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PI MU EPSILON JOURNAL

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The C. C. MacDuffee Award for Distinguished Service

J. C. Eaves, The University of Kentucky

1. Introduction: Pi Mu Epsilon's first recipient of The C. C. MacDuffee Award for Distinguished Service was selected at the annual meeting of the organization's officers, held on the Cornell campus at Ithaca, New York, September 1965. perhaps the discussions were somewhat prolonged but the unanimous choice to receive the honor of the first such award since its adoption was to be the presiding officer at all regularly scheduled meetings, the national president of Pi Mu Epsilon, Dr. J. Sutherland Frame; and he kept showing up on time.

Since The C. C. MacDuffee award is Pi Mu Epsilon's highest recognition it was decided late in the meetings that presentation would be most appropriate upon Dr. Frame's retirement from the presidency, this allowing ample time to arrange for the banquet and to lay adequate plans for the occasion.

It was the opinion of those on the Governing council that the award should be made "often enough to be recognized and seldom enough to be meaningful." Excerpts from the presentation notes are given below.

2. The First Presentation: "Members of Pi Mu Epsilon, distinguished guests: In making this presentation I call your attention to the following observations.

Last year, the Councilors General, cognizant of the fact that the awarding of The C. C. MacDuffee plaque for Distinguished Service is Pi Mu Epsilon's highest tribute and most prestidigious recognition, voted that, during the past decade, the most enduring and valuable proponent of its cause -- the promotion of mathematics -- is its retiring president. This group unanimously concurred in the opinion that some significant acknowledgment of gratitude was here due and that only the C. C. MacDuffee plaque befits this occasion.

"Our honoree is recognized as an outstanding scholar who exemplifies triumphantly the true ideals of this learned society. He is appreciative and productive of effective promotional action in the area of mathematics. His dedication over the past years supports our contention that he possesses the intellectual strength and organizational qualities embodying competent leadership. He is a motivating teacher and an inspiring speaker who maintains a learning environment for himself and his associates. These characteristics coupled with the curiosity of a researcher, the critical mind of a mathematician, and an unlimited concern for all aspects of service to Pi Mu Epsilon make him worthy of this award.

Professor and Mrs. J. S. Frame

"Here is a man whose service spans nine years as our President and numerous years prior to this time expounding the cause in other capacities. He has installed almost 50% of our chapters, 51 of the 120, these including ten alpha chapters. This growth is more significant when measured in terms of the 30,000 increase in membership witnessed during the period 1951-1966. These last years have brought an inauguration of the matching funds for recognition awards within local chapters, and book awards for the presentation of superior papers. Finally, Pi Mu Epsilon became a fully grown mathematics organization when Dr. Frame initiated the first papers session at the Michigan State meeting.

"This man has brought encouragement to hundreds of prospective mathematics students many of whom continued their interests to become productive scholars. All of this has not been without its hardships. Traveling the equivalent of nearly four times around the earth to see that "Chapters got their Charters" must have accounted for the consumption of gallons of stale coffee, bouillon, undercooked egges, overcooked toast, airport delays, and lost baggage. Surviving this, smiling, is one blessed with tolerance and a measure of devotion to service which would compliment any of us. Pi Mu Epsilon shall always be indebted to him.

I am very pleased that it falls my honor to present this, the First C. C. MacDuffee Award for Distinguished Service to one to whom I can say, "Dr. Frame, only our highest award expresses our sincere appreciation for your past devotion, your prudent judgment, and your continued wise counsel and loyalty. Only our highest award expresses the esteem with which you are regarded by our members. May this plaque find a prominent spot in your home or office. Take unrestrained pride without embarrassment in the message it bears, for those who see it will know that herein dwells one who has pursued his calling not only in a superbly successful manner but with unrelenting vigor and unselfish devotion.

Ladies and Gentlemen: The first recipient of our highest award, Dr. J. Sutherland Frame."

3. The Second Presentation: Ladies and Gentlemen: You have just been briefed on the true significance of the C. C. MacDuffee Award. I need not repeat these facts in making a second presentation tonight.

"Dr. Richard V. Andree has served our organization in a multitude of capacities, faithfully bringing forth workable new ideas and the energy to pursue them to fruition, this for many years. He has done so with genuine interest and unselfish motives He has been active in promoting mathematics wherever the opportunity exists and his efforts in advancing Mu Alpha Theta, the

Professor and Mrs. Richard V. Andree

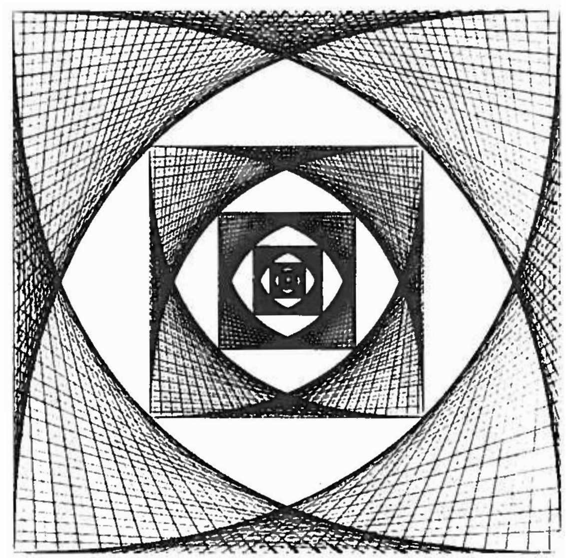
international honor society for high school and Junior College mathematics students has fed many top students into Pi Mu Epsilon. It was through his foresight and wisdom that Pi Mu Epsilon gave support to Mu Alpha Theta during the trying time of MAT's organizational days.

It is not necessary to present the achievements and success this able servant has enjoyed in his promotion of mathematics. His abundantly impressive and valuable pioneering ventures in all directions and at all levels are well known. He seems to thrive on projects which promote scholarly study and investigations, and this, particularly among the young scholars supports and reinforces the primary objectives of our organization. His guiding philosophy never seems to be "We must move forward," but rather, "We must move. Our movement will be forward only."

"On behalf of Pi Mu Epsilon, the Councilors General concurring unanimously, it is a stimulating experience to present this, the second such high recognition to be announced, the C. C. MacDuffee Award for Distinguished Service, to one who earned it through devotion and love, not through labor; not by the dangerous and damaging drudgery of a duty but through the pleasure of Service to Mathematics and to his fellow man.

"Ladies and Gentlemen, the second individual to receive our highest tribute, Dr. Richard V. Andree."

(at Rutgers, 30 August, 1966)



THE STUDY OF A RECURSIVE SEQUENCE

James Wingert, John Carroll University

The problem which will be discussed in this paper appeared as an advanced problem in the June-July, 1965, issue of the American Mathematical Monthly and is stated as follows: Given the following 41 terms of a sequence 12211212212211211221211212211211212212211.. . , determine a simple generating relationship for this sequence and determine whether or not the sequence is cyclic.

Since the first and last elements are identical, one obvious solution would be to merely keep repeating the first 40 terms. Another idea occurred to me when I noticed that every third term was a 2. Upon deleting these 2's an interesting pattern appears. 12 11 12 12 11 11 21 11 12 11 11 12 12 11 ... As you can see, there is a 12 followed by a 11, two 12's followed by two 11's. The 21 appears to have the function of interchanging the roles of the 11's and the 12's. Hence there is a 11 followed by a 12, two 11's followed by two 12's. However, since the last term is a 11 the pattern is broken. A sequence very similar to this one appeared in an article by Marston Morse and Gustav Hedlund in the Duke Mathematics Journal; however, I was unable to apply all of its properties to this case.

The generating relationship which I have used was discovered in the following way. I began by counting the number of elements as they appeared in groups. There was ONE 1, TWO 2's TWO 1's, ONE 2, ONE 1, TWO 2's, ONE 1 TWO 2's, TWO 1's, etc. As can be seen, these numbers are repeating the numbers of the sequence. This led me to the two rules that form the generating relationship. The first rule concerns the number of elements generated and the second rule concerns the kind of elements generated, that is whether they are 1's or 2's. The number of elements generated depends upon the generating element. It will generate one or two elements depending upon whether it is a 1 or a 2. The kind of elements generated depend upon the last element generated. If it is a 1, the next term generated will be either a 2 or a 22. If the last element generated is a 2, the term generated will be either a 1 or a 11.

I have prepared a few examples to illustrate this. In the first example the generating element is a 2, so two elements must be generated. The last element generated is a 1. Hence, the generated element is 22. In the second example the generating element is a 2, the last element generated is a 2, so the generated term is 11. Finally, if the generating element is a 1 and the last element generated is a 2, the generated term is a 1.

