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## PI MU EPSHON JOURNAL

 THE OFEICIAL PUBYTCATIONOF THE HONORARY MATHEMATICAL FRATERNITY

RUTH W. STOKES, Editor
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Pi Mu Epsilon Journal, volume 2:

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## MODEL ILLUSTRATING THE BRACHISTOCHRONE

The model shows five possible paths. They are, when numbered from the back of the model toward the front: (a)a segment of a straight line; (b)an arc of a circle having radius equal to four times the radius of the rolling circle generating the cycloid: (c) one half of the cycloid (the middle path shown in the picture); (d) a compound curve consisting of two straight line segments connected by an arc of a circle smaller than that in (b) above; (e) part of one branch of an equilateral hyperbola pictured in the first quadrant and tangent to the horizontal and vertical axes.

## A HISTORY OF THE BRACHISTOCHRONE

William S. Kimball<br>University of Vermont

## 1. At the Birth of the Calculus of Variations

The calculus of variations is usually thought of as starting with the researches of the Bernoulli brothers and their contemporaries on the Brachistochrone. Briefly stated this is the plane curve down which a frictionless body will fall, between two points not on the same vertical line, in minimum time. Its name is due to John (Jean) Bernoulli and derives from the Greek words Brachistos, shortest, and Chronos, meaning time.

The essential feature that marks this problem a case of the calculus of variations is that a definite integral, here the time of fall, has to be minimized by selecting the critical path, i.e., an infinite set of points before the consequent desired minimum integral can be evaluated. Ordinary minimum and maximum problems are determined usually by one or two critical values of the independent variable on which some quantity to be minimized is functionally dependent and varies accordingly until its minimum is specified by the critical value of the independent variable. By contrast it is an infinite set of points comprising a line or curve in the role of an independent variable that is critical for a minimum or maximum in the calculus of variations proper.

Realization of this aspect of this kind of minima emerged from the discussions of the Brachistochrone problem especially at first by James and John Bernoulli, and later Euler and Lagrange gave it a more general form by introducing the "variations ${ }^{\text {w }}$ fix $=\boldsymbol{\xi}(x, y)$ and fiy $=\boldsymbol{\eta}(\mathrm{x}, \mathrm{y})$ along a path of integration, whence came the name, Calculus of Variations.

The Brachistochrone is not the first calculus of variations problem ever tackled. The ancients worked by precalculus methods on the problem of finding the maximum
area enclosed by a plane curve of fixed length, which is characteristic of the type requiring calculus of variations methods. Also, Newton's problem to find the meridian curves of a surface of revolution of minimum resistance was solved by him using essentially calculus of variations methods, and this is sometimes referred to as the oldest problem in the calculus of variations.

Even the Brachistochrone (shortest time) problem itself, apart from its correct solution, was tackled by Galileo before the invention of the calculus. He obtained no proved solution but surmised that the time of fall, down a circular arc was shorter than the sums of the times along the successive straight line sides of any inscribed polygon connecting the same two end points. For Galileo's remarks see his Dialog uber die beiden haubtsuchlichtsen Weltsysteme (1630), Straus pp. 471-472 and Dialogues concerning two New Sciences (1638), Crew and DeSalvia p. 239. But the flare-up in mathematical circles over the Brachistochrone brought recognition that one is dealing here with a new science, calculus of variations, which has ever since held its important place as a branch of advancing and advanced calculus.

As a touch of the then contemporary viewpoint we may quote from John Bernoulli's Proclamation at Groningen, January, 1697. "Jean Bernoulli, public professor of mathematics, pays his best respects to the most acute mathematicians of the entire world.
"Since it is known with certainty that there is scarcely anything which more greatly excites noble and ingenious spirits to labors which lead to the increase of knowledge than to propose difficult and at the same time useful problems through the solution of which, as by no other means, they may attain to fame and build for themselves eternal monuments among posterity, so I should expect to deserve the thanks of the mathematical world if, imitating the example of such men as Mersenne, Pascal, Fermat - and others who have done the same before me, I should bring before the leading analysts of this age some problems on which, as upon a touchstone, they could test their methods, exert their powers, and, in case they brought anything to
light, could communicate with us in order that everyone might publicly receive this desired praise from us.
"The fact is that half a year ago in the June number of the Leipzig Acta I proposed such a problem whose usefulness linked with beauty will be seen by all who successfully apply themselves to it. Six months from the day of publication was granted to geometers, at the end of which, if no one had brought a solution to light, I promised to exhibit my own. This interval of time has passed and no trace of a solution has appeared. Only the celebrated Leibniz, who is so justly famed in the higher geometry, has written me that he has, by good fortune, solved this, as he himself expressed it, very beautiful and hitherto unheard of problem; and he has courteously asked me to extend the time limit to next Easter in order that in the interim the problem might be made public in France and Italy and that no one might have cause to complain of the shortness of the time allotted. I have not only agreed to this commendable request, but I have decided to announce myself the prolongation (of the time interval) and shall now see who attacks this excellent and difficult question and, after so long a time, finally masters it. For the benefit of those to whom the Liepzig Acta is not available, I here repeat the problem.
"Mechanical-Geometrical Problem on the Curve of Quickest Descent.
"1. 'To determine the curve jointing two given points, at different distances from the horizontal and not on the same vertical line along which a mobile particle, acted upon by its own weight and starting its motion from the upper point, descends most rapidly to the lower point.'
"The meaning of the problem is this: Among the infinitely many curves which join the two given points, or which can be drawn from one to the other, to choose the one such that, if the curve is replaced by a thin tube or groove, and a small sphere is placed in it and released, then this (sphere) will pass from one point to the other, in the shortest time.
"In order to exclude all ambiguity, let it be expressly understood that we here accept the hypothesis of Galileo, of whose truth, when friction is neglected, there is now no
reasonable geometer who had doubt: 'The velocities actually acquired by a heavy falling body are proportional to the square roots of the heights fallen through.' However, our method of solution is entirely general, and could be used under any other hypothesis whatever."

## 2. As a Problem in Mechanics

The two explanatory paragraphs following the above Bernoulli statement of the problem signalize it as being in his mind a problem in the dynamics of falling bodies, where gravity is the force operating in a conservative mechanical system where the total energy does not change with the time, and furthermore restricted to frictionless forces having a potential, to which the calculus of variations applies. Thus Galileo's approach to the problem and his mechanics of frictionless descent under gravity are implicitly included in its statement.

The acceleration component of gravity along the path of descent is given by $g \sin \mathbf{a}=\mathbf{g} \frac{d \mathbf{y}}{\mathrm{ds}}$ where a is the inclination of the path shown in Figure 1. As a time derivative this may be written:

$$
\begin{equation*}
\frac{d^{2} s}{d t^{2}}=\frac{d V}{d t}=\frac{d V}{d s} \frac{d s}{d t}=V \frac{d V}{d s}=g \frac{d y}{d s}=g \sin \alpha \tag{1}
\end{equation*}
$$

which gives upon integration:

$$
\begin{equation*}
\mathrm{V}^{2}=\left(\frac{\mathrm{ds}}{\mathrm{dt}}\right)^{2}=2 \mathrm{gy} \tag{2}
\end{equation*}
$$

where the constant of integration is zero since the origin is taken as the starting point. I we solve for dt, we may set up the integral to be minimized for this problem:

$$
\begin{equation*}
T=\int d t=\frac{1}{\sqrt{2 g}} \int \frac{d s}{\sqrt{y}} \tag{3}
\end{equation*}
$$

For simplicity the constant may be dropped:

$$
\begin{equation*}
J=\int \frac{\mathrm{ds}}{\sqrt{\mathrm{y}}} \tag{4}
\end{equation*}
$$

There is more mechanics in these four equations than appears on the surface. Thus equation (2) is the familiar velocity formula for freely falling bodies, which here refers to any frictionless path of descent. It applies here because the working force component (1), with gravity reduced by the factor sin a, must act through proportionately greater distances, $\mathrm{ds}=$ dy csc a , which exactly compensate the reduction in force due to the factor sin a . The force component normal to the path does not work because no displacements occur in this direction, and, with no friction, the energy levels are the same as for freely falling bodies. Thus the kinetic energy equals precisely the loss, mgy, of potential energy, which latter at the start where $\mathrm{V}=0$, equals the total energy constant E for this conservative system:

$$
\begin{aligned}
\text { K.E. }= & \mathbf{m} \mathbf{V}^{2}=m g y=E^{-}-W(y), \\
& W(y)=E^{-} \operatorname{mgy} .
\end{aligned}
$$

So these equations refer to any frictionless path of decent, and we seek the path that minimizes (3), and (2) is merely the control factor that prescribes compliance with frictionless mechanics and, by itself, says nothing about the path to be selected so as to provide minimum time of descent.

## 3. The Lagrange Method for Conditional Maxima and Minima

We will use the Lagrange method for conditional minima to minimize the integrals (3) and (4) of the brachistochrone. First, let us illustrate the Lagrange method by applying it to the problem of finding the maximum rectangular box, $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})=\mathrm{V}=\mathbf{x y z}$, that can be inscribed in the first octant of the ellipsoid, given as a constant restricting condition:

$$
\left.\begin{array}{l}
F(x, y, z)=V=x y z  \tag{5}\\
C(x, y, z)=\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}+\frac{z^{2}}{c^{2}}-1=0
\end{array}\right\}
$$



Fig. 2
The maximum box inscribed in the first octant of an ellipsoid.

