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CONTENTS

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THE ROLE AND NATURE OF MATHEMATICS

Preston C. Hammer, University of Wisconsin

Introduction.

Mathematics has recently achieved, along with certain other branches of learning, a new status. The reasons are numerous but they are associated with the successes of fields formerly confined to the remote reaches of erudite scholarship in applications of concern to all of us. The causes which have led us to re-examine the mathematics curriculum are not necessarily those springing from the highest motives attributed idealistically to science and scientific enterprise. A central factor is the success of the Russians. It is a sobering fact that societies will often make the effort to achieve great things only under duress, for example, in the interest of so-called defense.

Today teachers of mathematics are being called on to teach concepts which a short time ago were saved for graduate students and research mathematicians. In retrospect, it can be wondered, how did we underestimate so thoroughly the capacity of the young or, conversely, how did we overestimate equally thoroughly the inherent difficult of mathematics? Whatever may be the causes, it has become clear that the principal educational problems concerning mathematics rest on decisions as to how much should be taught and in what sequence, and not primarily on the capacity of the pupils to grasp it.

In view of the upheaval in progress it is the more important that teachers in secondary and elementary schools acquire a broad perspective of mathematical activity. In this essay I attempt to record some of the conclusions I have reached in years of activities and thinking related to mathematics--research, computing and teaching. I do not claim to be an authority. "Authorities", who would impose their limited outlook on others are the principal curse of science and mathematics as well as other areas. A man who believes that his outlook should be accepted because he has accepted it should be treated for illness.

There are several attitudes concerning mathematics which have been indicated as authoritative which will not bear even casual scrutiny. It is true that many successes have been scored in mathematics by eminent mathematicians who have seemingly deluded themselves rather thoroughly concerning the nature of mathematics. Thus one famous mathematician undertakes to present mathematical creativity as a prerogative principally of a few great men. This is not only a false concept, it reflects ignorance concerning the nature of creativity. It reflects also an attempt to maintain a high priesthood of mathematics based on false precepts and sheer arrogance.

Now I am going to put together a chain of statements for your consideration. I believe that these statements can stand the test of your thought but you should not accept them without considering quite carefully their implications.

The most neglected existence theorem in mathematics is the 502
existence of people. Mathematics was created by people and it bears their imprint. It is not infallible nor have its precepts always been wise. Take, for example, the often heard slogan "mathematics for the sake of mathematics" a paraphrasing of "art for the sake of art". A mathematician who claims to work on mathematics for its own sake does not know what he is doing. There is no sake of mathematics, nor of art either. I may work on mathematics because it amuses me, because I believe it to be useful, to establish a reputation or any reason but certainly not for its sake!

Mathematics as a Necessity of Civilization.

Mathematics arose from the needs of organized societies of people. Imagine a primitive tribe living by hunting and collecting the natural harvest of forest and field. Rudimentary forms of counting are needed to communicate numbers important to the tribe. This may be the number of animals in a herd; the number of people in a hostile tribe. Also needed are measures of size, strength, distance and time however crudely formulated they may be. A certain primitive awareness of similarities of shapes must be present in efforts to duplicate arrow-heads and implements. It is also important to have some means of describing location involving both distance and direction. Thus even in a primitive society certain intuitive concepts which later developed into mathematics are necessary. Moreover, this primitive tribe needs something of virtually all the great branches of specialized mathematics.

If I now consider instead an advanced civilization such as Babylonia or Ancient Egypt, I find that the mathematical necessities have increased enormously. The counting of large populations, armies, and herds must lead to the idea of the necessity of extended counting--i.e., the natural numbers. Distances must be measured systematically and locations described with some precision. Why? Because there is now private property--real estate which must be surveyed, roads, streets and canals to be built and these must be planned. Thus metric geometry is necessary. Calendars are important and astronomy becomes a field of specialization. We inherit, for better or worse, the Babylonian sexagesimal system of measuring time and angles.

Geometry is also important in the planning of houses, palaces, bridges and ships. The mechanics of building, of construction, and so on, must be planned in advance.

Weights, measures, money all must receive some consideration. Accounting of some form must be initiated for purposes of taxation and management of wealth.

Thus in Babylonia, as in Ancient Egypt, there was much demand for the commodity we know as applied mathematics although the only area which we might recognize as mathematically developed was that of calculation with a certain amount of algebra. That these civilizations could have used much more of modern mathematics is obvious.

Since society was organized, this meant specialization and, for some, leisure. Thus there are indications that certain Babylonians delved into mathematical problems out of sheer curiosity rather than directly to achieve an immediate objective. This spirit of exceeding the necessities has led throughout the years to much of the significant work in mathematics.

Pythagoras and Euclid.

As a conceivable example of doing mathematical work, I select Pythagoras first and then go on to Euclid. The Theorem of Pythagoras, stating that in a right triangle the sum of the squares on the sides is equal to the square on the hypotenuse is one of the best known ones in mathematics. I do not know how Pythagoras actually did the work, but I will assume that he did it like research is done today.

First of all, how did Pythagoras come to consider such a problem? The answer might be that he was aware of previous experience with special forms of the problem. The Egyptians knew, for example, of the 3, 4, 5 right triangle and used it in surveying. The isosceles right triangle may also have been part of this experience. However, Pythagoras was not satisfied with these results for special right triangles, he wanted a result applying to all right triangles. This endeavor to achieve completeness marks much of mathematical activity.

What did Pythagoras do? He draws various right triangles; he becomes convinced that if he knows the lengths of the legs that the length of the hypotenuse is determined, that is, it is a function of the lengths of the legs. Perhaps his experiments are carried out over a period of several years, intermixed with other activities. There comes a time, however, when Pythagoras guesses the answer. Now he has a proposition to prove or disprove. His experiments may have already led him to a method of proof. However, he may have spent much more time in devising a demonstration which could be used to convince everyone of the validity of his conclusion. He tries various procedures, based on his experience, and finally arrives at a rather short demonstration that he presents to his colleagues and students. This demonstration takes the form of deductive proof. In a few minutes Pythagoras convinces his audience of the validity of his theorem. It does not take long since they already have agreed on the background of geometry assumed by Pythagoras.

I now draw several conclusions from this pseudo-example. The first is that mathematics, Bertrand Russell to the contrary notwithstanding, is not deductive science. The selection of what to work on is not deductive. In this case Pythagoras drew on experience and this is as true of mathematics today as it was then. Granted the selection of a problem, the establishment of what conclusion to draw from the hypotheses is not a deduction. Granted the complete phrasing of the theorem, the establishment of a demonstration is not done deductively. Once a "path" has been established from hypotheses to conclusions this path is not unique and deduction as such did not lead to it. Only when the work is being cast in the final demonstration does it actually appear as a deduction! There are some 50 or more different proofs of the Theorem of Pythagoras. Why did he select the one he did? Partly because of his experience and the limitations of his knowledge!

When you present a proposition to pupils you cannot take the same amount of time as it took to guess and establish it in the first place. No, not even Pythagoras would go through a full description of his work, his hopes, fears and elation at achieving the result. Why? Simply because it would take too long. Our education system is, in part, supposed to bring the pupil through the important accomplishments of millenia of hard-working people. He cannot relive these millenia and must be given them in capsule form. Nevertheless, it is wrong to give the impression that the work did not take place, that Pythagoras dashed off his theorem as the exercise it appears. It is also wrong to create the impression that the pupil himself might not create something or that the creators of the mathematics were essentially different from himself.

Let me now go to Euclid. Euclid, in my estimate, was one of the most percipient of known mathematicians although his influence, as an authority after his death,' had its enormously detrimental effects. Euclid was adequately supported and had leisure to work on mathematics. He was surrounded by learned scholars, a good library, and probably had assistants in such quantity as he could use. Behind him was already a large collection of geometrical propositions. Euclid studied this geometry; he knew it well. He was distressed by the cyclic use of propositions for proofs. That is, a proposition A might be proved on the basis of propositions B, C, and D, say. On the other hand, proposition A might have been used in the proof of B, C, or D. The recognition of this and his success in avoiding such circularity in geometry is the main contribution of Euclid.

What Euclid did was to select, based on his experience with geometry and other experiences, those propositions which seemed to him correct but not provable on the basis of simpler propositions. Such propositions he called axioms. He attempted to introduce enough axioms to characterize geometry and imply the important results already known. He also attempted never to introduce as an axiom a proposition which could be proved on the basis of his previously selected ones. For this reason the famous

Parallel Axiom was not introduced until rather late, since Euclid was not convinced that it was necessary although, since he failed to prove it on the basis of the preceding axioms he did finally include it.

Thus the axiomatic method in mathematics originated with Euclid and it has turned out to be a powerful implement of research. Observe that Euclid's work was primarily not deductive. He could not deduce which propositions to choose as axioms, nor did he deduce which deductions to use in making demonstrations. The summary of his achievements were recorded in his Elements but the work involved was not recorded. Thus Euclid's Geometry **does** not reveal how to do mathematics; it gives a form of **presenting** it after it is done.

The treatment of Euclid's Geometry as a model **of reasoning** is one of the reasons for the slow development of mathematics. In effect, it 505
has been used largely to prevent reasoning by its use as an authority. The geometry taught as a model of thinking has actually been used as a mental strait jacket. A heritage can be a curse as well as a blessing.

Logic, Theorems and Proofs.

In the foregoing section I **may** have given the impression that a proof is somewhat independent of **people. In** this section I attempt to destroy such an illusion. Attempts to suggest that mathematics is part of a safe, secure, logical structure existing independently of human experience are erroneous.

Consider a proposition or theorem in the usual sense. It can be written schematically $H \geqq C$ where H stands for hypotheses, C for conclusions and the statement reads H is logically stronger than or equal to C. Thus I may write schematically H - C to indicate the logical difference between Hypotheses and Conclusions. Since only when H is necessary and sufficient for C does H = C, I have the result that almost all mathematical theorems represent a logical **loss!** Theorem after theorem represents loss after loss. Is mathematics **simply** an accumulation of logical losses? **No!** A good theorem provides information to people; it contains an element of surprise. What may be logically **true** is not necessarily known to be true. Mathematicians are engaged in the production of information in establishing that one state of affairs implies another state of affairs.

