PI MU EPSILON Journal

VOLUME 4 NUMBER 2 REPRINT

CONTENTS

SPRING 1965

A GENERALIZATION OF MULTIPLICATION OF COMPLEX NUMBERS

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1. Introduction. In order to construct the field of complex numbers, we consider ordered pairs (a,b) of real numbers. Our definitions of addition and multiplication are motivated by properties of the expression $(a,b) = a + bi$ where $i^2 = -1$. If instead, we set $i^2 = a + \beta i$ where a and β are real numbers, and assert that in scalar multiplication the real number x be associated with the ordered pair $(x,0)$, we define multiplication by

(1) $(a,b) \circ (c,d) = (ax + abd,\ ad + bc + \beta bd),$

from which it follows that

(2) $x(a,b) = (xa,\ xb).$

The following definitions must also be made:

(3) $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

(4) $(a,b) \oplus (c,d) = (a + c,\ b + d).$

A set S of ordered pairs which fulfills conditions (1), (2), (3), and (4) constitutes a commutative ring with the identity element $(1,0)$. In an attempt to preserve the axioms for a field, we ask if for any choice of α and β all non-zero elements have inverses. If

$$(a,b) \circ (c,d) = (ac + abd,\ ad + bc + \beta bd) = (1,0),$$

then by equation (3)

$$ac + abd = 1$$

and

$$ad + bc + \beta bd = 0.$$

In order to solve these simultaneous equations for the value of (c,d), the determinant

$$\begin{vmatrix} a & \alpha b \\ b & a + \beta b \end{vmatrix} = a^2 + \beta ba - \alpha b^2$$

must be different from zero.

Let $a^2 + \beta ba - ab^2 = 0$. If $b = 0$, the only element without an inverse is the ordered pair $(0,0)$ which is the zero element of S. Suppose then that $b \neq 0$. Multiplying by $(b^2)^{-1}$, we have

$$\left(\frac{a}{b}\right)^2 + \frac{\beta a}{b} - \alpha = 0$$

which implies that

$$\frac{a}{b} = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$$

or

$$a = \left(\frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}\right) b.$$

Elements of the form $\left(\frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2} b,\ b\right)$ therefore do not have inverses and these are the only elements without inverses.

The nature of this ordered pair is such that it depends upon the value of the discriminant $\beta^2 + 4\alpha$. Here we will consider the three possibilities for $\beta^2 + 4\alpha$: CASE I, the set S_1 of all rings where $\beta^2 + 4\alpha < 0$; CASE II, the set S_2 of all rings where $\beta^2 + 4\alpha > 0$; and CASE III, the set S_3 of all rings where $\beta^2 + 4\alpha = 0$.

2. The Negative Discriminant. When the discriminant is less than zero, we see that all non-zero elements of rings belonging to S_1 have inverses, for when b is real, $\frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2} b$ is complex, and the only ordered pair of the form $\left(\frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2} b,\ b\right)$ with both elements real is $(0,0)$, which is the zero element. Thus for any choice of α and β such that $\beta^2 + 4\alpha < 0$ the conditions for a field are satisfied. The complex number system is a particular example of this case where $\alpha = -1$ and $\beta = 0$. Then

$$(a,b) \odot (c,d) = (ac - bd,\ ad + bc).$$

THEOREM 1. All fields of ordered pairs of real numbers satisfying equations (1), (2), (3), and (4) such that $\beta^2 + 4\alpha < 0$ are isomorphic to the complex numbers.

Proof. Let C be the set of all complex numbers generated by the linearly independent ordered pairs $(1,0)$ and $(0,1)$ and with multiplication defined by equation (1) with $\alpha = -1$ and $\beta = 0$, and let C' be the set of all elements generated by $(1,0)$ and $\left(\beta\sqrt{\frac{-1}{\beta^2 + 4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2 + 4\alpha}}\right)$ with multiplication defined by equation (1) with arbitrary α and β such that $\beta^2 + 4\alpha < 0$.

We describe the mapping $C \longleftrightarrow C'$ by

$$(1,0) \longleftrightarrow (1,0)$$

and

$$(0,1) \longleftrightarrow \left(\beta\sqrt{\frac{-1}{\beta^2 + 4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2 + 4\alpha}}\right).$$

This determines a one-to-one mapping of C onto C' given by

$$[a(1,0) + b(0,1)] \longleftrightarrow [a(1,0) + b\left(\beta\sqrt{\frac{-1}{\beta^2 + 4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2 + 4\alpha}}\right)],$$

and therefore

$$[c(1,0) + d(0,1)] \longleftrightarrow [c(1,0) + d\left(\beta\sqrt{\frac{-1}{\beta^2 + 4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2 + 4\alpha}}\right)].$$

To prove that C is isomorphic to C', we must show that addition, multiplication, and scalar multiplication are preserved under the mapping. $C \longleftrightarrow C'$. Since scalar multiplication follows directly from the definition of multiplication, equation (1), we need only show that addition and multiplication are preserved.

The preservation of addition is a consequence of the following calculations:

$$[a(1,0) + b(0,1)] \oplus [c(1,0) + d(0,1)]$$

$$= [(a+c)(1,0) + (b+d)(0,1)] \longleftrightarrow [(a+c)(1,0) + (b+d)\left(\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)]$$

$$= [a(1,0) + b\left(\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)] \oplus [c(1,0) + d\left(\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)].$$

A demonstration of the preservation of multiplication completes the proof of the theorem.

$$[a(1,0) + b(0,1)] \odot [c(1,0) + d(0,1)] = [(a,0) + (0,b)] \odot [(c,0) + (0,d)]$$

$$= (a,b) \odot (c,d) = (ac - bd,\ ad + bc) = [(ac - bd,0) + (0,ad + bc)]$$

$$= [(ac-bd)(1,0) + (ad+bc)(0,1)] \longleftrightarrow [(ac-bd)(1,0) + (ad+bc)\left(\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)]$$

$$= [(ac-bd,0) + \left(\{ad+bc\}\left\{\beta\sqrt{\frac{-1}{\beta^2+4\alpha}}\right\},\ \{ad+bc\}\left\{-2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right\}\right)]$$

$$= \left(ac-bd + \{ad+bc\}\left\{\beta\sqrt{\frac{-1}{\beta^2+4\alpha}}\right\},\ \{ad+bc\}\left\{-2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right\}\right)$$

$$= \left(a + b\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2b\sqrt{\frac{-1}{\beta^2+4\alpha}}\right) \odot \left(c + d\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2d\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)$$

$$= \left[(a,0) + \left(b\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2b\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)\right] \odot \left[(c,0) + \left(d\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2d\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)\right]$$

$$= \left[a(1,0) + b\left(\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)\right] \odot \left[c(1,0) + d\left(\beta\sqrt{\frac{-1}{\beta^2+4\alpha}},\ -2\sqrt{\frac{-1}{\beta^2+4\alpha}}\right)\right]$$

D. J. Hansen [2] adds to definitions (1), (2), (3), and (4) used here the condition that the classical definition for the modulus of an ordered pair be preserved along with the statement that

$$|(x,y) \odot (u,v)| = |(x,y)| \odot |(u,v)|.$$

He then proves that multiplication must be defined by

$$(a,b) \odot (c,d) = (ac - bd,\ ad + bc)$$

in order to obtain a ring and preserve lengths. It is easily shown by using equation (1) for multiplication of ordered pairs that the only values for α and β which fulfill the conditions regarding moduli are -1 and 0 respectively.

3. *<u>The Positive Discriminant</u>. System S_2, where $\beta^2 + 4\alpha$ is greater than zero, is not composed of fields because inverses do not exist for elements* (c,d) *along the intersecting lines* $c = \dfrac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}\,d$.

THEOREM 2. *All commutative rings of ordered pairs of real numbers satisfying equations* (1), (2), (3), *and* (4) *such that* $\beta^2 + 4\alpha > 0$ *are isomorphic to each other.*

<u>Proof</u>. To define a mapping of a ring A_1 onto a ring A_2 where A_1 and $A_2 \in S_2$, the identity element maps onto itself

$$(1,0) \longleftrightarrow (1,0)$$

and we set

$$\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right) \longleftrightarrow \left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)$$

Elements $(1,\ 0)$ *and* $\left(\dfrac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \dfrac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)$ *are linearly independent and generate all elements of* A_1, *thus we define the one-to-one mapping*

$$[a(1,0) + b\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \longleftrightarrow [a(1,0) + b\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)],$$

and therefore

$$[c(1,0) + d\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \longleftrightarrow [c(1,0) + d\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)].$$

Here *again we may omit the proof of the preservation of scalar multiplication under the mapping and show only that addition and multiplication are preserved.*

Addition is preserved under the mapping $A_1 \longleftrightarrow A_2$ *because*

$$[a(1,0) + b\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \oplus [c(1,0) + d\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)]$$

$$= [(a+c)(1,0) + (b+d)\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \longleftrightarrow [(a+c)(1,0) + (b+d)\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)]$$

$$[a(1,0) + b\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)] \oplus [c(1,0) + d\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)].$$