Hence the sequence is built up as follows: A 1 generates itself. Since it is the last element generated, the next term will be either 2 or 22. Since in either case the second element of the sequence is a 2 and since a 2 generates two elements, the second term generated will be 22 and we have . The second 2 becomes the last element generated and the generating element. This will generated a 11 and we have 12211. The second 1 becomes the generating element and the third 1 becomes the last element generated. This will generate a 2 and we have 122112' The third 1 becomes the generating element and the third 2 becomes the last element generated. This will generate a 1 and we have 1221121. The third 2 becomes the generating element and the fourth 1 becomes the last element generated. This will generate a 22 and we have 122112122. The fourth 1 becomes the generating element and the fifth 2 becomes the last element generated. This will generate a 1 and we have 1221121221' As you can see, each element in turn becomes the generating element, but not every element is a last element generated.

In order to make the sequence easier to read I have used the following code: A = 1, B = 11, C = 2, and D = 22. Now the given 41 terms begin like this: ADBCADADBCB and so on. This code was used because letters will generate letters. An A is a 1 and a 1 can generate either a 1 or a 2. Hence, an A can generate either an A or a C. A B is a 11 and each 1 can generate either a 1 or a 2. Hence a B can generate either a CA or an AC. Likewise a C can generate a B or a D and a D can generate either a DB or a BD.

I have programed the generating relationship on an LPG-30 computer and have printed out the first 1800 terms in the code just described. This was done in order to get some idea if the sequence would cycle. According to a theorem on sequence, if three consecutive blocks of letters can be found that are identical, the sequence is cyclic. However, the printed terms give no proof if the sequence is not cyclic. I checked to see if the sequence had cycled in the following way. The first four letters of the sequence were ADBC. I counted the number of letters between each succeeding pair of blocks ADBC. Since the last three numbers are 16, 6, 14 (See Appendix) and they occur only once in this order, the sequence has not yet cycled. In fact, there are numbers which are progressively larger, first 2, then 6, then 8 and finally as high as 24. This would seem to suggest that the sequence is not cyclic, although I have not been able to prove it as yet. In this year's July-July issue of the Monthly a solution to this problem has been published and the sequence has been proven non-cyclic.

While working on this sequence I discovered several properties of it.

Property I: There are never more than two successive 1's or 2's.

The proof of this property comes immediately from the way the sequence was defined, for there were never more than two elements generated at one time. Property I leads to several facts about the code we have used. There can be no double letters AA = 11 = B, CC = 22 = D, BB = 1111, DD = 2222. There can be no combinations AB = BA = 111 or CD = DC = 222, since both violate property I. By this we can see that an A and a B must be followed by either a C or a D and a C and a D must be followed by an A or a B. Since the first letter is an A, all odd numbered terms are A's or B's and all even numbered terms are C's or D's. There can be no combinations ACA, CAC, DBD or BDB because they violate Property I. This can be shown as follows. ACA = 121. Somewhere in the sequence there would have to be a 1 to generate the first 1, another 1 to generate the 2 and a third 1 to generate the last 1. Hence three consecutive 1's were necessary to generate ACA and therefore it cannot exist in the sequence. In a similar manner the other combinations can be shown to violate Property I.

Property II: In any group of five consecutive elements there is at least one double (either a 11 or a 22).

The two contradicting cases are if the five elements are 12121 or 21212. A 12121 would be written as 1CAC1 and a 21212 would be written as 2ACA2. Since neither of these combinations can exist in the sequence, the property holds.

Property III: In any block of N consecutive elements there are at least $K-1$ doubles if $N = 4K$, and there are at least K doubles if $N = 4K+1$, $N = 4K+2$, or $N = 4K+3$.

In order to prove this property I will first show that any block of length $4N+1$ for any integer N contains at least N doubles.

When $N = 1$ this statement is true by Property II.

Now assume that this is true for integers 1, 2, ..., N. Now $4(N+1)+1 = 4N+5 = (4N+1)+4$. In the first $4N+1$ elements there are at least N doubles by hypothesis. The last five elements contain at least one double by Property II. Therefore there are at least $N+1$ doubles in the block of length $4(N+1)+1$. Hence, by induction, the property holds for all N.

Now given a block of length N either $N = 4K$, $N = 4K+1$, $N = 4K+2$ or $N = 4K+3$ for some integer K. If $N = 4K+1$, then there are at least K doubles by what has been shown above. If $N = 4K+2$ or if $N = 4K+3$, there are at least K doubles, since the addition of one or two elements will not affect the first case. If $N = 4K$, then there must be at least $K-1$ doubles, since $4K = 4(K-1)+1+3$ and the first $4(K-1)+1$ elements contain at least $K-1$ doubles. Hence Property III has been proven.

APPENDIX

The number of letters between successive blocks of the form ADBC are as follows:

2, 6, 4, 8, 2, 8, 2, 8, 2, 8, 4, 12, 8, 2, 2, 8, 14, 4, 14, 2,
8, 6, 4, 12, 16, 14, 8, 2, 6, 4, 14, 8, 2, 2, 14, 8, 4, 14, 14,
8, 2, 6, 4, 8, 12, 8, 2, 8, 2, 18, 20, 16, 6, 4, 14, 8, 4, 6,
2, 8, 20, 14, 14, 4, 6, 2, 8, 14, 2, 6, 4, 24, 14, 8, 4, 6, 4,
8, 2, 2, 8, 14, 8, 12, 4, 6, 8, 4, 12, 14, 4, 8, 20, 2, 8, 14,
4, 12, 16, 14, 14, 16, 6, 14.

UNDERGRADUATE RESEARCH PROJECT

Proposed by Paul Samuel, South Minneapolis, Minnesota.

Investigate problems of inscribing equilateral triangles in a given triangle:

(1) Can an equilateral triangle always be inscribed in a given triangle? If not, under what conditions?

(2) If an equilateral triangle can be inscribed in a given triangle, in how many ways can this be done?

(3) Under what conditions is a given point P on a side of a given triangle a vertex of an inscribed triangle? Can P be the vertex of infinitely many inscribed equilateral triangles? Under what conditions?

(4) Suppose there exists an inscribed equilateral triangle with P as a vertex. Can the other two vertices be determined by a euclidean construction (straight-edge and compass in a finite number of steps)?

MOVING??

Please be sure to let the Pi Mu Epsilon Journal know! Send your name and complete new address with zip code to:

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MATRICES OF SYMMETRIES AND REDUCTION FORMULAS

Ali R. Amir-Moez, Texas Technological College

A symmetry or reflection with respect to a line through the origin or the origin itself introduces interesting techniques for reduction formulas in trigonometry. In this note we would like to give a few examples.

1. Definitions and notations: We shall choose a rectangular coordinate system. Each vector A has its beginning at the origin. To each vector corresponds an ordered pair (x,y). Sometimes we write the row matrix $(x\ y)$ for this vector. A linear transformation f on the plane is a function whose domain is the set of vectors in the plane and its range is a set of vectors in the plane such that

$$\begin{cases} f(A+B) = f(A) + f(B) \\ f(cA) = cf(A) \end{cases},$$

where c is a real number [1]. This means that f transforms a sum of two vectors to the sum of their transforms and a multiple of a vector to the same multiple of its transform. Indeed a good example is symmetry (reflection) with respect to a line through the origin (Fig. 1). We observe that the symmetrical of a vector A with respect to the line OP is $f(A) = B$, where B has the same length as A and the line AB is perpendicular to OP. The reader may verify that a symmetry is a linear transformation.

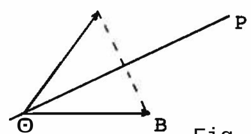


Fig. 1

2. Matrix of a linear transformation: There are two unit vectors $(1,0)$ and $(0,1)$ respectively on the x-axis and on the y-axis. If

$$f(1,0) = (a_{11},a_{12}) \quad \text{and} \quad f(0,1) = (a_{21},a_{22}),$$

then we define

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

to be the matrix of f with respect to the given coordinate system. We shall not go into the idea of a product of two linear transformations and corresponding matrix product. For more information we refer the reader to [1]. The transform of a vector means

$$f(x,y) = (x',y').$$

This is obtained through the matrix multiplication

$$(x'\ y') = (x\ y)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Indeed, this is the same as the set of equations

$$\begin{cases} x' = a_{11}x + a_{21}y \\ y' = a_{12}x + a_{22}y \end{cases}$$

To verify this we observe that

$$(x,y) = x(1,0) + y(0,1)$$

and

$$f(x,y) = xf(1,0) + yf(0,1)$$

$$= x(a_{11},a_{12}) + y(a_{21},a_{22}) = (x',y').$$

Thus

$$(x',y') = (a_{11}x + a_{21}y,\ a_{12}x + a_{22}y).$$

3. Matrices of symmetries: In general the matrix of a symmetry may not be very interesting. But one observes that if a vector is on the axis of symmetry, then it is transformed into itself. If a vector is perpendicular to the axis of symmetry, then it is transformed into its negative. We shall discuss a few examples.

