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TAUBERIAN THEOREMS FOR ABEL SUMMABILITY

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1. INTRODUCTION.

First we consider Abel's summability method. Assume that the power series
\[ \sum_{n=0}^{\infty} a_n x^n \]
converges for \(|x| < 1\) and for each \(|x| < 1\) define \(f\) by
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n. \]
Assume further that \(\lim_{x \to 1^-} f(x)\) exists and let
\[ \lim_{x \to 1^-} f(x) = a. \]
The number \(a\) is the A-sum of the series \(\sum_{n=0}^{\infty} a_n\). We say in this case that \(a\) is A-summable and we write:
\( (A) \sum_{n=0}^{\infty} a_n = a. \)

The following properties are obvious:
(i) If \((A) \sum_{n=0}^{m} a_n = a\) and \((A) \sum_{n=0}^{m} b_n = b\), then
\((A) \sum_{n=0}^{m} (a_n + b_n) = a + b. \)
(ii) If \((A) \sum_{n=0}^{m} a_n = a\), then for any \(c\) we have
\((A) \sum_{n=0}^{m} ca_n = ca. \)
(iii) If \((A) \sum_{n=0}^{m} a_n = a\), \((A) \sum_{n=0}^{m} b_n = b\) and \(a_n \leq b_n\) for all \(n\), then
\(a \leq b. \)

Thus the A-sum of an A-summable series has the most important properties of the sum of a convergent series.

We show next that the A-sum of a series \(\sum_{n=0}^{\infty} a_n\) coincides with the ordinary sum of the series \(\sum_{n=0}^{\infty} a_n\) whenever this series is convergent. On the other hand it is easy to show that there exists series which are not convergent, but which are A-summable. We have for example:
\[ \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \]
and so \((A) \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}\), while \(\sum_{n=0}^{\infty} (-1)^n\)
does not converge. From these results and the following theorem it follows that the concept of the A-sum of a series actually extends the concept of the usual sum in a consistent way.

THEOREM 1 (Abel's Theorem). If \(\sum_{n=0}^{m} a_n = a\), then \((A) \sum_{n=0}^{m} a_n = a. \)

This theorem has various interpretations. One of these interpretations is the following continuity theorem for power series near the circle of convergence:

If \(\sum_{n=0}^{\infty} a_n\) converges, and if the function \(f\) is defined by:
\[ f(x) = \begin{cases} \sum_{n=0}^{\infty} a_n x^n, & |x| < 1 \\ \sum_{n=0}^{m} a_n, & x = 1 \end{cases} \]
then \(f\) is left continuous at 1, i.e.,
\[ \lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n = f(1). \]
Proof. If \( \lim_{n \to \infty} x_n = x \) is convergent, then obviously \( \lim_{n \to \infty} x_n = a \), converses for all \( |x| < 1 \) and we have only to show that \( \lim_{n \to \infty} f(x) = a \).

Let \( s_n = a_n \). For \( |x| < 1 \) we have

\[
\frac{f(x)}{1-x} = \sum_{n=0}^{\infty} a_n x^n \quad \text{and so} \quad f(x) = (1-x) \sum_{n=0}^{\infty} a_n x^n.
\]

Since \( a = (1-x) \sum_{n=0}^{\infty} a_n x^n \). for \( |x| < 1 \), we have

\[
|f(x) - a| \leq (1-x) \sum_{n=0}^{\infty} |a_n| |x|^n + \sum_{n=0}^{\infty} |a_n - a| (1-x) \sum_{n=0}^{\infty} x^n.
\]

Thus for all \( 0 < x < 1 \) we have

\[
|f(x) - a| \leq (1-x) \sum_{n=0}^{\infty} |a_n| + \sum_{n=0}^{\infty} |a_n - a|.
\]

It follows that \( \lim_{x \to 1^{-}} |f(x) - a| \leq \sup_{x \to 1^{-}} |a - a_n| \). Since \( a_n = a \) as \( n \to \infty \), the result follows by choosing \( m \) large enough.

The concept of A-limit and A-convergence of sequences can be introduced similarly. The number \( s \) is the A-limit of the sequence \( \{a_n\} \) if and only if:

\[ (A) \lim_{n \to \infty} a_n = s. \]

We write in this case simply \( (A) \lim_{n \to \infty} a_n = s \). Such a sequence is said to be A-convergent.

THEOREM 2. \( (A) \lim_{n \to \infty} a_n = s \) if and only if

\[
\lim_{x \to 1^{-}} (1-x) \sum_{n=0}^{\infty} a_n x^n = s.
\]

Proof. If \( (A) \lim_{n \to \infty} a_n = s \), then letting \( a_n = 0 \) and \( a_n = a_{n-1} \), we have

\[ (A) \sum_{n=0}^{\infty} a_n = s, \]

i.e.,

\[ (A) \lim_{n \to \infty} a_n = s \]

and so

\[ (A) \lim_{n \to \infty} a_n = s. \]

The analog of THEOREM 1 can be stated as follows:

THEOREM 3. If \( \lim_{n \to \infty} a_n = s \), then \( (A) \lim_{n \to \infty} a_n = s \).
THEOREM 5 (Tauber's Second Theorem). If \( \frac{b_n}{a_n} \to s \) as \( n \to \infty \) and \( \sum_{n=0}^{\infty} a_n = o(n) \) as \( n \to \infty \), then \( \sum_{n=0}^{\infty} b_n = s. \)

Proof. Define \( \nu_n = \sum_{k=1}^{n} b_k \) for \( n = 1, 2, \ldots \), and \( \nu_0 = 0 \). Then

\[
f(x) = \nu_0 + \sum_{n=1}^{\infty} \nu_n \left( \frac{x^n}{n + 1} \right) = \nu_0 + \sum_{n=1}^{\infty} \nu_n \left( \frac{x^n}{n + 1} \right)
\]

since \( \frac{x^n}{n + 1} = o(1) \) as \( n \to \infty \). We have

\[
f(x) = \nu_0 + \sum_{n=1}^{\infty} \frac{\nu_n x^n}{n + 1} = \nu_0 + \sum_{n=1}^{\infty} \frac{\nu_n x^n}{n + 1}.
\]

By hypothesis \( \nu_n = o(n) \) as \( n \to \infty \). Thus \( \nu_n / (n + 1) = o(1) \) as \( n \to \infty \) and by Tauber's First Theorem we obtain

\[
\sum_{n=1}^{\infty} \frac{\nu_n x^n}{n + 1} = o(1) \text{ as } x \to 0. \]

Therefore

\[
\sum_{n=0}^{\infty} b_n = s. \]

In 1910, J. Littlewood [2] replaced the condition \( a_n = o(1/n) \) by the more general \( a_n = o(1/n) \). Littlewood's proof was complex. Other proofs remained complex in spite of the number of researches [3], [4], [5], [6] devoted to it. In 1931, Karamata [7] essentially simplified the proof of Littlewood's theorem by means of the following theorem.