It follows that multiplication is preserved from:

$$[a(1,0) + b\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \odot [c(1,0) + d\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)]$$

$$= [(a,0) + \left(\frac{b\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2b}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \odot [(c,0) + \left(\frac{d\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2d}{\sqrt{\beta_1^2+4\alpha_1}}\right)]$$

$$= (a + \frac{b\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2b}{\sqrt{\beta_1^2+4\alpha_1}}) \odot (c + \frac{d\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2d}{\sqrt{\beta_1^2+4\alpha_1}})$$

$$= (ac+bd + (ad+bc)\left\{\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}}\right\},\ \frac{-2(ad+bc)}{\sqrt{\beta_1^2+4\alpha_1}})$$

$$= [(ac+bd,0) + (ad+bc\left\{\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}}\right\},\ \frac{-2(ad+bc)}{\sqrt{\beta_1^2+4\alpha_1}})]$$

$$= [(ac+bd)(1,0) + (ad+bc)\left(\frac{\beta_1}{\sqrt{\beta_1^2+4\alpha_1}},\ \frac{-2}{\sqrt{\beta_1^2+4\alpha_1}}\right)] \longleftrightarrow$$

$$[(ac+bd)(1,0) + (ad+bc)\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)]$$

$$= [(ac+bd,0) + \left((ad+bc)\left\{\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}}\right\},\ (ad+bc)\left\{\frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right\}\right)]$$

$$= (ac+bd + (ad+bc)\left\{\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}}\right\},\ (ad+bc)\left\{\frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right\})$$

$$= \left(\frac{a+b\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2b}{\sqrt{\beta_2^2+4\alpha_2}}\right) \odot \left(c + \frac{d\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2d}{\sqrt{\beta_2^2+4\alpha_2}}\right)$$

$$= [(a,0) + \left(\frac{b\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2b}{\sqrt{\beta_2^2+4\alpha_2}}\right)] \odot [(c,0) + \left(\frac{d\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2d}{\sqrt{\beta_2^2+4\alpha_2}}\right)]$$

$$= [a(1,0) + b\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)] \odot [c(1,0) + d\left(\frac{\beta_2}{\sqrt{\beta_2^2+4\alpha_2}},\ \frac{-2}{\sqrt{\beta_2^2+4\alpha_2}}\right)].$$

This completes the proof of the theorem.

In connection with S_2, *let us keep equations (2),* (3), *and (4) and consider another multiplication of ordered pairs defined by*

$$(5) \qquad (a,b) \odot (c,d) = (ac, bd).$$

It is easily verified that with these definitions of addition and multiplication the set A' of all such ordered pairs form a commutative ring with an identity element (1,1).

THEOREM 3. *The ring A' of ordered pairs of real numbers satisfying equations (2),* (3), (4), *and (5) is isomorphic to every commutative ring of ordered pairs of real numbers satisfying equations* (1), (2), (3), *and (4) such that* $\beta^2 + 4\alpha > 0$.

Proof. Since by THEOREM 2, all rings in $\mathcal{S}_2$ are isomorphic to each other, let us choose $\alpha = 1$ and $\beta = 0$ and call this ring A. Our proof then consists of showing that A and A' are isomorphic to each other.

For $A \longrightarrow A'$ we have

$$(1,0) \longleftrightarrow (1,1),$$

and since in ring A $(0,1)^2 = (1,0)$ and in ring A' $(1,-1)^2 = (1,1)$ we may set

$$(0,1) \longleftrightarrow (1,-1).$$

In A, the elements $(1,0)$ and $(0,1)$ are linearly independent as are $(1,1)$ and $(1,-1)$ in A', so we define the mapping $A \longrightarrow A'$ by

$$[a(1,0) + b(0,1)] \longleftrightarrow [a(1,1) + b(1,-1)]$$

and therefore

$$[c(1,0) + d(0,1)] \longleftrightarrow [c(1,1) + d(1,-1)].$$

Addition is preserved under this mapping because

$$\begin{aligned}
&[a(1,0) + b(0,1)] \oplus [c(1,0) + d(0,1)] \\
= \; &[(a+c)(1,0) + (b+d)(0,1)] \longleftrightarrow [(a+c)(1,1) + (b+d)(1,-1)] \\
= \; &[a(1,1) + b(1,-1)] \oplus [c(1,1) + d(1,-1)].
\end{aligned}$$

Multiplication is preserved as follows:

$$\begin{aligned}
&[a(1,0) + b(0,1)] \odot [c(1,0) + d(0,1)] \\
= \; &[(a,0) + (0,b)] \odot [(c,0) + (0,d)] \\
= \; &(a,b) \odot (c,d) = (ac + bd, \; ad + bc) \\
= \; &[(ac+bd)(1,0) + (ad+bc)(0,1)] \longleftrightarrow [(ac+bd)(1,1) + (ad+bc)(1,-1)] \\
= \; &(ac + bd + ad + bc, \; [ac + bc] - [ad + bc]) \\
= \; &(a + b, \; a - b) \odot (c + d, \; c - d) \\
= \; &[(a,a) + (b,-b)] \odot [(c,c) + (d,-d)] \\
= \; &[a(1,1) + b(1,-1)] \odot [c(1,1) + d(1,-1)].
\end{aligned}$$

Again the preservation of scalar multiplication follows from the preservation of multiplication, and the proof of the theorem is therefore completed.

4. <u>The Zero Discriminant</u>. In order for $\beta^2 + 4\alpha$ to be zero, α and β must both be zero, or α must be less than zero. Let $\alpha = -u$, then $\beta = \pm\sqrt{-4\alpha}$ or $\beta = \pm 2\sqrt{u}$. Let $\beta = -2\sqrt{u}$. Using these values for α and β, we find that elements of the form $(d\sqrt{u}, d)$ do not have inverses, and we see that elements of $\mathcal{S}_3$ are not fields.

THEOREM 4. All commutative rings of ordered pairs of real numbers satisfying equations (1), (2), (3), and (4) such that $\beta^2 + 4\alpha = 0$ are isomorphic to each other.

<u>Proof.</u> Let $\alpha = -u$ and $\beta = -2\sqrt{u}$. We describe the mapping of a ring R_1 onto a ring R_2, where R_1 and $R_2 \in \mathcal{S}_3$ by

$$(1,0) \longleftrightarrow (1,0)$$

and

$$(\sqrt{u_1}, 1) \longleftrightarrow (\sqrt{u_2}, 1).$$

Elements $(1,0)$ and $(\sqrt{u_1}, 1)$ are linearly independent and generate all other elements of the ring R_1, and thus we define a one-to-one mapping of $R_1 \longleftrightarrow R_2$ by

$$[a(1,0) + b(\sqrt{u_1}, 1)] \longleftrightarrow [a(1,0) + b(\sqrt{u_2}, 1)],$$

and therefore

$$[c(1,0) + d(\sqrt{u_1}, 1)] \longleftrightarrow [c(1,0) + d(\sqrt{u_2}, 1)].$$

To show that addition is preserved under the mapping, we readily see that the sum of two elements in R_1 maps into the sum of two elements in R_2.

$$\begin{aligned}
&[a(1,0) + b(\sqrt{u_1}, 1)] \oplus [c(1,0) + d(\sqrt{u_1}, 1)] \\
= \; &[(a+c)(1,0) + (b+d)(\sqrt{u_1}, 1)] \longleftrightarrow [(a+c)(1,0) + (b+d)(\sqrt{u_2}, 1)] \\
= \; &[a(1,0) + b(\sqrt{u_2}, 1)] \oplus [c(1,0) + d(\sqrt{u_2}, 1)].
\end{aligned}$$

The preservation of multiplication is shown by:

$$\begin{aligned}
&[a(1,0) + b(\sqrt{u_1}, 1)] \odot [c(1,0) + d(\sqrt{u_1}, 1)] \\
= \; &[(a,0) + (b\sqrt{u_1}, b)] \odot [(c,0) + (d\sqrt{u_1}, d)] \\
= \; &(a + b\sqrt{u_1}, b) \odot (c + d\sqrt{u_1}, d) \\
= \; &(ac + [ad + bc]\sqrt{u_1}, \; ad + bc) \\
= \; &[ac(1,0) + (ad+bc)(\sqrt{u_1}, 1)] \longleftrightarrow [ac(1,0) + (ad+bc)(\sqrt{u_2}, 1)] \\
= \; &(ac + (ad + bc)\sqrt{u_2}, \; ad + bc) \\
= \; &(a + b\sqrt{u_2}, \; b) \odot (c + d\sqrt{u_2}, \; d) \\
= \; &[(a,0) + (b\sqrt{u_2}, b)] \odot [(c,0) + (d\sqrt{u_2}, d)] \\
= \; &[a(1,0) + b(\sqrt{u_2}, 1)] \odot [c(1,0) + d\sqrt{u_2}, 1)].
\end{aligned}$$

It is easily shown that addition and multiplication are also preserved when $\beta = 2\sqrt{u}$, and therefore the proof of the theorem is completed.

Since all definitions of multiplication such that $\beta^2 + 4\alpha = 0$ are isomorphic, let us consider now the specific ring $R_3 = \{(a,b)\}$ where $\alpha = 0$ and $\beta = 0$. Here

$$(a,b) \odot (c,d) = (ac, \; ad + bc).$$

If we set this product equal to $(0,0)$, we see that either a must be zero or c must be zero, and if a is zero, then c must also be zero. Conversely, if $a = c = 0$, then $(a,b) \odot (c,d) = (0,0)$. Thus the set $I = \{(0,b)\}$ is the set of all divisors of zero.