I. Symmetry with respect to the x-axis: Here one observes that $(1,0)$ is on the axis of symmetry and $(0,1)$ is perpendicular to the axis of symmetry. Thus the matrix of this transformation is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

II. Symmetry with respect to the angle bisector of the first quadrant: Here a simple geometric observation (Fig. 2) implies that

$$f(1,0) = (0,1)$$

and

$$f(0,1) = (1,0).$$

Thus the matrix of this symmetry is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

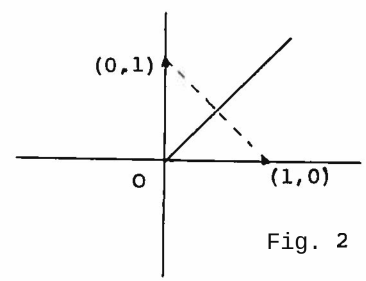


Fig. 2

III. Symmetry with respect to the origin: The symmetrical of any vector with respect to the origin is its negative. Thus the matrix of this symmetry is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4. Application to reduction formulas: Let us look for $\cos(-t)$ and $\sin(-t)$ in terms of functions of t. It is clear that the vector $(\cos[-t], \sin[-t])$ is the symmetrical of $(\cos t, \sin t)$ with respect to the x-axis. Thus

$$(\cos[-t] \;\; \sin[-t]) = (\cos t \;\; \sin t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= (\cos t \;\; -\sin t)$$

Therefore $\cos[-t] = \cos t$ and $\sin[-t] = -\sin t$.

Next we look for $\cos(\frac{\pi}{2} - t)$ and $\sin(\frac{\pi}{2} - t)$ in terms of functions of t. Here the vector $(\cos[\frac{\pi}{2} - t], \sin[\frac{\pi}{2} - t])$ is symmetrical of $(\cos t, \sin t)$ with respect to the angle bisector of the first quadrant. Thus

$$(\cos[\tfrac{\pi}{2} - t] \;\; \sin[\tfrac{\pi}{2} - t]) = (\cos t \;\; \sin t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= (\sin t \;\; \cos t).$$

This implies that $\cos(\frac{\pi}{2} - t) = \sin t$ and $\sin(\frac{\pi}{2} - t) = \cos t$.

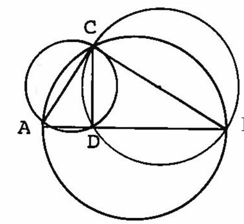
Indeed, one can obtain many other formulas similarly. For example, for functions of $\pi - t$ and $\pi + t$ we respectively use the symmetry with respect to the y-axis and the symmetry with respect to the origin.

[1] A. R. Amir-Moez, Matrix Techniques, Trigonometry, and Analytic Geometry, Edwards Brothers, Inc., Ann Arbor, Michigan, 1964.

THE PYTHAGOREAN THEOREM

Dana W. Alien, University of California-Davis

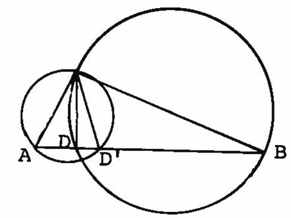
CONSTRUCTION

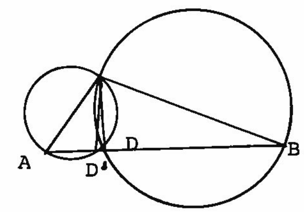


Consider the circle (AB) with diameter AB. Choose an arbitrary point C on the circumference and construct the chords AC and CB. Since the vertex of angle ACB is on the circumference and the sides are subtended by a diameter of the circle, angle ACB is a right angle. Therefore triangle ACB is a right triangle.

Using CB as a diameter, construct the circle (CB). Call D the point at which circle (CB) intersects AB. Construct CD, which is a chord of circle (CB). Triangle CDB is then a right triangle.

Similarly, using AC as a diameter, construct circle (AC), which intersects AB at D'. Points D and D' coincide, for





angle CDD' is a right angle and so is angle CD'D. Since the sum of the interior angles of triangle CDD' is equivalent to two right angles, angle DCD' is 0°. Consequently, D coincides with D'. Therefore, triangle ADC is a right triangle.

Since the sum of the interior angles of all triangles is equivalent, we have:

$$\angle CAB + \angle ABC + \angle BCA = \angle ACD + \angle CDA + \angle DAC$$

$$= \angle CDB + \angle DBC + \angle BCD.$$

And $\angle ACB = \angle ADC = \angle CDB$ because each is a right angle.

Therefore

$$\angle CAB + \angle ABC = \angle ACD + \angle DAC = \angle DBC + \angle BCD.$$

Since $\angle CAB$ coincides with $\angle DAC$ and $\angle ABC$ coincides with $\angle DBC$, $\angle ABC = \angle ACD$ and $\angle CAB = \angle BCD$.

Therefore

$$\angle CAB = \angle DAC = \angle BCD \text{ and } \angle ABC = \angle DBC = \angle CBD.$$

Consequently the triangles ACB, ADC, and CDB are similar, and

$$AB : BC :: AC : CD :: AD : DB : CB : CD.$$

Let $A(ACB) = \frac{1}{2}$ AC BC (the area of triangle ACB), and let

$A(AB) = \frac{\pi}{4}(AB)^2$ (the area of circle (AB)).

Then $\frac{A(AB)}{A(ACB)} = \frac{\frac{\pi}{4}(AB)^2}{\frac{1}{2}\cdot AC\cdot BC} = \frac{\pi}{2}\cdot\frac{AB}{AC}\cdot\frac{AB}{BC}$. Similarly,

$$\frac{A(AC)}{A(ADC)} = \frac{\pi}{2}\cdot\frac{AC}{AD}\cdot\frac{AC}{DC},$$

and $\frac{A(CB)}{A(CDB)} = \frac{\pi}{2}\cdot\frac{CB}{DC}\cdot\frac{CB}{DB}$.

From the similarity of the triangles ACB, ADC, and CDB:

$$\frac{AB}{AC} = \frac{AC}{AD} = \frac{BC}{CD}, \text{ and } \frac{AB}{CB} = \frac{AC}{CD} = \frac{BC}{BD}.$$

Therefore, $\frac{A(AB)}{A(ACB)} = \frac{A(AC)}{A(ADC)} = \frac{A(CB)}{A(ACB)}$,

or $A(AB) : A(AC) : A(CB) :: A(ACB) : A(ADC) : A(CDB)$.

Since $A(ACB) = A(ADC) + A(CDB)$, it follows immediately that

$$A(AB) = A(AC) + A(CB).$$

Multiplying this last equation by $\frac{4}{\pi}$, we have

$$(AB)^2 = (AC)^2 + (CB)^2,$$

and the proof is complete.

This method of proof may easily be extended to include the construction of all regular polygons on the sides of a right triangle; to show that the sum of the areas of the two polygons constructed on the legs of the right triangle is equal to the area of the polygon constructed on the hypotenuse. The use of a circle is the most general solution and as such involves a more intimate set of relationships.

THE FUNDAMENTAL THEOREM OF ALGEBRA

Paul J. Campbell, University of Dayton

In elementary courses in algebra the theorem that has become known as the Fundamental Theorem of Algebra is usually stated without proof. The proof is first encountered in an introductory course in complex variables after the development of a considerable number of concepts and theorems.

One advantage of the following proof of the Theorem is that an understanding of the proof requires only the most elementary knowledge of complex numbers and their vector representations. The proof, however, does make use of the concepts of "bound," "infimum," and "cluster point," and serves as an example of the application of the techniques they engender. Consequently, the level of the proof is approximately that of beginning advanced calculus.

Gauss in 1799 was the first to offer a correct formal proof of the Theorem. His predecessor Jean LeRond D'Alembert (1717 - 1783), however, gave an incomplete proof; and it is by means of the lemma devised by and named after D'Alembert that the Theorem will be proved. The general approach may be found in Huntington's paper [2], but a great deal of restructuring and simplification has been effected. The proof of D'Alembert's Lemma is essentially the one outlined in [1].

We begin with a basic definition:

Definition: A function f is a polynomial of degree n if and only if $f(z) = a_nz^n + \cdots + a_1z + a_o$, where for all i, a_i is a complex constant, and $a_n \neq 0$. The following is a statement of the theorem we shall prove:

THEOREM (The Fundamental Theorem of Algebra): If f is a polynomial of degree $n > 0$ whose domain is the set C of all complex numbers, then there exists a c in C such that $f(c) = 0$.

