PI MU EPSILON Journal

VOLUME 2—REPRINT NUMBER 5

CONTENTS

FALL 1956

Pi Mu Epsilon

Journal

Volume 2—Reprint Number 5

CONTENTS

Fall 1956

PI MU EPSILON JOURNAL

THE OFFICIAL UBLICATION

OF THE HONORARY MATHEMATICAL FRATERNITY

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

MANAGEMENT SCIENCE: A NEW FIELD FOR APPLIED MATHEMATICIANS*

By William Prager
Division of Applied Mathematics
Brown University

As the title indicates, my talk is concerned with a new field of scientific endeavour called "Management Science." I shall not attempt to define this term at this time; rather, I will discuss some typical problems from this field and indicate the type of analysis that is used in their solution.

When I began organizing this talk, I remembered an advertisement which I had repeatedly seen in some scientific journal. This advertisement stated that a certain company was looking for a special kind of mathematician and was willing to pay especially well for his services. It went on to specify more precisely the educational background and scientific inclinations of this special kind of mathematician and finally stated that he would work as top-level consultant to business executives. Among the scientific inclinations which the advertisement enumerated was a liking for mathematical puzzles.

Of course, no sooner had I decided that this advertisement would make an ideal starting point for this talk, than the company stopped running it, and despite several attempts I have not been able to locate the journal where I had seen it. Thus, I cannot project this advertisement onto the screen, as I should have liked to do, but I will nevertheless use it as a spring board.

What strikes us as curious in this advertisement is that a liking for, and presumably skill in dealing with mathematical puzzles should be listed as one of the qualifications of this mathematical consultant to management. Actually, this is not as striking as it may appear at first sight. Consider, for instance, the following well-known puzzle, which is attributed to Alcuin, the teacher of Charlemagne.

* Lecture given before the Brown Chapter of the Society of the Sigma Xi on April 19, 1956.

A farmer who is bringing a goat, a basket of cabbage, and of all dungs, a wolf to the market town has to cross a river using a boat that accommodates, in addition to himself, only one of his three commodities. How will he manage to ferry these across the river in view of the fact that the goat must not be left alone with either the wolf or the cabbage?

You probably have come across this puzzle before and may wonder whether the mathematical consultant could do much to help the fanner, short of trying various courses of action and discarding those that do not work, until he finally stumbles upon one that does work. While this procedure is quite adequate for the simple problem considered here, a systematic survey of all possibilities is indicated for more complex problems. In the present case, such a survey can be made as follows.

Using the initials **F**, C, G, and **W** to indicate farmer, cabbage, goat, and wolf, specify a 'state" of the system by listing the initials of those items that are on the near side of the river, and employ O to denote the desired final state in which all items are on the far side. Our task then is to make the system pass from the initial state FCGW through some intermediate states to the final state O. Disregarding all restrictions, we would have the 16 states listed in Fig. 1.

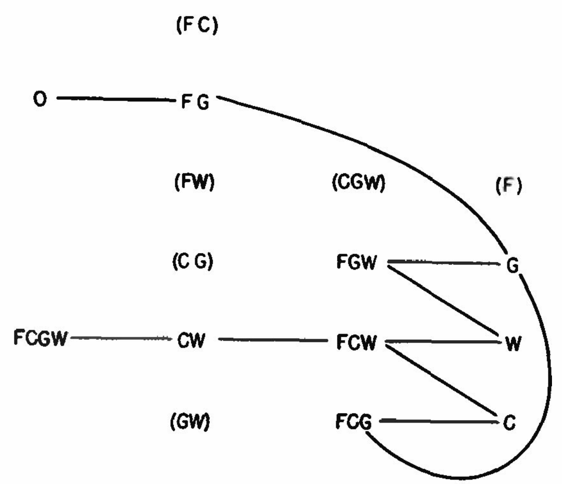


FIG. I

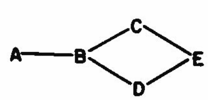
Now, states containing the letter groups CG or GW without also containing the letter F can be ruled out, because the goat would be on the near side with either the cabbage or the wolf without the supervision of the farmer. Similarly, states containing **F** without containing either **G** or **CW** can be ruled out, because the goat would then be on the far side with either the cabbage or the wolf without the supervision of the farmer. **We** see that these conditions rule out the 6 states that are enclosed in parentheses in Fig. 1.

We now connect two of the remaining states by a line if the system can be made to pass from one to the other by a single trip of the boat. Since the farmer operates the boat, states so connected must differ by the letter **F.** Also, since the farmer can transport at most one commodity at a time, states so connected differ at most by one letter in addition to the letter F. Following these rules, we obtain what the topologists call a ***linear graph*** (Fig. **1**), which has the states of the system as ***vertices*** and the possible managerial actions as ***arcs.*** This action graph shows that there are two ways of accomplishing the task, each requiring seven crossings of the river.

Linear graphs of this kind, indicating all possible actions and their results will obviously be useful in many management problems. Of course, the action graph for any real problem is likely to be much more complex than that for our artificial problem.

The following are typical questions that may be asked with respect to a given action graph. What is the number of paths leading from the given initial state to the desired final state? Are there ***essential states*** or ***essential actions*** in the sense that all solutions involve these states or actions? For the graph of our simple problem such questions can be answered by inspection, but computation may be required for a realistic action graph. For this purpose, the structure of the graph is described by a square array of zeros and ones. To each state there corresponds a row and a column of this array. If **X** and Y are two arbitrary states, a one or a zero is entered at the intersections of row X with column Y (and also at the intersection of column X with row Y) according to whether X and Y are joined by a single arc of the action graph or not. The square array of numbers obtained in this manner is called the ***structural matrix*** of the graph. Figure 2 shows a simple graph and its structural matrix.