Assuming the old-fashioned logic in which $H \geqq C$ is either true or false independent of human capability of establishing either, then it is easily seen that a proof has **no logical** function. That is, it has no effect whatever on the truth or falsity of the theorem. Why prove theorems? The answer is to convince people that the theorem is true. If you read the statements in a "proof" and fail to be convinced that the theorem is true, it has not been proved to you. On the other hand, if you are convinced by means however **bad** of the truth of a theorem, it is proved to you. Similarly, if you present a proof to pupils, it is not a proof to those who are not convinced. Efforts to gloss over this simple fact of life can only result in bad instruction.

Mathematical friends have said to me "You are right concerning a proof but how depressing it is". What is right may be depressing but, as teachers, we should try to know what is right. I know of no theorem established except through human acceptance; by vote, if you like. A definite unbalance in our reasoning powers is most useful here. A theorem can be shown to be false by one counterexample; it is true **only** if no counterexample can be constructed. We are stronger on negative than on positive decision. David with a counterexample is stronger than Goliath with a **theorem!** Mathematicians attempt to avoid **all** counterexamples of their theorems. It is amazing how well they seem to have succeeded.

Understanding a theorem and its proof requires active participation. This may take the form of trying to construct counterexamples, of trying to formulate a proof yourself, or of trying to improve on the theorem or its proof. There is generally too much practice in imitating correct 506
statements as compared to practice in detecting errors in false statements. A teacher should @ require a student always to give back the same proof as presented in class. To do so puts a penalty on thinking.

The Generality of Mathematics.

Much fuss has been made over the generality of mathematics. Jaundiced eyes are cast on generalizations often in the mistaken notion that mathematics is general enough. What are the facts? Mathematicians have borrowed the safest part of the language and attempted to construct a secure structure on it. All attempts to meet further demands must result in more general systems. Thus ancient mathematics did not provide models for games of chance but probability theory was initiated to do so. Euclidean geometry failed to describe effectively rather simple curves until Descartes introduced coordinates, thus giving a panorama of curves. Cantor then went further to meet the applications and introduced a theory of sets which admitted many more objects as geometrical figures.

Topology, as contrasted with Euclidean geometry, is rather general. Yet it is not general enough for the demands of computing theory and numerical analysis.

By reason of its intensive development of certain concepts mathematics has been considered an object language for sciences. However, the security sought and achieved has its price in the inapplicability of mathematics to any but comparatively simple situations.

You may quote examples of the applications of mathematics by the hundreds. I will be impressed but not overwhelmed. I know that in the simple process of attempting to pack as many dishes in a box a housewife is trying to solve a problem more difficult than has been solved in the far reaches of measure **theory!** I also know that when you accept responsibility for advising a student you have a problem in which mathematics is principally useless, the problem is too difficult1

Mathematics **has** an expanding area of influence but it can represent only a small portion of human activity. Mathematics is inherently less general than the common language!

creativity **and** Purity **in** Mathematics.

What is creative activity? Creative activity can occur in many unsalable forms. The recognition of a pattern, an analogy, the smoothing over of a quarrel, the phrasing of a sentence are examples of creative activity. Every normal person does many creative acts, but sometimes these are ones required of him and he is not specifically rewarded for them. It is creative to discover a relationship or proof in mathematics ~~no matter how~~ many have done it before. However, if a person wants to be paid for his creations be it in recognition, deference, or money then he has additional conditions to meet.

The tagging of only masterpieces of creativity as being creative is foolish and misleading. It is important that students be encouraged in their efforts and that their positive creations be rewarded. Creativity is not a prerogative of the qreat, it is almost a necessity of survival1

This brings me to the schism between pure and applied mathematics 507
accentuated by great fools. There is no mathematician today who is pure and known in one sense. Even so-called pure mathematicians publish their work for the application of receiving recognition and/or money. Moreover, they are naturally pleased when their results are used for any worthy purpose. If mathematics were useless society would not support it. The applied mathematician, on his part, may be disdainful of the results achieved by his "pure" colleague since he can't apply them immediately to his work. Sincere earnest work is usually going to be beneficial.

Incidentally, if you should meet someone corroded by the power of recognition, consider this. If he, (the powerful one), achieved his status because he had superior equipment, for this he deserves no credit; he had nothing to do with it. On the other hand, if he had but mediocre gifts but made the most of them would he like to be reminded of it? The quest for recognition has no merit in itself and it produces warped personalities in quantity.

Definitions and Axioms.

It is common practice to act as if a set of elements coupled with a set of propositions presented as axioms define a mathematical system. This may be an adequate artifice for research workers but teachers should know better. The notion that a set of words define something independent of the people who read them is destined to become the archaism it should long since have been. The axiom system as used provides a starting point, assuming you read the language in which it is written, for the unfolding of a subsequent body of theory and concepts. The theory is generally necessary to the understanding of the axiom system--and hence helps define it. From the sterile logical viewpoint, if the axiom system really described completely a mathematical area there is no point in the subsequent theory since all is implied by the axioms.

The same is true of definitions. In what sense does a definition read by two people mean the same thing to both? If it doesn't mean the same then does it define? The answer is that most of the pat definitions do not define in any precise sense. Use of a concept is necessary to grasp its meaning. This should affect our teaching by the realization that the so-called definitions are merely prologues. What follows should be the cultivation of an understanding of the meaning of these concepts. Can anyone define variable, constant, or function so that their meanings are clear to the reader? Not in any few words and certainly not without much experience with the concepts.

Attempts to define natural numbers have been rather dismal. Why? Because, as Euclid observed, you cannot start without assumptions and you cannot define words in terms of words without starting with some that are known. These known words have had to be learned by association and abstraction. In mathematics efforts are made to obtain precision of meaning and this is done by attempting to sort out proper and improper responses. The teacher plays a critical role in this endeavor. What is a number, say 2? It is principally an agreement among people! It is neither ink, chalk, nor appropriate sound packets.

Instruction.

It is impossible to consider all the implications of what I have been saying for instruction. Nevertheless perhaps a few paragraphs will give some opinions for you to disagree with.

First of all I consider the circumstances in which our secondary and elementary teachers now work tend to make the best in instruction almost impossible. First, the teachers need more time and opportunity to learn mathematics and participate in the applications of mathematics. They also need more time to consider individual pupils to find better ways for them to approach whatever they are learning. The pupil is supposed to digest in a comparatively short time the heritage of millenia. He is supposed to learn to communicate in the language of mathematics. Almost inevitably, however, he spends most of his time listening and reading instead of thinking, writing and speaking. This is partly because the pupils outnumber the teachers by too great a factor. Thus the teacher finds it comfortable, if not necessary, to force one point-of-view on the pupil, to require that he do things in the precise fashion laid down. This tends to produce good parrots but not good students.

I have heard repeatedly the injunction to parents "Please don't help your children, it will only confuse them". Actually, it may be that the teacher cannot cope with ideas and methods with which he is not already familiar. The modern teacher needs to erase the image of himself as an oracle. In any classroom the pupils almost inevitably know things not known to the teacher. A good teacher should manage to capitalize on that source of information.

The applications of mathematics have tended to be slighted even by collegiate instructors. This attitude does not reflect strength but weakness. Knowing what mathematics does is part of knowing mathematics. On the other hand, repeated discussions of situations illustrating a mathematical concept without coming to grips with the concept essentially denies the reason for the efficiency of mathematics--in expressing the essence of many situations without actually being any one of them.

Education is, in its very nature, a form of thought and action control. To minimize the detrimental effects of such control, it is necessary to permit and encourage challenges. Thus, a mathematics text-book represents the author's viewpoint and experience and constitutes a form of thought-control. Teacher and pupil alike should not be reliant on that one source of information. They should compare with other books, be alert for inaccuracies, for better statements and proofs and so on. To require a pupil to learn one method is one thing, to forbid him to learn another is inexcusable.

There is much pressure these days to divert all the so-called gifted students into channels of science and engineering. In part, I suspect this is because the easiest people to teach are those capable of learning without instruction. The secondary teachers in particular need to consider the fact that it may actually not be the best to advise a pupil to go into such areas however apt he is. One of our greatest needs now is for better statesmen on all levels of government. Another need is for teachers. I have heard research mathematicians, who would never advise a good student to become a teacher, lambaste the quality of teaching. In my opinion, they have better students than they deserve!

Finally, let me close this section with an observation on the economics of teaching. As a guess, I should say that the time of the pupil is worth $1 per hour on the average. This value of time is too frequently ignored. Hence, teachers and pupils deserve the best possible in prepared materials. Yet generally most materials are inadequately researched and are written by one or two people in their spare time! The resultant loss is enormous. We can afford to do better.

Conclusion.

Mathematics is not deductive science and neither is logic. There is no logical excuse either for mathematics or logic. Mathematics was created by people who, generally speaking, were much concerned about the durability of their work. They very much need to know what they are talking about and they have shown a high degree of concern for the truth of their statements.

Any attempt to separate mathematics from its applications is foolish. Creative mathematical activity is not a prerogative of a few any more than creative art is. Mathematics has had amazing successes and yet remains, in its present state, applicable to principally simple problems.

The good teacher in whatever field returns to society much more than he or she is paid. Good teaching must be founded on an understanding of and appreciation of the subject matter, of the pupils, and of society. I hope these observations, in some way, promote better teaching in mathematics.

I am indebted to Professor John D. Hancock of Alameda State College for suggesting that I should write up a lecture on which this paper is based, and also for improvements he has suggested in the manuscript.

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Appendix. What Others Say.

In the body of this essay, after some deliberation, I decided against quoting other people to support my viewpoint. My viewpoint is rather different from that of most mathematicians who have similar inclinations, and it seemed a poor policy to suggest a complete accord by restricted quotations. However, there are several authors who agree, at least on the essential features, that mathematics and logic are not simply to be divorced from the activities of people in society. Among these men are Professors Nathan A. Court and George Polya. Another outstanding thinker on such matters was Professor Gerritt Mannoury, of the University of Amsterdam, whose books unfortunately have not been translated into English. In the closing paragraph of his book [3] Professor Mannoury labels as pure superstition the notions of mathematics as absolute, perfectly exact, general and autonomous or, in short, being true and eternal! This statement nicely puts the finger on mathematical fantasy.

