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METHOD OF SECANTS FOR THE SOLUTION OF EQUATIONS

J. C. Peck, University of Southwestern Louisiana

One of the standard methods for finding the solution of equations $f(x) = 0$ is the "cobweb" or "spiral staircase" method of iteration. This method works fairly well if conditions are "just right;" however, if these conditions are not properly satisfied, the method is uneconomical to apply or may even fail to converge to a root.

Using the fact that a straight line approximates a curve over a small interval, a variation of the method of iteration can be derived which eliminates many of the conditions which must be satisfied for the standard method. The new method assumes that the curve is nearly linear over a small interval and can be approximated by a secant line over this interval.

Before discussing the method of secants it is best that we give a short summary of the standard method of iteration. Let us assume that we have an equation $f(x) = 0$ with real roots. This equation can be expressed as:

$$g(x) = h(x) \qquad \text{where } f(x) = g(x) - h(x).$$

We next form the two equations:

$$y = g(x) \quad \text{and} \quad y = h(x). \tag{1}$$

The value of x at the intersection of the graphs of these two equations is the value of x that makes $f(x) = 0$ in the original equation. (Fig. 1) Now the problem becomes a problem of finding the solution to the simultaneous equations (1).

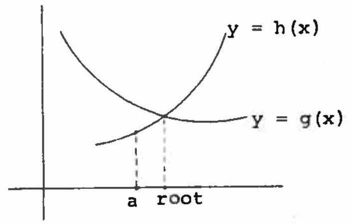


Fig. 1

Let us now assume that we have an approximation a to the root of the equation $f(x) = 0$. We now wish to improve this approximation. Our first step in refining this approximation is to solve one of the equations (1) for x. Solving $y = h(x)$ for x yields $x = H(y)$. We now have the two equations:

$$y = g(x), \tag{2}$$

and

$$x = H(y). \tag{3}$$

We then substitute our approximation a into (2) and determine y_1 such that $y_1 = g(a)$. (Fig. 2) This value of y is then substituted into (3) and a new value of x is determined. This new value of x is again substituted into (2) and the process is continued until two successive values of x are within the desired tolerance.

This method works very well for many cases; however, unless special care is taken in selection of the equations (1), it also fails for many cases. It is quite possible to choose these two equations in such a manner that convergence will occur at a slow rate (Fig. 3) or will not

Fig. 2

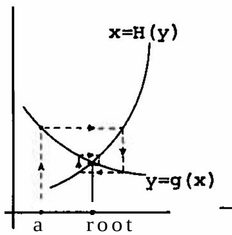


Fig. 3

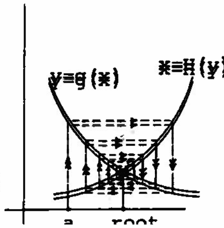


Fig. 4

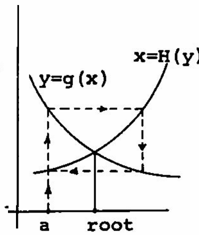
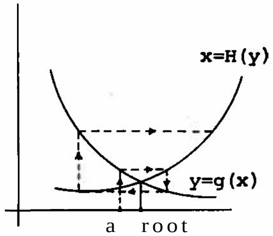


Fig. 5



occur at all. (Fig. 4) Care must also be exercised when deciding which one of the equations (1) is to be solved for x. If the wrong equation is chosen, divergence will occur. (Fig. 5)

The following method is a variation of the method of iteration which forces convergence to occur much more rapidly and also finds roots of many equations that the normal iterative method fails to determine.

The method begins to differ from the normal iterative method explained above at the point where one of the equations (1) is solved for x. With this new method <u>either</u> equation (1) can be solved for x and convergence will occur. Solving $y = h(x)$ for x yields:

$$y = g(x) \tag{4}$$

and

$$x = H(y) . \tag{5}$$

We shall refer to the graph of (4) as curve 1 and the graph of (5) as curve 2.

Substituting our initial approximation a into our original equations $y = g(x)$ and $y = h(x)$, we get two points: $C(a,g(a))$ on curve 1 and $E(a,h(a))$ on curve 2. (Fig. 6) Substituting $g(a)$ into (5) yields a point $F(j,g(a))$ on curve 2, where $j = H(g(a))$. One more substitution of j into (4) yields a point $D(j,g(j))$ on curve 1.

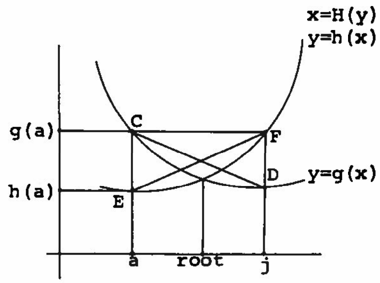


Fig. 6

One can easily see that the slope of $\overline{CD}$ is:

$$M_1 = \frac{g(j) - g(a)}{j - a}$$

The slope of $\overline{EF}$ is:

$$(fa = \frac{h(j) - h(a)}{j - a}$$

Consider a portion of Fig. 6. (Fig. 7) Other expressions for slopes M_1 and M_2 are:

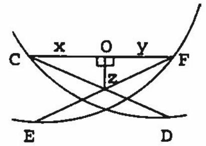


Fig. 7

$$M_1 = \frac{z}{-x} , \qquad M_2 = \frac{z}{y}$$

or

$$z = -M_1 x , \qquad z = (fay$$

$$-M_1 x = (fay .$$

Adding $M_2 x$ to both sides yields:

$$(fax - M_1 x = M_2 y + (fax$$

$$(M_2 - M_1)x = M_2 (y + x)$$

$$\frac{x}{x + y} = \frac{M_2}{M_2 - M_1} .$$

Now $\frac{x}{x + y}$ is the fraction of the distance from C to F where the perpendicular from 0 is drawn. This ratio will tell us how far along the line $\overline{CF}$ we should "slide" to take our next approximation for the root. Since the distance from C to F is the same as the distance from a to j, our next approximation a' will be:

$$a' = a + \frac{M_2}{(fa - M_1} (j - a) . \tag{6}$$

We then repeat the process by using a' to find $g(a')$ which is used to find $H(g(a'))$ which we can again call j. These results can then be substituted back into (6) to yield a new a'. This process can be expressed more compactly as:

$$a_{i+1} = a_i + \frac{M_2}{M_2 - M_1} (H(g(a_i)) - a_i) .$$

The ratio $\frac{M_2}{M_2 - Mi}$ need not be calculated for every iteration; however, convergence will occur in fewer steps if the ratio is recalculated for each iteration. When the difference $(H(g(a_i)) - a_i)$ becomes sufficiently close to zero, we know' that we have found the root.

We have observed only the case where the slopes are of opposite sign in the foregoing discussion. When the slopes are both positive, we have a graph somewhat like Fig. 8. In this case the ratio $\frac{M_2}{Ha - M_1}$ is a fraction with value greater than one so that when the difference $(j - a)$ is multiplied by the ratio and added to a, our new approximation has a value greater than j. A similar argument holds for the case where both slopes are negative.

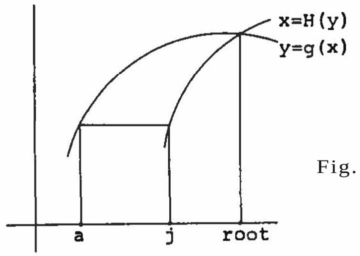


Fig. 8

Several comparison tests were run between the standard method of iteration and the method of secants variation of the standard method. Using the FORTRAN language and an IBM 1620 computer for the calculations, roots accurate to 8 decimal digits were sought for the equations $x^2 - x - 30 = 0$ and $x^2 - e^x = 0$. The methods compared as follows:

	Standard Method	Secant Method
$x^2 - x - 30 = 0$		
$g(x) = 6/x$; $h(x) = (x-1)/5$		
Root	6.00000000	6.00000000
Initial approximation	9.00000000	9.00000000
Number of iterations	83	5
$x^2 - e^x = 0$		
$g(x) = x^2$; $h(x) = e^X$		
Root	diverged	-.70346742
Initial approximation	3.00000000	3.00000000
Number of iterations	stopped at 97	8

In every case tested the method of secants converged more rapidly than did the standard method; however, special cases can be devised where this probably will not hold true. In general though, it appears that the method of secants is a definite improvement over the standard "cobweb" method of iteration.

A GENERALIZATION OF THE STRUCTURE OF THE SENTENTIAL CALCULUS

Dennis Spellman, Temple University

I. Introduction.

All formulae of the sentential calculus may be expressed in terms of conjunction, disjunction, and negation. If conjunction and disjunction are interpreted as operations defined on the truth-values "T" and "F" occurring in their truth-table definitions (or equivalently of substitute symbols, e.g. "1" and "0"), and if negation is considered a function of these elements, then the resulting system is a Boolean algebra. In this paper we shall consider a system having a remnant of Boolean structure, as well as, containing a sub-system, the structure of which is identical with that of the propositional calculus.

Let $U = [0,1]$.

We define the following operations on U:

$$p \oplus q = p + q - pq$$
$$p \odot q = pq.$$

We make the following definition:

If $p \in U$, then $\bar{p} = 1 - p$.

11. Interpretations of $(u, \oplus, \odot)$.

Let $p, q \in U$. Let p be the probability of P and q be the probability of Q, and let P and Q be independent events. $p \oplus q$, $p \odot q$, and $\bar{p}$ can be interpreted as the probability of $P \vee Q$, $P \wedge Q$, and $\sim P$ respectively where "$\vee$" represents disjunction, "$\wedge$" represents conjunction, "$\sim$" represents negation. At the endpoints we have the operation tables below.

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

We note also that $\bar{1} = 0$ and $\bar{0} = 1$. Consider the following tables:

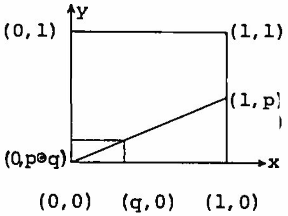
P	$\sim$P
T	F
F	T

P	Q	P $\vee$ Q
T	T	T
F	T	T
T	F	T
F	F	F

P	Q	P $\wedge$ Q
T	T	T
F	T	F
T	F	F
F	F	F

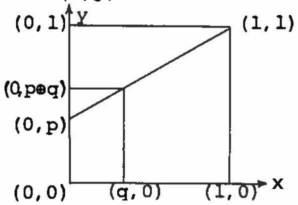
We see that the sub-system defined by the first set of "operation tables" is indeed structurally identical to the sentential calculus.

$p \odot x = px$ --$p, x \in [0,1]$--can be interpreted graphically as a line segment having range $[0,p]$ and passing through the points $(0,0)$ and $(1,p)$.



We see that the line segment can also be characterized as the one originating from the vertex $(0,0)$ of the unit square $[(0,0),(1,0),(1,1),(0,1)]$ and terminating at a height p on the opposite vertical side.

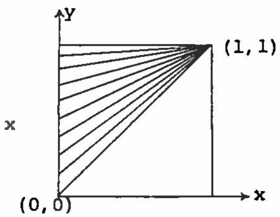
$p \oplus x = (1-p)\cdot x + p$ --$p,\ x \in [0,1]$-- can be interpreted graphically as a line segment having range $[p,1]$ and passing through the points $(1,1)$ and $(0,p)$.



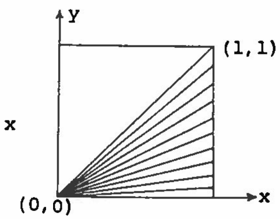
Similarly, the line segment $p \oplus x$ can also be characterized as the one originating from the vertex $(1,1)$ of the unit square and terminating at a height p on the opposite vertical side.