We note that no polynomial equations may be solved in the proof, explicitly or implicitly. This fact would seem to preclude the use of the modulus function,

$$|a + bi| = \sqrt{(a^2 + b^2)},$$

which assumes a positive solution to the polynomial equation

$$z^2 - (a^2 + b^2) = 0.$$

The proof of the existence of such a solution is established independently of the Theorem in, for example, Fulks' Advanced Calculus (p. 53). With its foundation thus assured, we will use the modulus function freely in the proof.

The proof requires three lemmas; we assume for each of them the same hypotheses concerning f that we use in the statement of the Theorem.

~~Lemma 1:~~ $w(z) = |f(z)|$ is continuous.

Proof: we assume from elementary complex variable theory that f is a continuous function of z. Then we need only show that the modulus of a continuous function is continuous. Let an $\epsilon > 0$ be given. Then, since f is continuous, for any given z_o there exists a $\delta > 0$ such that $|f(z) - f(z_o)| < \epsilon$ whenever $|z - z_o| < \delta$. But $||f(z)| - |f(z_o)|| \leq |f(z) - f(z_o)|$, so that $||f(z)| - |f(z_o)|| < \epsilon$ whenever $|z - z_o| < \delta$. Hence, $w = |f|$ is continuous.

Lemma 2: If Z is a subset of C and $w(Z)$ is a bounded set, then Z is bounded.

Proof: Suppose, on the contrary, that Z is unbounded. This means there exists a sequence $\{z_i\}$, z_i in Z for all i, such that for every $M > 0$ there exists a positive integer m such that $|z_m| > M$. Consider $w(z)$:

$$w(z) = |f(z)| = |a_n| \cdot |z^n| \cdot \left| 1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_o}{a_n z^n} \right|$$

Now, $|(1 + \ldots)| \geq 1 - \dfrac{|a_{n-1}|}{|a_n| \cdot |z|} - \cdots - \dfrac{|a_o|}{|a_n| \cdot |z^n|}$.

Let $A = \underset{0 \leq i \leq n}{\text{maximum}} \dfrac{|a_i|}{|a_n|}$, and let P be any positive number. Then if M is greater than the larger of $2nA$ and $2P/|a_n|$, there exists an m such that $|z_m| > M$ and

$$\frac{|a_{n-1}|}{|a_n||z_m|} + \cdots + \frac{|a_o|}{|a_n||z_m|^n} < \frac{1}{2}.$$

Hence, $|(1 + \ldots)| > 1 - \frac{1}{2} = \frac{1}{2}$, so

$$w(z_m) > |a_n| M^n (\tfrac{1}{2}) \geq (\tfrac{1}{2}) | a_n| [2P/| a_n|]^n \geq P.$$

Thus, for any given P the sequence $w(z_i)$ has a term greater than P. Therefore, $w(Z)$ is unbounded, contrary to hypothesis.

Lemma 3: (D'Alembert's Lemma) If $f(a) \neq 0$, then there exists an h such that $|f(a + h)| < |f(a)|$.

Proof: We write out $f(a + h)$ in order of increasing powers of h: $f(a + h) = f(a) + Ah^m + Bh^{m+1} + \ldots + a_n h^n$, where A, B, etc., may depend on a but do not depend on h, and where $1 \leq m \leq n$ and $A \neq 0$.

$$f(a + h) = f(a) + Ah^m + Ah^m \left[\frac{B}{A} h + \cdots + \frac{a_n}{A} h^{n-m}\right]$$
$$= f(a) + Ah^m + \Delta.$$

Now, h is determined by two parameters: its modulus, and its argument, so that

$$h = |h| \exp i [\arg(h)].$$

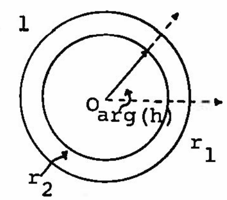
We will restrict $|h|$ and $\arg(h)$ so that $|f(a + h)| < |f(a)|$. The restrictions are:

(1) $0 < |Ah^m| < |f(a)| \neq 0$

(2) $\left|\frac{B}{A} h + \cdots + \frac{a_n}{A} h^{n-m}\right| < 1$

(3) $\arg(Ah^m) = \arg[f(a)] + \Pi$, or
$\arg(h) = \frac{1}{m}\{\arg[f(a)] + \Pi - \arg(A)\}$.

Fig. 1



The left-hand sides of the first and second restrictions are both moduli of polynomials in h; by Lemma 1, they are continuous functions of h. The two of them are both 0 at $h = 0$, and the two right-hand sides are both constants. Hence, there exist r_1 and r_2 such that if $|h| < r_1$, the first restriction is satisfied, and if $|h| < r_2$, the second one is. Therefore, choose r_o to be the lesser of r_1 and r_2. Then if $|h| < r_o$, both restrictions are satisfied.

Fig. 2 shows how the three restrictions accomplish their goal of keeping $|f(a + h)| < |f(a)|$. The first circle is drawn with the origin as its center and radius $|f(a)|$. The center of the second circle is the point representing the sum $f(a)+Ah^m$, and its radius is $|Ah^m|$. The third restriction establishes that this point lies on the line through the vector from the origin to the point representing $f(a)$; the first restriction assures us that the second circle is contained within the first. Finally, the second restriction makes it mandatory for the point representing $f(a+h) = f(a)+Ah^m+\Delta$ to lie within the inner circle, and *a fortiori* within the outer one. But the radius of the outer circle is $|f(a)|$; hence, any h satisfying the three restrictions (that is, any h on the open segment marked in Fig. 1 will also satisfy $|f(a+h)| < |f(a)|$].

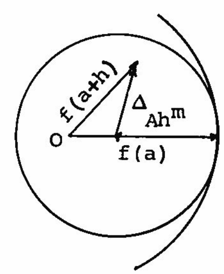


Fig. 2

At last we come to the main argument.

Proof of the Theorem: Contrary to the conclusion of the theorem, suppose that for all z it is true that $w(z) = |f(z)| > 0$. Let $A = \inf_z (w)$. There are then two conceivable cases:

(a) $w > A$ for all z, or

(b) There exists a c such that $w(c) = A$.

In his attempt to prove the Theorem, D'Alembert failed to realize the possibility of Case (a).

Case (a): $w > A$ for all z.

(1) Using the definition of A, we construct a sequence of values $\{w_n\}$ as follows:

(a) Choose $B > A$.

(b) By the definition of A as infimum, there exists a z_1 such that

$$\frac{A+B}{2} > w_1 = w(z_1) \geq A.$$

(c) By hypothesis, equality is impossible, so

$$\frac{A+B}{2} > w_1 > A.$$

(d) Using w_1 as a new B, iterate the process to obtain a monotone decreasing sequence $\{w_n\}$ converging to A.

(2) Consider the corresponding sequence $\{z_n\}$.

(a) $A < w_n < \frac{A+B}{2}$ for all n implies that $\{z_n\}$ is bounded for all n (Lemma 2).

(b) Therefore, $\{z_n\}$ has at least one cluster point c (Bolzano-Weierstrass Theorem). By definition, every neighborhood of c contains an infinite number of points of $\{z_n\}$.

(3) Consequently, since w is continuous (Lemma 1), every neighborhood of $w(c)$ contains an infinite number of points of $\{w_n\}$. Hence, $w(c)$ is a cluster point of $\{w_n\}$.

(4) Inasmuch as $\{w_n\}$ possesses a limit, however, the cluster point must be the limit point: i.e., $w(c) = A$. This result contradicts our supposition that $w > A$ for all z. Case (a) is impossible; Case (b) must hold.

Case (b): There exists a c such that $w(c) = A$.

(1) Suppose $A > 0$. Then according to Lemma 3 (D'Alembert's), there exists a z_0 such that

$$w(z_0) < w(c) = A,$$

contrary to the definition of A as infimum.

(2) Therefore, $A = 0$. Then $w(c) = 0$, which is true if and only if $f(c) = 0$.

I should like to express my thanks to Dr. Ralph Steinlage for his help and encouragement.

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AN r^{th} ROOT ALGORITHM

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THEOREM. Let r, x_1, A, be positive real numbers such that $0 < A^{1/r} < x_1$, and $r > 1$. Then the sequence $\{x_i\}$ defined by $x_{i+1} = \frac{1}{r}[(r-1)x_i + \frac{A}{x_i^{r-1}}]$ converges to $A^{1/r}$. Moreover,

$$|x_{i+1} - A^{1/r}| < \left(\frac{r-1}{r}\right)^i |x_1 - A^{1/r}|, \quad i = 1,2,\ldots .$$

Proof. Let $u = A^{1/r}$. We first show that $x_i > u$ implies $x_{i+1} > u$, for $i = 1,2,\ldots$. Assume that for some integer k, we have $x_k > u$.

$$x_{k+1} - u = \frac{1}{r}[(r-1)x_k + \frac{u^r}{x_k^{r-1}}] - u$$

$$= \frac{x_k}{r}[(\frac{u}{x_k})^r - r(\frac{u}{x_k}) + (r-1)]$$

Now $0 < \frac{u}{x_k} < 1$. For $0 \le x \le 1$, define the function f to be such that $f(x) = x^r - rx + (r - 1)$. Then $\frac{d}{dx} f(x) = r(x^{r-1} - 1) < 0$, on $0 \le x < 1$. $f(0) = r - 1 > 0$. $f(1) = 0$. Thus $f(x) > 0$ for all x on $0 < x < 1$, and the inductive step follows from the continuity of f. Since $x_1 > u$, we conclude that $x_i > u$, $i = 1,2,\ldots$.