THEOREM 6. If \( \nu_n \sim 0 \) for \( n \to \infty \), then

\[
f(x) = \nu_0 + \sum_{n=1}^{\infty} \nu_n \left( \frac{x^n}{n + 1} \right) = \nu_0 + \sum_{n=1}^{\infty} \nu_n \left( \frac{x^n}{n + 1} \right)
\]

then

\[
\sum_{n=0}^{\infty} b_n = s. \]

Proof. By the Weierstrass approximation theorem if \( g \) is continuous on \([0, 1]\) then for any \( \epsilon > 0 \), there exists a polynomial \( Q \) such that

\[
\max \{ g(x) - Q(x) \} < \frac{\epsilon}{2} x. \]

For all \( x \in [0, 1] \) we have

\[
\sum_{n=0}^{\infty} a_n \sim \frac{1}{1 - x} \quad \text{as } x \to 1^{-}. \]

Thus we have constructed polynomials \( p \) and \( P \) such that

\[
p(x) < g(x) < P(x) \quad \text{for all } x \in [0, 1], \quad \text{and}
\]

\[
\int_{0}^{1} \left[ g(x) - p(x) \right] dx < \epsilon \quad \text{and} \quad \int_{0}^{1} \left[ P(x) - g(x) \right] dx < \epsilon. \]

Next suppose that \( g \) is continuous on \([0, 1]\) except at \( c \in (0, 1) \) where \( g(c-\epsilon) < g(c) < g(c+\epsilon). \) We can still construct polynomials \( p \) and \( P \) satisfying (6) and (7) above. Let \( 6 < \min \{ c, c-\epsilon \} \) and define:

\[
\Phi(x) = \begin{cases} 
\frac{g(x) - \frac{1}{4} \epsilon}{\epsilon}, \quad & x < c - \frac{6}{4} \\
\max \{ f(x), g(x) + \frac{1}{2} \epsilon \}, \quad & c - \frac{6}{4} \leq x \leq c \\
\frac{g(x) + \frac{1}{2} \epsilon}{\epsilon}, \quad & x > c.
\end{cases}
\]

\[
\Psi(x) = \begin{cases} 
\frac{g(x) - \frac{1}{4} \epsilon}{\epsilon}, \quad & x < c \\
\min \{ f(x), g(x) + \frac{1}{2} \epsilon \}, \quad & c \leq x \leq c + 6 \\
\frac{g(x) + \frac{1}{4} \epsilon}{\epsilon}, \quad & x > c + 6.
\end{cases}
\]

where \( \Phi \) and \( \Psi \) are linear functions such that:

\[
\Psi(x) = g(x) - \frac{1}{4} \epsilon, \quad \text{and} \quad \Phi(x) = g(x) + \frac{1}{4} \epsilon.
\]

Clearly \( \phi \) and \( \Phi \) are continuous and \( Q(x) \leq g(x) < Q(x) \). We can find polynomials \( R \) and \( R \) such that

\[
[\Phi(x) - R(x)] < \frac{1}{4} \epsilon \quad \text{and} \quad [\Psi(x) - Q(x)] < \frac{1}{4} \epsilon.
\]

Thus (6) and (7) have been satisfied. Let \( \Omega \) be the larger of the two values

\[
\max \{ f(x), g(x), f(x) - \frac{1}{2} \epsilon \}, \quad \text{and} \quad \max \{ f(x), g(x) + \frac{1}{2} \epsilon \}.
\]

Then as

\[
p(x) - g(x) = \frac{1}{4} \epsilon + \Phi(x) + \Psi(x) - g(x),
\]

we have

\[
\int_{0}^{1} \left[ p(x) - g(x) \right] dx < \frac{1}{4} \epsilon, \quad \text{and} \quad \int_{0}^{1} \left[ f(x) - g(x) \right] dx < \frac{1}{4} \epsilon.
\]

Similarly

\[
p(x) - p(x) = \frac{1}{4} \epsilon + Q(x) - \Psi(x) + \Phi(x) - g(x)
\]

and so

\[
\int_{0}^{1} \left[ p(x) - g(x) \right] dx < \frac{1}{4} \epsilon, \quad \text{and} \quad \int_{0}^{1} \left[ f(x) - g(x) \right] dx < \frac{1}{4} \epsilon.
\]

Thus (6) and (7) have been satisfied.

Next we show that the hypothesis (5) implies that

\[
\lim_{x \to 1^{-}} \sum_{n=0}^{\infty} a_n x^n \Phi(x) = \int_{0}^{1} P(t) dt
\]

for any polynomial \( P \). It is sufficient to consider the case \( p(x) = x' \).
Thus we have to show
$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \frac{1}{k+1} \int_0^1 x^x \, dx.$$ 

Let $\eta(x) = (1-x) \sum_{n=0}^{\infty} a_n x^n - 1$. Then $\eta(x) = 0$ as $x \to 1^-$. We have
$$\begin{align*}
|\eta(x)| & = \left| \sum_{n=0}^{\infty} \frac{1}{(k+1)^n} \sum_{m=0}^{n} a_m x^m - \frac{1}{k+1} \right| \\
& = \left| \frac{1}{1+x} + \cdots + \frac{1}{x^k} \right| \\
& \leq \frac{1}{1+x} + \cdots + \frac{1}{x^k} - \frac{1}{k+1} \\
& < 0 \quad \text{as} \quad x < 1.
\end{align*}$$

Next we show
$$\lim_{x \to 1^-} \left(1-x\right) \sum_{n=0}^{\infty} a_n x^n g(x) = \int_0^x g(t) \, dt,$$
for any $g$ which is continuous everywhere except at $c \in [0,1]$ where $g(c) < g(c^+)$. Let $P$ and $P'$ be the polynomials having properties (6) and (7). Since $a \geq 0$ for $n = 0, 1, \ldots$, and $g(x) \leq P(x)$ for $x \in [0,1]$, we have
$$\lim_{x \to 1^-} \left(1-x\right) \sum_{n=0}^{\infty} a_n x^n g(x) \leq \lim_{x \to 1^-} \left(1-x\right) \sum_{n=0}^{\infty} a_n x^n P(x)$$
$$\leq \int_0^1 P(t) \, dt < \int_0^1 (t) \, dt + \epsilon.$$

Since $\epsilon'$ can be chosen arbitrarily small we have
$$\lim_{x \to 1^-} \left(1-x\right) \sum_{n=0}^{\infty} a_n x^n g(x) \leq \int_0^x g(t) \, dt.$$

By a similar argument we obtain
$$\lim_{x \to 1^-} \left(1-x\right) \sum_{n=0}^{\infty} a_n x^n g(x) \geq \int_0^x g(t) \, dt,$$
and (9) follows.

Finally, define $g$ as follows:
$$g(t) = \begin{cases} 0, & t \in [0, e^{-1}] \\ 1/t, & t \in [e^{-1}, 1]. \end{cases}$$

Then $\int_0^1 g(t) \, dt = \frac{1}{2}$. By (9), given $\epsilon > 0$ we can choose $s$ such that
$$\left| (1-x) \sum_{n=0}^{\infty} a_n x^n g(x) - 0 \right| < \epsilon.$$

Then for $n \geq \frac{\log n}{\log 2}$, we have
$$\left| (1-x) \sum_{n=0}^{\infty} a_n x^n g(x) - 0 \right| < \epsilon,$$
and
$$\lim_{n \to \infty} \frac{1}{n} \sum_{n=0}^{\infty} a_n = 1.$$

Now we can give a simple proof of Littlewood’s theorem.

