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One of the standard methods for finding the solution of equations \( f(x) = 0 \) is the "cobweb" or "spiral staircase" method of iteration. This method works fairly well if conditions are "just right;" however, if these conditions are not properly satisfied, the method is uneconomical to apply or may even fail to converge to a root.

Using the fact that a straight line approximates a curve over a small interval, a variation of the method of iteration can be derived which eliminates many of the conditions which must be satisfied for the standard method. The new method assumes that the curve is nearly linear over a small interval and can be approximated by a secant line over this interval.

Before discussing the method of secants it is best that we give a short summary of the standard method of iteration. Let us assume that we have an equation \( f(x) = 0 \) with real roots. This equation can be expressed as:

\[
g(x) = h(x) \quad \text{where} \quad f(x) = g(x) - h(x).
\]

We next form the two equations:

\[
y = g(x) \quad \text{and} \quad y = h(x).
\]

The value of \( x \) at the intersection of the graphs of these two equations is the value of \( x \) that makes \( f(x) = 0 \) in the original equation. (Fig. 1)

Now the problem becomes a problem of finding the solution to the simultaneous equations (1).

Let us now assume that we have an approximation \( a \) to the root of the equation \( f(x) = 0 \). We now wish to improve this approximation. Our first step in refining this approximation is to solve one of the equations (1) for \( x \). Solving \( y = h(x) \) for \( x \) yields

\[
x = H(y).
\]

We now have the two equations:

\[
y = g(x),
\]

and

\[
x = H(y).
\]

We then substitute our approximation \( a \) into (2) and determine \( y_1 \) such that \( y_1 = g(a) \). (Fig. 2) This value of \( y \) is then substituted into (3) and a new value of \( x \) is determined. This new value of \( x \) is again substituted into (2) and the process is continued until two successive values of \( x \) are within the desired tolerance.

This method works very well for many cases; however, unless special care is taken in selection of the equations (1), it also fails for many cases. It is quite possible to choose these two equations in such a
manner that convergence will occur at a slow rate (Fig. 3) or will not occur at all. (Fig. 4) Care must also be exercised when deciding which one of the equations (1) is to be solved for $x$. If the wrong equation is chosen, divergence will occur. (Fig. 5)

The following method is a variation of the method of iteration which forces convergence to occur much more rapidly and also finds roots of many equations that the normal iterative method fails to determine.

The method begins to differ from the normal iterative method explained above at the point where one of the equations (1) is solved for $x$. With this new method either equation (1) can be solved for $x$ and convergence will occur. Solving $y = h(x)$ for $x$ yields:

$$x = H(y),$$

(4)

and

$$x = y = g(x),$$

(5)

We shall refer to the graph of (4) as curve 1 and the graph of (5) as curve 2.

Substituting our initial approximation $a$ into our original equations $y = g(x)$ and $y = h(x)$, we get two points: $C(a, g(a))$ on curve 1 and $E(a, h(a))$ on curve 2. (Fig. 6) Substituting $g(a)$ into (5) yields a point $F(j, g(a))$ on curve 2, where $j = H(g(a))$. One more substitution of $j$ into (4) yields a point $D(j, g(j))$ on curve 1.

One can easily see that the slope of $CD$ is:

$$M_4 = \frac{g(j) - g(a)}{j - a}$$

The slope of $EF$ is:

$$f_0 = \frac{h(j) - h(a)}{j - a}$$

Consider a portion of Fig. 6. (Fig. 7) Other expressions for slopes $M_5$ and $M_6$ are:

$$M_5 = \frac{z}{-x}, \quad M_6 = \frac{z}{y}$$

or

$$z = -M_5 x, \quad z = (fay)$$

$$-M_5 x = (fay).$$

Adding $M_6 x$ to both sides yields:

$$(fay - M_5 x = M_6 y + (fay)$$

$$(M_5 - M_6) x = M_6 (y + x)$$

$$\frac{x}{x + y} = M_6 - M_5.$$

Now $\frac{x}{x + y}$ is the fraction of the distance from $C$ to $F$ where the perpendicular from $0$ is drawn. This ratio will tell us how far along the line $CF$ we should "slide" to take our next approximation for the root. Since the distance from $C$ to $F$ is the same as the distance from $a$ to $j$, our next approximation $a'$ will be:

$$a' = a + \frac{M_6}{M_5 - M_6} (j - a).$$

(6)

We then repeat the process by using $a'$ to find $g(a')$ which is used to find $H(g(a'))$ which we can again call $j$. These results can then be substituted back into (6) to yield a new $a'$. This process can be expressed more compactly as:

$$a_{i+1} = a_i + \frac{M_6}{M_5 - M_6} (H(g(a_i)) - a_i).$$

The ratio $\frac{M_6}{M_5 - M_6}$ need not be calculated for every iteration; however, convergence will occur in fewer steps if the ratio is recalculated for each iteration. When the difference $|H(g(a_i)) - a_i|$ becomes sufficiently close to zero, we know that we have found the root.

We have observed only the case where the slopes are of opposite sign in the foregoing discussion. When the slopes are both positive, we have...
a graph somewhat like Fig. 8. In this case the ratio \( \frac{M_b}{B_a} \) is a fraction with value greater than one so that when the difference \( (j - a) \) is multiplied by the ratio and added to \( a \), our new approximation has a value greater than \( j \). A similar argument holds for the case where both slopes are negative.

Several comparison tests were run between the standard method of iteration and the method of secants variation of the standard method. Using the FORTRAN language and an IBM 1620 computer for the calculations, roots accurate to 8 decimal digits were sought for the equations \( x^2 - x - 30 = 0 \) and \( x^2 - e^x = 0 \).

The methods compared as follows:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Standard Method</th>
<th>Secant Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - x - 30 = 0 )</td>
<td>6.00000000</td>
<td>6.00000000</td>
</tr>
<tr>
<td>( g(x) = 6/x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h(x) = (x-1)/5 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Root</td>
<td>9.00000000</td>
<td>9.00000000</td>
</tr>
<tr>
<td>Initial approximation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of iterations</td>
<td>83</td>
<td>5</td>
</tr>
</tbody>
</table>

In every case tested the method of secants converged more rapidly than did the standard method; however, special cases can be devised where this probably will not hold true. In general though, it appears that the method of secants is a definite improvement over the standard "cobweb" method of iteration.
We see that the sub-system defined by the first set of "operation tables" as a line segment having range $[0, p]$ and passing through the points $(0,0)$ and $(1, p)$.

Similarly, the line segment $p \oplus x$ can also be characterized as the one originating from the vertex $(1, 1)$ of the unit square and terminating at a height $p$ on the opposite vertical side.

Let $p(x) = (1 - p) \cdot x + p \cdot x \in [0, 1]$ can be interpreted graphically as a line segment having range $[0, p]$ and passing through the points $(0,0)$ and $(1, p)$.

$\oplus (1, 0)$ $\triangleright \triangleright (0, 0)$ $\oplus (1, 0)$ $\triangleright \triangleright (0, 0)$

$\ominus (0, 0)$ $\triangleleft \triangleleft (0, 0)$ $\ominus (0, 0)$ $\triangleleft \triangleleft (0, 0)$

We see that the line segment can also be characterized as the one originating from the vertex $(0, 0)$ of the unit square $\{(0,0), (1,0), (1,1), (0,1)\}$ and terminating at a height $p$ on the opposite vertical side.

Let $P = \{(0,0), (1,0), (1,1), (0,1)\}$.

Similarly, the line segment $p \ominus x$ can also be characterized as the one originating from the vertex $(1, 1)$ of the unit square and terminating at a height $p$ on the opposite vertical side.