Because of the restricting condition we have only two independent variables, say $x$ and $y$, whose vanishing partialderivatives are the conditions for the maximum volume:

$$
\left.\begin{array}{l}
\frac{\partial F}{\partial x}=F_{x}+F_{z} \frac{\partial z}{\partial x}=0  \tag{6}\\
\frac{\partial F}{\partial y}=F_{y}+F_{z} \frac{\partial z}{\partial y}=0
\end{array}\right\}
$$

and from the restricting condition $C=$ constant, of (5), we have:

$$
\left.\begin{array}{l}
\frac{\partial C}{\partial x}=C_{x}+C_{z} \frac{\partial z}{\partial x}=0  \tag{7}\\
\frac{\partial C}{\partial y}=C_{y}+C_{z} \frac{\partial z}{\partial y}=0
\end{array}\right\}
$$

The elimination of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ from (6) and (7) gives:

$$
\left|\begin{array}{ll}
\mathbf{F}_{\mathrm{x}} & C_{x}  \tag{8}\\
\mathbf{F}_{\mathrm{z}} & C_{z}
\end{array}\right|=0 ;\left|\begin{array}{ll}
\mathbf{F}_{\mathrm{y}} & \mathbf{C}_{\mathrm{y}} \\
\mathbf{F}_{\mathrm{z}} & C_{z}
\end{array}\right|=0
$$

where the subscripts refer to partial derivatives of $F$ and $C$ as functions of three independent variables. Introduce these partial derivatives into (8) to obtain:
or

$$
\left.\begin{array}{c}
\frac{C}{x}_{F_{x}}=\frac{C_{y}}{F_{y}}=\frac{C_{z}}{F_{z}}=\frac{2 x}{a^{2} y z}=\frac{2 y}{b^{2} x z}=\frac{2 z}{c^{2} x y} ;  \tag{9}\\
\frac{x^{2}}{a^{2}}=\frac{y^{2}}{b^{2}}=\frac{z^{2}}{c^{2}}
\end{array}\right\}
$$

and hence, using $C$ of (5), we have the edges and volume of the maximum box:

$$
\begin{equation*}
x=\frac{\mathrm{a}}{\sqrt{3}} ; \mathrm{y}=\frac{\mathrm{b}}{\sqrt{3}} ; \mathrm{z}=\frac{\mathrm{c}}{\sqrt{3}} ; \mathrm{V}(\max )=\frac{a b c \sqrt{3}}{9} \tag{10}
\end{equation*}
$$

Now a striking algebraic curiosity appears when we obtain the unrestricted maximum of the sum:

$$
\begin{equation*}
\mathbf{I}(x, y, z)=\mathbf{F}+\lambda \mathbf{C} \tag{11}
\end{equation*}
$$

where $\mathbf{x}, \underline{y}$, and $\mathbf{z}$ are now taken as three independent variables instead of the two independent $\underline{x}$ and $\underline{y}$, with $\underline{\underline{z}}$ dependent on them as in (6) and (7); and $\bar{\lambda}$ is an undetermined constant, the so-called Lagrange undetermined multiplier.

The critical equations for the maximum of (11) are

$$
\left.\begin{array}{l}
\frac{\partial I}{\partial x}=F_{x}+\lambda C_{x}=0  \tag{12}\\
\frac{\partial I}{\partial y}=F_{y}+\lambda C_{y}=0 \\
\frac{\partial I}{\partial z}=F_{z}+\lambda C_{z}=0
\end{array}\right\} \text { or }-\lambda=\frac{F_{x}}{C_{x}}=\frac{F_{y}}{C_{y}}=\frac{F_{z}}{C_{z}}=\frac{a b c}{6} \sqrt{3}
$$

These are seen to give identically the vanishing determinants (8) with A determined along with $\underline{x}, \underline{\mathbf{y}}$ and $\mathbf{z}$ by the extra condition (5) as was done to get the $\overline{\text { c }}$ ritical values (10).

Similarly, in case of n independent variables and one restricting condition, $\mathbf{C}\left(\mathbf{x}_{1}, \mathbf{x}_{2}, \cdots, \mathbf{x}_{\mathrm{n}}\right)=0$, there are $\mathbf{n - 1}$ vanishing first order determinants like (8), and one dependent variable, say $\mathbf{x}_{\mathbf{n}}$. But the Lagrange method again applies, whereby the identical critical conditions are given by the vanishing partial derivatives of $I=\mathbf{J}+\boldsymbol{\lambda} \mathbf{C}$ with respect to all $\underline{n}$ of the now independent variables $\mathbf{x}_{\mathbf{1}}, * * *, \mathbf{x}_{\mathbf{n}}$ whereby we have $n$ critical conditions:

$$
\left.\begin{array}{l}
\frac{\partial \mathrm{I}}{\partial \mathrm{x}_{1}}=\mathrm{F}_{\mathrm{x}_{1}}+\lambda C_{x_{1}}=0 \\
\frac{\partial \mathrm{I}}{\partial \mathrm{x}_{\mathrm{n}}}=F_{x_{n}}+\lambda C_{x_{n}}=0
\end{array}\right\} \text { or }-\lambda=\frac{F_{x_{1}}}{C_{x_{1}}}=\cdots=\frac{F_{x_{n}}}{C_{x_{n}}}
$$

plus the restriction $\mathbf{C}\left(\mathbf{x}_{\mathbf{1}}, \cdots, \mathbf{x}_{\mathrm{n}}\right)=0$ to determine the $\underline{n}$ critical values of the $\underline{n}$ independent variables plus the now determined value of $\overline{\lambda,}$ as thus specified by the boundary condition, $\mathrm{C}=0$.

This Lagrange method came into mathematical recognition contemporaneously with the birth of the calculus of variations and is tied up with its history as part of Euler's rule for the isoperimetric case. The gist of it is that it provides the technique whereby all position coordinates may be treated as independent, as here, replacing (8) by the equivalent (12) and (13).

## 4. The Critical Equations and Their Solution

When we apply the Lagrange method to (3) we use a minus sign between $\mathbf{I}$ and $\underline{C}$ for convenience. Then

$$
\begin{equation*}
I=J-C=\int \frac{\sqrt{d x^{2}+d y^{2}}}{\sqrt{y}}-\int_{(0,0)}^{\left(x_{n}, y_{n}\right)} c_{1} d x+c_{2} d y \tag{15}
\end{equation*}
$$

where C is the restricting condition on the n independent, according to the Lagrange method, variables $d x=A x$ and $\mathrm{dy}=\Delta \mathrm{y}$ which here replace the $\mathrm{x}_{1}, \cdots, \mathbf{x}_{\mathrm{n}}$ of (13). The undetermined multipliers $\lambda_{1}$ and $\lambda_{2}$ are included in the small c's. Here C is evidently the definite integral condition whereby fixed end points $(0,0)$ and $\left(X_{n}, y_{n}\right)$ are specified.

The partial derivative with respect to $d x=A x$, taken here at the outer end of $\underline{x}$ is the same as for $\underline{x}$, and, when set equal to zero, gives

$$
\begin{align*}
& \frac{\partial I}{\partial x}=\frac{\partial J}{\partial d x}-\frac{\partial C}{\partial d x}=\frac{d x}{\sqrt{y\left(d x^{2}+d y^{2}\right.}}-c_{1}=0  \tag{16}\\
& \frac{\partial J}{\partial x}=\frac{\partial J}{\partial d x}=\sqrt{\frac{1}{\left.y^{\left(1+y^{\prime 2}\right.}\right)}=c_{1}}
\end{align*}
$$

Besides being differential equations in the usual sense, equations (16) are equivalent to, and replaceable by $\underline{n}$ equations that determine the $n$ critical values of the $n d x_{i}^{-}=\Delta \mathbf{x}_{\mathbf{i}}$, finite variables that approach zero in the limit in the usual
way as they increase in number, with their sum or integral always equal to $\underline{x}$. Thus equations (16) here are a special case of the more general (13).

We may note in passing that (16) is a first integral of Euler's critical condition for extrema in the calculus of variations, with $\mathbf{c}_{\boldsymbol{1}}$ for the constant of integration. To see this, write

$$
J=\int f\left(x, y, y^{\prime}\right) d x=\int \frac{\sqrt{1+y^{\prime 2}}}{\sqrt{y}} d x ; f\left(x, y, y^{\prime}\right)=\frac{\sqrt{1+y^{\prime 2}}}{\sqrt{y}}(17)
$$

Then, using this value of f , we have (16) in the form (18) where the subscripts indicate partial differentiation.

$$
\begin{equation*}
\frac{\partial J}{\partial x}=\frac{1}{\sqrt{y\left(1+y^{\prime 2}\right)}}=f-y^{\prime} f_{y^{\prime}}=c_{1} \tag{18}
\end{equation*}
$$

whose total derivative with respect to $¥$, taken along the path of integration, where $y^{\prime}=\frac{d y}{d x}$ and $y^{\prime \prime}=\frac{d y^{\prime}}{d x}$, gives:

$$
\left.\begin{array}{c}
-\frac{d}{d y} \frac{\partial J}{\partial x}=-\frac{1}{y^{\prime}} \frac{d}{d x} \frac{\partial J}{\partial x}=\frac{-1}{y^{\prime}}\left[f_{x}+y^{\prime} f_{y}+y^{\prime \prime} f_{y^{\prime}}\right.  \tag{19}\\
\left.-y^{\prime \prime} f_{y^{\prime}}-y^{\prime} \frac{d}{d x} f_{y^{\prime}}\right]=\frac{d}{d x} f_{y}-f_{y}=0
\end{array}\right\}
$$

Thus (16) appears as the first integral of Euler's classical condition (19) first published in his famous memoir of 1744. It is of interest that John Bernoulli, who died in 1748, was still alive at this time and that this basic landmark of the calculus of variations is contemporary with him and may be due in part to the impetus toward generalization in calculus of variations methods given by the brachistochrone discussions.