Where p is not held constant but allowed to vary over U, $p \oplus x$ and $p \odot x$ may be interpreted as families of line segments in the plane--as illustrated below.

$p \oplus x$



$p \odot x$



III. Properties of $(U, \oplus, \odot)$.

Let $p,q \in U$, i.e., $0 \leq p \leq 1$ and $0 \leq q \leq 1$.

$\therefore 0 \leq pq \leq q \leq 1$
$\therefore$ U is closed under $\odot$
$1 \geq 1 - p \geq 0$
$\therefore p \in U \Longrightarrow \bar{p} \in U$
$0 \leq q(1-p) \leq 1 - p$
$\therefore 0 \leq p \leq p + q(1-p) \leq (1-p) + p = 1$
$\therefore 0 \leq p + q - pq \leq 1$
$\therefore$ U is closed under $\oplus$.

Theorem 1. $\bar{\bar{p}} = p$.

Proof. $1 - (1 - p) = p$.

Theorem 2. $\forall\ p,q,r,\quad p \odot q = q \odot p,\quad (p \odot q) \odot r = p \odot (q \odot r),$
$0 \odot p = 0,\quad 1 \odot p = p.$

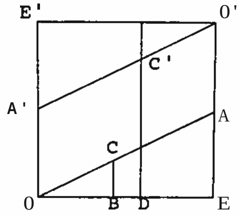
The theorem follows from the properties of multiplication of real numbers since $p \odot q = pq$ by definition.

Theorem 3. $\overline{p \odot q} = \bar{p} \oplus \bar{q}$.

Proof.
$$\overline{p \odot q} = 1 - pq$$
$$\bar{p} \oplus \bar{q} = (1-p) + (1-q) - (1-p)(1-q) = 2 - p - q - (1-p-q+pq)$$
$$= 1 - pq$$
$$\therefore \quad = \bar{p} \oplus \bar{q}.$$

Alternate Proof.

Case I. $0 < p,\ q < 1.$



Let $OEO'E' = [(0,0),(1,0),(1,1),(0,1)]$.
Let $EA = p$, $OB = q$, $OD = \bar{q}$, $OA' = \bar{p}$.
We recognize BC as $p \odot q$ and DC' as $\bar{p} \oplus \bar{q}$. Since $OA' = \bar{p}$ and $OD = \bar{q}$, we have $E'A' = p$ and $DE = q$.

$\angle E = \angle E' = \pi/2$ and $OE = O'E' = 1$
$\therefore \triangle OEA \cong \triangle O'E'A'$ by S.A.S.
$\therefore \angle AOE = \angle A'O'E'$

Since DE and $B'O'$ are opposite sides of rectangle $DEO'B'$, $DE = O'B'$. But $DE = q$, $\therefore O'B' = q$. Now $\angle OBC = \angle O'B'C' = \pi/2$ and $OB = O'B' = q$.
$\therefore \triangle OBC \cong \triangle O'B'C'$ by A.S.A.
$\therefore BC = B'C'$
But $DC' = 1 - B'C'$
$\therefore DC' = 1 - BC$
$\therefore \bar{p} \oplus \bar{q} = 1 - (p \odot q)$

Case II. $p \in U$.

$\overline{0 \odot p} = \bar{0} = 1$
$\bar{0} \oplus \bar{p} = 1 \oplus \bar{p} = 1 + \bar{p} - 1\bar{p} = 1$
$\therefore \overline{0 \odot p} = \bar{0} \oplus \bar{p}$
$\overline{1 \odot p} = \bar{p}$
$\bar{1} \oplus \bar{p} = 0 \oplus \bar{p} = 0 + \bar{p} - 0\bar{p} = \bar{p}$
$\therefore \overline{1 \odot p} = \bar{1} \oplus \bar{p}$

Consider the function $y = f(x) = \bar{x}$. Given any $y \in U$, $f: \bar{y} \to y$; hence, f is onto U. Let $\bar{x}_1 = \bar{x}_2$.
$\therefore 1 - x_1 = 1 - x_2$
$\therefore x_1 = x_2$
$\therefore$ f is one-one.

Moreover $x \to \overline{x}$ is an isomorphism from $(U, \odot)$ onto $(U, \oplus)$ since $f(p \odot q) = f(p) \oplus f(q)$.

Theorem 4. $\forall\, p,q,r,\quad p \oplus q = q \oplus p,\quad (p \oplus q) \oplus r = p \oplus (q \oplus r),\quad 1 \oplus p = 1,$ $0 \oplus p = p.$

Proof. The theorem follows immediately from Theorem 2 since $x \to \overline{x}$ is an isomorphism from $(U, \odot)$ onto $(U, \oplus)$.

Theorem 5.

A. $\overline{p \oplus q} = \overline{p} \odot \overline{q}$

B. $\overline{p_1 \odot p_2 \odot \cdots \odot p_n} = \overline{p_1} \oplus \overline{p_2} \oplus \cdots \oplus \overline{p_n}$

C. $\overline{p_1 \oplus p_2 \oplus \cdots \oplus p_n} = \overline{p_1} \odot \overline{p_2} \odot \cdots \odot \overline{p_n}$

Proof. A. Let $p = \overline{r}$, $q = \overline{s}$.

From Theorem 3, $\overline{r} \oplus \overline{s} = \overline{r \odot s}$.

$\overline{\overline{p}} \oplus \overline{\overline{q}} = \overline{\overline{p} \odot \overline{q}}$

$p \oplus q = \overline{\overline{p} \odot \overline{q}}$

$\overline{p \oplus q} = \overline{p} \odot \overline{q}$

B. since $\odot$ and $\oplus$ are both commutative and associative, generalized commutative and associative laws hold for both. Under the isomorphism $x \to \overline{x}$, the image of the generalized product: $p_1 \odot \cdots \odot p_n$, is the generalized sum of the images: $\overline{p_1} \oplus \cdots \oplus \overline{p_n}$.

C. Part A establishes that $x \to \overline{x}$ is also an isomorphism from $(U, \oplus)$ onto $(U, \odot)$.

. like in part B,

$\overline{p_1 \oplus \cdots \oplus p_n} = \overline{p_1} \odot \cdots \odot \overline{p_n}$.

$\forall\, p$, $1 \oplus p = 1$. $\therefore 1$ has no additive inverse. $1/2 \odot x = 1 \Rightarrow x = 2 \notin U$. . $1/2$ has no multiplicative inverse. In general, inverses do not exist.

We are familiar with the following three properties of the $\leq$ relation: if p, q, r are real numbers then

1. $p \leq p$
2. $p \leq q,\ q \leq r \implies p \leq r$
3. $p \leq q,\ q \leq p \implies p = q.$

The following theorems show that $\leq$ is an inclusion-like relation in $(U, \oplus, \odot)$.

Theorem 6. $\forall\, p,q,\quad 0 \leq p \odot q \leq p \leq p \oplus q \leq 1.$

Proof. Since $0 \leq p \leq 1$, $0 \leq q \leq 1$

$0 \leq pq = p \odot q \leq p$

and $0 \leq q(1 - p)$

$\therefore p \leq p + q(1 - p) = p \oplus q \leq 1$

$\therefore 0 \leq p \odot q \leq p \leq p \oplus q \leq 1.$

Theorem 7. $p_1 \leq p_2 \leq \cdots \leq p_n \iff \overline{p_n} \leq \cdots \leq \overline{p_2} \leq \overline{p_1}$.

Proof. $p_1 \leq p_2 \leq \cdots \leq p_n \iff 1 - p_1 \geq 1 - p_2 \geq \cdots \geq 1 - p_n$

Theorem 8. $r \leq p \odot q \leq s \iff \overline{s} \leq \overline{p} \oplus \overline{q} \leq \overline{r}$ and $r \leq p \oplus q \leq s \iff \overline{s} \leq \overline{p} \odot \overline{q} \leq \overline{r}$.

Proof. Let $r \leq p \odot q \leq s$.

$\therefore 1 - r \geq 1 - (p \odot q) = \overline{p \odot q} \geq 1 - s$

$\therefore \overline{s} \leq \overline{p} \oplus \overline{q} \leq \overline{r}$

Let $\overline{s} \leq \overline{p} \oplus \overline{q} \leq \overline{r}$.

$\therefore 1 - s \leq \overline{p \odot q} \leq 1 - r$

$\therefore s \geq \overline{\overline{p \odot q}} \geq r$

$\therefore r \leq p \odot q \leq s$

Similarly $r \leq p \oplus q \leq s \iff \overline{s} \leq \overline{p} \odot \overline{q} \leq \overline{r}$.

IV. Other Properties.

"$\not\equiv$" is to be read: "not identically equal to."

$p \oplus p = p + p - p^2 = 2p - p^2 \not\equiv p$

$p \odot p = p^2 \not\equiv p$

$p \oplus \overline{p} = p + (1 - p) - p(1 - p) = 1 - p + p^2 \not\equiv 1$

$p \odot \overline{p} = p(1 - p) \not\equiv 0$

$1/2 \odot (1/3 \oplus 1/4) = 1/4$

and $(1/2 \odot 1/3) \oplus (1/2 \odot 1/4) = 13/48$

$\therefore p \odot (q \oplus r) \not\equiv (p \odot q) \oplus (p \odot r)$

$1/2 \oplus (1/3 \odot 1/4) = 13/24$

and $(1/2 \oplus 1/3) \odot (1/2 \oplus 1/4) = 5/12$

$\therefore p \oplus (q \odot r) \not\equiv (p \oplus q) \odot (p \oplus r)$

We know that in a Boolean algebra $a \leq b' \implies a \odot b = 0$.

$1/3 \leq 3/4 = \overline{1/4}$

$1/3 \odot 3/4 \neq 0$

$\therefore p \leq \overline{q} \not\Rightarrow p \odot q = 0$.

V. From Another Viewpoint.

Another truth-functional connective is defined by the table below.

P	Q	P [illegible] Q
T	T	F
F	T	T
T	F	T
F	F	F

It is, perhaps, an unwieldy ingredient in the business of evaluating the truth-values of compound statements, but it is not at all artificial for it represents the exclusive sense of the word "or". Since all formulae of the propositional calculus can be expressed in the terms of conjunction and negation alone, they can certainly be expressed in terms of "exclusive disjunction", conjunction, and negation. Again we may consider "|||" and "∧" as operatives defined on "T" and "F" and consider "~" as a function of these truth-values. Consider the tables below.

$\|\|\|$	F	T
F	F	T
T	T	F

$\wedge$	F	T
F	F	F
T	F	T

$\equiv$	T	F
T	T	F
F	F	T

$\vee$	T	F
T	T	T
F	T	F

I/(2)

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

We see that the truth-values under the operations "|||" and "∧" are isomorphic to the integers modulo 2 and also to themselves under the operations "≡" and "∨" where "≡" is the biconditional. Since I/(2) is a Boolean ring with unit, the others are also.

We may make the following definitions of operations on U:

$$p \textcircled{\scriptsize |||} q = p + q - 2pq$$
$$p \textcircled{\scriptsize $\wedge$} q = p \odot q$$
$$p \textcircled{\scriptsize $\equiv$} q = 1 - p - q + 2pq$$
$$p \textcircled{\scriptsize $\vee$} q = p \oplus q \text{ .}$$

I shall state without proof some of the properties of $(U, \textcircled{\scriptsize |||}, \textcircled{\scriptsize $\wedge$})$ and $(U, \textcircled{\scriptsize $\equiv$}, \textcircled{\scriptsize $\vee$})$. U is closed with respect to all four operations.