I is an ideal in R_3 since with $(0,a)$ and $(0,b)$ elements of I and (c,d) any element of R_3,

(i) $(0,a) - (0,b) = (0, a-b) \in I$

and

(ii) $(c,d) \odot (0,a) = (0, ca) \in I.$

Moreover, since $(0,b) \odot (0,a) = (0,0)$ or $I' = [(0,0)]$, we see that I is a nilpotent ideal.

Now let K be the set of all real numbers of the form $(a,0)$, and we see that under the correspondence $[(a,b)] \longleftrightarrow (a,0) \longleftrightarrow a$, every equivalence class contains a representative of the form $(a,0)$. The ring R_3 modulo the ideal I is therefore isomorphic to the real numbers K; that is

$$K \cong R_3(I).$$

5. Conclusion. The values of α and β in the definition of multiplication of ordered pairs of real numbers, [Eq. (1)], determine three systems;

I. When $\beta^2 + 4\alpha$ is less than zero, fields in S_1 are isomorphic to the complex numbers, although the statement that the modulus of the product of ordered pairs is equal to the product of their moduli is retained only for the complex numbers, where $\alpha = -1$ and $\beta = 0$.

II. When $\beta^2 + 4\alpha$ is greater than zero, rings in system S_2 are isomorphic to the direct product of the reals with the reals.

III. When $\beta^2 + 4\alpha$ equals zero, giving a ring in S_3, the ring contains a nilpotent ideal such that the ring modulo this ideal is isomorphic to the reals.

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Prepared in a National Science Foundation Undergraduate Science Education Program in Mathematics at Colorado State University under the direction of Professor E. R. Deal.

GRAPHICAL REPRESENTATION OF THE CONCEPTS OF GAME THEORY

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1. Introduction. An interesting graphical solution of games may be found in The Theory of Games and Linear Programming by S. Vajda. According to this method, one player's strategies are represented by points in the plane, while his opponent's strategies are indicated by lines. The solution for a particular game depends upon where the strategy points lie in the plane; it is my intention to interpret carefully the geometrical aspects of solutions to various types of simple games.

	B_1	B_2
A_1	a_{11}	a_{12}
A_2	a_{21}	a_{22}

Fig. 1

Figure 1 is the game matrix for a 2x2 game. Two opponents, A and B, each have a choice of two strategies; a_{ij} represents the payoff for A's i^{th} strategy and B's j^{th} strategy. Player B's j^{th} strategy, B_j, will be represented by a point whose coordinates are (a_{1j}, a_{2j}). Once A learns B's choice, he should pick that strategy which maximizes the payoff a_{ij}. If he selects his first pure strategy A_1, he gains the value of the abscissa of B's strategy point; if A_2 is his choice, he gains the value of the ordinate. The graph of line L, $y = x$, determines two half-planes, $y < x$ and $y > x$. If B's strategy point lies in the former region, A should employ his first pure strategy, since $a_{1j} > a_{2j}$. If this point lies in the latter region, A's more sensible move is to use A_2, since $a_{2j} > a_{1j}$. If B's strategy point lies on L, then it makes no difference which strategy A uses; he may even mix them if he wishes, but the payoff to A remains constant. Naturally, B must exercise caution and select that strategy point which gives his opponent the least gain. Geometrically, this will be the point whose larger coordinate is smaller.

2. B's Optimal Strategy. We shall devote our attention to finding the minimizing player's strategy first.

THEOREM I. B's optimal strategy will be mixed if the line segment which connects his two strategy points

(1) has negative slope
(2) intersects L.

The point of intersection of this line segment with L divides the line segment into two parts. The lengths of these segments are the ratio of B's pure strategies used in his optimal strategy. In Figure 2, P_1P_2 has negative slope and intersects L. B's optimal strategy $(B_1 : B_2)$ is given by

$$(\overline{P_0P_2} : \overline{P_1P_0}) .$$

COROLLARY. If either condition is not fulfilled, then one of the end points of B's strategy line gives the solution. This occurs when the game has a saddle point, and B's optimal strategy will then be a pure one.

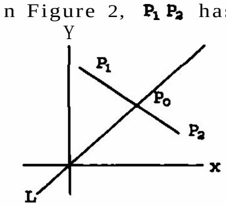


Fig. 2

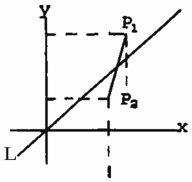


Fig. 3

Figure 3 illustrates a game in which condition (1) is lacking. When the line segment connecting B's strategy points has positive slope, the following method may be used to determine which strategy point he should select: From each of B's strategy points, let there be drawn a lower vertical line and a left horizontal line, called supports. B should choose that strategy point from which a support intersects L in a point whose coordinates are as small as possible. This method may also be used if both conditions of the theorem are not fulfilled.

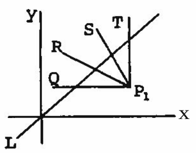
If condition (2) is lacking, then B should choose that point whose larger coordinate is smaller. If both points lie in the lower half-plane determined by $y = x$, then B should select the point whose abscissa is smaller. If both points lie in the upper half-plane, B's best choice is the point whose ordinate is smaller.

3. <u>**A's** Optimal **Strategy**</u>. Now we turn our attention to superimposing the **maximizing** player's **strategies** on the same diagram, so that we may obtain a complete solution for the game.

Player A's strategies are represented by straight lines whose equations are of the form $mx + ny = c$, where m and n are the frequencies with which he mixes A_1 and A_2. These strategy lines may be drawn from either of B's strategy points such that they intersect line L. The coordinates of this point of intersection give A's expected payoff.

The line $mx + ny = c$ must have a nonpositive slope (if it has a slope) since both m and n are non-negative; hence this line will either have negative slope or be parallel to one of the coordinate axes. If A knows B's choice, he should select that strategy line drawn from B's strategy point which intersects L in a point whose coordinates are as large as possible.

Let us find the algebraic expression for the coordinates of this point of intersection. Solving $x = y$ and $mx + ny = c$ simultaneously, we obtain $x = c/(m+n)$. But $m + n = 1$, and $c = mx + ny$, so the point of intersection has coordinates $(mx_j + ny_j, mx_j + ny_j)$, when A draws his strategy line from B's j^{th} strategy point. Since this is the expression for the payoff to A, his aim is to maximize the coordinates of this point, while B is determined to minimize them. B will select a strategy point so that A's best strategy line from this point to L has a point of intersection with L with the smallest coordinates possible. In general, this will be the point first met if a line parallel to A's strategy line sweeps upwards from the lower left of the plane to the upper right.



Whenever A selects the mixture $(m:n)$ for his two pure strategies A_1 and A_2, where $m + n = 1$, then the average payoff to A for B's j^{th} strategy will be $mx_j + ny_j$. In Figure 4, B has chosen P_1. There are an indefinite number of lines of the form $mx + ny = c$ through this point having non-negative slope. Which point of intersection with L has the largest coordinates?

If the line segment connecting B's strategy points satisfies both conditions of Theorem I, A cannot do better than to choose this line as his strategy line. In other words, his strategy line will have negative slope also, and the point of intersection of this line with L will be the value of this game. In this case, A cannot select a strategy line parallel to one of the axes, even though the payoff would be greater, because the payoffs are represented only by points lying on the line segment which joins B's strategy points. The equation of A's best line is $(a_{22}-a_{21})x + (a_{11} - a_{12})y = c$, and his optimal strategy will be mixed.

If one or both conditions of Theorem I do not hold, then A should select the strategy line parallel to one of the coordinates axes which is farther away from B's best point. His optimal strategy will be a pure one. When one strategy yields a greater gain (or lesser loss for the minimizing player) regardless of the other player's choice of strategies, that strategy is called a dominant one. If a player has a dominant strategy, his optimal strategy will be a pure one rather than a mixture.

In any case, A's optimal strategy is given by the coefficients of the strategy line $mx + ny = c$, where $(m:n)$ gives A_1 and A_2's ratio, respectively. It is seen that A's optimal choice is always the answer to B's optimal strategy, and vice versa. A solution always exists, and the value of the game is always given by the coordinates of some point on L.

4. <u>Summary</u>. The following points of interest have been noted:

(1) When the line connecting B's strategy points has negative slope, A has a dominant strategy, as long as this line does not intersect L.

(2) When the line connecting B's strategy points has positive slope, B has a dominant strategy; there are no restrictions on where this line may lie in the plane.

(3) When B's strategy points both lie in the lower half-plane determined by the line whose equation is $y = x$, then A's strategy line is always parallel to the y-axis.

(4) When B's strategy points both lie in the upper half-plane determined by the line whose equation is $y = x$, then A's strategy line is always parallel to the x-axis.

(5) A's strategy line has negative slope if and only if the line joining B's strategy points has negative slope and intersects L.

Doesn't it appear that the slope concept is the basis of this interesting analytic method?

5. <u>Application to **2xm** Games</u>. This method is suitable for **2xm** games, but instead of a straight line connecting B's strategy points, a convex polygon is drawn, having m or less sides. Some strategy points may lie in the interior of the figure, but all the vertices and points on the

boundary represent possible payoffs for **B's** various strategies. **B's** best mixture is determined by the straight line joining two vertices which intersects L in a point whose coordinates are as small as possible. This 2x2 **subgame** is then treated according to the rules previously given for 2x2 graphical solutions.