We now show that the sequence converges to u.

$$x_i - x_{i+1} = x_i - \frac{1}{r}\left[(r-1)x_i + \frac{u^r}{x_i^{r-1}}\right]$$

$$= \frac{1}{rx_i^{r-1}}(x_i^r - u^r) > 0, \quad i = 1,2,\ldots .$$

Hence $0 < u < \cdots < x_3 < x_2 < x_1$ and so $\{x_i\}$ converges.

Let the limit of $\{x_i\}$ be L.

$$L = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} \frac{1}{r}\left[(r-1)x_i + \frac{u^r}{x_i^{r-1}}\right]$$

$$L = \frac{1}{r}\left[(r-1)L + \frac{u^r}{L^{r-1}}\right], \quad \text{so that} \quad L^r = u^r.$$

But $0 < u \le L$.

Therefore, $L = u = A^{1/r}$.

Finally,

$$x_{i+1} - u = \frac{1}{r}\left[(r-1)x_i + \frac{u^r}{x_i^{r-1}}\right] - u$$

$$< \frac{1}{r}\left[(r-1)x_i + \frac{u^r}{u^{r-1}}\right] - u$$

$$= \frac{r-1}{r}(x_i - u)$$

Therefore,

$$|x_{i+1} - u| < \left(\frac{r-1}{r}\right)\left[\frac{r-1}{r}(x_{i-1} - u)\right] < \cdots < \left(\frac{r-1}{r}\right)^i |x_1 - u| .$$

<u>Remark</u>: For $r = 2$, the theorem yields Newton's well-known square root algorithm.

FORMAL POWER SERIES OVER A COMMUTATIVE RING WITH IDENTITY

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I. INTRODUCTION.

Let R be a commutative ring with identity. In the study of abstract algebra, a basic object of study is the ring of polynomials in one indeterminate X over R. This ring is denoted by R[X]. This paper provides the definition and some basic results concerning a generalization of the concept of a polynomial ring. The notation here is rather standard. We use $\in$ for "is a member of," $\le$ for "is a subset of," $<$ for "is a proper subset of," and 1 for the identity of the ring R.

II. DEFINITION OF R[[X]].

Consider sequences of elements of R of the following type, $\{r_i\}_{i=0}^\infty$. Let S denote the set of all such sequences. For $\alpha = \{r_i\}_{i=0}^\infty$, $\beta = \{s_i\}_{i=0}^\infty$, $\alpha = \beta$ if and only if $r_i = s_i$, $i = 0,1,2,\cdots$.

For α and β as above, $\alpha + \beta = \gamma = \{t_i\}_{i=0}^\infty$ where $t_i = r_i + s_i$, $i = 0,1,2,\cdots$, and $\alpha \cdot \beta = \{u_i\}_{i=0}^\infty$ where $u_i = \sum_{j+k=i} r_i + s_i$, $i = 0,1,2,\cdots$.

It is straightforward to verify that S is a commutative ring with identity $\{1,0,0,\ldots\}$ under + and $\cdot$. It is obvious that the mapping $\emptyset$ from R into S defined by $\emptyset(r) = \{r,0,0,\ldots\}$ is an isomorphism and therefore induces an imbedding of R in S. If for $r \in R$ we identify r and the sequence $\{r,0,0,\ldots\}$ and if we denote by X the element $\{0,1,0,0,\ldots\}$ of S, then we may easily see that any element $\alpha = \{a_i\}_{i=0}^\infty$ of S may be represented uniquely in the form $a_0 + a_1X + \cdots + a_nX^n + \cdots$. It is this representation of α with which we most commonly work. Further, it is clear from this representation of the elements of S that $S > R[X]$; that is, any polynomial $f(X) \in R[X]$ is merely a member of S all of whose coefficients are zero from some point on. It is this fact that motivates the notation. S is usually denoted R[[X]] and we call R[[X]] the ring of formal power series in one indeterminate X over R. The elements of R[[X]] are called the <u>formal power series</u> or simply <u>power series</u>.

Definition. If $\alpha = \sum_{i=0}^{\infty} r_i X^i \in R[[X]]$ and if $\alpha \neq 0$, by the order of α we mean the smallest nonnegative integer k such that $r_k \neq 0$. The order of 0 is not defined. If α has order k, we shall call r_k the leading coefficient of α.

III. SOME ELEMENTARY PROPERTIES OF R[[X]].

3.1. Proposition. R[[X]] is an integral domain if and only if R is.

Proof. If R[[X]] has no zero divisors, then $R \subset R[[X]]$ also has none. Conversely, let $\alpha, \beta \in R[[X]] - \{0\}$. Let the lead- leading coefficient of α be a_n and the leading coefficient of β be b_m. Since R is an integral domain $a_n b_m \neq 0$. But $a_n b_m$ is the leading coefficient of $\alpha\beta$. Hence, $\alpha\beta \neq 0$. Thus R[[X]] is an integral domain.

3.2. Remark. The corresponding result is also true of R[X].

Proof. The same arguments apply.

3.3. Proposition. An element $\alpha = \sum_{i=0}^{\infty} r_i X^i$ is a unit of R[[X]] if and only if r_0 is a unit of R.

Proof. Recall that b is a unit of R provided there exists an element $c \in R$ such that $bc = 1$. Now if α is a unit of R[[X]], it is obvious that r_0 is a unit of R since α a unit of R[[X]] means there exists $\beta \in R[[X]]$, $\beta = s_0 + s_1X + \cdots$, such that $\alpha\beta = 1$, and this equality implies $r_0s_0 = 1$; that is, r_0 is a unit of R.

For the converse, suppose r_0 is a unit of R. Then there exists an element $s_0 \in R$ such that $r_0s_0 = 1$. We proceed inductively to define a sequence $\{s_i\}_{i=0}^{\infty}$ in such a manner that $s_0r_0 = 1$ and such that $\sum_{j+k=i} r_js_k = 0$ for $i = 1,2,\cdots$. We have defined $s_0 = r_0^{-1}$. Having defined $s_0, s_1, \cdots, s_{n-1}$, we wish to define s_n so that $s_nr_n + s_{n-1}r_1 + \cdots + s_0a_n = 0$. Thus define $s_n = -r_0^{-1}(s_{n-1}r_1 + \cdots + s_0r_n)$. Then $s_0, s_1, \cdots, s_n$ satisfy the required conditions. Now $\beta = s_0 + s_1X + \cdots \in R[[X]]$ and by the choice of s_i's, $\alpha\beta = 1$ so that α is a unit of R[[X]].

3.4. Remark. $\alpha = r_0 + r_1X + \cdots + r_nX^n \in R[X]$ is a unit of R[X] if and only if r_0 is a unit of R and r_i, $1 \le i \le n$, is nilpotent; that is, there exists n_i, a positive integer, such that $r_i^{n_i} = 0$. The proof of this result is omitted since R[X] is not the principal topic of investigation here. It is worth noting that this result differs considerably from 3.3.

Definition. Let R be a ring and A, an ideal of R. By a basis S for A, we mean a subset S of A such that each element $b \in A$ is expressible as a finite sum of the form $r_1s_1 + r_2s_2 + \cdots + r_ns_n$ where $r_1,\cdots,r_n \in R$, and $s_1, \cdots, s_n \in S$. We write A = (S). If S is a finite set, we say the ideal A is finitely generated. If each ideal A of R is finitely generated, R is said to be a Noetherian ring.

3.5. Proposition. R is Noetherian if and only if R[[X]] is. No formal proof will be presented. The proof in one direction is easy. If R[[X]] is Noetherian then the mapping ϕ from R[[X]] onto R defined by $\phi(\alpha) = r_0$, where $\alpha = r_0 + r_1X + \cdots \in R[[X]]$, is a homomorphism of R[[X]] onto R. But a homomorphic image of a Noetherian ring is Noetherian. Hence, R is Noetherian. A proof that R Noetherian implies R[[X]] is Noetherian may be found in [2; 50].

3.6. Remark. Proposition 3.5 remains valid with R[[X]] replaced throughout by R[X]. One half of this result is the celebrated Hilbert basis theorem and the other half may be proved using the above argument.

It is well known that if R is a field, R[X] is a Euclidean domain in the terminology of [4], and as such is a principal ideal domain (PID); that is, an integral domain with identity in which each ideal is generated by a single element. For R[[X]] we have the following:

3.7. Proposition. Let R be a field. Then the set of all ideals of R[[X]] is $\{R[[X]], (X), (X^2), \cdots, (0)\}$ and the ideals of R[[X]] are related as follows: $R[[X]] \supset (X) \supset (X^2) \supset \cdots \supset (0)$.

Proof. It is obvious that each of the ideals listed is indeed an ideal. Let A be any non-zero ideal of R[[X]], and choose $\alpha \in A$, α of minimal order. Suppose $\alpha = r_kX^k + r_{k+1}X^{k+1} + \cdots = X^k(r_k + r_{k+1}X + \cdots)$. Since R is a field, Proposition 3.3 implies that $r_k + r_{k+1}X + \cdots$ is a unit of R[[X]]; that is $\alpha = X^k \cdot \epsilon$, ϵ a unit of R[[X]]. Therefore $X^k = \alpha \cdot \epsilon^{-1} \in A$. Thus $(X^k) \subseteq A$. conversely, let $\beta = s_nX^n + \cdots \in A$. α was of minimal order among all members of A. Hence, $n \ge k$. Thus, $\beta = X^k(s_nX^{n-k} + \cdots) \in (X^k)$, and $A \subseteq (X^k)$. This proves the first assertion, while the second is obvious.

Definition. An integral domain with identity in which the ideals are linearly ordered is called a valuation ring.

3.8. Corollary. If R is a field, R[[X]] is a valuation ring.

3.9. <u>corollary</u>. If R is a field, R[[X]] is a PID.

<u>Proof</u>. Obvious.

Let R be a Unique Factorization Domain (UFD) in the sense of [4]. Then it is well known that R[X] is also a UFD. That the corresponding result is not true for R[[X]] was shown by Samuel [3]. However Krull has shown in [1;780] that the following result is true.

3.10. <u>Proposition</u>. If R is a PID, R[[X]] is a UFD.

All of the above results are known. There remain, however, many open questions involving power series. For example, a characterization of zero divisors and nilpotent elements of R[[X]] has not been given. Also, many results known to the author could not be presented here, either for the sake of brevity or for the level of presentation. All the results contained in this paper were solved by the author as exercises in a course on commutative algebra. To the instructor of this class, Dr. Robert W. Gilmer, I am deeply indebted both for encouragement and aid in writing this paper.

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NATIONAL MEETING IN AUGUST 1967

Each chapter is encouraged to nominate either a delegate or a speaker for the National Pi Mu Epsilon Meeting to be held in conjunction with the international meeting of the Mathematical Association of America in Toronto, Canada, August 28-30, 1967.

Apply <u>at once</u> to national headquarters for travel funds for your delegate ($75 maximum) or speaker ($150 maximum). It is important that your best student speaker be given an opportunity to participate in this meeting and that YOUR chapter be represented. Write: Dr. Richard V. Andree, Pi Mu Epsilon, The University of Oklahoma, Norman, Oklahoma 73069.

A SHORT AXIOMATIC SYSTEM FOR BOOLEAN ALGEBRA

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The purpose of this paper is to set forth and explain a set of seven axioms for Boolean Algebra, to prove that they are equivalent to the ordinary axioms, and to show that the three axioms which peculiarly characterize the Boolean Algebra -- the axioms of complementation (union is defined by means of the complement) -- are independent.

Axioms, Definitions, Basic Theorems

A Boolean Algebra is a set X such that, for all $a, b, c, \ldots \in X$:

A. There is defined a (closed) binary operation (Intersection) such that:

<u>Axiom 1</u>: $a \cap (b \cap c) = (a \cap b) \cap c$ (Associative)
<u>Axiom 2</u>: $a \cap b = b \cap a$ (Commutative)
<u>Axiom 3</u>: a A a = a (Idempotent)

B. There exists an element $I \in X$ such that

<u>Axiom 4</u>: $a \cap I = a \ \forall a \in X$ (Identity)

C. There can be defined a function $\cdot'$ (Complementation) from X to itself such that:

<u>Axiom 5</u>: $(a')' = a \ \forall a \in X$
<u>Axiom 6</u>: $a \cap a' = I' \ \forall a \in X$
<u>Axiom 7</u>: $a \cap b = I' \rightarrow a \cap b' = a$.

Three definitions are in order to clarify matters:

<u>Def. 1</u>: $a \trianglelefteq b : a \cap b = a$ (Inclusion)

<u>Def. 2</u>: $a \cup b = (a' \cap b')'$ (Union)
(This definition of union is merely a rephrasing of DeMorgan laws.)

<u>Def. 3</u>: $\Theta = I'$ ("null set")

Three basic theorems will now be presented to complete the picture. (Here and hereafter, when a theorem has a very straightforward and trivial proof, I will save space by omitting the proof.)

THEOREM 1 (Uniqueness of I): At most one element of X satisfies the property of I (Axiom 4).

THEOREM 2 (Uniqueness of complementation): At most one function' from X to X can be defined satisfying Axioms 6 and 7.

<u>Proof</u>: Let $\cdot'$ and $\cdot^*$ be two such functions. Then for any $a \in X$, $\Theta = a \cap a' = a' \cap a = a \cap a^* = a^* \cap a$, and therefore, $a' \cap a^* = a' \wedge a^* \cap a' = a^*$, which implies $a' = a^*$ by commutativity.

THEOREM 3 (0 is "smallest" element): $0 \trianglelefteq a \ \forall\, a \in X$.
Proof: $0 \cap a = a \cap 0 = a \cap (a \cap a') = (a \cap a) \cap a' = a \cap a' = 0$.

We can now explain the meaning of the seven axioms. The first three axioms are easily shown to be equivalent to the assumption that X is a p. o. set (where $a \leq b \Leftrightarrow a \trianglelefteq b$) with a glb for every finite subset (glb $\{a,b\} = a \cap b$, etc.). The fourth axiom says that X has a greatest element under this p. o. The last three axioms imply X has a smallest element (THM 3), and state that X can be divided into pairs (of complements) such that, not only do the elements of such a pair not meet (i.e., they are "as incomparable as possible": their greatest and only lower bound is 0, a lower bound of everything), but each member of the pair contains everything in X that does not meet the other.

Proof of Equivalence with the Ordinary Axioms

First we will show that the ordinary axioms imply the system given above.

THEOREM 4: Axioms 1 - 7 and Definitions 1 - 3 are true in any system which satisfies the ordinary axioms of Boolean Algebra.

Proof: Axioms 1 - 6 and Definitions 1 - 3 are all statements or rephrasings of certain of the ordinary axioms of Boolean Algebra. And Axiom 7 is implied by the distributive law: $a \cap b = 0 \Rightarrow a \cap b' = (a \cap b') \cup (a \cap b) = a \cap (b' \cup b) = a \cap I = a$.

Now we will show that the implication runs the other way also. The only real difficulty is with the distributive laws.

A. Axioms of Intersection: These are given, as Axioms 1 - 4 and 6.

B. Axioms of Union:

THEOREM 5 (Associative): $a \cup (b \cup c) = (a \cup b) \cup c$
Proof: $a \cup (b \cup c) = a \cup (b' \cap c')' = (a' \cap (b' \cap c')'')' = (a' \cap (b' \cap c'))' = ((a' \cap b') \cap c')' = ((a' \cap b')'' \cap c')' = (a' \cap b')' \cup c = (a \cup b) \cup c$.

THEOREM 6 (Commutative): $a \cup b = b \cup a$.

THEOREM 7 (Idempotent): $a \cup a = a$.

THEOREM 8 (Identity): $a \cup 0 = a$.

THEOREM 9 (Complement): $a \cup a' = I$.

THEOREM 10 (Property of I): $a \cup I = I$.

C. The Distributive Laws: These turn out to follow from Axiom 7:

THEOREM 11 (First distributive law): $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$.
Proof: We will proceed by steps.

Lemma 1: $a \cap (a \cap b)' = a \cap b'$.
Proof: $0 = (a \cap b) \cap (a \cap b)' = a \cap (b \cap (a \cap b)')$

$= (a \cap (a \cap b)') \cap b$

$\therefore a \cap (a \cap b)' \cap a \cap b' = a \cap a \cap (a \cap b)' \cap b'$

$= (a \cap (a \cap b)') \cap b'$

$= a \cap (a \cap b)'$.

