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CONTENTS

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NETWORKS, HAM SANDWICHES, AND PUTTY'

Stewart S. Cairns

Mathematicians are frequently occupied with problems which, to a non-mathematician, are incomprehensible. On the other hand, they sometimes devote extraordinary efforts to proving the "obvious" or to investigating questions which appear to be merely amusing puzzles. The "obvious", however, occasionally turns out to be false and, if true, is often much more difficult to prove than many a result which is hard to understand. As for problems in the puzzle category, their entertainment value is sufficient justification for them. On the other hand, to appease those who must forever be doing something "useful", let it be noted that mathematical puzzles may have a direct bearing on practical problems. More generally, in pure mathematics as a whole, no one can possibly tell which abstraction will have practical value at some future date. A mathematician can thus justify himself, even from a utilitarian viewpoint, in working on those problems which he finds most fascinating. Mathematics thus done for fun may prove even more useful, in the long run, than much of that which is motivated by applications.

The problems discussed below are intended to be entertaining. All of them belong to the branch of geometry known as *topology,* where properties are studied which do not depend on size or shape. The examples will help to clarify this partial description of the subject.

1. *LINEAR GRAPHS*

This section contains some definitions and results for later use.

DEFINITIONS. A ***linear graph*** L is a figure consisting of ***edges*** and ***vertices*** in 3-dimensional space. A ***vertex*** is a point, and an ***edge*** is an arc joining two vertices, but not passing through a vertex or through a point on another edge. Vertices will generally be denoted by small letters and edges by capitals. If a vertex v is end point of an edge E, then v and E are said to be ***incident.*** A graph may contain ***isolated vertices;*** that is, vertices not incident with edges. We even admit graphs with no edges at all, consisting only of vertices, because such graphs are useful in certain proofs.

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All our reasoning will involve only the numbers of vertices and edges in a graph, and their incidence relations. It will not involve the form (straight or curvilinear) or the lengths of the edges. This is in the spirit of topology.

A vertex v of a graph L will be described as *odd* or *even* according as the number of edges with v for **an end** point is odd or even.

LEMMA 1.1 If a linear graph L is obtained from a linear graph K by adding one edge to **K,** then the number of odd vertices in L exceeds the number in K by +2, 0, or –2.

PROOF. Let E=pq be the edge of L which is not in K. If p is even in K it is odd in **L,** and conversely. The same holds for q. Hence, if p and q are both odd in **L,** then L has two more odd vertices than **K;** if they are both even, it has two less; and if one is odd and one even, it has the same number of odd vertices as K.

THEOREM 1.1 The number of odd vertices in a linear graph L is even.

PROOF. Let L_0 consist of the vertices of L, with all edges deleted. We can build up **L,** commencing with L_0 and inserting the edges one at a time. In L_0, the number of odd vertices is zero. With each added edge, the number of odd vertices is increased by 2, decreased by 2, or left unchanged (Lemma 1.1). Hence, it remains even, no matter how many edges may be inserted.

DEFINITIONS. A *path* is a sequence $\pi = (a, A, b, B, \ldots, Y, z)$ of alternating vertices and edges, in which each edge is preceded by one of its two end-points and followed by the other. A path π is said to *join* its first point to its last, or to *go from* its first point to its last. If the first point of π is the same as the last, π is described as *closed.* A *simple* path is one in which no edge or vertex appears more than once, except, perhaps, for the first point being the same as the last.

Intuitively, a *path* is a sequence of edges which can be continuously traced in the order named, but it is convenient to include the vertices in the definition. A *simple path* is such that a point tracing it from its first to its last point does not go through any point more than once; though it may return to its starting point, in which case we have a *simple closed path.*

DEFINITION. A linear graph L is *connected* if either (1) L has no edges and consists of a single vertex, or (2) L has more than one vertex and each two of its vertices can be joined by a path on L.

THEOREM 1.2 If, on a graph **L,** there exists a path π joining two given different vertices, a and **z,** then there exists a simple path on L from a to z.

PROOF. If $\pi = (a, A, b, \ldots, Y, z)$ is not simple, either an edge or a vertex appears at least twice in it. If an edge appears twice; then, since it is always immediately preceded and followed in π by its end points, at least one of the latter must appear twice.

Hence, if π joins a to $z \neq a$ and is not simple, then π is of the form $(\ldots p, P, q, \ldots, p, \ldots)$ where p may possibly be either a or z. Now let π' be obtained from π by deleting $P, q, \ldots, p$. Then π' has fewer edges than π, and is a path from a to z. If π' is not simple, we can similarly drop out part of it to obtain a path from a to **z,** with fewer edges than π'. Continuing thus, we arrive finally at a simple path from a to **z,** since there cannot be more deletions than there are edges in the original path.

2. *SOME CONNECTEDNESS PROPERTIES.*

LEMMA 2.1 Let L be a connected linear graph, and let K be obtained from L by deleting one edge **E.** Then K is either connected or consists of two separate connected linear graphs, **K'** and **K"**, each containing an end point of **E.**

PROOF. First suppose L has just two vertices p and q. If L has just one edge **E,** then $K = L - E$ consists of $K' = p$ and $K'' = q$. If L has more than one edge, then all its edges join p and **q,** so that $K = L - E$ is connected.

Now suppose L has more than two vertices. Let **p,q** be the end points of **E,** and let r be a third vertex of L. Since L is connected, some simple path π joins r to **q,** by Theorem 1.2. If **E** is not in π, then π joins r to q on $K = L - E$. If **E** is in π, then π is of the form $\pi = (r, \ldots, p, E, q)$. This is so because π ends with q, and an earlier appearance of **E** on π would mean a double appearance of **q,** contrary to the fact that π is simple. Dropping E, q from π, we are left with a path $\pi' = (r, \ldots, p)$ from r to p on $K = L - E$.

We have shown that each vertex of L other than **p,q** can be joined on $K = L - E$ to either p or q. If K is not connected, let **K'** be the graph made up of p and all vertices of K which can be joined to p on K, together with all edges of K incident with such vertices; and let **K"** be similarly defined with q replacing p. It is now easy to verify that K' and **K"** satisfy all the requirements of the Lemma.

LEMMA 2.2 If, in Lemma 2.1, all the vertices of L are even, then K is connected.

PROOF. If p and q are the end points of **E,** then they are the only odd vertices of K. If K were not connected, K' and K" would each have just one odd vertex, contradicting Theorem 1.1.

LEMMA 2.3 Suppose, in Lemma 2.1, that L has just two odd vertices, that **E** = pq is incident with at least one of them, and that $K = L - E$ is not connected. If both p and q are odd in L, then K has no odd vertices. If p is odd and q even in **L,** then K has just two odd vertices, one of them being **q,** both in the same connected part of K.

PROOF. Part of the lemma follows from the fact that each vertex of an edge of L changes from even to odd, or from odd to even, when that edge is deleted. The very last conclusion in Lemma 2.3 follows from Theorem 1.1.

3. *A PROPERTY OF TRACEABLE GRAPHS*

A familiar type of puzzle is to trace a figure composed of lines joining a number of points, without lifting the pencil or tracing any line more than once. This is sometimes possible (Figure 1) and sometimes impossible (Figure 2). A famous old puzzle of this nature related to the seven bridges of Koenigsberg, schematically shown in Figure 2, where a and b represent islands, the edges ef and cd lie along the banks of a river, and the other seven edges are bridges joining the islands to the banks and to one another. The problem was to plan a promenade in which each bridge would be crossed exactly once. It is easy to try the various possibilities and to conclude that no solution exists.* It was a much greater feat, however, to discover a simple test whereby one can quickly decide, merely by counting, whether a given linear graph can be thus traced. Such a test was found in 1736 by the Swiss mathematician Leonhard Euler, stimulated by the puzzle of the bridges.

DEFINITION. A linear graph L is traceable if some path on it, to be called a tracing path, contains each edge of L exactly once, and each vertex at least once. If a is the first point of such a path, and z (possibly the same as a) is the last, then L is traceable from a to z. This corresponds to the idea of traversing L continuously from a to z in such a way as to trace each edge of L exactly once.

LEMMA 3.1 If L is traceable and $\pi = (a, A, \ldots, Y, z)$ is a tracing path for L, then (a) if $z = a$, all vertices of L are even, and (b) if $z \neq a$, a and z are odd, but all other vertices of L are even.

PROOF. If $a \neq z$, then each time a occurs on π, except the first time, it is preceded by one edge and followed by another. By definition of tracing path, all the edges consecutive with a on π are different and they are the only edges of L incident with a. Hence, a is incident on L with exactly $2n + 1$ edges, an odd number, if $n + 1$ is the total number of times a appears on π. The proof is easy to complete, by applications of this type of reasoning to all the vertices in cases (a) and (b).

4. *EULER'S TRACEABILITY THEOREM*

THEOREM 4.1 (EULER). A linear graph L is traceable if, and only if, it is connected and (a) all its vertices are even or (b) exactly two of them are odd. In case (a), L is traceable from any one of its vertices, and each tracing path is closed. In case (b), L is traceable from either odd vertex to the other, and each tracing path has the two odd vertices for end points.

***The author has a theory that the dachshund was developed by citizens of Koenigsberg who walked the legs off their dogs in experimental promenades over the bridges.**

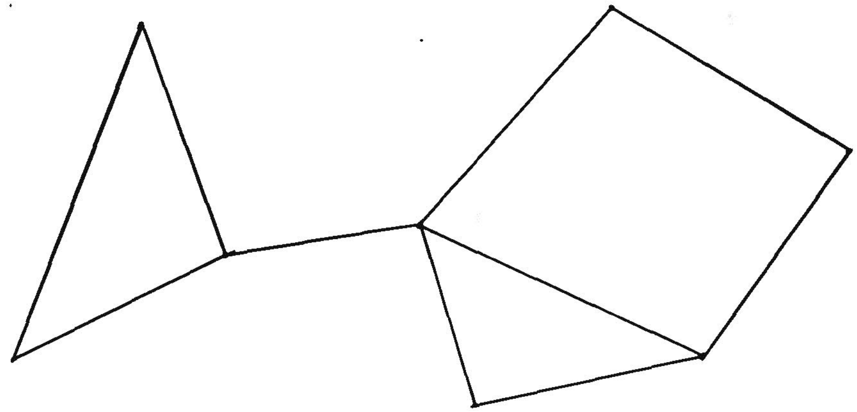

Figure 1. A traceable graph.

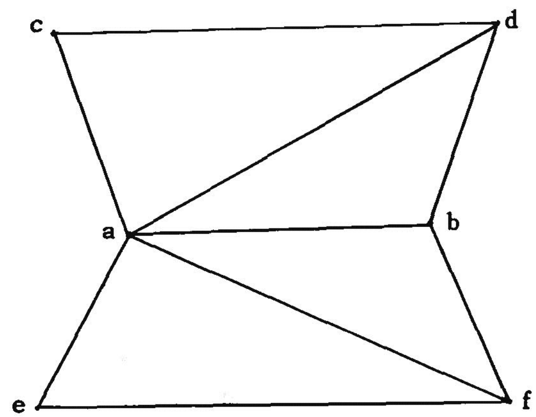

Figure 2. A non-traceable graph.

PROOF. First suppose L traceable and let π be a tracing path. If p and q are two different vertices of **L**, then both occur on π. Suppose the notation such that p occurs before q on π = (. . .p. . .q. . .). The subsequence of π from the first occurrence of p through an occurrence of q is a path joining p and q. Hence L is connected. Lemma 3.1 now completes a proof-of the "only if" part of Theorem 2.2.

We next show that if L is connected and satisfies hypotheses (a) and (b), it is traceable as stated. Our proof will be inductive in the number of edges of L.

CASE I. Suppose L is connected and has just one edge A, hence just two vertices, a and b. Then, condition (b) is fulfilled, and (a, A, b), (b, A, a) are the required tracing paths.

INDUCTIVE HYPOTHESIS. For some number n, Theorem 3.1 has been proved for all graphs with fewer than n edges.

Let p be either of the odd vertices of **L**, if L has odd vertices, and, if it has none, let p be an arbitary vertex of L. Let E = pq be an edge of L incident with **p**, and let K = L–E. First suppose K is connected. If p and q are both odd in **L**, they are even in K and, by inductive hypothesis, there exists a tracing path (q,Q,...,q) for K. But (p,E,q,Q,...,q) is then a tracing path for L. If p and q are both even in **L**, they are odd in **K**, and, by inductive hypothesis, there exists a tracing path (q,Q,...,p) for K. But (p,E,q,Q,...,p) is then a tracing path for L. The remaining possibility is that p be odd and q even in L. Let r be the second odd vertex in L. Then q, r are the only odd vertices in **K**, and a tracing path (q,Q,...,r) for K can, as above, be augmented to a tracing path (p,E,q,...,n). Finally, suppose K is not connected, and let **K'**, **K''** be the connected parts into which it falls (Lemma **2.1**), with p and q in **K'** and **K''** respectively. The case where **p,q** are both even in L cannot arise here, by Lemma 2.2, since p is required to be odd if L has odd vertices. All vertices of **K'** are even, **K'** is connected, and K' has fewer than n edges. Hence, by inductive hypothesis, if **K'** consists of more than the single vertex **p**, there is a tracing path for K' of the form (p,...,p). If K' has no edges, (p,...,p) will stand for just the vertex p. If q is odd in **L**, it is even in **K''**, and, as in the case of **K'**, there is a tracing path (q,...,q) for **K''**, which may reduce to the single vertex q. Then (p,...,p,E,q,...,q) is easily seen to be a tracing path for L from p to q. If q is even in **L**, it is odd in **K''** and, by Lemma 2.3, the odd vertex r of **L**, other than **p**, is an odd vertex of K''. By inductive hypothesis, there is a tracing path (q,...,r) for **K''**. But (p,...,p,E,q,...,r) is then a tracing path for L from p to r.

We have proved that L is traceable as specified in Theorem 4.1. It follows from Lemma 3.1 that all tracing paths are closed in case (a) and go from one odd vertex to the other in case (b).

5. *NETWORKS AND CIRCUITS*

Our next problem has a practical flavor. A ***network*** will mean a connected linear graph, with at least two vertices. A ***circuit*** will mean a simple closed path. This terminology suggests electrical considerations, which led the German physicist G. Kirchhoff in 1847 to deal with the questions treated below. His methods, unlike those here employed, were suggested by the physics of the situation.

A ***tree*** is a network on which no circuit exists.

LEMMA 5.1 A network L is a tree if, and only if, there exists exactly one simple path joining a and **z**, where a and *z* are two arbitrary different vertices of L.

PROOF. By definition of ***connected*** (§1), there is at least one simple path π from a to z (Theorem 1.2). First, suppose L is a tree, and suppose there is a second such path π'. Then π and π', since they are different, can be written in the forms π = (a,...,p,E,...,z) and π' = (a,...,p,F,...,z), where a,...,p (which may reduce to a) is the same on both paths, and where E $\neq$ F. Now let q be the first point beyond p on π which also appears on π'. There exists such a point, since z is on both paths. Then (p,E,...,q,...,F,p) is easily seen to be a circuit, where (p,E,...,q) is taken from π and (q,...,F,p) is from π' reversed. But no circuit exists on a tree. Hence there is just one simple path on a tree from a to z.

Now suppose L is a network which is not a tree. Then there exists a circuit π = (a,A,...,z,...,Z,a) on L. By definition of ***circuit***, the paths (a,A,...,z) and (a,Z,...,z), taken from π and its reverse, respectively, are two simple paths from a to z. This completes a proof of Lemma 5.1.

DEFINITION. If G is a linear graph, then a second graph **G'** is obtained by ***adding an edge*** E to **G'**, if (1) each edge and each vertex of G is in **G'**, (2) E is in **G'** but not in **G**, and (3) E and possibly one or both of its vertices are the only elements of **G'** not in G.

LEMMA 5.2. If **L'** is obtained from a network L by adding an edge **E**, then **L'** is also a network if, and only if, at least one vertex of E is in L. Also L' is a tree if, and only if, L is a tree and exactly one vertex of E is in L.

PROOF. If neither vertex of E is in **L**, then **L'**, is not connected, but otherwise, **L'** is connected and is therefore a network. If L is not a tree, it contains a circuit, which also belongs to L', so **L'** can not be a tree. If L is a tree and just one vertex of E is in **L**, then it is easy to verify that the characteristic property of a tree given in Lemma 5.1 is carried over from L to **L'**. Suppose L is a tree, and both vertices, a and **b**, of E are in L. Let π = (a,A,...,b) be the simple path from a to b on L (Lemma 5.1). Then (a, A,...,b,E,a) is a circuit on L', so that **L'** is not a tree. This completes a proof of Lemma 5.2.

LEMMA 5.3 Let L be a network, and let $m+1$ and n be the numbers of vertices and edges, respectively, in L. Then there exists a sequence $L_1, L_2, \ldots, L_n = L$ of networks made up of vertices and edges of L where (a) L_i has exactly i edges (b) L_{i+1} is obtained by adding an edge to L_i (c) $L_1, \ldots, L_m$ (also to be called $T_1,\ldots,T_m$) are trees, but $L_{m+1}, \ldots, L_n$ are not.

PROOF. Let L_1 be a tree consisting of an arbitrary edge, E_1, of L and its vertices, p_0 and p_1. Then $L_1 = T_1$ satisfies the following hypothesis for $k=1$.

INDUCTIVE HYPOTHESIS. For some positive integer k, there exists a tree, T_k, consisting of k edges $(E_1,\ldots,E_k)$ of L and k+1 vertices $(p_0,\ldots,p_k)$ of L.

If $k < m$, at least one vertex, p, of L is not in T_k, since L has $m+1$ vertices, and T_k has only $k+1$. Because L is connected, there is a path π from p_0 to p. Let p_{k+1} be the first vertex on π which is not on T_k. Then π is of the form $(p_0,\ldots,p_j,E_{k+1},p_{k+1},\ldots,p)$. By Lemma 5.2, the graph obtained by adding E_{k+1} to T_k is a tree, T_{k+1}, satisfying the inductive hypothesis with k replaced by $k+1$. Starting with T_1, we repeat the above step over and over, obtaining $T_1, \ldots, T_m$. Since T_k has k+1 vertices at each stage, the $m+1$ vertices of T_m are all the vertices of L.

If there remain edges of L not on T_m, denote them, in any order, by $E_{m+1}, \ldots, E_n$; and let them be added, one at a time, to complete the required sequence of networks.

THEOREM 5.1 Each network L contains at least one tree **T**, which has all the vertices of L for its vertices.

PROOF. This theorem follows from the proof of Lemma 5.3, with $T_m = T$. Unless L is itself a tree, there is more than one possibility for T.

A tree satisfying Theorem 5.1 will be called a ***maximal*** tree in L.

THEOREM 5.2 If m+1 is the number of vertices of a network L, then L has at least m edges. If L has exactly m edges, it is a tree, and conversely. If L has $n > m$ edges, then L can be reduced to a tree by the removal of n–m of its edges, suitably selected, but not by the removal of any smaller number of edges. The removal of more than $n-m$ edges necessarily disconnects L.

PROOF. Let T be a maximal tree on L. By Theorem 5.1 and the proof of Lemma 5.3, T has exactly m edges. Hence L has at least that many and is a tree if, and only if, it has no more. In the notation of the proof of Lemma 5.3, the removal of $E_{m+1},\ldots,E_n$ reduces L to a tree. There is generally a considerable freedom of choice in the selection of $E_{m+1},\ldots,E_n$.

Kirchhoff proved Theorem 5.2 in the form of a statement as to the number of connections which would need to be broken in an electrical network in order to break every circuit.

6. *PANCAKES*

We will precede our ham sandwiches with pancakes. As usual in mathematics, the objects with which we deal are idealizations, or abstractions, of real objects. If you are inclined to associate some sort of improvement with the word "idealization", then the word "abstraction" is better, since neither the pancakes nor the ham sandwiches will appear in an improved form.

The networks discussed above were one-dimensional, the pancakes will be two-dimensional (suggested by "flat as a pancake"), and the sandwiches, three-dimensional.

Consider a pancake P and, of course, some syrup, S. Both P and S will be regarded as plane regions (Figure 3). We co not care whether they overlap, as they generally do when served on a plate. However, they will be fixed regions, in the sense, for example that the syrup will not be allowed to flow around.

THE PANCAKE THEOREM. There exists, in the plane, a straight line which simultaneously bisects the areas of P and of **S.**

PROOF. We first remark that this is a topological theorem, in that it does not depend on the size or shape of P or S. If P and S happen to be circular, and not concentric, the line through their centers is the only one satisfying the theorem.

In the plane containing P and **S**, let (x,y) be a rectangular cartesian coordinate system. We will think of the positive x-axis as horizontal to the right and of the y-axis as pointing up. The projection of P onto the y-axis is a segment, say from $y = a$ to $y = b$ (Figure **3**), so that none of P is below $y = a$, none is above $y = b$, and each line $y = c$, where $a < c < b$, cuts across P. Let $P(c)$ denote the part of P below $y = c$. Using the symbol for a part of the plane to denote also its area, we let $p(c) = P_1(c)/P$; that is, $p(c)$ is the ratio of the area of P below $y = c$ to the entire area of P. (Figure 3). As c increases from a to b, $p(c)$ increases from 0 to 1. By a wellknown theorem concerning continuous functions, there is a value **m**, unique in the present case, for which $p(m) = 1/2$; that is, for which $y = m$ bisects P.

Now let $(x'; y')$ be the coordinate system obtained by rotating the (x,y)-axes through an angle θ, with the positive sense for angles counterclockwise, and let $m(\theta)$ denote the number such that the line $y' = m(\theta)$ bisects the area of **P.** For example, $m(0)$ is the m of the

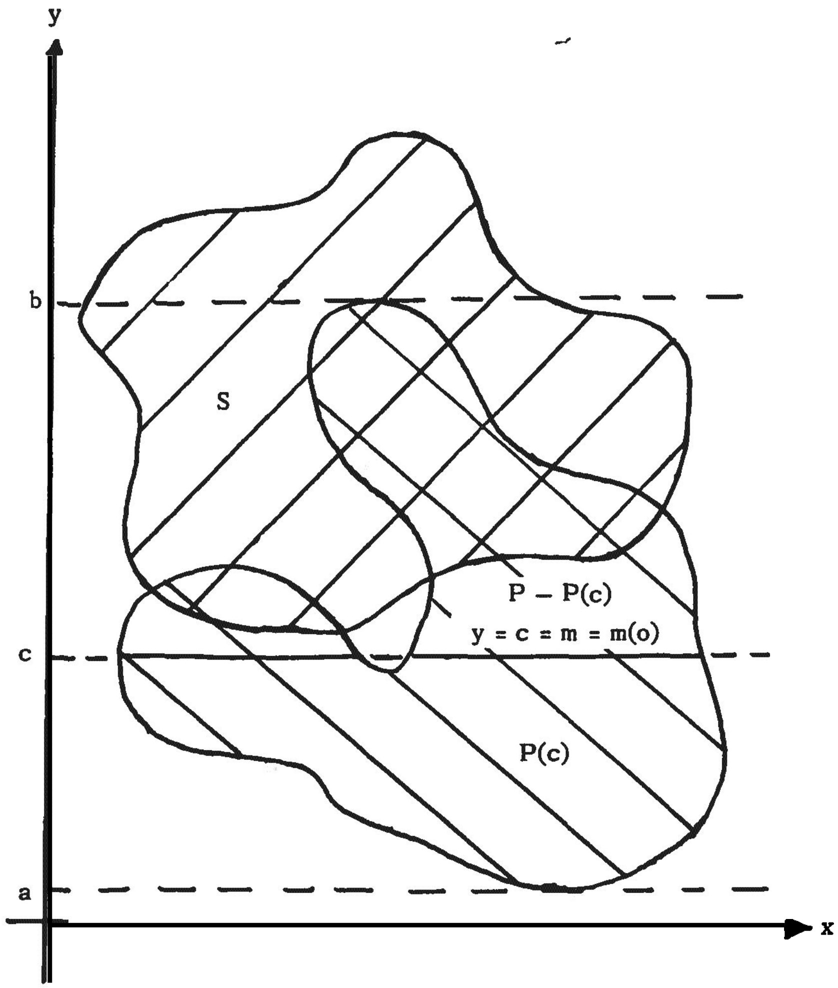

Figure 3. Equitable division of pancake and syrup.

preceding paragraph. Let $S_1(\theta)$ be the part of S on which $y' < m(\theta)$, and let $r(\theta) = S_1(\theta)/S$. In Figure 3, $r(0) = 0$, since $S_1(0) = 0$, none of S being below $y = m(0)$. We remark, but do not prove, that $r(\theta)$ is continuous in θ. Now let θ increase from 0 to π. When $\theta = \pi$, we have $x' = -x$, $y' = -y$. Since the sense of the y-axis is now reversed, $S_1(\pi) = S - S_1(0)$. Therefore,

$r(\pi) = \frac{S - S_1(0)}{S} = 1 - \frac{S_1(0)}{S} = 1 - r(0)$. Since $r(0) + r(\pi) = 1$, either $r(0) = r(\pi) = 1/2$, or one of the numbers $r(0)$ and $r(\pi)$ is less than 1/2 and the other greater than $1/2$. Since $r(\theta)$ assumes all values between $r(0)$ and $r(\pi)$ as θ increases from 0 to π, there exists a θ for which $r(\theta) = 1/2$. For such a θ, the line $y' = m(\theta)$ bisects both P and S, proving the theorem.

7. *HAM SANDWICHES.*

THE HAM SANDWICH THEOREM. Let **B, H,** and **C,** be three solid regions of Euclidean three-dimensional space. Then there exists at least one plane which bisects the volumes of **B,** of **H,** and of C.

PROOF. The letters **B, H,** and C are intended to suggest bread, ham and cheese. The theorem affirms that if one swings a sword in exactly the right plane, one can divide the bread, ham and cheese each equally between two people. For example, if **B,** H and C are spherical regions, a plane through their centers satisfies the requirements.

Our proof will be analogous to that of the Pancake Theorem, but we formulate it differently, partly for the sake of variety. Let S be the surface of a sphere enclosing **B,** H and C. Let p be a point on **S,** and let p' be the antipodal point. Let q be a point on the diameter pp', and let $P(p,q)$ be the plane through q normal to pp'. Then $P(p,q)$ separates the interior of S into two parts, one of which, to be denoted by $S(p,q)$, has p on its boundary. Let $B(p,q)$, $H(p,q)$, and $C(p,q)$, respectively, denote the parts of **B,** H and C in $S(p,q)$. Using the same symbol for a solid and for its volume, let $r(p,q) = B(p,q)/B$. This is the ratio of the volume of the part of B in $S(p,q)$ to the entire volume of B. As q moves along pp' from p to p', the ratio $r(p,q)$ increases from 0 to 1; and, just as in the proof of the Pancake Theorem, there is exactly one* position q_0 such that $r(p,q_0) = 1/2$.

In general, of course, the plane $P(p,q_0)$, which bisects B, will not also bisect **H** or C. When we have shown that it bisects both H and C for at least one choice of p, the Ham Sandwich Theorem will be proved.

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* This assumes B to be connected, corresponding to an openfaced sandwich. If B consists of two equal slices of bread, then each plane separating the two slices would bisect B, so that there might, for certain choices of **p,** be a whole segment full of points like q_0 on pp'.

Since q_o is determined p, the ratios $h(p) = H(p,q_o)/H$ and $c(p) = C(p,q_o)/C$ are also determined, and can be interpreted as functions of p.

LEMMA 7.1 Let H' be the set of all points on S where $h(p) = 1/2$, and let C' be the set of all points on S where $c(p) = 1/2$. Then the Ham Sandwich Theorem will follow if it is proved that H' and C' have a point in common.

PROOF. If $h(p) = 1/2$, then $P(p,q_o)$ bisects H. If $c(p) = 1/2$, $P(p,q_o)$ bisects C. By definition, $P(p,q_o)$ always bisects B. Therefore, if p is common to H' and C', $P(p,q_o)$ bisects B and H and C, as required.

LEMMA 7.2 The ratios $h(p)$ and $c(p)$ have the property,

$$h(p) + h(p') = 1 \qquad c(p) + c(p') = 1,$$

where p' is antipodal to p on S.

This lemma follows by the same sort of reasoning as the relation $r(0) + r(\pi) = 1$ in the proof of the Pancake Theorem; since interchanging p and p' is strictly analogous to reversing the positive direction of the y-axis in that proof.

LEMMA 7.3 If a point p belongs to H', its antipodal point p' is also on H'. If p is not on H', then any arc K from p to p' (half of a great circle, for example) goes through at least one point of H'. These two statements are true with C' replacing H' throughout.

PROOF. The proof is the same for C' as for H'. We give it only H'. The first sentence of the lemma follows from Lemma 7.2, which implies that $h(p') = 1/2$ if $h(p) = 1/2$. If p is not on H', then $h(p) \neq 1/2$, and Lemma 7.2 implies that 1/2 lies between $h(p)$ and $h(p')$.

As a tracing point moves along K from p to p', the functional values of h change continuously from $h(p)$ to $h(p')$. Hence each value between $h(p)$ and $h(p')$, in particular the value 1/2, must* be assumed somewhere on K. This proves the lemma.

In some cases, H' covers all of S. This would happen if, for example, the ham were wrapped about the bread in such a way that H and B both had central symmetry in some point q_o. Then the planes through q_o would be the bisecting planes of B, and also of H, so that $h(p)$ would equal 1/2 at every point on S. In such case, each point of C' would obviously be on $H' = S$, and the Ham Sandwich Theorem would follow, by Lemma 7.1. These statements are, of course, true with H' and C' interchanged.

Suppose, then, that H' does not cover all of S, and let p_o be a point on S at which $h(p_o) \neq 1/2$. We will think of p_o as being at the north pole of a globe, so that p'_o, is at the south pole. Then, by Lemma 7.3, each meridian of longitude, which is half of a great circle, from p_o to p'_o, intersects H'. It can be shown (we omit the proof) that there exists a closed curve H'_o on H' which intersects each meridian. Because, by Lemma 7.3, H' is symmetric in the center of S, it can also be required that the curve H'_o be symmetric in the center of S, but all we need is the property that H'_o pass through some pair of antipodal points, p and p'.

Each of the arcs of H'_o joining p to p' must then intersect C', by Lemma 7.3 with C' replacing H'. Since these arcs are entirely on H', their intersections with C' are common to H' and C', and the Ham Sandwich Theorem follows, by Lemma 7.1.

*Here, and hereafter, we omit detailed justifications of certain plausible statements, whose proofs would not be appropriate in this paper.

8. PUTTY.

Consider a peice of putty, originally spherical, which we proceed to deform' topologically. That is, intuitively speaking, the putty is molded without being torn apart, having holes punched through it, or being bent around and stuck back into itself. We will think of the piece of putty as being quite large. Indeed, we will regard it as a planet, on which explorers, in a space ship, have landed. Putty being a pretty dull substance, they quickly lose interest in its physical properties; but, since they have come so far, they decide to survey the planet, count its mountains and valleys, and make contour maps, so they will have some kind of report to take home. We suppose the planet small enough for this to be feasible.

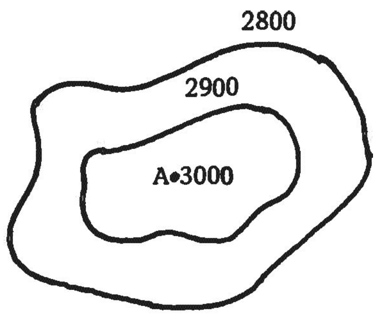
A contour map shows loci at a constant altitude, where ***altitude*** will mean distance, in meters, from the center of gravity, 0, of the planet. Near a mountain peak of altitude 3000, for example, a contour map might look like Figure 4a. Near the bottom of a valley, at altitude 1000, it might look like Figure 4b. The explorers also take note of mountain passes, near which a contour map might look like Figure 4c.

To simplify our analysis, we make the following assumptions: (1) If A is a peak, then no other point as high as A is within a certain distance of A. This rules out, for example, a plateau or a level ridge containing A. (2) If B is a pit (that is, the bottom point of a valley) then no point with as low an altitude as B is within a certain distance of B. (3) If C is a pass, then the contour locus at the level of C is, in a certain neighborhood of C, composed of two intersecting arcs, as shown in Figure 4c. (4) Of all the pits, peaks and passes, no two are at exactly the same altitude.

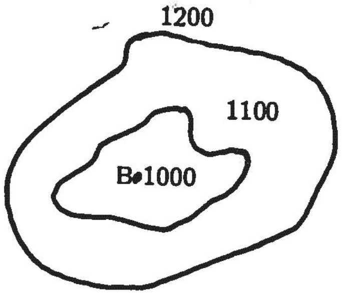
We are borrowing the terminology ***pits, peaks*** and ***passes,*** along with other ideas, from Professor Marston Morse, of the Institute for Advanced Study in Princeton, New Jersey.

THEOREM. Let N_o be the total number of pits on the planet, let N_1 be the number of passes and N_2 the number of peaks. Then $N_o - N_1 + N_2 = 2$.

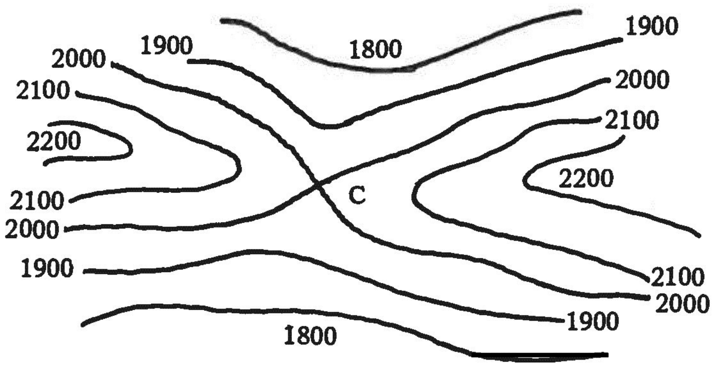
This remarkable theorem says that, under our assumptions, no matter how many mountains and valleys there may be, their total number exceeds by exactly 2 the number of passes.



(a) A peak



(b) A pit



(c) A pass

Figure 4. Contour lines near a peak A, a pit B and a pass C.

PROOF. To facilitate our proof, we assume the following miracle: Water appears at the bottom of the deepest valley and slowly rises, always covering all of the planet up to a certain contour line, until the entire planet is submerged. We mercifully allow our explorers to take off from the highest peak, a sort of putty Mount Ararat, just in time.

When the water first appears, it forms a tiny lake in the deepest valley. The next significant event, as the water rises, is the appearance of a second lake in the next deepest vally. Next, these first two lakes may merge across a mountain pass, or perhaps a third lake appears, at the bottom of the third deepest valley.

Let us now be systematic and see what effect the rising water has on the numbers of lakes and islands. (1) As the water goes up past the level of a pit, the number of lakes increases by 1. (2) When a peak is submerged, the number of islands decreases by 1. (3) As the water rises above the level of a pass, either (a) the number of lakes decreases by 1 (two different lakes merge) or (b) the number of islands increases by one (a lake merges with itself across a pass, changing a peninsula into an island).

Now let $N_0(c)$ and $N_2(c)$ be the numbers of pits and peaks, respectively, below some altitude c. Let $N_1'(c)$ be the number of passes below altitude c, at each of which two different lakes have merged, and let $N_1''(c)$ be the number of passes where a lake has merged with itself. We introduce an auxiliary result.

LEMMA. Let $J(c)$ be the number of islands (connected bodies of land) and let $L(c)$ be the number of lakes (connected bodies of water), at the time the water has reached altitude c, where c does not equal the level of any pit, peak, or pass. Then (8.1) $L(c) - J(c) = -1 + N_0(c) + N_2(c) - N_1'(c) - N_1''(c)$

Proof of Lemma. Let c_0 be a little less than the smallest altitude on the surface of the planet. When $c = c_0$, the surface is entirely dry, so that $L(c_0) = 0$, $J(c_0) = 1$, and $N_0(c_0) = N_2(c_0) = N_1'(c_0) = N_1''(c_0) = 0$. Therefore, (8.1) holds for $c = c_0$. As the water rises from level c_0 to some level c, the increase in the number of lakes is $N_0(c) - N_1'(c)$ by (1) and (3a), and the decrease in the number of islands is, by (2) and (3b), $N_2(c) - N_1''(c)$. Thus $L(c) - J(c)$ increases by $N_0(c) + N_2(c) - N_1'(c) - N_1''(c)$, and equation (8.1), since it holds for c_0, must hold for all greater altitudes.

Now let c exceed the greatest altitude on the planet so that the planet is covered by water. Then

$L(c) = 1$, $J(c) = 0$, $N_0(c) = N_0$, $N_2(c) = N_2$, $N_1'(c)n+N_1''(c) = N_1$

Substituting these values in (8.1), we have

(8.2) $1 = -1 + N_0 + N_2 - N_1$

which is equivalent to the conclusion of our theorem.

University of Illinois

SYLOW THEORY[1]

JOSEPH J. MALONE, JR.
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In the books which survey abstract algebra, many interesting and important topics have, of necessity, been omitted. It is the purpose of this paper to present one such topic - the Sylow theory of finite groups - for those who have not specialized in group theory.

The theorem of Lagrange states that the order of a subgroup of a finite group G is a factor of the order of G. But the converse of this is not true since, for instance, the tetrahedral group of order 12 has no subgroup of order 6. However the work of the Norwegian mathematician L. Sylow provides a partial answer to the problem of whether a given group has a subgroup whose order corresponds to an arbitrary factor of the order of the group.

To facilitate the discussion, six algebraic results to be used in this paper are given below. Note that multiplicative group notation will be used.

Result 1 The number of distinct conjugates of an element or subgroup of a finite group is equal to the index of the normalizer of the element or subgroup.

Result 2 If A and B are subgroups (of G) of orders a and b, having an intersection of order j, then C = AB contains exactly ab/j elements and C is a subgroup if and only if $AB = BA$.

Result 3 A commutative group of order g contains at least one (invariant) element of order p, where p is any prime factor of g.

Result 4 If G/K contains a subgroup A (of order a), then G contains a subgroup H of order $h = ak$, where k is the order of K. If A is invariant in G/K, H is invariant in G.

Result 5 If, in a subgroup G, a subgroup A is invariant under k elements of a subgroup B of order b, then the elements of B transform A into b/k distinct conjugates of A.

Result 6 If A and B are two invariant subgroups of G which have only the identity in common, then every element of A commutes with every element of B.

Definition If the order of a group G is divisible by p^m but by no higher power of p, where p is a prime and m is a positive integer, then any subgroup of G of order p^m is called a Sylov subgroup of G corresponding to p.

[1]Received by Editors Jan. 31, 1962.

Theorem 1 Every group G (of order g) possesses at least one Sylow subgroup corresponding to each prime factor of g.

Proof: The theorem holds when $g = 2$ since the Sylow subgroup need not be proper. We show that the theorem is valid for a group of order g if it is valid for all groups with order less than g. Let $g = p^m g'$, where $(g', p) = 1$. Resolve G into classes of conjugate elements so that $G = \bigcup_{i=1}^{m} (a_i)$, where (a_i) is the set of all elements conjugate to a_i in G. Also, $g = \sum_{i=1} k_i$, where k_i is the order of (a_i). By Result 1, N_i, the normalizer of a_i, has order $n_i = g/k_i$. Two cases must be considered.

1. Suppose one of the k_i, say k_j, is greater than one and that $(k_j, p) = 1$. Since $n_j k_j = g$ and p^m divides g, p^m must divide n_j. But n_j, the order of the group N_j, is less than g. From the induction hypothesis then N_j contains a Sylow subgroup of order p^m. Hence G contains a Sylow subgroup corresponding to p.

2. Suppose, that for every i, either $k_i = 1$ or p divides k_i. If $k_j = 1$, then a_j is invariant. There is at least one such element since the identity is invariant. Denoting the number of invariant elements (the order of the center of G) by q, we may write $g = p^m g' = q + ep$. Hence p must divide q. By Result 3, the center contains an element w of order p. Hence G has at least one element w which commutes with all elements of G and is of order p. Let P represent the cyclic group generated by w. Then P is an invariant subgroup of G of order p. Thus G/P has order $p^{m-1}g'$, which is less than g. By the induction hypothesis, G/P contains a Sylow subgroup of order p^{m-1}. But then, by Result 4, G contains a subgroup of order p^m.

Corollary 1a If p is a prime factor of the order of the group G, then G contains at least one element of order p.

Proof: Let H be a Sylow subgroup of order p^m. Let x be in H, x not the identity. Then x has order p^r, $0 < r \leq m$. Let $v = x \exp p^{r-1}$. Then v is of order p since $v^p = (x \exp p^{r-1})^p = x \exp p^r = 1$.

Theorem 2 No element (of the group G) whose order is a power of p and which is not contained in a given Sylow subgroup H corresponding to p can transform that Sylow subgroup into itself.

Proof: Let c be an element of order p^r, $0 < r \leq m$, which is not in H. Assume c transforms H into itself, Then $c^{-1}Hc = H$ and $Hc = cH$. If C is the cyclic subgroup of G generated by c, C has order p^r. From the assumption, $HC = CH$. By Result 2, HC is a subgroup. The intersection of H and C is a subgroup which, by the theorem of Lagrange, has a power of p as its order. Denote this order by p^n, $0 \leq n < m$, $0 \leq n < r$. Hence HC is of order $p^m p^r / p^n = p^{m+r-n}$. Since $r-n > 0$, there is a contradiction because p^m is the highest power of p which divides g, the order of G. Thus the assumption is false.

Theorem 3 All Sylow subgroups of G belonging to the same prime p are conjugate to one another. The number of them is $f = 1 + kp$, where k is a non-negative integer and f is a factor of g.

Proof: First we show that the number of Sylow subgroups conjugate to a given Sylow subgroup is $f = 1 + kp$. If H, the given Sylow subgroup, has no conjugates other than itself, then $f = 1 + 0p$. Next assume that H' is a subgroup conjugate to H but not identical with H. Denote the intersection of H and H' by H_1. Let p^{r_1}, $0 \leq r_1 < m$, designate the order of the subgroup H_1. Consider the conjugates of H' formed by transforming H' by the elements of H. By Theorem 2, the only elements of H which transform H' into itself are those of H_1, and there are p^{r_1} elements in H_1. Hence, by Result 5, H' is transformed by the elements of H into p^m/p^{r_1} or p^{m-r_1} distinct conjugates of H' (and hence of H) different from H. If this set does not contain all the conjugates of H, we select a conjugate H'', $H'' \neq H$, not in the set. H_2, the intersection of H and H'', is of order p^{r_2}, $0 \leq r_2 < m$, and we obtain p^{m-r_2} more distinct conjugates. None of these is contained in the previous set since this would imply, for some x and y in H, that $x^{-1}H'x = y^{-1}H''y$ and $(xy^{-1})^{-1}H'(xy^{-1}) = H''$. But this last equation contradicts the fact that H'' was not in the first set. A continuation of this process must eventually exhaust the conjugates of H. Thus the number of distinct conjugates of H must be $1 + p^{m-r_1} + p^{m-r_2} + \ldots + p^{m-r_n}$, $0 \leq r_i < m$. The one occurs in the sum because H is a conjugate of itself and H does not occur in any of the sets. Thus there are $f = 1 + kp$ conjugates of H.

We must show G contains no Sylow subgroup of order p^m other than those conjugate to H. Assume a subgroup K of order p^m and not conjugate to H does exist. We shall, as before, distribute the conjugates into sets. This is done by transforming the conjugates of K by the elements of H. Let K' be an arbitrary conjugate of K. If the intersection of H and K' is of order p^{e_1}, and so on, the number of conjugates in this set, following the process above, would be $p^{m-e_1} + p^{m-e_2} + \ldots + p^{m-e_n}$, $0 \leq e_i < m$. The important consideration here is that the sum does not have one as one of its terms. K is already accounted for since it can be taken as one of the K-primes. Also, one is not added for the transforming set H as before since H and K are not conjugate. But from the previous paragraph we see that K must be in a family of $1 + k_1p$ distinct conjugates. Since the number of distinct conjugates of K cannot be congruent to both 0 and 1 modulo p, we have a contradiction. Hence such a K does not exist.

Since f is the number of distinct conjugates of H, Result 1 tells us that $fn = g$, where n is the order of the normalizer of H. Hence f is a factor of g.

Corollary 3a A Sylow subgroup H corresponding to p is unique if and only if it is an invariant subgroup of G.

Proof: If H is unique, then it is in a set of $1 + 0p$ conjugates and is invariant in G, and conversely.

Corollary 3b If *K* is a subgroup (of G) of order p^r, $0 < r < m$, then K is contained in at least one of the Sylow subgroups corresponding to p.

Proof: If K is not in some H, use the method of the proof of the theorem and separate H and its conjugates into sets by using the elements of K as transformers. This would give the number of conjugates of H as $p^{r-e_1} + p^{r-e_2} + \ldots + p^{r-e_n}$, where e_i is the order of the intersection of K and the i-th conjugate of H. But by the theorem, the number of conjugates is $1 + kp$. Hence there is a contradiction and K must be contained in H or in one of the conjugates of H.

Theorems 1 and 3 together are sometimes called "Sylow's Theorem."

Theorem 4 If every Sylow subgroup of G is an invariant subgroup, then G is the direct product of its Sylow subgroups.

Proof: We denote the direct product G of n subgroups by $G = H_1 \times H_2 \times \ldots \times H_n$ and use the notation when the following three conditions are satisfied. It must be possible to express an arbitrary element u of G in the form $u = a_1a_2 \ldots a$, a_i, in H_i. The a_i must also satisfy the conditions that $a_ia_j = a_ja_i$, $i \neq j$, and that they are independent in the sense that $a_1a_2 \ldots a_n = 1$ if and only if each a is the identity.

We first show that any element u of G can be written as $u = a_1a_2 \ldots a$, a_i in the Sylow subgroup H_i (of order $p_i^{m_i}$) corresponding to the prime factor p_i of g. Note that $g = p_1^{m_1}p_2^{m_2} \ldots p_n^{m_n}$. Since the order of an element of a group must be a factor of the order of the group, the only element common to any two of the H_i is the identity. Consider the invariant subgroup generated by H_1 and H_2. By Result 2, it is of order $p_1^{m_1}p_2^{m_2}$. Since it has no element in common with H_3 except the identity, (H_1H_2) and H_3 generate a subgroup of order $p_1^{m_1}p_2^{m_2}p_3^{m_3}$. Continuing, we see that the Sylow subgroups generate a group of order $p_1^{m_1}p_2^{m_2} \ldots p_n^{mn} = g$. Since this subgroup is contained in G, it must be identical with G and any element of G may be written in the desired form.

Furthermore, there is a unique representation of each element of G as a product of the a_i since there are only g possible products and we have seen that these yield a group of order g. Thus $a_1a_2 \ldots a = 1$ if and only if each a_i is the identity.

Generalizing from Result 6, we have the condition of commutativity satisfied and the theorem follows.

Corollary 4a *A* commutative group is the direct product of its Sylow subgroups.

Proof: Every subgroup of a commutative group, is invariant.

Sylow theory, interesting in its own right, also has many applications in the theory of finite groups. It may be shown that there cannot be simple groups of certain orders or that some categories of groups must be commutative because of the properties of their Sylow subgroups. The theory of Sylow subgroups is also important in determining how many distinct groups of a given order exist. Examples of such applications may be found in the books given as references.

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(This article was written while the author was at St. Louis University and Rockhurst College.)

University of Houston

PROBLEM DEPARTMENT

Edited by
M. S. KLAMKIN
State University of New York
at Buffalo

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate, signed sheets within four months after publication. Address all communications concerning problems'to M. S. Klamkin, Avco Research and Advanced Development Division, T-430, Wilmington, Massachusetts.

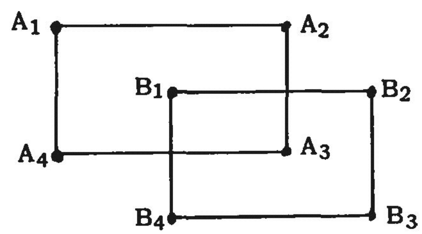
PROBLEMS FOR SOLUTION

149. Proposed by John Selfridge, University of Washington.

A game of bridge is dealt and each player has distribution **abcd** into suits **(e.g.,** each player has 4333). Is each suit distributed **abcd** among the players? In another deal each player has the same distribution as some suit. Does each suit have the same distribution as some player?

150. Proposed by D. J. **Newman,** Yeshiva University.

Given two overlapping parallel rectangles $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ and a quadratic polynomial $Q(x,y)$.



Show that Q cannot be >0 at A_1,A_2,A_3,A_4, and <0 at B_1,B_2,B_3, and B_4.

151. Proposed by K. S. Murray, New York City.

Three points are chosen at random with a uniform distribution from the three sides of a given triangle (one point to each side). What is the expected value of the area of the random triangle that is formed?

152. Proposed by Leo Moser, University of Alberta.
If ϕ denotes Euler's totient function, show that in every base

$$\frac{\phi(1)}{1}+\frac{\phi(1+1)}{11}+\frac{\phi(1+1+1)}{111}+\ldots = 1.111\ldots .$$

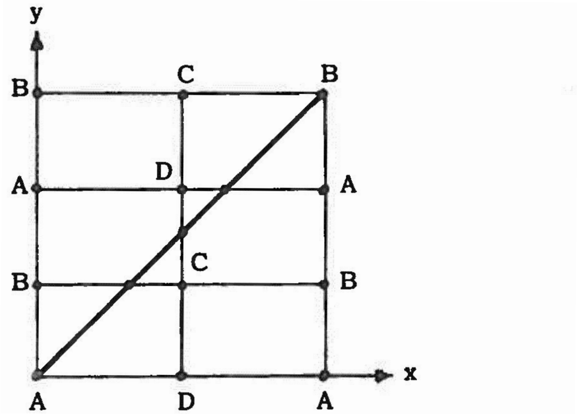
153. Proposed by M. S. Klamkin, University of Buffalo.
Show that the maximum area ellipse which can be inscribed in an equilateral triangle is the inscribed circle.

SOLUTIONS

116. Proposed by M. S. Klamkin, University of Buffalo.
Problem n. 147, due to **Auerback-Mazur,** in the "Scottish" book of problems is to show that if a billiard ball is hit from one corner of a billiard table having commensurable sides at an angle of $45°$ with the table, then it will hit another comer. Consider the more general problem of a table of dimension ratio m/n and initial direction of ball of $\theta = \tan^{-1}(a/b)$ (m,n,a, and b, are integers). Show that the ball will strike another corner after

$$\frac{an+bm}{(an,bm)}-2$$

cushions ((x,y) as usual denotes the greatest common divisor). Also, determine which other corner the ball will strike first.
Solution by the proposer.
We can consider the ball to move in one straight line if we surround the table with reflections of itself. For the case $a=b$, $m=2$, $n=3$, we obtain the following diagram.



From the diagram, it follows the ball first hits corner B after 3 cushions. In general, if the coordinates of the first corner that is hit are (qn,pm), then

$$\frac{a}{b}=\frac{pm}{qn}$$

where p and q are the smallest integers possible. Whence,

$$p=\frac{an}{(an,bm)},\quad q=\frac{bm}{(an,bm)}.$$

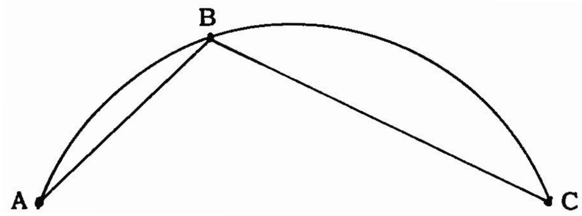
The number of cushions struck will be

$$(p-1)+(q-1) \quad\text{or}\quad \frac{an+bm}{(an,bm)}-2$$

The first corner hit will be
1. A if q and p are both even
2. B if q is even and p is odd,
3. C if q is odd and p is odd,
4. D if q is odd and p is even.

Also, solved by H. Kaye, Paul Myers and M. Wagner.

121. Proposed by M. S. Klamkin, University of Buffalo, and D. J. **Newman,** Yeshiva University.
Three circular arcs of fixed total length are constructed, each passing through two different vertices of a given triangle, so that they enclose the maximum area. Show that the three radii are equal.
Solutions by the proposers.
We first prove the result for two arcs of fixed total length say L.



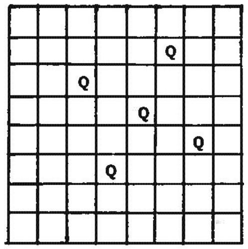
Change the angle B such that the length of the portion of the circumcircle through A,B and C (above AC) is also L. It now follows by the isoperimetric property of the circle that the sum of the areas of the two segments is maximized. Whence the two radii are the same and this will also be true for any number of arcs. It is assumed that the given arc length is greater than the perimeter of the given figure and small enough so that the circular arcs only intersect at the vertices.

The more general problem of determining a closed curve of given length which passes through a set of given points and encloses a maximum area was treated by Steiner (Ges. **Werke,** ii, 75-91). Also, solved by J. Thomas and F. **Zetto.**

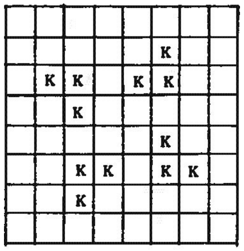
122. Proposed by Paul Berman, New York City.

What is the minimum number of queens which can be placed on an nxn board such that no queen is cowered by any other queen and such that the entire board is covered by all the queens?

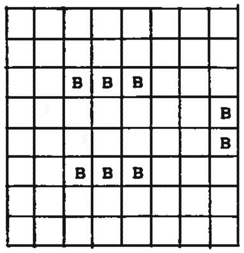
Editorial note: No solutions were sent in for this problem. However, the editor came across a similar problem in "The Theory of Graphs", by Claude Berge, Methuem and Company, 1962, Great Britain, p. 41, i.e., "In the game of chess, what is the smallest number of queens which can be placed on the board so that every square is dominated by at least one of the queens? This problem is the same as finding a minimum externally stable set for a graph with 64 vertices (the squares on the board), with $(x,y) \in U$ if and only if the squares x and y are on the same rank or file or diagonal.



B(Q) = 5



B(K) = 12



B(B) = 8

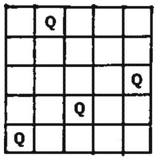
The coefficient of external stability is $\beta = 5$ for the queens; $\beta = 8$ for the bishops; $\beta = 12$ for the knights".

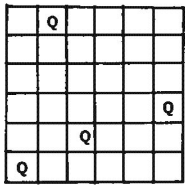
Although here there is not any condition that any piece may not cover any other piece, this condition is satisfied for the case of the queens and bishops but not the knights.

On communicating with the author, the following experimental table given by Kraitchik, for the minimum number of queens on an nxn board was received:

n	5	6	7	8	9	10	11	12	13	14	15	16	17
k(n)	5	5	5	5	5	5	5	6	7	8	9	9	9

The first two entries are probably typographical errors, since it can easily be shown that 4 queens suffice for n = 5,6, i.e.,





123. Proposed by John Selfridge, University of Washington.

A set of positive integers $\{a_r\}$ is said to be linearly independent if $\sum_1^n a_r c_r$ implies $c_r = 0$ where c_r are integers > -2.

1. For every n, show that there exists a linearly independent set
2. Is the set

$$\begin{array}{l} 1\,0\,0\ldots 0, \\ 1\,1\,0\ldots 0, \\ 1\,1\,1\ldots 0, \\ \ldots\quad ., \\ \ldots\quad ., \\ \ldots\quad ., \\ 1\,1\,1\ldots 1, \end{array}$$

(to the base 2) linearly independent?

3. Given a linearly independent set of length n, show that its least element is at least 2^{n-1}.
4. It is conjectured that there is only one linearly independent set of length n with its largest element $\leq \sum_1^{n-1} 2^i$.

Solution by the proposer.

2. Suppose $\sum_1^n a_r c_r = 0$ where $a_i = \sum_{j=1}^{i} 2^{n-j}$. In particular $a_1 = 2^{n-1}$, $a_{n-1} = 2^n-2$, and $a_n = 2^n-1 = a_1 + a_{n-1}/2$. Hence $\sum_1^{n-1} a_r c_r + (a_1 + a_{n-1}/2)\, c_n = 0.$ (1)

Since a is the only odd number in the set, its coefficient c_n must be even and therefore ≥ 0. Now let $a_i' = a_i/2$ $(i = 1,2,\ldots,n-1)$. Then from (1)

$$a_1'(c_1 + c_n) + \sum_2^{n-2} a_r' c_r + a_{n-1}'(c_{n-1} + c_n/2) = 0, \qquad (2)$$

and each coefficient of a_i' is > -2. We can now assume by induction that $a_1'\ a_2', \ldots, a_{n-1}'$ are linearly independent and therefore each of their coefficients in (2) must be zero. But then $c_1 + c_n = 0$, c_n even, and $c_1 > -2$ imply that $c_1 = c_n = 0$ and so each c_i is zero. Thus $a_1, a_2, \ldots, a_n$ are linearly independent. The solution to part 1 now follows.

3. Let a_i be the least element. Consider the 2^{n-1} sums $\sum_2^n a_i e_i$ where $e_i = 0$ or 1.

These sums must all lie in distinct residue classes modulo a. To prove this, suppose

$$\sum_2^n a_i e_i + ma, = \sum_2^n a_i e_i' \quad (m \geq 0).$$

Then $ma_1 + \sum_2^n a_i (e_i - e_i') = 0,$

and by linear independence $e_i = e_i'$ and $m = 0$. Hence $a_1 \geq 2^{n-1}$.

4. The conjecture is still open.

133. Proposed by Robin Robinson, Dartmouth College.

Prove that for any convex plane area there exists a non-obtuse positive angle θ_1, such that for every angle θ in the interval, $\theta_1 \leq \theta \leq \pi - \theta_1$, there is a pair of lines meeting at the angle θ and dividing the area into four equal parts, while for every angle outside this interval this is impossible. If θ is different from θ_1, $\pi - \theta_1$, or $\pi/2$, the pair of lines is not unique. Also, prove that there exists an area for which θ_1 is the desired non-obtuse positive angle.

Solution by the proposer.

1. Given any direction m, there is a unique line with direction m which halves the area.
2. For each of the halves in (1), the same lemma applies for any given direction n. As m is held fixed and n is allowed to vary, it is clear that the two lines with direction n must come into coincidence before n varies 180°. This position and the line of direction m divide the complete area into four equal parts.
3. For any m, the direction n determined in (2) is unique.
4. Denote by θ the smallest positive angle from m to n. As m is allowed to vary, θ varies continuously. When m varies through a 180° interval, that interval is mapped by this relationship onto part of the range $0 < \theta < 180°$, definitely not including its end-points. Since a continuous map of a compact set is compact, the one-dimensional set of values of θ has a minimum θ_1. Obviously, $\pi - \theta_1$ is the maximum (same pair of lines), and θ must take on all intervening values.
5. Since θ varies from any position at least down to its minimum, then up to its maximum, and then back to its starting point, it passes any intervening position at least twice. Only if $\theta = \pi/2$ can the two lines for these two positions be the same pair.
6. For an ellipse of semi-axes a and b, the pairs of lines are conjugate diameters, and $\theta_1 = 2 \arctan (b/a)$. Clearly then, for an ellipse θ_1 may take any value in the range $0 \leq \theta_1 \leq$ nil.
7. It is to be noted that the angle θ_1 associated with any area is of course an invariant of that area under similarity transformations. The circle is not the only area for which θ_1 is 90°; the square and regular octagon are also in this category.

BOOK REVIEWS

Edited by

FRANZ E. HOHN, UNIVERSITY OF ILLINOIS

Statistics, An Intuitive Approach. By G. H. Weinberg, and J. A. Schumaker. Belmont, California: Wadsworth Publishing Company, 1962. xii + 338 pp., $6.50.

As the title suggests, this is an unpretentious textbook on statistics for students having little mathematical knowledge. The authors have restricted themselves to a relatively small number of topics which can be developed and appreciated under an heurisitc treatment (no F-test or analysis of variance, for example). These topics are presented with great elabroation in the book; diagrams and tables are used freely and effectively. Proofs, as such, are not given, but students with poor backgrounds in mathematics will undoubtedly find these intuitive discussions as convincing and meaningful as those where formal proofs are given. As a rule, the discussions are exceptionally clear and well devised, particularly those introducing the mean, standard deviation, correlation coefficient, and linear regression.

On the negative side, the section on the Central Limit Theorem is poor, and the book would be better without it. Not only is it out of place here, but the theorems in this section are stated incorrectly and lead to other errors, such as the theorem on page 147 which implies that the binomial distribution is normal! Since the book seems pedagogically sound otherwise, these errors are particularly unfortunate.

University of Illinois — Gus Haggstrom

SCIENTIFIC METHOD: OPTIMIZING APPLIED RESEARCH DECISIONS. By Russell L. Ackoff; with the collaboration of Shiv K. Gupta and J. Sayer Minas. New York, John Wiley, 1962. xii + 464 pp., $10.25.

R. L. Ackoff is one of the more prolific writers on operations research. This, his latest book, is "...definitely slanted to the decision maker, the man of affairs, the manipulator of men, machines, and resources.". As such, much of the book is given to discussing the context of a problem and ways of gathering and utilizing the relevant data. The illustrations are taken from business and industry. The mathematics developed is mostly statistics on the level of the first year graduate student.

In Chapter 14: Implementation and Organization of Research" Ackoff discusses with great feeling and clarity his experiences as a researcher dealing with the man who sponsors research. While the material here can hardly be taught in a classroom, it is important as long as one believes with the author that operations research must be used in order to be worthwhile. One of his hints is never to do research for a man unless he pays for it, otherwise he will not be interested. Another, if you are going to propose reorganizing the factory, get the key personnel to work with you so they won't fight the proposed changes. As these themes are developed, Ackoff captures the challenge and excitment that would lure a young man into this branch of applied mathematics.

Champaign, Illinois — Jane I. Robertson

Elementary Vector Geometry. By Seymour Schuster. New York, Johy Wiley, 1962. XII t 213 pp., $4.95.

There is a rapidly growing "honors" market comprising bright high school students, college undergraduates, teachers, and non-professional mathematicians, among others. Prof. Schuster's attractive little book is ideal for this group.

His clear and deliberate presentation starts from the **concrete** geometric vector of the physicist and leads the reader steadily upward into vector algebra, covering linear independence, inner products, and cross products. At each stage, he introduces immediate applications to geometry (both plane and solid) as well as to analytics, complex numbers, and trigonometry (both plane and spherical). The applications are clearly presented without stinting on details, a feature which will undoubtedly be appreciated by most readers. The final chapter includes a very pretty vector treatment of the theorems of Desargues, **Pappus,** and Menelaus, along with other topics such as convexity, using vector language and techniques. There is also included a brief section on linear progtamming, but this is more or less "thrown in" without relating the topic to vectors. It is perhaps the only weak part of the book.

Prof. Schuster starts the book with an excellent discussion of the role of **abstraction** in mathematics and its importance to mathematicians. He makes the valid point that, for a beginner, a completely abstract approach is deadly, so one must assist him with the "crutch" of a geometric model. There can be no denying this point. However this reviewer also believes that *after* an abstract concept has been adequately motivated by geometric and physical models, these models should be set aside when one proves theorems about vectors. The proofs should be entirely algebraic and the geometric models can then be brought back to give fruitful interpretations and applications of the theorems.

In several of his definitions and theorems dealing with vectors, Prof. Schuster mixes geometric intuition with the algebraic analysis, thereby creating some minor flaws which an alert reader will undoubtedly discover. For example, the definition on page 14 uses the geometric notion of 'parallelism" and the intuitive notion of "sense" to define the vector **nA,** whenever the vector A is given. On the other hand, his definition of linear independence is purely algebraic (see pp. **22,23).** This makes it necessary to prove Theorem 3 **(pp. 24,25)** by mixing geometric and algebraic arguments. There results a small imperfection in the proof, at the point where it is asserted that line L intersects the **line** of action of vector A. An alert reader, seeking to prove this, will find himself hampered by the fact that linear independence is defined purely algebraically, while parallelism is a geometric notion, and an adequate connection between these two concepts has not been established. The difficulty is, of course, easily eliminated by defining parallelism in terms of linear dependence, **viz:** Definition: Vectors A and B are parallel, if and only if A and B are linearly dependent. To be sure, one does not throw this at the reader immediately. First, one leads up to it via diagrams and geometric intuition, but once motivated, it is adopted as a formal definition and the proofs are based upon it.

A few minor corrections are: p. 15 (middle) "m t n =O" should read "m + n > O"; p. 23 (Definition) "if" should read "if and only if"; p. 23 (Figure 17) the important label "D" is missing; p. 52 (line 21) "radius" should read "radians"; p. **115** (line -10) "three points" should read "three non-collinear points". Finally "the equation" should read "an equation" almost everywhere that this phrase appears.

Except for the few items noted, the book is remarkably free of errors, is clear, readable, and most enjoyable. It is eminently well suited to its intended audience and should function admirably in modern enrichment programs.

Bronx High School of Science — A. M. Glicksman

The Theory of Probability. By B. V. Gnedenko. Translated from the Russian by B. D. Seckler. New York, Chelsea Publishing Company, 1962, 459 pages, $8.75.

The first six chapters are described by the author as a text for an elementary course in probability and statistics. The chapter titles are 1. The Concept of Probability, 11. Sequences of Independent Trials, **III,** Markov Chains, IV. Random Variables and Distribution Functions, V. Numerical Characteristics of Random Variables, **VI.** The Law of **Large** Numbers. Some of the material requires advanced calculus; however, the remaining material and problems over it could be used to supplement a course in which less mathematical maturity is required. Prof. Gnedenko also suggests that the last five chapters might be used in special courses in probability and statistics. The chapter titles are **VII.** Characteristic Functions, VIII. The Classical Limit Theorem. IX. The Theory of Infinitely Divisible Distribution Laws, X. The Theory of Stochastic Processes, and XI. Elements of Statistics. Some knowledge of real and comples variable theory is needed for these topics.

When no unusual procedure is required to go from one step to another in a proof, the author rightly economizes by letting the reader fill in the details. Some of the material in this translation is not in the earlier Russian edition or the German translation. Changes include additions to the chapter on Markov chains and an entirely new section on the Poisson process. Most of the misprints are trivial.

For one who is interested in an introduction to advanced probability and statistics, this book is excellent.

University of Illinois. — **Leone** Low

Introductory **College** Mathematics, Third Edition. By William E. **Milne** and David R. Davis. Boston, Ginn and Company, 1962. xii + 579 pp., $7.50

What do you want in an introductory course in college mathematics? Is it for Commerce and Finance with Business Administration, terminal course for ill-prepared arts majors, or a foundation course for Engineering? Whatever your wish you need only make the proper selection of chapters to achieve your goal, and that with an abundance of material. All topics although covered adequately for an introuduction are not complete. This may be a benefit in disguise. A little knowledge although a dangerous thing may give incentive to additional study. The early introduction to the calculus (chapter three) and its influence on later developments is commendable. The suggestion of iteration techniques in solution of linear equations shows the modem influence. For those who wish a foundational development new material had been added in this edition to strengthen the algebraic fundamentals with additional emphasis on both linear and quadratic inequalities; It is a little unusual to find **Newton's** divided difference formulae in an introductory approach to approximations as well as **Maclaurin's** series, but they are introduced with a minimum of development.

The use of sets and set notation have been avoided as well as symbolic logic, which today are enjoying considerable interest. The trigonometric functions have not formally been extended to functions of real numbers, although radian measure is introduced.

The book is well written and should receive wide adoption because of its flexibility as to material needed. It certainly will provide a challenge to the capable student as will as prepare him for the mathematics needed in many fields.

Saint Louis University — John J. Andrews

Finite Mathematics with Business Applications. By J. G. Kemeny, A. Schleifer, Jr., J. L. **Snell,** and G. L. Thompson. Englewood Cliffs, N. J., Prentice-Hall, 1962. xii + 482 pp., $10.60.

Contrary to what might be expected, this is much more than the earlier book Introduction to Finite Mathematics, by three of the same authors, to which additions have been made; Finite Mathematics with Business Applications is definitely a new book. It certainly has the same **spirit and** style which made the earlier book so popular and useful, but it also has considerable expansion, many additions and deletions, **and** much rewriting.

The larger page size means that there is more material per page; in **addition,** there are about one hundred more pages. Even where the chapter heading and the name of the section are identical, there is often quite a difference between the contents of the two books, both as to topics and examples mentioned and as to the discussion of items in common. The very first sentence differs, and from there on there is scarcely a section of one which is identical, or even almost identical, to the other.

In addition to the minor and major differences in treatment of common topics, there are several new topics in the new book, and some which have been dropped. Some of the new topics are computer circuits, flow diagramming for computers and accounting, Monte **Carlo** methods (especially applied to decisions), finance and accounting. The chapter in the older book on Linear programming and the Theory of Games has been split into two **separate** chapters in the new book, one for each topic, and has been greatly expanded by the addition of several sections in each chapter. Some **of** the sections which have been deleted are: valid arguments, indirect methods of proof, measures as areas, permutation matrices, and subgroups of permutation groups. Naturally, the chapter on applications to behavioral science problems has been left out.

There is an increasing number of students in ecomonics who are interested in mathematics and vice versa. From comments made to me by members of the economics staff, this book should certainly help give the students in economics the mathematics they need for the economics **courses.** As one example we have been asked to include more set theory in ow introductory mathematics course since the economics students do not otherwise get enough for their statistics course; and sets is the second chapter of Finite ***Mathematics,*** with related logic topics being discussed in the first chapter. Both the topics and the examples should help convince the economics student that more mathematics is needed by him and is applicable to **his** field, as well as give him the basic elements of the needed knowledge.

DePauw University — Robert J. Thomas

Fibonacci Numbers. By N. N. Vorob'ev. New York, **Blasidell,** 1961. viii + 66 **pp.,,$0.95.**

This is Volume 2 of the translations from the Russian of the Popular Lectures in Mathematics series.

Fibonacci numbers are defined and studied with the aid of the recursion formula and a closed formula for the nth Fibonacci number. Very many arithmetic and number-theoretic properties are given, as are applications to the theory of continued fractions and to geometry.

As in the other volumes of the series, the exposition is self-contained and exceptionally clear. All necessary formulas and theorems from number theory and the theory of continued fractions are derived in the book. As a result, almost all of it is within the grasp of an able senior high school student.

University of Illinois — **Franz** E. Hohn

Curvature and Homology. By S. I. Goldberg. New York, Academic Press, 1962. xvii + 315 pp., $8.50.

This is an advanced treatment, by an acknowledged authority, emphasizing the relationship between the curvature properties of a Riemannian manifold and its topological structure. The reader will need a knowledge of linear algebra, real and complex variables, differential equations, **point-set** topology, algebraic topology, and Riemannian manifolds. The mature reader will find the hook a self-contained and informative treatment of developments in this field **of** differential geometry in the large.

University of Illinois — R. L. Bishop

An Introduction to Differentiable **Manifolds.** By Serge Lange. New York, Interscience, 1962. x + 126 **pp.,** $7.00.
Differential **Geometry** and Symmetric Spaces. By **Sigurdur** Helgason. New York, Academic **Press,** 1962. **xiv** + 486 pp., $12.50.

Both of these **books** are intended for beginning graduate students and beyond. All of Lange's book and the first chapter of Helgason's book (of ten chapters) are concerned with the basic material in the geometry of manifolds. This includes the definition of a manifold, tensor fields on a manifold, vector fields and their integral curves, differentiable mappings, Riemannian metrics, and the exponential mapping. **Lange** also includes a discussion of the **Frobenius** theorem and tubular neighborhoods. Helgason has a discussion of affine connections and parallelism and some more special but important theorems on sectional curvature and totally geodesic submanifolds. The latter topic is particularly relevant to the remainder of his book.

Aside from this similarity in the choice of basic material and the fact that both works are thoroughly up to date, there is little resemblance in treatment. Lange does everything as generally as possible, introducing more restrictions as he progresses. Thus, initially his manifolds are locally Banach spaces, not even Hausdorff. In the end he has narrowed down to Riemannian manifolds which are locally **Hilbert** spaces. Whether or not such generality is warranted is not known, because very little research has been done on **infinite** dimensional manifolds, nor have they been much used. Moreover, he gives no examples or problems to indicate any connection with other fields. The initial chapter on Banach spaces and the appendix on the spectral theorem are intended to fill in the necessary background for his readers.

Helgason's treatment of the basic **material** is extremely concise and restricted to the finite dimensional case. The emphasis is on the intrinsic **formulation** of concepts, but the expressions in **terms** of local coordinates are given and used whenever it will lead to a **quicked** exposition. Chapter two deals with **the** elementary theory of Lie groups **in** a manner which rivals the now classic volume I of Chevalley in both content and clarity. With a very good **initial** course in the theory of Lie groups and algebras is available here.

The **remainder** of the **book** deals with symmetric spaces, which are sufficiently general to include, **e.g.,** all of the classical Euclidean and non-Euclidean geometries, in all dimensions, the projective spaces, the **Grass**mann manifolds, and all compact Lie groups, but special enough so that a great deal can be said about their topology and analysis and so that they are completely classified. Their importance lies to a great extent in the extraction of **counter-examples** to conjectures, particularly in areas of synthesis between analysis and topology.

Helgason has included an extensive bibliography, a fair number of significant but difficult problems, and **historical** notes at the end of each chapter.

University of Illinois — Richard L. Bishop

Sets, Relations, and Functions. By James F. Gray. New York; **Holt,** Rinehart and Winston; 1962. Paperbound, ix t 143 pp., $2.50.

This is a series of 19 lessons, each beginning with a brief discussion of a small number of ideas and each containing a substantial **sequence** of almost programmed exercises designed to make the ideas intuitively clear and to fix them by means of varied practice exercises. Sets, relations, and functions are treated in the order stated in the title. The **definitions** are simply stated but precise, the explanations are brief but usually adequate, and the **exer-**cises are well designed. The material is all mathematically sound.

Secondary students studying a classical curriculum should find this a refreshing supplement to their regular work. The book should also be particularly useful to the teacher who has little or no preparation in "modem" mathematics as a first introduction to the type of materials now being introduced into the high school classroom. A special advantage in this connection is that the book can be read with a minimum of effort and that it proceeds by very gradual steps, so that the shock experienced by many newcomers to these subjects is minimized. However, for both groups, a few of these lessons will require supplementary explanation from someone who knows the material already.

University of Illinois **Franz** E. Hohn

Introduction to Calculus. By **Kazimierz Kuratowski.** Translated by **J. Musielak.** Reading, Mass.; Addison-Wesley; 1962. 315 pp., $5.00.

"**This** book contains notes of lectures on differential and integral calculus, prepared for publication, which (the author) held for many years in the University of Warsaw,

"**The** first volume contains differential and integral calculus of functions of one variable. Functions of two and more variables, partial derivatives, and multiple integrals will be treated in the second volume." (Preface to the Polish Edition, 1946.)

For the most part, this book is a real variables text. The subject is carefully developed with all theorems properly stated and fully (if concisely) proved. The examples and exercises are mathematical in nature, and physical applications are left to those who are interested in physical applications. In these respects, the book has little in common with the usual elementary calculus text.

With regard to foundations, however, one finds the usual weakness of an elementary text. Set theory is dealt with largely by ignoring it. The word "set" is used as a non-technical term, in spite of the fact that **Dedekind** cuts and bounded sets are discussed in the first chapter. Consequently, we find (Chapter 1) that "**if** a real number corresponds to each positive integer, then we say that an **infinite** sequence is defined," and (Chapter 2) "**if** to any x belonging to a certain set there corresponds a number y = **f(x),** then a function is defined over this set."

The other main weakness is that, for most students, the introductory section of Chapter one would not be very useful except as a review of previously covered material. The brief exposition is not improved in translation.

The reviewer feels that this text would be suitable as an elementary text for above average students with strong mathematical interests. However, the introduction should be supplemented.

University of Illinois Frans M. Djorup

Intermediate Algebra. By William **Wooton** and Irving Drooyan. **Belmont, California,** Wadsworth Publishing Company, **Inc.,** 1962 . xi t 334 pp.

This is a clearly written text treating the usual topics of **beginning** and intermediate algebra.

Chapters 1 through 6 provide an excellent review of **beginning** algebra. They start with a careful **definition of** constants and variables, the axioms of equality, and a description of the real numbers as on ordered field. Polynomial and fractional expressions, their properties, and **similar** properties of real numbers are developed next. Firstdegree equations and ine aualities in one variable are **precisely** treated by means of the concepts **of** open sentences, solution sets, and elementary transformations. As a minor suggestion we would prefer that symbols such as a, b, and c, be directly regarded as variables **rather** than as symbols "**representing** constants" as on page 82. Exponents, roots, radicals, and seconddegree equations in one variable complete the review.

Chapters 7 and 8 contain one of the best discussions of the function concept we have seen in **an** elementary algebra text. It is first defined as a correspondence or association and later as a set of ordered pairs. The rectangular coordinate system is defined, and the graphs of linear functions and linear equations and inequalities in two variables are discussed in detail. Quadratic functions, conic sections, and quadratic **inequalitites** in two variables are also graphed. Later, in Chapter 11, the exponential and logarithmic functions are studied.

Chapters 9 and **10** treat systems of linear and quadratic equations and sequences and series with clear exposition. Systematic elimination and determinants are presented as two equally **good** methods for computing solutions of systems of linear equations. We feel, however, that the method of elimination should be recommended, since it requires much less work, and that determinants should be regarded as a very important theoretical tool. We are disappointed that mathematical induction is not included, for it is an exciting topic of elementary algebra and the clear style of the authors would render it accessible to most students.

There are plenty of exercises, and many will challenge the good student. The **book** obviously is very carefully written. We summarize the style by quoting from the preface: "**The** routine manipulative procedures are developed through informal deductive and inductive arguments and every effort is made to lend plausibility to **formally** stated conclusions."

University of **Illinois** Gilbert L. Sward

Inequalities. By P. P. Korovkin. New York, Blaisdell, 1961. **vii** t 60 pp., $0.95.

The scope of this little book is restricted to two very general classes of algebraic inequalities: the class of exponential means and the class of inequalities related to $(1 + x)^{\alpha}$. **The** approach is to derive certain basic inequalities and then to derive a large number of other inequalities as consequences of these. The transformations employed to accomplish this are often ingenious, with the result that many inequalities that would otherwise be difficult to prove are obtained very easily.

Applications to maxima and **minima,** as well as to limits, are given. There are non-trivial exercises, **solutions** to which are provided in the final chapter.

The explanations are clear. The earlier part of the book is easily within the grasp of able advanced high school students. In some of the later proofs, computations are necessarily longer and more involved so that more than high school maturity would probably be required to appreciate them.

University of Illinois **Franz** E. Hohn

A **Survey** of Numerical Analysis. Edited by John Todd. New York, **McGraw-Hill, 1962.** xvi t **589** pp., **$12.50.**

This book is a record of lectures presented at a **1957** training program in numerical analysis for senior university staff. In this survey fourteen experts in the area of automatic computers have gathered and condensed considerable material. John Todd has done a masterful job of **editing.**

Contrary to the statement in the preface, this is **not a** text for the **beginning** numerical analysis student but an authoriatative reference work for any working mathematician or scientist. The book presents methods and techniques for finding numerical solutions to specific problems. Although the survey is not computer oriented, it gives accounts of current **practices** and limitations in solving problems using automatic computers.

University of Maine — Beverly M. **Toole**

Introduction to Calculus with Analytic Geometry. By R. V. Audree. New York, **McGraw-Hill, 1962,** xi t **360** pp., **$6.00**

The author has made good use of the old adage that a picture is worth a thousand words. The many illustrations used in the text add much to its usefulness. He has been successful in his stated attempt to transmit basic ideas rather than drill on techniques and tricks, even at the expense of the students' being able to set up problems that they do not have the **techniques** to solve.

The first two chapters cover basic algebraic and geometric theory with the third chapter devoted to tangents and limits. Chapters four through six are concerned with the development of the theory of differentiation and related applied problems. The next three chapters deal with the basic concepts of integration including the introduction of logarithmic functions through the use of integrals. Chapter **10** is entitled Trigonometric Functions and is involved with their differentiationand integration. Chapter **11** (twelve pages) covers techniques of integration.

The exposition in the text is extremely well done. There is an excellent discussion concerning division by zero, in fact, an excellent collection and correction of many of the misconceptions with which students often enter the calculus concerning such **things** as absolute values and square roots, as well as a splendid discussion of such calculus concepts as limits, derivatives, and the definite integral.

The text seems unusually free of errors, those observed being relatively minor, such as occurs in the table of contents in which the reader is referred to the theorem $\frac{d(kx^m)}{dx} = kx^{m-1}$, or on page **32** where the point **(-3,5)** is said to have coordinates **(-3,5),** or on page **251** where the bar is missing in the expression e'' \$?x.dx, to list a few.

Among other desirable features are **"self-tests"** at the end of each chapter, serving as a brief review; carefully stated problems as well as an excellent section of answers and hints complete with illustrations concerning the exercises.

In general, the book seems more than adequate for the use of the social scientists and N.S.F. participants for whom the author has written it although the statement that all the material should be covered in a four hour course may be rather optimistic. In addition, the reviewer would suggest another use – as a reference for the student of the traditional three semester analytic geometry and calculus sequence who **is having** difficulty in **this** area due to his lack of understanding of the basic underlying concepts.

St. Louis University — Raymond Freese

Mathematics for the **Physical** Sciences. By Herbert S. Wilf, New York, John Wiley, **1962.** xii **+ 284** pp., **$7.95**

The author of this **book** has tried to "provide the student with some heavy artillery in several fields of mathematics" and has left the application of these weapons to the student's discretion and interest. The artillery in question consists of chapters on **1.** Vector Spaces and Matrices, **2.** Orthogonal Functions, **3.** Roots of Polynomial Equations, **4.** Asymptotic Expansions, **5.** Ordinary Differential Equations, 6. Conformal Mapping, and **7.** Extremum Problems. While any such choice of topics can be disputed, these generally occupy central positions in the applications of mathematics to the physical sciences. Some **material** which has never appeared before in book form is welcome. An **example** is Wintner's extension of the **Picard** theorem on existence of solutions of ordinary differential equations. Another valuable feature is the author's emphasis on methods of calculation suitable for automatic computation.

The text of the book is smoothly and lucidly written. In addition, about forty pages are devoted to exercises and their solutions. The latter being indicated in the back of the book. Several slips which might cause difficulty were noticed by the reviewer. For example, on p. **27,** in the section on bordering Hermitian matrices, the special case where the bordering vector is an eigenvector of the bordered matrix is omitted, and Theorem **21** is **incorrect.** On p. **72** the definition of Fourier series is given incorrectly. On p. **144** an ordinary differential equation given as an exemplar of equations having a finite number of solutions actually has infinitely many solutions.

In spite of these difficulties, the author has provided a concise exposition of many of the central features of the topics covered here. This book will be valuable both as a text at the early graduate level and as a reference work. It is a worthwhile addition to the literature.

University of Illinois — Richard Jerrard

The Theory of Graphs and its Applications. By Claude Berge. Methuen, London; and Wiley, New York; **1962.** x t **247** pp., **$6.50**

This book is a translation, by Alison Doig, of **Théorie** des **graphes** et ses applications, Dunod, Paris, **1958,** and is one of the two books on the subject presently available in English, the other one being Ore's Theory of Graphs.

The purpose of this book, according to its introduction, is to develop the theory of graphs in such a way as **"to** provide the reader with a mathematical tool which can be used in the behavioural sciences, in the theory of information, cybernetics, games, transport networks, etc., as well as in **...** any other appropriate abstract discipline." Many of the classic puzzles of mathematics are expressible as problems in graph theory and can be found in this book. Examples of these are the **Königsberg** bridge problem, the problem of the three utilities, Kirkman's schoolgirl problem, the difficult crossing problem, the knight's tour, the problem of placing eight non-attacking queens on a **chessboard,** the four-colour **problem,** etc.

Throughout the twenty-one chapters are many helpful diagrams, often located at strategic points in the proofs. For the most part the book is self-contained, however, a familiarity with some of the basic terminology of set theory and matrix theory would be an asset in reading it.

University College, London — J. W. Moon

Mathematical Discovery (On understanding, learning, and teaching problem solving), vol. 1. By George Polya. New York, John Wiley and Sons, 1962. 15 + 216 pp., $3.95.

By the publication of this book Professor Polya has once again put in his debt all who are concerned with the teaching of mathematics. In this, the first of two volumes, he "combines the theoretical aim: to improve the preparation of high school mathematics teachers", "heuristic (being) the study that the present work attempts, the study of means and methods of problem solving." (from the Preface). That study is concentrated, in the present volume, on problems from the classical high school curriculum of algebra and plane and solid geometry, with a few glimpses into the calculus. It is enriched by his experience in conducting Seminars in Problem Solving various Institutes for teachers, whose formation he urges and for the conduct of which he offers suggestions in an Appendix. So much for the purpose of volume 1.

As for the method it is like that of a polished, subtle, and successful composer of music who wishes to enable others to learn the art of composition by a careful and detailed examination of representative smaller works which he dissects, analyses and puts together again, showing the patterns groups of them have in common so that others may use those patterns for works of their own doing. He spurs them on with cries of: "Generalise", "Can you use this?" "Devise some problem similar to, but different from (these) ...., especially such problems as you can solve", "Could you use its results -- or its method?", "Predict the result and check your prediction." The patterns he displays and discusses appear in Part One of this volume, in four chapters. In Chapter 1, geometrical patterns appear, the pattern of two loci, of similar figures, of auxiliary figures, applied to several problems of differing difficulty and followed by a long set of examples and comments, carefully graded. A short section on set theory without notation closes the chapter. Chapter 2 is devoted to the Cartesian pattern, i.e., the clothing of a problem with an algebraic form, with more examples. Chapter 3, Recursion, and Chapter 4, Superposition, carry on the algebraic development. Chapter 4 starts with the ingenious idea of developing Lagranges' interpolation formula as an example of superposition. Throughout, Polya suggests checks by special cases, returns to geometrical problems where appropriate as in the chapter on superposition, uses the works of the masters: Euclid, Descartes, Newton, Euler, Lagrange, Polya. Every page sparkles with a gem from Polya's collection, even the set of answers or pointers towards solutions of all problems which ends Part One. Part Two: Toward a General Method is concerned more nearly directly with P.'s study of heuristic, containing two chapters, one on the nature of problems, one on widening the scope (of the Cartesian method). Volume Two will continue this study, and we await its appearance. Impatiently? No, because the first volume has so much good listening. Eagerly? Yes, because we wish more of it.

University of Maine L. H. Swinford

Fourier Analysis on Groups, By Walter Rudin. New York, Wiley, 1962, ix + 285 pp., $9.50.

This excellent new book from the Interscience Tracts in Pure and Applied Mathematics deals with a subject which brings the reader to the forefront of an interesting but difficult field of research in mathematics. However, the material in this text is at a level of knowledge beyond that attained by most undergraduate and first year graduate mathematics students. The author assumes that the reader has a background of at least one course in each of the fields (point set topology, topological groups, Banach spaces and algebras, and measure theory) summarized in the appendices. Many facts about these background subjects are given, but, being in appendices, a complete list of results is not possible.

There are two major parts covered in the book. The first consists of the first two chapters, where the general theory of Fourier analysis on locally compact abelian groups and the structure theory of such groups are given. The Fourier analysis is presented from the Banach algebra point of view, for as the author states in his preface, "It seems appropriate to develop the material in this way, since much of the early work on Banach algebras was stimulated by Fourier analysis." Chapters 1 and 2 are introductory to the main theory presented and almost all of the information in them is well known.

The second section, consisting of Chapters 3 through 9, is presented here in book form for the first time. Many of the results proved here have appeared in the mathematical literature in recent years; for example, the bibliography lists papers published in 1961. In the hands of a graduate student with the proper background, this book brings the student to the point of choosing a thesis topic, since the reader will find many questions to ask himself throughout this second part.

It should be noted, for the reader who studies this book, that there are no exercises, However, the author's style of writing and of giving proofs lends itself to a more thorough understanding. In many places, there are details to be completed between steps in a proof in order to proceed and often a theorem is stated when its proof can be carried out in a manner analogous to the proof of a previous result. The author points these analogous situations out in the text.

It is recommended to the serious student that this book be read with the help and guidance of an instructor who will explain the ideas behind the concepts discussed and supply examples and exercises for the student.

University of Illinois Neal J. Rothman

Ordinary Differential Equations. By L. S. Pontryagin. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962. 296 pp., $8.75

A glance at the contents, namely, Introduction, Linear Equations with Variable Coefficients, Existence Theorems, Stability, and Linear Algebra, suggests that the first half of the book, roughly speaking, parallels to some extent the material found in standard books on differential equations, whereas the latter half contains material not usually found in a text meant for a first course in that subject. In reality, the book departs considerably from the usual text. For one, it stresses ideas, rather than methods. For another, its standard of rigor is high, demanding a rapid assimilation of new concepts introduced as they are needed.

According to the publishers, this volume is designed for a one-semester course in the junior or senior year, preferably after the course in calculus. It is the opinion of the reviewer that the average student having had just one year course in calculus is not mathematically mature enough to really master the material of which there is more than can be covered in one semester. The fact that there are no problems for the student to solve is a drawback. However, given an adequate first course in analysis, it should be possible and, of course, profitable, for the student so prepared to make the most of the wealth of material so elegantly presented by an outstanding Russian mathematician. No disdain is shown by Pontryagin for applications as is shown by his illustrations of the applications of mathematical theory to nontrivial engineering problems in electrical circuits, autonomous systems, stability of a centrifugal governor, and the design of vacuum tubes. In brief, this is a most satisfying book, combining as it does theory and applications in the right proportion.

University of Illinois E. J. Scott

CALCULUS: Analytical Geometry and Calculus, with Vectors. By Ralph Palmer Agnew. McGraw-Hill, New York, **1962.** xii **+738** pp., **$8.95.**

Professor Agnew has written a calculus text which is both mathematically rigorous and extremely entertaining. He introduces vectors in the second chapter (using the geometric approach) and the development of the rest of the book follows in a modern, logical and mathematically sound sequence. The topics covered include all of the usual ones (except ~~that~~, to the author's credit, the standard final chapter on differential equations has been omitted). An unusually thorough treatment of Riemann integrals is given, and a few non-standard topics, including a derivation of the Euler-Maclaurin formula, for example, are also discussed. Throughout the book, the author stresses theory rather than application.

Two features set the book apart from and in many instances above, the usual calculus text: the "tone" of the book and the problems. The author has adopted the attitude that calculus need not be approached with a grim face and a fearful mind. He has written in a narrative form which presents his ideas in a light, even breezy manner. The result is a text which is more enjoyable to read than most "popular" books on mathematics. Yet, despite the informal attitude, the book has more solid, serious mathematics in it than most other calculus books, and the material is presented in a rigorous development which should appeal, to all but the most inflexible of the "prove everything" school.

The problems at the end of each section are also unorthodox. Only a few of the usual "find $\frac{dy}{dx}$" and "evaluate" type of problems are found; instead, most of the problems extend or enlarge upon the ideas of the preceding section, or introduce new material, and they are often quite lengthy. In his preface, the author remarks that all of the problems should be read, and some of them should be worked, by each student; ideas introduced in the problems are sometimes discussed again in later sections. On the other hand, much of the material in the problems is presented for its own interest and the general knowledge of the student. We find, for example, a good introduction to the algebra or matrices in the problems of Chapter **2.** The problems, together with the numerous remarks and footnotes, provide the student with a wealth of mathematical ideas far beyond that ordinarily found in a calculus text.

In general, Professor Agnew's book can be recommended for the standard three-semester calculus course; in particular, the book is highly recommended for a group of students who are thinking about majoring in mathematics.

Saint Louis University — Edwin G. Eigel, Jr.

SOME APPLICATIONS OF MECHANICS TO MATHEMATICS. By V. A. **Uspenskii.** Translated by Halina Moss. New York; Blaisdell, **1961.** **vii + 58** pp., **$0.95**

This book is a lecture given to Russian students in Moscow in **1956.** Now that this English translation is abailable, the reader with a knowledge of analytical geometry and a little knowledge of elementary mechanics will find it an easy-to-read book on the application of mechanics to some problems in mathematics. The applications are primarily to conic sections with other applications to such fields as number theory and the evaluation of area. Note, however, on page 22 the apparently erroneous equating of time, **T,** and energy, E.

Uspenskii's book may not add considerably to the reader's mathematical knowledge, but will point out by its examples a method of thinking, aided by analogies from mechanics, that may be helpful in his problems.

Monsanto Research Corporation — Lawrence A. Weller

Studies in Modern Analysis. R. C. Buck, editor, Math. Assn. of America, **1962;** distributed by Prentice-Hall, **Englewood** Cliffs, N. J. vii **+ 182** pp., **$4.00.**

This book is a collection of four outstanding expository articles on modern analysis. The four papers are "A Theory of Limits" by E. J. **McShane,** "A Generalized **Weierstrass** Approximation Theorem" by M. H. Stone, "The Spectral Theorem" by E. R. **Lorch,** and "Preliminaries to Functional Analysis" by Casper Goffman.

The first paper treats limits of functions defined on directions, a direction being a family of sets with certain properties. The author shows that the familiar theorems about limits of sequences hold in this more general context. By suitably choosing the direction, a function on a direction can be an ordinary sequence, a double sequence, or the sums used to define a Riemann-**Stieltjes** integral.

The second paper is devoted to generalizations of the Weierstrass approximation theorem which states that a continuous function on a bounded closed interval of real numbers can be approximated uniformly by polynomials. In this paper, the author replaces the family of polynomials by families of real bounded continuous functions defined on a topological space and separating the points of the space. Applications are given for trignometric series, **Laguerre** functions, and **Hermite** functions.

The third paper is an excellent introduction to the problem of determining the structure of a linear transformation on a Hilbert space. Starting with the spectral theorem for linear transformations on a three-dimensional Euclidean space, the author makes the transition to infinitedimensional Hilbert space via completely continuous operators. The structure of a bounded symmetric operator on a Hilbert space is then determined.

The author of the last paper discusses two important concepts in analysis and their applications to differential and integral equations. The first concept is that of a contraction mapping of a complete metric space onto itself. It is shown that such a mapping has a unique fixed point. This result is then applied to prove the existence of solutions of integral equations. The second concept is that of compactness of a subset of the space of continuous functions on a closed bounded interval of real numbers. The Arzela-Ascoli theorem which gives a nedessary and sufficient condition for such compactness is discussed and applied to prove the existence of solutions of differential equations. The author concludes with a brief introduction to Hilbert spaces, Banach spaces, and Banach algebras.

The four papers included in this book were selected to expose the interplay between algebra, topology, and analysis. Although parts of some of the papers may be too sophisticated for the undergraduate, there is a good deal of material in this book which is accessible to the advanced undergraduate.

University of Illinois — L. L. Helms

Contribution to the Theory of Estimation from Grouped and Partially Grouped Samples. By Gunnar Kulldorff. New York, Wiley **1961, 144** pp., **$5.00.**

Maximum liklihood estimators are obtained for normal and exponential distribution function parameters when the sample information consists of the number of observations n_i, in a given interval (x_{i-1}, x_i), or is a combination of individual sample values and grouped values. Tables are given for optimum allocation of interval length for varying numbers of intervals and fixed sample size. The results are applicable to sample survey design.

University of Illinois — Leone Low

Diophontine Geometry. By Serge Lang. Interscience Tracts in Pure and Applied Mathematics, Number 11. New York, John **Wiley,** 1962. x + 170 pp., $7.45.

One of the most elegant, and at the same time most difficult, theorems on diophantine equations is the following one proved by **Siegel** in 1929: If **f(x,y)** is a polynominal in two variables withintegral coefficients such that the Riemann surface of the algebraic function defined **by the** equation **f(x,y)** = O has topological genus greater than zero, then the diophantine equation **f(x,y) = O** has at most a finite number of integral solutions.

Roth's theorem (1955) on rational approximations to algebraic numbers made it possible to simplify somewhat the logical structure of the proof of Siegel's theorem, and the advances in algebraic geometry of the last two decades have made it possible to generalize Siegel's theorem considerably. The present book is devoted to presenting these two developments. However, it does very little to make Siegel's theorem more accessible to the general mathematical public, for it is addressed to those who have scaled the Olympian heights of modern algebraic geometry.

University of Illinois — Paul T. **Bateman.**

Introduction **to** Matrices **and** Linear Transformations. By D. T. Finkbeiner. San Francisco, W. H. Freeman, 1960. ix + 246 pp., $6.50.

In this book Professor Finkbeiner states that one of his purposes is "to present a lucid and unified introduction to linear algebra at a level which can be understood by undergraduates who possess reasonable mathematical aptitude".

The author has elected an axiomatic approach emphasizing the structural aspects beginning with abstract systems. He uses linear transformations on **finite** dimensional vector spaces as motivation for learning about matrices. eigenvalues, eigenvectors, **diagonalization** theorems, and canonical forms.

The student who is introduced to mathematical rigor through this text will have to rely on faith and his instructor to carry him through the early chapters. However, as the vocabulary and concepts become more familiar he will see how all the pieces fall into place. Finkbeiner writes in a clear and concise style. In the exercises which occur at the end of each section, he includes problems which introduce the student to the forthcoming section. These exercises include both concrete examples as well as simple theorems to be proved. There are a handful of minor errors in the exercises.

Finkbeiner's second purpose in writing this text is "to prepare the student for advanced scientific work by developing his powers of abstract reasoning". Students with a background of one or two years of college mathematics will find a course in linear algebra with **Finkbeiner's** text invaluable in their further mathematics and engineering studies.

University of Maine — J. William **Toole**

An Introduction to Set Theory **and** Topology. By K. Kuratowski. Translated by Leo F. Boron., Reading, Massachusetts, Addison Wesley Publishing Company, 1962. 283 pp., $6.50.

The expressed purpose of Prof. Kuratowski's little volume is "to give an accessible presentation of the fundamental concepts of set theory and topology; special emphasis being placed on presenting the material from the viewpoint of its applicability to ... other branches of mathematics...". There is no doubt as to his success.

The book is divided into two parts; as might be gleaned from the title, the first deals with "Set Theory", the second is entitled "Topology". Part I begins with the propositional calculus and the algebra of sets, proceeds to axiomatics and elementary cardinal arithmetic, and ends with a clearly written chapter on well-ordering, transfinite induction, and other forms of the choice axiom. Part II covers most of the standard topics of point set topology, the emphasis being quite heavy on metric spaces (in fact, the definitions of Hausdorff space, normal space, etc., appear only in connection with theorems providing the corresponding properties for metric spaces). In addition, there are short chapters on continua and dimension theory, and a captivating chapter - "Cutting of the Plane" - culminating in the Jordon Curve Theorem, ends the book.

The one thom on the rosebush is the attempt to intorduce combinatorial topology in a few pages. Chapter XX, "Simplexes and Their Properties", was obviously written solely to present the Brouwer Fixed Point Theorem (via Spemer's Lemma) and demonstrate an application to differential equations. Well and good. But the following chapter, "Complexes, Chains, and Homologies", is not well conceived. The author defines abelian groups, then misleads the beginner with an oversimplified (and unsatisfactory) theory of complexes. For example, he goes to the trouble of defining the homology group of a complex, then remarks (without proof) that the **Betti** Numbers of a complex are invariant under homeomorphisms, making no mention of the invariance of the groups, and indeed, giving no hint at all as to the use **of** homology theory, but stopping short with the **definitions.** A redeeming factor for Chapter **XXI** is the bibliography.

Physically, the paper and typography are eye-pleasing, but the binding has a shoddy appearance (this seems to be de rigueur for books produced in Poland).

All things considered, this text **seems** to be just the ticket for the beginning student who is interested mainly in applications. The serious student of topology will also find pleasure here in examining some of the **less-often-**presented material. It might also be remarked that **Prof.** Kuratowski's new introductory opus abounds in interesting exercises of various degrees of difficulty.

The University of Chicago — D. A. Moran

Calculus of Variations. By L. E. Elsgolc. Reading, Mass., Addison-Wesley Publishing Company Inc., 1962. 178 pp., $4.50.

This is a clear exposition of those aspects of the calculus of variations which form a base for the intelligent use of this subject by engineers and physicists in various fields of mechanics and technology. Covered are: the derivation of the Euler equation for fixed and movable boundaries, notion of a field of extremals, the various conditions of Jacobi, **Legendre** and Weierstrass, mixed problems, constrained extrema, and direct methods such as those of **Ritz** and **Kantorovič.** The reader having a good background in advanced calculus will have little trouble in following the author in his presentation of the subject. Care is taken to clarify matters by contrasting, for example, the notion of function with that of functional, *of* differential and variation, of the theory of maxima and minima of ordinary functions with that of functionals. Also carefully defined is the meaning of closeness of two curves as well as the various orders of closeness. At the end of each chapter are problems with solutions given at the end of the book. Here, **the** reviewer would like to have seen more of them culled from actual problems arising in mechanics and technology. However, this detracts little from this carefully written book which could serve admirably as a text for a one semester course on the fundamentals of the calculus of variations.

University of Illinois — E. J. Scott

C. R. C. Standard Mathematical Tables, Twelfth Edition. Cleveland, Chemical Rubber Publishing Company, 1959. ix + 525 pp., $4.00.

Because of the vision of its publishers and editors, the C.R.C. Standard Mathematical Tables has grown from the very small early editions of the Mathematical Tables from the Handbook of Chemistry and Physics to the most broadly inclusive and most widely useful and inexpensive handbook of mathematical formulas and numerical tables in existence today.

Every student of mathematics or Science should obtain one of these handbooks early in his career and learn to use it as he would his dictionary: as a time saver, as a means of checking his work, and as a source of information and of inspiration. For such a student a copy of the C.R.C. tables would be a most appropriate and long appreciated gift, as many a worker in the fields of mathematics, statistics, science, and engineering can well attest from experience.

University of Illinois — Franz E. Hohn

Famous Problems of Elementary Geometry; From Determinant to Tensor; Introduction to Combinatory Analysis; Fermat's Lost Theorem (all reprinted in one volume). By F. Klein, W. F. Sheppard, P. A. MacMahon, and L. J. Mordell, respectively. New York, Chelsea, 1962. 321 pp., $1.95 (paper), $3.50 (cloth).

The first of these four famous little tracts proves the impossibility of trisecting the angle and of duplicating the cube with straight-edge and compasses, solves other algebraic construction problems, and establishes the transcendental character of π and e. If all high school mathematics teachers were to study the first fifteen pages - which is not hard to do since only mathematics through analytic geometry is required - they would be better armed to deal with the perennial attacks of angle trisectors.

The second tract is an elementary, easy-to-read introduction to the tensor concept, starting from the theory of determinants. However, the notation and terminology are not those used today.

The third tract concerns the theory of the distribution of "objects" into "boxes". The theory of symmetric functions is used to derive the various formulas. This material is as alive and as relevant in this age of "finite mathematics" as it was when it was written.

The last tract is an elegant presentation of the history of attempts to prove Fermat's last theorem as well as many of the results which were byproducts of these attempts.

Each of these little treatises is the work of a master and reflects his love of the subject as well as his eagerness to have the reader understand it. As a result, it is difficult to purchase elsewhere so much mathematical pleasure for so little money.

One is led to the nostalgic wish that the day of this style and this quality of elementary exposition by true experts might once again return, but the current crop of dollar-ninety-eight expositions are too often conceived in haste and delivered in confusion in order to supplement courses that, at all levels, spend so much time on a rigorous but thoroughly uninspiring examination of the number system that no time remains to do anything exciting or useful with it. One wonders as a consequence whether some of the "modern" mathematics programs may not convince the student that the mathematician leads a very dull life indeed and thus drive potential Kleins or Mordells to devote their creative energies to less sterile subjects. Reading parts one and four of this book, from which a good high school student could learn much, might well serve to prevent some such catastrophes.

University of Illinois — Franz E. Hohn

NOTE: All correspondence concerning reviews and all books for review Should be sent to PROFESSOR FRANZ E. HOHN, 375 ALTGELD HALL, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS.

BOOKS RECEIVED FOR REVIEW

Edited by

FRANZ E. HOHN, UNIVERSITY OF ILLINOIS

*R. P. Agnew: *Calculus.* New York, McGraw-Hill, 1962. xii + 738 pp., $8.95.

*R. V. Andree: *Introduction to Calculus with Analytic Geometry.* New York, McGraw-Hill, 1962. xi + 360 pp., $6.00.

W. G. Bickley and R. E. Gibson: *Via Vector to Tensor.* New York, Wiley, 1962. xvi + 152 pp., $4.50.

B. A. Fuchs and V. L Levin: *Functions of a Complex Variable and Some of Their Applications, Volume 11.* Reading, Mass.; Addison-Wesley, 1961. x + 286 pp., $7.00.

*S. L Goldberg: *Curvature and Homology.* New York, Academic Press, 1962. xvii + 315 pp., $8.50.

*J. F. Gray: *Sets, Relations, and Functions.* New York; Holt, Rinehart and Winston; 1962. Paperbound, ix + 143 pp., $2.50.

J. Heading : *An Introduction to Phase-Integral Methods.* New York, Wiley, 1962. vi + 160 pp., $4.50.

*S. Helgason: *Differential Geometry and Symmetric Spaces.* New York, Academic Press, 1962. xiv + 486 pp., $12.50.

*C. D. Hodgman: *C.R.C. Standard Mathematical Tables, Twelfth Edition.* Cleveland, Chemical Rubber Publishing Co., 1959. ix + 525 pp., $4.00.

*F. Klein; W. F. Sheppard; P. A MacMahon; L. J. Mordell: *Famous Problems of Elementary Geometry; From Determinant to Tensor; Introduction to Combinatory Analysis; Fermat's Last Theorem (resp.; in one volume).* New York, Chelsea, 1962. 321 pp., $1.95 paper, $3.50 cloth.

E. Kreyzig: *Advanced Engineering Mathematics.* New York, Wiley, 1962. xvii + 856 pp., $10.50.

*G. Kulldorf: *Contributions to the Theory of Estimation from Grouped and Partially Grouped Samples.* New York, Wiley, 1962. 144 pp., $5.00.

*S. Lang: *Diophantine.Geometry.* New York, Wiley, 1962. x + 170 pp., $7.45.

*S. Lang: *Introduction to Differentiable Manifolds.* New York, Wiley, 1962. x + 126 pp., $7.00.

W. Maak: *An Introduction to Modem Calculus.* New York; Holt, Rinehart and Winston; 1962. x + 390 pp., $7.00.

J. P. Marchand: *Distributions, an Outline.* New York, Wiley, 1962. xii + 90 pp., $4.75.

*L. S. Pontryagin: *Ordinary Differential Equations.* Reading, Mass.; Addison-Wesley; 1962. vi + 298 pp., $7.50.

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko: *The Mathematical Theory of Optimal Processes.* New York, Wiley, 1962. viii + 360 pp., $11.95.

W. Rudin: *Fourier Analysis on Groups.* New York, Wiley, 1962. ix + 285 pp., $9.50.

R. S. Varga: *Matrix Iterative Analysis.* Englewood Cliffs, N.J.; Prentice-Hall; 1962. xiii + 322 pp., $7.50 test edition, $10.00 trade edition.

*G. H. Weinberg and J. A. Schumaker: *Statistics, an Intuitive Approach.* Belmont, Cal.; Wadsworth; 1962, xii + 338 pp., $6.50.

*W. Wooton and I. Drooyan: *Intermediate Algebra.* Belmont, Cal.; Wadsworth; 1962. xi + 334 pp.

*See review, this issue.

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This section of the Journal is devoted to encouraging advanced study in mathematics and the sciences. Never has the need for advanced study been as essential as today.

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To point out the need of advanced study, the self-satisfaction of scientific achievement, the rewards for advanced preparation, the assistance available for qualified students, etc., we are publishing editorials, prepared by our country's leading scientific institutions, to show their interest in advanced study and in you.

Through these and future editorials it is planned to show the need of America's scientific industries for more highly trained personnel and their interest in scholars with advanced training.

Astronomy - "The science that treats of the heavenly bodies, their motions, magnitudes, distances, and physical constitution." Space, with stars in place, is a challenge being conquered by **todays** astronomers. This section of the journal is devoted to this fascinating field of scientific endeavor. Now through electronics the location of stars never seen are determined.

We are most fortunate to have in this issue articles from four leading universities whose astronomy departments are recognized for their contributions to research in this field. Harvard, Indiana, Ohio State, and Georgetown Universities have generously contributed to this program to encourage advanced study in mathematics and science.

To capable students the opportunities in "Astronomy" are unlimited, and preparation will pave the way.

The following lists contributing corporations with the issue in which their editorials appeared.

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Aeronutronics	Vol. 3, No. 2
Army Ballistic Missile Agency	Vol. 2, No. 10
AVCO, Research and Advanced Development	Vol. 2, No. 10
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General American Life Insurance Company	Vol. 2, No. 9
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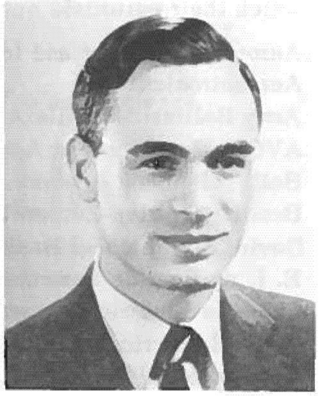
HARVARD UNIVERSITY

THE ROLE OF MATHEMATICS IN ASTRONOMY



DAVID LAYZER
Harvard College Observatory
Cambridge, Massachusetts

Mathematics is a kind of language - by ordinary standards a highly specialized language, capable of expressing only rather abstract ideas. These, however, may attain a degree of subtlety and complexity far beyond the powers of ordinary language.

The regularities that underlie natural phenomena find their natural expression in mathematics. A few of the very simplest could, perhaps, be described more or less adequately in plain English, but a wholesale translation of natural laws into nonmathematical language would be no more feasible than a translation of the works of Shakespeare into the vocabulary of a two-year-old. This is what is meant by the **often**-quoted remark that mathematics is the language of science.

It follows that the prospective natural scientist must first become proficient in mathematics. What kind of mathematics? Fifty years ago it was easy to define the boundaries of applied mathematics. They included the calculus, the theory of ordinary and partial differential equations, and parts of the theory of functions of a complex variable. Outside these boundaries lay the realm of pure mathematics, into which few scientists ventured. Nowadays it is harder to tell where pure mathematics leaves off and applied mathematics begins. Riemannian geometry, group theory, the theory of group representations, and the theory of finite- and infinite-dimensional vector spaces have long since joined the ranks of subjects that were once of interest only to the pure mathematician but are now studied routinely by scientists. When I was an undergraduate (concentrating in pure mathematics) I would not have believed that such subjects as topology, measure theory, or functional analysis would ever become interesting to natural scientists; but they have.

It is clear, then, that the risk of learning more mathematics than one is likely to need in a scientific career is not worth worrying about. I always advise by students to get as much pure mathematics as they can as soon as they can, and to take courses in modem algebra and functions of a real variable as well as in conventional analysis.

Although natural laws are **mathematical** in form, they are not about mathematics; they are about natural phenomena. Whereas the pure mathematician seeks to enlarge the boundaries and simplify the structure of the mathematical language itself, the scientist uses mathematics to discover and describe regularities underlying the structure of the physical world. Sometimes (though not very often) the necessary mathematics does not already exist and needs to be invented, but it is the content rather than the form of the mathematics that occupies the center of the scientist's stage.

Astronomy is the oldest, and in some ways the most comprehensive, of the natural sciences. Its aim is to describe and understand the structure, origin, and evolution of astronomical systems; of the fields and particles between astronomical systems; and of the universe as a whole. Modem astronomy draws heavily on the theories, techniques, and results of pure and applied physics. Conversely, **astronomical** problems and requirements have often stimulated fundamental developments in physics. The line between astronomy and physics is even more ill-defined than that which divides pure **from** applied mathematics.

The prospective astronomer should begin his education in physics as he has mastered the calculus - the language of elementary physics. I sometimes hear students express the view that, while theoretical astronomy is an essentially mathematical subject, mathematics and physics play a comparatively minor part in observational and experimental astronomy. The truth is that important work in astronomy, whether theoretical or observational, invariably springs **from** a deep understanding of the subject. Such an understanding must rest on a solid foundation of mathematics and physics.

New observational and experimental techniques - among them the techniques of space science -, as well as developments in such areas of theoretical physics as magnetohydrodynamics and **plasma** physics, have initiated a new period of growth in astronomy. The qualifications for an astronomical career - in particular the **mathematical** ones - are perhaps more stringent than they have ever been before; but the opportunities and the rewards have never been greater.

INDIANA UNIVERSITY

ASTRONOMY AND THE SPACE AGE

By John B. Irwin
Professor of Astronomy
Indiana University

The term "star gazer" is often associated with an egg-headed, absent-minded type of individual who might best be described as having his feet firmly planted in the clouds. In similar fashion, astronomy is often considered, even by professional astronomers who should know better, to be a most impractical or useless science.

Nothing could be further from the truth. Historians of science agree that astronomy was the first of the sciences - and this surely must have been so because of its intensely practical nature. These same historians point out that up to about the eighteenth century, astronomy was the most important science of all and was absolutely basic in the development of all science. Most of the mathematics, for example, which a student studies in order to achieve a Ph. D. in mathematics, came about or was discovered because of man's need to solve ***astronomical*** problems. The impact of the study of the heavens on philosophical and religious thinking can only be described as devastating. As one scientist has put it, if the human race had not been continually prodded and needled by the stars and planets, the chances are that the present population of the earth would consist of a few hundred thousand savages, living in caves and gnawing raw flesh from bones for their sustenance.

The three most immediate, basic and practical needs for human beings to exist at all have to do with food, clothing and shelter. If one scans the globe for those regions where people concentrate ***only*** on such practical necessities, it is in just those areas that one finds incredible poverty and misery. Such people are at the mercy of famine, flood, disease, superstition - and their fellow man. Little if any astronomy is known, used, or taught. In contrast, in those places where astronomy flourishes such as in the United States, Canada and western Europe, it is just here that the highest standards of living have been achieved. The pattern holds even within a country. In the United States there is relatively little astronomical activity in the deep South, for example, although there are some indications of change. Further, the top ten astronomy departments or observatories in this country are associated with just those universities that are considered to be the leaders in American education and research.

For the future the probabilities are that astronomy will play an even greater role in human affairs and destiny. This nation is committed to "sail the oceans of space". The space budget is now at six billion dollar: a year and is increasing exponentially. The cost of putting a man on the moon has been estimated at from twenty to forty billion dollars. We may expect in the near future that from a fourth to a third of the total national effort will be devoted to the space race.

Three things will happen if we don't train enough **Ph.D.'s** to meet this gigantic challenge. Our space programs will (1) take longer, (2) cost more, and (3) we will learn less than we should. The stakes **are** very high and it is most regrettable that NASA has not worked closely with the universities - where scientists are produced.

The present shortage of astronomers can only be described as appalling and will get worse before it gets better. It takes astronomers to train astronomers - but it also takes telescopes, which are expensive. Many if not most of our telescopes are either obsolete, or in impossible climates, or both. Astronomy is - temporarily I hope — caught in a strait jacket. We desperately need more modern telescopes in good climates to train future space researchers. Astronomy has become too important to depend for its support on the generosity of an occasional millionaire.

The astronomer is not primarily interested in going to the moon, although he will be deeply thrilled when men first walk on the bleak landscape. What he wants and what he will soon get are orbiting observatories a few hundred miles up, so that he can view and analyze the heavens free from the turbulence and absorption of the earth's atmosphere. Not only will he be able to penetrate to much greater depths in space because the star images are smaller by a factor of from ten to a hundred, but he will also be able to use those regions of the electromagnetic wave spectrum, especially those wavelengths in the ultraviolet shortward of 3,000 Angstroms, which the air completely absorbs. Fundamental atomic phenomena happen in the far ultraviolet that we can only vaguely guess at, if at all. Our experience in radio astronomy, which opened up celestial observations in the centimetermeter regions, has been an eye-opener. Most of the great discoveries of radio astronomy could not have been predicted ahead of time from previous knowledge gained at visual or photographic wavelengths.

The earth has been travelling through space at a velocity of 150 miles a second for four and a half billion years, coming from we know not where, going we know not where or for how long. In all that time it has intercepted electromagnetic radiation containing the secrets of the universe. Most of these "messages" from the stars and galaxies, **some** of them having been on their way for more than a billion years, have been absorbed in a split second a hundred miles overhead and have been lost forever. Properly intercepted and interpreted, using orbiting telescopes above the atmosphere, would initiate a voyage of discovery that would make Columbus look like a piker and would give man a knowledge and a power and a vision past all present understanding.

OHIO STATE UNIVERSITY

MATHEMATICAL PROBLEMS IN MODERN ASTRONOMY

Philip C. Keenan
Department of Astronomy
The Ohio State University

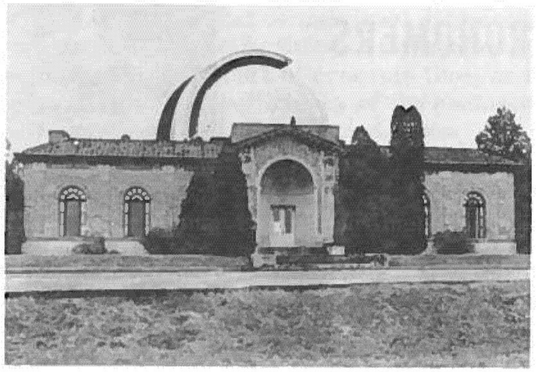
The widely-known role of mathematics in orbital astronomy has tended to overshadow its importance in astrophysics to-day, but in both fields there are theoretecal problems requiring for their solution something more than the more obvious techniques of applied mathematics.

Others writing in this space have brought out the special problems in celestial mechanics involved in the guidance of artificial additions to the solar system. It is not necessary here to do more than mention that in addition to the techniques of programming computations the ability to solve analytical problems involving extensions of classical mechanics comes into play.

If we turn to astrophysics we find that the problems to be solved are often similar to those of the other branches of theoretical physics. The famous problem of the solution of the integral equation for the flow of radiation through an atmosphere has led to a literature of hundreds of papers which might have been suspected to exhaust the subject, but new numerical solutions have been found necessary to deal with scattering in planetary atmospheres. Both in the atmospheres and throughout the bodies of stars the phenomena of convection and turbulence have assumed such importance that Chandrasekhar has been led to extend the theory of turbulent motion in his book "Hydrodynamic and Hydromagnetic Stability". This work has been concerned also with questions of the stability of configurations assumed by rotating masses, and for this problem Chandrasekhar and Lebovitz at Chicago have introduced a set of new tensors defining gravitational potentials and superpotentials, and have extended the sutdy of Jacobi ellipsoids. Such investigations are closely related also to the theory of pulsation of variable stars. To simplify the problem the treatment has usually been restricted to radial pulsations, but even these require systems of non-linear partial differential equations which, as Ledoux pointed out in Volume 51 of the Handbuch der Physik, have been solved only in very special cases. In connection with violent instability leading to stellar explosions there is the likelihood that it will be necessary to generalize the treatment to include non-radial osscillations. The few cases that have thus far been worked out suggest that this problem is sufficiently complex to delight a mathematical astronomer. The few problems mentioned above are only samples; it would be difficult to predict the directions in which the most exciting theoretical work will move within even the next few years.

Different but related problems are involved in the study of stellar motions and galactic sturcture. From the time of Gauss to the present some of the most fundamental work in statistical theory has been inspired by the necessity of deriving the best distribution functions from limited obseravtions (e.g., star counts) extended to large populations. Oustanding has been the work done at Swedish observatories and universities, where Charlier and Malmquist are among the best-known names. Current statistical research is directed toward a wide range of problems and can be exemplified by the general theory of spatial distribution of galaxies developed by Neyman. Miss Scott and others of the Berkeley group. In the Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (1961) they extend the theory to develop a statistical test for the stability of systems of galixies. In this connection a serious cosmological difficulty has been the question of the reality of the suspected large-scale clustering of galaxies.

What are the opportunities in college positions open to young men and women who are trained in applied mathematics and interested in astronomy? They are very good. Consider as fairly typical our Department of Astronomy at Ohio State. Although it was natural to start building our staff with enough experimental men to keep our observatories active, we found that by the time we had four or five people it was almost essential to have at least one man with primarily theoretical training. It is no longer true that the largest observatories can be staffed only with observers; it is the general experience that mathematical and observational astronomers are needed together. Thier activities cross-fertilize one another by the exchange of ideas and



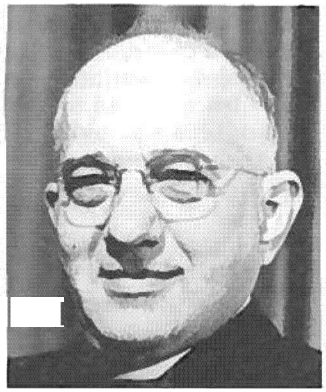
The Perkins Observatory

challenging problems. While it must be admitted that there are probably not over a dozen universities in this country that have astronomy staffs of four or more men, there is the interesting circumstance that a mathematical astronomer can work in a small department or group without being handicapped by lack of the expensive instrumentation needed for so much of the observational astronomy of to-day. The applied mathematician will usually find people with similar backgrounds working at the same institution in such fields as physics or engineering, and will rarely find itdifficult to obtain access to electronic computing equipment. Even the smaller colleges are setting up computing centers, where the essential programming of problems can be carried out even if the computation is so complicated that it proves worth while to obtain some additional machine time at one of the larger installations. such as the Numerical Computation Laboratory operated by the Department of Mathematics at Ohio State. Regardless of how such computing centers are organized, their purpose is to serve the whole university, and it is a reasonable estimate that about half the members of the average department of astronomy (or of any other of the mathematical sciences) are already making use of such facilities.

Finally, in the application of mathematics to astronomy women are making some of the finest contributions, so that the girls should not hesitate to enter this field!

GEORGETOWN UNIVERSITY

SPACE AGE ASTRONOMERS



Francis J. Heyden, S.J.
Georgetown College Observatory

The poet Keats expressed his surprise and delight on reading a very fine translation of Homer with the words: "Then felt I like some watcher of the skies when a new planet swims into his ken". Without a doubt most of the reading public whether they be engrossed in a translation of Homer or the Wall Street Journal still think of astronomers as watchers of the skies in search of some new planet. As far

as planets are concerned, the watchers of the skies have had this thrilling experience only twice within the past century at the discoveries of Neptune and Pluto. To be sure the astronomers who watched for these two planets had calculated and worked tirelessly for years in advance of their watching and the result was not so much a surprise as a reward for toil well spent.

In a way the space age caught the astronomer a bit off his guard. The present day astronomer has been wrapped up in studies of the ages of the stars, the detailed structure of the Milky Way and the nature of the universe. He has not specialized in the motions of moons or speculated in the performance of man-made satellites. He looked with interest on the International Geophysical Year and perhaps a bit cynically at the Vanguard Project, which was principally a great effort in engineering, but not astronomy. Astronomers, by and large, were not too deeply intrigued by the earth as an astronomical body and too few of them had devoted an extensive amount of their time to the study of celestial mechanics to care whether or not an artificial satellite ever rose in the southwest and set a few minutes later, in the northeast. The routine job of watching the motions of the planets and the moon belonged to the devoted few who kept the Nautical Almanacs up to date, or maintained the government supported time services.

The Space Age sprang from an engineering achievement, and many astronomers promptly assured the first success with the historical fact that Isaac Newton was the first to propose the launching of a satellite about four hundred years ago.

The picture is all quite different now. An entirely new class of astronomers has appeared. These are the men who by mastery of the large computers have explored the perturbing effects of air friction, radiation pressure and gravitational irregularities on fast moving earth satellites further than anyone before them. Among them also are found the electronic scientist who is giving the conventional astronomer a jolt with the accuracy he claims for measuring distances greater than the astronomical unit with radar beams bouncing off the planet Venus. This is an approach quite different from the precise measurements of angles that conventional astronomy has always used. Television cameras are being fitted to telescopes so that photographs which would have taken hours of exposure time, or might not even be possible because of the inefficiency of the photographic emulsions, are showing up details on the moon and in very distant galaxies never seen before.

All of these achievements have come from the great interest to reach out and explore space with the techniques that have become available from the rockets and electronic devices that appeared out of the war effort. Instead of hammering swords into plowshares, the peacetime applications have been directed to the skies above. The results have been more satisfying than victory over an enemy. Knowledge is always the exhaltation of man and the ability to penetrate space out to the orbits of the planets is far above brute force.

As a result there is no need to fear a sudden-collapse of interest. There are no limits to the frontiers of interplanetary space as far as items of interest are concerned. There are investigations already with within reach and which must be tried not only for sake of knowledge but for economic benefits. Weather forecasting by satellites is much more comprehensive and cheaper in the long run than world wide coverage with small weather stations. Communications satellites provide stronger bonds for understanding among nations than travel at any speed. It is hard to misunderstand the neighbor who sees and hears us. Besides there are still the unknown orbits for our probes to follow, sources of energy still untried and planets still very much unknown. The exploration of these is no longer a remote possibility.

Space exploration is all so new that the means that will be used for it are still growing with it. There is no field of engineering, no area of physics or chemistry that feels left out, or nation on the earth that does not want a share of the work for its scholars. One wonders at times if the future astronomers will be able to keep up with all of the fields that have merged with their own. It already appears beyond them because of the spread in the spectrum through which they can now gather the units of light energy from which they glean the fast growing information. A whitened harvest lies before them and now they are looking about them for the laborers to help reap it.

Because of the vast variety of data to be harvested the choice of the scholars will be wider than usual. Departments of astronomy in universities are already indicating this change in the titles they are choosing. Soon a candidate may find himself applying for admission to the Department of Earth, Space and Astronomical Sciences. The faculty for such a department becomes just as widespread in the interests and fields of its members. And yet this is perhaps the only way in which the training of such experts in the future can be attained with a certain amount of material in common and another portion highly specialized.

The educators are somewhat at a loss for a perfectly satisfactory solution to the problems involved in the educational program. The students too, are somewhat beleagured with very tempting offers for their services before they have reached the end of their academic careers. This is not good for them nor for the departments in which they are studying because every day spent in the earning of money is usually a day lost from study which will never be regained.

This means then that the crash programs of the space age will have to move forward with too few workers unitl a well trained group can be brought up by the slow process of graduate study. This is not conservatism, but rather a certain assurance that their advent into many new fields will not be simply a continuation but a definite acceleration.

NEWS AND NOTICES

Edited By
Mary L. Cummings, University of Missouri

Second Lt. James G. Voss, 10417 S. Maplewood, Chicago, recently completed the eight-week field artillery officers' orientation course at the U. S. Army Artillery and Missile Center, Fort Sill, Oklahoma. Lt. Voss is a graduate of DePaul University in 1962.

Dr. James J. Malone, Missouri Gamma, is now assistant professor of Mathematics at the University of Houston.

Mr. George Matusek, M.S., Missouri Gamma, is a Graduate Fellow in the Mathematics Department of the University of Missouri.

The 89th chapter of Pi Mu Epsilon was installed at Xavier University, Cincinnati, Ohio on Dec. 8, 1962. This chapter is the Ohio Theta.

NEWS AND NOTICES

A STATEMENT ON THE PEACE CORPS
FROM
R. SARGENT SHRIVER, JR., DIRECTOR

The United States is sending some of its most outstanding young men and women as Peace Corps Volunteers to the developing nations. As teachers, engineers, nurses, coaches and surveyors, and in community development work, these Volunteers are providing leadership and knowledge to people throughout the world.

Fraternities and sororities have prided themselves on their ability to attract and develop leadership. Responsibility, too, has come with this leadership.

Let me suggest that an even greater responsibility and challenge awaits you now. The chance to serve overseas, and thus to continue the work of more than 4,000 Peace Corps Volunteers now in the field, offers a rare fulfillment and experience. Inform yourself about the Peace Corps and how you may become a part of it after college. Contact the Peace Corps Liaison Officer on your campus, or write directly to PEACE CORPS, College and University Division, Washington 25, D.C.

CHAPTER ACTIVITIES

Edited By

Houston T. Karnes, Louisiana State University

EDITOR'S NOTE. According to Article VI, Section 3 of the Constitution: "The Secretary shall keep account of all meetings and transactions of the Chapter and, before the close of the academic **year,** shall send to the Secretary General and to the Director **General,** an annual report of the chapter activities **including** programs, results of elections, etc." The Secretary General now suggests that an additional copy of the annual report of each chapter be sent to the editor of this department of the ***Pi Mu Epsilon Journal.*** Besides the **information** listed above, we are especially interested in learning what the chapters are doing by way of competitive examinations, medals, prizes and scholarships, news and notices concerning members, active and alumni. Please send reports to Chapter Activities Editor Houston T. **Karnes,** Department of **Mathematics,** Louisiana State **University,** Baton Rouge 3, Louisiana. These reports will be published in the chronological order in which they are received.

REPORTS OF THE CHAPTERS

ALPHA OF CALIFORNIA, University of California at Los Angeles

The California Alpha Chapter held **seventeen** program meetings during the academic year **1961-62.** The following papers were presented:

"**The Structure** of the **Integers,**" by Dr. **Berri**

"Paradox of the Infinite," by Mr. S. Franklin

"**Choice** and Chance," **by Mr. R.** Wessner

"**Group** Structure in N-Dimension Space," by Mr. L. Robertson

"Geometry-Plane and Fancy," by Mr. E. Stiel

"Some Unsolved Problems in Point Set Topology," by Dr. **Berri**

"The Box Principle," by Professor Straus

"**How** to Invert Power Series," by Professor **Henrici**

"**A** Minimax in Elementary Geometry," by Professor **Horn**

"Minimal Surfaces and Some Applications of the **Dirichlet-Integral,**" by Professor **Tompkins**

"**Two** Point-Set Problems Arising in Connection with the Graph of an Equation," by Professor Green

"What is a Mathematical Theory?,*' (GSA Sponsored) by Professor **R.** Nevanlinna

"Computers and Number Theory," by Dr. J. **Selfridge**

"Weak Mathematical Systems," by Mr. J. Lindsay

"**Some Favorite** Problems from Analytic Mechanics," by Professor Valentine

"The Prevalence of Cocycles," by Professor Dye

"**The** Analytical Engine **Revisited,**" by Mr. F. Hollander

Two initiations were held during the year. Fifteen students were initiated at the fall initiation and twenty two students were initiated at the spring initiation.

The annual picnic was held May 27, 1962, at which time approximately one hundred students, faculty and families were in attendance.

Officers for 1961-62 were: Director, S. P. **Franklin;** Vice-Director, E. F. **Stiel;** Secretary, Betty Lou; Treasurer, Dr. M. **Berri;** Faculty Advisor, Dr. E. F. **Beckenbach;** Scholarship Committee, Dr. **O'Neill;** Dr. **Strauss,** G. Michels, D. **Gotlieb,** R. **Weiss;** Program Committee, T. **McLaughlin.**

The officers for **1962-63** are: Director, T. G. **McDaughlin;** Vice-Director, Betty Lou; Executive Vice-president, A. **Feldstein; Secretary,** Derek Fuller, Treasurer, Dr. T. **Klotz;** Faculty Advisor, Dr. E. F. **Beckenbach;** Scholarship Committee, Dr. Green, Dr. Valentine, S. **Berman,** F. **Eng,** T. **McCullough;** Program **Committee,** G. Senge.

ALPHA OF KANSAS, University of Kansas

The Kansas Alpha chapter held six meetings during the year, Four of these were business meetings. The remaining two were program meetings. At the program meetings the following papers were presented:

"**Helly's** Theorem on Convex Figures," by Neal Wagner

"**Minimal** Surfaces," by Phillip Roberts.

The officers for **1962-63 are:** Director, Raymond **Pippert;** Vice-Director, Fred **Womack;** Recording **Secretary,** Joanne Stover; Treasurer, Carol **Bassett;** Corresponding-Secretary, Wealthy Babcock; **Librarian,** Gilbert Ulmer.

FELLOWSHIP OPPORTUNITIES

For the past four years, the American Mathematical Society has published a special issue of the ***Notices*** of the society listing the various assistantships and fellowships in mathematics which are available at colleges and **universities** throughout the country. The January 1963 issue also contains a list of foundations and organizations other than colleges and universities which offer such aid for graduate and undergraduate study in mathematics. Students who are interested in obtaining financial assistance, especially for **graduate-level** study, should find this issue of the ***Notices*** most helpful.

INITIATES

ALABAMA ALPHA, University of Alabama (Fall **1962)**

Robert **Irvin Balch**
Barbara Jane **Ballard**
Mary Louise **Brassell**
Robert E. Bryant
Julian Ervin Bynum
Joseph Bibb **Cain, III**
George Ralph Carpenter
Chester Coen Carroll
Charlotte Anne **Cayley**
Alvin Joel **Connor**
Kenneth Cooper, **Jr.**
Leon E. Cooper
Charlorte Cross
Willie Joe Dean, **Jr.**
Mrs. Dorothy Dickman
John Frank Dollar
Barbara **Geissler**
Spencer L. **Glasgow, Jr.**
Paul Howell Glenn, **Jr.**
Peggy Guffin **Glenn**
William Austin Hall, **Jr.**
Larry G. Harper
John Randall **Holmes**
John Howard Horn, **Jr.**
Suzanne E. Laatsch
John Everett Lamar
Carolyn E. **Langston**
William T. **Mauldin**
Janice Moody
Lino Gutierrez Novoa
Thomas W. **Poe,** Jr.
Robert Glenn Priddy
Robert R. **Richwine, Jr.**
Alan Dale Sherer
Bobby R. Spencer
Tommy G. Thompson
Elizabeth Coleman Toffel
Henry Burton **Waites**
Mildred M. **Whatley**
Edward L. **Wilkinson**
David Louis Wood, **Jr.**
Jeong Sheng Yang

CALIFORNIA GAMMA, Sacramento State College (November **1962)**

John **Britton**
Gary Chaix
Lois **Gallaher**
Laura **Gleason**
Richard Henderson
Theodore **Lindberg**
Johnnye Lundgren
Richard Nickel
Kathryn Peters
Elizabeth Quackenbush
Edmond **Rottmiller**
Earl Stephens
Gerald Stevens
Bernard Schaaf

FLORIDA ALPHA, University of Miami (Fall **1962)**

Vicente Alvarez
Clemencia del Barrio
Zacarias Bramnick
Marjorie Hope Church
Truong-Hoang-Dan
C.L. Fishback, **Jr.**
Elliot C. **Friedwald**
William S. **Jennings**
Sandra L. Jones
Sherman Kane
Lynn Andrea Michael
Paul **Raynor** Penny
John F. **Quinn**
Robert H. Rucker
Maurice A. Scholar
Charlotte Lita Sieber
J.F.B. Shaw
Herbert Lee **Snitz**
Waino H. **Tervo, Jr.**
John B. Thuren

FLORIDA BETA, Florida State University (December **5, 1962)**

Charles H. **D'Augustine**
Frank **Whatley Gamblin,** Jr.
James Randall Gray
Herbert F. **Kreimer, Jr.**
Daniel W. **Litwhiler, Jr.**
George John Michael
Henry **Julian** Noble
Margaret C. **Norteman**
George **William Schultz**
James **Lardner** Semmes
Emile Boyd Roth
Andrew **M. Wasilewski**
John P. Russo
Louis **R.** Wirak

GEORGIA ALPHA, University of Georgia **(February 6, 1963)**

William S. **Bolton**
Thomas A. Bowman
Constance J. Boyd
John L. Bryant
Andrew C. **Conner**
Linda **J.** Cook
Marie Dyer
Edith Hand
Peter W. Harley, **III**
Susan **J.** Hill
James T. Hunt
Marjorie **J. Kingsley**
Lawrence Martin
Fayette **M. McElhannon**
Sam B. **Nadler, Jr.**
Choon Jai **Rhee**
Jan P. Richey
Susan L. Sevier
Mary L. Tharp
Marshall Waters, **III**
Margaret J. Wray
June F. Young
Alice **Youngblood**

ILLINOIS BETA, Northwestern University (November **1962)**

Stephanie I. **Ackley**
Wes. L. Anderson
Jack N. **Bezpalec**
James N. **Boyle**
George **J.** Cermak
Carol A. **Dilibert**
Roderick P. **Donaldson**
William **E. Drummond**
James O. Edwards
Edward A. **Euler**
John E. Farr
Robert B. **Feinberg**
Thomas L. **Flosi**
Allen R. Grahn
Anne C. Johnson
Kalman **J. Kaplan**
Norman A. **Karsten**
John F. **Kastner**
John S. **Keller**
John W. **Keller**
Edgar B. Klunder
Carol A. **Lidgen**
Conrad E. **Littig**
Bertrand J. Nisek
Robert L. Puette
Charles A. Rogers
Mitchell L. **Slotnick**
Sam A. **Sperry**
Ronald A. **Swanson**
Terry A. Weisshaar
Molly Wells
Jerry L. Zook

KANSAS GAMMA, University of Wichita (December **14, 1962)**

Laurence D. Bachman
Clark Duane **Dauner**
Richard S. Graves
Tommy S. **Mayfield**
Richard E. Neu
Melvin H. Snyder, **Jr.**
Hal W. Stephenson
William Frederic **Wenzlaff**
Robert C. **Wherritt**
Paul F. Williams
Gordon L. Wood
Max E. **Zent**

KENTUCKY ALPHA, University of Kentucky (January **17, 1963)**

Charles Richard **Eckel**
Walter P. Gerlach
John Michael Gibson
Carol **Anne** Harper
Harry L. Hurd
Robert Francis **McGuire**
Gordon **Mowat**
Silvio O. Navarro
Martine Nojin
James D. Powell
Alan Ross
Cohen Lee **Sharpe**
Jeanne Barbee Shaver
Joseph Lee **Stautberg**
Hope **Cornett Stidham**
Robert **Allan** Stokes
William H. **Zuber**

MISSOURI ALPHA, University of Missouri **(December 1962)**

M. Herman Acevedo
Stephen **J. Asher**
Gerald **Kent Bankus**
James K. **Ballou, Jr.**
Joseph E. **Barbay,** Jr.
Carl W. **Caldwell**
John E. **Connett**
Jerry G. **Crumpler**
Gerald Jay **Fishman**
Harold F. **Gebhardt**
Jerry A. Gentry
Dorothy Helen Godfrey
Douglas **M.** Goodman
George B. **Gordon, III**
James W. Gorham, Jr.
Carl W. **Guenther,** II
Jack L. Harlow
Ralph E. **Herndon, Jr.**
Ronald **Clair Higgins**
Paul E. Howard
Larry W. **Irminger**
Richard A. **Jesperson**
Rowland A. Jones
Emma Jean **Jundy**
Gordon D. **McLaren**
Ruth Ellen **Mellen**
James R. **Meyer**
Charlene Morgenstern
Thomas P. **Mullen**
David P. **Owsley**
John W. **Pridgeon**
Bobbette **K. Ranney**
Linda Lee Schick
Bernard Daniel Simon, III
L. Alvin **Spindler**
Lewis S. **Steenrod**
Raymond G. Smart
Martha Ann **Sudholt**
Edmund W. **Wilkenson**

NEBRASKA ALPHA, University of Nebraska (January **13, 1963)**

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Walter John Bauman
Clare Linda **Bentall**
Wayne H. **Bostic**
William Wallace Davis
John **Tallev Demel**
Randall **Kirk Heckman**
Jack Leroy Hershberger
Douglas **Matson** Howard
Douglas Lee Kreifels
Keith William **Kroon**
Donald Dean **Kummer**
Linda Lou **Larson**
Merlin Eugene **Lindahl**
James Kenneth **Linn**
Carol Sue **McKinley**
Curtis Wayne **Nicholls**
James **Paul Rutledge**
Gary Lee **Schrack**
Danny Lee Schwartz
Jack Walter Schwarz
Richard Donald Sudduth
Neil Nicholas **Wellenstein**
Leland **Yelland** Wilson

NEW JERSEY ALPHA, Rutgers, The Stare University (December 16, 1962)

Donald Guy Badami
Stuart Edward Beeber
Aaron E. Boorstein
Dominick A. Cardace
Albert H. Clark, Jr.
John Michael Clarke
Barry Druesne
Lee Arthur Duxbury
Roy Alan Feinman
Mark Joel Ganslaw
Gerald J. Goodman
Garry Joseph Huysse
Alan Clarke Jolley
Kenneth R. Lee
Dennis Ross Lowe
Joseph Benson Ludwig
David Mantel
Jay Edward Marowitz
Charles Robert Miller
David Charles Mollen
Fred Nisenholtz
Arthur Fytron Pabst
Ralph Edgar Portmore
Christian Przirembel
Donald L. Reisler
Murray L. Rothstein
Alvin Owen Sabo
Wallace Arden Smith
Silvio R. Verdile
Maxim H. Waldbaum

NEW YORK BETA, Hunter College (November 5, 1962)

Elena Bergonzi
Emily Bronstein
Rochelle Feiner
Kenneth Fogarry
Germana Glier
Charles Kahane, Prof.
Roberta Kleiner
Howard Levi, Prof.
Rita Pearlmutter
Diana Radovnikovich
Robert Schatten, Prof.
Sandra Schiffman
Stella Taras
Julia Wanzel, Mrs.

NEW YORK DELTA, New York University (January 2, 1963)

Martin D. Arnove
Herbert J. Bernstein
Karen Halbert
Eugenie Lee Hunsicker

NEW YORK EPSILON, St. Lawrence University (November 7, 1962)

Lynne Susan Eisenberg
Barbara Jane Greene
Nancy Selma Ludwig
Elizabeth Clare Neuse
Sharon Lee Perry
Donald Karl Rose
Peter Lawton Schofield
Thomas Donald Smith
Lee Alan Spector

NEW YORK GAMMA, Brooklyn College (November 19, 1962)

Ellen R. Axel
William H. Fellner
Teddy Fishman
Alexander Flamhols
Solomon Garfunkel
Libby R. Goldberg
Norman J. Hassan
Larry King
Edward H. Korte
Arthur R. Krowitz
Lawrence Landweber
Harvey Morgenstern
Sidney Richard
Bernard Septimus
Raezelle Snow
Harriett Weitsen
John Zechowski
Michael Zeiger
Ronald F. Hirshon

NEW YORK ETA, University of Buffalo (December 5, 1962)

Thurlow Adrean Cook
Sri G. Mohanty
Gilbert E. Prine
Richard N. Schmidt
Benjamin B. Sharpe
Keith Lawrence Shuert
Stevm L. Siegel
David Spencer
Andrew Jay Umen

NORTH CAROLINA ALPHA, Duke Universiry (December 1962)

Brenda Prue Balch
Karl Theodore Benson
Brent Francis Blackwelder
William Cudd Blackwelder
Spencer Ariss Bruckner, Jr.
Joseph William Cook, Jr.
David Alan Coolidge
Weldon Royall Cox
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Thaddeus George Dankel
Ellen Campbell Finley
Carol Ann Gerz
Richard Paul Guelcher
Alice Catherine Guerry
Mary Ann Hart
Edward Lee Keller
Lewis Waynick Kenyon
Charles William Phelan
Ronald Walter Rau
John Harger Roediger
James Russell Stites
Stewart Thorne Spencer
Kathleen Stettler
Bernell Kenneth Stone
Martha Davis Strayhorn
Peter Rockwell Ward
Terry Emerson Ward
William Edward Watson
John Neville Williamson
Richard Byrd Woods
Joseph Dawitt Wright
Richard W. Zaran

NORTH CAROLINA GAMMA, North Carolina State University (November 28, 1962)

Anthony James Barr
William Moncier Cox
Kenneth Elmer Cmss
Paul Alexander Helminger
Cynthia Graham Johnson
Walter Vann Jones
Malcolm Robert Judkins
Walter Allan Kester
Vello Alexander Kuuskraa
Thomas H. McLawhorn
Harmon Lindsay Morton
John Rutledge Rainey
Stephen Warner Sherman
William Henry Smyth
Arthur Conrad Springer
Alfred John Stamm
Clifton Bruce Suirt
David Wood Swain
Jack Owen Watson
James Miller Whisnant

OHIO DELTA, Miami University (October 25, 1962)

Marshall Barton
Robert W. Beyer
Sharon A. Boelter
Paul W. Carlton
Robert Kreunen
Robert E. Lover
David P. Mather
Mary Phillips
Norman Whitlatch

OHIO THETA, Xavier University (Charter Members) (Fall 1962)

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Benito Joseph Carimele
Robert Francis Cissell
David Clifford Flaspohler
Charles Matthew Geschke
Ralph Henry Johnson
Robert Joseph Kolesar
William James Larkin, III
Edward Lawrence Spitznagel, Jr.
Richard Edward Strenk

OHIO THETA, Xavier University (December 8, 1962)

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Walter Frederick Ludmann
James Koenig Luers
William Alfred McCafferty
Charles Thomas Uhl
Kenneth Paul Yanosko

PENNSYLVANIA BETA, Bucknell University (November 14, 1962)

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Diane K. DeLonge
Richard R. Edwards
Carol E. Heckel
Donald L. Herman
Frederick J. Hills
Jeannette G. Hogan
Phyllis A. Horine
Walter J. Landzettel, Jr.
Joyce M. Lattal
Elizabeth A. Lundry
Myra B. McFadden
Kathryn E. Meara
Edwin L. Meyer, Jr.
John S. Mowbray, Jr.
Janvier M. Ort
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William A. Urch

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John C. Bennett
David R. Burg
C. Frederick Burrell
Burton Decker Corwin
Thomas DiStefano
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Gordon C. Everstine
George F. Feissner
Michael D. Feit
James David Foley
Janes Barry Fry
Preston R. Gray, Jr.
Thomas W. Grim
Fred N. Heidorn
David P. Heintzelman
Kenneth W. Horch
John Bernard Holz
Andrew Jansons
Dean P. Kinard
John F. Loeber
Michael J. Maskornick
David B. Salerno
Robert R. Schwartz
Howard A. Seid
Allan P. Shumofsky
Martin R. Simsak
Henry A. Spindler, Jr.
Lawrence E. White
Richard A. Young

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Frances Chleboski
Samuel Geffner
Jerome Goldsrein
Alan Hodel
Kenneth Kloss
George Logan
Hazel Jo Mumaw
Alan Pollack
Andrew Y. Schoene
James Sharkey
William Simons
Stephen Strasen
Charles Thornton
Jack Williamson
Mary Jean Winter

PENNSYLVANIA ZETA, Temple University (June 10, 1962)

Ivars Dregers
Krendelle Ersner
Maier Fein
William Henry Harrison
Debbie Kaplan
Jerome Mersky
Joseph Albert Paone
Gary Marvin Sampson
Roy Snyder
David Zerling

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Roderick P. C. Caldwell
Marie J. Capozzo
Gail A. Dollar
Joseph C. Gallo
Cynthia E. Hill
Susan E. Porter
Manuel R. Renasco
Norman K. Roth
David B. Secor
David Silber
Edward A. Sylvestre
Richard C. Testa
Chris P. Tsokos
Carmine J. Vallese
Richard J. Zino

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John David Brown
Frances E. P. Cotten
George L. Hill
Alexander A. J. Hoffman
Tommy L. May
Donald L. Romary
James L. Shawn
Charles Duane Tabor
Henderson O. VanZandt
Margaret Reames Wiscamb

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Sandra Crowder
Elizabeth Edwards
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Glenda Nicholas
Ralph A. Palmieri, Jr.
Lea Perkins
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Carolyn Richardson
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Joseph D. Tatum
George W. Tiller
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Nancye Lee Webster
Dorothy Williams

WASHINGTON GAMMA, Seattle University (December 1962)

Daniel J. Costello
Paul H. Domres
Norman Cary Meyer
Sr. Mary Normandin, S.S.A.
Mary Kay Owens
Allison Frances Reed
Lloyd M. VonNormann

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James A. Bonesho
Sharon A. Christie
Margaret M. Garsombke
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Kathleen E. Kavanagh
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Rita M. McDonough
James J. Metzger
Paul G. Miller
Thomas E. Pauly
Thomas H. Payne
Ellen M. Proctor
John Nicholas Rachac
Peter Ryge
John R. Walker

WISCONSIN BETA, University of Wisconsin (December 1962)

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Linda Cooley
Paul Donis
Richard Goodrick
Peter Hirsch
William Lopez
John McCall
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Dora Helen Skypek
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