**Theorem 7 (Littlewood’s Theorem).** If $\{a_n\} \sum_{n=0}^{\infty} a_n = s$ and $a_n = 0(1/n)$ as $n \to \infty$, then
$$\sum_{n=0}^{\infty} a_n = s.$$
If we divide by \((1 - x)^p\) and integrate, we get

\[
\sum_{i=0}^{n} a_i x^{i+1} = \int_0^x f(t) dt \quad .
\]

Since \(f(x) = s\) as \(x \to 1^-\) we have for \(0 < 1 - 6 < 1 < x < 1\)

\[
\int_{1-6}^x f(t) dt \leq \int_{1-6}^x f(t) dt \leq \int_{1-6}^x \left( \frac{e - 1}{1 - t} \right) dt .
\]

Thus

\[
\int_0^x \frac{f(t)}{(1 - t)^x} dt \geq \int_0^x \frac{f(t)}{(1 - t)^x} dt + \left( \frac{e - 1}{1 - t} \right) \left| \frac{x}{1 - x} \right| .
\]

Multiplying by \(1 - x\) and subtracting \(s\) we get, since

\[
\left| (1 - x) \int_0^x \frac{f(t)}{(1 - t)^x} dt - s \right| \leq \epsilon + (1 - x) \left[ \frac{a_i + \epsilon + (1 - 6) f(1 - 6)}{(1 - t)^x} \right] ,
\]

i.e.,

\[
\left| (1 - x) \int_0^x \frac{f(t)}{(1 - t)^x} dt - s \right| \leq \epsilon + (1 - x)C \leq 2\epsilon \text{ for } 0 < x < 1 ,
\]

as \(f\) is bounded for such \(x\). Thus

\[
\lim_{x \to 1^-} (1 - x) \int_0^x \frac{f(t)}{(1 - t)^x} dt = s .
\]

From (10) and (11) it follows that

\[
(1 - x) \sum_{i=0}^{n} a_i x^i = s \text{ as } x \to 1^- .
\]

From this and

\[
(1 - x) \sum_{i=0}^{n} a_i x^i = s \text{ as } x \to 1^- \]

follows

\[
(1 - x) \sum_{i=0}^{n} (a_i - a_1) x^i = 0 \text{ as } x \to 1^- .
\]

Since

\[
s_1 - a_1 = \sum_{i=0}^{n} \left[ a_i - \frac{1}{n+1} s_i \right] = \frac{1}{n+1} \sum_{i=1}^{n+1} k_i a_i
\]

and

\[
-M \leq \frac{1}{n+1} \sum_{i=1}^{n+1} k_i a_i \leq M
\]

we have \((s_1 - a_1 + M) \geq 0\). Then from

\[
(1 - x) \sum_{i=0}^{n} (a_i - a_1) x^i = (1 - x) \sum_{i=0}^{n} (a_i - a_1) x^i + M \geq M
\]

as \(x \to 1^-\), and from THEOREM 6 we conclude

\[
\frac{1}{n} \sum_{i=0}^{n} (a_i - a_1 + M) \geq M \text{ as } n = n ,
\]

i.e.,

\[
\frac{1}{n} \sum_{i=0}^{n} (a_i - a_1) \geq 0 \text{ as } n = n .
\]

Since

\[
s_1 - a_1 = (1 + 1) a_1 - 1 a_1 - a_1 = i(a_1 - a_1 - 1) \]

we have

\[
\sum_{i=0}^{n} i(a_1 - a_1 - 1) = o(n) \text{ where } a_1 = 0 .
\]

Finally, from this condition and

(12) we have by THEOREM 5 that

\[
1 - n = \sum_{i=0}^{n} (a_i - a_1 - 1) = s .
\]

In 1913, Landau [8] weakened the Tauberian condition further, as follows:

THEOREM 9. If \(\lambda_n \sum_{n=0}^{\infty} a_n = s\) and

\[
\omega(n) = \lim_{n \to \infty} \max_{1 \leq k \leq n} \left| \sum_{i=0}^{n+k} a_i \right| = 0 \text{ as } n = 0 ,
\]

then

\[
\sum_{n=0}^{\infty} a_n = s .
\]

Proof. By hypothesis \(\omega(n)\) must exist for some \(n\). If \(n \geq 2\), \(\omega(2)\) exists

\[
\max_{1 \leq k \leq n} \sum_{i=0}^{n+k} a_i \leq \max_{1 \leq k \leq n} \sum_{i=0}^{n+k} \left| a_i \right| .
\]

If \(n < 2\), let \(m\) be the greatest integer such that \((1 + o)^{m+1} \leq 3\). Then for

\[
1 \leq p \leq 2n
\]

we have

\[
\sum_{i=0}^{n} a_i \leq \sum_{i=0}^{n} \left| a_i \right| + \sum_{i=0}^{n} \left| a_i \right| + \sum_{i=0}^{n} \left| a_i \right| + \sum_{i=0}^{n} \left| a_i \right| .
\]

and so

\[
\omega(2) = \lim_{n \to \infty} \max_{1 \leq k \leq n} \sum_{i=0}^{n+k} a_i \leq (m + 1) \left( \omega(2) + \left| a_i \right| \right) .
\]

Next we show

\[
\sum_{i=1}^{n+1} i a_i = o(n) \text{ as } n = n .
\]

Let \(\rho(n) = \max_{1 \leq k \leq n} \sum_{i=0}^{n+k} a_i \), \(n = \left[ \frac{n}{\lambda + 1} \right] \text{ and } \rho_n = \left[ \frac{n}{\lambda} \right] .
\]

Then

\[
\sum_{i=0}^{n+1} i a_i = \sum_{i=0}^{n+1} \left[ \sum_{i=0}^{n+1} i a_i - \sum_{i=0}^{n} i a_i \right] .
\]

We have by partial summation

\[
\sum_{i=1}^{n+1} i a_i = \sum_{i=1}^{n+1} i i(a_i - a_{i-1})
\]

\[
= \sum_{i=1}^{n+1} i i(a_i - a_{i-1}) + \sum_{i=1}^{n+1} i i(a_i - a_{i-1})
\]

\[
= - \sum_{i=1}^{n+1} i i(a_i - a_{i-1}) + \sum_{i=1}^{n+1} i i(a_i - a_{i-1})
\]

i.e.,

\[
\sum_{i=1}^{n+1} i a_i = \sum_{i=1}^{n+1} \left( t_n - n \right) i a_i + \sum_{i=1}^{n+1} \left( n - i \right) a_i .
\]

Since

\[
\sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i
\]

we have

\[
\sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i .
\]

Thus

\[
\sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n+1} a_i .
\]
and from (16) it follows that
\[
\left| n \sum_{n} a_n \right| \leq 2 \left( \frac{\omega(n)}{n} + 2n \rho \left( \frac{n}{\omega(n)} - 1 \right) \right) \leq 2n \rho(n, 2)
\]
since
\[
\frac{\omega(n)}{n} - 1 \leq \frac{\rho(n)}{n} \leq 2.
\]

Using this inequality we get from (15)
\[
\left| \sum_{n} a_n \right| \leq 2 \sum_{n \leq N} \left[ \frac{n}{2} \right] \rho \left( \frac{\left[ \frac{n}{2} \right]}{2} \right) \leq 2n \rho(n, 2) \sum_{n \leq N} \frac{1}{n^2}
\]
i.e.,
\[
\frac{1}{n} \left| \sum_{n} a_{1n} \right| \leq 4n \rho(n, 2).
\]

Consequently
\[
\lim_{n \to \infty} \frac{1}{n} \left| \sum_{n} a_{1n} \right| \leq 4 \lim_{n \to \infty} \rho(n, 2) \leq 4\omega(2) < \infty.
\]

Thus the relation (14) is proved.

By THEOREM 3, from (A) \( \sum_{n} a_n = s \) and \( \sum_{n} a_{1n} = O(n) \) we have
\[
(17)
\]
It remains to be shown that this and (13) imply \( \lim_{n \to \infty} a_n = s \).