But $0 = a \cap 0 = a \cap (b' \cap b) = a \cap a \cap b' \cap b$

$= (a \cap b') \cap (a \cap b)$

$\therefore a \cap (a \cap b)' \cap a \cap b' = a \cap a \cap b' \cap (a \cap b)'$

$= (a \cap b') \cap (a \cap b)' = a \cap b'$.

$\therefore a \cap (a \quad b)' = a \cap b'$.

Lemma 2: $(a \cap b) \cup (a \cap b') = a$.
Proof: $((a \cap b) \cup (a \cap b'))' = (a \cap b)' \cap (a \cap b')'$

$= (a \cap b)' \cap (a \cap b')' \cap (a \cap b)'$

$= (a \cap b)' \cap (a \cap b')' \cap (a \cap b'')'$

$= (a \cap b)' \cap (a \cap b')' \cap (a \cap (a \cap b')')'$

$= (a \cap b)' \cap (a \cap b')' \cap (a)'$

$= (a' \cap a \cap b)' \cap (a' \cap a \cap b')' \cap a'$

$= (0 \cap b)' \cap (0 \cap b')' \cap a' = (0)' \cap (0)' \cap a'$

$= I \cap I \cap a' = a'$.

$\therefore (a \cap b) \cup (a \cap b') = ((a \cap b)' \cap (a \cap b')')' = (a')' = a$.

~~Proof of~~ Theorem:

i) $a \cap ((a \cap b)' \cap (a \cap c)')' = a \cap (a \cap (a \cap b)' \cap (a \cap c)')'$

$= a \cap (a \cap (b)' \cap (c)')' = a \cap (b' \cap c')' = a \cap (b \cup c)$.

ii) $a' \cap ((a \cap b)' \cap (a \cap c)')' = a' \cap (a' \cap (a \cap b)' \cap (a \cap c)')'$

$= a' \cap (a' \cap (a' \cap a \cap b)' \cap (a' \cap a \cap c)')'$

$= a' \cap (a' \cap 0' \cap 0')' = a' \cap (a')' = 0$.

iii) $a \cap (b \cup c) = (a \cap (b \cup c)) \cup 0 = [a \cap ((a \cap b)' \cap (a \cap c)')'] \cup$
$[a' \cap ((a \cap b)' \cap (a \cap c)')'] = ((a \cap b)' \cap (a \cap c)')'$

$= (a \cap b) \cup (a \cap c)$. QED

THEOREM 12 (Second distributive law): $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.
<u>proof:</u> $a \cup (b \cap c) = (a' \cap (b \cap c)')' = (a' \cap (b' \cup c'))'$

$= ((a' \cup b') \cup (a' \cup c'))' = ((a \cup b)' \cup (a \cup c)')'$

$= ((a \cup b)'' \cap (a \cup c)'')'' = (a \cup b) \cap (a \cup c)$.

D. Properties of Inclusion and Complementation: These are listed below, though they habe been mentioned before. They have been proven, or their proofs are trivial.

THEOREM 13 (de Morgan's Laws): a) $(a \cap b)' = a' \cup b'$

b) $(a \cup b)' = a' \cap b'$.

Note 1 (extreme elements): $\forall a \in X$, a) $0 \trianglelefteq a$

b) $a \trianglelefteq I$

THEOREM 14 (partial, ordering)= $\forall a,b,c \in X$,

a) $a \trianglelefteq a$

b) $a \trianglelefteq b \wedge b \trianglelefteq c \Longrightarrow a \trianglelefteq c$

c) $a \trianglelefteq b \wedge b \trianglelefteq a \Longrightarrow a = b$.

Independence of the Complementation Axioms

The examples given here to prove independence are all subsets of power sets which are closed under finite intersection, and which contain the Identities of their respective power sets. Hence they satisfy Axioms 1 - 4.

THEOREM 15 (Independence of Axiom 5): Axioms 1 - 4, 6, and 7 do not imply Axiom 5.
<u>Proof</u>: Let $X = \{0,a,I\}$ where $0 = \emptyset$, $a = \{1\}$, and $I = \{1,2\}$. Define $I' = a^* = 0 \wedge 0' = 1$. Axiom 6 is seen to be satisfied; and so is Axiom 7, because $x \cap y = 0 \Longrightarrow x = 0$ or $y = 0$ for $x,y \in X$. But Axiom 5 obviously must fail, for $'$ is not 1-1.

THEOREM 16 (Independence of Axiom 6): Axioms 1 - 5 and 7 do not imply Axiom 6.
<u>Proof</u>: Let $X = \{0,I\} \cup \{L_N : N \in Z\}$, where $0 = \emptyset$, Z = the set of all integers, $I = Z$, and $L_N = \{n \in Z : n \leq N\}$. Define $I' = 0$, $0' = 1$, and $L_N' = L_N$ $\forall N \in Z$. Axiom 5 is obviously satisfied. Axiom 7 is satisfied, because $a,b \in X \wedge a \cap b = 0 \Longrightarrow a = 0$ or $b = 0$. But Axiom6 is not satisfied: $L_N \cap L_N' = L_N \neq 0$ $\forall N \in Z$.

THEOREM 17 (Independence of Axiom 7): Axioms 1 - 6 do not imply Axiom 7.
<u>proof</u>: Let $X = \{0,I,a_1,a_2,a_3,a_4\}$, where $0 = \emptyset$, $I = \{1,2,3,4\}$, and $a_i = \{i\}$.

Define $0' = 1$, $I' = 0$, and $a_i' = a_{(5-i)}$. Inspection shows both Axiom 5 and Axiom 6 satisfied, but Axiom 7 is not -- e.g., $a_1 \cap a_2 = 0$, but $a_1 \cap a_2' = a_1 \cap a_3 = 0 \neq a_1$.

Reference

Allendoerfer and Oakley: <u>Principles of Mathematics</u>.

PROBLEM DEPARTMENT

Edited by
M. S. Klamkin, Ford Scientific Laboratory

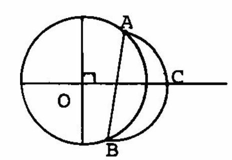
This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to M. S. Klamkin, Ford Scientific Laboratory, P. 0. Box 2053, Dearborn, Michigan 48121.

PROBLEMS FOR SOLUTION

187. Proposed by R. C. Gebhardt, Parsippany, N. J.
A semicircle ACB is constructed, as shown, on a chord AB of a unit circle. Determine the chord AB such that the distance OC is a maximum.



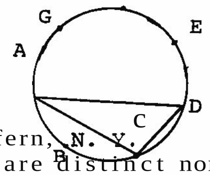
188. Proposed by Waldemar Carl Weber, University of Illinois. For any two real numbers x and y with $0 < x \leq y$, verify the following procedure for adding on a slide rule using the A, S, and T scales. First setting of slide:

A	opposite y	opposite x
T	set right index	read angle 9, $0 < \theta < \pi/4$

Second setting of slide:

A	opposite x		read x+y
S	set angle 9		opposite right index

189. Proposed by Leon Bankoff, Los Angeles, California.
If A, B, C, D, E, F, and G denote the consecutive vertices of a regular heptagon, show that CD is equal to half the harmonic mean of AC and AD.



190. Proposed by Joseph Arkin, Suffern, N. Y.
If w, v, t, n, u, q, k, and r are distinct non-zero integers, find infinitely many solutions to the diophantine equation

$$w^4 + v^4 + t^4 + n^8 = u^4 + q^4 + k^4 + r^8$$

where w, v, u, and q are each a hypotenuse of some Pythagorean right triangle.

191. Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.
Let P and P' denote points inside rectangles ABCD and A'B'C'D', respectively. If PA = a + b, PB = a + c, PC = c + d, PD = b + d, P'A' = ab, P'B' = ac, P'C' = cd, prove that P'D' = bd.

SOLUTIONS

155. Proposed by William J. LeVeque, University of Michigan.
Two mountain climbers start together at the base of a mountain and climb along two different paths to the summit. Show that it is always possible for the two climbers to be at the same altitude during the entire trip (assuming each path has on it a finite number of local maxima and minima).

Editorial note: The proposer notes that the problem is not original with him and he does not know the original proposer.

Solution by the proposer.
With no loss in generality, each path may be regarded as a plane polygonal path connecting the origin and the point (1,1), entirely contained in the unit square and having one ordinate for each abscissa. Suppose first that the ends are the only points of the paths at heights 0 or 1.

Represent one such path, P_1, in an (x^1,y)-plane, and the other, P_2, in an (x^2,y)-plane. For each y with $0 \le y \le 1$, there is a finite set of values $x_1^1(y), x_2^1(y), \cdots$ of x^1 for which $(x_i^1(y),y)$ is on P_1, and a corresponding set of values $x_j^2(y)$ of x^2. Plot all the points $(x_i^1(y), x_j^2(y))$ for all combinations of i and j, and for all y, in an (x^1,x^2)-plane, thus determining a point set S. S lies entirely in the open square $0 < x^1 < 1$, $0 < x^2 < 1$, except for the two points (0,0) and (1,1) on it. Two climbers are at the same height on the two paths if and only if their positions give a point of S, and the problem reduces to showing that S contains an arc connecting (0,0) and (1,1) in the (x^1,x^2)-plane.

