 should perhaps have been switched around, since chapter 4 seems to be the harder of the two. (d) In certain respects the book is not quite up to date. For instance, in chapter 13 the important methods introduced by A. T. James (Ann. Math. Stat. 1954), using invariant measures and exterior differential forms, are not even mentioned.

The book has been printed with considerable care. The print is excellent and there are relatively few printing errors. Taking everything into account, the book can be warmly recommended to everyone learning, using, or teaching multivariate normal analysis.

2. Roy's book is essentially a rehash of papers by the author, and by the author and co-workers, which appeared in the Annals of Mathematical Statistics and in Biometrika. These papers deal with the author's heuristic method for test and confidence interval construction, and with the extension of multivariate ideas into multinomial situations. One soon becomes aware of the fact that the greater part of these papers have been taken over bodily into the book, and after that the book has been filled up with a little more discussion, with some classical material (chapters 3, 4, and part of 12) and with a set of appendices which occupy about one third of the book. It is inevitable that putting a book together in this way will lead to some incoherence and inconsistencies. At the top of p. 121, for instance, the reader is referred to an earlier paper by the author instead of being referred to chapter 2. A peculiar inconsistency appears in chapter 15. After being instructed in section 15.1 that 'j' stands for 'variate', and 'i' for 'classification', in subsection 15.2.1 both 'i' and 'j' are 'variates', and in subsection 15.2.2 'j' is 'classification' and 'i' is 'variate'. Subsequent subsections are similarly inconsistent.

One wonders what the aim of Roy's book is. It is clearly not directed toward the beginning student, but if it is intended for the more advanced worker in the field, then it is not clear why the classical material has been included at all. A reference to T. W. Anderson's book would have been appropriate. It is regrettable that apparently no attempt has been made to prevent overlapping, and that the bookshelves of Anderson and Roy, which is the stranger since both were published by Wiley.

The printing was done in Calcutta by the printers of Sankhya. The print is acceptable but far from excellent, and printing errors are numerous. Last and least, the book has the typical awkward Sankhya size, which won't fit on ordinary bookshelves. In all, this reviewer sees little justification for Roy's book, since it is unsuitable for students, and the more advanced worker will feel happier reading the original papers.

University of Illinois

Robert A. Wijsman


This is a collection of nine expository papers describing experimental designs, many of recent origin, particularly suited for industrial research and describing the experiences of industrial research workers with them. Most well known experimental designs were first used in agricultural research. However, industrial experimentation differs from agricultural experimentation in important ways. Industrial experiments are usually of shorter duration and therefore sequential methods may be more easily used. The book contains a set of appendices which might be of interest to anyone who is considering employment in industrial research. The book contains an extensive bibliography.

University of Illinois

D. Burkholder


It is within a number of these chapters, however, that differences between the first and second editions occur. In the first two chapters, for example, new proofs of some of the operational rules have been given and the step as well as the unit impulse functions are introduced. Chapter three on elementary applications is essentially the same as before, except for more problems involving electrical circuits and a section on servo-mechanisms. The next six chapters, except for minor modifications, are the same as in the first edition. The last chapter has been expanded some-
what and includes topics such as the "generalized Fourier transforms". The table of transforms is substantially the same as in the first edition. The author has written an excellent revision in the same lucid manner characterized by its predecessor. It should serve as an important introduction to the growing field of operational mathematics.

University of Illinois  
E. J. Scott


As indicated in the preface, this book is a reprinting of some notes given, in 1943, by the late Professor Hurewicz. The very fact that these notes have not only survived, but even found their way into print, is itself a testimonial.

This small volume will be of value to students with some background in the techniques of solution of ordinary differential equations who wish to go more deeply into certain of the mathematical aspects of this interesting field. To read it profitably they will need to have reached a level where they can read analytic proofs involving uniform continuity, uniform convergence, elementary estimates, etc., with some understanding. Also, a familiarity with the rudiments of matrix theory will be helpful.

An appealing feature of this book is its economy of style. Despite its small size, it treats (1) questions of existence and uniqueness of ordinary differential equations, (2) systems, both linear and non-linear, of differential equations, (3) the singularities of a second order autonomous system of equations, and (4) the general structure of solutions of such systems. While these topics are discussed in other works, it is certainly convenient to have this brief, fairly self-contained, and highly lucid treatment.

Of course, brevity is not without certain shortcomings. The shortcomings here are that there are no examples and no exercises. In our opinion this prevents its use as a text, but it will still be very valuable for supplementary reading.

University of Illinois  
Robert G. Bartle


This excellent manual provides, for the "mature person, used to complex ("linguistic") structures," a brief and efficient discussion of the structure of the Russian language, a detailed reference outline to aid in the identification of the forms of nouns, adjectives, verbs, etc.; a vocabulary aid which groups in separate lists the prepositions, the connectives, the most frequently used adverbs, the most important phrases, etc.; and a guide to the technique of translation. The book concludes with a brief set of practice readings.

For the linguistically experienced and persistent do-it-yourself enthusiast, this book - together with a good dictionary - provides an effective means of learning to read scientific Russian without classroom instruction but not without effort. Pronunciation is ignored as much as possible here. Appropriately, conversation, colloquial and literary Russian, and the writing of Russian are not treated at all.

The person who wishes to learn to read Russian and who has the opportunity to take or audit a semester or more of the language will find this a particularly useful reference volume when he begins the serious reading of scientific work.

The reviewer also recommends N. F. Potapova: Russian, Elementary Course, Volumes I and II (Moscow, Foreign Languages Publishing House, 1954, about $2.00 per volume) and A. I. Smirnitsky: Russian-English Dictionary (Moscow, State Publishing House of Foreign and National Dictionaries, 1958, 951 pp., about $4.50) as superior aids for the one who tackles the job by himself. These may be ordered through your local book-seller. For the mathematician, the pamphlet: Russian-English Vocabulary with a Grammatical Sketch (American Mathematical Society, 190 Hope Street, Providence 6, Rhode Island, $1.50) is excellent.

Russian is a difficult but by no means impossible language to learn. Thus the do-it-yourself student must expect only those miracles that are the result of hardwork. However, the accomplishment is well worth the effort, for Russian is a language with a special beauty of its own. Moreover, it is decidedly sobering to observe at first hand, even if only to a limited extent, the maturity of the scientific work of a people whose intellectual efforts we so long have scorned because of political differences.

University of Illinois  
Franz E. Hohn


Imagine for a moment that you are driving from Long Island to Connecticut on a toll road one summer Sunday. At the toll booths are cars in queues waiting their turn to pay the toll and then move on. The mathematician in you might conjecture: At what rate do the queues grow? How long does it take each car to pay the toll? If there were more toll booths open, how long would a car have to wait? etc. The "toll booth" problem has its counterpart wherever people - or objects - wait for service.

Mr. Morse has written a timely book, Queues, Inventories and Maintenance, for "newcomers to the field of operations research". The newcomers are presumed to know mathematics on the first year graduate level, and to comprehend the scientific method. Mr. Morse explains that queues, inventories, and maintenance share a common mathematical formulation. He elaborates on the physical interpretation of the symbols in the equations. The interpreting is the substance of the book. The mathematics of matrix algebra and probability which underly the work are scarcely mentioned. Yet the book can be appreciated only superficially without the reader's having read, say, Feller's Probability Theory and its Applications.

The book will be most interesting and helpful to a person considering operating research as a profession. Mr. Morse has captured the delight of formulating very common real world situations mathematically. In Finite Queuing Tables there are good derivations and explanations of the equations used, as well as of the method of calculation. These tables should be of value to those engaged in queuing theory.

Jane I. Robertson


This is a book written by an expert and reflects this fact in every page. It is very readable and deals with the subject with the thoroughness that is characteristic of the author.
The author has taken Combinatorial Analysis in its broad sense of enumeration. Thus problems arising in numerous applications such as Statistics and Network Theory are treated alongside the classical combinatorial problems. However, as the title implies, only the most important tool - the enumerating generating function - is introduced and only such problems as lend themselves easily to this technique are considered.

These are mainly problems that are describable by linear difference equations. The contributions of Slepian (Jour. Math. Phys., 1953, pp 185 - 193) and Davis (Proc. Amer. Math. Soc., 1953, pp 486-495) are examples of problems that are not considered.

The concept of a generating function is introduced in chapter 1, after an elementary discussion of permutations and combinations. Chapter 2 is devoted entirely to generating functions, their elementary properties and their application to linear difference equations and related problems in statistics. The principle of inclusion and exclusion is discussed briefly in chapter 3. The next chapter contains a brief treatment of the enumeration of permutations by cycles. The important problem of distributions and the related problem of occupancy are given a thorough treatment in chapter 5. The following chapter is one of the best in the book, dealing mainly with the problem of enumerating the number of linear graphs of various types - trees, rooted trees, series-parallel two terminal graphs, labelled graphs, etc. Since the author has made many original contributions to these problems it is natural that this chapter should be excellent. The last two chapters concern the enumeration of permutations in which certain elements are restricted to occupy or not to occupy certain positions. The traditional combinatorial problems (probleme des menages, the rook problem, card matching, etc.) are discussed in detail and in a unified manner.

Within the scope considered in the book, the coverage is excellent. Virtually all the major contributions are included, and special emphasis is placed on those of Euler, Touchard, Bell, MacMahon and, of course, Riordan. There is a short but well-chosen bibliography in each chapter, with many references as recent as 1956. Each chapter is followed by a long list of well-chosen problems, many of which are far from trivial.

The faults that can be found with the book are mainly those ascribable to the subject itself and to the relatively small number of pages in the present text. Many non-specialists will find this book difficult to refer to, due to the special notations of combinatorial (which are necessary to keep the formulas brief). A table of symbols would have been a useful addition to the book. An abstract study such as this always suffers from lack of motivation. The author has tried to compensate for it by giving the origins of the various problems but space considerations have necessarily left these discussions very brief.

There are very few typographical errors in the book.

University of Toronto

Sundaram Seshu


This book is a very attractive introduction to the theory of linear operators and topological linear spaces that has been developed in this century. This field has grown from studies in linear differential and integral equations of the sort arising in potential theory and quantum physics, and is in turn applicable to these subjects. Taylor's book is written primarily for graduate students of mathematics and "is an introduction, not a treatise" which is "meant to open doors for the student." In our opinion it should be highly successful in this goal.

In the past five years at least a dozen books on this general area have appeared in this country or abroad. In a review of this nature it is not appropriate to contrast and analyze the varying features of these books. We shall content ourselves with expressing the opinion that Taylor is the most suitable one for use as a textbook in a first course in functional analysis, or for independent reading.

Each new circle of ideas is introduced carefully with considerable motivational discussion. Many examples and applications are given, so the importance and relevance of the ideas are demonstrated. The actual development is rigorously done but flows smoothly. Finally, there are numerous well-chosen exercises on almost every topic that is discussed.

In summary, we recommend this text most highly as an introduction to abstract analysis.

University of Illinois

Robert G. Bartle


Professor Hohn's book is designed to serve many purposes. It presupposes little background on the part of the student and could perhaps be used profitably in a course for students with no more formal background than a course in college algebra. On the other hand it contains material that is vital to the development of anyone seriously interested in mathematics or in the application of mathematics. Thus it could be used properly for a beginning graduate course.

In the preface the author states that the book makes no pretense of being complete. However, in the opinion of the reviewer, it covers a surprising number of details in matrix theory. It covers well the usual concepts in matrix algebra and determinant theory. Vector spaces and linear transformations are developed quite thoroughly. The last two chapters deal with characteristic-equations of matrices, and bilinear, quadratic and Hermitian forms. Three appendices are added. They treat the summation and product symbols, complex numbers and the general concept of isomorphism. The best feature of the book is the obvious care taken to introduce each topic in as natural a fashion as possible. The book is quite readable. Since the author considers the person with a need for matrix algebra as a tool, emphasis is placed on techniques for actually calculating such things as inverses of matrices and evaluating determinants. On the other hand, for the sake of the person interested in the theoretical aspects of the subject, the concepts of field and group are introduced. The chapter on linear transformations is well done and the isomorphism between the set of linear transformations on an n-dimensional vector space over a field and the set of n x n matrices with elements of the field is established very nicely.

The numbering system and notation is consistent and good throughout the book and makes for easy reference. The book is well indexed, listing symbols as well as terms. The exercises vary in difficulty and supplement well the material in the text. They are often used to point out areas of application for the topics treated.

In the opinion of this reviewer, the book could be used by the person who wants the material in the text as a tool or it could be used by the person wanting a good introduction to linear algebra. It treats both extremes well. It is a very readable book and this readability is effected by the care taken by the author to introduce new concepts naturally rather than formally. It should be well received.

St. Louis University

John W. Riner
BOOKS RECEIVED FOR REVIEW


*See review, this issue.

NOTE: All correspondence regarding book reviews should be addressed to FRANZ E. HOHN, 374 ALTGELD HALL, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS.
It is of interest to notice that of the many men who have made great contributions to the field of science a large percentage were men of wealth and position, and therefore had no monetary incentive to spur them forward in their great tasks. It is also interesting that many of the early researchers actually jeopardized their social and financial positions by pursuing their interests and presenting their theories with such vigor. In view of these observations, we are moved to look for the motivating force which drove these men to immortality.

This motivating force is not characteristic of scientists alone, nor of humans alone, but of all living creatures. It is known as curiosity. The small child is consumed by an overwhelming curiosity and thirst for knowledge, (much to the distress of some parents). As the child grows older, the curiosity gradually subsides until the pedestal of adulthood is surmounted from which vantage point we survey the world in blissful ignorance, entirely too sophisticated to trouble our minds over things we cannot understand.

Fortunately there are those in whom the spark of curiosity is not smothered, but rather grows into a bright flame to light the way for others who follow. Hence we have our Galileos, our Newtons, and our Einsteins.

Man now finds himself faced with history's greatest challenge - the conquest of Space. Never before has the nurturing of the spark of curiosity been so crucial to the welfare of mankind, for this age bears with it not only a technological challenge but also a tremendous philosophical challenge. There are forces in the world today which would thwart the very purpose of man's existence. These forces which are gathering momentum at a staggering rate must be met and subdued. This can only be accomplished by cultivating that with which all men are born - the thirst for knowledge.

The need for sound basic research is self evident. In many fields of scientific endeavor, the lack of basic research has seriously retarded progress to the extent that grave errors, which should have been easily avoided, have been made.
When I am asked for a definition of applied mathematics, I usually reply - without any expectation of ending the discussion - that applied mathematics is what I do. For our present purposes, let me broaden this to include the activities of research mathematicians on the Ph.D. level at the Bell Telephone Laboratories; there are, depending on exactly what areas of specialization you count, between 50 and 100 of us. There is, of course, much exciting work for those with less advanced training, but it will not be considered here.

What we do is, first and foremost, solid mathematical research, new mathematics which is of interest to the scientific community and publishable in an appropriate research journal. Industrial mathematical research takes no back seat in either number or content. The recent Survey of Research Potential in the Mathematical Sciences showed that industrial mathematicians on the Ph.D. level published at least as much as their academic colleagues - somewhat more, in fact, among recent graduates. Nor is there a difference in quality, as (to name just two out of many) Shannon's Information Theory and Rice's studies of noise will attest. The real distinctions, as we shall see, lie in the origin of the research, and in the nature of its successful conclusion.

Many people would, without further thought, claim that the primary function of a mathematician in industry is problem solving. This is rather far from the truth; the solution of a specific mathematical problem is only a small part of the picture. As a matter of fact, there seem to be five stages in the evolution and dispatch of a problem, and while there are no sharp boundaries between these, they do catch the mathematician in different attitudes. They are, to use a one-word title for each, recognition, formulation, solution, computation, explanation.

An industrial laboratory parades before the open mathematical mind a host of simultaneous and exciting activities. Every so often, you notice that there is in one of these activities something fundamental which is a stumbling block, which requires exploration, which is not understood. You don't as yet know what to try to do, but you have discovered a situation that needs thought; you have recognized that there is a problem.

Well then, see if you can formulate something precise to work on. You need to make a mathematical model which is simple enough to permit of mathematical analysis, and yet sufficiently close to the real situation to be relevant. This model building is probably the most difficult, and the most valuable, task of the industrial mathematician. To quote T. C. Fry, who has done this as successfully as anyone, "It is because the mathematician is expert in analyzing relations, in distinguishing what is essential from what is superficial in the statement of these relations, and in formulating broad and meaningful problems, that he has come to be an important figure in industrial research teams." To suppress the irrelevant details, and get to the heart of the matter, that is the formulation of the problem. And do not imagine that because someone else has brought the problem area to your attention, that he can therefore formulate the problem for you. Many times the original mathematical question which someone else has asked turns out to have been the wrong question. To avoid the irrelevant and misleading, a mathematician must usually do his own model-building.

Mathematical exploration and solution is the next step. The branches of mathematics which come into play here may be quite unexpected; number theory and algebraic topology have on occasion turned out to be as important as classical and abstract analysis. The key points are two: The relevant mathematics is unpredictable, and so is the difficulty of the problem. While everyone appreciates a neat and beautiful theory when it can be found, there are times when you want the answer and have to slug to get it. The neatness of textbook problems cannot be guaranteed by the real world; and yet you must not give up too soon, for neatness is frequently a symptom of real understanding.

The computation of interesting cases is to be recommended to any mathematician, industrial or otherwise, whenever it is meaningful, for planning a numerical program and understanding its output frequently suggests further directions for research. But when you hope that your mathematics says something significant about a real situation, the computing is almost essential, and a source of genuine satisfaction when all goes well.

At this stage, you are certainly tempted to call it a day and a job well done. And yet it is not so, for a beautiful theory in either head or notebook is still of no value to anyone else. The industrial mathematician must make his knowledge available to those who can use it, and this is a non-trivial process. The typical mathematical paper, written in a typical mathematical exposition, is not enough,
for this kind of presentation is too compact, and probably in the wrong language, for the customer. The significance of the result must be explained without assuming the specialized terminology of the mathematical field which was used to obtain it. What is needed is a leisurely exposition, full of motivation, in the customer's own terms, with detailed proofs relegated to the appendix unless it is the proof that provides the real understanding.

After this description of the nature of mathematical research in industry as it appears from the Bell Labs, the next step is to inquire into the character and training of the man for the job. When you look back over the description, three features stand out: Mathematics, physical world, exposition. First of all, it is clear that industry is looking for a man who knows his mathematics and is really good at it, one who has an interest in, and knows the fundamentals of, a great many fields. For you never know what you are going to get into; if your only interest is in topological groups, or even in hyperbolic partial differential equations, your path will not be easy. Thus the work is certainly mathematics on the Ph.D. level, but with this difference: The best man is a mathematical generalist, rather than a narrow specialist. And such a mathematician will have as broad an outlook as anyone in the company. Secondly, you ought to know the language, and the basic facts, of a number of other fields of science, for these are the source of many of your problems, and other scientists should be able to understand what you have to say. Much of this you can learn on the job, but a good background certainly helps. Thirdly, you should know how to speak and to write, and be willing to work at a clear presentation. A man likes to have his labors appreciated by those around him, and unhappy is he who cannot be understood. When you have spent months in cracking a beautiful problem, isn't it sensible to spend a few extra days striving for a beautiful write-up?

The most important question of all has been left for last. Is this sort of work a satisfying experience, and is it fun? There is no need to dwell on the question of money; industrial mathematical research is certainly well paid, but no scientist will be happy for any length of time if the intellectual rewards are not there. And what are these intangibles? You have, first of all, the thrill of discovery, of seeing pieces fall into place and form a theory. You are not without the satisfaction of teaching, for you have the opportunity to consult for other scientists and engineers, and to teach them what they want to know. Finally, every once in a while, you can actually see greater understanding, and faster technical progress, because of what you have been able to do. Each of these contributes to the satisfaction derived from industrial mathematical research.

In previous articles in this series of "Operations Unlimited", a general picture has been given of the need, the importance, and the application of mathematics to a broad spectrum of industry. In this article, I will take an opposite tack and give some specific problems which have arisen directly in the course of my work plus several interesting ones which have arisen as by-products. It should be kept in mind that this is only a small sample of only one person's work and consequently will probably not be truly representative. However, it should provide, in a specific manner, indications of the mathematical work being done in the missile industry.

Problem 1. The Satellite Solar Battery

A satellite of given shape is rotating in some random way in its orbit. We wish to determine the average flux of light from the sun which intercepts a specified portion of the satellite's surface. To simplify the discussion here, we will consider the two-dimensional analog. Corresponding results will hold in three dimensions. If the body is a smooth convex one and we consider that the entire length is light absorbing, then it follows from a known theorem in integral geometry that the average diameter (which is proportional to the light flux) is given by the diameter $D$ of a circle with the same perimeter. If only half the length is light absorbing, then it follows from a known theorem in integral geometry that the average diameter (which is proportional to the light flux) is given by the diameter $D$ of a circle with the same perimeter. If only half the length is light absorbing, it will follow by symmetry that, if the body is centro-symmetric, the average diameter will be $D/2$. An interesting problem is the converse one; if the average diameter is
one can have, in a plane perpendicular to the axis of an axial-symmetric shell, of the CG of the component parts (lying inside) of the shell due to possible variations in weight of ± p% of the component parts. This problem would be a simple one but for the fact that there are in the order of fifty component parts to be accounted for. Mathematically, the problem reduces to determining a vector \( \mathbf{E} \) lying on the surface of the N-dimensional hypercube \( \{\pm 1, \pm 1, \ldots, \pm 1\} \) \((N \geq 50)\) such that \( (A - E \cdot \mathbf{B} - \mathbf{E}) \mathbf{E}^2 \) is a maximum (here \( \mathbf{A} \) and \( \mathbf{B} \) are given vectors).

Problem 6. A Non-Linear Integral Equation

In a study of the distribution of particle sizes in a fog, the following non-linear integral equation has come up:

\[
\int_0^{\infty} r^3 \phi(\xi - r) \, dr = a \left\{ \int_0^{\infty} r^2 \phi(\xi - r) \, dr \right\}^b,
\]

where \( a \) and \( b \) are independent of \( \xi \). Extraordinarily, this non-linear equation has an elegant solution (assuming differentiability of \( \phi \)). Unfortunately, the solution did not correspond to experimental data.

Problem 7. On the Propagation of a Wave

Determine the equation of an axial-symmetric wave which propagates normally to itself with a velocity varying with distance from the axis. This leads to the P.D.E.

\[
\left( \frac{\partial F}{\partial t} \right)^2 = V(t) \left\{ 1 + \left( \frac{2}{3} \frac{\partial F}{\partial t} \right)^2 \right\},
\]

subject to the initial condition \( \mathbf{F}(r,0) = \mathbf{G}(r) \) (the initial wave front). This is solved by finding the envelope of the complete integral of the equation. If \( V(t) \) is constant, we obtain parametric equations for the class of axial-symmetric parallel surfaces.

Problem 8. Steady-State Temperature Solution in a Half-Space

Consider a half-space which is subject to a constant heat flux \( Q \) over a circle on the bounding plane and insulated elsewhere. It can be shown that the steady-state solution is given by

\[
\]
The problem here is to obtain numerical results. It can be shown that
\[
T_s(r, z) = \frac{Qa}{k} \int_0^\infty e^{-\lambda z} J_0(\lambda r) J_1(\lambda a) \, d\lambda /\lambda.
\]

For other \( r \) and \( z \), rapidly convergent series expansions are obtained. As a side result, we obtain the two interesting sums
\[
\frac{2}{\pi} = \sum_{m=0}^{\infty} \frac{(2m)!^2}{2^{4m}(2m+2)!^4} = 1 - \sum_{m=0}^{\infty} \frac{(2m)!^2}{2^{4m}(2m-1)!^4}.
\]

Problem 9. Unsteady-State Heat Transfer Due to A Source Between Two Parallel Insulated Planes

The physical problem here is to determine the temperature distribution due to a continuous source lying between two parallel insulated planes. Since the solution for a source in all-space is well known, the problem is solved very simply by the method of images. This leads to a solution which is very convenient for small time numerical calculations. However, for large times, the solution is not at all suitable since it is extremely slow in convergence. For this case, suitable asymptotic expansions are derived. The following are two interesting problems which came up in determining the asymptotic expansions:

a.
\[
\lim_{r \to 0^+} \left\{ -\log r + 2 \sum_{n=1}^{\infty} \frac{K_0(\frac{n \pi r}{H}) \cos \frac{n \pi z}{H} \cos \frac{n \pi h}{H} }{n} \right\},
\]

b. is the function \( F(x) = x^{-1}(x^2 + a^2)^{-1/2} \) completely monotone
\( (i.e., (-1)^m F^m(x) \geq 0) \) for \( a, x \geq 0 \)?

Problem 10. Evaluation of Some Integrals Arising in Corona Light Scattering

\[
\int_0^\infty \left\{ J_0(x)^2 - J_1(x)^2 \right\} \, dx = 2/\pi,
\]

\[
\int_0^\infty \left\{ J_0(x)^2 - J_2(x)^2 \right\} \, dx = 8/3 \pi.
\]

These results can be generalized in the one integral
\[
\int_0^\infty \left\{ J_m(x)^2 - J_n(x)^2 \right\} \, dx
\]
\( (m, n \text{ rational}) \) which can also be evaluated in closed form.

Problem 11. Solution of a Cubic with Complex Coefficients

This problem arises in stability considerations of a liquid being heated from below. The trigonometric solution for the cubic with real coefficients is extended to take care of this case.

Problem 12. An Extension of the Mean Value Theorem for Harmonic Functions

Consider the heat flow equation.

\[
\frac{\partial T}{\partial t} = \alpha \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right\}
\]

for a circular region subject to the initial condition* \( T(r, 9, 0) = 0 \), and boundary condition \( \oint T(R, \theta, t) \, d\theta = 0 \). It is easily shown that \( T(O, 9, t) = 0 \) (this is also true for an N-sphere). If \( T(R, \theta, t) \) is independent of \( t \), we obtain a steady-state solution which gives the mean value theorem for harmonic functions. One simple intuitive method of solution is to consider the problem for a regular polygon where the result is easily obtained by superposition. Then take the limit as the number of sides increases without bounds. The same results also hold for the Neumann problem.

* Professor L. Nirenberg has pointed out that this can be extended to \( \oint T(r, \theta, t) \, d\theta = 0 \).
Problem 13. On Sets Preserving an Axis of Symmetry
Under the Group of Affine Transformations

This problem arose in an unsuccessful attempt at solving a certain
heat flow problem. It is well known that conics transform into conics
under affine transformations. Consequently, an axis of symmetry
(actually two) is preserved. Are there any other sets having an axis
of symmetry which is preserved under the group of affine transforma-
tions? The following partial result has been obtained by D. J. Newman:

Consider a set, symmetric to both x and y axes, passing thru
(± a, 0), (0, ± b) and, except for these latter points, lying wholly inside the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \]
Then, in general, there is no axis of symmetry in
the transformed set.

Problem 14. A Numerical Solution of a Boundary Value Problem

This last problem has been solved "experimentally" using an I.B.M. 704 computer. This is typical of a wide class of problems in which
the mathematical theory available is not sufficient to resolve the
problem.

Consider the one-dimensional heat flow equation
\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \]
subject to following initial and boundary conditions:

I.C. \[ T(x, 0) = 0 \]

B.C. 1 \[ -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = Q(t) \]

B.C. 2 \[ \left. \frac{\partial T}{\partial x} \right|_{x=L} = 0 \]

The usual problem is to determine the temperature function \( T(x, t) \).
This can be done by using the difference equations
\[ T(x, t + \Delta t) = r \left\{ T(x + \Delta x, t) + T(x - \Delta x, t) \right\} + (1 - 2r) T(x, t) \]
where \( r = \frac{\Delta t}{\Delta x^2} \). In order that the numerical process be
stable \( r < 1/2 \). To increase the accuracy, we decrease the size of
the increment \( \Delta x \).

DO YOU KNOW

*Not All The Good Die Young*

Evariste Galois Died at the age of 21
Niels H. Abel Died at the age of 27
Blaise Pascal Died at the age of 39
George F. B. Riemann Died at the age of 40
Carl G. J. Jacobi Died at the age of 47
Johann F. C. Gauss Died at the age of 78
Jean-Victor Poncelet Died at the age of 79
Richard Dedekind Died at the age of 85
Isaac Newton Died at the age of 85
Albertus Magnus Died at the age of 87
Pi Mu Epsilon members are invited to a Dutch-treat luncheon or breakfast to be held during the annual summer meetings of the M.A.A. and the Society, at the University of Utah, Salt Lake City, August 31 – September 3, 1959.

The next delegate meeting of Pi Mu Epsilon will be at Michigan State University, East Lansing, Michigan, in the summer of 1960.

Dr. Bernard Derwort, Missouri Gamma, ’48, formerly of North American Aviation, Columbus, Ohio is now head of the Mathematics department at St. Thomas College, St. Paul.

Mr. Thomas K. Boehme, Oklahoma Alpha, ’52, who received his Master’s in mathematics from Oklahoma State University is now working on his Doctor’s at California Institute of Technology.

News items should be sent to Mary L. Cummings, Mathematics Department, University of Missouri, Columbia, Missouri.

NOTICE TO INITIATES

On initiation into Pi Mu Epsilon Fraternity, you are entitled to two copies of the Journal. It is your responsibility to keep the business office informed of your correct address, at which delivery will be assured. When you change address, please advise the business office of the Journal.

REPORTS OF THE CHAPTERS

ALPHA OF MONTANA, University of Montana
The Montana Alpha Chapter began the 1957-58 year by initiating ten new members. There were six program meetings and the year was closed with the annual Spring picnic. The following papers were presented during the year:

"Report on the Michigan Meeting on Differential Equations" by Dr. Louis J. Schmitroth
"Random Walks and the Gambler's Ruin" by Mr. Howard Reinhardt
"The Stone-Weierstrass Theorem" by Dr. William R. Ballard
"Veblen-Wedderburn Planes" by Dr. Wayne Cowell
"Boolean Functions" by Dr. Frederick Young
"Coordinatization of the Plane" by Mr. Donald Sward

Due to a change in course offerings, the traditional Pi Mu Epsilon prizes, formerly awarded to the three freshmen placing highest in entrance examinations, were combined in a single prize of $50.00 given to the high school student getting the top award in the Mathematics Section of the Montana Science Fair, which is held on the University Campus each Spring. This year the prize went to Lindsey Hess, Gallatin County High School of Bozeman.

The election of 1958-59 officers was postponed until the first meeting in the Fall.

ALPHA OF KANSAS, University of Kansas
The Kansas Alpha Chapter held seven meetings during 1957-58. Six of these meetings were business meetings. The seventh meeting was the annual initiation and the election of officers for 1958-59. At this session Professor Stief, of Zurich, Switzerland, presented a paper on "The Calculation of f(x)".

DELTA OF ILLINOIS, Southern Illinois University
The Illinois Delta Chapter held three regular program meetings during 1957-58. The following papers were presented:

"Mathematics in the Social Sciences" by Professor Mark Wilson
"Bendixson Theory of Non-linear Differential Equations" by Professor Wilson
"Normal Numbers" by Paul Phillips, chapter president

In April, the chapter sponsored a Mathematics Field Day, which was attended by about 600 students in the high schools of Southern Illinois. The high school students were given a competitive examination in the morning and in the afternoon attended a program prepared by chapter members. The talks, on the general topic of information, presented by the chapter members were:
BETA OF NEW YORK, Hunter College

The New York Beta Chapter held four program meetings and two initiation dinners during the year 1957-58. Twenty new members were initiated during the year.

The following papers were presented at the program meetings:

"The Theory of Strictly Determined Games" by Barbara Saul, Annabelle Siegel, and Frances Testa under the direction of Professor Marguerite Darkow.

"A Problem of Information Transfer in Biology" by Professor T. L. Hill.

ALPHA OF PENNSYLVANIA, University of Pennsylvania

The Pennsylvania Alpha Chapter held five program meetings during 1957-58. The following papers were presented:

"Computers" by Dr. Robert McNaughton of the Moore School of Electrical Engineering, University of Pennsylvania.

"The Theory of Non-Strictly Determined Games" by Lillian Scott, Natalie Horowitz, and Ruby Freed under the direction of Professor Marguerite Darkow.

"Calendar Problems" by Marianne Sinapi, Stephanie White, and Helen Kaufman under the direction of Professor A. Day Bradley.

"A Problem of Information Transfer in Biology" by Professor T. L. Hill.

ALPHA OF OREGON, University of Oregon

The Oregon Alpha Chapter held four program meetings, which included business sessions and social periods, during 1957-58. In addition the annual initiation was held and concluded with a picnic at which 175 members, families and friends were in attendance. Fifty-six new members were inducted at the initiation ceremonies. The chapter presented the library with a new book for the browsing room. The following papers were presented during the program meetings:

"Mathematics in Economics" by Professor Paul Simpson, Department of Economics.

"The Theory of Strictly Determined Games" by Barbara Saul, Annabelle Siegel, and Frances Testa under the direction of Professor Marguerite Darkow.

"A Problem of Information Transfer in Biology" by Professor T. L. Hill.
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**MISSOURI GAMMA, St. Louis University (December 6, 1958)**

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<td>Bucknell University</td>
<td>December 11, 1958</td>
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<td>October 27, 1958</td>
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