From the identity
\[
\frac{(n + 1 + 1)}{1 + 1} \frac{1}{n} \left( \sum_{k=0}^{n} a_k + (1 + 1 - 1) a_n + \sum_{k=1}^{n} s_{n+k} \right)
\]
we get
\[
a_n - a_{n+1} = \frac{1}{n} \left( a_{n+1} - a_{n} - \frac{1}{1 + 1} \sum_{k=1}^{n} (s_{n+k} - s_n) \right).
\]

Letting \( i = \lfloor n/6 \rfloor \) we have
\[
|a_n - a_{n+1}| \leq \frac{n}{n^2} |a_n - a_{n-1}| + \left( \frac{1}{1 + 1} \max_{1 \leq k \leq n} |s_{n+k} - s_n| \right).
\]

Thus,
\[
|a_n - s| \leq |a_n - a_{n+1}| + |a_{n+1} - s| = \frac{1}{6} \left( |a_{n+1} - a_{n-1}| + \max_{1 \leq k \leq n} |s_{n+k} - s_n| + |a_{n+1} - s| \right).
\]

Given \( \epsilon > 0 \). Since \( \omega(n) < 0 \) as \( n \to 6 \). Choose \( \delta < 2 \) such that \( \omega(n) < \epsilon \).

By (17) for any \( \delta > 0 \) and for any fixed \( \delta > 0 \), we can choose an \( \delta \) such that for \( n \geq N \) we have
\[
\frac{1}{6} |a_{n+1} - a_{n-1}| \leq \frac{1}{6} |a_{n+1} - a_{n} - s_n| \leq \frac{\delta}{20} + \frac{\delta}{20} = \epsilon.
\]

Thus, for \( n \geq N \) we have
\[
|a_n - s| \leq \epsilon + \max_{1 \leq k \leq n} |s_{n+k} - s_n| + \frac{\delta}{20} \leq \epsilon + \delta \leq 3\epsilon.
\]

And so
\[
\lim_{n \to \infty} |a_n - s| \leq \epsilon + \omega(n) + \epsilon \leq 3\epsilon.
\]

Thus
\[
\lim_{n \to \infty} a_n = s.
\]

In conclusion we note, that in these few pages we have proven four Tauberian theorems that took a number of mathematicians 34 years to prove. Only with hard work does mathematics progress.

This article is not original, but expository. Its results are known to those versed in series and summability, but the entire sequence of theorems presented can not be found in any one book.

**FOOTNOTES**


**BIBLIOGRAPHY**


Introduction. In the theory developed by Galois an interesting relationship exists between fields and their Galois groups. Every normal subgroup of the Galois group of a finite extension field $K$ corresponds to a unique normal subfield $L$ of the field $K$. It is this correspondence which I shall demonstrate in the following example. All the theory used in this paper can be found in [1], [2], [4], and [5].

**Cyclotomic Field.** The polynomial $f(x) = x^5 - x^2 - x + 1$ is irreducible over the field $Q$ of rational numbers, and is called the cyclotomic polynomial of index 15, since the zeros of $f(x)$ are the primitive fifteenth roots of unity. If a primitive fifteenth root, $e^{(1/15)}$, where $e^{2\pi i k/n}$, is adjoined to $Q$, a normal extension field $K_{15} = Q(e^{(1/15)})$ is formed. The field $K_{15}$ is called a cyclotomic field.

**Subfields of $K_{15}$.** It follows from the fact that a primitive $n$th root of unity generates all the $n$th roots of unity that the field $K_{15}$ contains all the fifteenth roots of unity. The cube roots of unity, $1$, $e^{(3/15)}$, $e^{(9/15)}$, and $e^{(12/15)}$, are among the fifteenth roots.

The cube roots of unity are the zeros of the irreducible polynomial $g(x) = x^3 + x + 1$ and the primitive fifth roots are the zeros of $h(x) = x^8 + x^5 + x + 1$. The extension fields $K_3 = Q(e^{(1/3)})$ and $K_5 = Q(e^{(1/5)})$ include all the zeros of these polynomials and are normal extension fields. Since these zeros belong to the field $K_{15}$, the fields $K_3$ and $K_5$ are normal subfields of $K_{15}$.

**The Automorphism Group.** Since $K_{15}$ is a normal extension of $Q$, we consider the automorphisms of the field $K_{15}$ which leave all elements of the field $Q$ fixed. This set of automorphisms is called the Galois group of the field $K_{15}$ over the field $Q$ and is denoted by $G(K_{15}, Q)$.

An arbitrary automorphism from the Galois group $G(K_{15}, Q)$ carries every zero of the polynomial $f(x)$ into a zero of the same polynomial, that is, this Galois group carries a primitive fifteenth root of unity into another primitive fifteenth root of unity. There are eight of these roots so there are eight automorphisms belonging to the Galois group corresponding to the cyclotomic field $K_{15}$. These automorphisms may be denoted as follows:

<table>
<thead>
<tr>
<th>$I$</th>
<th>$e^{(1/15)}$</th>
<th>$e^{(2/15)}$</th>
<th>$e^{(3/15)}$</th>
<th>$e^{(4/15)}$</th>
<th>$e^{(5/15)}$</th>
<th>$e^{(6/15)}$</th>
<th>$e^{(7/15)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
<tr>
<td>$A_7$</td>
<td>$e^{(1/15)}$</td>
<td>$e^{(2/15)}$</td>
<td>$e^{(3/15)}$</td>
<td>$e^{(4/15)}$</td>
<td>$e^{(5/15)}$</td>
<td>$e^{(6/15)}$</td>
<td>$e^{(7/15)}$</td>
</tr>
</tbody>
</table>

This is a commutative group but not a cyclic group. A basis for a commutative group is composed of all subgroups of $G(K_{15}, Q)$, besides the group itself, are:

- $S_1 = \{ I, A_0, A_1, A_2 \}$
- $S_2 = \{ I, A_0, A_1, A_3 \}$
- $S_3 = \{ I, A_0, A_1, A_4 \}$
- $S_4 = \{ I, A_0, A_1, A_5 \}$
- $E = \{ I \}$

The diagram at the right shows the inclusion relationship existing among these subgroups.

Now we shall make a study of the subfields corresponding to these subgroups. The elements of $K_{15}$ which remain unchanged by the automorphisms of a subgroup $S$ of $G(K_{15}, Q)$ form a normal subfield of $K_{15}$. The degree of this field is equal to the index of $S$ in $G(K_{15}, Q)$.

**A Basis for $K_{15}$.** A basis for the field $K_{15}$ is formed by raising a primitive fifteenth root of unity to the 0, 1, ..., $n-1$ powers where $n$ is the degree of the polynomial.

The basis formed by using $e^{(1/15)}$ is $\{ 1, e^{(1/15)}, e^{(2/15)}, e^{(3/15)}, e^{(4/15)}, e^{(5/15)}, e^{(6/15)}, e^{(7/15)}, e^{(8/15)}, e^{(9/15)}, e^{(10/15)}, e^{(11/15)}, e^{(12/15)}, e^{(13/15)}, e^{(14/15)} \}$. Every element of $K_{15}$ has a unique representation as a linear combination of this basis. A typical element
The Subfield Corresponding to $S_1$. An examination of the above table shows that if the automorphisms of $S_1$ are to leave $\beta$ unchanged the necessary condition is that $a_1 = a_3 = a_6 = a_9 = a = \sqrt[15]{1}$. The cube roots of unity $1, e(5/15)$, and $e(10/15)$ are fifteenth roots of unity. By examining the elements of $S_1$ we see that under each of them $1 - 1$, $e(5/15)$, and $e(10/15)$ leave all the elements of $K_1$ fixed, that is, $S_1$ is the Galois group of $G(K_1, \alpha)$.

The Subfield Corresponding to $T_1$. An examination of our table shows that if the automorphisms of $T_1$ are to leave $\beta$ unchanged the necessary condition is that $a_1 = a_2 = a_5 = 0$ and $a + a e^{(3/15)} + a e^{(6/15)} + a e^{(12/15)}$ leaves elements of the form $a + a e^{(3/15)} + a e^{(6/15)} + a e^{(12/15)}$ fixed.