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FIFTY YEARS IN THE PI MU EPSILON FRATERNITY

J. S. Frame, Director General
Michigan State University

1. Incorporation. The Pi Mu Epsilon Fraternity, incorporated on May 25, 1914, under the laws of the State of New York, is celebrating its golden anniversary as a national mathematics honorary fraternity with nearly 100 active chapters in 39 states and the District of Columbia. It is a non-secret organization whose purpose is the promotion of scholarly activity in mathematics among students and faculty in academic institutions, and among staffs of qualified non-academic institutions.

Its first Director General, Dr. Edward Drake Roe, Jr., had organized a Mathematical Club at Syracuse University in the fall of 1903 and had guided it through ten years of successful endeavor. At the club meeting on November 17, 1913, he proposed the establishment of a mathematical fraternity. Details were worked out in committees, and on March 2, 1914, a convention was held and a constitution was adopted. Specific names for the Fraternity were considered on March 23, and the Greek letters ΕΣΜ, "To promote scholarship and Mathematics", were adopted, but with the order of letters changed to ΠΜΕ. Those present then took the following pledge and signed their names as charter members of the Fraternity.

PLEDGE: I do solemnly promise to give my best efforts in the improvement of my scholarship in all my subjects and especially in mathematics, and to maintain a reserved silence concerning the obligations of the fraternity, and to cheerfully accept advice and admonition as long as I am a member of the fraternity. (The pledge has been altered in subsequent years.)

The fifty charter members included 8 faculty, 2 graduate students, 15 seniors, and 25 juniors and sophomores. The Mathematical Club was dissolved on April 27 after turning its assets over to the new fraternity. Five officers and four additional members of the executive committee of Pi Mu Epsilon were then elected. These became the incorporators of the fraternity and signed their names on May 20, 1914, to the certificate of incorporation, which was approved by Justice P. C. J. DeAngelis of the Supreme Court of New York.

Incorporators of Pi Mu Epsilon

Edward Drake Roe, Jr. (Director)	Florence A. Lane
Floyd Fiske Decker (Vice Director)	Helen Mary Barnard
Helen L. Applebee (Secretary)	Edward Jay Cottrell
Purley J. Bentley (Treasurer)	Adolph Sussman
Olive Evelyn Jones (Librarian)	

2. Early Years. The vote to establish the new fraternity might not have prevailed if high scholastic requirements had been set for charter members. Scholarship standards for election of new members in 1914-15 were discussed on October 3, 1914. Minimum general averages and mathe- 512
matics averages of 75 and 80 for sophomores and of 72 and 75 for juniors were adopted then, but higher minimum requirements were set at later meetings. Sophomores must now have an A average in mathematics and be in the upper quarter of their class in general average to be eligible for election to membership.

The young fraternity became a national organization after World War I when the second chapter was established with 23 charter members at the Ohio State University in October, 1919. First known as the Beta chapter," it became the Ohio Alpha Chapter when it was later decided to include the state name in chapter designations. The next three charters were granted by the Syracuse chapter to the University of Pennsylvania (1921), the University of Missouri (1922) and the University of Alabama (1922).

Records show that General Officers of the Fraternity were nominated (and they presumably were elected) in December, 1922, as follows:

Director General: Dr. E. D. Roe, Jr. (Syracuse)
Vice-Director General: Mr. W. V. Houston (Ohio State)
Secretary General: Dr. Warren G. Bullard (Syracuse)
Treasurer General: Miss Louisa Lotz (Pennsylvania)
Librarian General: Miss Mabel G. Kessler (Pennsylvania).

Under the new national organization chapters were chartered at Iowa State in 1923, at the University of Illinois on the tenth anniversary date of May 25, 1924, and at Bucknell University on March 5, 1925. Professor H. S. Everett of Bucknell was elected Secretary General in January, 1927, when Professor Warren A. Lyon withdrew his name after a tie vote for that office. Professor Everett replaced Professor Bullard, then on leave of absence because of cancer which soon claimed his life. Professor John S. Gold succeeded Professor Everett as Secretary in the fall of 1927. Dr. Roe and Miss Lotz continued as Director and Treasurer.

Dr. Roe expressed his strong feelings about the need for democracy in Pi Mu Epsilon. Opposing the appointment of a nominating committee. he suggested that each chapter send its nominations to the Bucknell chapter, which would serve as teller, and that the two highest candidates for each office be voted upon by the fraternity. Writing to Professor

Everett on February 13, 1926, he said, "All along I have endeavored to keep the management of Pi Mu Epsilon out of the hands of a few. Its government is democratic and I have aimed to prevent anything like an oligarchy .... The chapters have all the legislative powers, the council is merely executive and advisory ...." To Professor R. C. Archibald he wrote on December 9, 1926, "I have had the conception from the start of a fraternity uniting faculty and the most advanced students (normally above sophomore, though an exceptional sophomore may be eligible) and I have never departed from this ideal. I have always felt that a merely undergraduate fraternity would be only a half success - in accomplishing our whole purpose and ideal, the advancement of mathematics and scholarship."

A jeweled pin was presented to Dr. Roe by the Fraternity on the occasion of his retirement, just six months before his death in 1929, as a token of appreciation for his fifteen years of devoted service as Director General. Since 1949 this pin has been entrusted to the incumbent Director General, to be worn as a badge of office. 513

When Dr. Louis Ingold of Missouri became the second Director General in 1929, the Fraternity had 18 chapters. In 1936, with Professor John S. Gold of Bucknell as Secretary-Treasurer General, a policy was instituted of issuing all membership certificates from the national office. In 1937 the L. G. Balfour Company was designated as the official jeweler of the Fraternity. Royalties for fraternity jewelry sold to members have assisted in underwriting some of the expenses of the national office.

3. The Pi Mu Epsilon Journal. The establishment in 1949 of the Pi Mu Epsilon Journal was an important milestone in the history of the Fraternity. This journal aims to publish high quality articles by undergraduates, graduate students and others, that are of interest to the undergraduate student in mathematics, in addition to items such as chapter reports that may be of interest to the chapters. Those who have served as Editors-in-chief and business managers of the Journal are

Editors

Ruth Stokes (Syracuse) 1949-55
Franz Hohn (Illinois) 1955-57
Francis Regan (St. Louis Univ.) 1957-63
Seymour Schuster (Univ. of Minnesota) 1963-

Business managers

Howard C. Bennett (1949-54)
Henry W. Farnham (1954-55)
Echo Pepper (1955-57)
J. J. Andrews (1957-63)
Rita Vatter (1963-).

4. Affiliate Chapters. In 1957 the Constitution was amended to provide for the establishment of affiliate chapters of Pi Mu Epsilon at non-academic institutions, and the first such chapter was established at the General Electric Company, Evandale, Ohio. Affiliate chapters are intended to foster and promote an interest in mathematics, but do not elect persons to regular membership in Pi Mu Epsilon.

5. National Meetings. As the Fraternity has grown from a single club in 1914, to 18,857 members in 51 chapters in April, 1951, to over 45,000 members in nearly 100 chapters in May, 1964, it has become increasingly important to provide contacts between the members of different clubs at national meetings. Such meetings have been held almost every year since 1923. In 1952 and subsequent years, a session for student speakers has been arranged at the national meetings, and the chapters have been urged to send their best student speaker to

present a paper. Several of these papers have been subsequently published in the Pi Mu Epsilon Journal. Financial assistance by the national treasury of the Fraternity is given to student speakers (full fare) and to delegates (half fare) for one representative of each chapter who has not reached the Master's degree level prior to the commencement immediately preceding the meeting. The fiftieth anniversary meeting is scheduled on August 25, 1964, at the University of Massachusetts, in conjunction with meetings of the Mathematical Association of America.

APPENDIX I. GENERAL OFFICERS OF PI MU EPSILON.

	Director	Vice-Director	Secretary	Treasurer	Librarian
1914	E.D. Roe, Jr.	F. F. Decker	Helen Applebee	P.J. Bentley	Olive Jones

Local officers of the Syracuse chapter served as general officers until 1922.

1922	E.D. Roe, Jr.	W.V. Houston	W.G. Bullard	Louisa Lotz	Mabel Kessler
1927		T. Fort	H.S. Everett		Mabel G. Kessler
		"	John S. Gold	"	
1929	Louis Ingold	"	'	"	
1933	F.W. Owens	T.M. Putnam	"	E.R. Smith	[Librarian
			Secretary-Treasurer		office
1936	G.C. Evans	W. E. Milne	John S. Gold		discontinued)
1939	W.E. Milne	Lincoln LaPaz	"		
1942	Tomlinson Fort	H.H. Downing	"		
1945		E.H.C. Hildebrandt			
1948	C.C. MacDuffee	D.R. Holl	E.H.C. Hildebrandt		
					Editor-General
1951		W.M. Whyburn	J.S.Frame		Ruth Stokes (49-55)
1954	S.S. Cairns	J.S. Frame	R.V. Andree		Franz Hohn (55-57)
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1960		R. H. Bing			
1963	J.S. Frame	H.T. Karnes	R.V. Andree		Seymour Schuster (63-)

APPENDIX II. COUNCILORS GENERAL OF PI MU EPSILON.

1914 Florence A. Lane, Helen Mary Barnard, Edward Jay Cottrell, Adolph Sussman
1922 E. D. Hedrick, Roeven, Rasor
1926 R. D. Carmichael
1929 E. D. Roe, Jr., R. D. Carmichael, E. R. Hedrick, Mitchell
1933 E. R. Hedrick, T. Fort, C. S. Latmin, Louis Ingold
1936 W. C. Brenke, Alan Campbell, D. Lehmer, F. W. Owens
1939 H. H. Downing, W. W. Elliott, G. C. Evans, R. A. Johnson
1942 W. C. Brenke, P. J. Daus, E. H. C. Hildebrandt, W. P. Ott
1945 George Williams, C. A. Hutchinson, C. H. Richardson, E. R. Smith
1948 S. S. Cairns, T. Fort, J. S. Gold, A. H. Kempner 515
1951 S. S. Cairns, T. Fort, Sophia L. McDonald, Ruth W. Stokes, H. C. Bennett, ex. off.
1954 Wealthy Babcock, R. F. Graesser, S. L. McDonald, R. W. Stokes, H. S. Thurston, Henry W. Farnham, ex. off.
1957 R. F. Graesser, Harriet M. Griffin, E. H. C. Hildebrandt, R. L. San Soucie
1960 J. C. Eaves, Marion K. Fort, Jr., Ivan Niven, Angus E. Taylor
1963 Josephine Chanler, Roy Dubisch, Kenneth O. May, Francis Regan

APPENDIX III. Pi Mu Epsilon Chapters installed list was published with the Constitution and By-laws.