Since the structural matrix gives a complete description of the graph, many questions regarding the graph can be answered by performing numerical operations on this matrix. As an example, we mention the computation of the ***distance matrix*** in which the number at the intersection of row X and column Y indicates the smallest number of steps by which one can pass from one of these states to the other. The structural matrix M, its square M^2, and its cube M^3 are shown in Fig. 3. Unfortunately, time does not permit me to discuss the manner in which these powers are computed; die important point is that this computation can be performed without reference to the graph represented by the matrix M. Having computed these powers, we are ready to construct the distance matrix D. Firstly, we put zeros into all cells of the main diagonal (i.e. the diagonal joining the top left and bottom right corners). We then write ones where the matrix M has ones. We next write a two into any still open cell in which the matrix M^2 has a non-zero element. Finally, we write a three into any still open cell in which the matrix M^3 has a non-zero element. In our example all cells in the distance matrix are now filled. Had there still been any open cells, we would have had to compute M^4 and write a four into any still open cell in which M^4 has a non-zero element, etc.



$$
\begin{array}{c|ccccc}
 & A & B & C & D & E \\
\hline
A & & & & & \\
B & & & & & \\
C & & & & & \\
D & & & & & \\
E & & & & &
\end{array}
\quad
\begin{array}{c}
 \\ A \\ B \\ C \\ D \\ E
\end{array}
\begin{array}{c}
\begin{matrix} A & B & C & D & E \end{matrix} \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

FIG. 2

$$
\underset{M}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}
\quad
\underset{M^2}{\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 & 2 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 \end{bmatrix}}
\quad
\underset{M^3}{\begin{bmatrix} 0 & 3 & 0 & 0 & 2 \\ 3 & 0 & 5 & 5 & 0 \\ 0 & 5 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 4 \\ 2 & 0 & 4 & 4 & 0 \end{bmatrix}}
\quad
\underset{D}{\begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 \\ 2 & 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 1 & 0 \end{bmatrix}}
$$

FIG. 3

In Alcuin's problem, each of the crossings may be assumed to take the same time. In more complicated problems, different costs in terms of time or money may be attributed to die various managerial actions, and we may wish to find the solution of smallest total cost.

Alcuin's problem may be described as a problem of transportation with rather artificial constraints. The following more realistic transportation problem has been thoroughly discussed in recent years. A homogeneous product is produced in specified amounts at several production centers and consumed in specified amounts at several consumption centers, the total consumption equalling the total production. The cost of shipping a unit amount from a given production center to a given consumption center is supposed to be independent of the amount shipped between these centers. This ***specific transportation cost*** is known for all pairs of production and consumption centers. The shipping program that minimizes the total cost of transportation is to be determined.

Let it be asserted that a specified program is optimal. How can we test this assertion?

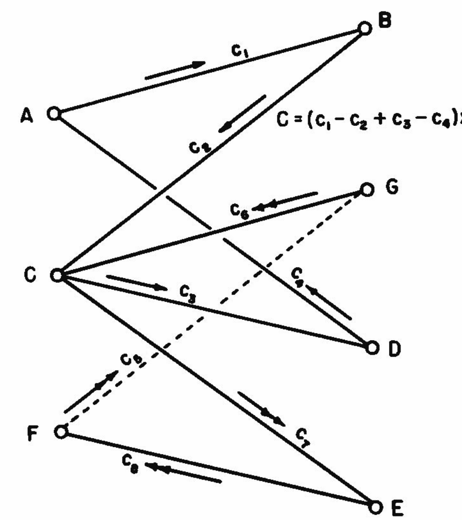


FIG. 4

The discussion of this question is facilitated by the use of a linear graph that represents the given shipping program. The vertices of this graph are arranged in two columns, the vertices on the left representing the production centers and those on the right representing the consumption centers **(Fig.4).** If the given program involves a shipment **from** a given production center to a given consumption center, these centers are joined by an arc of **the** graph.

Let us assume that this graph is connected, **i.e.,** that any two vertices are joined by a chain of arcs of the graph. We wish to find out how a change of the given program affects the total transportation cost. Suppose we try to increase the shipment **along AB** by an amount x. Since this would bring more of the product to B than will be consumed there, we must, at the same time, decrease the shipment along CB by the amount x. While we now have restored the balance at **B,** we have disturbed it at **A** and C, since more is shipped from **A** and less **from C** than is produced at these centers. We can restore the overall balance by decreasing the amount shipped along AD by x and, at the same time, increasing die amount shipped along CD by x. The total change of the program **can** then be interpreted as the superposition of a circular flow of intensity x along the circuit ABCD. The resulting change C in the total cost is

$$C = (c_1 - c_2 + c_3 - c_4)\ x,$$

where c_1, c_2, c_3, and c_4, are the specific costs for AB, CB, CD, and AD. The contents of the parentheses in the expression for **C** will be called the specific cost for die considered circuit. It is worth noting that **x** cannot be chosen arbitrarily large. Since the flows along **CB** and AD are decreased by **x** each, and since we cannot have negative shipments in any program, x must not exceed the smaller of the amounts that are shipped along **CB** and AD.

It is readily seen that any admissible change of a program amounts to the superposition of a circular flow along some circuit. Two kinds of circuits must be distinguished with respect to the given program. **A** circuit of die first kind exclusively consists of

routes that are used in the given program. **A** circuit of the second kind contains at least one route that is not used in the given program.

The circuit ABCD in Fig. 4 is of the first kind. While x has been taken as positive in the preceding discussion, there is no objection against negative values of x provided that the absolute value of x does not exceed the smaller one of the amounts that are shipped along AB and **CD.** If the specific cost for the circuit ABCD does not vanish, it is therefore always possible to choose the sign of x so that the change C in total cost is negative. In other words, the given program cannot be optimal if it contains a circuit of the first kind with non-vanishing specific cost.