Any point in the closed unit square U in the (x^1,x^2)-plane determines unique positions on the two paths. In particular, the point (1,0) places one climber at the top, the other at the bottom; the point (0,1) gives the reverse positions. An arc connecting (1,0) and (0,1) represents a recipe for getting one man down the mountain while the other ascends it; obviously, under any such prescription, the climbers are at the same height at some instant. That is, any arc in U connecting (0,1) and (1,0) intersects S. It follows that S connects boundary points of U, and hence connects the only two possible boundary points, (0,0) and (1,1).

If one (or both) of the paths has several points at height 0, it can be modified slightly so as to have minima at distinct heights very close to 0 (closer than any of the other minima except the beginning point), and a simple continuity argument shows that the lowest minimum can again be dropped to 0, then the next lowest, etc. The case of several maxima of height 1 can be handled similarly.

161*. Proposed by Paul Schillo, SUNY at Buffalo.
It is conjectured that the smallest triangle in area which can cover any given convex polygon has an area at most twice the area of the polygon.

Editorial note: This is a known result and is given in H. G. Eggleston, Problems in Euclidean Space, Pergamon, N. Y., 1957, p. 156:
"Theorem 9.5: Let Γ be a convex set. Then every triangle circumscribing Γ is of area greater than or equal to twice that of Γ if and only if Γ is a parallelogram."

177. Proposed by C. S. Venkataraman, Sree Kerala Varma college, Trichur, South India.
If s is the semi-perimeter and R, r, r_1, r_2, and r_3 are the circum-, in-, and ex-radii, respectively, of a triangle, prove that

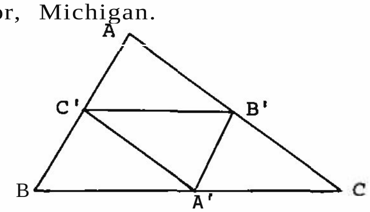
$$\frac{R}{r^2} \geq \frac{2s^2}{r_1 r_2 r_3} .$$

Solution by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.
We start with the known inequality, $R \geq 2r$, with equality if and only if the triangle is equilateral. It is also known that $\sqrt{r r_1 r_2 r_3} = K$ where K is the area of the triangle (see N. A. Court, College Geometry, p. 79). Since also $K = rs$, we have $r r_1 r_2 r_3 = r^2 s^2$. Finally,

$$\frac{R}{r^2} \geq \frac{2}{r} = \frac{2s^2}{r_1 r_2 r_3} .$$

Also solved by H. Kaye (Brooklyn, N. Y.), Paul Meyers (Philadelphia, Pa.), M. Wagner (N.Y.C.), F. Zetto (Chicago, Ill.) and the proposer.

178. Proposed by K. S. Murray, Ann Arbor, Michigan.
Show that the centroid of triangle ABC coincides with that of triangle $A'B'C'$ where A', B', and C' are the mid-points of $\overline{BC}$, TO, and $\overline{AB}$, respectively. Also, generalize the result.



Solution by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.
Since $AB'A'C'$ is a parallelogram, AA' bisects $B'C'$. Hence AA' is a median of both triangle ABC and $A'B'C'$. Hence the medians of both these triangles meet at the same point.

Generalization: Let $A_0, A_1, A_2, \ldots, A_r$ be the vertices of an r-simplex and let B_i be the centroid of the $(r-1)$-dimensional face opposite A_i, $i = 0, 1, \ldots, r$. Then the centroid of the r-simplex with vertices $B_0, \ldots, B_r$ is the same as the centroid of the original r-simplex.

Proof: We use the following facts. The medians of an r-simplex meet at the centroid and this point is $1/(r+1)$ of the way up from the base. [A median of an r-simplex is a line going from a vertex to the centroid of the opposite face.] Therefore points $B_1', B_2', \ldots, B_r'$ form

an r-simplex homothetic to the original one. Therefore median $B_0 B_0'$ is also a median of the medial r-simplex since it passes through the centroid of the r-simplex formed by $B_0, B_1', B_2', \ldots, B_r'$. So both sets of medians meet at the same point. Hence the r-simplex and its medial simplex have the same centroid.

Editorial note: There is a still further generalization and it is easily established by means of vectors. Although the generalization holds for an n-dimensional simplex, we only illustrate it for n = 3. Let $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ denote four linear independent vectors from some origin O to the four vertices A, B, C, and D, respectively, of the tetrahedron. Its centroid is then given by $(\vec{A} + \vec{B} + \vec{C} + \vec{D})/4$. We now consider another tetrahedron whose four vertices lie on the four faces of our initial tetrahedron and are given by

$$\frac{r\vec{A}+s\vec{B}+t\vec{C}}{r+s+t}, \quad \frac{r\vec{B}+s\vec{C}+t\vec{D}}{r+s+t}, \quad \frac{r\vec{C}+s\vec{D}+t\vec{A}}{r+s+t}, \quad \frac{r\vec{D}+s\vec{A}+t\vec{B}}{r+s+t},$$

where $r,s,t \geq 0$. The centroid of this latter tetrahedron coincides with that of the initial one. If we let all the weights r, s, t, be equal, we obtain the previous result.

Also solved by Paul Meyers (Philadelphia, Pa.), Philip Trauber (Brooklyn College), M. Wagner (N.Y.C.), G. Weeks (San Francisco, Calif.) and the proposer.

179. Proposed by Donald Schroeder, Seattle, Washington.
It is well known that

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2.$$

Generalize the above by finding integers a satisfying

$$\sum_{k=0}^{m} (a + k)^2 = \sum_{k=m+1}^{m} (a + k)^2 .$$

Solution by Michael F. Brunner (no listed address).
Squaring out and summing, we obtain the equation

$$a^2 - 2am^2 - 2m^3 - m^2 = 0.$$

Whence,

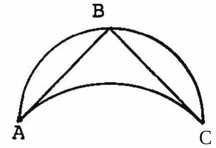
$$a = m^2 \pm m(m + 1).$$

Editorial note: Charles Ziegenfus, Madison College, Virginia, notes that the problem with solution occurs as No. 550 in the Nov., 1964, Mathematics Magazine.

Also solved by J. H. Cozzens (Kettelle Associates, Pa.), R. W. Feldman (Lycoming College, Pa.), E. Johnson (University of South Carolina), O. Marrero (Miami, Fla.), P. Myers (Philadelphia, Pa.), R. Prielipp (University of Wisconsin), S. Rabinowitz (Polytechnic Institute of Brooklyn), G. Weeks (San Francisco, Calif.) and the proposer.

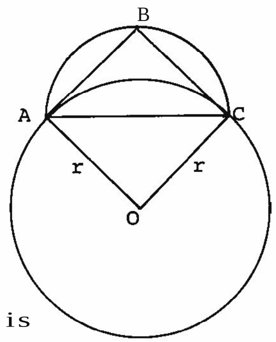
180. Proposed by R. C. Gebhart, Parsippany, N. J.

In the figure, AB = BC and angle ABC = 90°. The arcs are both circular with the inner one being tangent to $\overline{AB}$ at A and $\overline{BC}$ at C. Determine the area of the crescent.



Solution by B. W. King, Burnt Hills-Ballston Lake High School, N. Y.

Let r denote the radius of the circle determined by arc AC and 0 denote its center. It follows that ABCO is a square and that the radius of semicircle ABC is $r/\sqrt{2}$. Then, area of semicircle ABC = $\Pi r^2/4$, area of segment bounded by arc and chord AC = $\Pi r^2/4 - r^2/2$. Finally, the area of the crescent is



$$\frac{\Pi r^2}{4} - \left(\frac{\Pi r^2}{4} - \frac{r^2}{2} \right) = \frac{r^2}{2} = \frac{AB^2}{2} .$$

W. W. Wallace (Wisconsin State University) in his solution notes that since the area of the crescent equals that of triangle ABC, it follows that the sum of the areas of the two smaller segments AB and BC equals the area of the sector AC.

Also solved by J. H. Cozzens (Kettelle Associates, Pa.), R. C. Gebhardt (Parsippany, N. Y.), G. Jacobs (2 sol.) (Temple University), G. Mavrigian (2 sol.) (Youngstown University), S. Rabinowitz (Polytechnic Institute of Brooklyn), P. Trauber (Brooklyn College), M. E. Votypka (John Carroll University), M. Wagner (N.Y.C.), F. zetto (Chicago, Ill.) and the proposer.

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