DEFINING THE CHARACTERISTIC POLYNOMIAL WITHOUT DETERMINANTS

S. Cater, University of Oregon
(Oregon Alpha)

In this exposition we will give an adequate definition of the characteristic polynomial of an n by n matrix with entries in an algebraically closed field F without resorting to determinants, or to Jordan forms, or to ideals in the ring of polynomials over F. It will be apparent from our definition that the characteristic polynomial of (c_{ij}) will be determined in the expected way by the entries on the main diagonal of any triangular matrix similar to (c_{ij}). We will also define the characteristic polynomial of a linear operator T on an n-dimensional vector space V over F and give a simple proof of the Cayley-Hamilton equation. Our arguments will rest primarily on the uniqueness of the dimension of a vector space. We hope our development will also be of some intrinsic interest. Lemmas 1 and 2 are trivial, but we briefly sketch proofs of them for the sake of completeness.

Lemma 1. Let S_1, S_2 be commuting linear operators on V, let W be the null space of S_1S_2 and let W_1 be the null space of S_1. Then dim W $\le$ dim $W_1 + W_2$.

Proof. $[(V)S_1]S_2 = [(V)S_2]S_1 \subset (V)S_1$ and S_2 maps $(V)S_1$ into itself. Hence dim W = n − dim $(V)S_1S_2$ = n − [dim $(V)S_1$ − dim $W_2 \cap (V)S_1$]
= dim W_1 + dim $W_2 \cap (V)S_1 \le$ dim W_1 + dim W_2.

Lemma 2. There is a basis $[z_1, \dots, z_n]$ of V such that for any $i = 1, \dots, n$, the vector z_iT is in the span of the vectors $z_i, z_{i+1}, \dots, z_n$.

Proof. The proof is by induction on n. For n = 0 or 1 there is nothing to prove. Assume the Lemma is valid on any vector space over F of dimension $\le n - 1$. Select any nonzero $z \in V$. Then there are scalars r_i, not all 0, such that $\sum_0^n r_i(zT^i) = 0$. Since F is algebraically closed there are scalars c_i such that $z[(T - c_1)(T - c_2)\cdots(T - c_n)] = 0$. Since the product of nonsingular operators is nonsingular, it follows that there is a scalar c such that T − c is singular and dim $(V)(T - c) \le n - 1$; set m = dim $(V)(T - c)$. Select a basis $[z_1, \dots, z_n]$ of V such that $[z_{n-m+1}, \dots, z_n]$ constitutes a basis of $(V)(T - c)$ satisfying the desired property relative to the operator T − c. Then $[z_1, \dots, z_n]$ has the desired property relative to T − c, and hence relative to T. This completes the induction.

Observe that the matrix of T relative to the basis $[z_1, \dots, z_n]$ constructed in Lemma 2 is triangular: i.e., there are only zeros below the main diagonal. We call a triangular matrix (c_{ij}) a <u>triangulation</u> of T if there is a basis of V which, with (c_{ij}), gives rise to the operator T. Lemma 2 shows that T has at least one triangulation.

Theorem 1. For any $c \in F$, let V_c denote the null space of $(c - T)^n$ and let (c_{ij}) be a triangulation of T. Then $V_c = (0)$ for all but finitely many $c \in F$, $(c_{11} - x)(c_{22} - x)\cdots(c_{nn} - x) = \prod_{c\in F}(c - x)^{\dim V_c}$ and $\prod_{c\in F}(c - T)^{\dim V_c} = 0$ on V.

Proof. Let $[z_1, \dots, z_n]$ be a basis of V which, with (c_{ij}), gives rise to the operator T. Let $(d_{ij}(s))$ be the matrix of the operator $c_{ss} - T$ relative to $[z_1, \dots, z_n]$ for $s = 1, \dots, n$. Then $(d_{ij}(s))$ is triangular and $d_{ss}(s) = 0$ for all $s = 1, \dots, n$. The first column of $(d_{ij}(1))$ is zero, the first two columns of $(d_{ij}(1))(d_{ij}(2))$ are zero, the first three columns of $(d_{ij}(1))(d_{ij}(2))(d_{ij}(3))$ are zero, and so on. Finally all the columns of the product $(d_{ij}(1))\cdots(d_{ij}(n))$ are zero and $(c_{11} - T)(c_{22} - T)\cdots(c_{nn} - T) = 0$ on V.

Select $d \in F$ and suppose d occurs k times on the main diagonal of (c_{ij}) (possibly k = 0). An inspection of the matrix of the operator d − T relative to the basis $[z_1, \dots, z_n]$ shows that in the basis representation of the vector $z_i(d - T)^n$ the coefficient of z_i is $(d - c_{ii})^n$ and the coefficient of z_j is 0 for all $j < i$. Hence the vectors $z_i(d - T)^n$ for which $c_{ii} \ne d$ are linearly independent, dim $(V)(d - T)^n \ge n - k$ and dim $V_d \le k$. Since there are only n entries on the main diagonal of (c_{ij}) it follows that $V_c = (0)$ for all but finitely many $c \in F$ and $\sum_{c\in F} \dim V_c \le n$. But $\prod_{c=c_{ii}} (c - T)^n = 0$ by the preceding paragraph, and by repeated applications of Lemma 1 we have $\sum_{c\in F} \dim V_c = n$. Again because there are only n entries on the main diagonal of (c_{ij}) we have k = dim V_d, and clearly $(c_{11} - x)\cdots(c_{nn} - x) = \prod_{c\in F}(c - x)^{\dim V_c}$. This concludes the proof.

The characteristic polynomial of T we define to be $p(x) = \prod_{c\in F}(c - x)^{\dim V_c}$. Any operator similar to T has the same triangulations and, by Theorem 1, has the same characteristic polynomial. The Cayley-Hamilton equation $p(T) = 0$ also follows from Theorem 1. We define the characteristic polynomial of an n by n matrix with entries in F in the obvious manner. Clearly the characteristic polynomial of (c_{ij}) is determined by the entries on the main diagonal of any triangular matrix similar to (c_{ij}).

By employing the determinants of n by n matrices with entries in F[x] and the multiplicative property of the determinant, we can easily show that our definition of the characteristic polynomial is equivalent to the conventional definition. Our definition suffers the weakness of being useless in computing characteristic polynomials.

In conclusion we will employ dimension to show that in Theorem 1, V is the direct sum of all the nonzero V_c. Since we know $\sum_{c\in F} \dim V_c = n$ it suffices to prove that the subspaces V_c are linearly independent. Fix $d \in F$ and set $W = (V)(d - T)^n$. Then $WT \subset W$, W is annihilated by $\prod_{c\ne d}(c - T)^{\dim V_c}$ and $\sum_{c\ne d} \dim (V_c \cap W) \ge$ dim W = n − dim V_d by the proof of Theorem 1. But dim $(V_c \cap W) \le \dim V_c$, $\sum_{c\ne d} \dim V_c = n -$ din V_d, so dim $(V_c \cap W) = \dim V_c \subset W$ for all $c \ne d$. Again by the proof of Theorem 1, $\sum_{c\in F} \dim (V_c \cap W) = \dim W = \sum_{c\ne d} \dim (V_c \cap W)$, dim $(V_d \cap W) = 0$ and V_d can contain no nonzero vector in the span of all the V_c, $c \ne d$. This concludes the proof. 518

AN EQUIVALENT DEFINITION OF VECTOR PRODUCT AND TOPOLOGICAL CONSIDERATIONS

Robert J. Buck*, California Institute of Technology

1. INTRODUCTION.

In accordance with a suggestion of A. G. Fadell [1], the vector (cross) product of two vectors, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, may be defined as follows:

Definition. $a \times b = f(a,b)$, where the function f satisfies the following conditions:

i. f: $E_3 \times E_3 \to E_3$, where $E_3 \times E_3$ denotes the cartesian product of Euclidean 3-space with itself.

ii. $|f(a,b)| = [(|a||b|)^2 - (a\cdot b)^2]^{\frac{1}{2}}$, where $a\cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$.

iii. $f(a,b)\cdot a = f(a,b)\cdot b = 0$.

iv. $f(i,j) = k$, where $i = (1,0,0)$, $j = (0,1,0)$, and $k = (0,0,1)$.

v. f is continuous; that is, for every $\epsilon > 0$, there exists a $\delta > 0$, such that $|a - a_0| < \delta$ and $|b - b_0| < \delta$ implies $|f(a,b) - f(a_0, b_0)| < \epsilon$ for all (a_0, b_0) in $E_3 \times E_3$.

We show that the definition above agrees with the usual representation of the vector product in the sense that a necessary and sufficient condition that a function, f, satisfies properties i. - v. above, is that for all pairs, (a,b), in $E_3 \times E_3$ we have

$$(1.0) \qquad f(a,b) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

In other words, the class F, of functions satisfying properties i. - v., consists of precisely one member, namely that function defined by (1.0).

At the expense of being redundant, we formulate our problem as a theorem:

Theorem I. A necessary and sufficient condition that $f \in F$, is that

$$f = \left\{\left((a,b), \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right) : (a,b) \in E_3 \times E_3\right\}.$$

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The proof of the sufficiency follows from a straightforward, algebraic argument, as will be seen. The necessity, however, is less immediate and involves topological considerations. It is the proof of the necessity, then, which will be the main concern of this paper.

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2. PROOF OF THEOREM I.