Where $p$ is not held constant but allowed to vary over $U$, $p \oplus x$ and $p \ominus x$ may be interpreted as families of line segments in the plane— as illustrated below.

### III. Properties of $(U, \oplus, \ominus)$

Let $p, q \in U$, i.e., $0 \leq p \leq 1$ and $0 \leq q \leq 1$.

- $\ominus$ is closed under $\ominus$.
- $1 \cdot x = 1 - p \geq 0$.
- $p \oplus u = \ominus p \in U$.
- $0 \leq q(1 - p) \leq 1 - p$.
- $0 \leq p \leq p + q(1 - p) \leq (1 - p) + p = 1$.
- $\ominus y$ is closed under $\ominus$.

\[ x \ominus p = x(1 - p) \]

### Theorem 1

Let $P = \{(0,0), (1,0), (1,1), (0,1)\}$.

Let $A = p$, $B = q$, $C = q$, $D = q$, $E = p$.

We recognize $BC$ as $p \ominus q$ and $DC$ as $p \ominus q$. Since $OEA' = p$ and $OAB = q$, we have $E' - A' = p$ and $E' - q$.

Let $\angle A' = \angle A' = \angle A' = \angle A' = \angle A'$.

Let $\angle A' = \angle A' = \angle A' = \angle A' = \angle A'$.

Let $E = O'B' = O'B'$.

Let $E = q$, $O'B' = q$. Now $\angle A' = \angle A' = \angle A'$.

Consider the function $y = f(x) = x$. Given any $y \in U$, $f: \mathbb{R} \to \mathbb{R}$; hence, $f$ is onto $U$. Let $x_1 = x_2$.

Let $x_1 = 1 - x_2$.

Let $x_1 = x_2$.

Let $f$ is one-one.

Theorem 2

Let $P = \{(0,0), (1,0), (1,1), (0,1)\}$.

Let $A = p$, $B = q$, $C = q$, $D = q$, $E = p$.

The theorem follows from the properties of multiplication of real numbers since $p \ominus q = pq$ by definition.

Theorem 3

Let $P = \{(0,0), (1,0), (1,1), (0,1)\}$.

Let $A = p$, $B = q$, $C = q$, $D = q$, $E = p$.

The theorem follows from the properties of multiplication of real numbers since $p \ominus q = pq$ by definition.
Moreover $x \approx \mathbb{T}$ is an isomorphism from $(U, \circ)$ onto $(U, \odot)$ since $f(p \odot q) = f(p) \odot f(q)$.

**Theorem 4.** $\forall p, q, r$. $p \odot q = q \odot p$. $(p \odot q) \odot r = p \odot (q \odot r)$. $1 \odot p = 1$, $0 \odot p = p$.

Proof. The theorem follows immediately from Theorem 2 since $x \approx \mathbb{T}$ is an isomorphism from $(U, \circ)$ onto $(U, \odot)$.

**Theorem 5.**

A. $p \odot q = p \odot q$  
B. $p \odot q \odot r \odot \cdots \odot q = p \odot q \odot r \odot \cdots \odot q$  
C. $p \odot q \odot r \odot \cdots \odot q = p \odot q \odot r \odot \cdots \odot q$

Proof. A. Let $p = \mathbb{T}$, $q = \mathbb{T}$. From Theorem 3. $p \odot q = p \odot q$.

IV. Other Properties.

"#" is to be read: "not identically equal to."

V. From Another Viewpoint.

Another truth-functional connective is defined by the table below.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \odot Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

It is, perhaps, an unwieldy ingredient in the business of evaluating the truth-values of compound statements, but it is not at all artificial for it represents the exclusive sense of the word "or". Since all formulas of the propositional calculus can be expressed in the terms of conjunction and
negation alone. They can certainly be expressed in terms of "exclusive
disjunction", conjunction, and negation. Again we may consider \( \mathcal{R} \) and
"A" as operatives defined on \( \mathcal{T} \) and \( \mathcal{P} \) and consider "\( \mathcal{R} \)" as a function
of these truth-values. Consider the tables below.

<table>
<thead>
<tr>
<th>( \mathcal{R} )</th>
<th>( \mathcal{P} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{R} )</td>
<td>( \mathcal{P} )</td>
</tr>
<tr>
<td>( \mathcal{P} )</td>
<td>( \mathcal{R} )</td>
</tr>
</tbody>
</table>

\( \mathcal{I}(2) \)

We see that the truth-values under the operations \( \mathcal{R} \) and \( \mathcal{P} \) are isomorphic
to the integers modulo 2 and also to themselves under the operations "\( + \)"
and "\( \cdot \)" where "\( - \)" is the biconditional. Since \( \mathcal{I}(2) \) is a Boolean ring with
unit, the others are also.

We may make the following definitions of operations on \( \mathcal{U} \):

\[ p \oplus q = p + q - 2pq \]
\[ p \odot q = p - q + 2pq \]

I shall state without proof some of the properties of \( (\mathcal{U}, \oplus, \odot) \) and
\( (\mathcal{U}, \oplus, \odot) \). \( \mathcal{U} \) is closed with respect to all four operations.

Theorem. \( p \oplus q = \neg \odot q \quad \text{and} \quad p \odot q = \neg \oplus q \).

Now we see that \( x \) is an isomorphism from \( (\mathcal{U}, \oplus, \odot) \) onto
\( (\mathcal{U}, \oplus, \odot) \), and vice-versa. Hence, we need explore the properties of only
\( (\mathcal{U}, \oplus, \odot) \).

Theorem. A. In general inverses do not exist.

B. \( \odot \) does not distribute over \( \oplus \).

C. \( p \odot p \neq 0 \).

In the same way that interpretations of \( p \oplus q \) and \( p \odot q \) are given,
interpretations can be given to \( p \oplus q \) and \( p \odot q \). At the endpoints of
\( (\mathcal{U}, \oplus, \odot) \) we have the operation tables below:

\[
\begin{array}{c|cc}
\oplus & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad \begin{array}{c|cc}
\odot & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

This sub-system is isomorphic to \( \mathcal{I}(2) \) and the other two rings.

I should like to express my thanks to Dr. Leon Steinberg for his help
and encouragement.

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2. W. V. Quine, Mathematical Logic, Harper and Row, New York and Evanston,
1962.


4. N. H. McCoy, Introduction to Modern Algebra, Allyn and Bacon, Boston,
1960.
Thousands of years ago mathematics was simply an ordinary occupation. When a person said he was a mathematician, no one shuddered, or stuck his fingers in his ears, or gave him a look of thorough distaste. Rather, this announcement was met with giggles, snickers, and even chortles. Why should the people not have laughed? All that mathematicians did was to sit around drawing circles and triangles, muttering “\(\alpha\) squared plus \(\beta\) squared equals \(\gamma\) squared.” It hardly seemed an appropriate pastime for the grossly retarded. Naturally, mathematicians felt they were fully competent, even more severely struck and became schizophrenic. Of course, their work was hampered by their lack of mental stability, and mathematics went into a catastrophic decline, causing it to fall into even greater disrepute than before.

This was almost the end of the field of mathematics.

However, one of the saner mathematicians, in a lucid moment, realized that mathematics was doomed unless this group of pathetic psychotics received immediate help. So he took the group to the nearest accredited psychotherapist to see if therapy could not help them.

Fortunately, the therapist was able to help these wretches. He immediately put them in group therapy, and it was here that mathematics was actually saved. After a few sessions, they had gained insight enough into their problems to consider a solution. Their problem (ignoring the extraneous factors of incorrect toilet training, Oedipus complexes, and such) was that they were suffering from deep insecurity and thwarted snobbery. Their profession certainly gave them no prestige; they had either to change jobs or do something to mathematics.