Because the specific cost for any circuit of the first kind must vanish for an optimal program, it is possible to assign to each center an accounting price for the unit amount of the product in such a manner that the specific cost for any route used in the program equals the difference of the accounting prices at the consumption and production ends of the route. Since the program is supposed to have a connected graph, this rule furnishes unique accounting prices at all centers once the accounting price at an arbitrary reference center has been chosen.

Let us now consider a route that is not used in the given program, **e.g.** the route FG in Fig. 4. Since the graph of the program is supposed to be connected, it contains a chain of arcs (GC, CE, **EF)** leading from the end to the origin of the unused route. This route and this chain therefore form a circuit of the second kind. The intensity of any circular flow along this circuit that can be superimposed on the given program must be positive, because the resulting shipment along the previously unused route **FG** must be positive. If the specific cost for this circuit were negative, a decrease in transportation cost would result from the superposition of the circular flow. If the given program is optimal, the specific cost for the considered circuit must therefore be non-negative. When this condition is expressed in terms of the accounting prices at the vertices of the circuit, it is found that the difference of the accounting prices at the end and origin of the unused route cannot exceed the specific cost for this route.

In other words, the profit that could be obtained from shipping along this route does not suffice to pay for the shipping. We thus have the following result: If the given program is optimal, an accounting price can be associated with each center in such a manner that the specific cost for any route used in the program equals the difference of the accounting prices at the endpoints of the route, and the specific cost for any route not used in the program is not smaller than the difference of the accounting prices at the endpoints of the route. It can be shown that this theorem remains valid if the graph of the shipping program is not connected.

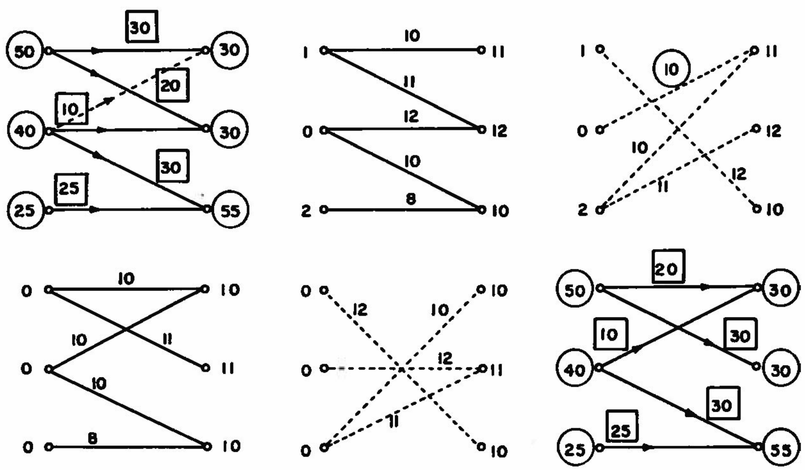


FIG. 5

Figure 5 illustrates the practical use of this theorem. There are three production centers and three consumption centers. The circled numbers in the diagram at the upper left indicate the amounts produced and consumed, and the numbers in squares indicate a feasible shipping program. In the next diagram, accounting prices are worked out from the specific transportation costs of the routes used in the program. In the diagram at the upper right, the price differentials for the unused routes are compared to the specific shipping costs for these routes. Only for one route is the price differential found to exceed the (circled) shipping cost.

To include this route into a revised program, we superimpose a circular flow involving the two topmost vertices in each column. The intensity of this flow is chosen as large as is possible without leading to negative shipments. The resulting program is shown in the lower right. Its accounting prices are worked out in the lower left, and tested in the lower center diagram. It is found that the price differentials for all unused routes exceed the shipping costs. The program at the lower right is therefore optimal.

Let us now modify our problem slightly by assuming that the transportation cost for the bottom route is 10 instead of 8. This means that the accounting price at the lower right vertex should be 0 instead of 2. Testing the unused routes we then find that for two of these routes the price differential just equals the shipping cost. This means that the program shown in the lower right can be modified by including shipments along these routes without changing the total transportation cost. For the modified problem, we therefore do no longer have a unique optimal program. This lack of a unique solution is frequently encountered in this type of problem.

Capacity restrictions for the shipping routes would not complicate the solution of our problem. For an optimal program any route used to capacity may then have a price differential that exceeds the shipping cost, because even such a favorable price differential cannot attract more traffic to a route that is already used to capacity.

We may render our problem somewhat more realistic by assuming that the total productive capacity of the plants exceeds the total demand and that the manufacturing costs differ from plant to plant. We must then decide how much to produce at each plant and how to ship from each plant in order to minimize the total cost of production and distribution. This problem may be reduced to the previous one by the following tricks. Firstly, we increase the specific shipping cost along any route by the specific manufacturing cost at its origin. This takes care of the differentials in manufacturing cost. Secondly, we add a fictitious consumption center called the "dump" and list as its demand figure the excess of total productive capacity over total demand.

This restores the balance between production and consumption. Finally, we connect each plant to the dump by a route whose specific transportation cost is set equal to zero. Indeed, the goods that are dumped in our artificial system with balanced production and consumption are not produced at all in the actual system and therefore cause neither production nor transportation costs.

It is a common experience in the physical sciences that a mathematical method proves useful far beyond the field for which it was first devised. To show that this is also true in management science, let us consider a number of problems that can be treated along the lines developed for the transportation problem.

The Quartermaster Corps must often evaluate the most economical way of awarding contracts to a number of suppliers of military equipment. The simplest situation that can arise in this connection is this: specified quantities of a certain item are needed at each of a number of depots, and each bidder states the prices at which he can deliver the item to the various depots. In this form, the problem can be solved by a straightforward application of the method discussed above. Of course, since the total manufacturing capacity of all bidders may exceed die total demand, a dump has to be included in the model. A bidder may state, however, that he is not interested in any award that does not cover a stated minimum number of items. In this case, the first solution is worked out as before, ignoring this bidder's minimum figure. If this first program should award the bidder more than his stated minimum, we have the solution of our problem. If, however, the first program awards less than the stated minimum to this bidder, we have to explore two alternatives: remove this bidder completely from the program or increase his award to his minimum figure. The costs of these two alternative programs have to be determined so that the cheaper program can be selected. If there are many bids with minimum figures, an exhaustive survey of all possibilities may require a considerable computing effort, but there is no short cut to the solution known at present.