6. Examples.

	B_1	B_2
A_1	1	2
A_2	3	1

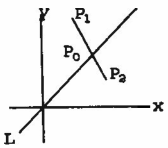


Fig. 5

Figure 5 illustrates the type of solution when **B's** strategy line satisfies Theorem I. A's best strategy line coincides with the line joining B's strategy points, so the coefficients of the variables of the equation of this line give A's optimal strategy. A's optimal strategy is **(2:1)**, **B's** is **(2:1)**, and the value of the game is **5/3**.

This game is an example of a pure strategy for both players. **B's** strategy line lies in the lower half-plane, so his opponent's best strategy line is parallel to the y-axis. **A's** optimal strategy is **(1:0)**, **B's** is **(0:1)**, and the value of the game is 1.

	B_1	B_2
A_1	2	1
A_2	-2	0

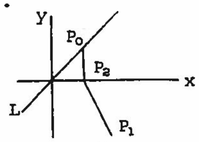
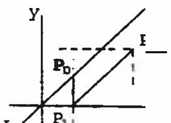


Fig. 6

B's strategy line again lies in the lower half-plane, but has positive slope. Notice the vertical and horizontal supports drawn to determine **B's** optimal strategy. A's optimal strategy is **(1:0)**, **B's** is **(1:0)**, and the value of the game is 1.

	B_1	B_2
A_1	1	3
A_2	0	2

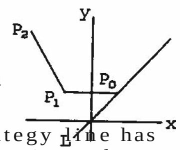


Fig

Figure 8 shows a strategy line for B in the upper half-plane with negative slope. In this case, it is a horizontal line that maximizes player A's gains. A's optimal strategy is **(0:1)**, **B's** is **(1:0)**, and the value of the game is 1.

	B_1	B_2
A_1	-1	-2
A_2	1	3

Fig. 8



	B_1	B_2
A_1	-2	1
A_2	1	3

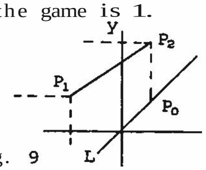


Fig. 9

In Figure 9, since **B's** strategy line has positive slope, supports are again used to determine his most advantageous strategy. A's optimal strategy is **(0:1)**, B's is **(1:0)**, and the value of the game is 1.

Figure 10 illustrates the case where the line connecting B's strategy points has positive positive slope and intersects L. The method already cited for a line with positive slope is used to solve this game. A's optimal strategy is **(1:0)**, **B's** is **(1:0)**, and the value of the game is 0.

	B_1	B_2
A_1	0	1
A_2	-1	2

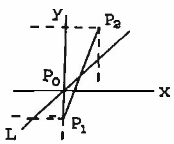


Fig. 10

Should one of **B's** strategy points lie on L, this point will be **B's** best choice. But his opponent may play a pure strategy, or mix his strateqies in any ratio he desires, since an indefinite number of lines may be drawn through this point with nonpositive slope. The value of the game is 2, **B's** optimal strategy is **(1:0)**, and A's might be **(1:0)**, **(0:1)**, or **(1:1)**, for example.

	B_1	B_2
A_1	2	1
A_2	2	3

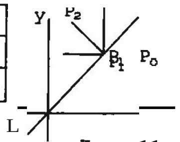


Fig. 11

Mathematics is truly timeless and transcends all ages; by drawing on the past for algebraic and geometrical concepts, the very modern problems of game theory find a unifying and meaningful solution.

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This paper was written while a participant in a National Science Foundation Undergraduate Science Education Program in Mathematics.

THE ETYMOLOGY OF MATHEMATICAL TERMS

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Mathematics has a special language all its own. One of the difficulties that many students find in the successful study of mathematics is understanding the meaning of the many technical terms which are unique to mathematics. Their meaning is better understood after studying their origin. To better understand the origin of our mathematical terms it is important to understand the development of the English language.

When the Romans came to Britain in 55 B.C. bringing the Latin language, they found the people speaking Celtic. Following this, the Jutes, Angles, Saxons, and Frisians came and brought their languages to add to the Latin and Celtic already there. Later, the Scandinavians ruled England and brought their language. This combination is known as Anglo-Saxon or Old English. Today our number words, one, two, three, etc., and words of measurement, such as foot and inch, come from this source. That these words have been retained from the Old English is a good indication that they were used frequently by the common people and therefore were passed on from generation to generation. With the Norman conquest of 1066 French words were introduced. Technical and scholarly words of Latin and Greek origin were added later.

Thus the main sources for the mathematical words of today are Anglo-Saxon (words in use before 1066), Latin through French, directly from Latin, and directly from Greek.

Many of the words used in mathematics have Latin roots, and for people who have studied Latin the mathematical usage of these words "makes sense." The derivations of these are easily found in any good dictionary. A dictionary, however, gives only the bare facts for the sources of these words, while we are more interested in words whose origins are not so obvious and which have interesting stories behind them. Some of these words will be discussed in detail.[1]

ARITHMETIC—from Old French arismetique, Latin arithmetica, Greek arithmetikē technē. The Greek words meant number science and were used to describe what we call theory of numbers. For calculating, the Greeks used the word logistikē. It was not until the sixteenth century that arithmetic was used for both subjects. It is also interesting to note that in the Middle Ages the word took on an extra r, arithmetric, as if it had something to do with the Greek word metron (a measure). This extra r was found in Italy until the time of printing and to some extent

[1]The Oxford English Dictionary, which is an authoritative reference for the etymology of words, was found to be very useful. Much of the information which follows was taken from this source.

in Germany and France.[2] Webster's New World Dictionary uses the r in its etymology from the Old French, but the Webster's Third New International Dictionary does not.

ALGEBRA—from Italian algebra, adapted from the Arabic al-jebr (reunion of broken parts) from jabara (to reunite, bonesetting). This word is one of the few words which has an Arabic origin. In the ninth century al-Khowarizmi wrote a treatise with the title al-jebr w'al muqabalah. The word algebra came from this title. The al-jebr was said to mean the transposing of a negative quantity (reuniting all broken parts), and al-mucabalah means the transposing of a negative quantity and the combining of terms. Later, it was learned that al-jebr is similar to an Assyrian term meaning equal in rank. The Arabs may have used this Assyrian term to apply to an equation.[3] So it may be that the Arabs took their word al-jebr and used it as the transliteration of an Assyrian word meaning equation. It has been reported that during the Middle Ages a sign was seen in a barbershop in England which read "Algebra and Blood-Letting." It is well-known that blood-letting was one of the services performed by barbers, but what does algebra mean? It is more likely that the barber set broken bones than that he was a scholar who helped students with their mathematics.

The ancient game of archery furnished two mathematical words. The word ARC comes from the Latin arcus (bow, arch, curve). The medieval writers used the word to describe any part of the circumference of a circle. No doubt this word was chosen because they were familiar with the bow which formed a part of a circle. The word CHORD comes from the Greek chord? (gut, string of a musical instrument). The gut was used for a bowstring. The similarity of the mathematical chord as the straight line segment joining points on a curved line and the bowstring is readily seen. So we have arc from the bow and chord from the bowstring.

CIRCUMFERENCE—from French circonference, or adapted from Latin circumferentia, from circum (round) and ferre (to bear). Literally this would mean "to bear around," but the word arose as a translation of the Greek word meaning "outer surface" or "periphery" so means the boundary line of a circle.

COROLLARY—adapted from Latin corollarium (money paid for a garland, a gift, gratuity), from corolla (a little crown). A corollary is a theorem which is established by the proof of another theorem. In other words it is a bonus in the investigation, or a gift.

ELLIPSE--adapted from Greek elleipsis, noun of action from elleipein (to come short). PARABOLA—from Latin parabola, from Greek parabole

[2]David Eugene Smith, History of Mathematics (Boston: Ginn and Company, 1925), II, 7-8.

[3]Vera Sanford, A Short History of Mathematics (Boston: Houghton Mifflin Company, 1930), pp. 144-145.

(side by side, juxta-position, application). HYPERBOLA--from Latin hyperbola, adapted from Greek hyperballein (to exceed), from hyper (over) and ballein (to throw). When the Pythagoreans were constructing a figure that was to be equal in area to a figure of a different shape, they called it ellipsis if the base of the new figure was shorter than the old figure. It was called parabole if the two bases were equal and hyperbole if the new base was longer (had some left over). Many years later Apollonius followed the same idea when he called an ellipse that curve whose latus rectum is shorter than the side of the rectangle on the abscissa equal to the square of the ordinate. He called it a parabola if the latus rectum was equal to the side of the rectangle and a hyperbola if the latus rectum was longer than the side of the rectangle.[4]

EXPONENT—from Latin exponentem, present participle of exponere (to put forth, set forth, display, declare or publish) from ex (out) and ponere (to put, place). This is an example of the etymology being somewhat distant from the mathematical meaning. Perhaps it could be said that we are declaring the fact that a certain number is being used more than once as a factor. Or perhaps it refers to the position where the exponent is written.

LINE--from Old English lin, perhaps an adaptation of the Latin linea (linen thread), from linum (flax). This word has been in use in some form for a long time. Because of its constant use, it has had different forms--Middle English ligne and the Old High German lina are examples. A geometric line is an abstraction. Apparently this was best represented to the Ancients by a tightly drawn thread.