<u>Theorem</u>. $\overline{p \textcircled{\scriptsize |||} q} = \bar{p} \textcircled{\scriptsize $\equiv$} \bar{q}$ and

$\overline{p \textcircled{\scriptsize $\equiv$} q} = \bar{p} \textcircled{\scriptsize |||} \bar{q}$.

Now we see that $x \to \bar{x}$ is an isomorphism from $(U, \textcircled{\scriptsize |||}, \textcircled{\scriptsize $\wedge$})$ onto $(U, \textcircled{\scriptsize $\equiv$}, \textcircled{\scriptsize $\vee$})$, and vice-versa. Hence, we need explore the properties of only $(U, \textcircled{\scriptsize |||}, \textcircled{\scriptsize $\wedge$})$.

<u>Theorem</u>. $\forall$ p, q, r, $p \textcircled{\scriptsize |||} q = q \textcircled{\scriptsize |||} p$, $(p \textcircled{\scriptsize |||} q) \textcircled{\scriptsize |||} r = p \textcircled{\scriptsize |||} (q \textcircled{\scriptsize |||} r)$, $0 \textcircled{\scriptsize |||} p = p$, $1/2 \textcircled{\scriptsize |||} p = 1/2$, $1 \textcircled{\scriptsize |||} p = \bar{p}$.

<u>Theorem</u>. A. In general inverses do not exist.
B. $\textcircled{\scriptsize $\wedge$}$ does not distribute over $\textcircled{\scriptsize |||}$.
C. $p \textcircled{\scriptsize |||} p \not\equiv 0$.

In the same way that interpretations of $p \textcircled{\scriptsize $\vee$} q$ and $p \textcircled{\scriptsize $\wedge$} q$ are given, interpretations can be given to $p \textcircled{\scriptsize |||} q$ and $p \textcircled{\scriptsize $\equiv$} q$. At the endpoints of $(U, \textcircled{\scriptsize |||}, \textcircled{\scriptsize $\wedge$})$ we have the operation tables below.

$\textcircled{\scriptsize \|\|\|}$	0	1
0	0	1
1	1	0

$\textcircled{\scriptsize $\wedge$}$	0	1
0	0	0
1	0	1

This sub-system is isomorphic to I/(2) and the other two rings.

I should like to express my thanks to Dr. Leon Steinberg for his help and encouragement.

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A MYTHICAL HISTORY OF THE JARGON OF MATHEMATICS

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Thousands of years ago mathematics was simply an ordinary occupation. When a person said he was a mathematician, no one shuddered, or stuck his fingers in his ears, or gave him a look of thorough distaste. Rather, this announcement was met with giggles, snickers, and even chortles. Why should the people not have laughed? All that mathematicians did was to sit around drawing circles and triangles, muttering "alpha squared plus beta squared equals gamma squared." It hardly seemed an appropriate pastime for the grossly retarded. Naturally, mathematicians felt they were fully competent, and the constant disparagement began to undermine their morale. Soon they were breaking out in spots and showing neurotic tendencies like tics, paranoia, persecution complexes, and disassociative reactions. Several were even more severely struck and became schizophrenic. Of course, their work was hampered by their lack of mental stability; and mathematics went into a catastrophic decline, causing it to fall into even greater disrepute than before.

This was almost the end of the field of mathematics.

However, one of the saner mathematicians, in a lucid moment, realized that mathematics was doomed unless this group of pathetic psychotics received immediate help. So he took the group to the nearest accredited psychotherapist to see if therapy could not help them.

Fortunately, the therapist was able to help these wretches. He immediately put them in group therapy, and it was here that mathematics was actually saved. After a few sessions, they had gained insight enough into their problem to consider a solution. Their problem (ignoring the extraneous factors of incorrect toilet training, Oedipus complexes, and such) was that they were suffering from deep insecurity and thwarted snobbery. Their profession certainly gave them no prestige; they had either to change jobs or do something to mathematics.

Thanks to the psychotherapist, they were able to devise a way to save their beloved profession. He had talked to them at one of the sessions (about their improvement, historians think), and no one understood a syllable. Hot damn! If only they could manage to garble their speech in the same way, they could cure their problems1

At the very next session they began to work on what was to become a most successful jargon. Since they could not use jargon for the existing concepts, they decided to start branching out, making up anything which seemed reasonable; for now they would have jargon to hide any inconsistencies. The few who had not yet recovered from their psychoses were put to work creating the new jargon since they spent most of their time babbling nonsense anyway.

Of course, a plan of such divine simplicity worked marvelously. One needs only to look at the work done in a small section of plane analytic geometry to realize the genius of these men. A retired butcher had found his way into mathematics. Having been a butcher all his life, he kept in practice with his cleaver by attacking wooden cones with it. One particular day, one of his words to use was "conic." "Cone," "conic:" there was a resemblance, so he created the conic sections (curves created by slicing a cone in various ways). He was quite inventive, and he used his day's quota and requested three additional lists. (This won for him a solid gold cleaver and the steer of his choice.)

Out of the conic sections came the hyperbola, parabola, circle, ellipse, a straight line, a pair of straight lines, and a point. All of these figures had common properties or properties common to at least some. The hyperbola has all the qualities--axes of symmetry, one being conjugate, the other transverse; two branches; two asymptotes: a latus rectum for either branch; and eccentricity. It also has two ridiculous definitions, neither of which makes much sense. How often does a person feel compelled to talk about the locus of a point whose distance from two fixed points has a fixed difference? But the jargon was working. The definitions and properties sounded impressive, perhaps made sense to a few, and baffled everyone else.

The men in calculus had a problem different from that in plane analytic geometry. They had been assigned to this branch because they had no ability at all in drawing. All sections of geometry had refused to take them, and these unartistic men were trying to create some form of mathematics which required no drawing. Weeks went by, and still they had nothing to call calculus but the name on their door. When they were asked to double as the janitorial staff, they decided they must attack the problem differently.

Since none of the men could conceive of doing any mathematics without a picture, they decided to borrow sketches from other fields. Perhaps if these blighted artists could add a new line or two and give a reason for doing so, they could claim that they had finally invented calculus. Anyone can draw a line with a ruler, so they tried it. They even found something to describe what they had done: they had taken the derivative of the equation of the figure and shown its geometric interpretation.

Now that they had started, they were creating volumes of work. The original derivative was called the first derivative because they had found second and third derivatives. Naturally, these were not simply first or second derivatives but first or second derivatives taken with respect to a given variable. The derivative was obtained by differentiation, not by derivation. The mathematicians even founded a concept which they named 'the partial derivative," taken with respect not to a partial variable but simply a given variable.

The men were so excited about actually creating calculus that they turned in their mops permanently and had a celebration. Feeling sublimely daring from all the wine, these dedicated mathematicans boldly created a concept even more ludicrous than that of differentiation--antidifferentiation. When they were sober, they realized that they had committed a ghastly error by simply putting a negating prefix in front of the name of the former concept and changed the latter's name to integration. Since they had been able to differentiate all sorts of functions--algebraic, logarithmic, trigonometric--they might integrate these functions. When they integrated they had an integral, but they had all types of integrals. There was the definite integral, the indefinite integral, also the improper, the iterated single, the double, and the triple integral. They even claimed that they could find areas, volumes, first moments, moments of inertia, and work by integration.

Work in all branches of mathematics was as successful as it was in plane analytic geometry and calculus. The jargon was created, and the resourceful mathematicians were able to find concepts for it. Much of the vernacular is quite impressive today because it is either utterly misleading or means absolutely nothing unless the listener has a Master's degree in mathematics. A logarithm sounds like a pulsating piece of felled tree, the deleted neighborhood a problem for the N.A.A.C.P. Who was the witch of Agnesi? Are you a surd? What is Lipschitz's condition? After a person casts out nines, does he have a Baire function? Is society safe from the standard deviate? Do shrinking and stretching transformations hurt? Is the sheet of a surface usable on a bed? Is the class called Life Drawing a person's first Baire class? Do amicable numbers love each other? will the annihilator get you?

It is quite apparent that jargon has absolutely solved the problem of the ancient mathematicians. Since they unleashed their powerful vernacular on the public, the guffaws and chortles have been replaced by awful sighs and reverent silence. Who now would dare to laugh when a mathematician opens his mouth? Who understands but the smallest part of what a mathematician utters? Who listens?

ELEMENTARY MATRICES EXPRESSED IN TERMS OF THE KRONECKER DELTA

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The Kronecker Delta is defined as: $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. It is possible to express an elementary matrix entirely in terms of the Kronecker Delta, using it to represent the elements of the matrix under consideration. The process of finding the precise K-Delta expression for a given elementary matrix is essentially one of trial and error. It is known that the statement must contain the general term δ_{ij}, and other terms to express any non-zero off diagonal elements and any diagonal elements equal to zero.

By way of example, consider the elementary matrix P_{pq} defined as an n x n matrix such that every element on the principle diagonal is 1 except for the p^{th} and q^{th} rows. In these rows the diagonal element equals zero, while the element in the p^{th} row and q^{th} column equals 1, as does that in the q^{th} row and p^{th} column. All other off diagonal elements equal zero.

Consider the following expression written in terms of K-Deltas:

$$P_{pq} = [\delta_{ij} + (\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})]$$

To demonstrate that the above statement is true, it is necessary to show the following three properties: (1) that one can express any P_{pq} by the expression; (2) that $P_{pq}P_{pq} = I = [\delta_{ij}]$; (3) that premultiplication of a matrix A by P_{pq} (in the K-Delta form) does change the p^{th} and q^{th} rows of A.

(1) consider $P_{pq} = [\delta_{ij} + (\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})]$:

if $i \neq p$ or q, then the expression equals $[\delta_{ij}]$,

if $i = p$, $P_{pq} = [\delta_{pj} + (0 - 1)(\delta_{pj} - \delta_{qj})] = [\delta_{qj}]$,

if $i = q$, $P_{pq} = [\delta_{qj} + (1 - 0)(\delta_{pj} - \delta_{qj})] = [\delta_{pj}]$.

It is clear that our K-Delta form can express any P_{pq}.

(2) Let $P_{pq}P_{pq} = [d_{ij}][d_{ij}] = \sum_{k=1}^{n} d_{ik}d_{kj}$

$$= \sum_{k=1}^{n} [\delta_{ik}+(\delta_{iq}-\delta_{ip})(\delta_{pk}-\delta_{qk})][\delta_{kj}+(\delta_{kq}-\delta_{kp})(\delta_{pj}-\delta_{qj})]$$

$$= \sum_{k=1}^{n} [\delta_{ik}\delta_{kj} + \delta_{ik}(\delta_{kq}-\delta_{kp})(\delta_{pj}-\delta_{qj}) + \delta_{kj}(\delta_{iq}-\delta_{ip})(\delta_{pk}-\delta_{qk})$$
$$+ (\delta_{iq}-\delta_{ip})(\delta_{pk}-\delta_{qk})(\delta_{kq}-\delta_{kp})(\delta_{pj}-\delta_{qj})].$$

Expanding the second term:

$$\delta_{ik}(\delta_{kq}\delta_{pj} - \delta_{kj} - \delta_{kj} - \delta_{kp}\delta_{qj}) = (\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj}).$$

Clearly the expansion of the third term will yield the same result. Now factoring $(\delta_{iq} - \delta_{ip})(\delta_{pj} - 6_{qj})$ and expanding the last term:

$$P_{pq}P_{pq} = [\delta_{ij} + [(\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})](1 + 1 + \delta_{pq} - 1 - 1 + \delta_{qp})].$$

But $6_{pq} = 6_{qp} = 0$ since $q \neq p$.

Therefore, $P_{pq}P_{pq} = [\delta_{ij}]$.