Another problem that falls into the same class is the so-called assignment problem. A number of men are to be assigned

to an equal number of jobs in such a manner that each job is handled by a single man. Each man qualifies for each job; a man's efficiency in handling a given job is rated by the cost of having this job performed by him. What is the most efficient assignment of the men to the jobs? Clearly, this problem has the same pattern as the transportation problem. Each man may be considered as a center producing work and each job as a center consuming work, each man producing one unit and each job consuming one unit of work. An assignment policy may then be treated as a shipping program for work, the efficiency ratings corresponding to the specific transportation costs. Since each job is to be handled by a single man, the "assignment graph" has as many disconnected branches as there are jobs. This fact, however, does not prevent us from using the pricing method developed for the transportation problem. A variation of this problem concerns the assignment of machine tools to the jobs that have to be performed in a shop. Here the number of tools that are suitable for a given job will be limited; on the other hand, a given tool can be assigned to various jobs in succession.

Another problem that can be cast into the mold of the transportation problem concerns production scheduling for a manufacturing company with a seasonally fluctuating sales pattern. These fluctuations in sales must cause some fluctuations in production rate or inventory. The problem is to find the program that minimizes the total expense for overtime pay and storage. We can treat this as a transportation problem in which amounts are shipped from the initial inventory and from the regular production or the overtime production of each month to the sales of any later month or to the final inventory. The specific transportation cost along any route includes not only the cost of producing the unit amount on regular time or overtime, but also the cost of carrying the unit amount in inventory from the month in which it is produced to the month in which it is sold. To create a uniform basis for the comparison of prices that have to be paid at various times, all costs are best discounted to the starting date of the schedule. Since nothing can be sold before it has been produced, an infinite cost must be attributed to any route that leads back in time. Since the total capacity for regular and overtime production will exceed the total demand if die solution of the problem is not to be trivial, a dump must be included in the model. For a schedule covering 13 months, we may thus have 25 production centers (initial inventory and regular and overtime production for each month) and 14 consumption centers (sales for each month, final inventory, and dump).

Dest. / Orig.	1	2	3	Av.
1	10 / 20	11 / 30	12 / 0	50
2	10 / 10	12 / 0	10 / 30	40
3	10 / 0	11 / 0	8 / 25	25
Req.	30	30	55	115

FIG. 6

Before leaving this group of problems, let us briefly consider their mathematical characteristics taking the transportation problem of Fig. 5 as a typical example. The data of this problem are presented in Figure 6. The three rows of the central part of this table correspond to the three origins, and the three columns of this part correspond to the three destinations. The quantities available at each origin are indicated at the end of each row, and the quantities required at each destination are given at the bottom of each column. The specific shipping costs for the various routes are written in the upper left corners of the nine cells. The problem demands that we write non-negative numbers into the bottom right corners of the cells in such a manner that the numbers in each row or column add up to the prescribed total and that the sum of the products of the two numbers in each cell is as small as possible. Mathematically speaking, we wish to minimize

a linear function of non-negative unknowns, which must satisfy a number of equations, the number of unknowns (9 in our example) exceeding that of equations (6). Problems of this type are known as linear programming problems. The following is an example of a linear programming problem that is more complex than the transportation problems considered so far.

An oil refinery has given quantities of three different crudes available and wishes to meet given demands for three products as far and as profitably as is possible. The product yields and the profit for each crude are known, and so is the loss (in customer good will) caused by the failure to meet the demand for each product. What is the most profitable production program?

Prodct / Crude	1	2	3	Av.	Profit
1	.5 / x	.3 / x	.2 / x	100	15
2	.3 / y	.4 / y	.2 / y	80	20
3	.2 / z	.3 / z	.4 / z	120	10
Req	100	70	90		
Loss	3	1	2		

$$\text{Net Profit} = .15x + 20y + 10z - 3(100 - .5x - .3y - .2z)$$
$$- (70 - .3x - .4y - .3z) - 2(90 - .2x - .2y - .4z)$$
$$= 17.2x + 21.7y + 11.7z - 550$$

FIG. 7

Figure 7 contains the numerical data for a specific problem of this kind. The rows labeled 1, 2, 3 correspond to the three crudes and the columns labeled 1, 2, 3 correspond to the three products. The available amount of each crude (in some appropriate unit, e. g. thousands of barrels) is given in the column labeled Av. and the amount required of each product is listed in the row labeled *Req.* The numbers in the upper left corners of the cells indicate the yields of the various crudes (in barrels per barrel). Profit and loss figures are given in the last column and the last row.

A solution of the problem is specified by three positive numbers x, y, and **z** specifying the amounts used of each crude. These numbers are subject to six restrictions, three of which stem from the limited availability of crudes and three from the limited market capacity for products. **We** wish to find positive numbers x, y, and **z** satisfying these availability and capacity restrictions and maximizing the net profit stated at the bottom of Figure 7.