Sufficiency:

Lemma I. If for all pairs, (a,b), in $E_3 \times E_3$, we have

$$f(a,b) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{then } f \in F.$$

Proof: The function, f, obviously satisfies property i. Now, since

$$\begin{aligned}
(2.0) \qquad |f(a,b)| &= [(a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2]^{\frac{1}{2}} \\
&= [(a_1 b_1)^2 + (a_1 b_2)^2 + (a_1 b_3)^2 + (a_2 b_1)^2 + \\
&\quad + (a_2 b_3)^2 + (a_3 b_1)^2 + (a_3 b_2)^2 + (a_3 b_3)^2 - (a_1 b_1)^2 \\
&\quad - (a_2 b_2)^2 - (a_3 b_3)^2 - 2(a_2 b_2)(a_3 b_3) - 2(a_1 b_1)(a_3 b_3) \\
&\quad - 2(a_1 b_1)(a_2 b_2)]^{\frac{1}{2}} \\
&= [(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2]^{\frac{1}{2}} \\
&= [(|a||b|)^2 - (a\cdot b)^2]^{\frac{1}{2}},
\end{aligned}$$

we see that f satisfies property ii. Further,

$$\begin{aligned}
f(a,b)\cdot a &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (a_1, a_2, a_3) \\
&= a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) \\
&= 0 \\
&= b_1(a_2 b_3 - a_3 b_2) + b_2(a_3 b_1 - a_1 b_3) + b_3(a_1 b_2 - a_2 b_1) \\
&= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} (b_1, b_2, b_3) = f(a,b)\cdot b,
\end{aligned}$$

and thus property iii. is satisfied. Condition iv. is obvious. Let $a_0 = (a_{01}, a_{02}, a_{03})$ and $b_0 = (b_{01}, b_{02}, b_{03})$ be any two fixed vectors in E_3. By the triangle inequality we have,

$$\begin{aligned}
|f(a,b) - f(a_0, b_0)| \le\ & |a_2 b_3 - a_3 b_2 - a_{02} b_{03} + a_{03} b_{02}| \\
& + |a_3 b_1 - a_1 b_3 - a_{03} b_{01} + a_{01} b_{03}| \\
& + |a_1 b_2 - a_2 b_1 - a_{01} b_{02} + a_{02} b_{01}|.
\end{aligned}$$

We shall show that for any $\epsilon > 0$, there exists a $\delta_1 > 0$, such that 521
$|a - a_0| < \delta_1$ and $|b - b_0| < \delta_1$ imply that

$$|a_2 b_3 - a_3 b_2 - a_{02} b_{03} + a_{03} b_{02}| < \epsilon/3,$$

leaving δ_2 and δ_3 for the remaining summands to be found in the obvious manner. Let $Q = (|a_{02}| + |a_{03}| + |b_{02}| + |b_{03}|)$ and $\delta_1 = [-3Q + (9Q^2 + 24\epsilon)^{1/2}]/12$. Clearly, for $0 < d < \delta_1$, we have,

$$6d^2 + 3Qd - \epsilon < 0, \text{ or}$$

$$(2.1) \qquad d(|a_{02}| + |a_{03}| + |b_{02}| + |b_{03}| + 2d) < \epsilon/3.$$

If $|a - a_0| \le d < \delta_1$ and $|b - b_0| < d < \delta_1$, then it follows that $|a_3| \le d + |a_{03}|$ and $|b_3| \le d + |b_{03}|$; and thus by (2.1),

$$d(|a_{02}| + |b_{02}| + |a_3| + |b_3|) < \epsilon/3.$$

But we have,

$$|a_2b_3 - a_3b_2 - a_{02}b_{03} + a_{03}b_{02}| = |b_3(a_2 - a_{02}) + a_3(b_{02} - b_2)$$
$$+ a_{02}(b_3 - b_{03}) + b_{02}(a_{03} - a_3)|$$
$$\leq d(|a_{02}| + |b_{02}| + |a_3| + |b_3|).$$

Consequently condition v. is satisfied and the lemma is established.

Necessity:

Lemma **II.** If $f \in F$, then for all pairs, (a,b), in $E_3 \times E_3$,

$$f(a,b) = t\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \text{ where } t = \pm 1.$$

Proof: Let $f(a,b) = c = (c_1, c_2, c_3)$. Then by property **iii.**,

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \text{ and}$$
$$b_1c_1 + b_2c_2 + b_3c_3 = 0.$$

If $c \neq (0,0,0)$, then it follows that:

$$c_1 = t(a_2b_3 - a_3b_2),$$
$$c_2 = t(a_3b_1 - a_1b_3), \text{ and}$$
$$c_3 = t(a_1b_2 - a_2b_1);$$

where t is any non-zero real number. By property ii., we must have

$$|(c_1, c_2, c_3)| = [(|a||b|)^2 - (a\cdot b)^2]^{\frac{1}{2}},$$

or by (2.0),

$$|t^2|[(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2]$$
$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2.$$

Hence we see that $t = \pm 1$ and $f(a,b) = t\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, if $f(a,b) \neq (0,0,0)$. On the other hand, if we have $f(a,b) = (0,0,0)$, then

$$|f(a,b)|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$
$$= 0.$$

Thus,

$$a_2b_3 - a_3b_2 = t(a_2b_3 - a_3b_2) = 0 = c_1,$$
$$a_3b_1 - a_1b_3 = t(a_3b_1 - a_1b_3) = 0 = c_2, \text{ and}$$
$$a_1b_2 - a_2b_1 = t(a_1b_2 - a_2b_1) = 0 = c_3$$

for any real t; and the lemma follows.

Lemma **III.** Let $f \in F$. Then for all pairs, (a,b), in $E_3 \times E_3$, $f(a,b) = (0,0,0)$ if and only if $a = (0,0,0)$ or $b = (0,0,0)$, or else $a = kb$, where k is any real number.

Proof: By Lemma **II**, if $a = (0,0,0)$ or $b = (0,0,0)$ or $a = kb$, then $f(a,b) = (0,0,0)$. Conversely, suppose $a \neq (0,0,0)$, $b \neq (0,0,0)$ and $f(a,b) = (0,0,0)$. Then by condition **ii.** and formulas (2.0),

$$a_2b_3 - a_3b_2 = 0,$$
$$a_3b_1 - a_1b_3 = 0, \text{ and}$$
$$a_1b_2 - a_2b_1 = 0.$$

It follows from these equations, that if b_i $(i = 1, 2, \text{ or } 3)$ is not zero, then a_i is not zero. Further, $k = a_i/b_i$, depending upon which of b_1, b_2, or b_3 is not zero.

The preceding lemma is not essential for the proof of Theorem **I**; however, since we are interested in the set of vector pairs whose image under a function in F is not zero, the lemma will be useful in defining the zero's of members of **F**.

Let $Z = \{(a,b): (a,b) \in E_3 \times E_3, (f)(f \in F \text{ implies } f(a,b) = 0)\}$**. Then by Lemma **III**, we have,

$$Z = \{(a,b): (a,b) \in E_3 \times E_3, \ (a = 0) \vee (b = 0) \vee (a = kb)\},$$

where it is understood that k is any real number. We now denote the set $(E_3 \times E_3) - Z$ by **S**. If f is any function in F, and if

$$A = \{(a,b): (a,b) \in S, f(a,b) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\}, \text{ and}$$

$$B = \{(a,b): (a,b) \in S, f(a,b) = -\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\},$$

then it is clear that $S = A \cup B$ and $A \cap B = \emptyset$. Further, we may prove:

Lemma IV. For all pairs, (a,b), in S, there exists a $\delta > 0$, such that for all (c,d) in S, if $(a,b) \in A$, $|a - c| < \delta$, and $|b - d| < \delta$, then $(c,d) \in A$.

Proof: Suppose, on the contrary, that there exists a pair, (a,b), in **S** such that for all $\delta > 0$, there exists a pair, (c,d), in S such that $(a,b) \in A$, $|a - c| < \delta$, $|b - d| < \delta$, and $(c,d) \in B$. Then in particular, for each positive integer n, there exists a pair, (c,d), in **S** such that $(a,b) \in A$, $|a - c| < 1/n$, $|b - d| < 1/n$, and $(c,d) \in B$. For each n, let us define the set K_n by the following:

$$K_n = \{(x,y): (x,y) \in B, |a - x| < 1/n, |b - y| < 1/n\}.$$

Clearly, each set, K_n $(n = 1, 2, \ldots)$, contains an infinite number of points of **S**. By the Axiom of Choice [2], there exists a collection, T, defined as follows:

$$T = \{(K_n, (x,y)): n \in J, (x,y) \in K_n,$$
$$[(r,s) \, K_j, (u,v) \, K_n, (r,s) \# (u,v) \text{ implies } K_j \neq K_n]\},$$

where J denotes the set of positive integers. Hence, we may define a function, C, on J as follows: $C(n) = (x,y)$, where $(K_n, (x,y)) \in T$. Then the set, $\{C(n): n \in J\}$, is a net in **S** and, by our supposition, converges to (a,b). In order that f be continuous, it is necessary and sufficient that if the net $\{C(n): n \in J\}$ converges to (a,b), then the net $\{f(C(n)): n \in J\}$ converges to $f(a,b)$ [3]. We shall establish the lemma by showing that, in fact, $\{f(C(n)): n \in J\}$ converges to $-f(a,b)$. For notational purposes, we let $C(n) = (x_n, y_n)$. As in the proof of Lemma **I**,

$$|f(C(n)) + f(a,b)| \leq |x_{n3}y_{n2} - x_{n2}y_{n3} + a_2b_3 - a_3b_2|$$
$$+ |x_{n1}y_{n3} - x_{n3}y_{n1} + a_3b_1 - a_1b_3|$$
$$+ |x_{n2}y_{n1} - x_{n1}y_{n2} + a_1b_2 - a_2b_1|,$$

and for any $\epsilon > 0$, we find an integer N_1, such that $n > N_1$ implies that $|x_{n3}y_{n2} - x_{n2}y_{n3} + a_2b_3 - a_3b_2| < \epsilon/3$. Let N_1 be least integer, greater than or equal

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** Here and in the following, the symbols "0" and "(0,0,0)" are used interchangeably whenever confusion between the scalar zero and the zero vector is unlikely.

$$[9R + (81R^2 + 72)^{\frac{1}{2}}]/2\epsilon,$$

where $R = [|a_2| + |a_3| + |b_2| + |b_3|]$. Then for $n > N_1$, we have,

$$\epsilon n^2 - 9n - 18 > 0.$$

Proceeding as in the proof of Lemma I, we find that, indeed, $\{f(C(n)) : n \in J\}$ converges to $-f(a,b)$.