(3) using the notation of (2), we have:

$$P_{pq}A = \sum_{k=1}^{n} d_{ik}a_{kj} = \sum_{k=1}^{n} [\delta_{ik} + (\delta_{iq} - \delta_{ip})(\delta_{pk} - \delta_{qk})][a_{kj}],$$

if $i \neq q$ or p, ten $\sum_{k=1}^{n} d_{ik}a_{kj} = [a_{ij}]$,

if $i = p$, then $\sum_{k=1}^{n} d_{ik}a_{kj} = [\delta_{pk}+(0-1)(\delta_{pk}-\delta_{qk})][a_{kj}] = [a_{qj}]$,

if $i = q$, then $\sum_{k=1}^{n} d_{ik}a_{kj} = [\delta_{qk}+(1-0)(\delta_{pk}-\delta_{qk})][a_{kj}] = [a_{pj}]$.

Thus we know that the K-Delta representation under consideration does indeed correctly express P_{pq}.

ADDITION PROGRAM FOR TURING MACHINE

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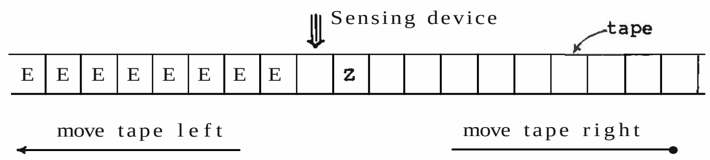
Many people consider that the Turing machine is the abstract prototype of a digital computer. Therefore, anyone who plans to master computer science might find it useful to first become acquainted with this theoretical device conceived by A. M. Turing in 1937. [1]

For our purposes the Turing machine consists of an infinite tape divided into squares called "cells" and a device called the "sensing device" which is able to sense the symbol in the current cell, that is the cell where the sensing device is presently resting, and replace this symbol with another symbol or leave it as is. The symbols are E, Z, 0, and 1, though a person could introduce any finite number of symbols. The tape can be moved to the right one cell at a time or to the left one cell at a time, or left at the current cell.

We assume that all data on the tape is given in binary notation with the lowest order bit to the extreme right of the word which may consist of an arbitrary number of cells (i.e., the usual order in which numbers are written). The tape is originally assumed to have all E's in the cells left of a given cell and a Z in the cell immediately to the right of this given cell. The symbols in these cells are never changed. The remaining cells have either a 0 or a 1 in them and these symbols may be replaced by either a 0 or a 1. The given cell also has either a 0 or a 1 in it which may be replaced by either a 0 or a 1.

One could imagine the tape to appear as follows:



Note: Blank cells have a 0 or a 1 in them.
The "given cell" is positioned at the "sensing device." We move the tape not the sensing device.

Later on we will find it convenient to name the cells. It must be remembered that the machine does not recognize names for the cells; it is only for our convenience. The "given cell" above is called the "AC"; the cell containing "Z" (the next cell to the right) is called the "origin" or "zero cell"; the remaining cells are numbered 1, 2, 3, ... to the right from the "origin." The following diagram illustrates this:

								AC	0	1	2	3	4				
E	E	E	E	E	E	E	E		Z								

Note: Blank cells have a 0 or a 1 in them.

The "current symbol" is defined to be the symbol in the cell at the sensing device. Each cell is referred to as a "location." The numbered cells are referred to by their number, e.g. loc. 27; this is the 27^{th} cell right of the origin. When we say that we move the tape to loc. N, we mean we position the tape so that loc. N is at the sensing device.

A specific set of instructions which the machine is asked to execute is called a "program", and the n instructions of a given program are consecutively numbered 1, 2, 3, through n.

Following Arden's [2] method, we will have only three basic instructions which the Turing machine can execute:

(a) If the current symbol is x, move the tape one cell to the left and execute instruction y. Otherwise, execute the next instruction. This instruction is denoted symbolically IFL x,y where x is a symbol and y is an instruction number.

(b) If the current symbol is x, move the tape one cell to the right and execute instruction y. Otherwise, execute the next instruction. This instruction is denoted symbolically IFR x,y where x is a symbol and y is an instruction number.

(c) If the current symbol is x, replace it with z, then execute the next instruction. Otherwise, just execute the next instruction. This instruction is denoted symbolically IFS x,z where x and z are symbols.

We have deviated slightly from A. M. Turing's [1] and subsequent authors' definition of a Turing machine in order to make its analogy with the modern day computer more apparent: of course, the machine described above is not a stored-program computer. For example, Turing does not number his cells, thus he has no "locations."

A. M. Turing presents an argument in [1] that a machine such as described above is capable of performing any calculation that a computer is able to perform. So, let us see how we would add two four-digit binary numbers together. To do this with only the three basic instructions requires one hundred or more instructions and is very difficult to follow. To avoid this, we introduce "macro instructions." Each macro instruction must be defined by a program of previously defined macro instructions or basic instructions.

We will now list the macro instructions and give their defining programs in a few cases:

(a) IFL ,y — Move the tape to the left one location and then execute instruction y.

1. IFL 0,y
2. IFL 1,y
3. IFL Z,y
4. IFL E,y

Since this is the first program which has been presented, we will explain it briefly to the reader. We know when this program is executed that the tape is moved to the left one cell and instruction y is executed. since each symbol appears in one of the four instructions.

(b) IFR ,y — Move the tape to the right one location and then execute instructions y.

(c) TRA ,y — Transfer to instruction y. (Execute instruction y next.)

1. IFL ,2
2. IFR ,y

(d) HALT — The machine halts. (The state of the machine remains the same indefinitely without affecting the tape; used to stop the machine, e.g. at the end of the program.)

1. TRA ,1
2. ....

(e) IFS ,x — Replace the current symbol with x and then execute the next instruction. (Note, the current symbol can not be Z or E and x can not be Z or E.)

1. IFS 0,x
2. IFS 1,x
3. ....

(f) MTR N,y — Move the tape to the right N locations and then execute instruction y.

1. IFR ,2
2. IFR ,3
3. ....

i. IFR ,(i+1)
(i+1). ....

N. IFR ,(N+1)
(N+1). TRA ,y

(g) **MTL M,y** Move the tape to the left M locations and then execute instruction y.

(h) **TRA x,y** If the current symbol is x, execute instruction y. Otherwise, execute the next instruction.

(i) **MTP N,y** Move the tape to loc. N and then execute instruction y.

1. TRA E,3
2. MTR 1,1
3. TRA Z,5
4. MTL 1,3
5. MTL N,y

(j) **MTA ,y** Move the tape to the AC and then execute instruction y.

(k) **CLA N,y** Put the symbol which is presently in loc. N into the AC, leave the tape positioned at the origin (loc. 0), and then execute instruction y. Note, the symbol in loc. N is not changed.

(l) **STO N,y** Put the symbol which is presently in the AC into loc. N, leave the tape positioned at the origin, and then execute instruction y. Note, the symbol in the AC is not changed.

1. MTA ,2	5. MTP 0,y
2. TRA 0,6	6. MTP N,7
3. MTP N,4	7. IFS ,0
4. IFS ,1	8. MTP 0,y

We will now introduce three instructions which give us the body of our addition program. The first instruction adds the contents of the AC and the contents of a location. The second instruction adds the contents of two locations. The third is a combination of the first two and is the only instruction of the three which actually appears in the addition program.

(a) **ADA N,y** Let x be the symbol in loc. N and let z be the symbol in the AC. Add x and z, write it as a double bit binary sum. Put the left-hand bit of the double bit sum obtained into the AC and put the right-hand bit of the double bit sum obtained into loc. N. Leave the tape positioned at the origin and execute instruction y.

x \ z	0	1
0	00	01
1	01	10

double bit sums

1. MTA ,2	5. IFS ,0
2. TRA 0,6	
3. MTP N,4	6. MTP 0,y
	7. IFS ,1
4. TRA 0,7	8. MTA ,5

We shall explain this program briefly. First we move the tape to the AC and test to see if there is a 0 there by instructions 1 and 2. If there is a 0 there, the left-hand column of the table is pertinent and we wish to leave the 0 in the AC and leave whatever symbol is in loc. N there. So we merely move the tape to the origin and execute instruction y (by executing Instruction 6). If there is not a 0 there, we go on to instructions 3 and 4 which move the tape to loc. N and test to see if there is a 0 in loc. N. If there is, we wish to put a 0 into the AC and a 1 into loc. N. Instructions 7, 8, and 5, operating in that order, accomplish this. If there is not a 0 in the AC, we wish to put a 1 into the AC and a 0 into loc. N. Instruction 5 alone accomplishes this since there is already a 1 in the AC.

(b) **ADN N,M** Add the symbol in loc. N to the symbol in loc. M. If the sum is less than or equal to 1, the sum is placed in loc. N and the AC remains unchanged. If the sum is greater than 1, a 1 is put into the AC and a 0 into loc. N. In both cases the tape is left at the origin and the next instruction executed.

1. MTP M,2	6. CLA M,8
2. TRA 0,8	7. IFS ,1
3. MTP N,4	8. MTP 0,9
4. TRA 0,7	9. ....
5. IFS ,0	

The dots in instruction 9 indicate that the next instruction is to be executed.

Again we will briefly explain the program. The first two instructions move the tape to loc. M and test for a 0. If there is a 0 there, the sum of the symbols in loc. N and loc. M is the symbol in loc. N, so we merely move the tape to the origin and execute the next instruction (by executing Instruction 8). If there is a 1 in loc. M, we go on to instructions 3 and 4 which move the tape to loc. N and test for 0. If there is a 0 there, the sum would be 1, so we transfer to instruction 7 which puts a 1 into loc. N and then we move the tape to the origin and execute the next instruction (by executing Instruction 8). If there is no 0 in loc. N, we wish to put a 1 into the AC and a 0 into loc. N. To do this we go to instruction 5 which puts a 0 into loc. N, then instruction 6 brings the 1 in loc. M into the AC and we are finished.

It must be noted that except in the case where there is a 1 in both loc. N and loc. M, the AC was left unaltered.

(c) ANM N,M   Add the symbol in the AC, the symbol in loc. N, and the symbol in loc. M together. This sum is less than 4. Express it in a binary form (two bits), put the left-hand bit into the AC and the right-hand bit into loc. N. Leave the tape at the origin and execute the next instruction.

1. ADA N,2
2. ADN N,M
3. ....

To do this we use the first two instructions. Let all sums be expressed as double bits. The first instruction adds the symbol in the AC and the symbol in loc. N and puts the right-hand bit of the sum into loc. N and the left-hand bit of the sum into the AC. The second instruction adds the right-hand bit of the previous sum now in loc. N and the symbol in loc. M and puts the right-hand bit of this sum (the sum of the symbol now in loc. N and the symbol in loc. M) into loc. N. If the left-hand bit of this sum is 0, it does nothing to the AC; thus leaving the left-hand bit (highest order bit) of the previous sum of the symbols in the AC and loc. N undisturbed (there is no carry for this sum). If, on the other hand, the left-hand bit of this sum (the sum of the symbol now in loc. N and the symbol in loc. M) is 1, it puts a 1 into the AC. In this case, the left-hand bit of the previous sum, that is the sum of the symbols in the AC and loc. N, is lost but it would have been 0 anyway because the total sum, that is of the AC, loc. N, and loc. M, has to be less than 4.

The following program adds a number A and a number B where A is in locations N-3, N-2, N-1, N; and B is in locations M-3, M-2, M-1, M; and stores the results in N-4, N-3, N-2, N-1, N (where the highest order bit is in N-3 or M-3 for A and B respectively). Note, N and M must be > 4 and $|M - N| > 4$.