Mathematically, the availability and capacity restrictions are stated by inequalities. For instance, the availability restriction for die first crude reads $x \leq 100$. With three unknowns, we must expect that for the optimal program only three of the six restrictions will be fulfilled in the form of equalities, **e.g.** only two **crudes** may be fully used and only one of the products made in sufficient quantity to meet the demand. If we knew which of the six restrictions are fulfilled in this manner, all that the **determination** of the optimal program would require would be the solution of three simultaneous linear equations with three unknowns. Actually, even in our extremely simple problem, there are 20 ways of choosing three of the six restrictions. To try all possibilities and find the one which yields the greatest profit without leading to a violation of some of the remaining three restrictions, would be very time-consuming. For a more realistic example, this procedure would involve a prohibitive amount of computing, even for a modem electronic computer. The so-called simplex method of linear programming progresses in a systematic manner from one admissible combination to another that yields a larger profit; after a finite number of steps an optimal solution is reached. In recent years an amazing number of practical problems have been discovered that are reducible to problems in linear programming. Many of these have been successfully treated by the simplex method. In other cases, the number of unknowns has been so great that the computational effort required by the simplex **meth**od proved to be prohibitive. Much of the current research in this area is directed towards special methods that do not possess the generality of the simplex method but take full advantage of the special features of the problem and so reduce the amount of computation required for its solution.

The programming considered so far is called "linear" because the quantity (cost or profit) that is to be minimized or maximized is assumed to be a linear function of the unknowns. For many problems this assumption would represent an oversimplification of the actual situation. For instance, if we deal with a transportation problem in which the "cost" of shipping along a route is the time it takes the shipment to reach its destination, congestion on a route would increase the "cost" of shipping along this route. Whereas we had a linear programming problem when the shipping cost was independent of the amount shipped, we have a non-linear programming problem when this cost depends on the amount shipped. While some general principles of non-linear programming have been established, no general and powerful method is as yet available for the solution of large scale problems of this kind.

Another important feature of practical programming problems should be mentioned at this time. In all preceding examples, the demands have been treated as known with certainty. Actually, such certainty is possible only under exceptional conditions; in many practical problems the demand for a certain product can only be described in a statistical manner. A given program then is no longer associated with a definite profit but only with a certain profit expectation. The task of maximizing the expected profit is a problem in stochastic programming.

In attempting to give you an idea of mathematical problems in management science, I had to restrict myself to a few examples that could be discussed with a minimum of formal mathematics. While these may, to some extent, have conveyed an oversimplified picture of the field, I hope that they have brought out two important facts. Firstly, the applied mathematician will find a fruitful area of research in management science. Secondly, his acquaintance with the applications of mathematics to the physical sciences will be of limited usefulness in this new area which often requires a radically different approach.

The last remark is worth elaborating. The fact that managerial action is often directed towards minimizing cost or maximizing profits would seem to suggest calculus as one of the principal mathematical tools of management science. Actually, none of the management problems considered in this talk could be solved by the usual techniques of calculus. In the transportation problem, for instance, the cost of the optimal program is a minimum not because the amounts shipped along the various routes could not be modified to yield a program of still smaller cost, but because such modifications would violate the availability and requirement restrictions at some centers of production or consumption. Mathematically speaking, the linear cost function admits a minimum only because the variables are restricted to a convex domain.

In the history of mathematics, the requirements of the applications have often led to the development of new mathematical disciplines. The outstanding example for this is, of course, calculus which stems from Newton's work on celestial mechanics. While calculus has proved invaluable in the physical sciences, it does not seem to be destined to take the same dominant position in the social sciences. As management science can be said to straddle the fence between physical and social sciences, it may well provide the impetus for the development of the kind of mathematics that meets the needs of the social sciences.

* * * * *

The department devoted to chapter activities shows that some chapters have a most commendable program. Do the others just not bother sending in reports?

OVERHEARD ON THE BUS

"My brother is a mathematician and I think he's crazy. He works for hours and is happy when he gets ***nothing*** for an answer."

OBJETS DE MATH

by Albert Wilansky
Lehigh University

A. Examination questions culled from fiction:

1. In Booth Tarkington: "The Lorenzo Bunch," Chapter II, Arlene says: "I stayed on ... five years after Roy and I were married, ...; but I quit when little Ola was four years old ... Ola's going on thirteen now. I was only twenty when Roy and I were married, ... Roy's almost thirty-four now ... ." How old is Arlene?

2. In Mark Twain: "Tom Sawyer," Chapter IX, we read: "It (the graveyard) had a crazy board fence around it, which leaned inward in places, and outward the rest of the time, but stood upright nowhere." Prove that the fence is discontinuous.

B. ***Theorems from fiction:***

1. In Robert Louis Stevenson: "The Bottle Imp," we read of a bottle which (a) is extremely advantageous to own, (b) must be sold by each owner for less than he paid, (c) brings damnation to anyone who dies owning the bottle. Theorem: ***Nobody would buy this bottle.*** Proof by induction: Nobody would buy it for 1¢. Suppose that nobody would buy it for k¢ or less. Then nobody would buy it for (k + 1)¢ since he could not sell it.

C. ***Some grading problems.***

1. One of my students gave the infinite series expansion for sine x with exactly one term wrong. I wish to weight each term in the series equally, what should be his grade?

2. I recently gave my class the problem of writing the largest possible number in 5 seconds. Here are the entries written by the various members of the class:

* * * * *

Professor W. S. Beckwith, former corresponding secretary of the Georgia Alpha Chapter of Pi Mu Epsilon, has retired and is now teaching temporarily at the University of Tennessee.

Smith: 10000000

Jones: $10^{10^{10^{10}}}$

Robinson: 2!!!!!!!

Johnson: $\infty - 1$

Jackson: a million, million million

Brown: n + 1, where n is the largest gotten by the others

Hill: n + 2, where n is the largest gotten by the others

Simson: the largest number which can be written in 5 seconds.

Who is the winner?

STORIES OF FAMOUS MATHEMATICIANS

SCRATCH THIS ONE

For years we've enjoyed telling this story:

"Queen Victoria was so pleased on her first reading of Lewis Carroll's Alice in Wonderland that she requested that the author send her, without fail, a copy of his next book. In due course, she received a copy of his Treatise on the Theory of Determinants."