Note that Lemma IV remains true when the symbol "A" is replaced by the symbol "B" in the lemma; that is, an argument analogous to that given for the proof of Lemma IV suffices to prove the following: 524

Lemma IV'. For all pairs, (a,b), in S, there exists a $6 > 0$, such that for all (c,d) in S, if $(a,b) \in B$, $|a - c| < 6$, and $|b - d| < 6$, then $(c,d) \in B$.

By condition iv. of our proposed definition, the set A, as defined above, is non-empty for every f in F. If we further assume that for some function, f, in F, the set B is non-empty, then the discussion immediately preceding Lemma IV and the lemma itself give us:

Lemma V. Let S have the relativized product topology of $E_3 \times E_3$. If there exists a function, f, in F such that the set,

$$B = \{(a,b) : (a,b) \in S,\ f(a,b) = -\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\},$$

is non-empty, then the pair, (A,B), is a separation of S.

Lemma VI. Let X and Y be topological spaces, and let g be an open, monotone mapping of X onto Y. If X is separated, then Y is also separated.

Proof: Let $X = A \cup B$, where $A \# \emptyset \# B$, $A \cap B = \emptyset$, and both A and B are open in X. Further, let A' and B' be defined as follows:

$$A' = \{y : y \in Y,\ g^{-1}(y) \subset A\}, \text{ and}$$
$$B' = \{y : y \in Y,\ g^{-1}(y) \subset B\}.$$

Since g is monotone and $A \cap B = \emptyset$, we must have $Y = A' \cup B'$ and $A' \cap B' = \emptyset$. Furthermore, if A', say, were empty, then $g^{-1}(Y) = B$, but this contradicts the fact that $A \# \emptyset$. Since neither A nor B is empty, $A' \# \emptyset \# B'$. Clearly, $A' \subseteq g(A)$. Conversely, if $y \in g(A)$ and $y \notin A'$, then $g^{-1}(y) \subset B$, and $g^{-1}(y) \cap A \# \emptyset$, which is a contradiction. Hence $A' = g(A)$ and $B' = g(B)$, from which it follows that both A' and B' are open in Y. Thus the pair, (A',B'), separates Y.

It is now clear that the assumption that there exists a function, f, in F such that for at least one pair, (a,b), in S,

$$f(a,b) = -\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

will lead to a contradiction, if we can exhibit an open, monotone function mapping S onto a connected space.

Let D_1 be the projection, P_1, restricted to S, of $E_3 \quad E_3$ onto its first coordinate space, E_3. We then have,

Lemma VII. $D_1(S) = E_3 - \{(0,0,0)\}$.

Next, we may show: 525

Lemma VIII. The set S is open in the product topology of $E_3 \times E_3$.

Proof: Let f be any member of F. Then since $E_3 \times E_3 - S = f^{-1}(0,0,0)$, which is closed by the continuity of f, S must be open in $E_3 \times E_3$.

Lemma IX. The function, D_1, is an open mapping of S onto $E3 - \{(0,0,0)\}$.

Proof: Let V be any open subset of S. Then by Lemma VIII and the definition of the relativized topology for S, V is open in $E_3 \times E_3$. Then $D_1(V) = P_1(V)$, which is open in E_3 [3]. Further, $D_1(V) \subset E_3 - \{(0,0,0)\}$ and $E_3 - \{(0,0,0)\}$ is open in E_3. Hence, $D_1(V)$ is open in $E_3 - \{(0,0,0)\}$.

Lemma X. The function, D_1, is monotone.

Proof: Let $a \in E_3 - \{(0,0,0)\}$ and let (a,x) and (a,y) be any two points of $D_1^{-1}(a)$. Then by Lemma III, the points, x and y, are elements of $E_3 - [c : c = ka,\ k \in \text{Reals}]$. We have two cases. If y is not coplanar with x and the line,

$$L = \{c : c = ka,\ k \in \text{Reals}\},$$

then we may define a function g as follows: for all $u \in [0,1]$, $g(u) = (1 - u)x + uy$, which is continuous on $[0,1]$. If for some u and some real number t, $g(u) = ta$, then it is clear that x, y, and L are coplanar, contradicting our assumption. On the other hand, if x, y, and L are coplanar, then let q be any point not on plane $P = \{c : c = k_1 x + k_2 y,\ k_1, k_2 \in \text{Reals}]$. Further, let r be any fixed point in the open interval $(0,1)$. Then we define the following function:

$$g(u) = \begin{cases} x + (u/r)(q - x), & \text{if } u \in [0,r] \\ q + ((u - r)(y - q))/(1 - r), & \text{if } u \in [r,1], \end{cases}$$

which is continuous on $[0,1]$. If for some u in $[0,r]$ or some u in $[r,1]$, and if for some real number t, $g(u) = ta$, then

$$(u/r)q = ta + ((u/r) - 1)x, \quad \text{or}$$
$$(1 - u)q/(1 - r) = ta + (r - u)y/(1 - r),$$

both of which contradict our assumptions. In any event, there exists a continuous function, g, defined on $[0,1]$, such that, $g(0) = x$ and $g(1) = y$ and such that, if $u \in [0,1]$, then $g(u) \in E_3 - L$. We now define a function, h, as follows: for all $u \in [0,1]$, $h(u) = (a, g(u))$. Clearly, h is continuous and hence D_1 is monotone.

Lemmas V - X show that the assumption that F contains more than one function implies that $E_3 - \{(0,0,0)\}$ is separated. But it can be shown [2], that for each $n \geq 2$, the space $E_n - 0$ is connected, Q.E.D.

3. ADDITIONAL THEOREMS. 526

It is apparent from the proof of Lemma X, above, that the inverse image of a point under D_1 is arcwise connected; and that this follows from the arcwise-connectedness of the space, $E_3 - L$. The situation generalizes to the cartesian product of arbitrarily many spaces.

For emphasis primarily, we make the following definition: a function will be called arcwise connected if, and only if, the inverse image of the function at a point is arcwise connected. Let A be any indexing set and let $\times\{Y_a : a \in A\}$ denote the cartesian product of the spaces, Y_a. Then we have, Theorem II. If for each $a \in A$, the space Y_a is arcwise connected, then for each $a \in A$, the projection, P_a, of $\times\{Y_a: a \in A\}$ onto its a-th coordinate space, Y_a, is arcwise monotone.

Proof: Let $c \in Y_b$, and let x and y be any two points of $P_b^{-1}(c)$. Since each coordinate space is arcwise connected, for each $a \in A$, there exists a continuous function, g_a, mapping the closed unit interval onto a subset of Y_a, such that $g_a(0) = x_a$ and $g_a(1) = y_a$, where x_a and y_a denote the a-th coordinates of the points x and y. Note that we must have for all $u \in [0,1]$, $g_b(u) = c$. By the Axiom of Choice, we may construct the following collection:

$$G = \{g_a: a \in A, \ [g_d \neq g_e \text{ implies } d \neq e]\}.$$

Let m be a function defined on the closed unit interval such that, for all $u \in [0,1]$, $m(u)_a = g_a(u)$, i.e., the a-th coordinate of $m(u)$ is $g_a(u)$. Then it follows that m is a continuous mapping of $[0,1]$ onto $m([0,1])$, which is contained in $\times\{g_a([0,1]): a \in A\}$ [3]. Since $g_b([0,1]) = \{c\}$, $P_b^{-1}(c)$ is arcwise connected. Q.E.D.

It is interesting to note that the proof given for Theorem II may be applied to obtain the following result:

Theorem III. A necessary and sufficient condition that the product space, $\times\{Y_a: a \in A\}$ be arcwise connected, is that for each $a \in A$, the space Y_a is arcwise connected.

Proof: We merely outline the proof.

Necessity: For each $a \in A$, the projection P_a is continuous [3].

Sufficiency: For each pair of points, x and y, in $\times\{Y_a: a \in A\}$, we may form the collection:

$$G_{x,y} = \{g_a: a \in A, \ [g_d \neq g_e \text{ implies } d \neq e]\},$$

where g_a is a continuous mapping of $[0,1]$, onto a subset of Y_a such that, $g_a(0) = x_a$ and $g_a(1) = y_a$. Proceeding as in the proof of Theorem II, we find that $\times\{Y_a: a \in A\}$ is arcwise connected. Q.E.D.

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Edited by

M. S. Klamkin

State University of New York
at Buffalo

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate, signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to Professor M. S. Klamkin, Division of Interdisciplinary Studies, University of Buffalo, Buffalo 14, New York.

PROBLEMS FOR SOLUTION

159.* Proposed by David L. Silverman, Beverly Hills, California.

If A_n denotes the largest integer divisible by all the integers less than its n^{th} root, show that $A_2 = 24$ and $A_3 = 420$. Find a general formula for A_n.

160. Proposed by Sidney Kravitz, Dover, New Jersey.

"I have here," said the editor, "a cryptarithm which shows a two digit number being multiplied by itself. You will note that the subproducts are not shown, only the number being squared and the final product."

"Well," said the reader, "I've tried to solve this cryptarithm but the solution is not unique. It is possible that I might be able to give you the answer if you told me whether the number being squared is odd or even."

"The number being squared is odd," said the editor.

"Good," said the reader. "I was hoping you would say that. I now know the answer."

What is the solution to this unique cryptarithm?

161.* Proposed by Paul Schillo, University of Buffalo.

It is conjectured that the smallest triangle in area which can cover any given convex polygon has an area at most twice the area of the polygon.

162. Proposed by M. S. Klamkin, University of Buffalo.

If a surface is one of revolution about two axes, show that it must be spherical.

143. Proposed by M. S. Klamkin, University of Buffalo.
If λ is a rational approximation to the $\sqrt{N}$ (assumed irrational), find an always better rational approximation.

Solution by David L. Silverman, Beverly Hills, California.
The arithmetic mean of λ and N/λ will not always be a better approximation but their harmonic mean $\lambda' = 2N\lambda/(N + \lambda^2)$ will be. (Editorial note: To take care of the possibility of λ being negative, replace λ by $|\lambda|$).