1.	MTA	,2	5.	ANM	N-2, M-2
2.	IFS	,0	6.	ANM	N-3, M-3
3.	ANM	N,M	7.	STO	N-4,8
4.	ANM	N-1,M-1	8.	HALT	

Instructions 1 and 2 put a 0 into the AC since there is no initial carry. Instructions 3, 4, 5, and 6 add the ones' bit, twos' bit, fours' bit, and eights' bit of the binary numbers A and B and the previous carry together respectively putting the lowest order bit of each sum into locations N, N-1, N-2, and N-3 respectively, and putting the highest order bit of each sum, that is the carry, into the AC. Finally, instruction 7 puts the final carry into location N-4. Thus the sum of A and B appears in locations N-4, N-3, N-2, N-1, N. Note, by merely adding extra ANM's we could increase the maximum magnitude of A and B.

We have shown that a simple machine such as described above can perform addition of positive integers. Subtraction could also be handled easily by writing an instruction which would replace a number by its complement.

To subtract a number B from a number A, we would replace B by its complement and add it to A. For example, if we assume that we are working with a maximum of three-digit binary numbers, the complement of the binary number 0101 (5) is 1011 which is the first number, 0101, subtracted from binary 10000 (16). The assumption that we are only working with three-digit binary numbers is required since we must have a way of telling whether a number is a complement quantity or not. Note, the four-digit representation of binary five is also binary eleven. For this purpose we reserve the extreme left-hand bit of a four-bit word; if it is a 1, the next three bits are a three-digit complement quantity; if it is a 0, the next three bits are a regular three-digit quantity. We could also put negative numbers on the tape in their complement form.

We have not mentioned multiplication or division or decimal quantities. Multiplication and division could be taken care of by still more complicated programs using addition and subtraction as developed above and a shift instruction. Decimal quantities could be handled by the manner in which the data was placed on the tape. That is, one could presume that the decimal point lies after the first ten places of all numbers.

Thus, it is reasonable to say that the Turing machine can perform any calculation that a computer can perform. But is a computer capable of calculating everything that a Turing machine is able to calculate?

Recently many authors including M. O. Rabin and D. Scott [3] have come to the conclusion that the Turing machine with its infinite tape is too general to serve as an actual model for digital computers. They base their argument on the fact that the Turing machine has an unlimited amount of memory space on its tape while an actual computer would have a limited amount of memory space.

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This paper was presented at the National Pi Mu Epsilon Meeting at Cornell University, September 1, 1965, by James Woeppel.

A MULTIPLICATION–FREE CHARACTERIZATION OF RECIPROCAL ADDITION

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Motivation.

Reciprocal addition, which we shall represent by the symbol "o," is defined, in terms of the real number system, by

$$a \circ b = 1/(1/a + 1/b) = ab/(a + b), \qquad (a,\ b,\ a+b \neq 0)$$
$$a \circ 0 = 0$$

and is commonly used with an ideal infinity element ω with the properties $a \circ \omega = a$ and $a + \omega = \omega$. In such applications as parallel–series problems in electronics, for instance, the operations of addition and reciprocal addition frequently occur together (the constants 0 and ω corresponding to the two extremes of short and open circuit values) exclusive of the use of multiplication; it is therefore of interest to develop an axiomatic characterization of + and o independently of the notion of multiplication.

We begin with a listing of some properties of R^*, the real number system with ω and the operation o adjoined.

I. We use the mapping

$$f(x) = 1/x, \quad 0 \neq x \neq \omega; \qquad f(0) = \omega; \qquad f(\omega) = 0$$

to show that R^* is a dual system:

$a,\ b,\ a+b \neq 0,\ \omega \Rightarrow$

(1) $f(-a) = 1/(-a) = -1/a = -f(a)$

(2) $f(a + b) = 1/(a + b) = (1/a)(1/b)/(1/a + 1/b)$
$= 1/a \circ 1/b = f(a) \circ f(b)$

$f(a \circ b) = 1/(1/(1/a + 1/b)$
$= 1/a + 1/b = f(a) + f(b)$

(3) $f(ab) = 1/ab = (1/a)(1/b) = f(a)f(b)$

The argument for special elements proceeds by definitial patching thus:

(1) $-\omega = \omega$

(2) $a \circ \omega = a,\ a + \omega = \omega,\ a \circ 0 = 0,\ a \neq 0 \Rightarrow a \circ (-a) = \omega$

(3) $a \cdot \omega = \omega \circ \omega = \omega.$

II. + and o are correlated independently of multiplication by the relation

$$a + (a \circ b) = a + \frac{ab}{a + b} \qquad (a,\ b,\ a+b \neq 0,\ \omega)$$
$$= (a^2 + 2ab)/(a + b)$$
$$= a(a + 2b)/(a + b)$$
$$= 2a(2a + 4b)/(2a + (2a + 4b))$$
$$= 2a \circ (2a + 4b)$$

We now proceed to investigate a system Γ with special elements o and ω, with a monary operation ' , and with binary operations + and o , satisfying axioms I – VI for all elements a, b, c in Γ.

Definition 1.

Let N be the set of positive integers; $\forall n \in N$,

$$1 \cdot a = a, \quad (n + 1)a = na + a$$
$$a/1 = a, \quad a/(n + 1) = a/n \circ a .$$

Axioms.

I. $a + b = b + a;\quad a \circ b = b \circ a$

II. $(a + b) + c = a + (b + c);\quad (a \circ b) \circ c = a \circ (b \circ c)$

III. $a + 0 = a;\quad a \circ \omega = a$

IV. $0' = 0;\quad \omega' = \omega;\quad 0 \neq a \neq \omega \Rightarrow (a + a' = 0$ and $a \circ a' = \omega)$

V. $a \circ 0 = 0$

VI. $a + (a \circ b) = 2a \circ (2a + 4b)$

Theorem I. $\forall$ a, b $\in \Gamma$,

1. $(2a/2) = a$
2. $2a = 0 \Rightarrow a = 0$
3. $a = a' \Rightarrow (a = 0$ or $a = \omega)$
4. $a + \omega = \omega$
5. $a \circ b = 0 \Rightarrow (a = 0$ or $b = 0)$
   $a + b = \omega \Rightarrow (a = \omega$ or $b = \omega)$
6. $\Gamma \neq \{0\} \Rightarrow 0 \neq \omega$
7. $a \circ b = \omega \Rightarrow a' = b;$
   $a + b = 0 \Rightarrow a' = b$
8. $(a + b)' = a' + b';\quad (a \circ b)' = a' \circ b'$
9. $2(a/2) = a$
10. $2(a \circ b) = 2a \circ 2b;\quad (a + b)/2 = a/2 + b/2$

Proof I.1.

$$a = a + (a \circ 0) = 2a \circ (2a + 4 \cdot 0) \qquad \text{(III, V; VI)}$$
$$= 2a \circ 2a = (2a)/2 \qquad \text{(III, def)}$$

Proof 1.2.

If $2a = 0$, then

$$a = (2a)/2 = 0 \circ 0 = 0. \qquad \text{(I.1; hyp, def; III)}$$

Proof 1.3.

If $a = a'$ and $a \neq \omega$, then

$$2a = a + a' = 0 \qquad \text{(IV, hyp)}$$
$$a = 0 . \qquad \text{(I.2)}$$

Therefore, $a = a' \Rightarrow (a = 0$ or $a = \omega)$.

Proof 1.4.

(1) If $a = 0$, then
$a + \omega = 0 + \omega = \omega.$ (hyp, III)

(2) (Lemma) If $r \neq \omega$ and $r + s = 0$, then
$r' = r' + 0 = r' + (r + s)$ (III, hyp)
$= (r' + r) + s$ (II)
$= 0 + s = s.$ (IV, hyp; III)

(3) (Lemma) If $2a = \omega$, then
$a = 2a \circ 2a = \omega \circ \omega = \omega$. (I.1, hyp, III)

(4) (Lemma) If $a \# \omega$, then
$(2a)' = (2a)' + 2 \cdot 0$
$= (2a)' + 2(a + a')$ (III, def; IV, hyp)
$= ((2a)' + 2a) + 2a' = 2a'$. (II; IV, hyp; 111)

(5) If $0 \neq a \neq \omega$, then
$a + (a \circ a') = 2a \circ (2a + 4a')$, (VI)
$a + \omega = 2a \circ ((2a + 2a') + 2a')$ (IV, hyp: def, II)
$= 2a \circ ((2a + (2a)') + (2a)')$ (4)
$= 2a \circ (2a)'$. (IV, 3, hyp; III)

(6) Therefore $a \# \omega \Rightarrow a + \omega = \omega$. (1,5)

If $a = \omega$, then $a + \omega = 2\omega$; suppose $2\omega \# \omega$, then

(7) $\omega = 2\omega + \omega = 3\omega$. (hyp, 6 with $a = 2\omega$; def)
$2\omega = \omega + \omega = \omega + (\omega \circ \omega)$ (def, III)
$= 2\omega \circ (2\omega + 4\omega)$ (VI)
$= 2\omega \circ (3\omega + 3\omega)$ (def, II)
$= 2\omega\ 0\ (\omega + \omega)$ (7)
$= 2\omega \circ 2\omega = \omega$ (def, I.1)

Contradiction, therefore $\forall a \in \Gamma$, $a + \omega = \omega$.

~~Proof 1.5.~~

If $a \circ b = 0$ and $b \# 0$, then
$a = a \circ \omega = a \circ (b \circ b') = (a \circ b) \circ b'$
$= 0 \circ b' = 0$. (III; IV, hyp; II; hyp; V)

Therefore, $a \circ b = 0 \Rightarrow (a = 0$ or $b = 0)$.

Similarly, if $a + b = \omega$ and $b \# \omega$, then
$a = a + 0 = a + (b + b') = (a + b) + b'$
$= \omega + b' = \omega$. (III; IV, hyp; 11; hyp; I.4)

~~Proof 1.6.~~

If $\Gamma \# \{0\}$, $\exists X \in \Gamma$: $X \# 0$; suppose $0 = \omega$, then
$0 \neq X = X + 0 = X + \omega = \omega$. (hyp, III, hyp, I.3)

Contradiction, therefore $\Gamma \# \{0\} \Rightarrow 0 \# \omega$.

~~Proof 1.7.~~

If $a \circ b = \omega$ and $\Gamma = \{0\}$, then $a' = 0 = b$.

(8) If $a \circ b = \omega$ and $\Gamma \neq \{0\}$, then $a \# 0$ since
$a = 0 \Rightarrow 0 = 0 \circ b = a \circ b = \omega$. (V, hyp, hyp)

Contradiction by Theorem 1.6.

$a' = a' \circ \omega = a' \circ (a \circ b) = (a' \circ a) \circ b$
$= \omega \circ b = b$ (III; hyp; 11; IV, 8; 111)

(9) Similarly, if $a + b = 0$, and $\Gamma \# \{0\}$, then $a \# \omega$ since
$a = \omega \Rightarrow \omega + b = a + b = 0$. (I.3, hyp, hyp)

Contradiction by Theorem I.6.

$a' = a' + 0 = a' + (a + b) = (a' + a) + b$
$= 0 + b = b$ (III; hyp; 11; IV, 9; 111)

~~Proof 1.8.~~

If $a = 0$, then
$(a \circ b)' = (0 \circ b)' = 0' = 0 = 0 \circ b' = 0' \circ b'$
$= a' \circ b'$. (hyp, V, IV, V, IV, hyp)

If $a \# 0$ and $b \# 0$, then
$(a \circ b) \circ (a' \circ b') = (a \circ a') \circ (b \circ b')$
$= \omega \circ \omega = \omega$. (I, II; IV, hyp; III)

Therefore, $(a \circ b)' = a' \circ b'$ by Theorem I.7.

If $a = \omega$, then
$(a + b)' = (\omega + b)' = \omega' = \omega = \omega + b' = \omega' + b'$
$= a' + b'$. (hyp, 1.4, IV, 1.4, IV, hyp)

If $a \# \omega$ and $b \# \omega$, then
$(a + b) + (a' + b') = (a + a') + (b + b')$
$= 0 + 0 = 0$. (I, II; IV, hyp: III)

Therefore $(a + b)' = a' + b'$ by Theorem I.7.

~~Proof 1.9.~~

suppose $0 \# a \# \omega$:
$a + (a \circ a'/2) = 2a \circ (2a + 4(a'/2))$ (VI)
$a + (a \circ a') \circ a' = 2a \circ (2a + 4(a'/2))$ (def, II)
$a + (\omega \circ a') = 2a \circ (2a + 4(a'/2))$ (IV, hyp)
$a + a' = 2a \circ (2a + 4(a'/2))$ (III)
$0 = 2a \circ (2a + 4(a'/2))$ (IV, hyp)

Therefore either $2a = 0$ or $2a + 4(a'/2) = 0$ by Theorem I.5.

But $2a = 0 \Rightarrow a = 0$ by Theorem 1.2, and $a \# 0$, therefore
$0 = 2a + 4(a'/2) = 2a + (4(a/2))'$ (I.8)
$2a = ((4(a/2))')' = 4(a/2) = 2(2(a/2))$ (1.7, def)
$(2a)/2 = (2(2(a/2)))/2$, $a = 2(a/2)$. (I.1)

Finally:
$2(0/2) = (0 \circ 0) + (0 \circ 0) = 0 + 0 = 0$, (def, 111, III)
$2(\omega/2) = (\omega \circ \omega) + (\omega \circ \omega) = \omega + \omega = \omega$. (def, 111, 1.6)

Proof I.10.

$2(a \circ b) = 2((2a)/2 \circ (2b)/2)$ (I.1)
$= 2((2a \circ 2b)/2)$ (def, II)
$= 2a \circ 2b$. (I.9)

$(a + b)/2 = (2(a/2) + 2(b/2)/2$ (I.9)
$= (2(a/2 + b/2))/2$ (def, II)
$= a/2 + b/2$ (I.1)

~~Meta-Theorem~~.

Γ is a dual system; that is, for each theorem T in Γ, the statement T* formed by replacing each occurrence of $+$, $\circ$, 0 and ω with $\circ$, $+$, ω and 0 (respectively) is also a theorem in Γ.

Proof.

Axioms I - IV are self-dual. Theorem 1.2 is the dual of Axiom V. The dual of Axiom VI is proved as follows: For any given x and y in Γ, let $a = x/2$, $b = y/4$, then

$$2a = 2(x/2) = x \qquad \text{(I.9)}$$
$$4b = 2(2b) = 2(2(y/4))$$
$$= 2(2((y/2)/2)) = 2(y/2) = y.$$
(def, 11; hyp; def, 11; I.9; I.9)

Making the appropriate substitutions in Axiom VI then gives $\forall\, x, y \in \Gamma$;

(10) $$x/2 + (x/2 \circ y/4) = x \circ (x + y)$$

The meta-theorem is completed by noting that if T is any theorem with a proof P, then T*, the dual of T, will have a proof P*, each step of which is the dual of the corresponding step in P.

At this point it would be nice to know how close Γ comes to representing the notion of a field. In one direction, beginning with any field, we may construct all of the axioms of Γ just as we did for the real numbers, with the one exception that the derivation of VI requires the principle 1.2: $2a = 0 \Rightarrow a = 0$, thus eliminating fields such as $I/(2)$. In the other direction, however, the answer seems less clear: Theorem II contains some partial results.

Definition ii.

We classify Γ by writing for each $k \in N$,

$\Gamma \in S_k$ iff $\forall\, a \in \Gamma$, $k(a/k) = a$

$\Gamma \in T_k$ iff $\forall\, a \in \Gamma$, $(ka)/k = a$.

(11) Lemma.

If $\Gamma \in S_p \cap S_q$, then $\forall\, a \in \Gamma$

$$pq(a/pq) = p(q((a/p)/q)) \qquad \text{(def, II)}$$
$$= p(a/q) \qquad (\Gamma \in S_q)$$
$$= a \qquad (\Gamma \in S_p)$$

so that $\Gamma \in S_{pq}$. Similarly, $\Gamma \in T_p \cap T_q \Rightarrow \Gamma \in T_{pq}$.

(12) Lemma.

If $\Gamma \in S_q \cap T_q$, then $\forall\, p \in N$;

$$(pa)/q = (p(q(a/q)))/q \qquad (\Gamma \in S_q)$$
$$= (q(p(a/q)))/q \qquad \text{(II, def)}$$
$$= p(a/q) \qquad (\Gamma \in T_q)$$

(13) Lemma.

$\Gamma \in S_p \Rightarrow (a/p = \omega \Rightarrow a = \omega)$, since if $a/p = \omega$, then
$a = p(a/p) = p\,\omega = \omega.$ $(\Gamma \in S_p$, I.4)

Similarly, $\Gamma \in T_p \Rightarrow (pa = 0 \Rightarrow a = 0)$.

(14) Lemma.

If $\Gamma \in S_{k-1}$, then $(ka = 0 \Rightarrow a = 0) \Rightarrow (a/k = \omega \Rightarrow a = \omega)$, for suppose

1) $ka \neq 0$, 2) $a/k = \omega$, and 3) $a \# \omega$, then

$$\omega = a \circ a/(k-1) \qquad \text{(def, hyp 2)}$$
$$a = (a/(k-1))' = a'/(k-1) \qquad \text{(1.7, I.8)}$$
$$(k-1)a = (k-1)(a'/(k-1)) = a' \qquad (\Gamma \in S_{k-1})$$
$$ka = (k-1)a + a = a' + a = 0. \qquad \text{(def; IV, hyp 3)}$$

Contradiction by hypothesis 1.

Therefore $(a \neq 0 \Rightarrow ka \# 0) \Rightarrow (a/k = \omega \Rightarrow a = \omega)$.

(15) Basic Lemma for Theorem 11.

$\forall\, a \in \Gamma$, $\forall\, k \in N$; If $\Gamma \in \bigcap_{n=1}^{k-1} (S_n \cap T_n)$ and $ka = 0 \Rightarrow a = 0$, then $\Gamma \in S_k \cap T_k$.

We shall prove $\Gamma \in S_k$ and show $\Gamma \in T_k$ by a duality argument. We begin with proofs for special cases:

(16) If $a = 0$, then $k(0/k) = (k \cdot 0)/k = 0$, (III, V)
if $a = \omega$, then $k(\omega/k) = (k\omega)/k = \omega$. (III, I.4)

(17) The cases for $k = 2$ and $k = 4$ follow directly from the fact that $\Gamma \in \bigcap [S_2, S_4, S_8, \cdots]$, which in turn follows from Theorem I.9 and Lemma 11.

(18) We next prove the case for $k = 3$ as follows: if $3a \# 0$ and $a \# \omega$, then $a/6 \neq \omega$ since if $a/6 = \omega$, then

$$a/3 = 2((a/3)/2) = 2(a/6) = 2\omega = \omega.$$

Contradiction by Lemma 14.

$$2a \circ a = a \circ (a + a) = a/2 + (a/2 \circ a/4)$$
$$= a/2 + a/6 \qquad \text{(def; 10; def, II)}$$
$$(2a \circ a) + a'/6 = a/2 + a/6 + a'/6 = a/2$$
(II; 1.6; IV, $a/6 \neq 0$; III)
$$2(2a \circ a) + 2(a'/6) = 2(a/2) = a \qquad \text{(11, def; 1.9)}$$
$$2(2((2a \circ a)/2)) + 2((a'/3)/2) = \qquad \text{(1.9; def, II)}$$
$$4((2a)/2 \circ a/2) + a'/3 = \qquad \text{(def, 11; I; I.9)}$$
$$4(a/3) + a'/3 = \qquad \text{(I.1, def)}$$
$$3(a/3) + (a/3 + (a/3)') = \qquad \text{(def, 11; 1.8)}$$
$$3(a/3) = a \qquad \text{(IV, 14, hyp; 111)}$$

(19) Finally, by induction for any $k > 4$, choose $0 \neq a \neq \omega$, then

(20) $a/2(k-2) \neq \omega$ by 13, since $\Gamma \in S_{2(k-2)}$ by Theorem 1.9, hypothesis, and Lemma 11.

(21) $(k-4)a' \neq 0$ by 13 since $\Gamma \in T_{k-4}$.

(22) $(k-4)a \# \omega$ by Theorem 1.5 since $a \neq \omega$ by hypothesis, and therefore $(k-4)a/k \neq \omega$ by 14.

Having dispensed with the preliminary details, we make the following computation:

$$a + [a \circ ((k-4)a')/2(k-2)] = 2a \circ [2a + 4[((k-4)a')/2(k-2)]]$$
[VI with $b = ((k-4)a')/2(k-2)$]

$$= 4[a/2 \circ [a/2 + ((k-4)a')/2(k-2)]] \qquad (1.9,\ I.10)$$

$$= 4[a/2 \circ [(k-2)(a/2(k-2)) + ((k-4)a')/2(k-2))]] \qquad (\Gamma \in S_{k-2})$$

$$= 4[a/2 \circ [(k-2)(a/2(k-2)) + (k-4)((a'/2)/(k-2))]] \qquad (I.10)$$

$$= 4[a/2 \circ [(k-2)(a/2(k-2)) + (k-4)((a'/2)/(k-2))]] \qquad (12)$$

$$= 4[a/2 \circ [(k-2)(a/2(k-2)) + (k-4)(a/2(k-2))']] \qquad (I.10,\ I.8)$$

$$= 4[a/2 \circ 2(a/2(k-2))] \qquad (IV,\ 20)$$

$$= 4[a/2 \circ a/(k-2)] \qquad (\text{def}, 11;\ 1.9)$$

Therefore,

$$a + [a \circ ((k-4)a')/2(k-2)] = 4(a/k) \qquad (\text{def}, II)$$

$$a + [((k-4)a)/(k-4) \circ ((k-4)a')/2(k-2)] = \qquad (\Gamma \in T_{k-4})$$

$$a + ((k-4)a')/k = \qquad (\text{def}, II;\ IV, 21;\ III)$$

$$a + [((k-4)a)/k]' = 4(a/k) \qquad (I.8)$$

Adding $((k-4)a)/k$ to both sides gives

$$a = 4(a/k) + ((k-4)a)/k \qquad (\text{def}, 11;\ IV, 22;\ III)$$

$$= k(a/k) \qquad (\text{def}, II)$$

Therefore, if $\Gamma \in \bigcap_{n=1}^{k-1} (S_n \cap T_n)$ and $ka = 0 \Rightarrow a = 0$, then

$\Gamma \in S_k$ by 16, 17, 18, and 19.

To arrive at $\Gamma \in T_k$, we note that the above proof takes the form "if P and Q, then R" [where P is $\Gamma \in \bigcap T_n$), Q is $ka = 0 \Rightarrow a = 0$, and R is $\Gamma \in S_k$)]. Since P and P* are identical, and since P and Q $\Rightarrow$ Q* by 14, P and Q $\Rightarrow$ P* and Q*; finally, P* and Q* $\Rightarrow$ R* by the meta-theorem, so that P and Q $\Rightarrow$ R* and $\Gamma \in T_k$.

<u>Definition iii</u>. $\Gamma^+ = \Gamma - \{\omega\}$.

~~Theorem 11~~.

With respect to suitable definitions of multiplication,

1. If for some prime integer p and for all $a \in \Gamma^+$, $pa = 0$, then Γ^+ is a vector space over $I/(p)$, the ring of integers modulo p.
2. If Γ^+ has a prime number of elements p, then Γ is derivable from $I/(p)$.
3. If $\forall a \in \Gamma^+$ and $\forall n \in N$, $na = 0 \Rightarrow a = 0$, then Γ^+ is a vector space over the field of rational numbers.

<u>Proof II.1</u>.

Assume $pa = 0$ for all $a \in \Gamma^+$, for some prime integer p.

(23) (Lemma) We note that Γ^+ is a prime order abelian group with respect to addition as a result of Axioms I–IV and the above assumption.

(24) Considering $[n] \in I/(p)$ as an equivalence class of n, we define, $\forall a \in \Gamma^+$, $\forall [n] \in I/(p)$,

$$[n]a = na.$$

$[n]a$ is well defined since if $[n] = [m]$, then $(n-m) = 0 \pmod p$ and therefore $(n-m)a = 0$, $na = ma$.

Theorem 11.1 now follows from the fact that any additive p-order Abelian group forms a vector space with respect to definition 24 of multiplication.

~~Proof 11.2.~~

Assume Γ^+ has p elements for some prime integer p.

Then, with respect to addition, Γ^+ is a cyclic group, so that by Theorem II.1, it forms a one dimensional vector space with respect to definition 24 of multiplication.

(25) To extend the scalar-field properties of Γ^+, we choose an arbitrary normalizing constant $a \in \Gamma^+ - \{0\}$, and define
$x \cdot y = [k(x)][k(y)]a$ where
$[k(x)]a = x$, $[k(y)]a = y$.

(26) We also define $1/x$ by
$1/x = y \iff x \cdot y = a$.

(27) (Lemma) $\forall n < p$, $\forall a \in \Gamma$, $(na)/n = n(a/n) = a$ since, by induction, we have first $\Gamma \in (S_2 \cap T_2)$ as mentioned in Lemma 15; then if $\Gamma \in \bigcap_{n=1}^{k-1} (S_n \cap T_n)$ for some $k < p$, then $a \neq 0 \Rightarrow ka \neq 0$ by Lemma 23 and therefore $\Gamma \in S_k \cap T_k$ by Lemma 15. The induction stops with $k = p - 1$.

(28) (Lemma) If $n \neq 0$, then

$$(na)\cdot(a/n) = [k(na)](k(a/n)]a) \qquad (25, \text{def } i)$$

$$= [n](a/n) \qquad (25, 24, 25)$$

$$= n(a/n) = a. \qquad (n \in [n], 27)$$

Therefore $1/(na) = a/n$ by definition 26.

Finally, we choose $x, y \in \Gamma^+ - \{0\}$ such that $x' \neq y$ and compute

$$x \circ y = 1/(1/x) \circ 1/(1/y) \qquad (26)$$

$$= [1/k(1/x)a] \circ [1/k(1/y)a] \qquad (25)$$

$$= a/k(1/x) \circ a/k(1/y) \qquad (28)$$

$$= a/[k(1/x) + k(1/y)] \qquad (\text{def } i, II)$$

$$= 1/[k(1/x) + k(1/y)]a \qquad (27)$$

$$= 1/[k(1/x)a + k(1/y)a] \qquad (\text{def } i, II)$$

$$= 1/[1/x + 1/y]$$

So that 0 is algebraically determined by the field properties of Γ^+ and the conventions for special elements (e.g. $a + \omega = \omega$).

~~Proof 11.3.~~

Assume $\forall a \in \Gamma^+$, $\forall k \in N$, $ka = 0 \Rightarrow a = 0$.

(29) We define $\forall k \in N$,

$$(-k)a = ka', \quad a/(-k) = a'/k, \quad 0 \cdot a = 0.$$

(30) (Lemma) By analogy with Lemma 27 of the previous proof, we conclude $\Gamma \in \bigcap_{k=1}^{\infty} (S_k \cap T_k)$, so that, in view of definition 29, $\forall k \in I$, $k(a/k) = (ka)/k = a$.

(31) $\forall a \in \Gamma^+$, $\forall p, q \in I$, $q \neq 0$, we define

$$(p/q)a = (pa)/q \; .$$

The above multiplication by rational numbers is well defined since if $p_1/q_1 = p_2/q_2$; if $p_1, q_1, p_2, q_2, \in I$, and if $q_1, q_2 \neq 0$, then $p_1 = p_2 q_1/q_2$.

$$\begin{aligned}
(p_1/q_1)a &= ((p_2 q_1/q_2)a)/q_1 && (31)\\
&= ((p_2 q_1 a)/q_2)/q_1 && (31)\\
&= [[q_1(p_2 a)]/q_2]/q_1 && (\text{def } i, II)\\
&= [q_1[(p_2 a)/q_2]]/q_1 && (12, 30)\\
&= (p_2 a)/q_2 = (p_2/q_2)a && (30, 31)
\end{aligned}$$

Assuming the normal field properties for rational numbers, the field-[vector space] correlation laws are proved as follows:

(32) $1 \cdot x = x$ (def i)

(33)
$$\begin{aligned}
(p/q)[(r/s)x] &= [p[(rx)/s]]/q && (31)\\
&= [[p(rx)]/s]/q && (12, 30)\\
&= [(pr)x]/qs && (\text{def } i, II)\\
&= (pr/qs)x = [(p/q)(r/s)]x && (31)
\end{aligned}$$

(34)
$$\begin{aligned}
(p/q)(x + y) &= [p(x + y)]/q && (31)\\
&= p[(x + y)/q] && (12, 30)\\
&= p[(q(x/q) + q(y/q))/q] && (30)\\
&= p[(q(x/q + y/q))/q] && (\text{def } i, I, II)\\
&= p(x/q + y/q) && (30)\\
&= p(x/q) + p(y/q) && (\text{def } i, I, II)\\
&= (px)/q + (py)/q && (12, 30)\\
&= (p/q)x + (p/q)y && (31)
\end{aligned}$$

(35)
$$\begin{aligned}
(p/q + r/s)x &= [(ps + qr)x]/qs && (31)\\
&= [(ps)x + (qr)x]/qs && (\text{def } i, II)\\
&= [1/qs][(ps)x + (qr)x] && (31, 32)\\
&= [1/qs][(ps)x] + [1/qs][(qr)x] && (34)\\
&= (p/q)x + (r/s)x && (33)
\end{aligned}$$

RESEARCH PROBLEMS

Proposed by GEORGE BRAUER, University of Minnesota.

The real-valued functions on $(-\infty,\infty)$ is a semi-group, if we define the product of two such functions by

(1) $$f \times g = g(f(t)) \; .$$

The function x acts as identity and each constant function c has the properties:

(2) $$c \times f = f(c) \; ,$$

(3) $$f \times c = c \; .$$

If we denote the set of real-valued functions by S, and the set of constants by I, we have

(4) $$I \times S \subseteq I \; , \qquad S \times I = I \; .$$

Moreover, if we define $f \geq g$ to mean that $f(t) \geq g(t)$ for all t, then S is right partially ordered, i.e.,

(5) $$f \geq h \implies f \times g \geq f \times h \; .$$

Problem: Obtain properties of abstract right partially ordered semi-groups. What algebraic properties, in addition to (1) - (5), would constitute a characterization of the semi-group of real functions? What properties would characterize the sub semi-groups?

Proposed by SIM KAH, University of Minnesota.

Examine, heuristically, the geometry of geodesics on a cylinder in order to obtain a set of axioms of incidence and order. Using these axioms, deductively develop a geometry and attempt to find models, other than the cylinder, to which this geometry applies.

PROBLEM DEPARTMENT

Edited by
M. S. Klamkin, Ford Scientific Laboratory

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to M. S. Klamkin, Ford Scientific Laboratory, P. 0. Box 2053, Dearborn, Michigan 48121.

PROBLEMS FOR SOLUTION

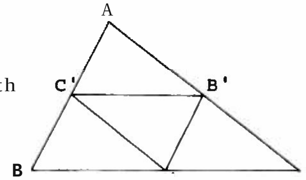
177. Proposed by C. S. VENKATARAMAN, Sree Kerala Varma College, Trichur, South India.

If s is the semi-perimeter and R, r, r_1, r_2, and r_3 are the circum-, in-, and ex-radii respectively of a triangle, prove that

$$\frac{R}{r^2} \geq \frac{2s^2}{r_1 r_2 r_3} .$$

178. Proposed by K. S. MURRAY, New York City.

Show that the centroid Of A ABC coincides with the centroid of A A'B'C' where A', B', and C' are the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively.



Generalize to higher dimensions.

179. Proposed by DONALD SCHROEDER, Seattle, Washington.

It is well known that

$$3^2 + 4^2 = 5^2,$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2 .$$

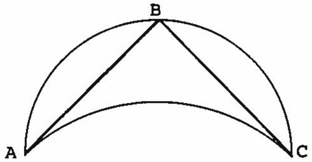
Generalize the above by finding integers a satisfying

$$\sum_{k=0}^{m} (a+k)^2 = \sum_{k=m+1}^{2m} (a+k)^2 .$$

180. Proposed by R. C. GEBHART, Parsippany, N. J.

In the figure, AB = BC and $\angle ABC = 90$. The arcs are both circular with the inner one being tangent to $\overline{AB}$ at A and $\overline{BC}$ at C.

Determine the area of the crescent.



181. Proposed by DONALD W. CROWE, University of Wisconsin and M. S. KLAMKIN, Ford Scientific Laboratory.

Determine a convex curve circumscribing a given triangle such that

1. The areas of the four regions (3-segments and a triangle) formed are equal and
2. The curve has a minimum perimeter.

SOLUTIONS

165. Proposed by D. J. NEWMAN, Yeshiva University.

Express cos θ as a rational function of $\sin^3 0$ and $\cos^3 θ$.

Solution by R. C. Gebhart, Parsippany, N. J.

Since

1. $4 \cos^3 0 = 3 \cos 9 + \cos 3θ$,
2. $256 \cos^3 9 \sin^6 θ = 6 \cos θ - 8 \cos 3θ + 3 \cos 79 - \cos 98$,
3. $64 \cos 9 \sin^6 θ = 5 \cos 9 - 9 \cos 3θ + 5 \cos 5θ - \cos 7θ$,
4. $256 \cos^9 θ = 126 \cos 9 + 84 \cos 39 + 36 \cos 59 + 9 \cos 79 + \cos 9θ$,
5. $64 \cos θ \cos^6 0 = 35 \cos 9 + 21 \cos 30 + 7 \cos 59 + \cos 79$,

cos 9 is a linear combination of the left-hand sides of equations 1-5, i.e., $\cos A \equiv A \cos^3 θ + B \cos^3 θ \sin^6 9 + C \cos θ \sin^8 θ + D \cos^9 θ + E \cos θ \cos^6 θ$.

It then follows that

$$A = 2,\ B = C = D = 1, \text{ and } E = -2.$$

Whence,

$$\cos θ = \frac{2 \cos^3 9 + (\cos^3 θ)(\sin^3 θ)^2 + (\cos^3 θ)^3}{1 - (\sin^3 θ)^2 + 2(\cos^3 θ)^2}$$

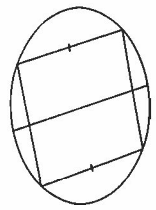
Also solved by K. S. Murray, P. Myers, and F. Zetto.

167. Proposed by M. S. KLAMKIN, Ford Scientific Laboratory.

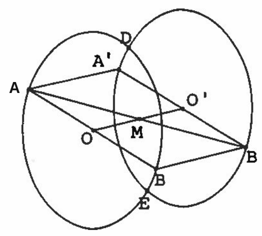
Given a centrosymmetric strictly convex figure and an intersecting translation of it; show that there is only one common chord and that this chord is mutually bisected by the segment joining the centers.

Solution by the proposer.

If the two ovals intersect in more than two points, then each oval would contain at least three parallel and equal length chords. The length and direction being the length and direction of the translation. This is impossible since the middle chord must be larger than the outer equal ones by strict convexity.



If A and B denote any pair of centrosymmetric points of the first oval and A' and B' the corresponding points of the translated oval then AA'B'B is a parallelogram. It then follows that the entire figure (both ovals) is centrosymmetric about point M, the midpoint of the segment joining the two centers. Since D and E are the only two double points, they must correspond to each other, i.e., be centrosymmetric with respect to M. Note that this generalizes the well known result for two intersecting congruent circles.



Also solved by M. Wagner.

168*. **Proposed** by *JERRY* TOWER, North High School (student), Columbus, Ohio.

Determine x asymptotically if

$$\log x = n \log \log x.$$

Solution by Sydney Spital, California State Polytechnic College.

The given equation may be rewritten as

(1) $$x^{1/n} = \log x$$

whose limiting solution is easily seen to be $x = e$. To determine x asymptotically, we let $x = e(1 + \delta)$ and assume (to be justified subsequently) that $\delta = O(\frac{1}{n})$. Substituting back into (1), yields

$$e^{1/n}\{1 + O(\tfrac{1}{n^2})\} = 1 + \delta + O(\tfrac{1}{n^2}).$$

Thus,

$$\delta = e^{1/n} - 1 + O(\tfrac{1}{n^2}).$$

Since $e^{1/n} = 1 + O(\frac{1}{n})$, our previous assumption is justified and the desired asymptotic behavior is given by

$$x = e^{1+1/n} + O(\tfrac{1}{n^2}) = e(1+1/n) + O(\tfrac{1}{n^2}).$$

Also solved by Paul J. Campbell, K. S. Murray and F. Zetto.

169. Proposed by JOE KONHAUSER, University of Minnesota.

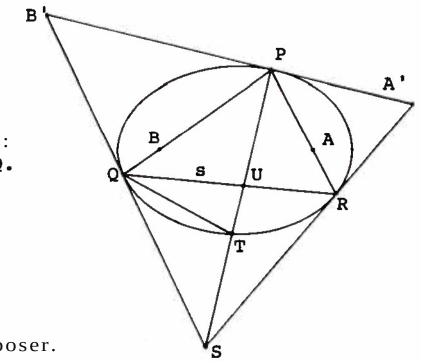
From an arbitrary point P (not a vertex) of an ellipse, lines are drawn through the foci intersecting the ellipse in points Q and R. Prove that the line joining P to the point of intersection of the tangents to the ellipse at Q and R is the normal to the ellipse at P.

Editorial Note: The proposer notes that he does not know the source of the problem and he has not been able to locate it in any of the books he has examined.

Solution by S. Schuster, University of Minnesota.

$s = QR$ is the polar of S. Then $H(PT,US)$; and $H(QP,QT;QU,QS)$ is a harmonic set of lines. The projectivity (defined by the conic) between the lines on Q and the lines on P makes the correspondences: $QP \to PB'$, $QT \to PT$, $QU \to PR$, $QS \to PQ$. $\therefore H(PB',PT;PR,PQ)$. But since $\angle A'PQ \cong \angle B'PR$, we have $PT \perp PB'$. Other interesting properties of the figure are:

$BB' \perp PQ$, $AA' \perp PR$,
$A'B \perp BS$ and $B'A \perp AS$.



Also solved by Paul Myers and the proposer.

Editorial Note: It would be of interest to supply a nonprojective proof.

170. Proposed by C. S. VENKATARAMAN, Sree Kerala Varma College, India.

Prove that a triangle ABC is isosceles or right-angled if

(1) $$a^3 \cos A + b^3 \cos B = abc.$$

The solutions by Paul J. Campbell, University of Dayton, and Marilyn Mantel, University of Nebraska were essentially identical and are given as follows:

Replacing cos A and cos B by the law of cosines, i.e.,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

(1) reduces to (after some algebraic manipulations)

$$(a^2 + b^2 - c^2)(a^2 - b^2)^2 = 0,$$

which implies the desired result.

Sydney Spital, California State Polytechnic College and the proposer use the projection formula

$$a \cos B + b \cos A = c$$

to obtain

$$(a^2 - b^2)(b \cos B - a \cos A) = 0$$

which implies the desired result.

Also solved by H. Kaye, P. Myers, M. Wagner and F. Zetto.

171. Proposed by MURRAY S. KLAMKIN, Ford Scientific Laboratory.

For $0 < \theta \leq \pi/2$, it is well known that the inequality

$$\frac{\sin \theta}{\theta} > \cos^m \theta$$

holds for $m = 1$. What is the smallest constant m for which it holds?

Solution by the proposer.

It follows from the power series expansions that $m \geq \frac{1}{3}$. We now show that it suffices to have $m = \frac{1}{3}$, i.e.,

(1) $$\sin^2 \theta \tan \theta > \theta^3 .$$

Since the l.h.s. and the r.h.s. of (1) are equal at $\theta = 0$, it suffices to show that the derivative of the l.h.s. is greater than the derivative of the r.h.s. of (1), i.e.,

(2) $$2 \sin^2 \theta + \tan^2 \theta > 3\theta^2 .$$

Similarly to establish (2), it suffices to show

(3) $$4 \sin \theta \cos \theta + 2 \tan \theta \sec^2 \theta > 6\theta ,$$

or

(4) $$4 \cos 2\theta + 2 \sec^4 \theta + 4 \tan^2 \theta \sec^2 \theta > 6 .$$

(4) follows since it can be rewritten as

$$(\sec^4 \theta)(\sin^4 \theta)(3 + 4 \cos^2 \theta) > 0 .$$

Sydney Spital, California State Polytechnic College, establishes the above result by showing that

$$m(\theta) = \frac{\log (\theta \csc \theta)}{\log \sec \theta}$$

is monotone decreasing on $(0, \pi/2]$ and then the desired minimum value of m is given by

$$\lim_{\theta \to 0} m(\theta) = \frac{1}{3} .$$

Also solved by Paul J. Campbell, H. Kaye, Paul Myers, and Ricky Pollack.

BOOK REVIEWS

Edited by
Roy B. Deal, Oklahoma State University

General Theory of Functions and Integration. By Angus E. Taylor. New York, Blaisdell Publishing Company, 1965. xvi + 437 pp., $12.50.

A rather thorough self-contained treatment of certain aspects of analysis, basically concerned with the development of Lebesgue measure, then the Lebesque integral, followed by treatment of the Daniell integral and then measure, establishing the inter-relatedness of the two, and providing the necessary topological background in the beginning with many interesting related concepts such as signed measures, L^p spaces, vector valued functions of bounded variation, and Stieltjes integrals developed throughout.

Prelude to Analysis. By Paul C. Rosenbloom and Seymour Schuster. Englewood Cliffs, New Jersey, Prentice-Hall, 1966. xix + 473 pp.

A unique textbook in which a vast number of important mathematical concepts are skillfully woven into pre-calculus material, using a variety of pedagogical techniques to establish these concepts at an elementary level. It does not contain trigonometry, however, as the authors feel that this should be developed by the use of calculus.

A Concept of Limits. By Donald W. Hight. Englewood Cliffs, New Jersey, Prentice-Hall, 1966. xii + 138 pp., $3.95.

A brief but rather complete treatment of the elementary theorems of limits of real sequences and real functions with some history and many examples and exercises designed to lead the reader with a high school algebra and trigonometric background to a thorough understanding of the fundamental concepts.

Linear Statistical Inference and its Applications. By C. Radhakrishna Rao. New York, John Wiley, 1965. xviii + 522 pp., $14.95.

Perhaps the most comprehensive modern treatment of statistical theory and methods available, treating analysis of variance, estimation, large sample theory, statistical inference, multivariate analysis with many facets of the modern fundamental concepts of each, and a rather thorough treatment of mathematical prerequisites such as vector spaces, including Hilbert Spaces, probability and measure theory, and necessary excerpts from analysis. It presupposes a mathematical maturity at about the advanced calculus level and some knowledge of elementary statistical theory and methods.

Vector Analysis. By Barry Spain. Princeton, New Jersey, Van Nostrand, 1965. ix + 114 pp., $5.95.

A brief but thorough treatment of the mathematics of classical vector analysis from a classical point of view with a brief introduction to classical tensor analysis, omitting all applications.

Topological Structures. By Wolfgang J. Thron. New York, Holt, Rinehart and Winston, Inc., 1966. x + 240 pp.

A modern self-contained treatment of general topology, with historical and motivation remarks, which proceeds from a very elementary introduction to some rather difficult advanced theorems in twenty-three brief chapters. Although the book is brief, there is an excellent bibliography and enough difficult but important theorems and exercises to warrant a year's study. In addition to the classical topics of general topology, there are modern treatments of quotient spaces, nets, filters, uniform spaces, proximity spaces, topological groups, and paracompactness and metrization.

Introduction to Basic Fortran Programming and ~~Numerical Methods.~~ By William Prager.

Based on lecture notes prepared for a one-semester course for students of applied mathematics with limited mathematical background, covering Fortran programming with applications to computing with polynomials, interpolation, quadrature, solution of equations and integration of ordinary differential equations.

Albert Einstein and the Cosmic World Order. By Cornelius Lanczos. New York, Wiley-Interscience, 1965. vi + 139 pp.

An outgrowth of a public lecture series at the University of Michigan in the spring of 1962 called "The Place of Albert Einstein in the History Of Physics," written by one of the outstanding mathematics expositors and a former associate of Einstein.

Modular Arithmetic. By Burton W. Jones.

An excellent book for secondary school teachers, outstanding high school students, or for college students who might wish to use it as an elementary secondary reference on modular systems.

BOOKS RECEIVED FOR REVIEW

Henry E. Fettis and James C. Caslin. Ten Place Tables of the Jacobian Elliptic Functions. Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, 1965. iv + 562 pp.

Fisher and Ziebur. Calculus and Analytic Geometry. Englewood Cliffs, New Jersey, Prentice-Hall, 1965. xiii + 768 pp.

John N. Fujii. Numbers and Arithmetic. New York, Blaisdell Publishing Company, 1965. xi + 559 pp., $8.50.

Einar Hille. Analytic Function Theory, Volume 11. New York, Ginn and Company, 1962. xii + 496 pp.

J. Nagata. Modern Dimension Theory. New York, John Wiley, 1965. 259 pp., $11.00.

L. B. Rall, editor. Error in Digital Computation, Voiume II. New York, John Wiley, 1965. x + 288 pp., $6.75.

Seaton E. Smith, Jr. Explorations in Elementary Mathematics. Englewood Cliffs, New Jersey, Prentice-Hall, 1966. xv + 280 pp., $5.95.

Hans F. Weinberger. A First Course in Partial Differential Equations with Complex Variables Transform Methods. New York, Blaisdell Publishing Company, 1965. ix + 446 pp.

Herbert S. Wilf. calculus and Linear Algebra. New York, Harcourt, Brace and World, Inc., 1966. xiv + 408 pp., $8.50.

NOTE: All correspondence concerning reviews and all books for review should be sent to PROFESSOR ROY B. DEAL, DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA, 74075.

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