Now Warren Weaver, writing about Lewis Carroll in the Scientific American, indicates that this story is not true. While we have a lot of respect for the truth, we nevertheless regret being obliged to abandon one of our favorite mathematical anecdotes. The article, in the April, 1956, issue, page 116, is worth looking up in case you haven't read it. The title is "Lewis Carroll: Mathematician."

BIRKHOFF'S STATURE

The following paragraphs are reprinted with permission from "MY TILT WITH ALBERT EINSTEIN,' as related by Professor Carlos Graef Fernandez, Director of Physics in the National University of Mexico, and former Professor of Relativity at Harvard, to Samuel Kaplan.

This story concerns the late Professor George D. Birkhoff, although the article itself is principally about Dr. Einstein. We wish we had space to reprint the entire article, which appeared in American Scientist, V. 44, 1956, pp. 204-211.

"Einstein is dead. How that profoundly sad event carries the memory back to my unforgettable meeting with the supreme scientist of our time!

"My heart beat fast as I stood before 112 Mercer Street in Princeton, New Jersey. I was going to defend the ideas of my dead friend, Prof. George D. Birkhoff, against those of Prof. Albert Einstein. But perhaps you do not know who Birkhoff was. Know then that Birkhoff, chairman of the Department of Mathematics, Harvard University, was one of the ten greatest mathematicians of all time!

"And permit me to say that Birkhoff did not minimize the importance of his extraordinary powers, as you may judge from this exchange between him and Prof. Luis Enrique Erro, Director of the National Astrophysics Observatory of Mexico.

"'Prof. Birkhoff' said Erro, 'I hope that in the future the United States Government will continue to send us savants of your stature.'

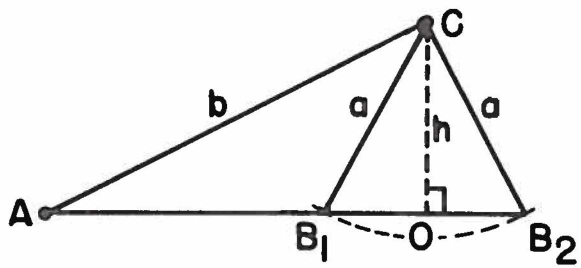
"'Prof. Erro,' was Birkhoff's surprising answer, 'in the States 1 am the only one of my stature.'

"To which I might add the words of Dr. Norbert Wiener, the present greatest American mathematician who, when making his obituary address before Birkhoff's body in Harvard University's chapel, said: 'He was the first among us and he accepted the fact. He was not modest.'"

AREA OF A TRIANGLE - AMBIGUOUS CASE

by G. D. Thaxton
(Freshman class, University of Richmond)

A survey of several books on plane trigonometry reveals that they give no formula for the area which will cover all aspects of the ambiguous case of two sides and the angle opposite one of them. This note gives a formula which will do this.



In the following discussion we use "B" to refer to either B_1 or B_2 in the above figure.

The area of triangle ACO is $\frac{1}{2} b^2 \sin A \cos A$.

In order to determine the area of triangle COB, we note that by the law of sines $b \sin A = a \sin B$ from which we obtain $\cos B = \pm \frac{\sqrt{a^2 - b^2 \sin^2 A}}{a}$. Using this value for cos B, we obtain the area of triangle COB as $\frac{1}{2} b \sin A \sqrt{a^2 - b^2 \sin^2 A}$ and the area of triangle ACO as

(1) $\frac{1}{2} b^2 \sin A \cos A \pm \frac{1}{2} b \sin A \sqrt{a^2 - b^2 \sin^2 A}$.

We may now summarize the cases as follows:

I. If $a > b$, then choosing the plus sign in (1) gives the area of the single solution.

II. If $a < b$ and $a^2 - b^2 \sin^2 A > 0$, then (1) gives the areas of the two solutions.

III. If $a < b$ and $a^2 - b^2 \sin^2 A = 0$, then (1) gives the area of the one right triangle solution.

IV. If $a < b$ and $a^2 - b^2 \sin^2 A < 0$, the imaginary value of the radical indicates that no triangle satisfying the conditions exists.

V. If angle A is $90°$ and $a > b$, choosing the plus sign in (1) gives the area of the right triangle solution.

VI. If angle A is $> 90°$ and $a > b$, then choosing the negative sign in (1) gives the area of the obtuse triangle solution.

SOUNDS FUNNY, BUT IT'S TRUE

"A zero of order zero is a regular point at which the function is not zero." (From a book on complex variables.)

"A thing is obvious mathematically *after you see it.*"

- Dean R. D. Carmichael

Presumably one shouldn't end a sentence with *a* preposition. But what about two or more? A little boy, who had wanted to be read to before going to sleep, asked his mother when she came upstairs, "What didn't you bring that book I wanted to be read to out of up for?"

PROBLEM DEPARTMENT

Edited by
Leo Moser, University of Alberta

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate, signed sheets within five months after publication. Address all communications concerning problems to Leo Moser, Mathematics Department, University of Alberta, Edmonton, Alberta, Canada.

PROBLEMS FOR SOLUTION

88. ***Proposed** by A. R. Aumalis and R. B. Wright, University of Nebraska*

A mathematics professor witnessed a hit and run accident. The police asked whether he recalled the license number of the fleeing car. The professor said, "No, but I did observe that the last four digits constituted the cube of the first two digits and that the sum of all six digits was odd."

At this point a student with die professor piped up, "But sir, did you not also observe that the greatest prime divisor was less than one hundred!"

Would this information enable you to obtain the license number?

89. ***Proposed** by R. B. Wright, University of Nebraska*

Evaluate the product

$$\prod_{n=1}^{\infty}\left(\cos\frac{2}{2n-1} + i\,(-1)^{n+1}\sin\frac{2}{2n-1}\right)$$

where $i^2 = -1$.

90. ***Proposed** by Vern Hoggatt, San Jose State College*

Prove that

$$\lim_{n\to\infty}\sum_{j=1}^{n-1}\frac{1}{(j(n-j))^{1/2}} = \pi .$$

91. ***Proposed** by Nathaniel Grossman, California Institute of of Technology*

Prove that

$$\sum_{d/n}\sigma\left(\frac{n}{d}\right)\phi(d) = n \cdot \tau(n)$$

where $\tau(n)$ denotes the number of divisors of n, $\sigma(n)$ is the sum of the divisors of n and $\phi(n)$ is the Euler Totient function.

92. ***Proposed** by Leon Bankoff, Los Angeles, California*

It has been said that algebra is but written geometry and geometry is but diagramatic algebra. (Sophie Germain, *Memoire sur les Surfaces Elastiques).* In the spirit of this quotation, show geometrically that

$$\sum_{n=2}^{\infty}\binom{n}{2}^{-1} = 2 .$$

86. ***Proposed by C. A. Grimm, South Dakota School of Mines and Technology***

For a, b, and x integers $(b > a)$ show that

$$x^3 + 3(a-b)x^2 + 3(a^2-b^2)x + a^3 - b^3 \neq 0 .$$

Solution by the proposer

The left hand side can be written in the form

$$x^3 + (x+a)^3 = (x+b)^3 .$$

Since a and b are integers and $b > a$ this is just Fermat's equation with $n = 3$ for which the non-existence of solutions is well known.

Also solved by N. *Grossman.*

SOLUTIONS

26. ***Proposed by Pedro A. Piza, San Juan, Puerto Rico***

For positive integers $\underline{n}$ and $\underline{c}$, let the number $[n:c]$ be defined by the relation

$$[n:c] = \frac{2^{c-1}\binom{2n-c}{c-1}}{c}.$$

Show that the numbers $[n:c]$ satisfy the recurrence relation

$$[n:c] = \frac{2(2n-c)}{c}\,[n-1 : c-1] \qquad (1)$$

and the formula

$$\frac{2^{2n}-1}{2n+1} = \sum_{c=1}^{n} [n:c]. \qquad (2)$$

Solution by J. R. Pounder, University of Alberta

The first result follows directly from

$$(1) \qquad \frac{1}{m}\binom{m}{c} = \frac{1}{c}\binom{m-1}{c-1}, \qquad (c,m \geq 1) .$$

To prove the second result we equate coefficients of x^{2n+1} in the MacLaurin expansions of each side of the identity

$$\log(1 - 2x) + \log(1 + x) = \log(1 - x - 2x^2) .$$

This gives

$$\frac{2^{2n+1}-1}{2n+1} = \sum_{r=n+1}^{2n+1} \frac{1}{r}\binom{r}{2n+1-r} \cdot 2^{2n+1-r}$$

$$= \sum_{0}^{n} \frac{1}{2n+1-c}\binom{2n+1-c}{c} 2^c .$$

Transferring the first term to the left hand side, dividing by 2, and using (1), we get the required result.

Also partially solved by M. *Lieber.*

79. ***Proposed by C. W. Trigg, Los Angeles City College***

Find the bounding values of the ratio of the sides a and c of a triangle in order that the median to one side and the symmedian to the other side may be concurrent with the internal bisector of the included angle.

Solution by the Proposer

Say the median is drawn to a, thus bisecting a, and the symmedian is drawn to c thereby dividing it in the ratio $a^2:b^2$. Then the internal bisector of B divides b in the ratio $c:a$. Hence, if the lines are to be concurrent, by the converse of Ceva's theorem, $a^2c = b^2a$. Thus the necessary condition is $b^2 = ac$. Now $b > |c - a|$, so $b = ac > c^2 - 2ac + a^2$, or $5a^2/4 > c^2 - 3ac + 9a^2/4$, and $\sqrt{5}\,a/2 > |c - 3a/2|$. It follows that $\frac{1}{2}(3 + \sqrt{5}) > c/a > \frac{1}{2}(3 - \sqrt{5})$.

This is equivalent to

$$\tfrac{1}{2}(3 + \sqrt{5}) > \frac{a}{c} > \tfrac{1}{2}(3 - \sqrt{5}).$$

81. Proposed by Leon Bankoff, Los Angeles, California

Show that

$$1 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots = 2(1 - 1/2^2 + 1/3^2 - 1/4^2 + \ldots).$$

Solution by B. Lachapelle, Cornell University

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + 2 \sum_{n=1}^{\infty} \left(\frac{1}{2n}\right)^2$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

from which the required result follows.

Solution by C. L. Gape, University of Buffalo

Consider the Fourier expansion of $f(x) = x^2$ in the interval $-\pi \leq x \leq \pi$, namely:

$$x^2 = \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^2}$$

Letting $x = 0$ we have

$$(1) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \pi^2/12 \; .$$

Letting $x = \pi$ we have

$$(2) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \; .$$

The required result follows from *(1)* and *(2)*.

Also solved by *J. C. Mathews, L. Miller, S. Robinson, C. R. B. Wright,* and the *proposer.*

84. Proposed by C. A. Nicol, University of Texas

If n is a positive integer, does there exist a positive integer k such that the sequence $k + 1, 2k + 1, 3k + 1, \ldots, nk + 1$ consists only of composite integers?

Solution by the proposer

Let $k = (n+1)! + 1$.

Then $k + 1 = (n+1)! + 2$

$2k + 1 = (2)(n+1)! + 3$

$3k + 1 = (3)(n+1)! + 4$

....

$nk + 1 = (n)(n+1)! + n+1,$

and these are obviously each composite.

REMARK. Calculations indicate that the value of k given here is much larger than is necessary. It would be interesting to find the smallest value of k for which this sequence would consist only of composite integers.

Q: "What is an expert?"

A: "Well, as everybody knows, x stands for an unknown, and a spurt is a drip under pressure."

BOOK REVIEW

B. E. **Meserve,** ***Fundamental Concepts of Geometry.*** Addison-Wesley, Cambridge, Mass. 1955. 9 + 321 pp. $7.50.

According to the Preface, this book is based upon the geometrical part of a course entitled "Fundamental Concepts of Mathematics" which was founded by J. W. Young and carried on by successive generations of other professors, including the author. It is the result of "selecting, modifying and expanding sequences of topics from the two volumes of ***Projective Geometry"*** by Veblen and Young. The topics selected are essentially the same as those in ***The Foundations of Geometry*** by *G.* de B. Robinson (Toronto, 1940); but the reviewer is disappointed to find the modifications and expansions less satisfactory. The modifications sometimes consist in omitting an essential step in a proof **(e.g.,** in the proof on p. 84 that $R(ABC) = R(ABT)$; cf. Veblen and Young I, p. 85). The author tacitly assumes (on pp. 64-66) that the tangents of a point conic form a line conic. He repeats (on p. 59) the one serious error of Veblen and Young (p. 41): the use of duality to prove the converse of **Desargues'** Theorem (cf. Coxeter, ***The Real Projective Plane,*** 2nd. edition, p. 13, Theorem 2.26).

In the introductory section on logic, the author rightly stresses the distinction between ***contrary*** and ***contradictory.*** Unhappily he gives a new and complicated definition for "contradictory" on p. 5, although he reverts to the simple and adequate definition in his statement of **Aristotle's** second law on p. 6. On p. 8, and again on p. 10, he declares that it is "desirable" that no two postulates be contrary; but surely the word "desirable" should have been "essential."

To establish the equivalence of synthetic **and** analytic geometries, it is necessary to derive each from the other. But he has reduced the synthetic introduction of coordinates to the barest sketch. The fascinating idea of adding and multiplying points on a line (or on a conic) is discarded as being "long and tedious" (p.89). His nearest approach to a proof that a line has a linear equation is "the assumption that on a plane the points (0, 0), (b, 1), (2b, 2), ... , (ab, a), ... are collinear" (p. 95). He is on surer ground when dealing with analytic geometry as a self-contained subject; but the proof given for **Desargues'** Theorem (p. 127) could have been more elegant (cf. Coxeter, op. **cit.,** p. 191, § 12.3).

Some misunderstandings in connection with homothetic transformations (p.166) could have been avoided by giving first a brief account of the projective theory of homologies and elations. For instance, on p. 151 the author correctly points out that, in affine geometry, "the principles of duality do not apply;" but on p. 174, and again on p. 182, he declares that "the plane dual of a point reflection is a line reflection." Incidentally, it is unfortunate that the well established term "dilatation" has been contracted to "dilation."

Chapter 7 provides a well written outline of the history of geometry. However, it does not seem quite fair (on p. 231) to accuse Euclid (who proved that there is no greatest prime) of tacitly assuming that "all sets of objects are finite." Again, **Euclid's** explicit assumption that a line is not re-entrant ((ii) on p. 232) has somehow become confused with his "tacit assumption that a straight line containing a vertex B and an interior point of a triangle ABC must also contain a point of the line segment AC" (p. 253), which remains valid not only in hyperbolic geometry but also in elliptic. Apart from these minor blemishes, the treatment of non-Euclidean geometry, though admittedly brief, is satisfactory. For instance, the remark is well made (on p. 283) that "Euclidean geometry of three-space imposes an elliptic geometry on the ideal plane $x_4 = 0$. In this sense, a thorough understanding of Euclidean geometry in three-space requires an understanding of elliptic plane geometry."

In the chapter on Topology, the treatment of surfaces is misleading. We read (on p. 299) that "Any closed connected surface with a boundary is homeomorphic to a disk with b holes ... and is said to have *Betti* number b." But how can this description be applied to such a surface as a torus with a hole? Having used the word "surface" in the sense of "orientable surface," the author goes on to describe the Möbius strip, which thus inevitably appears as a surface that is not a surface. An accurate classification of both orientable and nonorientable surfaces could have been given in the same number of pages (cf. Lefschetz, Introduction to Topology (Princeton, 1949), pp. 73-78).

The printing has been well done, the figures are clear, and there is a full index. One of the remarkably few misprints is "Grassman" on p. 262 (and again on p. 315).

H. S. M. Coxeter

WE NEED ADS

It costs a good deal to publish an issue of the Pi Mu Epsilon Journal. Whereas we obtain some money from subscriptions, much of our bill must be paid by the Secretary-Treasurer General. If those of you who are authors would encourage your publishers to place ads with us, that would help a great deal. The rates are reasonable: $25 for a full page, $15 for a half page. Can you help us?

BOOKS RECEIVED FOR REVIEW

Brixey, J. C. and Andree, R. V.: Fundamentals of College Mathematics, New York, Holt, 1954, $6.25.

Brixey, J. C. and Andree, R. V: Modern Trigonometry, New York, Holt, 1956, $3.50.

Friedman, B.: Principles and Techniques of Applied Mathematics, New York, John Wiley and Sons, 1956, $8.00.

Graves, L. M.: The Theory of Functions of Real Variables, New York, McGraw-Hill, 1956, $7.50.

Meyer, H. A. (Editor): *Symposium* on Monte *Carlo* Methods, New York, McGraw-Hill, 1956, $7.50.

Miller, K. S.: Engineering Mathematics, New York, Rinehart & Co., 1956, $6.50.

Neilson, K. L.: Methods in Numerical Analysis, New York, Macmillan, 1956,

Newell, H. E., Jr.: Vector Analysis, New York, McGraw-Hill, 1956, $5.50.

Niven, I.: Irrational Numbers, New York, John Wiley and Sons, 1956, $3.00.

We would appreciate offers from our readers to review the books in the above (and future) lists.

----the Editor

Q: What is a lemma?

A_1: A lemma is ½ (dilemma).

A_2: No. The idea is, "If you lemma prove this first, then I can prove the main theorem."