The Subfield Corresponding to $S_2$. An examination of the table shows that if the automorphisms of $S_2$ are to leave $\beta$ unchanged the necessary condition is that $a = a_4 = a_7 = a_8 = 0$ and $a + e^{(5/15)}$ leaves all the elements of $K_1$ fixed. The automorphisms of $S_2$ are fifteenth roots of unity, by examining the elements of $S_2$ we see that under each of them $1 - 1$, $e(5/15)$, and $e(10/15)$ leave all the elements of $K_1$ fixed, that is, $S_1$ is the Galois group of $G(K_1, \alpha)$.

The Subfield Corresponding to $T_2$. An examination of the table shows that if the automorphisms of $T_2$ are to leave $\beta$ unchanged the necessary condition is that $a = a_2 = a_5 = 0$ and also when $e^{(3/15)} e^{(12/15)}$ leaves $K_1$ fixed.

The Subfield Corresponding to $T_3$. For $\beta$ to remain unchanged by the elements of $T_3$, it is necessary that $a = a_1 = a_3 = a_6 = a_9 = a = \sqrt[5]{1}$. Thus elements of the form $a + a e^{(5/15)} + a e^{(12/15)} + a e^{(15/15)}$ remain fixed under $T_3$. The composite subfield $L_3 = G(K_3, \alpha)$ is of degree four corresponds to $T_3$.

The Subgroup Corresponding to $T_4$. The sum of $e^{(1/15)}$, a primitive fifteenth root of unity, and its inverse $e^{(14/15)}$ equals $2 \cos 2\pi/15$. The sum is unchanged by the elements of $T_4$, that is, $L_4 = Q(2 \cos 2\pi/15)$ corresponds to $T_4$. The degree of $L_4$ is four since the order of $T_4$ is two.

The Subfield Corresponding to $T_5$. The elements of $K_5$ which remain unchanged under the automorphisms of $S_5$ form a field $X$. The author is making a further study of the nature of these elements.

Relationship of Subfields in $K_5$. The inclusion relation among the subfields of $K_5$ has the same structure as that of the subgroups of $G(K_5, \alpha)$. The relationship is, however, in the inverse order to that of the corresponding subgroups.
In this note we generalize two identities due to Euler involving square \( \{1\}, 2771 \) and triangular \( \{1, 2841\} \) numbers, to polygonal numbers of arbitrary order. The development parallels that used by Euler \([1]\).

In 1636, Fermat gave the general form of the \( u \)-th polygonal number of order \( m+2 \), denoted here by \( P(u, \in) \), as

\[
P(u, \in) = \frac{1}{2} m(\in^2 - \in) + u.
\]

We obtain immediately from (2) the following result:

**LEMMMA.** For any integral value of \( m \),

\[
P(m+1, \in) - P(m, \in) = 1 + mn, \quad n = 0, 1, 2, \ldots.
\]

1. **GENERALIZATION OF THEOREM 345 (\( \{1, 2771\} \)).** Let the differences in (2) be used as the powers of \( x \) in forming the infinite product

\[
P(\in) = (1 + x)(1 + x^{2m})(1 + x^{2m+1}) \ldots
\]

We now use Euler's device of introducing a second parameter \( a \). Let

\[
K(a) = K(a, x) = (1 + ax)(1 + ax^{2^{m}})(1 + ax^{2^{m+1}}) \ldots
\]

where \( c_0 = c_0(x) \) is independent of \( a \). Clearly

\[
K(a) = \frac{1}{1 - ax}\frac{1}{1 - ax^2}\ldots,
\]

or

\[
1 + c_0 a + c_2 a^2 + \ldots = (1 + ax)(1 + c_0 ax^m + c_5 a^{2m} + \ldots).
\]

Hence, equating coefficients of \( a \), we obtain

\[
c_1 - x + c_0 x^2, \quad c_3 = c_2 a^{2m+1} + c_0 x^{2m}, \ldots,
\]

\[
c_0 = c_{-1} x^{(n-1)m+1} + c_0 x^{2m}, \ldots,
\]

REFERENCES


so that

\[ c_n = \frac{x^{(n-1)m+1}}{1 - x^m} \]

(3)

or

\[ c_n = \frac{x^{(n-1)m+1}(2m+1+\ldots+(n-1)m+1)}{k(1 - x^m)(1 - x^{2m})\ldots(1 - x^{km})} \]

But

\[ 1 + (m+1) + (2m+1) + \ldots + [(n-1)m+1] \]

\[ = x^m P(s,m) \]

Thus (3) may be written as

\[ (3') \]

\[ c_n = \frac{x P(s,m)}{k(1 - x^m)(1 - x^{2m})\ldots(1 - x^{km})} \]

It follows that

\[ (4) \]

\[ (1 + ax)(1 + ax^2m)(1 + ax^2m + \ldots) \]

\[ = 1 + \frac{2m+2}{1 - x^m} + \frac{3m+3}{1 - x^m(1 - x^{2m})} + \ldots \]

For the special case of \( s = 1 \), (4) becomes

\[ (5) \]

\[ (1 + x)(1 + x^2m)(1 + x^2m + \ldots) \]

\[ = 1 + \frac{x}{1 - x^m} + \frac{2m+2}{1 - x^m} + \frac{3m+3}{1 - x^m(1 - x^{2m})} + \ldots \]

In another form, (5) is

\[ (5') \]

\[ \prod_{j=0}^{\infty} (1 + x^{2jm}) = 1 + \sum_{n=0}^{\infty} \frac{P(s,m)}{(1 - x^m)^n} \]

These infinite series and products are all absolutely convergent for \( |x| < 1 \). For the special case of \( n = 2 \), \( (5') \) becomes Euler's identity for self-conjugate partitions \([\frac{1}{2}; 277, 279]\).

2. GENERALIZATION OF THEOREM 354 \([\frac{1}{2}; 284]\). An altered form of one of Jacobi's identities \([\frac{1}{2}; 283]\) may be written as

\[ (6) \]

\[ \prod_{n=0}^{\infty} \left( (1 + x^{2kn+k}) (1 + x^{2kn+2k}) \right) = \sum_{n=0}^{\infty} x^{kn^2+hn} \]

with \( |x| < 1 \). But (1) can be rearranged as

\[ \frac{1}{2} mu^2 + (1 - \frac{1}{2} m) u \]

Thus, by setting \( k = \frac{1}{2} m \) \( h = 1 - \frac{1}{2} m \), and \( n = u \) in (6), and noting from (2) that \( n \) is non-negative, we now have for the right-hand side of (6):

\[ \sum_{n=0}^{\infty} x P(n,m) \]

this is the infinite series with polygonal-number exponents, which is equivalent to an infinite product. If \( n = 1 \) \( k - h = 1/2 \), we have Euler's identity \([\frac{1}{2}; 284]\) involving the triangular numbers.

REFERENCE


RESEARCH PROBLEMS

This section is devoted to suggestions of topics and problems for Undergraduate Research Programs. Address all correspondence to the Editor.

Proposed by LEO MOSER.

If two numbers are expressible as sum of two squares then their product is so expressible. If \( n = 1, n + 1 \) are each so expressible then so are \( 2n, n^2 + 1 \) since the first is \( (n - 1)(n + 1) \), the second is \( n^2 + 1 \) and the third \( n^2 + 1^2 \). Since \( 8 = 2^2 + 2^2, 9 = 3^2 + 0^2 \) and \( 10 = 3^2 + 1^2 \) it follows that there are infinitely many triples of consecutive numbers each expressible as sum of two squares. On the other hand, since no number leaving remainder of 3 on division by 4 is so expressible, no 4 consecutive numbers are so expressible. Perhaps one could prove, however, that apart from every 4th number, longer blocks are expressible. For examples is it true that there exist infinitely many blocks of 7 consecutive numbers, 6 of which can be represented as sum of two squares?

Proposed by PAUL C. ROSENBLUM.

Differential actuations \( \Rightarrow \) Differential Geometry.

A space curve is determined, to within a rigid motion, by the curvature and torsion as functions of arc length, \( \kappa(s) \) and \( \tau(s) \). The curve can be constructed by solving the Riccati equation

\[ \frac{d\phi}{ds} = -\kappa^2 (1 + 2\kappa \tau - \phi^2), \]

or an equivalent linear differential equation of the second order.

In principle, therefore, theorems on such differential equations can be interpreted in terms of the geometry of space curves. Investigate such interpretations and find the geometric implications of the theorems on differential equations.

References: Struik, Differential Geometry
Eisenhart, Differential Geometry
Coddington and Levinson, Theory of Ordinary Differential Equations

Given: Semi-circle 0 with diameter AB and equilateral triangle PAB; C and D are trisection points of AB (i.e., A'C = CD = DB).

Prove: E and F are trisection points of DB.

Note: A synthetic proof is desired.


If $D^2 f(x)/x = f'(x)/x^{4+1}$, show that $D^2 f(x) = x^{4+1} f(x)$.

174. Proposed by C. S. Venkataraman, Sree Kerzala Vanna College, Trichur, South India.

Find the locus of a point which moves such that the squares of the lengths of the tangents from it to three coplanar circles are in arithmetic progression.

175. Proposed by R. C. Gebhart, Parsippany, New Jersey.

The twenty-one dominoes of a set may be denoted by \((1,1), (1,2), \ldots, (1,6), (2,6), \ldots, (6,6)\).

(a) Is there any arrangement of these, end-to-end with adjacent ends matching, such as \(3,1 | 1,1 | 1,6 | 6,4 \ldots\), such that all twenty-one dominoes may be involved?

(b) What conditions must a general set of dominoes satisfy in order that such an arrangement in (a) exists?

Note: A related problem would be to find the largest and the smallest chain which can be formed with a given set of general dominoes.

176. Proposed by M. S. Klamkin, Ford Scientific Laboratory.

Determine all continuous functions \(F(x)\) in \([0,1]\), if possible, such that \(F(x^0) = F(x)^k\) and

(a) \(F(0) = F(1) = 0\).

(b) \(F(0) = F(1) = 1\).

(c) \(F(0) = 0, F(1) = 1\).

(d) \(F(0) = 1, F(0) = 0\).

SOLUTIONS


Given two overlapping parallel rectangles \(A_1A_2A_3A_4\) and \(B_1B_2B_3B_4\) and a quadratic polynomial \(Q(x,y)\).

Show that \(Q\) cannot be divided by \((x,y)^2\).

SOLUTION

Address all communications concerning problems to Mr. M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Michigan 48121.
"Good," said the reader. "I was hoping you would say that. I now know the answer."

What is the solution to this unique cryptarithm?

Solution by Charles W. Trigg, San Diego, California.

Consider all the following possible patterns with their solutions:

1. \(A^2 = ACDE; \quad 42^2 = 1764, \quad 40^2 = 2304, \quad 43^2 = 8649.\)
2. \(AB^2 = AD; \quad AB = 53, \quad 57, \quad 59, \quad 79, \quad or \quad 54, \quad 72, \quad 84.\)
3. \(AS = ABCDE; \quad AB = 52 \ or \ 87.\)
4. \(AS = ACD; \quad AB = 95 \ or \ 96.\)
5. \(AS = CD; \quad AB = 35, \quad 65, \quad 85, \quad or \quad 46.\)
6. \(S \oplus CD; \quad AB = 45, \quad 81, \quad 91, \quad or \quad 56.\)
7. \(AH = ABDE; \quad AB = 36, \quad 86, \quad or \quad 51, \quad 61, \quad 71.\)
8. \(AH = ADB; \quad AB = 76.\)
9. \(AB = 41 \ or \ 75.\)
10. \(AB = CD; \quad AB = 32, \quad 78, \quad or \quad 82.\)
11. \(AB = CD; \quad AB = 73 \ or \ 89.\)
12. \(AB = ABCDE; \quad AB = 64.\)
13. \(AB = BC; \quad AB = 74.\)
14. \(AB = ADE; \quad AB = 97.\)
15. \(AB = ADB; \quad AB = 39.\)
16. \(AB = AC; \quad AB = 98.\)
17. \(AB = AC; \quad AB = 43 \ or \ 69.\)
18. \(AB = AC; \quad AB = 68.\)
19. \(AB = ABCDE; \quad AB = 83.\)
20. \(AB = ACDE; \quad AB = 97.\)
21. \(AB = ACDE; \quad AB = 37 \ or \ 49.\)
22. \(AB = ACDE; \quad AB = 43 \ or \ 69.\)
23. \(AB = ACDE; \quad AB = 52 \ or \ 87.\)
24. \(AB = ACDE; \quad AB = 33 \ or \ 44.\)
25. \(AB = ACDE; \quad AB = 55 \ or \ 66.\)
26. \(AB = ACDE; \quad AB = 99.\)
27. \(AB = ACDE; \quad AB = 77.\)

The three digit possibilities are given by

\(ABP: \quad CDB(16,31); \quad ACD(13,14); \quad CDE(17,29,18,24); \quad CDB(15,21); \quad CAB(25); \quad BCD(26); \quad BAC(27); \quad CBD(28); \quad CAD(23); \quad CDA(19).\)

The three digit possibilities are given by

\(ABP: \quad ACD(11,22); \quad ACD(13,14); \quad CDE(17,29,18,24); \quad CDB(16,31); \quad ACDE(11,22).\)

From the question asked and the answer given it follows that the particular pattern must lead to only one odd value of \(AB\) and more than one even value. This corresponds to (1) and the value \(93^2 = 8649.\)


162. Proposed by M. S. Klamkin, Ford Scientific Laboratory.

If a surface is one of revolution about two axes, show that it must be spherical.

Solution by Sidney Spital, California State Polytechnic College.

Denote the two axes of revolution by \(A\) and \(B\) and their intersection by \(O\). Consider a plane through \(O\) normal to \(A\). Its intersection with the surface is a circle all of whose points are equidistant from \(O\). Revolve this circle about \(B\). It sweeps out a spherical zone all of whose points are equidistant from \(O\). Now revolve this zone about \(A\), thus increasing the width of the spherical zone. By continued rotations about alternating axes, the entire sphere will be covered.

Solution by the proposer.

1. Also solved similarly as above, but one has to first prove that the two axes intersect. Assuming the surface is bounded, it follows by symmetry that the centroid of the figure must lie on each axis and thus the axes must intersect. Also, the surface could be a spherical annulus.

2. Analytically the functional form of a surface of revolution about the axis

\[ x = \frac{a}{l}, \quad y = \frac{b}{m}, \quad z = \frac{c}{n} \]

is given by

\[ (x - a)^2 + (y - b)^2 + (z - c)^2 = G(lx + my + nz) \]

This is obtained by noting that the circular cross-sections of the surface to the axis can be gotten either by intersections of the surface with spheres

\[ (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \]

centered on the axis, or by planes

\[ lx + my + nz = p \]

which are \(l\) to the axis. Then there has to be some functional relationship between \(p\) and \(z\), say \(c^2 = G(p)\). Since the two axes intersect, we can choose a coordinate system whose origin is the point of intersection and such that the two axes of revolution are symmetric with respect to the \(z\)-axis and to the \(y\)-axis. Then the equation of the surface is given by both

\[ x^2 + y^2 + z^2 = F(nx + lx), \]

\[ x^2 + y^2 + z^2 = G(nz - lx), \]

Choose \(x\) and \(z\) as independent variables (\(y\) will then be the dependent one). For all points \((x,y,z)\) on the surface,

\[ F(nx + lx) = G(nz - lx). \]

Since \(x\) and \(z\) are independent variables, so are \(nx + lx\) and \(nz - lx\). The only way a function of one independent variable can be equal to a function of another independent variable is for both functions to be constant. Whence,

\[ x^2 + y^2 + z^2 = constant, \]

which is a sphere.

Also solved by James Opelka (incompletely), M. Wagner and P. Zetto.

Editorial Note: The geometric solution suggests a new problem. Given two axes of revolution meeting at a given angle. Now starting with a given point of the figure, how many alternate rotations about the two axes successively does it take to generate the entire surface of the sphere? If the two axes are orthogonal, the number will be two if the point is on an axis (not the center) and three for any other point.

163. Proposed by Seymour Schuster, University of Minnesota.

Can any real polynomial be expressed as the difference of two real polynomials each of which having only positive roots?

Solution by the proposer.

Assume, without loss of generality, that the leading coefficient of the given polynomial \(P(x)\) of degree \(n\) is unity. We can then write
"Good," said the reader. "I was hoping you would say that. I now know the answer."

What is the solution to this unique cryptarithm?

Solution by Charles W. Trigg, San Diego, California.

Consider all the following possible patterns with their solutions:

1. \( A^2 = CDEA; \quad A^2 = 1764, \quad 4a^2 = 2304, \quad 9a^2 = 8649. \)
2. \( A^2 = CDEF; \quad AB = 53, \quad 57, \quad 59, \quad 79, \quad or \quad 54, \quad 72, \quad 84. \)
3. \( A^2 = BCDE; \quad AB = 52 \text{ or } 87. \)
4. \( A^2 = ACDB; \quad AB = 95 \text{ or } 96. \)
5. \( A^2 = CDBE; \quad AB = 35, \quad 65, \quad 85, \quad or \quad 46. \)
6. \( A^2 = CDBE; \quad AB = 45, \quad 81, \quad 91, \quad or \quad 56. \)
7. \( A^2 = CDCE; \quad AB = 36, \quad 86, \quad or \quad 51, \quad 61, \quad 71. \)
8. \( A^2 = CDEB; \quad AB = 34, \quad 58, \quad or \quad 47, \quad 67. \)
9. \( A^2 = CADA; \quad AB = 76. \)
10. \( A^2 = BCDA; \quad AB = 41 \text{ or } 75. \)
11. \( A^2 = CDDE; \quad AB = 32, \quad 78, \quad or \quad 82. \)
12. \( A^2 = CBDE; \quad AB = 73 \text{ or } 89. \)
13. \( A^2 = BCDA; \quad AB = 64. \)
14. \( A^2 = CBAD; \quad AB = 74. \)
15. \( A^2 = BACD; \quad AB = 63. \)
16. \( A^2 = CADE; \quad AB = 98. \)
17. \( A^2 = CDAE; \quad AB = 37 \text{ or } 49. \)
18. \( A^2 = CDDE; \quad AB = 43 \text{ or } 69. \)
19. \( AB = CADC; \quad AB = 68. \)
20. \( AB = CADE; \quad AB = 83. \)
21. \( AB = ACDA; \quad AB = 97. \)
22. \( AB = CACD; \quad AB = 39. \)
23. \( AB = CADA; \quad AB = 92. \)
24. \( AA = CDEF; \quad AB = 33 \text{ or } 44. \)
25. \( AA = CADE; \quad AB = 55 \text{ or } 66. \)
26. \( AA = ADEC; \quad AB = 99. \)
27. \( AA = CDED; \quad AB = 77. \)

The three digit possibilities are given by

\( A^2 = CDB(16,31); \quad ACD(13,14); \quad CDE(17,29,18,24); \quad CCB(15,21); \quad CAB(25); \quad BCB(26); \quad BAC(27); \quad CBO(28); \quad CAD(23); \quad CDA(19). \)

\( A^2 = ACO(11); \quad CDO(22). \)

From the question asked and the answer given it follows that the particular pattern must lead to only one odd value of \( AB \) and more than one even value. This corresponds to \((1)\) and the value \( 9a^2 = 8649. \)

Also solved by H. Kaye, Paul Meyers, K. S. Murray, M. Wagner, F. Zetto and the proposer.

162. Proposed by M. S. Klamkin, Ford Scientific Laboratory.

If a surface is one of revolution about two axes, show that it must be spherical.

---

The two axes of revolution by \( A \) and \( B \) and their intersection by \( 0. \) Consider a plane through \( 0 \) normal to \( A. \) Its intersection with the surface is a circle all of whose points are equidistant from \( 0. \) Revolve this circle about \( B. \) It sweeps out a spherical zone all of whose points are equidistant from \( 0. \) NW revolve this zone about \( A \) thus increasing the width of the spherical zone. By continued rotations about alternating axes, the entire surface will be covered.

Solution by the proposer.

1. Also solved similarly as above, but one has to first prove that the two axes intersect. Assuming the surface is bounded, it follows by symmetry that the centroid of the figure must lie on each axis and thus the axes must intersect. Also, the surface could be a sphere.

2. Analytically the functional form of a surface of revolution about the axis

\[
\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}
\]

is given by

\[
(x-a)^2 + (y-b)^2 + (z-c)^2 = G(tx + ny + nz).
\]

This is obtained by noting that the circular cross-sections of the surface to the axis can be gotten either by intersections of the surface with spheres

\[
(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2
\]

centered on the axis, or by planes

\[
lx + my + nz = p
\]

which are \( z \) to the axis. Then there has to be some functional relationship between \( p \) and \( r, \) say \( r = G(p). \) Since the two axes intersect, we can choose a coordinate system whose origin is the point of intersection and such that the two axes (of revolution) are symmetric with respect to the \( z \)-axis and to the \( y \)-axis. Then the equation of the surface is given by both

\[
x^2 + y^2 + z^2 = G(nz + lx),
\]

\[
x^2 + y^2 + z^2 = G(nz - lx).
\]

Choose \( x \) and \( z \) as independent variables \( y \) will then be the dependent one. For all points \((x, y, z)\) on the surface,

\[
F(nz + lx) = G(nz - lx).
\]

Since \( x \) and \( z \) are independent variables, so are \( nz + lx \) and \( nz - lx. \) The only way a function of one independent variable can be equal to a function of another independent variable is for both functions \( G \) to be constant. Whence,

\[
x^2 + y^2 + z^2 \text{ a constant,}
\]

which is a sphere.

Also solved by James Opelka (incompletely), M. Wagner and F. Zetto.

Editorial Note: The geometric solution suggests a new problem. Given two axes of revolution meeting at a given angle. NW starting with a given point of the figure, how many alternate rotations about the two axes successively does it take to generate the entire surface of the sphere? If the two axes are orthogonal, the number will be two if the point is on an axis (not the center) and three for any other point.

163. Proposed by Seymour Schuster, University of Minnesota.

Can any real polynomial be expressed as the difference of two real polynomials each of which having only positive roots?

Solution by the proposer.

Assume, without loss of generality, that the leading coefficient of the given polynomial \( P(x) \) of degree \( n \) is unity. We can then write

\[
P(x) = x^n - Q(x)
\]

where \( Q(x) \) is of degree \( n-1 \) and has only positive roots. If, for example, the \( n \) roots of \( Q(x) \) are the same, say \( a \), then we obtain a polynomial of the desired form. Otherwise, if the roots of \( Q(x) \) are \( b_1, b_2, \ldots, b_{n-1} \), then

\[
P(x) = x^n - (b_1 b_2 \cdots b_{n-1}) x + (b_1 b_2 \cdots b_{n-1})
\]

is a polynomial of the desired form. The only remaining case is when all the roots of \( Q(x) \) are different. In this case, \( Q(x) \) can be factored into \( (x - a_1)(x - a_2) \cdots (x - a_{n-1}) \) where \( a_1, a_2, \ldots, a_{n-1} \) are the roots of \( Q(x) \). Then

\[
P(x) = x^n - (a_1 a_2 \cdots a_{n-1}) x + (a_1 a_2 \cdots a_{n-1})
\]

is a polynomial of the desired form. Thus, any real polynomial can be expressed as the difference of two real polynomials each of which having only positive roots.
\[
\frac{P(x)}{\prod_{i=1}^{n} (x - k_i)} = \{-\lambda + \sum_{i=1}^{n} \frac{b_i}{x - k_i}\} = \{-\lambda - 1 + \sum_{i=1}^{n} \frac{b_i}{x - \ell_i}\}
\]

where \(a_i, b_i > 0\) and \(n_1 + n_2 = n\) (the \(k_i\) and \(\ell_i\) are the positive integers \(1, 2, \ldots, n\)). Let

\[
f_1(x) = \lambda + \sum_{i=1}^{n_1} \frac{a_i}{x - k_i}, \quad f_2(x) = \lambda - 1 + \sum_{i=1}^{n_1} \frac{b_i}{x - \ell_i}.
\]

\(f_1(x)\) has \(n_1\) positive poles and \(n_2\) zeros. Now consider the graph of \(f_1(x)\). By continuity, there must be a zero in between each pair of consecutive polynomials which accounts for \(n_1\) of the zeros (which are positive). The \(n_2\)th zero is in the interval \((-\infty, \min \{k_i\})\). Since \(f_1(x)\) is negative just to the left of \(\min \{k_i\}\), this zero will be positive (by continuity) if \(f_1(0) > 0\). This can be insured by taking \(A\) sufficiently large. Similarly, \(f_2(x)\) has \(n_2\) positive zeros. Then

\[
P(x) = f_1(x) \prod_{i=1}^{n} (x - k_i) - f_2(x) \prod_{i=1}^{n} (x - \ell_i)
\]
gives an affirmative answer to the question.

Also solved by Robert J. Hursey, Jr. and K. S. Murray.

164. Proposed by F. Zetto, Chicago.

Which numbers of the form 300...007 are divisible by 377

Solution by Charles Ziegenbus, Madison College.

Let

\[
N = \sum_{i=0}^{n} c_i 10^i,
\]

where \(0 \leq c_i \leq 999\). Since \(10^3 \equiv 1 \pmod{377}\) for \(n \geq 1\), we see that \(N\) is divisible by 377 if and only if \(\sum_{i=0}^{n} c_i \) is divisible by 377. In the special case of 300...007, if the 3 occupies the \((3k + 2)\)th position, \(k = 0, 1, 2, \ldots, 4\) then 300...007 is divisible by 377.

R. C. Gebhardt, Parsippany, New Jersey, and Robert L. Winkler, University of Chicago, in their solutions note that, equivalently, the number of zeros must be divisible by 3. Gebhardt also gives the following table:

\[
\begin{align*}
37 & = (37)(1) \\
30007 & = (37)(811) \\
30000007 & = (37)(810811) \\
300000000007 & = (37)(810810811)
\end{align*}
\]

Also solved by H. Kaye, P. Myers, D. Smith, M. Wagner and the proposer.

** ***** **

SEX IN THE MODERN MATHEMATICS CURRICULUM

--- a letter to Professor Paul C. Rosenbloom

The following is an unaltered letter received by Professor P. C. Rosenbloom from the Director of Instruction of School District No. 6, Greenfield, Wisconsin.

An elementary introduction to real analysis with precise definitions, rigorous proofs, biographical sketches, and a wide variety of levels of problems, some designed to give research orientation. Based on lectures to students from freshman to graduate level, often non-mathematics majors, and covers the topology, differentiation, and integration of finite-dimensional Cartesian spaces, as well as the Riemann-Stieljes integral, infinite series, manifolds, differentials, line and surface integrals, and Green's and Stokes' Theorems.


With one exception, this volume consists of papers deemed by the authors and the editors of the Handbook of Mathematical Psychology to be especially relevant to approximately half the chapters of the Handbook. These articles partition naturally into six categories: computers, language, social interaction, sensory processes, preference and utility, and Bayesian statistics. Articles on measurement, psychophysics, reaction time, learning, and the stochastic processes that are relevant to the remaining chapters of the Handbook were included in Volume I of the Readings in Mathematical Psychology, which was published about a year earlier.


A modern treatment of the foundations of Euclidean and non-Euclidean Geometry with incidence properties for affine and projective geometry as well.


A modern elementary Cours d'Analyse of functions of one variable with sufficient complex analysis to develop the theory of elementary transcendental functions, treating rigorously, with historical perspective and many examples and problems, the topics normally covered in an integrated course in calculus and analytical geometry.


The young reader should perhaps know that these old classics contain a rich source of complicated classical results in classical algebra and elementary complex function theory which are often useful.


A concise presentation of basic facts about multidimensional Gaussian distributions (or multivariate normal) for those with basic knowledge in linear algebra, probability theory, and advanced calculus, including some applications to Gaussian Noise.


An introduction of the same fine caliber as the other Carus Monographs. Presupposes elementary modern algebra, particularly some matrix theory. Many counting arguments of an elementary, but difficult nature are used, which seems characteristic of the subject.

First Course in Mathematical Logic. By Patrick Suppes and Shirley Hill.

An outgrowth of the famous experiments in teaching logic to selected elementary school students, which develops for utilization in the study of mathematics the sentential inference, inference with universal quantifiers, and applications, of the theory of inference developed, to the elementary theory of commutative groups. Existential quantifiers are not discussed in this volume.


Professor Polya continues his illuminating heuristic discussions on the ways and means of discovery, and a 43 page chapter on "Learning, Teaching, and Learning Teaching."


A beginning graduate text which is done so thoroughly, however, that a good undergraduate student with some elementary general topology, modern algebra, and a modern advanced calculus course can gain an excellent introduction to modern analysis from it.
By confining himself to less general situations, the author is able to obtain the fundamental theorems of both Schwartz and Mikusinski for readers with an elementary knowledge of functional analysis.


A comprehensive treatment of ordinary differential equations for those in mathematics, physics, and engineering with a knowledge of matrix theory and modern advanced calculus, with an impressive collection of classical and modern theorems and theories on the qualitative stability and asymptotic behavior of solutions.


Based on lecture notes for a course at UCLA, and a summer institute for numerical analysis sponsored by the National Science Foundation. This book covers quite well the fundamental facets of numerical analysis with many modern algorithms, including linear algebra, eigenvalue problems, and machine language.


This little book in the Foundations of Philosophy Series discusses questions of truth, existence, and knowledge attainment in mathematics, focusing on geometry and numbers from both literalistic and non-literalistic use with some interesting comments on axiomatized and formalized systems, the synthetic a priori, the logistic thesis, the paradoxes, constructivity, and Gödel's theorem.


A translation of the last edition (fourth) of Bieberbach's well-known Einfluhrung in die Kofonne Abbildung, Berlin 1949, covers the fundamental facets of conformal mapping including a proof of Riemann's mapping theorem and many examples. For those with a bare introduction to the theory of complex variables, including use of the Cauchy integral theorem.

NOTE: All correspondence concerning reviews and all books for review should be sent to PROFESSOR ROY B. DEAL, DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA, 74075.


Serge Lang: A First Course in Calculus. Reading, Massachusetts; Addison-Wesley, 1964. xii + 258 pp., $6.75.