Solution by the proposer.
An always better rational approximation is given by

$$\lambda' = \frac{N + (N + 2)|\lambda|}{N + 2 + |\lambda|} .$$

Since

$$|\lambda' - \sqrt{N}| = ||\lambda| - \sqrt{N}|\left(\frac{N + 2 - \sqrt{N}}{N + 2 + |\lambda|}\right) < ||\lambda| - \sqrt{N}| \le |\lambda - \sqrt{N}| .$$

Generalizations of this result to a class of always better approximations and also to the roots of a class of equations other than $x^2 = N$ will be given in a subsequent paper.

Also solved by Michael Goldberg, H. Kaye and Paul Meyers.

145. Proposed by David L. Silverman, Beverly Hills, California.
For what integers a and b $(0 < a < b)$ are the roots of

$$x^4 + (a + b)x^3 + (a + ab + b)x^2 + (a^2 + b^2)x + ab = 0,$$

integers?

Solution by K. S. Murray, New York City.
The equation factors into

$$(x^2 + bx + a)(x^2 + ax + b) = 0.$$

Thus,

$$b = m + n = rs,$$
$$a = mn = r + s, \quad (m, n, r, s > 0).$$

Assume $r \ge s$, $m \ge n$; then

$$2r \ge mn \text{ and } 2m \ge rs,$$

which implies that $4 \ge ns$. This leads to the unique solution $a = 6$, $b = 5$ or vice-versa.

Also solved by Bob Emmett, H. Kaye, Paul Meyers, John Stout, M. Wagner, F. Zetto and the proposer.

146. Proposed by C. W. Trigg, Los Angeles City College.
Find a set of three-digit numbers, each of which is a permutation of the same three digits, which when divided by the sum of the digits yields two pairs of alternate integers.

Solution by the proposer.
If $\overline{ABC}/(A + B + C) = M$ and $\overline{ACB}/(A + B + C) = M + 2$, then $\overline{CB} - \overline{BC} = 2(A + B + C)$. Since $\overline{ABC} \equiv A + B + C \equiv \overline{ACB}$ (mod 9), $\overline{CB} - \overline{BC} \equiv 0$ (mod 9), and thus also $2(A + B + C) \equiv 0$ (mod 9).
The case $A + B + C = 18$ is impossible. For $A + B + C = 9$, the unique set is 1, 3, 5 and the solution is

$135/9 = 15$, $153/9 = 17$ and $513/9 = 57$, $531/9 = 59$.

There are three other sets of digits, each of which leads to a single pair of alternate integers: $324/9 = 36$, $342/9 = 38$; $648/18 = 36$, $684/18 = 38$; and $702/9 = 78$, $720/9 = 80$.

Also solved by K. S. Murray, M. Wagner and F. Zetto.

147. Proposed by Leo Moser, University of Alberta.
Show that the maximum number of terms of different form in a polynomial of degree n in k variables is the same as the maximum number of terms of different form in a polynomial of degree k in n variables.

Solution by Frank Bongiovanni, University of Buffalo.
The number of combinations of n things taken r at a time when each may be taken as often as we please is the same as the number of homogeneous products of degree r which can be formed from the n letters say a, b, c, $\cdots$, k. The sum of these products is the coefficient of x^r in the expansion of

$$(1 + ax + a^2x^2 + \cdots)(1 + bx + b^2x^2 + \cdots)\cdots(1 + kx + k^2x^2 + \cdots).$$

By setting $a = b = \cdots = k = 1$, the number of such products is then the coefficient of x^r in the expansion of $(1 + x + x^2 + \cdots)^n$ or $(1 - x)^{-n}$. This gives the number as $\binom{n + r - 1}{r}$. If we now add an extra letter z to a, b, $\cdots$, k, this will give us all the terms of different form in a polynomial in z of degree r. This number is then $\binom{n + r}{r}$ which is symmetric in n and r.

Also solved by K. S. Murray, David L. Silverman, John Stout, M. Wagner, F. Zetto, and the proposer.

148. Proposed by M. S. Klamkin, University of Buffalo.
If a convex polygon has three angles of 60°, show that it must be an equilateral triangle.

Solution by Edward L. Spitznagel, Jr., University of Chicago.
The sum of the interior angles of an n-gon is $180°(n - 2)$. Since all the interior angles of a convex polygon are less than 180° we have the inequality

$$3(60°) + (n - 3)(180°) > (n - 2)(180°)$$

unless $n = 3$. Thus, $n = 3$ and triangle is equilateral.

Also solved by H. Kaye, K. S. Murray, Paul Meyers, M. Wagner and the proposer.

Editorial note: A simple extension of this result is the following: If n of the interior angles of a convex polygon add up to $(n - 2)(180^\circ)$, then the polygon must be precisely an n-gon. Put this way, the result is not particularly surprising. Another special case is that if a convex polygon has four right angles it mast be a rectangle.

~~BOOK REVIEWS~~

Edited by

Franz E. Hohn, University of Illinois

Principles of Abstract Algebra. By R. W. Ball. New York; Holt, Rinehart and Winston; 1963. ix + 290 pp., $6.00.

The author's preface states: "This book presents an approach to abstract algebra that is directed to undergraduate students at an intermediate level. For most students this would come after a year of beginning calculus, although it could well be studied earlier." The reviewer agrees with the author's appraisal of his book.

Considering how carefully this book is written, the reviewer regrets that more material is not presented in the chapters on rings and groups. For example, a ring is defined and many excellent examples are given throughout the text, but little more is done with rings. Chapters 9 and 10, devoted to groups, cover the topics of binary operations, groups, and the laws of exponents, finite cyclic groups, finite groups, and reduced groups of residues.

The author gives a good discussion of the real number system, the complex number system, and polynomials. Additional chapters treat the theory of equations, real roots of real polynomial equations, rings of matrices, and systems of linear equations. Several of the theorems in the last ten chapters (11-20) are stated but not proved (for example, the completeness of the real number system).

In view of the choice of topics, this book would be ideally suited as a text for future high school teachers, or for those students not yet ready for a more rapid, more detailed treatment of these topics.

University of Illinois — Hiram Payley

General Stochastic Processes in the Theory of Queues. By Vaclav E. Benev. Reading, Mass., Addison-Wesley, 1963. viii + 88 pp., $5.75.

Principal results for queues with one server and order of arrival service-time are deduced by methods that are relatively new in queueing theory. Although this cannot be considered an elementary book on the subject, introductory sections have a fine intuitive presentation. Later sections contain an elegant mathematical treatment of delay using very general conditions on the interarrival-times and service-times. Explicit references are given for almost every result in analysis that is used, but results in queueing theory which appear in other recent books on the subject are usually just stated.

University of Illinois — Leone Y. Low

Gravitation: An Introduction to Current Research. By L. Witten, et al. New York, Wiley, 1962. x + 481 pp., $15.00.

This is a collection of articles devoted to the current status of our knowledge and theories about gravitation from the point of view of general relativity. The chapter headings are: 1. Experiments on Gravitation; 2. Exact Solutions of the Gravitational Field Equations; 3. The Equations of Motion; 4. The Cauchy Problem; 5. Conservation Laws in General Relativity; 6. Gravitational Radiation; 7. The Dynamics of General Relativity; 8. The Quantization of Geometry; 9. A Geometric Theory of the Electromagnetic and Gravitational Fields; 10. Geometrodynamics; 11. Relativistic Cosmology.

Chapters 1 and (to a somewhat lesser extent) 10 are descriptive and require no specialized background for their comprehension. The reader will need a command of tensor analysis to pursue profitably the remaining chapters.

The first chapter, which consists of a discussion of recent and not so recent experiments designed to throw light on the nature of gravitation, is noteworthy for its description of the great observational difficulties involved in the attempt to detect at solar eclipse the predicted outward displacement of star images from the sun's disk. Many appear to be unaware of the fact that unavoidably large observational errors are inherent in this experiment, rendering the results less reliable than commonly believed. Chapter 10 describes recent attempts to formulate classical physics and quantum mechanics entirely in terms of geometry--a notion (in the case of classical mechanics) that goes back as far as Riemann.

The remaining chapters represent advances along the more customary lines of general relativity theory and are valuable for bringing the reader up-to-date in this field.

University of Illinois — Ray G. Langebartel

~~Studies in Modern~~ Algebra. A. A. Albert, Editor. (Vol. 2, M. A. A. Studies in Mathematics.) Englewood Cliffs, N. J., Prentice-Hall, 1963. 190 pp., $4.00.

The first half of this book consists of two articles by S. MacLane on "recent advances in algebra." It is quite interesting to compare these, for a twenty-four year interval separates their dates. Four papers on non-associative algebra comprise the second half: What is a loop?, by R. H. Bruck; The four and eight square problem and division algebras, by C. Curtis; A characterization of the Cayley numbers, by E. Kleinfeld; Jordan algebras, by L. Paige. There is also an introduction by A. A. Albert which summarizes these articles.

Each of the authors has written most lucidly, and this book is accessible to anyone who enjoys the Mathematics Monthly.

University of Illinois — Joseph Rotman

Representation Theory of Finite Groups. By C. W. Curtis and I. Reiner. New York, Wiley, 1963. xiv + 686 pp., $20.00.

This is the most important single publication on representation of finite groups and on the part of the theory of rings and algebras that is related to these representations. There is no other book as comprehensive or as instructive. The presentation is modern; for example, it takes into account the use of the concepts of homological algebra in the theory of rings and algebras. There are several introductory chapters which could serve as a basis of a good graduate course in the subject. The later chapters serve well as an introduction to the . literature and contain basic material which can be of use to all working in this or related fields. The topics covered include on the more elementary level: the Wedderburn structure theory for rings and algebras, elements of algebraic number theory, group characters, and their application. Continuing on, Brauer's characterization of generalized characters and the theory of splitting fields are developed. Nonsemisimple rings and algebras, the theories of integral and modular representations, are other main topics that are covered.

This book is suited for the advanced graduate student and for research workers. Less experienced mathematicians will find the first chapters accessible.

University of Illinois — John H. Walter

An Introduction to Linear Programming and the Theory of Games. By M. Glicksman. New York, Wiley, 1963. x + 131 pp., $2.25 (paper), $4.95 (cloth).