From (16) we have:

$$
\left.\begin{array}{c}
\sqrt{\mathrm{y}\left(1+\mathrm{y}^{\prime 2}\right)}=\frac{1}{\mathrm{c}_{1}}=\sqrt{2 \mathrm{r}} \\
\mathrm{y}^{\prime}=\sqrt{\frac{2 \mathrm{r}-\mathrm{y}}{\mathrm{y}}} ; \mathrm{dx}=\sqrt{\frac{\mathrm{y}}{2 \mathrm{r}-\mathrm{y}}} \mathrm{dy}
\end{array}\right\}\left\{\begin{array}{l}
\frac{1}{\mathrm{c}_{1}^{2}}=2 \mathrm{r}=\begin{array}{l}
\text { Diameter } \\
\text { of the wheel. }
\end{array} \tag{20}
\end{array}\right.
$$

Under the substitutions:

$$
\left.\begin{array}{rl}
y & =r(1-\cos \theta) \\
2 r-y & =r(1+\cos \theta)  \tag{21}\\
d y & =r \sin \theta d \theta)
\end{array}\right\}
$$

we find:

$$
\left.\begin{array}{rl}
d x & =r \sqrt{\frac{1-\cos 0}{1+\cos \theta}} \sin \theta d \theta=2 r\left(\sin \frac{\theta}{2}\right)^{2} d \theta=r(1-\cos \theta) d \theta \\
x & =r(\theta-\sin \theta) \\
y & =r(1-\cos \theta) \\
y^{\prime} & =\frac{d y}{d x}=\frac{\sin \theta}{1-\cos 0}=\cot \frac{\theta}{2}=\sqrt{\frac{2 r-y}{y}}=p(x, y)
\end{array}\right\}
$$

The constant of integration is zero because the curve starts from the origin where $\mathbf{x}=0$ as 0 well as $\underline{y}$ and $\underline{V}$ and hence also 0 . The equations (22) are seen by Fig. 3 to represent a cycloid. The abscissa $x$, is the distance, $\mathbf{r} \theta$, which the wheel has rolled from the origin minus the projection, $r \sin \theta$, of its radius on the x-axis. And the ordinate $\underline{y}$ is the radius $\mathbf{r}$, plus (for 0 in the second quadrant) its projection on the $y$-axis.

From (2) and (22) we have for the length element and the velocity:

$$
\begin{align*}
& \mathrm{ds}^{2}=\mathrm{d} x^{2}+\mathrm{dy}^{2}=2 \mathrm{r}^{2}(1-\cos 0) \mathrm{d} \theta^{2}  \tag{23}\\
& \mathrm{~V}^{2}=2 \mathrm{gy}=2 \mathrm{gr}(1-\cos \theta)
\end{align*}
$$

Hence for (3) we have

$$
\begin{equation*}
T=\int \frac{d s}{V}=\sqrt{\frac{r}{g}} \theta=\frac{J}{\sqrt{2 g}} \tag{24}
\end{equation*}
$$

showing the minimum time as $\sqrt{r / g}$ times the angular range of the auxiliary circle or wheel which rolls under the $x$ axis as in Fig. 3, while the point ( $\mathbf{x}, \mathbf{y}$ ) on its rim traces the cycloid (22) which is the Brachistochrone curve for this problem.

## 5. The Brachistochrone and the Tautochrone

John Bernoulli was greatly delighted to find that the brachistochrone was the same cycloid as Huygens' tautochrone (path of equal times), or curve down which a pendulum bob must swing if its period of vibration is to be independent of the amplitude of the swing. He says, "You will be petrified with astonishment when I say that precisely this cycloid, the tautochrone of Huygens is our brachistochrone." Later he adds: "Before I conclude, I cannot refrain from again expressing the amazement which I experienced over the unexpected identity of Huygens' tautochrone and our brachistochrone. Furthermore, I think it is noteworthy that this identity is found only under the hypothesis of Galileo, so that even from this we may conjecture that nature wanted it to be thus."

We may show that our cycloid is the path of equal time or period, no matter what the amplitude, or length of arc through which the pendulum bob swings. To do this, first compute the minimum time T of (24) from the position of rest at the origin to the lowest point on the cycloid whose coordinates are ( $\pi \mathbf{r}, 2 \mathrm{r}$ ) as in Fig. 3.

$$
\begin{equation*}
T=\frac{P}{4}=\sqrt{\frac{r}{g}} \int_{0}^{\pi} d \theta=\pi \sqrt{\frac{r}{g}} \tag{25}
\end{equation*}
$$

Here $T$ is set equal to $\mathbf{P} / 4$ because this time includes only a quarter period for a particle if allowed to swing up on the right of the lowest point at $x=T I T$ to where the velocity is zero again on the $x$-axis and then back again through the lowest point $s=\pi r$ and up again to the original position of rest at the origin.

As pointed out by John Bernoulli, this tautochrone problem is again a problem in mechanics where the height of fall is proportional to the square of the acquired velocity.

Hence instead of (2) we have:

$$
\begin{equation*}
\mathrm{V}^{2}=2 g\left(\mathrm{y}-\mathrm{y}_{0}\right) \tag{26}
\end{equation*}
$$

where $2 \mathrm{gy}_{\mathbf{0}}$ comes in from (1) as the constant of integration, and $y^{-} y_{0}$ appears in Fig. 4 as the general height of fall that, in the absence of friction, gives the velocity (26) at any height y - Yo below the starting level at yo. This cycloid of Fig. 4 is the same (22) as for Fig. 3 so that ds is again given here by (23), and if we introduce the halfangle formula of trigonometry, we obtain from (23) and (26) above:

$$
\begin{align*}
& \mathrm{ds}=2 \mathrm{r} \sin \frac{0}{2} \mathrm{~d} \theta \\
& \begin{aligned}
& \mathrm{V}^{2}=2 \mathrm{gr}\left(1-\cos \theta-1+\cos \theta_{0}\right)=4 \mathrm{gr}\left(\sin ^{2} \frac{\theta}{2}-\sin ^{2} \frac{\theta_{0}}{2}\right) \\
&=4 \mathrm{gr}\left(\cos ^{2} \frac{\theta_{0}}{2}-\cos ^{2} \frac{\theta}{2}\right)
\end{aligned}
\end{align*}
$$

and hence the time of descent along the cycloid is:

$$
\begin{align*}
& T=\frac{P}{4}=\int \frac{d s}{V}=\sqrt{\frac{r}{g}} \int_{\theta_{0}}^{\pi} \frac{\sin \frac{\theta}{2} d \theta}{\sqrt{\cos ^{2} \theta_{0}-\cos ^{2} \theta} 2}  \tag{28}\\
& =-2 \sqrt{\frac{r}{g}} \int_{u_{0}}^{0} \frac{d u}{\sqrt{u_{0}^{2}-u^{2}}}=2 \sqrt{\frac{r}{g}} \int_{0}^{s_{0}} \frac{d s}{\sqrt{s_{0}^{2}-s^{2}}}=\pi \sqrt{\frac{r}{g}}
\end{align*}
$$

which is seen to be the same identical quarter period as (25) and here obtained as the time of descent from any $\mathrm{y}_{0}=\mathbf{r}\left(1-\cos \boldsymbol{\theta}_{\mathbf{0}}\right)$ to $\mathrm{y}=2 \mathrm{r}$ at the lowest point of equilibrium where $\boldsymbol{\theta}=I T$ and $u=\cos \frac{0}{2}=0$, whereby we have checked the fact that the quarter period and hence also the period $P$ is the same no matter what height $2 \mathrm{r}^{-} \mathrm{y}_{0}$ along the cycloid is fallen through during the pendulum's swing. Hence the cycloid (22) is the tautochrone, or path of equal periods as well as the brachistochrone.


Fig. 4
The Cycloid with Radius of Curvature $\mathbf{4 r} \sin \frac{\boldsymbol{\theta}}{\mathbf{2}}$ and the Circle of Radius 4 r for which the Simple Pendulum's Period is the same as for the Cycloid as Tautochrone.

On the right of (28) we have introduced from (29) the displacement $\mathbf{s}$ along the cycloid (22) measured from the lowest position where $0=I T$ and $\mathbf{s}=0$ according to (29), $\mathbf{s}=-4 \mathrm{ru}=-4 \mathrm{r} \cos {\underset{2}{\mathrm{~A}}}_{\tilde{2}}$. Here $\mathbf{s}_{\mathbf{0}}$ is negative on the left of the pendulum's equilibrium position where $0=\pi$ and $\mathbf{s}=0$ and positive on the right, but the time integral (28) is always positive with the sign of the radical taken so that ds and $\mathrm{V}=\mathrm{ds} / \mathrm{dt}$ have the same sign. Note also that the differential equation in $\underline{s}$ and $\underline{t}$ that is integrated by (28) shows simple harmonic motion, as in the next section:

$$
\left.\begin{array}{r}
\frac{d s}{\sqrt{s_{0}^{2}-s^{2}}}=\sqrt{\frac{g}{4 r}} \frac{d s}{V}=\sqrt{\frac{g}{4 r}} d t=\omega d t \\
s=s_{0} \sin \omega t=s_{0} \sin \sqrt{\frac{g}{4 r}} t
\end{array}\right\} \omega^{2}=\frac{g}{4 r}
$$

6. The Tautochrone has Simple Harmonic Motion as for a Simple Pendulum of Length $4 r$.

The extent of the displacement $s$ along the cycloidal path from its lowest position of equilibrium is given for the tautochrone by integrating ds of (27):

$$
\begin{equation*}
s=2 r \int_{\pi}^{\theta} \sin \frac{0}{2} d \theta=-4 r \cos \frac{9}{2} \tag{29}
\end{equation*}
$$

To find the acceleration along the path, we note from (27) that

$$
\begin{align*}
V & =2 \sqrt{g r} \sqrt{\cos ^{2} \frac{\theta_{0}}{2}-\cos ^{2} \frac{\theta}{2}} \\
d V & =\sqrt{g r} \frac{\sin \frac{0}{2} \cos \frac{0}{2} d \theta}{\sqrt{\cos ^{2} \frac{\theta_{0}}{2}-\cos ^{2} \frac{\theta}{2}}} \tag{30}
\end{align*}
$$

and from (28)

$$
\begin{equation*}
\mathrm{dt}=\sqrt{\frac{r}{g}} \frac{\sin \frac{\theta}{2} d \theta}{\sqrt{\cos ^{2} \frac{\theta_{0}}{2}-\cos ^{2} \frac{\theta}{2}}} \tag{31}
\end{equation*}
$$

From (30) and (31) we have the acceleration as the ratio of $d V$ to dt, and taking account of (29):

$$
\begin{equation*}
\frac{d^{2} s}{d t^{2}}=\frac{d V}{d t}=g \cos \frac{\theta}{2}=-\frac{g s}{4 r}=-\left(\frac{2 \pi}{p}\right)^{2} s=-\omega^{2} s \tag{32}
\end{equation*}
$$

whereby the acceleration along the path is shown proportional to the displacement but opposite in sign such as characterizes simple harmonic motion, where the proportionality factor is the square of the angular velocity $\omega=2 \pi / \mathbf{P}$ of the point on the reference circle whose projection on the axes gives simple harmonic motion. Hence the period is

$$
\begin{equation*}
P=2 \pi \sqrt{\frac{4 r}{g}}=2 \pi \sqrt{\frac{1}{g}} \tag{33}
\end{equation*}
$$

which by comparison with theperiod formula for the simple pendulum of length 1 , shows that the period is the same as
for a simple pendulum of length 4 r and this formula (33) is seen to check (25).

## 7. Fields of Extremals First Given for the Brachisto-

 chrone by John BernoulliAfter his discovery of the brachistochrone, John Bernoulli's bent for generalization impelled him to the consideration of families of cycloids all passing through the same vertex, here the origin, or starting point from rest along which a frictionless particle will descend in minimum time to any other point in the lower half plane.

Such a family of non-intersecting minimizing curves we now call a field of extremals, the latter term referring to solutions of Euler's condition. The integral which is minimized when taken along these paths is a line integral, because dependent in general upon the path of integration, and the integrand of such integrals is a vector $R$, whose two components in the plane are the corresponding two partial derivatives of $\mathbf{J}$ in the direction of the reference axes; $R=i\left(f{ }^{-} y^{\prime} f_{y^{\prime}}\right)+j f_{y^{\prime}}$. The first of these partial derivatives is shown by (16) and (18) above:

$$
\left.\begin{array}{l}
J=\int M\left(x, y, y^{\prime}\right) d x+N\left(x, y, y^{\prime}\right) d y=\int\left(f-y^{\prime} f y^{\prime}\right) d x  \tag{34}\\
+f y, d y \equiv \int \frac{d x^{2}+d y^{2}}{\sqrt{y}} \equiv \int \frac{d x}{\left.\sqrt{y\left(1+y^{\prime 2}\right.}\right)}+\frac{y^{\prime} d y}{\left.\sqrt{y\left(1+y^{\prime 2}\right.}\right)}
\end{array}\right]
$$

When the cycloidal slope $\mathbf{p}(\mathbf{x}, \mathbf{y})$ of (22) replaces $\mathbf{y}^{\prime}$ in (34) we have the Hilbert invariant integral of $\mathbf{I}$.

$$
\begin{align*}
H=J(\min ) & =\int \frac{d x}{\sqrt{y\left(1+p^{2}\right)}}+\frac{p d y}{\sqrt{y\left(1+p^{2}\right)}}=\int \frac{\theta d r}{\sqrt{2 r}}+\sqrt{2 r} d \theta  \tag{35}\\
M_{\theta} & =\frac{1}{\sqrt{2 r}}=N_{r}
\end{align*}
$$

shown here referred to the curvilinear coordinates of Fig. 5, as well as $\underline{x}$ and $y$. The transformations that give the right member come from (20) and (22), noting that

$$
\left.\begin{array}{l}
\sqrt{y\left(1+p^{2}\right)}=\sqrt{2 r} \\
d x=\operatorname{dr}\left(O^{-} \sin 0\right)+r d \theta(1-\cos 0) \\
d y=d r\left(1^{-}-\cos \theta\right)+r d \theta \sin 0
\end{array}\right\}
$$



Fig. 5
The Field of Cycloids as Curvilinear Coordinates
These curvilinear coordinates r and $\boldsymbol{\theta}$ are shown in Fig. 5 as the family of cycloids, each of which, $r=$ constant is intersected just once by each of the straight lines $\boldsymbol{\theta}=$ constant in its quadrant and having slope, $\tan \phi=\mathbf{y} / \mathbf{x}$ $=\left(1^{-} \cos \boldsymbol{\theta}\right) /(0 \cdot \sin \boldsymbol{\theta})$, whereby each pair $(\mathbf{r}, \boldsymbol{\theta})$ locates uniquely any point in the lower half plane, just as ( $\mathbf{x}, \mathrm{y}$ ) do. This is characteristicof fields of extremals, as shown here with $\underline{r}$ for the family parameter of this field of extremal cycloids, non-intersecting among themselves except where they touch at the origin or vertex.

The first of (35) may also be shown* invariant by introducing $\mathrm{p}=\sqrt{(2 \mathrm{r}-\mathrm{y}) / \mathrm{y}}$ of (22) and using the partial derivatives $\mathbf{r}_{\mathbf{x}}$ and $\mathbf{r}_{\mathbf{y}}$ given by the familiar equation for the cycloid, $\mathrm{x}_{\mathrm{x}}=\mathrm{r} \operatorname{arc} \cos (\mathrm{r}-\mathrm{y}) / \mathrm{r}-\sqrt{2 \mathrm{ry}-\mathrm{y}^{2}}$.

The invariant character of (34), using $p$ of (22) for $y^{\prime}$, illustrates the Beltrami identity in operation. According to this, when $\mathbf{y}^{\prime}=\mathbf{y}^{\prime}(\mathbf{x}, \mathrm{y})$ then

$$
\begin{equation*}
N_{x}-M_{y} \equiv \frac{d}{d x} f_{y^{\prime}}-f_{y} \tag{37}
\end{equation*}
$$

and when $\mathbf{y}^{\prime}=\mathbf{p}(\mathbf{x}, \mathbf{y})$, given by solving Euler's condition, then (37) will vanish and thus (34) becomes (35) an invariant

[^0]or Hilbert integral. This Beltrami identity (37) is one of the landmarks of the calculus of variations. It was discovered by Beltrami in 1868 and thirty years later rediscovered and applied by Hilbert as above.

## 8. Transversal Curves for the Brachistochrone First Given by John Bernoulli as Synchrones

Transversal curves are those which satisfy the transversality condition:
$\left(f{ }^{-} y^{\prime} f_{y^{\prime}}\right) \boldsymbol{\delta x}+\mathrm{f}_{\mathrm{y}^{\prime}} \boldsymbol{\delta} \mathbf{y}=\mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{y}) \boldsymbol{\delta} \mathrm{x}+\mathrm{N}\left(\mathrm{x}, \mathrm{y}, \mathrm{y}^{\prime}\right) \boldsymbol{\delta} \mathrm{y}=0$
seen above as the condition for perpendicularity between the transversals of slope $\delta \mathbf{y} / \boldsymbol{\delta} \mathrm{x}$ and the vector integrand of direction $\tan \beta=\mathbf{N} / \mathbf{M}$. This latter, however, is unique for a Hilbert integral equal to $\mathrm{J}(\mathrm{min})$ along extremals of a field like the cycloids of Fig. 5, so that a unique family of transversals is indicated.

Also, integrals of the type:
$J=\int Q(x, y) d s \equiv \int Q(x, y)\left(\frac{d x}{\sqrt{1+y^{\prime 2}}}+\frac{y^{\prime} d y}{\sqrt{1+y^{\prime 2}}}\right)=\int M d x+\operatorname{Ndy}(39)$
are seen to have a vector integrand in the same direction as the path of integration of slope $y$ ':

$$
\begin{equation*}
\tan \beta=\frac{\mathrm{N}\left(\mathrm{x}, \mathrm{y}, \mathrm{y}^{\prime}\right)}{\mathrm{M}\left(\mathrm{x}, \mathrm{y}, \mathrm{y}^{\prime}\right)}=\mathrm{y}^{\prime}=\mathrm{p}(\mathrm{x}, \mathrm{y}) \tag{40}
\end{equation*}
$$

AS $J$ for our problem is a case of (39) with $Q(x, y)=1 / \sqrt{y}$, we see that for the brachistochrone the transversality condition (38) is the perpendicularity condition between field extremals of slope $p$ and transversals of slope $\boldsymbol{\delta} y / \boldsymbol{\delta} x$.

When $\mathbf{J}$ is integrated in a direction at right angles to its vector integrand, the integral will be zero, and hence, with no change in $\mathbf{J}$ along these transversal curves, the latter are loci of constant time, $T=J / \sqrt{2 g}$. It is this aspect of the problem that interested John Bernoulli who sought the points which a particle falling from rest at the origin will reach in the same time as along other cycloidal paths from the vertex at the origin. Hence, he called his transversal curves synchrones, or curves of the same time; that is, loci of points to be reached in the same time by particles
falling from the origin. This is the first historic illustration of Kneser's Transversal Theorem, a generalization to the effect that extrema of calculus of variations integrals or Hilbert integrals are always the same if taken anywhere between two given transversal curves or surfaces. This follows because transversal curves are always normal, by the transversality condition, to the vector integrand which, hence, gives a zero value if integrated along these curves. See Kneser, Lehrbuch der Variations-rechnumg, 1900, p. 48.