Thanks to the psychotherapist, they were able to devise a way to save their beloved profession. He had talked to them at one of the sessions (about their improvement, historians think), and no one understood a syllable. Hot damn! If only they could manage to garble their speech in the same way, they could cure their problems!

At the very next session they began to work on what was to become a most successful jargon. Since they could not use jargon for the existing concepts, they decided to start branching out, making up anything which seemed reasonable for now they would have jargon to hide any inconsistencies. The few who had not yet recovered from their psychoses were put to work creating the new jargon since they spent most of their time babbling nonsense anyway.

Of course, a plan of such divine simplicity worked marvelously. One needs only to look at the work done in a small section of plane analytic geometry to realize the genius of these men. A retired butcher had found his way into mathematics. Having been a butcher all his life, he kept in practice with his cleaver by attacking wooden cones with it. One particular day, one of his words to use was “conic.” “Cone,” “conic”: there was a resemblance, so he created the conic sections (curves created by slicing a cone in various ways). He was quite inventive, and he used his day’s quota and requested three additional lists. (This won for him a solid gold cleaver and the steers of his choice.)

Out of the conic sections came the hyperbola, parabola, circle, ellipse, a straight line, a pair of straight lines, and a point. All of these figures had common properties or properties common to at least some. The hyperbola has all the qualities—axes of symmetry, one being conjugate, the other transverse; two branches; two asymptotes: a latus rectum for either branch; and eccentricity. It also has two ridiculous definitions, neither of which makes much sense. How often does a person feel compelled to talk about the locus of a point whose distance from two fixed points has a fixed difference? But the jargon was working. The definitions and properties sounded impressive, perhaps made sense to a few, and baffled everyone else.

The men in calculus had a problem different from that in plane analytic geometry. They had been assigned to this branch because they had no ability at all in drawing. All sections of geometry had refused to take them, and these unartistic men were trying to create some form of mathematics which required no drawing. Weeks went by, and still they had nothing to call calculus but the name on their door. When they were asked to double as the janitorial staff, they decided they must attack the problem differently.

Since none of the men could conceive of doing any mathematics without a picture, they decided to borrow sketches from other fields. Perhaps if these blighted artists could add a new line or two and give a reason for doing so, they could claim that they had finally invented calculus. Anyone can draw a line with a ruler, so they tried it. They even found something to describe what they had done: they had taken the derivative of the equation of the figure and shown its geometric interpretation.

Now that they had started, they were creating volumes of work. The original derivative was called the first derivative because they had found second and third derivatives. Naturally, these were not simply first or second derivatives but first or second derivatives taken with respect to a given variable. The derivative was obtained by differentiation, not by derivation. The mathematicians even founded a concept which they named “the partial derivative,” taken with respect not to a partial variable but simply a given variable.
The men were so excited about actually creating calculus that they
turned in their mops permanently and had a celebration. Feeling sublimely
daring from all the wine, these dedicated mathematicians boldly created a
concept even more ludicrous than that of differentiation—antidifferentiation. When they were sober, they realized that they had committed a ghastly error
by simply putting a negating prefix in front of the name of the former con-
cept and changed the latter’s name to integration. Since they had been able
to differentiate all sorts of functions—algebraic, logarithmic, trigono-
metric—they might integrate these functions. When they integrated they had
an integral, but they had all types of integrals. There was the definite
integral, the indefinite integral, also the improper, the iterated single,
the double, and the triple integral. They even claimed that they could find
areas, volumes, first moments, moments of inertia, and work by integration.

Work in all branches of mathematics was as successful as it was in plane
analytic geometry and calculus. The jargon was created, and the resourceful
mathematicians were able to find concepts for it. Much of the vernacular is quite impressive today because it is either utterly misleading or means
absolutely nothing unless the listener has a Master’s degree in mathematics.

A logarithm sounds like a pulsating piece of felled tree, the deleted neigh-
bordhood a problem for the NAACP. Who was the witch of Agnesi? Are you
a surd? What is Lipschitz’s condition? After a person casts out nines, does
he have a Baire function? Is society safe from the standard deviate? Do
shrinking and stretching transformations hurt? Is the sheet of a surface usable on a bed? Is the class called Life Drawing a person’s first Baire
class? Do amicable numbers love each other? Will the annihilator get you?

It is quite apparent that jargon has absolutely solved the problem of the ancient mathematicians. Since they unleashed their powerful vernacular
on the public, the guffaws and chortles have been replaced by awful sighs
and reverent silence. Who now would dare to laugh when a mathematician
opens his mouth? Who understands but the smallest part of what a mathema-
tician utters? Who listens?

ELEMENTARY MATRICES EXPRESSED IN TERMS OF THE
KRONECKER DELTA

Mark E. Christie, Bowdoin College

The Kronecker Delta is defined as: \( \delta_{ij} = 1 \) if \( i = j \), \( \delta_{ij} = 0 \) if \( i \neq j \). It is possible to express an elementary matrix entirely in terms of the
Kronecker Delta, using it to represent the elements of the matrix under con-
sideration. The process of finding the precise K-Delta expression for a
given elementary matrix is essentially one of trial and error. It is known
that the statement must contain the general term \( \delta_{ij} \) and other terms to
express any non-zero off diagonal elements and any diagonal elements equal
to zero.

By way of example, consider the elementary matrix \( P_{pq} \) defined as an
n x n matrix such that every element on the principle diagonal is 1 except
for the \( p^{th} \) and \( q^{th} \) rows. In these rows the diagonal element equals zero,
while the element in the \( p^{th} \) row and \( q^{th} \) column equals 1, as does that in
the \( q^{th} \) row and \( p^{th} \) column. All other off diagonal elements equal zero.

Consider the following expression written in terms of K-Deltas:

\[ P_{pq} = [\delta_{ij} + (\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})] \]

To demonstrate that the above statement is true, it is necessary to show
the following three properties: (1) that one can express any \( P_{pq} \) by the expression; (2) that \( P_{pq} P_{pq} = I - \delta_{ij} \); (3) that premultiplication
of a matrix \( A \) by \( P_{pq} \) (in the K-Delta form) does change the \( p^{th} \) and \( q^{th} \) rows of \( A \).

(1) Consider \( P_{pq} = [\delta_{ij} + (\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})] \);

if \( i \neq p \) or \( q \), then the expression equals \( \delta_{ij} \),
if \( i = p \), \( P_{pq} = [\delta_{iq} + (0 - 1)(\delta_{pj} - \delta_{qj})] = \delta_{qj} \),
if \( i = q \), \( P_{pq} = [\delta_{iq} + (1 - 0)(\delta_{pj} - \delta_{qj})] = \delta_{pj} \).

It is clear that our K-Delta form can express any \( P_{pq} \).

(2) Let \( P_{pq} P_{pq} = [\delta_{ij}] \delta_{ij} = \sum_{k=1}^{n} \delta_{ik} \delta_{kj} \)

\[ = \sum_{k=1}^{n} [\delta_{ik}(\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})] \delta_{kj} + (\delta_{iq} - \delta_{ip})(\delta_{pj} - \delta_{qj})] \delta_{kj} \]
Expanding the second term:

\[ \delta_{ik} (\delta_{ik} - \delta_{kj} - \delta_{kq} - \delta_{qj}) = (\delta_{ik} - \delta_{ip}) (\delta_{pj} - \delta_{qj}). \]

Clearly the expansion of the third term will yield the same result. Now factoring \((\delta_{iq} - \delta_{ip}) (\delta_{jp} - \delta_{qj})\) and expanding the last term:

\[ p_{ij} q_{ij} = (\delta_{ij} + (\delta_{ij} - \delta_{ip})(\delta_{pj} - \delta_{qj})(1 + 1 - \delta_{pq} - 1 + \delta_{pj})). \]

But \(\delta_{jq} - \delta_{qp} = 0\) since \(q \neq p\).