MATHEMATICS--adapted from French mathematique or its source, Latin mathematicus, adapted from Greek mathematikos, from mathema (science), from mathein (that which is learned). Originally, mathematics was any subject which required a formal course of instruction to be learned. Subjects like music and art could be learned individually so were not called mathematics. The Pythagoreans were probably the first to limit the word to geometry and arithmetic. Those Pythagoreans who had learned the Pythagorean theory of knowledge were called mathematicians. Others, who merely knew the rules of conduct, were called hearers. In Plato's time the general use of the term was common although there was a tendency to limit it to the subject we know today. By the time of Aristotle the restriction, as we know it today, had been established.[5]

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[4]Alfred Hooper, Makers of Mathematics (New York: Random House, 1948), pp. 40-41; Howard Eves, An Introduction to the History of ~~Mathematics~~ (New York: Rinehart and Company, Inc., 1953), p. 148.

[5]Ivor Thomas, Selections Illustrating the History of Greek Mathematics ~~with an~~ English Translation (Cambridge: Harvard university Press, 1951) I, 3.

POSTULATE--from Latin postulare (to demand), translation of Greek aitemata (to demand). Euclid used aitemata. Postulates were statements which teachers demanded that the student accept as a foundation on which to build.[6]

QUADRATIC--adapted from Latin guadratus, past participle of guadrare (to square). The idea of the square of an unknown was conceived long before a symbol was invented for it. The Greeks called it tetraqonas arithmos (four-angled number).[7] The Latin word for four was quattuor. It would seem that the Latin writers used the idea of a four-angled number in their word quadare (to square); thus a "quadratic equation" means an equation whose unknown is squared.

ROOT--from Old English rot, from Old Norse rot. The original stem wrot is connected with the Latin radix and the Old English wyrt (root, herb, plant). Root is a translation of the Arabic gidr, which in turn, is a translation of Sanskrit mula (root of a vegetable and square root of a number). The Arabs thought of a square number as growing out of a root, which accounts for their choice of gidr in translating from Sanskrit to Arabic. In works translated from Arabic into Latin the Latin radix (root of a plant) was used. The Arabs also used the word in describing the value of the unknown in an equation, thus "root of an equation." The Latin writers thought of a square root of a number as the side of a geometric square. Thus, scholars writing in Latin "found" the side, while scholars writing in Arabic "extracted" or "pulled out" the root. Both usages have been preserved in English since today we "find" the root of an equation and "extract" the root of a number.[8]

SINE--adapted from Latin sinus (a bend, bay, the hanging fold of the upper part of a toga, the bosom of the garment). This is an excellent example of a word whose etymology has no connection with the mathematical usage. In the sixth century Aryabhata, a Hindu, called what we call sine ardha-jya (half chord). It was shortened to jya (chord). The Arabs transliterated this to jiba. Since it was an Arabic custom to omit vowels, this was written jb. The word jiba was a technical word so was not very well-known. When Gherardo of Cremona (about 1150) was translating Arabic works, he came accros jb. The only Arabic word he knew which could be abbreviated jb was jaib meaning "bosom." He must have been puzzled by this, but he dutifully translated it into the Latin word sinus (bosom). From this came our word sine.[9]

TRAPEZOID--adapted from Modern Latin trapezoides, from Greek trapezoeides (table-like), from trapeza (table), from tetra (four) and pous (foot) and eidos (form). The definition of a trapezoid today in America is a quadrilateral with two sides parallel. Euclid did not use

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[6]Smith, op. cit., p. 280.
[7]Ibid., p. 394.
[8]Ibid., p. 150.
[9]Eves, op. cit., p. 196.

this word, but it was introduced by another Greek, Proclus, to mean a quadrilateral with no sides parallel. He used trapezium to mean a quadrilateral with two sides parallel. The definitions of Proclus are used in all continental European languages. However, in the eighteenth century in England the meanings were changed. This practice has continued in America but was changed back to the original meaning in England in the nineteenth century. Thus, a quadrilateral with two sides parallel is called a trapezoid in America, but in England the same figure is called a trapezium.

The recent developments in mathematics have seen the addition of many new terms. It is interesting to note that these are no longer words with Latin roots but on the whole are short Anglo-Saxon words. For example, **ring, field,** and **set** are Anglo-Saxon words which have abstract meanings in mathematics. It would be interesting to know how these words happened to be chosen instead of those of Latin origin. Perhaps the use of the vernacular instead of scholarly words will make the mathematical concepts easier for the average person to grasp.

This is only the beginning of a very fascinating study and one which never ends. When one becomes aware of words themselves as well as their usage he opens a new area to explore.

RAABE'S TEST

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Raabe's test, developed by J. L. Raabe in 1832, is a test for the convergence and divergence of infinite series. Although Raabe's test is easy to use, it is not as effective as Gauss's test, Kummer's test or Maclaurin's integral test. However, these tests are also not easy to use. In this paper it is shown that Raabe's test is more effective than the ratio test, and that the proof of Raabe's test is of medium difficulty. Thus, Raabe's test should be given a position in more of the textbooks on advanced calculus and real variables.

First we prove Raabe's test.

THEOREM. (Raabe's Test) Given an infinite series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$ for every n. Then if

$$(1)\quad \lim_{n\to\infty} n\left[\frac{a_{n+1}}{a_n} - 1\right] \begin{cases} \leqq -\alpha < -1, & \text{the series converges} \\ = a = -1, & \text{the test is indecisive} \\ \geqq a > -1, & \text{the series diverges} \end{cases}$$

Proof. Assume $-\alpha < -1$. Let $\epsilon = (\alpha-1)/2 > 0$. From (1) for every n greater than some large N

$$n\left[\frac{a_{n+1}}{a_n} - 1\right] < \frac{\alpha - 1}{2} - \alpha = \frac{-\alpha - 1}{2}$$

$$na_{n+1} - na_n - a < -\left(\frac{\alpha + 1}{2}\right)a_n - a_n$$

$$\left(\frac{\alpha - 1}{2}\right)a_n < (n - 1)a_n - na_{n+1} \quad \text{for } n > N$$

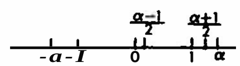


Fig. 1

Then

$$\sum_{n=N+1}^{\infty} a_n < \frac{2}{\alpha - 1} \sum_{n=N+1}^{\infty} [(n-1)a_n - na_{n+1}] = \frac{2}{\alpha - 1} Na_{N+1} = \text{const.}$$

Since the tail, $\sum_{n=N+1}^{\infty} a_n$ converges, the series, $\sum_{n=1}^{\infty} a_n$ converges.

Next assume $\alpha = -1$. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and the series $\sum_{n=2}^{\infty} \frac{1}{n} (\log n)^2$ is convergent, yet Raabe's test yields -1 in both cases. Thus the indecisiveness has been proven.

Next assume $\alpha > -1$. Let $\epsilon = (1 + \alpha)/2$. From (1) there exists some large M such that for every $n > M$,

$$n\left[\frac{a_{n+1}}{a_n} - 1\right] > \alpha - \frac{1+\alpha}{2}$$

$$\frac{a_{n+1}}{a_n} - 1 > \frac{1}{n}\left[\frac{\alpha - 1}{2}\right] > -\frac{1}{n}$$

$$\frac{a_{n+1}}{a_n} > 1 - \frac{1}{n}$$

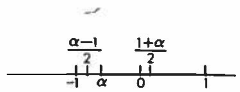


Fig. 2

Thus $na_{n+1} > (n-1)a_n$ and $(n-1)a$ is a monotone increasing sequence for $n > M$. As n, $a > 0$, it follows that there exists some constant $p > 0$ such that for every $n > M$, $(n-1)a_n > p$. But then $a > p/(n-1)$ for all $n > M$ and thus

$$\sum_{n=M+1}^{\infty} a_n > p \cdot \sum_{n=M+1}^{\infty} \frac{1}{n-1} .$$

Divergence is obvious.

Thus, Raabe's test is easy enough to prove to warrant use in textbooks on advanced calculus and real variables. Next we compare the ratio test with Raabe's test.

THEOREM.
(a) If the ratio test shows that the series $\sum_{n=1}^{\infty} a_n$ is convergent (divergent), then Raabe's test shows the series $\sum_{n=1}^{\infty} a_n$ is convergent (divergent).
(b) Both converses of (a) are false.

Thus, Raabe's test is superior to the ratio test. Whenever the ratio test works, Raabe's test does. But there are series for which Raabe's test works and the ratio test does not.

Proof. Consider the set of all infinite series $\sum_{n=1}^{\infty} a_n$ where the ratio test proves that the series converges. Then for any member $\sum_{n=1}^{\infty} a_n$ of this set

$$\lim_{n\to\infty}\left[\frac{a_{n+1}}{a_n} - 1\right] = k - 1 < 0 .$$

As $\lim_{n\to\infty}\left[\frac{a_{n+1}}{a_n} - 1\right] \neq 0$,

$$\lim_{n\to\infty}\left[\frac{a_{n-1}}{a_n} - 1\right]\left[\lim_{n\to\infty} n\right] = (k-1)\cdot \lim_{n\to\infty} n = -\infty$$

i.e.

$$\lim_{n\to\infty} n\left[\frac{a_{n+1}}{a_n} - 1\right] = -\infty .$$

Given any infinite series $\sum_{n=1}^{\infty} a_n$ where the ratio test proves divergence, we can show with a similar proof that

$$\lim_{n\to\infty} n\left[\frac{a_{n+1}}{a_n} - 1\right] = +\infty .$$

And (a) follows.

To complete this proof of superiority, we need only find a divergent and a convergent infinite series for which Raabe's test works, but for which the ratio test does not work. This may be demonstrated with the series $\sum_{n=2}^{\infty} 1/\log n$ and $\sum_{n=1}^{\infty} 1/n^2$.

$$\lim_{n\to\infty} \frac{\log n}{\log (n+1)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Yet

$$\lim_{n\to\infty} n\left[\frac{\log n}{\log (n+1)} - 1\right] = 0 \quad \text{and}$$

$$\lim_{n\to\infty} n\left[\frac{n^2}{(n+1)^2} - 1\right] = -2 .$$

And (b) follows.

Illustrating this point further, there are infinitely many series of the form

$$\sum_{n=1}^{\infty} \frac{a_k n^k + \cdots + a_1 n + a_o}{b_m n^m + \cdots + b_1 n + b_o} \qquad m,\ k > 0$$

where the ratio test fails and Raabe's test works. Other examples are

$$\sum_{n=2} 1/n^2 (\log n)^k \quad \text{for } k = 0,1,2$$

$$\sum_{n=2}^{\infty} 1/n \log n, \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$$

We conclude that Raabe's test is effective, convenient, superior to the ratio test, and that it deserves a position in any undergraduate advanced calculus or real variables course.

1. K. Knopp, Theory and Application of Infinite Series. (2nd ed.), trans. from 2nd Ger. ed. and rev. In accordance with the 4th Ger. ed. by R. C. H. Young (London: Blackie and Son Limited, 1951).

ADDITIVE SUBGROUPS OF THE RATIONAL NUMBERS

Alan Schwartz, University of Wisconsin

In this paper we will associate with each additive subgroup of the rational numbers a function on the prime numbers with range in the non-negative integers together with ∞. By means of these functions we can classify the additive subgroups of the rational numbers into isomorphism types, and also determine whether or not a given subgroup has a minimal set of generators.

DEFINITION. Let G be a subgroup of Q (the additive rationals) and suppose that G contains Z. Then G/Z is well defined and is, in fact, a union of finite cyclic groups. Let S be the set of orders of elements of G/Z. For each prime p, let g(p) be the highest power of p occurring as a divisor of some element of S if such exists; otherwise, let $g(p) = \infty$. We call g the <u>divisibility function</u> of G.

EXAMPLES.

(i) $G = Q$; S consists of all the natural numbers, hence $g(p) = \infty$ for every prime p.

(ii) G = the group of rational numbers that can be written as fractions with denominators being powers of 2; S consists of all the non-negative powers of 2, hence $g(2) = \infty$ and $g(p) = 0$ for every other prime.

(iii) G = the group generated by $\{1, .1, .01, .001, \ldots\}$; S consists of all numbers of the form $2^m 5^n$ for m and n non-negative integers, then $g(2) = g(5) = \infty$ and $g(p) = 0$ for all other primes.

(iv) G = the group generated by $\{2^{-1}, 2^{-2}, 2^{-3}, \ldots; 3^{-1}, 5^{-1}, 7^{-1}, \ldots\}$; S consists of all the non-negative powers of 2 together with all the prime numbers.

Thus, given a group G containing Z, we have its divisibility function. The following lemma gives us a relation between the group and its divisibility function.

LEMMA. Let G be a subgroup of Q containing Z, and let g be the divisibility function of G, then G is generated by $\{p^{-g(p)}$: p prime and $g(p) < \infty\} \cup \{p^{-1}, p^{-2}, p^{-3}, \ldots$: p prime and $g(p) = \infty\}$.

<u>Proof</u>. Let $x \in G$. If $x \in Z$ we are done, so assume that $x \notin Z$. Then $x + Z$ has finite, non-zero order in G/Z. Say it has order s; i.e., $sx + Z = Z$. Hence $sx = m$ for some integer m and $x = m/s$. s can be factored into powers of distinct primes, and s^{-1} can be expressed as a linear combination of the reciprocals of these powers of primes with integral coefficients, and this completes the proof.

<u>Remark</u>. The above lemma also shows us that every function on the primes with the non-negative integers together with ∞ as range is the divisibility function of some subgroup of Q containing Z.

We next examine the relation between the divisibility functions of isomorphic groups. The following lemma will prove useful.

LEMMA. Let G and H be non-trivial subgroups of Q; then G and H are isomorphic if and only if $H = rG$ for some non-zero rational number r.

Proof. If $H = rG$ the isomorphism is obvious. Conversely, we note that G contains at least one integer n. Let $r = \frac{f(n)}{n}$ and the result follows immediately.

DEFINITION. Let g and h be two divisibility functions. We shall say g and h are equivalent $(g \sim h)$ if for each prune p, $g(p)$ and $h(p)$ are both finite or both infinite, and $g(p) = h(p)$ except for finitely many p at which $g(p)$ and $h(p)$ are finite.

THEOREM. Let G and H be subgroups of Q containing Z, and let g and h be their divisibility functions. Then G and H are isomorphic if and only if g and h are equivalent.

Proof. Suppose $g \sim h$. Let $e(p) = h(p) - g(p)$ if $h(p) < \infty$, and $e(p) = 0$ if $h(p) = \infty$. Then $e(p) = 0$ except for finitely many p. Let $r = \Pi p^{-e(p)}$ (where the product is taken over all primes p). r is a rational number since all but finitely many of the factors are 1. A simple calculation shows that $H = rG$ and hence that H and G are isomorphic.

Conversely, suppose H and G are isomorphic. Then there must be a rational number r such that $H = rG$. r can be factored into positive and negative powers of primes and the result will follow by working the first part of this proof backwards.

We can extend the definition of divisibility function to all non-trivial subgroups of Q in the following manner. Let G be a non-trivial subgroup of Q and let n be the smallest positive integer contained in G, then $\frac{1}{n}G$ must contain Z and thus have a well defined divisibility function g. We then take g to be the divisibility function of G as well. The above theorem is now seen to be true for all non-trivial subgroups of the rational numbers.

COROLLARY. There are $2^{\aleph_0}$ isomorphism types of subgroups of Q.

Proof. There are no more than $2^{\aleph_0}$ since that is also the number of sets of rational numbers. That there are at least that many follows from consideration of those divisibility functions whose ranges consist solely of 0 and ∞.

THEOREM. Let G be a non-trivial subgroup of Q, and let g be the divisibility function of G. Then G has a minimal set of generators if and only if one of the following conditions holds:

(i) $g(p) < \infty$ for all primes p

(ii) $0 < g(p) < \infty$ for infinitely many primes p.

Proof. Without loss of generality, we can assume that G contains Z. If g satisfies (i), then G is generated by $\{p^{-g(p)} : p \text{ prime}\}$ and this set is clearly minimal. If g satisfies (ii), let $\{r_j : j = 1,2,3,\ldots\}$ be the countable set $\{p^{-m} : p \text{ prime and } g(p) = \infty;\ m = 0,1,2,\ldots\}$. Then if we let $\{q_j\}$ denote the primes such that $0 < g(q_j) < \infty$, a minimal set of generators for G is $\{r_j q_j^{-g(q_j)} : j = 1,2,3,\ldots\}$.

If G is such that neither (i) nor (ii) hold, then $g(p) = \infty$ for at least one prime and $0 < g(p) < \infty$ for only finitely many primes. Let $\{x_j : j = 1,2,3,\ldots\}$ be a set of generators for G. We write x_j as the reduced fraction $m_j \Pi p^{-e(p,j)}$ where the product is taken over all primes, m_j is an integer and $e(p,j)$ is non-negative for each prime. Let $d = g.c.d. (m_1, m_2, m_3, \ldots)$ (greatest common divisor), and let $d_n = g.c.d\ (m_1, m_2, \ldots, m_n)$. We can pick n_0 so large that $d_{n_0} = d$ and so that for each prime p such that $0 < g(p) < \infty$, $e(p,j) \geq g(p)$ for some $j \leq n_0$. Then it is easy to verify that $\{x_j : j \neq n_0 + 1\}$ generates G, hence $\{x_j\}$ is not a minimal set of generators.

Remark. It is interesting to note that even though some subgroup may not have a minimal set of generators, it can be embedded in a larger subgroup which does have a minimal set of generators: consider examples (ii) and (iv). (ii) has no minimal set of generators whereas (iv) is generated by $\{2^{-j} p_{j+1}^{-1} : p_j \text{ is the } j^{th} \text{ prime}\}$ and this set of generators is seen to be minimal.

I would like to express my appreciation to Professor L. Levy of the University of Wisconsin without whose helpful suggestions this paper would not have taken form.

RESEARCH PROBLEMS

This section is devoted to suggestions of topics and problems fox Undergraduate Research Programs. Address all correspondence to the Editor.

Proposed by M. **S. KLAMKIN.**

Analysis.

Simpson's rule for approximating a definite integral is given by

$$\int_a^b F(x)\,dx \approx \frac{b-a}{6}\left\{F(a) + F\left(\frac{a+b}{2}\right) + F(b)\right\}.$$

For a symmetric interval (i.e., $a = -b$), the rule is exact if and only if $F(x)$ is a quadratic polynomial plus an odd function among the class of differentiable functions.

The general case also holds exactly for cubic polynomials. (This is related to finding the volumes of a general class of solids called prismatoids.) Are there any odd functions other than $F = cx^3$ which make Simpson's rule exact (for all values of a and b)?

Proposed by **J.** D. E. **KONHAUSER.**

Seven men are seated at a circular table. Upon signal, the men rise and mingle.

(A) Given the original seating arrangement, **reseat** the men in such a way that the number of men (counted in either direction) separating each pair of men is different from that in the original seating arrangement.

(B) What is the smallest number of men for which such reseating arrangements can be found?

(C) For what values of n, do solutions of the above n-man problem exist?

(D) If the desired reseating arrangements exist, are they unique?

Proposed by S. SCHUSTER.

Algebra and Number Theory.

The proof that the circle cannot be squared rests on the fact that π is transcendental. However, it would be sufficient to establish the weaker result that π cannot be achieved through any finite sequence of quadratic extensions of the rational field. Can you find a relatively simple proof of the weaker result? Is there a general class of transcendental numbers that can be proven to be non-constructible without invoking transcendentality?

PROBLEM DEPARTMENT

Edited by
M. S. Klamkin, University of Minnesota

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate, signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to Professor M. S. Klamkin, Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.

PROBLEMS FOR SOLUTION

168*. Proposed by Jerry Tower, North High School (student), Columbus, Ohio.
Determine x asymptotically if

$$\log x = n \log \log x.$$

169. Proposed by Joe Konhauser, University of Minnesota.
From an arbitrary point P (not a vertex) of an ellipse lines are drawn through the foci intersecting the ellipse in points Q and R. Prove that the line joining P to the point of intersection of the tangents to the ellipse at Q and R is the normal to the ellipse at P.

Editorial **Note:** The proposer notes that he does not know the source of the problem and he has not been able to locate it in any of the books he has examined.

170. Proposed by C. S. Venkataraman, Sree Kerala Varma College, India.
Prove that a triangle ABC is isosceles or right-angled if

$$a^3 \cos A + b^3 \cos B = abc.$$

171. Proposed by Murray S. Klamkin, University of Minnesota.
For $0 < \theta \le \pi/2$, it is well known that the inequality

$$\frac{\sin\theta}{\theta} > \cos^m \theta$$

holds for $m = 1$. What is the smallest constant m for which it holds?

SOLUTIONS

154. Proposed by Kenneth Kloss, Carnegie Institute of Technology.
For a number in $(0,1)$, does there exist a base so that in this new system of enumeration the first two digits are the same?

Solution by Bob Prielipp, Madison, Wisconsin.
If the number is of the form $1/n$, $n = 2,3,4,\ldots$, then

$$\frac{1}{n} = 0.111\cdots \text{ [base } (n+1)].$$

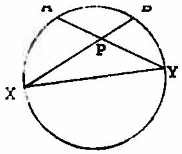
If the number is of the form m/n (rational), then

$$\frac{m}{n} = 0.mmm\cdots \text{ [base } (n+1)].$$

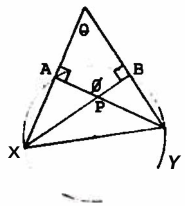
If the number is irrational, it can be approximated by a rational number (correct to a sufficient number of decimal places) and then use the above.

Also solved by H. Kaye, P. Myers, M. Wagner, F. Zetto, and the proposer.

156. Proposed by K. S. Murray, New York City.
If A and B are fixed points on a given circle and XY is a variable diameter, find the locus of point P.



Solution by Sidney Spital, California State Polytechnic College.
Let $\widehat{AB} = a$, then $\angle P'XB = \alpha/2$ and $\theta = (\pi - \alpha)/2$, $\emptyset = (\pi + \alpha)/2$.
Since $\emptyset$ is constant, the locus of P is a circle. The symmetric case (diameter parallel to chord AB) shows that the circles are orthogonal.



Also solved by L. Carlitz, Leroy J. Dickey, Theodore Junqreis, Marvin S. Levin, Charles W. Trigg, M. Wagner, and the proposer.

Editorial Notei Carlitz also refers to the following known theorem (Johnson's, Modern **Geometry**, p. 42, Theorem F): If AB is a diameter of a circle and if any two lines AC and BC meet the circle again at P and Q, respectively, then the circle CPQ is orthogonal to the given circle.

157. Proposed by John Selfridge, Pennsylvania State College.
Prove $n^{a^{a^a}} - n^{a^a}$ is divisible by $2^{a^{a^a}} - 2^{a^a}$.

Solution by Theodore Junqreis, New York University.

$$\begin{aligned} n^{2^{2^2}} - n^{2^2} &= n^4(n^6+1)(n+1)(n^2-n+1)(n-1)(n^2+n+1) \\ &= A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6 . \\ 2^{2^{2^2}} - 2^{2^2} &= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13. \end{aligned}$$

(1) For the factor 2^4, consider the two **cases:** If n is even, n^4 contains 2^4; if n is odd, A_1, A_3, and A_5 are even and since A_3 and A_5 are consecutive, one is divisible by 4, thus supplying four factors of 2.

(2) For the factor 3^2, consider the three cases: If $n = 3m$, n^4 contains 3^2; if $n = 3m + 1$, A_5 and A_6 each are divisible by 3; if $n = 3m + 2$, A_3 and A_4 each are divisible by 3.

(3) For the factor 5, consider the five cases: $n = 5m$, $5m + 1$, $5m + 2$, $5m + 3$, $5m + 4$. For each case one of the A_i is divisible by 5.

(4) and (5) treat the cases for 7 and 13 as in (3).

Also solved by L. Carlitz, H. Kaye, Kenneth M. Maloney, Bob Prielipp, L. Smith, Charles W. Trigg, and the proposer.

158. Proposed by M. S. Klamkin, University of Minnesota.
If $P(x)$ is an n^{th} order polynomial such that $P(x) = 2^x$ for $x = 1,2,3,\ldots,n+1$, find $P(n+2)$.

Solution by the proposer.
Since

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n},$$

it follows that

$$P(x) = 2\left[\binom{x-1}{0} + \binom{x-1}{1} + \cdots + \binom{x-1}{n}\right].$$

Thus,

$$\begin{aligned} P(n+2) &= 2\left[\binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{n}\right] \\ &= 2\left[2^{n+1} - \binom{n+1}{n+1}\right] = 2^{n+2} - 2. \end{aligned}$$

Similarly,

$$\begin{aligned} P(n+3) &= 2\left[2^{n+2} - \binom{n+2}{n+1} - \binom{n+2}{n+2}\right] \\ &= 2^{n+3} - 2n - 6. \end{aligned}$$

This problem is related to the number of parts space (E_n) can be divided into by m n-dimensional spheres, every pair intersecting. Also, the problem can be extended by considering

$$P(x) = \binom{x}{0} + \binom{x}{1}a + \cdots + \binom{x}{n}a^n$$

which reduces to $(1+a)^x$ for $x = 0,1,\ldots,n$.

Also solved by L. Carlitz, Patrick G. Carr, Peter A. Deninno, Theodore Jungreis, H. Kaye, Stephen L. Nemerofsky, L. Smith and M. Wagner.

BOOK REVIEWS

Edited by
Roy B. Deal, Oklahoma State University

Mathematical Models in Physical Sciences. Edited by Stefan Drobot and Paul A. Vierback. Englewood Cliffs, New Jersey; Prentice-Hall; 1962. 193 pp.

This is a book-length compilation of the proceedings of an NSF conference held at the University of Notre Dame in 1962. All contributors are prominent in their field. The theme of the conference is "mathematics is a powerful instrument for developing interdisciplinary research."

The eleven papers considered a wide range of subjects from a stochastic approach to cosmology to some properties of certain non-linear transformations.

All the papers had one quality in common. They were introduced by clear and simple explanations of certain physical problems and the need for a model for their study. In each of the papers an attempt is made to deal with empirical ideas. Mathematical treatment is held to a minimum, although the book contains valuable philosophical and scientific ideas for the most advanced scientist it is primarily worthwhile to the novice researcher. The reason is that one can gain insight into how powerful tools are developed as a result of rather simple ideas.

The trouble with reading the book is that after completing it, the reader has eleven more subjects in which he would like to specialize.

Robert G. McIntyre

Proceedings of the Symposium on Time Series Analysis. Edited by Murray Rosenblatt. New York, John Wiley, 1963. 497 pp., $16.50.

This is a collection of twenty-eight loosely related papers which were presented by prominent workers in the field, who discussed the present state of knowledge and considered current basic problems in theory and application of time series analysis. Although the papers, as a whole, are well-written and thought provoking, they are not intended for the undergraduate or beginning graduate student. A more sophisticated student will find the results and the bibliographies helpful.

A wide range of topics is included, and a hint of the variety is given by the titles of the sub-divisions into which the expositions are placed: Regression Analysis, Zeros of Processes and Related Questions, Meteorological Problems, Structural Problems, Spectral Analysis, Signal Detection, and Estimation and Prediction. This is a fine book for those working in any of these areas.

Leone Y. Low

Partially Ordered Algebraic Systems. By L. Fuchs. Reading, Mass., Addison-Wesley, 1964. ix + 299 pp., $7.00.

This book is a survey of the theory of partially ordered groups, rings, fields and semi groups, and much attention is given to the fully and lattice-ordered structures. The book is reasonably self-contained and contains an extensive bibliography of the articles and books written in this area. The author considers the non-abelian case as well as the abelian situation, and many things are done in great generality. The book is written in a reasonable style, but the author supposes that the reader has a good knowledge of abstract algebra. No exercises are included, but there is a long list of unsolved problems for the research-minded individual. I would recommend this book to any advanced graduate student who is interested in learning something about the algebraic aspects of partially ordered systems. It is an Important addition to the mathematical literature.

University of Illinois David Sachs

Köpfchen, Köpfchen! By B. A. Kordemski. (German Translation from the Russian by Dr. Klemens Junge.) Leipzig, Urania Verlag, 1964. 330 pp., DM 12.

The first 214 pages of this book pose 323 problems, some old, some new, some easy, some challenging. There are many clever and amusing illustrations in the form of sketches. Frequently the sketches are in two colors: black and pink. Appropriately, the first problem is a clever one about two observant young Pioneers. The second part of the book is devoted to solutions of the problems posed in the first part.

Since the range of subjects is very wide, many puzzle enthusiasts will wish to add this volume to their collections. The price (only about $3) is modest for such a well-printed, cloth-bound, attractive book as this one.

University of Illinois Franz E. Hohn

~~Calculus of~~ Variations. By L. E. Elsgolc. Reading, Mass., Addison-Wesley, 1962. 178 pp., $4.50.

This translation from the Russian is one of the most concise and lucid treatments of the calculus of variations to appear in the English language. 'The aim of this book is to provide engineers and students of colleges of technology with the opportunity of becoming familiar with the basic notions and standard methods of the calculus of variations." The book does this and even more by including a chapter on sufficiency conditions for an extremum for the student interested in the more advanced aspects of the calculus of variations.

In chapter one the theory of maxima and minima of functions of ordinary calculus is recalled and the analogous notions for functionals of the calculus of variations is introduced. The fundamental lemma of the calculus of variations is proven and the Euler equation is established, yielding the extremals of the functional $\int_{x_0}^{x_1} F(x,y,y')\,dx$. Functionals depending on functions with two or more dependent variables and their derivatives, functionals involving derivatives of higher order, and functionals depending on functions of several independent variables are considered and the corresponding Euler equation is derived for each. As a direct application, Hamilton's principle (called the principle of Ostrogradski-Hamilton by the author) of mechanics is stated and illustrated.

Chapter two deals with functions with variable or movable boundaries and the transversality conditions are established.

Chapter three is devoted to the notion of a field of extremals, the Jacobi condition, and the Weierstrass function; all are needed for the sufficiency condition for an extremum.

In chapter four, variation problems with side conditions are discussed. Using the Lagrange Multiplier the isoperimetric problems are solved.

And finally in chapter five, direct methods of solving variational problems are introduced. These include the method of finite differences and the Ritz method.

Worked examples are plentiful throughout the book and unsolved problems are included at the end of each chapter.

One nitpick is the use of the name Ostrogradski by the author. In establishing the necessary condition for the extrema of a functional depending on functions with two independent variables, i.e., finding the extrema of double integrals, the author calls the resulting partial differential equation the Ostrogradski equation, "after the famous Russian mathematician M. B. Ostrogradski, who discovered it first in 1834." This equation is commonly called the Euler-Lagrange equation.

According to Todhunter, the facts are that Euler, about a century before Ostrogradski, was the first to treat the variation of a double integral in his treatise on the calculus of variations in Integral Calculus. Euler made an error in this work. Lacroix's work, published first in 1797 contained the same error of Euler. This error was later corrected by Poisson in a memoir presented in November, 1831. Finally, on 24 January, 1834, a memoir was communicated by M. Ostrogradski to the Academy of Science of St. Petersburg. In this paper, Ostrogradski points out the error of Euler and confirms the correct results of M. Poisson. He claims no discovery of his own.

The author also attaches the name of Ostrogradski to Hamilton's principle, named after Sir William Rwan Hamilton (1805-1865) To tell the student of mechanics that Hamilton's principle must n w be called the Ostrogradski-Hamilton principle is like telling Americans that Columbus did not discover America; but our friends to the north have done that tool

Despite the nitpick, the book is well-written and well organized. Anyone interested in learning something of the calculus of variations will find it of value.

Research Analysis Corporation — Richard H. Gramann

Foundations of Differential Geometry, Vol. I. By Shoshichi Kobayashi and Katsumi Nomizu. New York, Interscience, 1963. xi + 329 pp., $15.00.

This is a tract on modern global differential geometry, written very concisely and intended for graduate study. By "global" is meant that the objects of study are who differentiable manifolds and their additional structures, not just the neighborhoods of a point. This does not mean that local properties are neglected but only that they are placed in a larger context.

The material is arranged roughly in order of the restrictiveness of the structure: differentiable manifolds and their tensor analysis, Lie groups, fibre bundles, connections on principal fibre bundles, linear connections, Riemannian connections, Riemannian curvature and space forms, and transformations of these structures. The amount of material included is enormous, well-chosen, and central to the main areas of present research. Although preference is shown for certain notations and formulations, care is taken to explain other notations currently used and their interrelations, so that the student will develop access to most of the other important sources.

University of Illinois — R. L. Bishop

University Mathematics. By R. C. James, Belmont, Calif., Wadsworth, 1963. xiii + 924 pp.

Professor James has written an extraordinary book for beginning students of the calculus. It is an unusual work for a number of reasons, most of them good ones, and will no doubt stir up much controversy and debate wherever it is adopted as a text. The book contains a prodigious amount of material. Chapter 1 is an introduction to the ideas and techniques of the calculus. The remaining chapters cover sets, logic, probability, continuity, limits, analytic and vector geometry, area and integration, transcendental functions, ordinary differential equations, calculus of several variables, vectors and curves, series, linear processes, multiple integrals, and vector theorems. The majority of colleges will find enough material here for a minimum of four semesters of work.

Perhaps the most striking feature of the book is its high degree of mathematical sophistication. The reader is presumed to be intelligent and mature (too much so in the case of many beginning students), and willing to do much of the reasoning himself. The theorems are well presented with, in general, excellent accompanying text to motivate discussion, and with non-trivial illustrations. The problems at the end of each section are exceptional. They are numerous and varied and will test the understanding and imagination of the best as well as the average student. Most unusual in a beginning text, there is a careful treatment of area and volume along with the development of the definite integral.

In spite of the fact that this is a generally superior work, it does contain a number of drawbacks. What has been already mentioned as one of the book's outstanding features may also be its greatest liability as far as general adoption is concerned. The approach may be far too sophisticated for all but the most select of pupils. Considering the present level of mathematics teaching in the high schools, there is no doubt that the majority of entering college students find it a difficult book to read and comprehend. Chapter 1, while laudable in its intent, falls considerably short of success in its execution. The informal treatment of the calculus, though non-rigorous, is far too taxing for the neophyte in its argumentation. In Chapter 5, the author presents an admirably unified exposition of the limit concept using systems of stages but his timing is not good. For pedagogical reasons, this generalization of the limit definition seems best understood after some of the particular limit processes have been presented to the student and he has had a chance to work with them in detail. Finally, the book contains a large number of minor errors, typographical and otherwise.

Here is a text that would seem to be ideally suited for the special section of a calculus course consisting of superior students. There is much to teach in this book and with enough time to spend both professor and student should find it highly rewarding.

Carleton College — Arthur L. Gropen

Elementary General **Topology.** *By* **Theral** *O. Moore. Englewood Cliffs, New Jersey; Prentice-Hall; 1964.*

This is one of the best of the recent crop of textbooks in elementary point set topology. Its quality approaches that of the very fine book of **Kelley's.** *In fact, the author is very clearly influenced in his presentation and choice of examples by* **Kelley's** *book. However, this text is easily accessible to the undergraduate with some experience in abstract* **mathematics--e.g.** *a good algebra or real variables* **course--while** *not being too easy for the more advanced student beginning to learn topology. It is very well and carefully written and organized and should serve as an excellent text for self study as well as for conventional classroom use.*

The subjects covered are the usual ones--elementary set theory, separation axioms, mappings, compactness, **Peano,** *metric, product, function spaces. There is also a very nice elementary treatment of nets. The author merely skims set theory (he does not pretend to do otherwise). There are few proofs for other than very elementary theorems about set theory. However, the main useful theorems are carefully explained and the ideas of set theory receive clear exposition.*

There are many interesting examples, some included among the exercises. The exercises are well-chosen to illustrate the proofs and cover a wide range of difficulty. They also serve to introduce material not covered in the text. The author includes discussions of the results of some of the more difficult exercises. This has its obvious bad points, but will be a help to the self studier and non-specialized instructor.

As a final recommendation, I point out that I found surprisingly few misprints or errors of fact or logic. This is indeed a very satisfactory little book, which no instructor should be afraid to present to his undergraduates or ashamed to present to his graduate students.

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