Since $\underline{T}$ and $\underline{J}$ are constant along these transversal curves, we easily obtain their equations in curvilinear coordinates by setting (24) equal to a constant:

$$
\begin{equation*}
\mathbf{r} \boldsymbol{\theta}^{\mathbf{2}}=\text { constant } \tag{41}
\end{equation*}
$$

which gives the equation of the family of transversals in about as simple a form as could be asked for in terms of the curvilinear coordinates of Fig. 5, i.e., $\underline{r}$ and $\boldsymbol{\theta}$ of the cycloid's auxiliary circle. We may graph a particular transversal of (41) for any particular time (24), say, T $=\pi \sqrt{r_{0} / g}$, of fall to the lowest point $\left(\pi r_{0}, 2 r_{0}\right)$ on the cycloid of parameter $r_{0}$, shown in Fig. 6. Then (41) becomes:

$$
\left.\begin{array}{rl}
r \theta^{2} & =r_{0} \pi^{2}=m \pi  \tag{42}\\
m & =\pi r_{0}=x(\max )
\end{array}\right\}
$$

In graphing (42), we recall from Fig. 5 that all points in the lower half-plane are given as $\theta$ ranges from $-2 \pi$ to $+2 \pi$, while $\underline{r}$ ranges from $\underline{0}$ toinfinity. Hence, the smallest value of $\underline{r}$ occurs where $\theta^{-}=2 \pi$ in (42) and the cycloidal coordinate curve meets the $x$-axis:

$$
\left.\begin{array}{l}
r(\min )=\frac{m}{4 \pi}=\frac{r_{0}}{4}  \tag{43}\\
x=2 \pi r=\frac{m}{2} ; y=0
\end{array}\right\}
$$

To find where (42) cuts the y -axis we need (42) in parametric form, giving $x$ and $y$. Thus replace $\tau$ of (22) by $\mathbf{x}$ of (42) to obtain:

$$
\left.\begin{array}{l}
\mathrm{x}=\frac{\pi \mathrm{m}}{\theta^{2}}(\theta-\sin 0)=\pi \mathrm{m}\left(\frac{1}{0} \cdot \frac{\sin \theta}{\theta^{2}}\right) \\
\mathrm{y}=\frac{\pi \mathrm{m}}{\theta^{2}}(1 \cdot \cos 0)
\end{array}\right\}
$$

These are the parametric equations of the transversals for the field of cycloids from the origin, that is, Bernoulli's synchrones. For $\boldsymbol{\theta}=0$, both of (44) are indeterminate forms which evaluate in the usual way to $\left(0, \mathrm{~m} \frac{\pi}{2}\right)$, showing that (42) and (44) cut the $y$-axis at distance $\pi / 2$ times $m$, the maximum $\mathbf{x}$ which is here the family parameter of the synchrones, (44).

This $¥$-intercept $\mathrm{m} \frac{\mathrm{IT}}{2}$ must also be the distance covered by a freely falling body in the time, IT $\sqrt{r_{0} / g}$ in question given by (24) because with $\theta=0$ and $\mathbf{r}=$ infinity, the auxiliary circle for the cycloidal coordinate becomes a straight


Fig. 6
Two of Bernoulli's Synchrones, (41), (42) or (44), transversal to the field of cycloids (22). The time for each synchrone is $\pi \sqrt{\mathrm{r}_{0} / \mathrm{g}}=\mathrm{P} / 4$ of (25), the same as to ( $\pi \mathrm{r}_{0}, 2 \mathrm{r}_{0}$ ), lowest point on the cycloid of parameter $r_{0}$. The range $\underline{R}$ between $\mathbf{x}(\max )$ and $\mathbf{x}(\min )$ is $2 \pi r_{0}$, the same as for the $\mathbf{c y} \mathbf{y}^{-}$ cloid, and their maximum $y$ is the same as for freely falling bodies, and is $\pi / 4$ times this range $R$, that is $y(\max )=$ $g t^{2} / 2=\pi^{2} r_{0} / 2=2 \pi^{2} r_{0} / 4$.
line, or circle of infinite radius, coincident with the $y$ axis, with no deflection from the vertical. Thus the synchrones of the brachistochrone problem again check the mechanics of freely falling bodies:

$$
\begin{equation*}
\mathrm{y}=\frac{1}{2} \mathrm{~g} t^{2}=\frac{1}{2} \mathrm{~g} \pi^{2} \frac{\mathrm{r}_{0}}{\mathrm{~g}}=\left(\pi \mathrm{r}_{0}\right) \frac{\pi}{2}=\mathrm{m} \frac{\pi}{2}=\frac{\pi}{4}\left(2 \pi r_{0}\right) \tag{45}
\end{equation*}
$$

That the family (44), or (42), actually comprises the orthogonal trajectories of the field of extremals (22) is easy to show by finding the slope of (44),

$$
\left.\begin{array}{c}
d x=\frac{\pi m}{\theta^{3}}\left(2 \sin 0^{-} \theta(1+\cos 0)\right) d \theta  \tag{46}\\
d y=\frac{\pi m}{\theta^{3}}(\theta \sin \theta+2 \cos \theta-2) d \theta \\
\frac{d y}{d x}=-\frac{1-\cos \theta=-\tan ,}{\sin 0}
\end{array}\right\}
$$

showing slopes for (44) that are negative reciprocals of those of (22). Here the ratio of dy to dx above is shown equal to $\boldsymbol{z}^{\prime}$ of (46) with the help of the identity:

$$
\frac{\sin 0}{1-\cos 0}=\frac{1+\cos 0}{\sin 0}=\sqrt{\frac{1+\cos 0}{1-\cos 0}}
$$

The equations of the synchrones (44) are of special interest, being, it is believed, new to science. The field parameter of the field of cycloids is $\mathbf{r}$, the radius of each one's auxiliary circle. Ten of these cycloids are here shown by Fig. 6, having the five parameters for the two quadrants, $\mathrm{r}=\mathrm{r}_{0} / 4 ; \mathrm{r}_{0} / 2 ; \mathrm{r}_{0} ; 2 \mathrm{r}_{0} ; 4 \mathrm{r}_{0}$, drawn to scale. These intersect the larger synchrone at the ten points where the parametric coordinates are $\left(r_{0} / 4, \pm 2 \pi\right), r_{0} / 2$, $\pm \pi \sqrt{2}),\left(\mathrm{r}_{0}, \pm \pi\right),\left(2 \mathrm{r}_{0}, \pm \pi / \sqrt{2}\right.$ and $\left(4 \mathrm{r}_{0}, \pm \pi / 2\right.$. The cartesian coordinates of these points of intersection are given by the introduction of the above values of 0 in the synchrone equations (44), and are seen to be the same $(\mathbf{x}, \mathbf{y})$ as when these values of $\theta$ are introduced into the cycloid equations (22), using the corresponding r's. The cartesian coordinates of the six points of intersection that do not include radicals are: $(\mathbf{x}, \mathrm{y})= \pm \mathrm{m} / 2,0),( \pm \mathrm{m}$,
$\left.2 r_{0}\right),\left( \pm 4 r_{0}(\pi / 2-1), 4 r_{0}\right)$. Since all points in the halfplane are given as $\mathbf{O}$ ranges from $\mathbf{O}$ to $\pm 2 \pi$ with $\underline{\mathbf{r}}$ ranging from zero to infinity, any values of 0 greater than 2 r will be trivial for the Brachistochrone problem and its field of extremals. But as a mathematical curiosity it may be noted that the synchrones (44) under increase of 0 past $\pm 2 \pi$, crowd into the corner at the origin. Thus even multiples of $\pi$ for 0 give always $y=0$ and, by (46), a horizontal tangent, and $x=m / 4, m / 6, m / 8, \cdots$, for $\theta=4 \pi, 6 \pi$, $8 \pi, \cdots$. Odd multiples of $\pi$ for 0 in (44) give: $(x, y)$ $=\left(m / 3,2 r_{0} / 9\right),\left(m / 5,2 r_{0} / 25\right),\left(m / 7,2 r_{0} / 49\right), \cdots$, for $0=3 \pi, 5 \pi, 7 r, \cdots$, with a vertical tangent by (46) at these relative maximum points for $y$, being cusps where from (46) $\mathbf{y}^{\prime \prime}=d y^{\prime} / \mathbf{d x}$ also is infinite.

John Bernoulli devised a geometrical method of constructing his synchrones by studying the mechanical aspects of the problem. See Source Book of Mathematics by David Eugene Smith, 1929, p. 655, for Bernoulli's figure of a synchrone crossing a field of extremals. He says: "But if one had chosen to state the problem in this purely geometrical fashion: 'to find the curve which cuts at right angles, all the cycloids with common vertex'; then the problem would have been very difficult for geometers." Nowadays, however, to those of us versed in the technique of modern analysis, the problem of deriving the equation of the synchrones (42) and (44) is easy enough.

Watch for Mina Rees' article on "New Frontiers for Mathematicians" to appear soon in this journal.

## PERFECT NUMBERS*

In this discussion the word integer will mean a positive whole number such as $1,2,3,4, \cdots$.

If a positive integer R divides a positive integer N , then R is called a divisor of N . For example, the divisors of 18 are $\mathbf{1 , 2 , 3 , 6 , 9}$ and 18. An integer is said to be an improper divisor of itself. All other divisors are called proper divisors. The proper divisors of 18 are 1, 2, 3, 6 and 9 (not 18).

An integer N is said to be perfect if the sum of the proper divisors of N is N . For example: the proper divisors of 6 are 1,2 and 3 , and since 1 t 2 t $3=6$, six is a perfect number. The number 28 is a perfect number, since the proper divisors of 28 are $1,2,4,7$ and 14 ; with sum $1+2$ t 4 t 7 t $14=28$. The four smallest perfect numbers are $6,28,496$, and 8128 . The fifth perfect number is $33,550,336$. The eighth perfect number contains 19 digits.

Perfect numbers were studied by the ancient Greeks and are still of interest today. They hold their secrets well.

No one knows whether an odd integer can be perfect. None has been found, yet no one has proved they do not exist. Since 1949 it has been shown that if an odd perfect number exists, it must be greater than 10 billion and must be equal to either $12 m$ t 1 or to 36 mt 9 for some whole number m . Still no odd perfect number has been discovered.

More is known about even perfect numbers. For example, every even perfect number must end in either 28 or 6. Another interesting fact is that the sum of the reciprocals of the divisors of an even perfect number must equal 2. The perfect number 6 has divisors $1,2,3,6$, and their reciprocals total $1 \mathbf{t} \frac{1}{2} \mathbf{t} \frac{1}{3} \mathbf{t} \frac{1}{6}=2$. In a similar fashion, the divisors of 28 are $1,2,4,7,14,28$ and their recipro-
*From Fundamentals of College Mathematics, by J. C. Brixey and R. V. Andree (Holt Co.), Section 25-7. Perfect Numbers.
cals total $1+\frac{1}{2}+\frac{1}{4} \mathbf{t} \frac{1}{5} \mathbf{t} \frac{1}{18} \mathbf{t} \frac{1}{28}=2$. This fact has been proved true for evety even perfect number.

An integer $\mathbf{P}$, greater than $\mathbf{1}$, is said to be prime if its only positive divisors are $\mathbf{1}$ and $\mathbf{P}$. Examples of prime numbers are $2,3,5,7,11,13,17,19,--$. An important theorem on perfect numbers states that $2^{P-1}\left(2^{P}-1\right)$ is an even perfect number if and only if $2^{P}-1$ is prime, and that every even perfect number is of this form. If, for example, $P=2$, then $2^{2}-1=3$ is prime. Thus $2^{2-1}\left(2^{2}-1\right)=2(3)=6$ is perfect. If $\mathbf{P}=3$, then $\mathbf{2}^{\mathbf{-}} \mathbf{1}=7$ is prime, and $2^{3-1}\left(2^{4}-1\right)$ $=2^{2}(7)=28$ is perfect. If $\left.\mathbf{P}=4,2^{P}-1\right)=\mathbf{2}^{\mathbf{4}} \mathbf{1}=15$ is not prime and hence $\mathbf{P}=4$ does not lead to a perfect number.

The first four perfect numbers were discovered by the end of the first century. By 1870 only four more had been found. Between 1870 and 1950 four additional even perfect numbers were discovered. Considering all the facts and formulae known about perfect numbers, it may surprise you to learn that in the 2,000 years prior to 1951 only 12 perfect numbers had been discovered. Since then five more perfect numbers have been found, using the SWAC (giant brain) computing machine at the National Bureau of Standards Institute for Numerical Analysis at U.C.L.A. They are $2^{520}\left(2^{521}-1\right), 2^{606}\left(2^{607}-1\right), 2^{1278}\left(2^{1278}-1\right), 2^{2002}\left(2^{2203}-1\right)$, $2^{2280}\left(2^{2281}-1\right)$ This last number contains 1372 digits.

Now it is known that $N=2^{\mathrm{P}-1}\left(2^{\mathrm{P}}-1\right)$ is perfect for $\mathbf{P}=2,3,5,7,13,17,19,31,61,89,107,127,521,607,1279$, 2203 , and 2281 , and that N is not perfect for any other $\mathbf{P}$ less than 2300 . However, the principal problems, namely, "How many perfect numbers are there?" and "Do odd perfect numbers exist?" are still unsolved mysteries which await you, or one of your contemporaries.

PROBLEM DEPARTMENT<br>Edited by<br>Leo Moser, University of Alberta

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate, signed sheets within five months after publication. Address all communications concerning problems to Leo Moser, Mathematics Department, University of Alberta, Edmonton, Alberta, Canada.

## PROBLEMS FOR SOLUTION

## 73. Proposed by Victor Thebault, Tennie, Salthe, France

Construct three circles with given centers such that the sum of the powers of the center of each circle with respect to the other two is the same.

## 74. Proposed by H. Helfenstein, University of Alberta

Prove that every convex planar region of area $\pi$ contains two points two units apart.

## 75. Proposed by Leon Bankoff, Los Angeles, California

A line parallel to hypoteneuse $A B$ of a right triangle ABC passes through the incenter I. The segments included between I and the sides $A C$ and $B C$ are designated by $m$ and $n$. Show that the area of the triangle is given by

$$
\frac{m n\left(m+\sqrt{m^{2}+n^{2}}\right)\left(n+\sqrt{m^{2}+n^{2}}\right)}{m^{2}+n^{2}}
$$

76. Proposed by J. J. Dodd, Aurora, Illinois

Prove that $18!+1$ is divisible by 437.
77. Proposed by P. L. Chessin, Baltimore, Maryland

If $A$ is the $k^{\text {th }}$ positive root of $\tan x=x$, prove that

$$
\sum_{k=1}^{\infty}{\frac{1}{\lambda_{k}}}_{2}^{2}=\frac{1}{10}, \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_{k}^{4}}=\frac{1}{350}
$$

Note 1. Reprinted below are all the problems published in this journal more than one year ago and for which no solutions have as yet been received. Solutions or partial solutions for these problems would be most welcome.

## 8. Proposed by R. T. Hood, University of Wisconsin

Consider the stereographic projection of a sphere onto a plane tangent to it at its south pole $S$, the center of projection being north pole $N$. Prove that every great circle on the sphere not passing through N is mapped into a circle whose center is on the line through N which is perpendicular to the plane of the great circle.

## 9. Proposed by the Problem Editor

If the bases of a prismatoid are equal in area, then so are the sections equidistant from the midsection.

## 26. Proposed by Pedro A. Piza, San Juan, Puerto Rico

For positive integers $n$ and $c$, let the number $n$ : $\mathbf{c}$ be defined by the relation

$$
[n: c]=2^{c-1} \frac{\binom{2 n-c}{c-1}}{c}
$$

Show that the numbers $[\mathbf{n}: \mathbf{c}]$ satisfy the recurrence relation

$$
\begin{equation*}
[n: c]=\frac{2(2 n-c)}{c}[n-1: c-1] \tag{1}
\end{equation*}
$$

and the formula

$$
\begin{equation*}
\frac{2^{2 n}-1}{2 n+1}=\sum_{c=1}^{n}[n: c] \tag{2}
\end{equation*}
$$

## 37. Proposed by Victor Thebault, Ternie Sarthe, France

Find all pairs of three digit numbers M and N such that $\mathbf{M} \cdot \mathbf{N}=\mathbf{P}$ and $\mathbf{M}^{\prime} \cdot \mathbf{N}^{\prime}$ and $\mathbf{P}^{\prime}$ are the numbers $\mathbf{M} \cdot \mathbf{N}$ and $\mathbf{P}$ written backwards. For example,

$$
122 \times 213=25986
$$

and

$$
221 \times 312=68952
$$

## 50. Proposed by Pedro Piza, San Juma, Puerto Rico

Prove that the integer $2 \mathrm{n}+1$ is a prime if, and only if, for every value of $\mathbf{r}=1,2,3, \cdots,[\sqrt{n / 2}]$ the binomial


## SOLUTIONS

## 61. Proposed by C. S. Venkataraman, Varma College,

 Trichur, South IndiaProve that

$$
(n!)^{1 / n} \prod_{p \leqq n}{\underset{p}{ } \frac{1}{p-1} \leqq}_{\left(1+\frac{\pi(n)}{g(n)}\right)^{g(n)}, ~, ~}^{\text {n }}
$$

where $\pi(n)$ denotes the number of primes not exceeding $n$, and

$$
g(n)=\Sigma \frac{1}{p-1},
$$

$p$ running through all primes not exceeding $n$.

## Solution by the proposer

Let $[\mathrm{x}]$ denote the largest integer not exceeding x , and $\boldsymbol{\lambda}$ the highest power of p which divides n ! Then,

$$
\lambda=\left[\frac{n}{\mathrm{p}}\right]+\left[\frac{\mathrm{n}}{\mathrm{p}^{2}}\right]+\ldots<\mathrm{n}\left(1 / \mathrm{p}+1 / \mathrm{p}^{2}+\ldots\right)=\mathrm{n} /(\mathrm{p}-1) .
$$

Since $n!=\Pi p^{\lambda}$ it follows that

$$
n!<\prod_{p \leq n} p^{n /(p-1)}
$$

Taking the $\boldsymbol{n}^{\text {th }}$ root of both sides yields the first part of the desired inequality. To prove the second part we must make use of the following known theorem on inequalities.

If $a_{1}, a_{1}, \cdots, a_{r}$ and $b_{1}, b_{2}, \cdots, b_{r}$ are positive, then

$$
\left(\begin{array}{c}
\left.\frac{\sum_{i=1}^{r} a_{i} b_{i}}{\sum_{i=1}^{r} a_{i}}\right)^{i} \stackrel{r}{\stackrel{r}{\sum} a_{1} a_{i}} \quad \prod_{i=1}^{r} b_{i}^{a_{i}} . \\
\geqq
\end{array}\right.
$$

Taking $\mathbf{a}_{\mathbf{i}}=\mathbf{1} /\left(\mathbf{p}_{\mathbf{i}} \mathbf{- 1}\right)$ and $\mathbf{b}_{\mathbf{i}}=\mathrm{p}_{\mathbf{i}}$, this reduces to

$$
\left(\frac{\sum_{p \leq n} \frac{p}{p-1}}{g(n)}\right)^{g(n)} \geqq \prod_{p \leqq n} p^{\frac{1}{p-1}} .
$$

Since $p /(p-1)=1+1 /(p-1)$, the result follows.

## 62. Proposed by N. S. Mendelsohn, University of Manitoba

The members of a bridge club decided to hold a tournament to extend over several days of playing, but, as the club rooms were too small to hold all the members, the tournament was programmed so that only part of the members would be scheduled to play on any given night. In order to make the tournament as equitable as possible for
the players the schedule was drawn up according to the following principles:
(a) Any two of the members were scheduled to appear together at exactly one day's play.
(b) For any two days' play there was to be one member, but not more than one, who participated in both days' play.
(c) The schedule for any day's play was to include at least four players.
(d) As a tribute to the club executives, the players scheduled to play on the first day were the president, vice president, secretary and treasurer.

How many members participated in the tournament, how many days of play were scheduled and how was the schedule arranged?

## Solution by K. A. Vrons, University of Illinois

Let A, B, C and D meet the first day. Let A meet with a, ..., $a_{k}$ on a following day. Since any meeting involving $B$ must also involve precisely one of the players $A$, $\mathbf{a}_{\mathbf{1}}, \ldots, \mathbf{a}_{\mathbf{k}}$, we see that B plays exactly $\mathbf{k}+\mathbf{1}$ times. Hence A meets with exactly $k$ others whenever he meets after the first day. Similarly, let B, C, D meet with $\ell, \mathrm{m}$, n others, respectively, whenever they meet after the first day.

Since B meets with a total of $3+\mathrm{kl}$ others, we have $\mathrm{N}=4+\mathrm{k} \ell$ is the total number of players. By symmetry, $\mathrm{k}=\ell=\mathrm{m}=\mathrm{n}$, so $\mathrm{N}=4+\mathrm{k}^{2}$. But $\mathrm{a}_{1}$ meets just four times (one each with $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D ), each time with k players. Hence $N=1+4 k$. By $N=1+4 k=4+k^{2}$, we get $k=3$, hence $\mathrm{N}=13$. The number of days involved is $\mathrm{G}=1+4((13-4) / 3)$ $=13$.

Numbering the players 1, 2, ... , we may have:

| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 2 | 5 | 8 | 11 | 5 | 6 | 7 | 5 | 6 | 7 | 5 | 6 | 7 |
| 3 | 6 | 9 | 12 | 8 | 9 | 10 | 9 | 10 | 8 | 10 | 8 | 9 |
| 4 | 7 | 10 | 13 | 11 | 12 | 13 | $i 3$ | 11 | 12 | 12 | 13 | 11 |

nith olvious interpretithon.

Note: There are a number of other such configurations. This method is readily adaptable to the generalization: "Suppose $\mathrm{k}+1$ players meet the first day and at least $\mathrm{k}+1$ players each day thereafter. Then the number of players is $\mathrm{N}=\mathrm{k}+\left(\mathrm{k}^{-1}\right)^{2}$, if there exists a solution for the given k . $n$ This problem is essentially that of finding the finite projective geometries of the plane.

Also solved by the proposer.

Note 2: In consideration of the fact that the November 1954 issue of the Journal was delayed in the printing, the Problem Editor, wishing to give our readers the announced five months (or more) time to submit solutions, is withholding until the fall issue solutions of new problems proposed in the last issue.

A junior forestry stu lent 1 in a differential calculus class made the following sta ement concerning a third degree equation:
"The slope is a quad ratic equaiion. First it goes one way then the other."
(Continued from Number 1 of Volume 2, this journal)


Richard Vernon Andree Secretary-Treasurer General

RICHARD VERNON ANDREE, Assistant Professor of Mathematics at the University of Oklahoma, was born on December 16,1919, in Minneapolis, Minnesota. He attended high school in Jackson, Michigan, and then studied at the University of Chicago, receiving his B.S. degree in 1942. He received his Ph.M.in 1945 and his Ph.D. degree in 1949, in mathematics, from the University of Wisconsin.

While doing graduate work at the University of Wiscon-
*Photograph of the new editor-in-chief of this journal, together with that of his business manager (not yet appointed), will appear in the November 1955 issue.
sin, he was a teaching assistant in mathematics, 1942-46, acting instructor, 1946-49. He has been assistant professor of mathematics at the University of Oklahoma since his appointment, 1949.

Dr. Andree founded the O. U. MATHEMATICS LETTER in December, 1951, and has been its editor ever since that date. He was elected secretary-treasurer of the Oklahoma Section of the MAA in 1951, and was re-elected in 1952, 1953 and 1954.

Dr. Andree is a member of a number of professional, honorary and scientific societies, including National Council of Teachers of Mathematics, Oklahoma Educational Association, Sigma Xi, Pi Mu Epsilon, Oklahoma Anthropological Society, Oklahoma Mineral and Gem Society, American Cryptogram Association, MAA, AMS, AAAS. Though he has several fields of interest, his special interest is in the theory of numbers. He is the author of numerous articles in mathematical journals such as Mathematics Magazine, Scripta Mathematica, American Mathematical Monthly and Pi Mu Epsilon Journal. In the last mentioned of this list of journals there appeared in Vol. 1, No. 2, his article on ${ }^{\alpha}$ The Number of Solutions of the Totient Equation $\phi(x)$ $=\mathrm{B}^{\prime \prime}$.

He is co-author (with Prof. Brixey) of two books, Fundamentals of College Mathematics (1953) and Modern Trigonometry (1954).

Dr. Andree was married in 1944, has two children, and he is a member of the Presbyterian Church.
Note. Photograph and most of the above information was kindly furnished us by the O.U. Bureau of Public Relations.

ROY FRENCH GRAESSER, Professor of Mathematics and Head of Mathematics Department, University of Arizona, was born October 31, 1892, at Creighton, Nebraska. Son of Henry Lee and Ida (French) Graesser. He attended high school at Burlington, Iowa; received his A.B. degree from the University of Illinois in 1919 , his A.M. in 1922,


Roy F. Graesser Councilor General
and his Ph.D. in 1926, all from the University of Illinois. He married Lois Mizpah MacClement, August 24, 1921.

His teaching experience began with the year 1919-1920, as teacher of mathematics in high school, Sullivan, Ill.; then he joined the faculty of the University of Illinois, at Urbana, with rank of assistant professor of mathematics, which position he filled from 1920 to 1922 and from 1924 to 1926. During the two intervening years he served as a statistician. On September 1,1926, he joined the faculty of the Universityof Arizona, as assistant professor of mathematics; became associate professor in 1932; and professor and head of the department in 1938. His special fields of interest are in mathematical analysis and statistics.

Dr. Graesser holds membership in the Central Association of Science and Mathematics Teachers, American Association for the Advancement of Science, Mathematical Association of America, American Mathematical Society, Sigma Xi, Phi Kappa Phi, and Pi Mu Epsilon.

His publications include: *A Certain General Type of Neumann Expansions and Expansions in Confluent Hypergeometric Functions," Oct., 1927, American Journal of Mathematics; "The Golden Section, ${ }^{\text { }}$ in The Pentagon, Spring issue, 1944; "Models of the Regular Polyhedrons," in The Mathematics Teacher, Dec., 1945; "Some Mathematics of the Honey Comb," School, Science and Mathematics, April, 1946; "An Experimental Determination of e, ${ }^{\text {D Sch. Sci. and }}$ Math., Jan., 1947; a review of Rider's 'First Year Mathematics for Colleges, Mathematics Magazine, Jan.-Feb., 1951; "Note on the Sums of Powers of the Natural Numbers, Sch. Sci. and Math., May, 1951; note on "Integration by Parts," Sch. Sci. and Math., Nov., 1951; "An Application of the Point-of-Division Formula,' Sch. Sci. and Math., Feb., 1952; "The Date of Easter," Sch. Sci. and Math., May,

1952; review of Mood's "Introduction to the Theory of Statistics," Math. Mag., May-June, 1952; "The Quadratrix, a Simple but Remarkable Curve," Sch. Sci. and Math., Dec., 1952; "Teachers*Marks and the Fluctuation of Sampling,' Sch. Sci. and Math., March, 1953; "Technological Applications of Statistics," by Tippett, reviewed in the Math. Mag. for Nov.-Dec., 1953; "On Teaching of the Slide Rule," Sch. Sci. and Math., Dec.,1953; "The Parabola of Surety," Math. Mag., March-April, 1954.
--University of Arizona Press Bureau, Tuscon.

HERBERT STANLEY THURSTON, Professor and Head of Mathematics Department, University of Alabama, was born at Sandford, Nova Scotia, September 13, 1896. He attended elementary and high school in Nova Scotia, and after matriculation attendedthe Provincial Normal College. He holds a B.Sc. degree from Acadia University, his master's from Brown University and a Ph.D. from the University of Wisconsin.

He taught in high schools in Nova Scotia for eight years prior to 1924. He was an instructor in mathematics at Brown University. 192430. In 1930, he accepted a position in the University of Alabama, as assistant professor (1930-37), then


Herbert S. Thurston Councilor General he was associate professor (193746), after which he became professor and head of the mathematics department (1946--). He has taught in the University's Mathematics Department since that time with the exception of one year on the faculty of Biarritz American University in Biarritz, France, in 1945-46.

Dr. Thurston became a mamber of Pi Mu Epsilon in 1930. He holds membership in Sigma Xi; MAA; AMS; Cir-
colo Mathematico of Palerrno, Italy. Special fields: Matrices; p-adic numbers. A teacher of algebra, he has published several papers in American and European journals.

Dr. Thurston is a member of the Baptist Church. He lists as the members of his household a wife, a son, a daughter and two cocker spaniels. Your editor has had the very great pleasure for many years of knowing personally all the members of this delightful family (with the exception of the two spaniels).

Note. Photograph and most of the above information on Dr. Thurston kindly furnished by UNIVERSITY NEWS BUREAU, Box 2008, University, Alabama.

## A Correction

Since the publication of the Directory of Pi Mu Epsilon Fraternity, last November, we have received notice that the Oklahoma Beta chapter has a new corresponding secretary. Dr. R. B. Deal of the Mathematics Department, Oklahoma A. \& M. College, Stillwater, Okla., has succeeded Professor James H. Zant, who in that office has served so long and faithfully.

FRANZ E. HOHN
EDITOR-IN-CHIEF of PI MU EPSILON JOURNAL

Just as copy for this issue of the Journal was going to the printer a telegram from Secretary-Treasurer R. V. Andree was received confirming the unanimous election of our new Editor-in-Chief of the Pi Mu Epsilon Journal. The Fraternity has chosen for this important post Professor Franz E. Hohn of the University of Illinois. His credentials show that he is admirably well qualified to make an efficient and most successful editor of the Journal.

At the University of Illinois he holds a unique position, in that he is an associate professor both of mathematics and of electrical engineering, dividing his time between these two departments located respectively in the College of Liberal Arts and Sciences and the College of Engineering, and personifying the importance of mathematics to engineering.

He was born Sept. 4, 1915, at Warrenton, Mo., where his father was Professor of German in Central Wesleyan College. He attended Wesleyan and then McKendree College, where he received his bachelor's degree in mathematics in 1936. He received a master's degree in mathematics from the University of Illinois in 1937 and doctorate in 1940.

He taught at Illinois, University of Arizona, Guilford College in North Carolina, University of Maine, and then in 1948 returned to Illinois in his present capacity. In 1951-52 under a Ford Foundation Fellowship he studied the application of mathematics, and in 1953-54 was a research guest in the Bell Telephone Laboratories. He is writing a book on "Matrix Algebraw and is author of papers in professional journals.

He is a member of Pi Mu Epsilon and past president of the University of Illinois chapter, member of Sigma Xi, American Mathematical Society, American Mathematical

Association, American Association of University Professors, and Institute of Radio Engineers. He is married and has four children.

We feel that the Fraternity is to be congratulated on securing the services of such an able person to edit their official publication. Along with our hearty congratulations to Professor Hohn, we offer him our very best wishes for success in his new undertaking when he takes over his formal duties July 1, 1955.

[^1]
## LETTERS TO THE EDITOR AND STAFF

## "Dear Dr. Stokes:

"Dr. Frame has informed me of my election as a councillor General for the three-year term beginning July 1, 1954, . . . . .
"In view of my election it would seem fitting that I support the official journal of the fraternity more faithfully than in the past. If you can send me the issues published last year and put me on your subscription list for the current and future years, I will be glad to forward a check on receipt of the bill.
(Signed)
H. S. Thurston

23 March 1954
University of Alabama ${ }^{\text {n }}$
"Dear Professor Stokes:
"Thank you so much for sending me several copies of the issue of the Pi Mu Epsilon Journal in which Miss Taylor's generous article about me appears. I confess I am shy about letting my friends see it, but my family really delights in it. I do appreciate your kindness.
"Very sincerely yours,
(Signed)
Mina Rees
30 June 1954
Dean of Faculty, Hunter College ${ }^{\text {n }}$
"Dear Dr. Stokes:
"I appreciate receiving the November 1954 issue of The Pi Mu Epsilon Journal, as well as the previous issues which you have sent me.
"The reports of the chapters, the news notes with the other contributions help one keep up on the activities and programs of the fraternity in general.
"Sincerely yours,
(Signed)
E. H. C. Hildebrandt

31 December 1954

NOTE. The Editor would like to remind the corresponding secretary for each chapter of the importance of sending in the annual report of chapter meetings. Be sure to send a copy of your report to the department devoted to chapter activities when you send your report to the Secretary General. This department is now edited by Professor Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana. He is doing a fine job but of course his material for publication is limited to the material sent him. Last year, from all the 56 active chapters, we received less than two dozen annual reports.

Please resolve to send in your annual reports before the end of the current academic year.
"Dear Professor Farnham,
(Business Manager of the Journal):
"Thank you for your letter of March 5. It is always a pleasure to learn when our customers are well pleased. I should like to add that much of the credit is due to the preparation of master copy by Miss Hegendorfer.
"Please do continue our ad in the Journal -- the same arrangements are fine.

> "Cordially yours,

CUSHING - MALLOY, INC.
(Signed)
J. H. Malloy
B. E. Cushing, President
J. H. Malloy, Vice-President ${ }^{\text {w }}$
"Dear Professor Frame:
"I herewith tender my resignation as Business Manager of the Pi Mu Epsilon Journal, effective as of June 1955.
*As Professor Stokes is resigning as Editor-in-Chief of the Journal after the completion of work on the next issue and her successor will undoubtedly be located elsewhere, to have the Business Managership remain in Syracuse would lead to practically an impossible situation. To say the least, it will be much more convenient and efficient to choose a new Business Manager at the same location as the new Editor. Therefore, I will leave you, the new Editor, and the Fraternity the freedom of choosing a new Business Manager.
"It has been a pleasure to serve you and Miss Stokes this year.
"Sincerely yours, (Signed) Henry W. Farnham
26 February 1955
Business Manager"
"Dear Miss Stokes:
"It is hard to see an able colleague give up a post that has been loyally filled. You have done a wonderful job on bringing the Pi Mu Epsilon Journal into existence. I'm sure the entire fraternity joins me in saying thank you for your devoted service. I know it is not an easy task. I've had a similar one in bringing out the O . U . Mathematics Letter -- a much smaller publication.
"Cordially,
(Signed)
Richard V. Andree
Secretary-Treasurer General
University of Oklahoma ${ }^{\text {w }}$
"Dear Miss Stokes:
"The Department of Mathematics of the University of Oklahoma has had many requests for copies of our series
of 15 -minute radio broadcasts on mathematics. Arrangements have been made to tape record the talks. The December 1953 issue of the O. U. Mathematics Letter contains directions for securing copies of these tapes, and a list of available tapes.* If you think that such an announcement would be of interest to your readers, please feel free to use it.

4 December 1953
"Sincerely,
(Signed)
Richard V. Andree"
*Please see O. U. Math. Letter, Vol. III, No. 2, for a list of topics for seventeen of these radio broadcasts.

NOTE. At a recent meeting of the Onondago County Mathematics Teachers an announcement about the availability of these tape recordings was made. Many of those present seemed interested in using the tapes in their mathematics clubs, and they copied the address. No doubt Professor Andree will be hearing from them in the near future.

One Hundred Mathematical Curiosities. By William R. Ransom. 212 pages, $8 \frac{2}{2} \times 11$ ins., with 225 diagrams. January, 1955. Price $\$ 3.00$. Published by J. Weston Walch, Box 1075, Portland 1, Maine.

This is a book which a college undergraduate or an advanced high school student can enjoy very much. Readers on this level will meet mathematical problems that are old friends and will be rewarded by an introduction to many new ones.

Its organization is not such as to facilitate a short description, but a small selection will illustrate the book's scope. The "curiosities" include: If a hen and a half lays an egg and a half in a day and a half, how many eggs will six hens lay in seven days; the problem of the shortest path of a spider on the walls of a room; geometrical constructions; the theory of the sun dial; non-euclidean geometry; Desargue's theorem; Fermat's theorem on $x^{n}+y^{n}=z^{n}$, for $\mathrm{n}=3$; and a whole host of other things. Each problem is subjected to a careful analysis, and solutions are presented for all problems in the book except the four color problem [the publishers unwisely claim that all problems are accompanied by complete solutions].

Throughout the book very simple considerations are scattered among more complicated ones, so that the reader who finds difficulty in understanding any one problem may expect to find one of the next within the range of his mathematical maturity. However, the average difficulty of the problems does increase toward the back of the book. Thus, near- page 20 is to be found a problem on trains passing each other and one on the construction of the $20^{\circ}$ isosceles triangle, while later pages contain material such as the construction of a regular seventeen sided polygon, a discussion of the Mobius strip, and some remarks on cross ratio. With just a few exceptions, all the problems and
their solutions can be understood by students who have mastered high school algebra, geometry and trigonometry; indeed, the early problems and some others can be understood easily on the basis of ninth grade algebra. There are two that require an elementary application of the calculus and some involving the exponential function. Fortunately, however, the number $e$ and the relation $\mathbf{e}^{\mathbf{i} \theta}=\cos 6+i \sin 9$ are discussed in some detail on pages 56 to 58 , where the material is probably accessible to the really good high school student.

While I have not attempted to check all the tables and numerical computations in the book, I am happy to say that a sample proved to be correct. However, there are here, as in most books, some questionable statements and errors of exposition. The author states on page 43 that there is no closed algebraic formula for the solution of what amounts to a quartic equation; presumably the author did not have Ferrari's solution of the quartic in mind. On page 18 the author states that until 1864 no strictly geometrical method was known by which a straight line could be generated; I am certain that Euclid would have quarreled with the spirit of this statement, and I am not sure that this is what was really intended. On page 72 the quantity $\boldsymbol{r}$ is not defined until long after it is used, and on page 74 line 11 it is necessary to conclude that what is written as a property of Brocard points is really their definition. The book also contains an occasional gross error of grammatical usage. Nevertheless these objections are not really critical and it is unnecessary to list more of them. It is obvious from the spirit in which the book is written that Professor Ransom has enjoyed gathering and explaining his "Curiosities" and the reader is invited to share this pleasure.

## Erik Hemmingsen

Associate Professor of Mathematics Syracuse University

## Abstract Set Theory. By Abraham A. Fraenkel. NorthHolland Publishing Company, Amsterdam, 1953. xii + 479 pp .38 .00 florins ( $\$ 10.00$ )

Abstract Set Theory is an introduction to the theory of abstract sets (i.e., cardinal numbers, order types and ordinal numbers) intended, according to its author, "for undergraduates in mathematics, graduate students in philosophy, and high school teachers." It covers the classical material together with some important additions and also "pays attention to the foundations of the theory and to matters of principle in general, including points of logical or general philosophical interest."

We are told in the preface that "a more profound treatment of the foundations of set theory and of the contiguous fields of mathematics, including the progress of research during the last thirty years, will be given in the forthcoming book Foundations of Set Theory" due to appear about 1955. This sequel to Abstract Set Theory will deal with the antinomies of the transfinite, axiomatic methods of basing set theory, logistic attitudes from Principia Mathematica to the present day, intuitionism and neo-intuitionism, axiomatics in general and meta-mathematics.

Among the outstanding features of Abstract Set Theory should be mentioned the following: seven "principles" ${ }^{\text {sim- }}$ ilar to Zermelo's axioms, which are introduced as they are needed; two distinctly different proofs of the equivalence theorem; an interesting discussion of "infinitesimals ${ }^{\text {n }}$; two proofs of the comparability theorem, one based on order and the other based on equivalence; and a discussion of definition by transfinite induction. There are many exercises and examples and an extensive bibliography ( 132 pp .)

The book is divided into three chapters, the first dealing with the concept of cardinal numbers, the second with equivalence and cardinals, and the third with order and similarity, order-types and ordinal numbers.

I noticed seven misprints, but none of them are misleading, with the possible exception of the word "not", which should not appear in line 15 on page 63. However, the meaning in several places is ambiguous due to the use
of the word "any" rather than the word "every." For example, this occurs on page 232 in the third line of definition V , and again on page 235 in the tenth line from the bottom of the page. Compare these instances with the correct use of "any" on page 279 in the first sentence of the third paragraph. With this one very minor exception, I found Abstract Set Theory to be an unusually lucid and inspiring book, well suited for individual study as well as for use as a text. Abstract Set Theory is certainly due to become a classic.

Gary H. Meisters<br>Iowa State College


[^0]:    ${ }^{*} r_{x}=-p r_{y}=p /(p \theta-2) ;$ and $N_{x}=M_{y}=-r_{y} /(2 r)^{3 / 2}$ with $\mathbf{r}, \mathrm{p}$ and $\theta$ given according to (22).

[^1]:    *It was not possible for us to print a photograph of Professor Hohn in this issue of the Journal. It will appear in the Fall 1955 issue.