Therefore, \(P_{pq} P_{pq} = [\delta_{ij}]\).

(3) Using the notation of (2), we have:

\[ p_{ij} = \sum_{k=1}^{n} d_{ik} S_{kj} = \sum_{k=1}^{n} (\delta_{ik} - \delta_{ip})(\delta_{pk} - \delta_{qk}) S_{kj}, \]

if \(i \neq q \) or \( p, \) then \(\sum_{k=1}^{n} d_{ik} S_{kj} = [S_{ij}]; \)

if \(i = p, \) then \(\sum_{k=1}^{n} d_{ik} S_{kj} = [\delta_{pk} + (0-1)(\delta_{pk} - \delta_{qk})] S_{kj} = [a_{kj}], \)

if \(i = q, \) then \(\sum_{k=1}^{n} d_{ik} S_{kj} = [\delta_{qk} + (1-0)(\delta_{pk} - \delta_{qk})] S_{kj} = [a_{pj}]. \)

Thus we know that the K- Delta representation under consideration does indeed correctly express \(P_{ij}.\)
The "current symbol" is defined to be the symbol in the cell at the sensing device. Each cell is referred to as a "location." The numbered cells are referred to by their number, e.g., loc. 27; this is the 27th cell right of the origin. When we say that we move the tape to loc. N, we mean we position the tape so that loc. N is at the sensing device.

A specific set of instructions which the machine is asked to execute is called a "program," and the n instructions of a given program are consecutively numbered 1, 2, 3, through n.

Following Arden's [2] method, we will have only three basic instructions which the Turing machine can execute:

(a) If the current symbol is x, move the tape one cell to the left and execute instruction y. Otherwise, execute the next instruction. This instruction is denoted symbolically IFL x, y where x is a symbol and y is an instruction number.

(b) If the current symbol is x, move the tape one cell to the right and execute instruction y. Otherwise, execute the next instruction. This instruction is denoted symbolically IFR x, y where x is a symbol and y is an instruction number.

(c) If the current symbol is x, replace it with z, then execute the next instruction. Otherwise, just execute the next instruction. This instruction is denoted symbolically IFS x, z where x and z are symbols.

We have deviated slightly from A. M. Turing's [1] and subsequent authors' definition of a Turing machine in order to make its analogy with the modern day computer more apparent: of course, the machine described above is not a stored-program computer. For example, Turing does not number his cells, thus he has no "locations."

A. M. Turing presents an argument in [1] that a machine such as described above is capable of performing any calculation that a computer is able to perform. So, let us see how we would add two four-digit binary numbers together. To do this with only the three basic instructions requires one hundred or more instructions and is very difficult to follow. To avoid this, we introduce "macro instructions." Each macro instruction must be defined by a program of previously defined macro instructions or basic instructions.

We will now list the macro instructions and give their defining programs in a few cases:

(a) IFL , y Move the tape to the left one location and then execute instruction y.
1. IFL 0, y
2. IFL 1, y
3. IFL 2, y
4. IFL 3, y

(b) IFR , y Move the tape to the right one location and then execute instructions y.
1. IFR 0, y
2. IFR 1, y
3. IFR 2, y
4. IFR 3, y

(c) TRA , y Transfer to instruction y. (Execute instruction y next.)
1. TRA 0, y
2. TRA 1, y
3. TRA 2, y
4. TRA 3, y

(d) HALT The machine halts. (The state of the machine remains the same indefinitely without affecting the tape; used to stop the machine, e.g., at the end of the program.)
1. TRA 0
2. TRA 1
3. TRA 2
4. TRA 3

(e) IFS , x Replace the current symbol with x and then execute the next instruction. (Note, the current symbol can not be Z or E and x can not be 7 or E.)
1. IFS 0, x
2. IFS 1, x
3. IFS 2, x

(f) MTR N, y Move the tape to the right N locations and then execute instruction y.
1. IFR 0, y
2. IFR 1, y
3. IFR 2, y
4. IFR 3, y

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We shall explain this program briefly. First we move the tape to the AC and test to see if there is a 0 there by instructions 1 and 2. If there is a 0 there, the left-hand column of the table is pertinent and we wish to leave the 0 in the AC and leave whatever symbol is in loc. N there. So we merely move the tape to the origin and execute instruction y (by executing Instruction 6). If there is not a 0 there, we go on to instructions 3 and 4 which move the tape to loc. N and test to see if there is a 0 in loc. N. If there is, we wish to put a 0 into the AC and a 1 into loc. N. Instructions 7, 8, and 5, operating in that order, accomplish this. If there is not a 0 in the AC, we wish to put a 1 into the AC and a 0 into loc. N. Instruction 5 alone accomplishes this since there is already a 1 in the AC.

We will now introduce three instructions which give us the body of our addition program. The first instruction adds the contents of the AC and the contents of a location. The second instruction adds the contents of two locations. The third is a combination of the first two and is the only instruction of the three which actually appears in the addition program.

(a) ADA N,y

Let x be the symbol in loc. N and let y be the symbol in the AC. Add x and y, write the result as a double bit sum obtained into the AC and put the right-hand bit of the double bit sum obtained into loc. N. Leave the tape positioned at the origin and execute instruction y.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>1</td>
<td>01</td>
<td>10</td>
</tr>
</tbody>
</table>

\{ double bit sums \}

The dots in instruction 9 indicate that the next instruction is to be executed.

Again we will briefly explain the program. The first two instructions move the tape to loc. N and test for a 0. If there is a 0 there, the sum of the symbols in loc. N and loc. M is the symbol in loc. 8, so we merely move the tape to the origin and execute the next instruction (by executing Instruction 8). If there is a 1 in loc. M, we go on to instructions 3 and 4 which move the tape to loc. N and test for 0. If there is a 0 there, the sum would be 1, so we transfer to instruction 7 which puts a 1 into loc. N and then we move the tape to the origin and execute the next instruction (by executing Instruction 8). If there is no 0 in loc. N, we wish to put a 1 into the AC and a 0 into loc. N. To do this we go to instruction 5 which puts a 0 into loc. N, then instruction 6 brings the 1 in loc. N into the AC and we are finished.

It must be noted that except in the case where there is a 1 in both loc. N and loc. M, the AC was left unaltered.
(c) \text{ANM} N, M

Add the symbol in the AC, the symbol in loc. N, and the symbol in loc. M together. This sum is less than 4. Express it in a binary form (two bits), put the left-hand bit into the AC and the right-hand bit into loc. N. Leave the tape at the origin and execute the next instruction.

1. \text{ADA} N, 2
2. \text{ADN} N, M
3. ....

To do this we use the first two instructions. Let all sums be expressed as double bits. The first instruction adds the symbol in the AC and the symbol in loc. N and puts the right-hand bit of the sum into loc. M and the left-hand bit of the sum into the AC. The second instruction adds the right-hand bit of the previous sum now in loc. N and the symbol in loc. M and puts the right-hand bit of this sum (the sum of the symbol now in loc. N and the symbol in loc. M) into loc. N. If the left-hand bit of this sum is 0, it does nothing to the AC, thus leaving the left-hand bit (highest order bit) of the previous sum of the symbols in the AC and loc. N undisturbed (there is no carry for this sum). If, on the other hand, the left-hand bit of this sum (the sum of the symbol now in loc. N and the symbol in loc. M) is 1, it puts a 1 into the AC. In this case, the left-hand bit of the previous sum, that is the sum of the symbols in the AC and loc. N, is lost but it would have been 0 anyway because the total sum that is of the AC, loc. N, and loc. M, has to be less than 4.

The following program adds a number A and a number B where A is in locations N-3, N-2, N-1, N and B is in locations M-3, M-2, M-1, M and stores the results in N-4, N-3, N-2, N-1, N (where the highest order bit is in N-3 or M-3 for A and B respectively). Note, N and M must be \(N \geq 4\) and \(|M - N| > 4\).

1. \text{MTA} N, 2
2. \text{HFS} N, 0
3. \text{ANM} N, M
4. \text{AM} N-1, M-1
5. \text{ANM} N-2, M-2
6. \text{ANM} N-3, M-3
7. \text{STO} N-4, 8
8. \text{HALT}

Instructions 1 and 2 put a 0 into the AC since there is no initial carry. Instructions 3, 4, 5, and 6 add the ones' bit, twos' bit, fours' bit, and eights' bit of the binary numbers A and B and the previous carry together respectively putting the lowest order bit of each sum into locations N, N-1, N-2, and N-3 respectively, and putting the highest order bit of each sum, that is the carry, into the AC. Finally, instruction 7 puts the final carry into location N-4. Thus the sum of A and B appears in locations N-4, N-3, N-2, N-1, N. Note, by merely adding extra ANM's we could increase the maximum magnitude of A and B.

We have shown that a simple machine such as described above can perform addition of positive integers. Subtraction could also be handled easily by writing an instruction which would replace a number by its complement.

To subtract a number B from a number A, we would replace B by its complement and add it to A. For example, if we assume that we are working with a maximum of three-digit binary numbers, the complement of the binary number 0101 (5) is 1011 which is the first number, 0101, subtracted from binary 10000 (16). The assumption that we are only working with three-digit binary numbers is required since we must have a way of telling whether a number is a complement quantity or not. Note, the four-digit representation of binary five is also binary eleven. For this purpose we reserve the extreme left-hand bit of a four-bit word if it is a 1, the next three bits are a three-digit complement quantity; if it is a 0, the next three bits are a regular three-digit quantity. We could also put negative numbers on the tape in their complement form.

We have not mentioned multiplication or division or decimal quantities. Multiplication and division could be taken care of by still more complicated programs using addition and subtraction as developed above and a shift instruction. Decimal quantities could be handled by the manner in which the data was placed on the tape. That is, one could presume that the decimal point lies after the first ten places of all numbers.

Thus, it is reasonable to say that the Turing machine can perform any calculation that a computer can perform. But is a computer capable of calculating everything that a Turing machine is able to calculate?

Recently many authors including M. O. Rabin and D. Scott [3] have come to the conclusion that the Turing machine with its infinite tape is too general to serve as an actual model for digital computers. They base their argument on the fact that the Turing machine has an unlimited amount of memory space on its tape while an actual computer would have a limited amount of memory space.

\textbf{REFERENCES}


This paper was presented at the National Pi Mu Epsilon Meeting at Cornell University, September 1, 1965, by James Woeppep.
A MULTIPLICATION-FREE CHARACTERIZATION OF RECIPROCAL ADDITION

James Williams, Carleton College

Motivation.

Reciprocal addition, which shall represent by the symbol "o," is defined, in terms of the real number system, by

\[ a \circ b = \frac{1}{1/a + 1/b} = ab/(a + b), \quad (a, b, a+b \neq 0) \]

and is commonly used with an ideal infinity element \( \omega \) with the properties

\[ a \circ \omega = a, \quad a + \omega = \omega. \]

In such applications as parallel-series problems in electronics, for instance, the operations of addition and reciprocal addition frequently occur together (the constants 0 and \( \omega \) corresponding to the two extremes of short and open circuit values) exclusive of the use of multiplication: it is therefore of interest to develop an axiomatic characterization of + and o independently of the notion of multiplication.

We begin with a listing of some properties of \( \mathbb{R}^* \), the real number system with \( \omega \) and the operation \( \circ \) adjoined.

1. We use the mapping

\[ f(x) = 1/x, \quad 0 \neq x \neq \omega \]

\[ f(0) = \omega \]

\[ f(\omega) = 0 \]

to show that \( \mathbb{R}^* \) is a dual system:

\[ a, b, a+b \neq 0, \quad \Rightarrow \]

<table>
<thead>
<tr>
<th>Property</th>
<th>Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( f(-a) = -f(a) )</td>
</tr>
<tr>
<td>(2)</td>
<td>( f(a+b) = (1/a)(1/b)/(1/a + 1/b) )</td>
</tr>
<tr>
<td>(3)</td>
<td>( f(ab) = 1/ab - (1/a)(1/b) = f(a)f(b) )</td>
</tr>
</tbody>
</table>

The argument for special elements proceeds by definitional patching thus:

11. + and o are correlated independently of multiplication by the relation

\[ a + (a \circ b) = a + ab, \quad (a, b, a+b \neq 0, \omega) \]

We now proceed to investigate a system \( \Gamma \) with special elements \( o \) and \( \omega \), with a monary operation \( \ast \), and with binary operations \( + \) and \( \circ \), satisfying axioms I - VI for all elements \( a, b, c \) in \( \Gamma \).

Definition I.

Let \( N \) be the set of positive integers: \( \forall n \in N, \quad 1 \cdot a = a, \quad (n+1)\cdot a = na + a \)

1. \( a + a = a \)
2. \( a \cdot a = a \)
3. \( a + (a \cdot a) = 2a \)
4. \( a \cdot (a + a) = a + a \)
5. \( a \cdot a = a \cdot a \)
6. \( a = a \cdot a \)
7. \( a \cdot a = a \cdot a \)
8. \( a = a \cdot a \)
9. \( a = a \cdot a \)
10. \( a = a \cdot a \)

Theorem I.

1. \( (a \cdot b) = a \cdot b \cdot a \)
2. \( a = a \cdot a \)
3. \( a = a \cdot a \)
4. \( a = a \cdot a \)
5. \( a = a \cdot a \)
6. \( a = a \cdot a \)
7. \( a = a \cdot a \)
8. \( a = a \cdot a \)
9. \( a = a \cdot a \)
10. \( a = a \cdot a \)

Proof 1.2.

If \( 2a = 0 \), then \( a = 2a/2 = 0 \).

Proof 1.3.

If \( a = a' \) and \( a \neq \), then \( 2a = 2a + a' = 0 \).

Therefore, \( a = a' \).

Proof 1.4.

(1) \( a = a' \)
(2) \( a = a' \)

(hyp; def; III)
Proof 1.5.

If \( a = b \), then \( a + a = 0 \) and \( b + b = 0 \).

(1.1, hyp, III)

Proof 1.5.

If \( a \# b \), then \( a + a = 0 \) and \( b = 0 \).

(1.1, hyp, III)

Fact 1.6.

If \( a \# 0 \) and \( b \# 0 \), then \( a + b \# 0 \).

(V, hyp, hyp)

Proof 1.6.

If \( a \# 0 \) and \( \Gamma \notin \{0\} \), then \( \Gamma \notin \{0\} \).

(1.1, hyp, III)

Proof 1.6.

If \( a \# 0 \) and \( \Gamma \notin \{0\} \), then \( a \# 0 \).

(1.1, hyp, III)

Meta-Theorem.

\( \Gamma \) is a dual system; that is, for each theorem \( T \) in \( \Gamma \), the statement \( T^* \) formed by replacing each occurrence of \( +, \circ, 0 \) and \( \omega \) with \( \circ, +, 0 \) and \( \omega \) (respectively) is also a theorem in \( \Gamma \).
Proof.

Axioms I – IV are self-dual. Theorem 1.2 is the dual of Axiom V. The dual of Axiom VI is proved as follows: For any given x and y in T, let
\[ a = x, b = y, \]
then
\[ 2a = 2(x^2) - x \]
\[ 2b = 2(2b) = 2(2(y^2)) \]
\[ 2y = 2(y^2) - y. \]

Making the appropriate substitutions in Axiom VI then gives
\[ x^2 + (x^2 y^2) = x x y. \]

The meta-theorem is completed by noting that if T is any theorem with a proof P, then T* is the dual of T, will have a proof P*, each step of which is the dual of the corresponding step in P.

At this point it would be nice to know how close T comes to representing the notion of a field. In one direction, beginning with any field, we may construct all of the axioms of T just as we did for the real numbers, with the one exception that the derivation of VI requires the principle 1.2:
\[ 2a = 0 \Rightarrow a = 0, \]
thus eliminating fields such as \( \mathbb{Q}/(2) \). In the other direction, however, the answer seems less clear: Theorem II contains some partial results.

Definition 11.

We classify \( T \) by writing for each \( k \in \mathbb{N} \):
\[ T \in S_k \iff a \in T, k(a/k) = a. \]
\[ T \in T_k \iff a \in T, (ka)/k = a. \]

(11) Lemma.

If \( T \in S_{k} \cap S_{q} \), then \( \forall a \in T \):
\[ \frac{aq}{(a/q)} = p(a/q), \]
\[ \frac{aq}{(a/q)} = p(a/q). \]

so that \( T \in S_{pq} \). Similarly, \( T \in T_{p} \cap T_{q} \Rightarrow T \in T_{pq} \).

(12) Lemma.

If \( T \in S_{k} \cap T_{q} \), then \( \forall p \in \mathbb{N} \):
\[ \frac{pa}{(a/q)} = p(a/q), \]
\[ \frac{pa}{(a/q)} = p(a/q). \]

Similarly, \( T \in T_{p} \Rightarrow (pa = 0 \Rightarrow a = 0) \).

(13) Lemma.

If \( T \in S_{k} \Rightarrow (a/p = \omega \Rightarrow a = \omega), \) since if \( a/p = \omega \) then
\[ a = p(a/p) = p \omega = \omega. \]

Similarly, \( T \in T_{p} \Rightarrow (pa = 0 \Rightarrow a = 0) \).

(14) Lemma.

If \( T \in S_{k-1} \), then \( (ka = 0 \Rightarrow a = 0) \Rightarrow (a/k = \omega \Rightarrow a = \omega) \), for suppose
1) \( ka \neq 0 \)
2) \( a/k = \omega \)
3) \( a \neq \omega \)

\[ \omega = a/(ka - 1) \]
\[ a = a'/k - 1 \]
\[ a = a'/(k - 1) \]
\[ a = a'/k - 1 \]
\[ a = a'/k - 1 \]

Contradiction by hypothesis 1.

Therefore \( a \neq 0 \Rightarrow ka \neq 0 \Rightarrow (a/k = \omega \Rightarrow a = \omega) \).

(15) Basic Lemma for Theorem 11.

If \( T \in T_{k} \) then \( \forall T_{k} \in T_{n+1} \) if \( T \in T_{k}, \) and \( ka = 0 \Rightarrow a = 0, \) then \( T \in T_{k} \).
We shall prove \( T \in S_{k} \) and show \( T \in T_{k} \) by a duality argument.
We begin with proofs for special cases:

(16) If \( k = 0 \), then \( k(0/k) = k(0/k) = 0, \)
if \( k = \omega \), then \( (k/k) = (k/k) = 0, \).

(17) The cases for \( k = 3 \) and \( k = 4 \) follow directly from the fact that \( T \in S_{3}, T \in S_{4}, \) which in turn follows from Theorems 1.2 and 1.3.

(18) We next prove the case for \( k = 3 \) as follows: if \( 3a \neq 0 \) and a \( \neq \omega \), then \( a/6 \neq \omega \). Since if \( a/6 = \omega \), then
\[ a/3 - 2(1(a/3)/2) = 2(a/6) = 2a = \omega. \]

Contradiction by Lemma 14.

(19) Finally, by induction for any \( k > 4 \), choose \( 0 \neq a \neq \omega, \) then

(20) \( a/2(k - 2) \neq \omega \) by 13, since \( T \in S_{2}(k-2) \) by Theorem 1.9, hypothesis, and Lemma 11.

(21) \( k - 4 \neq a \neq 0 \) by 13 since \( T \in T_{k-4} \).

(22) \( k - 4 \neq a \neq \omega \) by 15, since \( a \neq \omega \) by hypothesis, and therefore \( (k - 4)a/k \neq \omega \) by 14.
Having dispensed with the preliminary details, we make the following computation:

\[ a + \left( k - 4 a \right) / \left( k - 2 \right) + 2 a \left( k - 4 a \right) / \left( k - 2 \right) = 4 a / \left( k - 2 \right) + 4 a / \left( k - 2 \right) = \]

Therefore,

\[ a + \left( k - 4 a \right) / \left( k - 2 \right) = 4 a / \left( k - 2 \right) \]

Adding \( (k - 4)a \) to both sides gives

\[ a - 4 a / \left( k - 2 \right) + (k - 4)a / \left( k - 2 \right) = 0 \]

Therefore, if \( \Gamma \in \mathbb{T}_k \) and \( k a = 0 \Rightarrow a = 0 \), then

\[ \Gamma \in \mathbb{T}_k \] by 16, 17, 18, and 19.

To arrive at \( \Gamma \in \mathbb{T}_k \), we note that the above proof takes the form "if P and Q, then R" where P is \( \Gamma \in \mathbb{T}_k \), Q is \( k a = 0 \Rightarrow a = 0 \), and R is \( \Gamma \in \mathbb{T}_k \). Since P and P* are identical, and since P and Q \( \Rightarrow \) Q* by 14, P and Q \( \Rightarrow \) P* and Q*; finally, P* and Q* \( \Rightarrow \) R* by the meta-theorem, so that P and Q \( \Rightarrow \) R* and \( \Gamma \in \mathbb{T}_k \).

Definition 41. \( \mathbb{T}^+ = \Gamma - \{ \omega \} \).

Theorem 42.

With respect to suitable definitions of multiplication,

1. If for some prime integer \( p \) and for all \( a \in \mathbb{I}_p \), \( p a = 0 \), then \( \Gamma^+ \) is a vector space over \( \mathbb{Z}_p \), the ring of integers modulo \( p \).
2. If \( \Gamma^+ \) has a prime number of elements \( p \), then \( \Gamma \) is derivable from \( \mathbb{Z}_p \).
3. If \( \mathbb{W} \in \mathbb{T}^+ \) and \( \mathbb{W} \in \mathbb{N} \), \( n a = 0 \Rightarrow a = 0 \), then \( \Gamma^+ \) is a vector space over the field of rational numbers.

Proof 43.

Assume \( p a = 0 \) for all \( a \in \mathbb{T}_p \), for some prime integer \( p \).

(23) (Lemma) We note that \( \mathbb{T}_p \) is a prime order abelian group with respect to addition as a result of Axioms I-IV and the above assumption.

(24) Considering \( [n] \in \mathbb{I}/(p) \) as an equivalence class of \( n \), we define \( \mathbb{V} a \in \mathbb{T}_p \), \( \mathbb{V} [n] \in \mathbb{I}/(p) \).

\[ [n] a \in \mathbb{V} \]

where \( [n] a \) is well defined since \( [n] = [m] \), then \( n - m = 0 \mod p \) and therefore \( (n - m) a = 0 \), \( n a = m a \).

Theorem 11.1 now follows from the fact that any additive \( p \)-order Abelian group forms a vector space with respect to definition 24 of multiplication.

Assume \( \Gamma^+ \) has \( p \) elements for some prime integer \( p \).

Then, with respect to addition, \( \Gamma^+ \) is a cyclic group, so that by Theorem 11.1, it forms a one-dimensional vector space with respect to definition 24 of multiplication.

(25) To extend the scalar-field properties of \( \Gamma^+ \), we choose an arbitrary normalizing constant \( a \in \mathbb{I}^+ \neq \{ 0 \} \), and define \( x - y = [k(x)][k(y)] a \) where \( [k(x)] a = -x, [k(y)] a = -y \).

(26) We also define \( 1/x \) by \( 1/x = y \Rightarrow x - y = a \).

(27) (Lemma) \( \mathbb{V} n \leq p, \mathbb{V} a \in \mathbb{T}_p \), \( (n a) / n - n(a/n) = a \) since, by induction, we have first \( \mathbb{T} \in \mathbb{S}_2 \cap \mathbb{T}_2 \) as mentioned in Lemma 15; then if \( \mathbb{T} \in \mathbb{S}_k \cap \mathbb{T}_k \) for some \( k \leq p \), then \( a \# 0 \Rightarrow (n a) \# 0 \) by Lemma 23 and therefore \( \mathbb{T} \in \mathbb{S}_k \) by Lemma 15. The induction stops with \( k \geq p - 1 \).

(28) (Lemma) If \( n \neq 0 \), then \( (n a)^* (a/n) = [k(na)] [k(a/n)] a \) (25) \( \{ \}

\[ n(a/n) = n(a/n) = a \]

\( n \in \{ n \}, 27 \)

Therefore \( 1/(n a) = a/n \) by definition 26.

Finally, we choose \( x, y \in \mathbb{T}^+ \neq \{ 0 \} \) such that \( x \neq y \) and compute

\[ x \circ y = 1/(1/x) \circ 1/(1/y) \]

\[ = 1/(1/(x/y)) \circ 1/(1/(y/x)) \]

\[ = a/(k(x)/y) \circ a/(k(y)/x) \]

\[ = a/(k(x)/y) \circ a/(k(y)/x) \]

\[ = 1/(k(x)/y) + 1/(k(y)/x) \]

\[ = 1/(k(x)/y) + 1/(k(y)/x) \]

\[ = 1/(1/(x + 1/y)) \]

\[ = 1/(1/(x + 1/y)) \]

\[ = 1/(1/(x + 1/y)) \]
So that 0 is algebraically determined by the field properties of \( \Gamma^+ \) and the conventions for special elements (e.g. \( a + \omega - \omega \)).

Proof. 11.3.

Assume \( \forall a \in \Gamma^+ \), \( \forall k \in \mathbb{N} \), \( ka = 0 \Rightarrow a = 0 \).

(29) We define \( \forall k \in \mathbb{N} \)
\[
(-k)a = ka', \quad a' = \frac{a}{k}, \quad 0 \cdot a = 0.
\]

(30) (Lemma) By analogy with Lemma 27 of the previous proof, we conclude
\[
\Gamma \in i, a_i \in (\mathbb{Q} \cup \mathbb{R}), \quad \text{so that, in view of definition 29,} \quad \forall k \in \mathbb{I},
\]
\[
k(a/k) = (ka)/k = a.
\]

(31) \( \forall a \in \Gamma^+, \forall p, q \in \mathbb{Q}, q \neq 0 \), we define
\[
(p/q)a = (pa)/q.
\]

The above multiplication by rational numbers is well defined since if
\[
P_i/q_i = P_k/q_k
\]
then \( P_i - P_k = P_i/q_i - P_k/q_k \).

\[
(p_i/q_i)a = \frac{((p_i/q_i)a)/q_i}{q_i} = \frac{((p_i/q_i)a)/q}{q} = (p_i/q_i)a (\text{def } i, i, i, i, i, ii, ii, ii)
\]

Assuming the normal field properties for rational numbers, the field-vector space correlation laws are proved as follows:

\[
1 \cdot x = x \quad \text{(def i)}
\]

\[
(p/q)(r/s)x = \frac{[(pr)(rs)]s}{q} = \frac{[(pr)(rs)]s}{q} = [pr][rs]x = \frac{[(p/q)(r/s)]x}{q} (\text{def } i, ii, ii, ii)
\]

\[
(p/q)(x + y) = \frac{[(p/q)(x + y)]}{q} = \frac{[(p/q)(x + y)]}{q} (\text{def } i, i, i, i, i, i, ii, ii, ii)
\]

\[
(p/q + r/s)x = \frac{[(p/q + r/s)x]}{q} = (p/q + r/s)x = \frac{[(p/q + r/s)x]}{q} (\text{def } i, ii, ii, ii)
\]

RESEARCH PROBLEMS

Proposed by GEORGE BRAUER, University of Minnesota.

The real-valued functions on \((-\infty, \infty)\) is a semi-group, if we define the product of two such functions by

\[
f \cdot g = g(f(t))
\]

The function \( X \) acts as identity and each constant function \( c \) has the properties:

\[
c \cdot f = f(c),
\]

\[
f \cdot c = c
\]

If we denote the set of real-valued functions by \( S \), and the set of constants by \( I \), we have

\[
I \times S \subseteq S, \quad S \times I = I.
\]

Moreover, if we define \( f \geq g \) to mean that \( f(t) \geq g(t) \) for all \( t \), then \( S \) is right partially ordered, i.e.,

\[
f \geq h \Rightarrow f \cdot g \geq f \cdot h
\]

Problem: Obtain properties of abstract right partially ordered semi-groups. What algebraic properties, in addition to \((1) \sim (5)\), would constitute a characterization of the semi-group of real functions? What properties would characterize the sub semi-groups?

Proposed by SIM KAH, University of Minnesota.

Examine, heuristically, the geometry of geodesics on a cylinder in order to obtain a set of axioms of incidence and order. Using these axioms, deductively develop a geometry and attempt to find models, other than the cylinder, to which this geometry applies.
PROBLEM DEPARTMENT

Edited by
M. S. Klamkin, Ford Scientific Laboratory

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Michigan 48121.

PROBLEMS FOR SOLUTION

177. Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India.

If \( s \) is the semi-perimeter and \( R, r, r_1, r_2, \) and \( r_3 \) are the circum-, in-, and ex- radii respectively of a triangle, prove that

\[
R^2 = \frac{2a^2}{r_1 r_2 r_3}.
\]


Show that the centroid of \( \triangle ABC \) coincides with the centroid of \( \triangle A'B'C \) where \( A', B', \) and \( C' \) are the midpoints of \( \overline{BC} \), \( \overline{CA} \), and \( \overline{AB} \), respectively.

Generalize to higher dimensions.


It is well known that

\[
s^2 + d^2 = s^2 - 16^2 + 14^2.
\]

Generalize the above by finding integers \( a \) satisfying

\[
\sum_{k=0}^{a} (a+k)^2 = \sum_{k=1}^{m} (a+k)^2.
\]

180. Proposed by R. C. Gebhart, Parsippany, N. J.

In the figure, \( AB = HC \) and \( \angle ABC = 90^\circ \). The arcs are both circular with the inner one being tangent to \( AB \) at \( A \) and \( HC \) at \( C \).

Determine the area of the crescent.

181. Proposed by Donald W. Crowe, University of Wisconsin and M. S. Klamkin, Ford Scientific Laboratory.

Determine a convex curve circumscribing a given triangle such that

1. The areas of the four regions (3-segments and a triangle) formed are equal and
2. The curve has a minimum perimeter.

SOLUTIONS


Express \( \cos \theta \) as a rational function of \( \sin^3 \theta \) and \( \cos^3 \theta \).

Solution by R. C. Gebhart, Parsippany, N. J.

Since

1. \( 4 \cos^3 \theta = 3 \cos \theta + \cos 3 \theta \),
2. \( 256 \cos^3 \theta = 9 \sin \theta - 6 \cos \theta - 8 \cos 3 \theta + 3 \cos 7 \theta - \cos 9 \theta \),
3. \( 64 \cos 9 \sin \theta = 5 \cos \theta - 9 \cos 3 \theta + 5 \cos 5 \theta - \cos 7 \theta \),
4. \( 256 \cos 9 \theta = 126 \cos \theta + 84 \cos 3 \theta + 3 \cos 5 \theta + 9 \cos 7 \theta + \cos 9 \theta \),
5. \( 64 \cos \theta \cos^3 \theta = 35 \cos \theta + 21 \cos 3 \theta + 7 \cos 5 \theta + 5 \cos 7 \theta + \cos 9 \theta \),

It then follows that

\[
A = 2, \quad B = C = D = 1, \quad \text{and} \quad E = -2.
\]

Whence,

\[
\cos \theta = \frac{2 \cos^3 \theta + 3 \cos^3 \theta \sin^3 \theta + \cos^3 \theta}{1 - (\sin \theta)^2}.
\]

Also solved by K. S. Murray, P. Myers, and F. Zetto.

57. Proposed by M. S. Klamkin, Ford Scientific Laboratory.

Given a centro-symmetric strictly convex figure and an intersecting translation of it; show that there is only one common chord and that this chord is mutually bisected by the segment joining the centers.

Solution by the proposer.
If the two ovals intersect in more than two points, then each oval would contain at least three parallel and equal length chords. The length and direction being the length and direction of the translation. This is impossible since the middle chord must be larger than the outer equal ones by strict convexity.

If A and B denote any pair of centrosymmetric points of the first oval and A' and B' the corresponding points of the translated oval then AA'B'B is a parallelogram. It then follows that the entire figure (both ovals) is centrosymmetric about point M, the midpoint of the segment joining the two centers. Since D and E are the only two double points, they must correspond to each other, i.e., be centrosymmetric with respect to M. Note that this generalizes the well known result for two intersecting congruent circles.

Also solved by M. Wagner.

169. Proposed by JERRY TOWER, North High School (student), Columbus, Ohio.

Determine x asymptotically if
$$\log x = n \log \log x.$$  

Solution by Sydney Spital, California State Polytechnic College.

The given equation may be rewritten as
$$e^{\frac{\log x}{n}} = \log x$$
whose limiting solution is easily seen to be $x = e$. To determine $x$ asymptotically, we let $x = e(1 + \epsilon)$ and assume (to be justified subsequently) that $6 \geq 0(\frac{1}{n})$. Substituting back into (1), yields
$$e^{\frac{\log x}{n}} [1 + 0(\frac{1}{n})] = 1 + \delta + 0(\frac{1}{n}).$$
Thus,
$$6 = e^{2n} - 1 + 0(\frac{1}{n}).$$

Since $e^{2n} - 1 + 0(\frac{1}{n})$, our previous assumption is justified and the desired asymptotic behavior is given by
$$x = e^{1+\frac{1}{n}} + 0(\frac{1}{n}) = e(1+1/n) + 0(\frac{1}{n}).$$

Also solved by Paul J. Campbell, Y. S. Murray and P. Zetto.

170. Proposed by C. S. VENKATARAMAN, Sree Kerala Varma College, India.

Prove that a triangle ABC is isosceles or right-angled if
$$a^3 \cos A + b^3 \cos B = abc.$$  

The solutions by Paul J. Campbell, University of Dayton, and Marilyn Mantel, University of Nebraska were essentially identical and are given as follows:

Replacing \( \cos A \) and \( \cos B \) by the law of cosines, i.e.,
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac},$$
(1) reduces to (after some algebraic manipulations)
$$(a^3 + b^3 - c^3)(a^2 - b^2)^2 = 0,$$
which implies the desired result.

Sydney Spital, California State Polytechnic College and the proposer use the projection formula
$$a \cos B + b \cos A = C.$$
171. Proposed by MURRAY S. KLAMKIN, Ford Scientific Laboratory.

For $0 < \theta \leq \pi/2$, it is well known that the inequality

$$\frac{\sin \theta}{\theta} > \cos^2 \theta$$

holds for $m = 1$. What is the smallest constant $m$ for which it holds?

Solution by the proposer.

It follows from the power series expansions that $m > \frac{1}{3}$. We now show that it suffices to have $m = \frac{1}{3}$, i.e.,

$$\sin^2 \theta \tan \theta > \theta^2.$$  \hspace{1cm} (1)

Since the l.h.s. and the r.h.s. of (1) are equal at $\theta = 0$, it suffices to show that the derivative of the l.h.s. is greater than the derivative of the r.h.s. of (1), i.e.,

$$2 \sin^2 \theta 0 + \tan^3 \theta > 3\theta^2.$$  \hspace{1cm} (2)

Similarly to establish (2), it suffices to show

$$4 \sin \theta \cos \theta + 2 \tan \theta \sec^2 \theta > 60,$$

or

$$4 \cos 20 + 2 \sec^4 \theta + 4 \tan^3 \theta \sec^2 \theta > 6.$$  \hspace{1cm} (3)

or

$$4 \cos 20 + 2 \sec^4 \theta + 4 \tan^3 \theta \sec^2 \theta > 6.$$  \hspace{1cm} (4)

(4) follows since it can be rewritten as

$$(\sec^4 \theta) (\sin^4 \theta) (3 + 4 \cos^2 \theta) > 0.$$  

Sydney Spital, California State Polytechnic College, establishes the above result by showing that

$$m(\theta) = \frac{\log (\theta \csc \theta)}{\log \sec \theta}$$

is monotone decreasing on $[0, \pi/2]$ and then the desired minimum value of $m$ is given by

$$\lim_{\theta \to 0} m(\theta) = -\frac{1}{3}.$$  

Also solved by PAUL J. Campbell, H. Kaye, Paul Myers, and Ricky Pollack.

Also solved by H. Kaye, P. Myers, M. Wagner and F. Zetto.

A modern self-contained treatment of general topology, with historical and motivation remarks, which proceeds from a very elementary introduction to some rather difficult advanced theorems in twenty-three brief chapters. Although the book is brief, there is an excellent bibliography and enough difficult but important theorems and exercises to warrant a year's study. In addition to the classical topics of general topology, there are modern treatments of quotient spaces, nets, filters, uniform spaces, proximity spaces, topological groups, and paracompactness and metrization.


Based on lecture notes prepared for a one-semester course for students of applied mathematics with limited mathematical background, covering Fortran programming with applications to computing with polynomials, interpolation, quadrature, solution of equations and integration of ordinary differential equations.


An outgrowth of a public lecture series at the University of Michigan in the spring of 1962 called "The Place of Albert Einstein in the History Of Physics," written by one of the outstanding mathematics expositors and a former associate of Einstein.

Modular Arithmetic. By Burton W. Jones.

An excellent book for secondary school teachers, outstanding high school students, or for college students who might wish to use it as an elementary secondary reference on modular systems.
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