This well-written monograph lives up to the promise of its bright, eye-catching cover. Basic concepts of convex sets, game theory, and linear programming are explained in detail and are illustrated with attractive, simple figures, graphs, and tableaux. Written at the sophomore level and using only tools and concepts of algebra and analytic geometry, this book should be of interest, not only to the bright under-graduate mathematics student, but also to social scientists who are interested in a simple, though rigorous, development of applications.

Elementary proofs of the fundamental extreme point theorem for convex polygons, the fundamental duality theorems of linear programming, and its corollary, the minimax theorem, are included. Definitions and theorems are numbered, and their use is illustrated. The simplex method in linear programming is used to maximize or minimize functions subject to constraints, and to solve m x 2 matrix games. The amusing examples and problems help to heighten interest throughout the book.

The only criticisms are the misprints on pages three and four (24 should be substituted for 28) and the author's not discussing dominated strategies in matrix games.

University of Illinois — Leone Y. Low

Sets, Logic, and Axiomatic Theories. By Robert R. Stoll. San Francisco and London, W. H. Freeman and Co., 1961. Paperbound, x + 206 pp., $2.25.

This book is presented as a text for a one semester, undergraduate course for students who plan to study abstract mathematics, and for prospective high school mathematics teachers. Such students probably would have had no experience with mathematical proofs, except in high school geometry. They would be apt to find the proofs and the concise set notation difficult at first, but these would be valuable to them later. Certainly a student who knew the material in this book would be well prepared to continue in abstract mathematics. The author suggests that a good high school student might find the book stimulating. This seems doubtful. Although no special background is necessary for reading the book, the level of abstraction used in it requires a certain amount of maturity.

Naive set theory is presented in the first chapter as a prerequisite tool for further study in abstract mathematics. This chapter is designed as an expanded version of the "Chapter 0" which appears in many textbooks. Besides set operations, it includes functions, equivalence relations, and ordering relations.

Chapter two deals with the statement or propositional calculus and the first order predicate or functional calculus in terms of validity. The concept of a theorem in one of the calculi is not used in this chapter. The statement calculus is introduced in terms of truth tables and tautologies. The rules of inference, the regularity theorem, and the deduction theorem are all presented as preserving validity. Quantifiers are introduced with examples of translation from ordinary English sentences to formulas of the predicate calculus. The predicate calculus is then presented in much the same way as the statement calculus, although in less detail.

Chapter three introduces axiomatic theories. Groups, affine geometry, and the Peano axioms are used as examples. The ideas of consistency, completeness, and independence are defined. The statement and predicate calculi are presented as axiomatic systems, and the results of chapter two are given in terms of theorems, then the consistency and completeness of these calculi are discussed. Finally, the concepts of meta-languages and object-languages are given.

Chapter four is the reward for the other three chapters, especially one and three. It treats Boolean algebra as an example which ties together all the ideas presented in the previous chapters. Two different axiom sets are given, and are shown to be equivalent, for a Boolean algebra. A one-to-one correspondence between the congruence relations and the homomorphisms of a Boolean algebra is proved. Atoms and ideals are used to characterize Boolean algebras as being isomorphic to algebras of sets. As a final achievement in unifying the subject matter of the book, a direct relationship between statement calculi and Boolean algebras is demonstrated and the suggestion made of investigating validity in a statement calculus in terms of congruence relations in a Boolean algebra.

The book contains many examples and exercises which are interesting for themselves as well as illustrating the ideas being presented.

University of Illinois M. K. Yntema

Matrix Iterative Analysis. By R. S. Varga. Englewood Cliffs, N. J.: Prentice-Hall, 1962. xiii + 322 pp., $7.50 text edition, $10.00 trade edition.

This excellent book is primarily concerned with the analysis of matrix problems arising in the numerical solution of elliptic partial differential equations. It is designed for use as a text by first year graduate students in mathematics. Moreover it will serve as a valuable reference book for workers in this field.

The principal emphasis here is on theory, not practice. There is a thorough treatment of the convergence of matrix iterative schemes, with many theorems on this subject being proved, but there are only a few practical applications discussed. The backbone for a lot of this work is the Perron-Frobenius theory of non-negative matrices which is discussed in Chapter 2.

This reviewer was particularly impressed by the nice way that the author used graphs for illuminating various discussions. With some very elementary ideas from graph theory the author characterizes the structure of matrices arising in this work, gaining clarity and saving words thereby.

Another impressive feature of this book is the bibliography and discussion that follows each and every chapter. Each of these desserts contains a short historical account of the development of ideas presented in the chapter along with references to the original papers.

A list of chapter headings follows:

Chapter 1 -- Matrix Properties and Concepts
Chapter 2 -- Non-Negative Matrices
Chapter 3 -- Basic Iterative Methods and Comparison Theorems
Chapter 4 -- Successive Overrelaxation Iterative Methods
Chapter 5 -- Semi-Iterative Methods
Chapter 6 -- Derivation and Solution of Elliptic Difference Equations
Chapter 7 -- Alternating-Direction Implicit Iterative Methods
Chapter 8 -- Matrix Methods for Parabolic Partial Differential Equations
Chapter 9 -- Estimation of Acceleration Parameters.

University of Illinois Lloyd D. Fosdick

Generalized Analytic Functions. By I. N. Vekua. Reading, Mass.; Addison-Wesley, 1962. xxix + 668 pp., $14.75.

The present book is another volume of the distinguished Adiwes International Series under the editorship of A. J. Lohwater. It gives a systematic and thorough account of the subject of generalized analytic functions--a subject cultivated by Professor Vekua and his school in Russia and by Professor Lipman Bers and his students in the United States. The basic theme is the study of the partial differential equation

$$\frac{1}{2}(w_x + iw_y) + Aw + B\bar{w} = F,$$

where the coefficients are complex-valued, and its intimate connection with the theory of analytic functions. The first part is devoted to the general theory. The second half treats applications of the theory to problems of surface theory and the membrane theory of shells.

This treatise presupposes the usual material of what would generally be beginning graduate courses on real and complex analysis. The present translation has made available to an enlarged group of readers a significant exposition of this important field.

University of Illinois Maurice Heins

Understanding the Slide Rule. By R. F. Graesser. Paterson, New Jersey: Littlefield and Adams, 1963. 141 pp., $1.50 (paper).

This is a carefully written book reflecting the experience of a patient, understanding teacher who has taught the use of the slide rule to engineering students for many years. It should be widely useful to those who wish to learn the use of the instrument.

Those whose mathematical preparation is weak will be particularly helped by the fact that the book presumes almost no previous knowledge of algebra and none at all of trigonometry. All necessary facts are either reviewed or developed simply but sufficiently. Even basic facts of arithmetic are reviewed for those who need such a refresher. Those whose skills are more fluent can of course skip such material.

The emphasis here is on understanding and on the development of genuine skill, for which latter purpose many exercises, with answers, are given.

Finally, the fact that the explanations apply equally well to whatever type of slide rule the student is apt to have, guarantees the usefulness of this book for self-study or in the classroom under nearly all conditions.

University of Illinois Franz E. Hohn

Mathematical Diversions. By J. A. H. Hunter and J. S. Madachy. Princeton, Van Nostrand, 1963. vii + 178 pp., $4.95.

This little book was written by two well-known puzzle enthusiasts. As is the case in most such books, many of the problems are well-known, others are not. The problems include 40 "story teasers" whose solutions lead to diophantine equations or to the analysis of geometrical figures. Solutions to most of these are given.

The mathematical naiveté of the authors is revealed in almost every chapter. One regrets to read "as for the fourth dimension, even mathematicians admit they cannot visualize it," which appears to support the colloquial impression that the higher dimensions belong to some mystical domain of unreality. The authors say (p. 74) of the equation 01 = 1, "This can be proved, but the rigid proof is outside our scope here". It is hard to realize that anyone accomplished in mathematical problem solving would not be aware of the fact that it is necessary to define 01 and convenient to define it as 1.

The treatment of inferential problems in Chapter 4 by Boolean algebra is particularly poor, as is the treatment of probability problems in Chapter 10. These chapters will be incomprehensible to the newcomer and useless to the informed. In Chapter 10 one learns that the probability zero "... implies quite definitely that it is impossible ..." and hence that in the case of infinite sets, "infinitely small" and "zero" must be distinguished. One hopes that the authors have only tried too hard to be informal.

At times facts are stated without proof, even though proofs would be simple and instructive. Some proofs are so informal as to be erroneous. For example, one argument assumes that because each of two sets of lines is infinite. the sets must have a line in common.

Altogether, one arrives at the sad conclusion that this is a hastily prepared volume, not worthy of either the authors or the publishers. A careful reading of the manuscript by a competent critic could easily have eliminated most of the grounds for criticism and made the book acceptable as well as interesting. Some kinds of economy are, in the long run, expensive.

University of Illinois — Franz E. Hohn

Differential Equations: Geometric Theory, Second Edition. By Solomon Lefschetz. New York, Wiley-Interscience, 1963. x + 390 pp., $10.00.

This is an advanced text on differential equations which makes extensive use of topology and matrix methods. A reader who is well grounded in real analysis and linear algebra will find here a rigorous and exciting treatment of many important topics. The book begins with existence theorems and linear systems, treats stability, Liapunov's direct method, and second order equations and systems of equations. This volume is essential reading for the serious student of differential equations, whether he is in pure or applied mathematics.

University of Illinois — Franz E. Hohn

Generalized Functions and Partial Differential Equations. By A. Friedman. Englewood Cliffs, New Jersey; Prentice-Hall; 1963. xii + 340 pp., $7.50.

The first six chapters of this volume offer a more complete study of the theory of generalized functions than has hitherto been available in English. The treatment proceeds from the general to the particular so, for example, the material on distributions appears only after a rather abstract treatment of generalized functions. This approach, while it has the advantage of offering the material in concise form to those with a knowledge of measure theory, point set topology, and functional analysis, places the book at the advanced graduate level and beyond.

Chapters seven to eleven apply the techniques of the first six chapters to some problems in the theory of partial differential equations, with emphasis on the Cauchy or initial value problem for equations with space-independent coefficients. Much is presented here which was recently available only in research journals.

University of Illinois — Julius Smith

NOTE: All correspondence concerning reviews and all books for review should be sent to PROFESSOR FRANZ E. HOHN, 375 ALTGELD HALL, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS.