PI MU EPSILON Journal

VOLUME 4 Fall 1967 NUMBER 7

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PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF THE HONORARY MATHEMATICAL FRATERNITY

Kenneth Loewen, Editor

Chapter reports, books for review, problems for solution and solutions to problems, and news items should be mailed directly to the special editors found in this issue under the various sections. Editorial correspondence, including manuscripts, should be mailed to THE EDITOR OF THE PI MU EPSILON JOURNAL, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73069.

PI MU EPSILON JOURNAL is published semi-annually at The University of Oklahoma.

SUBSCRIPTION PRICE: To individual members, $1.50 for 2 years; to non-members and libraries, $2.00 for 2 years. Subscriptions, orders for back numbers and correspondence concerning subscriptions and advertising should be addressed to the PI MU EPSILON JOURNAL, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73069.

THE PME MOTTO: MEANING AND ETYMOLOGY

Panos D. Bardis

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> "Sosicrates says that Pittacus, after cutting off a small fragment, declared that the half was more than the whole."*
>
> Diogenes Laertius, Pittacus, 75.

Ten paedeusin cae ta mathematica epispeudein: to promote scholarship and mathematics.

1. Ten: feminine gender, singular number, accusative (objective) case of definite, or prepositive, article ho, he, to: the.

In his Definitions, XIV, 1-6, Hero of Alexandria writes: "In presenting to you as briefly as possible, O most illustrious Dionysius, an outline of basic technical terms in geometry, I will take as the starting point, and will base the entire organization on, the teaching of Euclid, the author of Elements, dealing with theoretical geometry" (see Panos D. Bardis, "Hero, the Da Vinci of Ancient Alexandria: His Aeolosphaera and Other Inventions," School Science and Mathematics, June 1965, pp. 535-542).

2. Paedeusin: feminine gender, singular number, accusative case of noun paedeusis: culture, education, educational system, instruction, learning, scholarship, training.

Plato states that "an excellent breeding and education, if emphasized constantly, generates good natures in the polity" (Republic, 424a).

From paes: child.

King Nestor, the military genius and garrulous Methuselah of Pylos, said to King Diomedes of Argos, the second greatest Greek hero---after Achilles---during the Trojan War: "Besides, you are so young, you could even be my child, my youngest born" (Homer, Iliad, IX, 57-58).

English words: pedagog, pedagogy (child leading), pediatrician, pediatrics (child medicine), pedobaptism, pedocephalic pedogenesis (introduced by Von Baer in 1828), pedology, pedomorphism, and many others.

3. Cae: conjunction: and. Also transliterated kai.

In Euclid's Elements, XIII, Scholium 1, we read the following regarding Theaetetus (415-369 B.C.), the Athenian astronomer, mathematician, philosopher, and one of the pupils of Socrates: "In this book, namely, the 13th, are described the so-called five Platonic solids, which, however, are not his, but three of the aforementioned five solids are of the Pythagoreans, that is, the cube and the pyramid and the dodecahedron, the octahedron and the icosahedron being of Theaetetus."

English word: triakaidekaphobia (fear of the number 13).

4. Ta: neuter gender, plural number, accusative case of definite article ho, he, to: the.

The famous Cattle Problem of Archimedes (?), which the noted inventor and mathematician of Syracuse solved in epigrams and sent to the mathematicians of Alexandria in a letter to Eratosthenes, closes with these four lines: "0 stranger, if you find out these things and add them up in your mind, giving all the relations among these quantities, you will depart gloriously and victoriously, knowing that you have been adjudged great in this kind of wisdom."

5. **Mathematica:** neuter gender, plural number, accusative case of adjective mathematicus: fond of learning, mathematical.

Mathematice episteme: mathematical science.

In the Nicomachean Ethics, Aristotle asserts that, "As far as conduct is concerned, the basic principle coincides with the pursued goal, which is analogous to the hypotheses (here the philosopher means propsitions] of mathematics" (VII, viii, 17-18).

From manthanein: to comprehend, to know, to learn, to perceive.

Mathema: knowledge, something learned, lesson, science. Also, mathematical science, especially, arithmetic, astronomy, geometry (see Panos D. Bardis, "Symmentrical Consonance of Play, Rhythm, and Harmony: An Essay on Plato's Mathematics," School Science and Mathematics, January 1963, pp. 52-67).

English words: mathematical, mathematician, mathematics, polymath, and so forth.

6. Epispeudein: infinitive of verb epispeudo: I hasten onward, I promote, I urge on.

From epi and speudein.

.. a. Epi: preposition: by, on, over, upon, and the like.

In dealing with the Pythagorean theorem, Euclid refers to two of the lines of the celebrated "windmill" figure with these words: "two straight lines, namely, AG, AE, not lying on the same side, make the adjacent angles equal to two right ones" (Elements, I, 47).

English words: epiblast, epicalyx, epicanthus, epicardium, epicene, epicenter, epicrisis, epidemic, epidermis epididymis, and myriad others. In mathematics: epicycloid, epitrochoid, epitrochoidal, and many more.

b. Speudein: infinitive of verb speudo: I hasten, I press on, I promote, I quicken.

Herodotus informs us that Queen Tomyris of the Massagetae sent the following message to King Cyrus when he was marching against her kingdom: "0 king of the Medes, cease from promoting what you are promoting" for vou cannot know if these tasks will be for your advantage when finished" (Histories, I, 206).

NOTE

*All quoted passages have been translated by the present author.

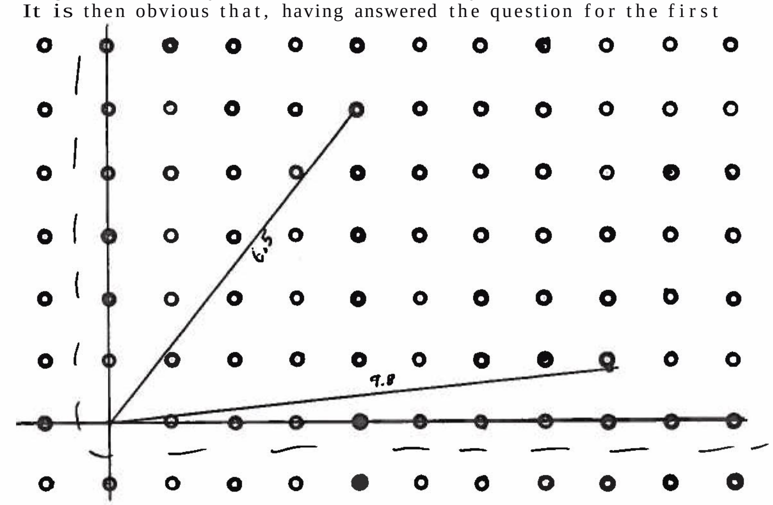
ON NOT SEEING THROUGH THE FOREST FOR THE TREES

Prof. David C. Kay, The University of Oklahoma
(Talk presented to the Oklahoma Alpha chapter)

In a moment I am going to acquaint you with a problem which will lead us quite naturally into two different fields of mathematics---- Minkowski geometry, and number theory. It is an extremely entertaining problem, and it may actually have an application outside mathematics which renders it particularly heart-warming. The problem is in the form of two questions, the second of which makes sense only if the first has a "No" answer.

Consider the xy-plane and the set of all lattice points, points of the form (m,n) where m and n are integers. Now imagine an infinite forest whose trees all have the same radius, say r, and centered at the lattice points (m,n). Now I fell the tree at $(0,0)$ and stand on the stump. The question is, can I see through the forest? Geometrically of course, I am asking whether I can pass a line through $(0,0)$ which does not touch any circle centered at a lattice point and having radius r.

I have prepared a little sketch of the situation for $r = 1/10$ which I'll let you see. Now if we examine the problem carefully for this case we observe several simplifying principles. One is that as I rotate about the origin seeking a line of vision out of the forest, it is not really necessary to consider all directions. If I merely answer the question for the rays whose angles with the x-axis range from 0 to $\pi/4$, I will be able to apply that answer to all the other remaining sectors, since the sector of rays from $\pi/4$ to $\pi/2$ contains the reflected image in the line $y = x$ of that contained by the sector from 0 to $\pi/4$. It is then obvious that, having answered the question for the first



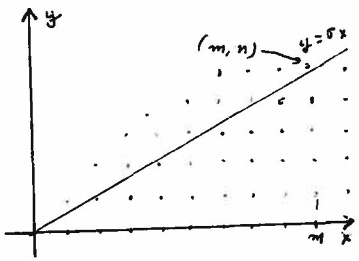
quadrant, I have answered the question for all quadrants. So it suffices to consider only the sector from 0 to $\pi/4$ as indicated by the dotted lines in the diagram I have given you.

Be making a few shrewd calculations, we can actually observe an upper bound in the case $r = 1/10$, and it is approximately 9.8. A few of the longer lines of vision are shown, labelled with their approximate lengths. We may conjecture, therefore, that the answer must be "no."! I cannot see through the forest for the trees. But might we not be jumping to conclusions based on insufficient evidence? What if r is exceedingly small and the trees in the forest are merely very skinny toothpicks? Is it then conceivable I might see through the forest in some carefully chosen direction? It is obvious that I shall be able to seealot farther than I could when $r = 1/10$, but how much farther?

The second question is, assuming that my vision is blocked by some tree in every direction, what is the forthest I can see? Ideally, the solution to both problems would be found by obtaining an expression for the function $f(\sigma,r)$ which measures the distance from the origin to the first tree which the line $y = \sigma x$ touches, if any. If none touch we could define $f(\sigma,r) = \infty$ for that σ and that r. But when we try to write down what we know about $f(\sigma,r)$ it becomes clear that there must be an easier way.

But the, another thought pops readily into mind. Suppose we rephrase the question slightly. Given a line $y = \sigma x$, $0<\sigma<1$, does a lattice point (m,n) come arbitrarily close to this line, or closer than r, by making some choice of the positive integers m and n? The answer is obviously "yes" if σ is rational. But suppose σ is irrational. Let us get an idea how close the lattice point (m,n) is from the line $y = \sigma x$. We can easily see that the difference between the ordinates σm and n, or

$$|\sigma m - n|$$



is a measure of how close (m,n) is to $y = \sigma x$. If this is less than r then the line $y = \sigma x$ will definitely be cut off by the tree centered at (m,n). So our problem becomes that of choosing positive integers m and n so as to minimize

$$|\sigma m - n|$$

where σ is an orbitrarily given irrational, $0<\sigma<1$.

As an example of the true nature of the problem, I have made a few calculations for $\sigma = \sqrt{2}$. Now the rational

$$1.4142 = \frac{14,142}{10,000}$$

approximates $\sqrt{2}$ (which is 1.41421435...) to within .0001435,.. So

$$\left|\sigma - \frac{n}{m}\right| = \left|\sqrt{2} - \frac{14,142}{10,000}\right| = .0001435... \quad \text{but } |m\sigma - n|,$$

the quantity we are interested in, equals .1435.... which is not very small. Thus, closer and closer approximations to σ by rational numbers $\frac{n}{m}$ does not necessarily make $|m\sigma - n|$ small. The choice $n = 99$, $m = 70$ happens to work better for this example, for then,

$$\begin{aligned} n &= 99.000000...\\ m\sigma = 70 \cdot \sqrt{2} &= 98.995004...\\ |m\sigma - n| &= 0.004996... \end{aligned}$$

which gives us a difference of .005 approximately, as compared with .143. But how can we obtain values for n and m which make this smaller than, say 10^{-8}?

Curiously enough this problem appears to be solved in Niven's book "An Introduction to the Theory of Numbers" following a discussion of Farey fractions. Perhaps you have heard of these fractions. For each positive integer n one forms all possible fractions p/q between zero and unity, inclusive, such that $q \leq n$, and reduce to lowest terms, placing them in numerical order. For example, for $n = 5$ we have

1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 0/1, 1/1

and placing thorn in numerical order, we have

{0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1}

This would be called the 5th sequence of Farey fractions. All preceding sequences may be obtained by deleting certain fractions. For example, the 4th sequence would be

{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1}

The 3rd, and 2nd are:

{0/1, 1/3, 1/2, 2/3, 1/1}, {0/1, 1/2, 1/1}.

Given a positive integer n, however, I can form the nth sequence independent of the others.

Farey fractions have a number of interesting properties which will enable us to prove the result we are trying to obtain, namely, that $|m\sigma - n| < r$ for some choice of m and n.

The first of these is easily spotted from our examples* Note that the cross-products of numerator times denominator of two consecutive fractions in any sequence always differ by unity. Picking a few at random:

(1/5, 1/4) ---------- $5 \cdot 1 - 1 \cdot 4 = 1$
(3/5, 2/3) ---------- $5 \cdot 2 - 3 \cdot 3 = 1$
(2/3, 3/4) ---------- $3 \cdot 3 - 2 \cdot 4 = 1$

This is no accident. Recalling the rule of inequality for fractions with positive numerator and denominator,

$$\frac{a}{b} < \frac{c}{d} \quad \text{iff} \quad ad < bc,$$

we see that, since we have arranged the fractions in numerical order for each sequence, we must minimize the difference

$$\frac{x}{y} - \frac{a}{b} = bx - ay$$

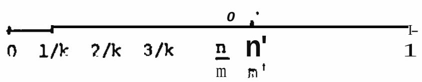
given $\frac{a}{b}$. Since $bx - ay$ is a positive integer, the least it can be is one. Hence $bc - ad = 1$ if there exists a fraction in our list for ~~which~~ this difference is unity. It may be that we cannot find such a fraction, and if we were to prove this property of Farey fractions, we would have settled this issue. However, it may be proved, indeed from so basic a principle as Enclid's algorithm, that we can find a fraction with this property.

A second property, a consequence of the first, is that if we take three consecutive Farey fractions a/b, c/d, e/f then c/d = (a + e)/(b + f). For example, consider (3/5, 2/3, 3/4). We have (3 + 3)/(5 + 4) = 6/9 = 2/3. We can add to the seeming trivia of the hour by observing that, since $bc - ad = 1$ and $de - cf = 1$, then

$$\begin{aligned} bc - ad &= de - cf\\ bc + cf &= ad + de\\ c(b + f) &= d(a + e)\\ \therefore \frac{c}{d} &= \frac{a + e}{b + f}. \end{aligned}$$

Farey fractions have a bit of unexpected power, as we shall now see. Observe that if we plot the kth sequence of Farey fractions, since we have contained in this list 1/k, 2/k, 3/k,... (k - 1)/k, the intervals between consecutive fractions is less than 1/k. Consequently, any real number σ between 0 and 1 can be approximated to within $1/k$ by a Farey fraction in the kth sequence. Let $r > 0$ be given. There is a positive integer k such that $\frac{1}{\sqrt{k}} < r$. Let us form the kth sequence of Farey fractions.

Choose the largest one which is less than or equal to σ, say n/m. Now I look at the next Farey fraction in this sequence, call it n'/m'. I know that $mn' - nm' = 1$. Hence



$$\frac{n'}{m'} - \frac{n}{m} = \frac{mn' - nm'}{m'm} = \frac{1}{m'm}$$

But by our above observation this difference cannot exceed $1/k$, so

$$\frac{1}{m'm} \le \frac{1}{k}.$$

Hence one of m, m' must be at least as large as $\sqrt{k}$. Let us say that $m' \le \sqrt{k}$. Observing the figure, we have, certainly,

$$\sigma - \frac{n}{m} < \frac{n'}{m'} - \frac{n}{m} = \frac{1}{m'm} \le \frac{1}{\sqrt{k}m}$$

Hence $$m\sigma - n < 1/\sqrt{k}$$

or $$|m\sigma - n| < r$$

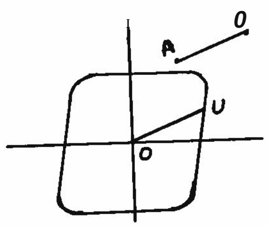
That is, we have found, using Farey fractions, a lattice point (n,m) that is closer to the line $y = \sigma x$ than r.

Interesting as all this is I have just shown you the wrong way to do this problem. For there is a way to answer both questions we are asking in the same breath. At this point Minkowski, if he were here, would be laughing us right out of this room that we could be so inept. If I may be allowed to paraphrase his jeering comments, they would probably go something like "You bungling idiots! I can not only solve the first question about the trees in the forest, but the second as well and include what you said about $|m\sigma - n|$ being arbitrarily small, as a corollary!"

lie would feel especially jilted because it so happens that one of his famous theorems applies to the situation at hand. Minkowski made some innovations to number theory by his use of convex bodies and his study of a certain distance function.

IF C is a convex curve ymmetric about the origin, define

$$d(A,B) = \frac{AB}{OU}$$



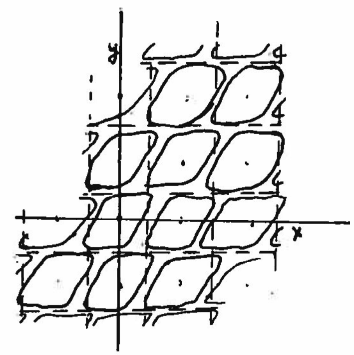
whoro AB is the euclidean distance from A to B and OU is the euclidean radius of C parallel to line AB. Tho function $d(A,B)$ turns out to be a metric for tho plane, and one may proceed to study the various properties this distance concept has. Such a atudy belongs to the area of mathematics known as Minkowskian geometry. It is clonr that if C is any circle then our metric is oucllddonn. Rut it Is not too hard to reason that if C is any ellipse, then the resulting Minkowskian geometry coincides with Euclidean geometry.

A different sort of idea which Minkowski applied to the successful solution of many difficult problems of number theory has come to be known as Minkowski's Theorem:

Theorem: Given a convex set in t'e plane symmetric about one lattice point and area greater than or equal to 4, that convex set must contain at least two further lattice points.

To make this plausible, Minkowski reasoned as follows: Let S be a convex body whose area is $K \ge 4$ and centered at $(0,0)$. Clearly we prove the theorem if we prove that S includes at least one other lattice point, for it will automatically contain its reflection in $(0,0)$. Using a mapping like $x' = ax$, $y' = ay$ it is clear we may "enlarge" S by a factor a until it just touches one other lattice point, and its reflection in $(0,0)$, besides $(0,0)$. Then shrink this new set back by a factor $1/2$. The resulting set S' has an area of $(a^2/4)K$. Let S' be translated to form a system of congruent symmetric convex sets in the plane centred at each and every lattice point. It is clear that these sets look something like this:

They are non-overlapping but certain pairs will touch at certain points. By considering unit squares also centered at each lattice point, since the convex sets do not completely fill up the plane---there are certain gaps left---we must have



$$\frac{a^2}{4}K \le 1$$

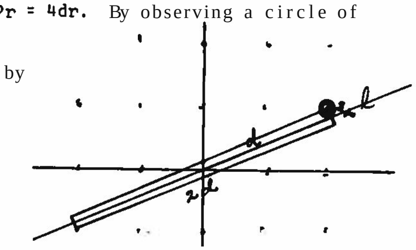
That is (using $k > 4$)

$$\frac{a^2}{4} \le \frac{1}{k} < \frac{1}{4} \quad \text{or} \quad a < 1.$$

Hence, the factor by which we had to "stretch" the original set S so it just touched two further lattice points was < 1, bearing witness to the fact that it already contained them in their interior.

You can easily make this rigorous, but I'm sure you get the idea Minkowski had in mind.

Well, I've kept you in suspense long enough. How should the forest problem he solved? Consider a line l in any direction passing through $(0,0)$. Let a rectangle be described symmetrically about l as axis, having width $2r$ (twice the radius of our trees) and length 7d where d is the distance we are seeing from $(0,0)$ in the direction of l. The area of this rectangle is $7d \cdot 2r = 4dr$. By observing a circle of radius r centered at one of the vertices, it is obvious that our vision will be blocked by one of the trees if and only if this rectangle contains a lattice point. Hence, if my vision is not blocked in this direction, by Minkowski's theorem, the area of the rectangle has to be less than or eaual to four.



Thus $4dr < 4$ or $d < \frac{1}{r}$.

Therefore, the distance I can see in any direction does not exceed the reciprical of the radius of the trees. It's Touche' once again by an elementary theorem of geometry.

UNDERGRADUATE RESEARCH PROJECT

Proposed by Kenneth Loewen, Universitv of Oklahoma

Complex numbers may be described as a two dimensional vector space over the real numbers with multiplication given by

$$(a + bi)(c + di) = (ac - b)$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)\, i;$$

where a, b, c and d are real numbers.
Quaternions may be defined as a two dimensional vector space over the complex numbers with multinlication given by

$$(m + jn)(p + qj) = (mp - \bar{n}q) + (m\bar{q} + np)j;$$

with m, n, p, q complex numbers and if $m = a + bi$ then $\bar{m} = a - bi$. etc. It is customary to write $ij = k$; so that if $n = c + di$, then $m + nj = a + bi + cj + dk$.
In turn a Cayley algebra may be defined as a two dimensional vector space over the quaternions with multiolication

$$(u + ve)(x + ye) = (ux - \bar{y}v) + (yu + v\bar{x})e.$$

Here u, v, x, y are quaternions so that if $u = a + bi + cj + dk$ $\bar{u} = a - bi - cj - dk$. Order is important is this definition since quaternion multiplication is not commutative.

Question: What sort of systems arise if this process is carried out beqinninq with a specific finite field instead of the reals?

The editor solicits contributions for this deoartment. Any problems which would be suitable undergraduate research are welcome.

1968 NATIONAL MEETING

A two day meetinq of Pi Mu Epsilon will be held in conjunction with the regular summer meetings of the Mathematical Association of America and the American Mathematical Society at Madison, Wisconsin sometime the last week in August.

Your chapter is encouraged to nominate your best speaker who will NOT have a masters degree by April, 1968, as a speaker for the national meeting. Nominations should be mailed to DR. RICHARD V. ANDREE, PI MU EPSILON, THE UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73069.

SOME USEFUL RESULTS IN APPLICATIONS OF THE POWER SERIES TRANSFORM TO THE SOLUTION OF DIFFERENCE EQUATIONS

Ray A. Gaskins, Virginia Polytechnic Institute

1. Introduction. During the past several years difference equations have come into their own as a technique for problem solving in many areas of science and engineering such as stochastic processes, electrical engineering and engineering mechanics. It is desirable, therefore, to develop a systematic method for solving the various types of difference equations which arise. One such method for handling a wide range of difference equations is the power series transform.

The power series transform as a tool for solvinp difference equations is the discrete counterpart of the Laplace Transform which is used in solving differential equations. The transform method of handling difference equations is superior to more conventional methods in that it is able to handle non-homogeneous equations and equations with variable coefficients. It is also possible to obtain directly a particular solution without first obtaining the general solution.

Equipped with a good table of transforms and a knowledge of partial fractions one is able to deal swiftly with complicated difference equations.

2. Partial Fractions. In order to facilitate the use of the power series transform in solving difference equations it is essential that one be able to resolve proper rational fractions into partial fractions. To this end the following shortcut is introduced.

Consider the proper rational fraction $N(x)/D(x)$, where the polynomial $N(x)$ is of lower order than the polynomial $D(x)$:

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(x-a)^m (x^2+bx+c)^\delta \prod_{i=1}^{n} (x-a_i)}, \quad \delta=0,1 \tag{2.1}$$

$$= \sum_{i=1}^{m} \frac{A_i}{(x-a)^i} + \sum_{j=1}^{n} \frac{B_j}{x-a_j} + \delta \frac{Ax+B}{x^2+bx+c}. \tag{2.2}$$

where.

$$A_i = \frac{d^{m-i}}{} \left. \frac{N(x)}{(m-i)!\,(x^2+bx+c)^\delta \prod_{j=1}^{n} (x-a_j)} \right|_{x=a}.$$

$$B_j = \lim_{x \to a_j} \frac{(x-a_j)N(x)}{(x-a)^m (x^2+bx+c)^\delta \prod_{i=1}^{n} (x-a_i)}.$$

A and B are found by placing A_i $(i=1,m)$ and B_j $(j=1,n)$ in (2.2) & setting the result equal to (2.1).

Several examples will be given in the following sections.

3. <u>The Power Series</u> <u>Transform</u>. Let $\{y_k\}$ represent the sequence $(y_0, y_1, \ldots, y_k, \ldots)$. We define the power series transformation of the sequence $\{y_k\}$ to be:

$$P\{y_k\} = \sum_{k=0}^{\infty} \frac{y_k}{s^k}. \qquad \text{(T.1)}$$

From the properties of power series we have by the ratio test that (T.1) converges for $s>s_0 = \lim_{k\to\infty}\left|\frac{y_{k+1}}{y_k}\right|$ if the limit exists.

As a shorthand notation we shall write $Y(s)$ to denote the closed form of the transform, while $P\{y_k\}$ will stand for the series representation.

Consider, for example, the sequence $\{1\} = (1,1,\ldots,1,\ldots)$ and its transform $P\{1\} = \sum_{k=0}^{\infty} 1/s^k$ which has the closed form $Y(s) = s/(s-1)$, for $s>1$.

The power series transform is a linear operator, i.e.

$$P\{Ay_k + Bz_k\} = AP\{y_k\} + BP\{z_k\} \qquad \text{(T.2)}$$

which follows immediately from the series definition.

Consider the transform of the sequence $\{y_{k+n}\}$

$$P\{y_{k+n}\} = \sum_{k=0}^{\infty}\frac{y_{k+n}}{s^k} = s^n\sum_{k=0}^{\infty}\frac{y_k}{s^k} - \sum_{k=0}^{n-1}\frac{y_k}{s^{k-n}}$$

$$= s^n P\{y_k\} - \sum_{k=0}^{n-1}\frac{y_k}{s^{k-n}}. \qquad \text{(T.3)}$$

The Laurent series $P\{y_k\}$ is uniformly convergent for $s>s_0$ and is therefore a continuous function of s for that range. It may be differentiated term by term to give:

$$P^{(n)}\{y_k\} = \sum_{0}^{\infty}\frac{(-1)^n k(k+1)\ldots(k+n-1)y_k}{s^{k-n}}$$

$$= \frac{(-1)^n}{s^n} P\{k(k+1)\ldots(k+n-1)y_k\}.$$

Hence,

$$P\{k(k+1)\ldots(k+n-1)y_k\} = (-1)^n s^n Y^{(n)}(s). \qquad \text{(T.4)}$$

Thus knowing $Y(s)$, the closed form of $P\{y_k\}$, we may obtain by (T.4) many useful transforms.

EXAMPLE 1. Find $P\{k\}$. Let $y_k=1$ and $n=1$ in (T.4).

$$P\{k\} = -sY'(s) = -s\frac{d}{ds}\left[\frac{s}{s-1}\right] = \frac{s}{(s-1)^2}.$$

Now let us consider the transform of the sequence $\{r^k\}$

$$P\{r^k\} = \sum_{k=0}^{\infty}\frac{r^k}{s^k} = \frac{r/s}{r/s-1} = \frac{s}{s-r};$$

which converges and is a continuous function of r as well as s for $r>s>s_0$. Differentiating $P\{r^k\}$ n times with respect to r we obtain:

$$P^{(n)}\{r^k\} = \sum_{k=0}^{\infty}\frac{k(k-1)\ldots(k-n+1)r^{k-n}}{s^k}$$

$$= P\{k(k-1)\ldots(k-n+1)r^{k-n}\},$$

And since $P^{(n)}\{r^k\} = Y^{(n)}(s)$ we get the identity:

$$P\{k(k-1)\ldots(k-n+1)r^{k-n}\} = \frac{s}{(s-r)^{n+1}}. \qquad \text{(T.5)}$$

EXAMPLE 2. Find $P\{k(k-1)\}$. Let $n=2$ and $r=1$ in (T.5).

$$P\{k(k-1)\} = s/(s-1)^3.$$

With the preceeding theory and examples behind us, we are in a position to handle the transformation of almost any difference equation one might encounter. A table of transforms is given at the end of section 5.

4. <u>The Inverse</u> <u>Transform</u>. Since the Laurent series representation of $P\{y_k\}$ is unique, there is a one-to-one correspondence between $\{y_k\}$ and $P\{y_k\}$, hence the inverse operator P^{-1} defined such that:

$$P^{-1}[Y(s)] = \{y_k\}$$

is also unique. We have by (T.2) that P^{-1} is a linear operator.

EXAMPLE 3. $P^{-1}[s^2/(s-1)^2] = P^{-1}[s/(s-1)] + P^{-1}[s/(s-1)^2]$

by linearity and partial fractions. Hence

$$P^{-1}[s^2/(s-1)^2] = \{1+k\}.$$

5. Applications of the Transform to the Solutions of Difference Equations. We consider difference equations of the type:

$$a_n y_{k+n} + a_{n-1} y_{k+n-1} + \dots + a_0 y_k = f_k \, ; \qquad (5.1)$$

where $a_i \, (i=1,n)$ and f_k may or may not be functions of k.

Since the difference equation is actually a sequence, we may apply the operator P to it and obtain an algebraic equation in $Y(s)$ and s. Solving for $Y(s)$ and using partial fractions to factor the right hand side, we apply the operator P^{-1} to both sides and obtain a solution for y_k.

EXAMPLE 4. In stochastic processes the usual random walk model with absorbing barriers at $x=0$ and $x=a$ may be represented by the difference equation $q_k = pq_{k+1} + qq_{k-1}$ for $1 \le k \le a-1$, where p is the probability of a move one unit to the left, $q=1-p$, and q_k is the probability that x reaches $x=0$ before $x=a$, given that initially $x=k$. Substituting k+1 for k and applying P:

$$s(Y(s) - q_0) = ps^2(Y(s) - q_0 - q_1/s) + qY(s);$$

$$Y(s) = \frac{-sq_0 + ps^2 + pq_1 s}{ps - s^2 + q}; \quad \text{since } X_0 \text{ is absorbing, } q_0 = 1.$$

Using partial fractions with $\delta=0, m=0, n=2$:

$$Y(s) = \frac{s(q_1 - q/p)}{s-1} + \frac{s(q_1 - 1)}{s-q/p},$$

$$P^{-1}[Y(s)] = (q_1 - q/p)\{1\} + (q_1 - 1)\{(q/p)^k\};$$

$$q_k = (q_1 - q/p + (q_1 - 1)(q/p)^k).$$

Letting $k=a$ we find that since $q_a = 0$

$$q_1 = \frac{q/p - (q/p)^a}{1 - (q/p)^a}, \text{ hence } q_k = \frac{(q/p)^a - (q/p)^k}{(q/p)^a - 1}.$$

EXAMPLE 5. Let $(k+1)y_{k+1} + (k-n)y_k = 0$, with $y_0 = 1$, then

$$P\{(k+1)y_{k+1}\} + P\{ky_k\} - nP\{y_k\} = 0;$$

$$-s^2 Y'(s) - sY'(s) - nY(s) = 0;$$

$$\frac{-Y'(s)}{Y(s)} = \frac{n}{s} - \frac{n}{s+1};$$

$$Y(s) = \frac{C(s+1)^n}{s^n}.$$

But $\lim_{s \to \infty} P\{y_k\} = y_0 = \lim_{s \to \infty} Y(s)$.

Hence, $C = y_0 = 1$ and by (T.10)

$$P^{-1}[Y(s)] = \{\tbinom{n}{k}\}, \quad 0 \le k \le n,$$

$$y_k = \tbinom{n}{k}.$$

TABLE OF POWER SERIES TRANSFORMS

	SEQUENCE	TRANSFORM
(T.1)	$\{y_k\}$	$Y(s)$
(T.2)	$\{Ay_k + Bz_k\}$	$AY(s) + BZ(s)$
(T.3)	$\{y_{k+n}\}$	$s^n Y(s) - \sum_{k=0}^{n-1} \frac{y_k}{s^{k-n}}$
(T.4)	$\{k(k+1)\dots(k+n-1)y_k\}$	$(-1)^n s^n Y^{(n)}(s)$
(T.5)	$\{\frac{k(k-1)\dots(k-n+1)}{n!} r^{k-n}\}$	$\frac{s}{(s-r)^{n+1}}$
(T.6)	$\{(k+n)y_{k+n}\}$	$-s^{n+1} Y'(s) - \sum_{k=0}^{n-1} \frac{ky_k}{s^{k-n}}$
(T.7)	$\{r^k y_k\}$	$Y(s/r)$
(T.8)	$\{\sum_{i=0}^{k} y_i\}$	$\frac{s}{s-1} Y(s)$
(T.9)	$\{\sum_{i=0}^{k} y_i z_{k-i}\}$	$Y(s) \cdot Z(s)$
(T.10)	$\{\tbinom{n}{k}\}$	$\frac{(s+1)^n}{s^n}$
(T.11)	$\{\frac{w^k \sin k\phi}{w \sin \phi}\}$	$\frac{s}{s^2 - 2wbs + w^2}$, $b = \cos \phi < 1$
(T.12)	$\{w^k \cos k\phi\}$	$\frac{s^2 - wbs}{s^2 - 2wbs + w^2}$, $b = \cos \phi \le 1$

~~TABLE OF POWER SERIES~~ TRANSFORMS (con't)

	SEQUENCE	TRANSFORM
(T.13)	$\{\frac{w^k \sinh k\phi}{w \sinh \phi}\}$	$\frac{s}{s^2 - 2wbs + w^2}$, $b = \cosh \phi > 1$
(T.14)	$\{w^k \cosh k\phi\}$	$\frac{s^2 - wbs}{s^2 - 2wbs + w^2}$, $b = \cosh \phi > 1$

REFERENCE:

McFadden, Leonard, Clemens, Paul F. and Campbell, Hugh G., The Power Series Transform and Applications To Solution of Difference Equations, Edwards Brothers Inc., 1963.

MATHEMATICAL CONFERENCES

The New Mexico Institute of Mining and Technology, Socorro New Mexico (enrollment about 500) has found that a mathematical conference is a good way of stimulating interest in mathematics among undergraduate students. Papers are presented both by students and professional mathematicians. Briefing sessions are held in connection with the conference to supply background on the material presented in the conference. Up to 2 hours may be allowed for each paper.

Students from other schools attend. The first year there were 12 papers presented and students from 12 institutions in three states attending. Before beginning this program the number of students in math was very small but recently as many as 24 percent of upperclassmen enrolled in the institution were math majors.

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The Governing Council of Pi Mu Epsilon has approved an increase in the maximum amount per chapter allowed as a matching prize from $70.00 to $25.00. If your chapter presents awards for outstanding mathematical papers and students, you may apply to the National Office to match the amount spent by your chapter--i. e., $30.00 of awards, the National Office will reimburse the chapter for $15.00 etc.--up to a maximum of $25.00. Chapters are urged to submit their best student papers to the Editor of the Pi Mu Epsilon Journal for possible publication.

ON HULA HOOPS

Dennis Spellman, Temple University

Several months ago L. Aileen Hostinsky of Connecticut College gave a lecture at Temple University entittled "On Groups, Loops, and Hoops"--that's right ! The purpose of this paper is to investigate the properties of hoops - especially a class of hoops-which we shall call hula hoops.

I. Quasi-groups:

Definition 1: Let G be a non-empty set. Then a binary operation $\circ$ defined on G is a mapping from the Cartesian product $G \times G$ into G. If $(a,b) \in G \times G$, its image is denoted $a \circ b$.

Definition 2. Let G be a non-empty set on which a binary operation $\circ$ is defined. G is a quasi-group under $\circ$ if there exist unique solutions in G to each of the equations $a \circ x = b$ and $y \circ a = b$ where a and b are fixed elements of G. A quasi-group G is a group if and only if it has the associative property. i.e., if and only if $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a,b,c \in G$. (See (1) p. 38 Theorem 1.4)

Definition 3: If G_1 and G_2 are quasi-groups, then the direct product of G_1 and G_2 is the Cartesian product $G_1 \times G_2$ on which a binary operation is defined as follows $(a_1,a_2) \circ (b_1,b_2) = (a_1 \circ b_1, a_2 \circ b_2)$ for all $(a_1,a_2), (b_1,b_2) \in G_1 \times G_2$. If C_1 is the unique solution of $a_1 \circ x = b_1$ in G_1, and C_2 is the unique solution of $a_2 \circ x = b_2$ in G_2 then (C_1,C_2) is the unique solution of $(a_1, a_2) \circ x = (b_1,b_2)$ in $G_1 \times G_2$. Similarly the equation $y \circ (a_1,a_2) = (b_1,b_2)$ has a unique solution in $G_1 \times G_2$. Thus, the direct product of two quasi-groups is a quasi-group.

Definition 4: Let G_1 and G_2 be quasi-groups. Let f be a mapping of G_1 into G_2. Then f is a homomorphism if $f(a \circ b) = f(a) \circ f(b)$ for all $a,b \in G_1$. If the homomorphism f is a one-one mapping of G_1 onto G_2, f is an isomorphism, and G_1 is ismorphic to G_2. In this case we write $G_1 \cong G_2$.

Definition 5: Let G_1 and G_2 be quasi-groups. Let g be a one-one mapping of G_1 onto G_2. Then g is an anti-ismorphism if $g(a \circ b) = g(b) \circ g(a)$ for all $a,b, \in G_1$. Evidently the concipts of isomorphism and anti-isomorphism coincide if G_1 and G_2 are commutative.

Definition 6: Let G be a quasi-group. Then $a \in G$ is idenpotent if $a \circ a = a$.

II. Hoops

Definition 7: A non-empty set H on which a binary operation $\circ$ is defined is a hoop (or medial system) if the following properties are satisfied:

H 1) H is a quasi-group under $\circ$

H 2) $a \circ a = a$ for all $a \in H$
(every element of H is idempotent)

H 3) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$ for all $a,b,c,d \in H$
(the medial property).

Theorem 1: If H is a hoop, then for all $a,b,c \in H$

1) $a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$ and

2) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$
(the right and left self-distributive laws).

Proof of (1):

$$a \circ (b \circ c) = (a \circ a) \circ (b \circ c) \quad \text{by (H 2)}$$
$$= (a \circ b) \circ (a \circ c) \quad \text{by (H 3)}.$$

(2) can be proven similarly.

(i) An example of a hoop is the set K of real numbers on which a binary operation is defined as the arithmetic mean i.e., if $a,b \in K$, then $a \circ b = \frac{a+b}{2} = \frac{1}{2}a + (1-\frac{1}{2})b$.

(ii) We still have a hoop if we change the binary operation into a weighed arithmetic mean, i.e., if w_1 and w_2 are positive real numbers, then we can define $a \circ b = \frac{w_1 a + w_2 b}{w_1 + w_2} = \frac{w_1}{w_1 + w_2} a + (1 - \frac{w_1}{w_1 + w_2}) b$.

We shall prove these assertions later (in greater generality). Another example of a hoop is the singleton $H = \{e\}$ on which the binary operation is defined by $e \circ e = e$. Such a hoop is called trivial. We note that a trivial hoop is also a group and satisfies the following properties:

E) there exists $e \in H$ such that $e \circ x = x \circ e = x$ for all $x \in H$ and
A) $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a,b,c \in H$.

Theorem 2: No non-trivial hoop satisfies properties (E) or (A).

Proof: Let H be a hoop and $a \in H$.
Let $e \in H$ be an identity element.
$a \circ a = a$ by (H 2)
and $a \circ e = a$ by hypothesis
Both a and e satisfy the equation $a \circ x = a$.
$\therefore a = e$ by (H 1)
Since a was arbitrary, $H = \{e\}$.
Hence, no non-trivial hoop satisfies (E).

A hoop H satisfying (A) is a group since (H 1) and (A) are satisfied simultaneously.
$\therefore H$ has an identity element.
$\therefore H$ is trivial.
Hence, no non-trivial hoop satisfies (A).

Definition 8: $\Gamma = \{R : R \text{ is a ring with unity } 1 \neq 0\}$.

Definition 9: Let $R \in \Gamma$, $\alpha \in R$ is medial in R provided:
1. α is invertible and
2. $\bar{\alpha} = 1 - \alpha$ is invertible.

We note $\bar{\alpha}$ is medial in R if and only if α is medial in R. If $R_1, R_2 \in \Gamma$, then their direct sum $R_1 \oplus R_2$ is an element of Γ. Moreover, if α_1 and α_2 are medial in R_1 and R_2 respectively, then (α_1, α_2) is medial in $R_1 \oplus R_2$ since $(\alpha_1, \alpha_2)^{-1} = (\alpha_1^{-1}, \alpha_2^{-1})$ and $\overline{(\alpha_1, \alpha_2)}^{-1} = (\bar{\alpha}_1^{-1}, \bar{\alpha}_2^{-1})$. (Direct sums are discussed in (1) pp. 61-62, 77 abelian groups, (2) pp. 17-18 rings).

Definition 10: Let $R \in \Gamma$. Let α be medial in R and let M be a module over R. $M^{(\alpha)}$ is the system $(M, \circ)$ where the binary operation is defined by $a \circ b = \alpha a + \bar{\alpha} b$ for all $a, b \in M$. (Modules are discussed in (1) pp. 145-147). We are now prepared to state our chief result.

Theorem 3: Let $R \in \Gamma$. Let α be medial in R, and let M be a module over R. Then $M^{(\alpha)}$ is a hoop.

Proof: Let $a, b \in M^{(\alpha)}$.
Consider the equation $a \circ x = b$.
this translates into $\alpha a + \bar{\alpha} x = b$.
We may verify by direct substitution that

$x = (\bar{\alpha}^{-1} b - \bar{\alpha}^{-1} \alpha a)$ is a solution.

$$\alpha a + \bar{\alpha}(\bar{\alpha}^{-1} b - \bar{\alpha}^{-1} \alpha a) = \alpha a + \bar{\alpha}\bar{\alpha}^{-1} b - \bar{\alpha}\bar{\alpha}^{-1} \alpha a$$
$$= \alpha a + b - \alpha a$$
$$= b.$$

Suppose x_1 and x_2 are any solutions of $a \circ x = b$.

Then $a \circ x_1 = b$ and $a \circ x_2 = b$.

$\therefore a \circ x_1 = a \circ x_2$

$\therefore \alpha a + \bar{\alpha} x_1 = \alpha a + \bar{\alpha} x_2$

$\therefore \bar{\alpha} x_1 = \bar{\alpha} x_2$

$\therefore \bar{\alpha}^{-1} \bar{\alpha} x_1 = \bar{\alpha}^{-1} \bar{\alpha} x_2$

$\therefore x_1 = x_2$

Thus the solution of $a \circ x = b$ is unique.
Similarly, there exists a unique solution of $y \circ a = b$.

$\therefore M^{(\alpha)}$ satisfies (H 1).

Let $a \in M^{(\alpha)}$

$$
\begin{aligned}
a \circ a &= \alpha a + \bar{\alpha} a \\
&= (\alpha + \bar{\alpha})a \\
&= 1a \\
&= a.
\end{aligned}
$$

$\therefore M^{(\alpha)}$ satisfies (H 2). Let a, b, c, d $\in M^{(\alpha)}$

$$
\begin{aligned}
(a \circ b) \circ (c \circ d) &= \alpha(a \circ b) + \bar{\alpha}(c \circ d) \\
&= \alpha(\alpha a + \bar{\alpha} b) + \bar{\alpha}(\alpha c + \bar{\alpha} d) \\
&= \alpha^2 a + \alpha\bar{\alpha} b + \bar{\alpha}\alpha c + \bar{\alpha}^2 d
\end{aligned}
$$

Rearranging terms and regrouping, we have

$(a \circ b) \circ (c \circ d) = (\alpha^2 a + \bar{\alpha}\alpha c) + (\alpha\bar{\alpha} b + \bar{\alpha}^2 d)$

$\alpha\bar{\alpha} = \bar{\alpha}\alpha$ because

$\alpha(1 - \alpha) = \alpha - \alpha^2 = (1 - \alpha)\alpha$

$$
\begin{aligned}
\therefore (a \circ b) \circ (c \circ d) &= (\alpha^2 a + \alpha\bar{\alpha} c) + (\bar{\alpha}\alpha b + \bar{\alpha}^2 d) \\
&= \alpha(\alpha a + \bar{\alpha} c) + \bar{\alpha}(\alpha b + \bar{\alpha} d) \\
&= \alpha(a \circ c) + \bar{\alpha}(b \circ d) \\
&= (a \circ c) \circ (b \circ d)
\end{aligned}
$$

Finally $M^{(\alpha)}$ satisfies (H 3) and is a hoop.

definition 11: Let $R \in \Gamma$. $\Omega(R) = \{M^{(\alpha)} = M$ is an R-module and α is medial in $R\}$. A hoop H is a hula hoop iff $H \in \bigcup_{R \in \Gamma} \Omega(R)$. We have a simple corollary to theorem 3.

Corollary 4: If $R \in \Gamma$ and $M^{(\alpha)} \in \Omega(R)$, then the identity mapping $i: M^{(\alpha)} \to M^{(\bar{\alpha})}$ is an anti-isomorphism.

Proof: Let a, b $\in M$. Let $M^{(\alpha)} = (M, \circ)$ and $M^{(\bar{\alpha})} = (M, \bullet)$.

$$
\begin{aligned}
i(a \circ b) &= i(\alpha a + \bar{\alpha} b) \\
&= \alpha a + \bar{\alpha} b \\
&= \bar{\alpha} b + \bar{\bar{\alpha}} a \\
&= b \bullet a \\
&= i(b) \bullet i(a).
\end{aligned}
$$

If $M^{(\alpha)} = M^{(\bar{\alpha})}$ is commutative, then the identity mapping is just the identity isomorphism. (iii) For example the additive group K of real numbers is a module over the ring K of real numbers, and 1/2 is medial in K. The commutative hoop $K^{(\alpha)} = K^{(\bar{\alpha})} = K^{(1/2)}$ is just the one we met before in example (i). (iv) Consider the four element field F

F

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

(See (3) pp. 158-159 Example 2).

$a + b = 1$; $a^{-1} = b$ and $b^{-1} = a$. $\therefore F^{(a)}, F^{(b)} \in \Omega(F)$. Let's denote the hoop operations of $F^{(a)}$ and $F^{(b)}$ by $\circ$ and $\star$ respectively. Then $x \circ y = ax + by$ and $x \star y = bx + ay$. $F^{(a)}$ and $F^{(b)}$ are non-commutative since $0 \circ 1 = 1 \star 0 = a \cdot 0 + b \cdot 1 = b$ and

$1 \circ 0 = 0 \star 1 = a \cdot 1 + b \cdot 0 = a.$

Now consider the mapping $f: F^{(a)} \to F^{(b)}$ defined by $f(x) = x^2$. We can look at the multiplication table of F to write out f explicitly.

$$
f \quad
\begin{array}{ccc}
0 & \to & 0 \\
1 & \to & 1 \\
a & \to & b \\
b & \to & a
\end{array}
$$

We note that f is a one-one mapping of $F^{(a)}$ onto $F^{(b)}$.

$$
\begin{aligned}
f(x \circ y) &= f(ax + by) \\
&= (ax + by)^2 \\
&= a^2x^2 + 2abxy + b^2y^2 \\
&= a^2x^2 + 0 + b^2y^2 \quad \text{since F is of characteristic 2.}
\end{aligned}
$$

$$
\begin{aligned}
\therefore f(x \circ y) &= a^2x^2 + b^2y^2 \\
&= bx^2 + ay^2 \\
&= bf(x) + af(y) \\
&= f(x) \star f(y)
\end{aligned}
$$

Hence, f is an isomorphism, and the non-commutative hoops $F^{(a)}$ and $F^{(b)}$ are isomorphic. However, hula hoops come in anti-isomorphic pairs which in general need not be isomorphic. (v) For example $Z^{(2)}, Z^{(4)} \in \Omega(Z_5)$ where Z_5 denotes the integers modulo 5.

$2 + 4 = 1$; $2^{-1} = 3$ and $4^{-1} = 4$.

Lemma 5: $Z_5^{(2)} \neq Z_5^{(4)}$.

Proof: Let $h: Z_5^{(2)} \to Z_5^{(4)}$ be an arbitrary homomorphism. We denote the hoop operations of $Z_5^{(2)}$ and $Z_5^{(4)}$ by $\circ$ and $\star$ respectively.

$$
\begin{array}{lll}
 & h(0 \circ 1) & = h(2 \cdot 0 + 4 \cdot 1) \\
 & & = h(4) \\
1. & \therefore h(0 \circ 1) & = h(4) \\
\text{But} & h(0 \circ 1) & = h(0) \star h(1) \\
 & & = 4h(0) + 2h(1) \\
2. & \therefore h(0 \circ 1) & = 4h(0) + 2h(1) \\
3. & \therefore h(4) & = 4h(0) + 2h(1) \\
\text{Now} & h(0 \circ 4) & = h(2 \cdot 0 + 4 \cdot 4) \\
 & & = h(16) \\
 & & = h(1) \\
4. & \therefore h(0 \circ 4) & = h(1) \\
\text{But} & h(0 \circ 4) & = h(0) \star h(4) \\
 & & = 4h(0) + 2h(4) \\
5. & \therefore h(0 \circ 4) & = 4h(0) + 2h(4) \\
6. & \therefore h(1) & = 4h(0) + 2h(4)
\end{array}
$$

Subtracting (3) from (6) we get

$$
\begin{array}{lll}
7. & h(1) - h(4) & = 2h(4) - 2h(1) \\
8. & \therefore 3h(1) & = 3h(4)
\end{array}
$$

9. Multiplying (8) by 3^{-1} (=2) we have $h(1) = h(4)$.
$\therefore$ h is not one-one.

But $h: Z_5^{(2)} \to Z_5^{(4)}$ was an arbitrary homomorphism.

$$\therefore Z_5^{(2)} \not\cong Z_5^{(4)}$$

If H and K are hoops, then the direct product $H \times K$ is readily seen to be a hoop also. It is easily seen that if M_1 is a module over R_1 and M_2 is a module over R_2, then the group $M_1 \oplus M_2$ is a module over the ring $R_1 \oplus R_2$ where the scalar multiplication is defined by $(\lambda_1, \lambda_2)(a_1, a_2) = (\lambda_1 a_1, \lambda_2 a_2)$ for all $(\lambda_1, \lambda_2) \in R_1 \oplus R_2$ and all $(a_1, a_2) \in M_1 \oplus M$.

Theorem 6: Let $R_1, R_2 \in \Gamma$, $M_1^{(\alpha_1)} \in \Omega(R_1)$, and $M_2^{(\alpha_2)} \in \Omega(R_2)$. Then $M_1^{(\alpha_1)} \times M_2^{(\alpha_2)} = (M_1 \oplus M_2)^{(\alpha_1, \alpha_2)}$. Thus, the direct product of two hula hoops is a hula hoop.

Proof: Let (a_1, a_2), (b_1, b_2) be elements of the cartesian product $M_1 \times M_2$. First let's consider them as elements of $M_1^{(\alpha_1)} \times M_2^{(\alpha_2)}$ and denote the hoop operation by $\circ$.

Then $(a_1, a_2) \circ (b_1, b_2) = (a_1 \circ b_1, a_2 \circ b_2)$
$= (\alpha_1 a_1 + \bar{\alpha}_1 b_1, \alpha_2 a_2 + \bar{\alpha}_2 b_2)$.

Now let's consider (a_1, a_2) and (b_1, b_2) as elements of $(M_1 \oplus M_2)^{((\alpha_1, \alpha_2))}$ and denote the hoop operation by $\cdot$.

Then $(a_1, a_2) \cdot (b_1, b_2) = (\alpha_1, \alpha_2)(a_1, a_2) + (\overline{\alpha_1, \alpha_2})(b_1, b_2)$

$= (\alpha_1, \alpha_2)(a_1, a_2) + (\bar{\alpha}_1, \bar{\alpha}_2)(b_1, b_2)$
$= (\alpha_1 a_1, \alpha_2 a_2) + (\bar{\alpha}_1 b_1, \bar{\alpha}_2 b_2)$
$= (\alpha_1 a_1 + \bar{\alpha}_1 b_1, \alpha_2 a_2 + \bar{\alpha}_2 b_2)$.

$\therefore (a_1, a_2) \circ (b_1, b_2) = (a_1, a_2) \cdot (b_1, b_2)$
for arbitrary $(a_1, a_2), (b_1, b_2) \in M_1 \times M_2$.

$\therefore M_1^{(\alpha_1)} \times M_2^{(\alpha_2)} = (M_1 \oplus M_2)^{((\alpha_1, \alpha_2))}$

If in theorem 6 $R_1 = R_2 = R$ and $\alpha_1 = \alpha_2 = \alpha$, then the group $M_1 \oplus M_2$ is a module over R as well as over $R \oplus R$, scalar multiplication being defined by $\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2)$ for all $\lambda \in R$ and all $(a_1, a_2) \in M_1 \oplus M_2$. In the same spirit as above, it is easily seen that $(M_1 \oplus M_2)^{(\alpha)} = (M_1 \oplus M_2)^{((\alpha, \alpha))}$. Hence, we have the following corollary.

Corollary 7: If $R \in \Gamma$ and $M_1^{(\alpha)}, M_2^{(\alpha)} \in \Omega(R)$, then $M_1^{(\alpha)} \times M_2^{(\alpha)} = (M_1 \oplus M_2)^{(\alpha)}$

Definition 12: Let R be a ring and M a module over R.
$A(M) = \{\lambda \in R: \lambda x = 0 \text{ for all } x \in M\}$.

$A(M)$ is non-empty since 0 is always a member; furthermore, $A(M)$ is easily seen to be an ideal in R which we shall call the **M-annihilator** in R. (Ideals are discussed in (2) pp. 21–26, and simple rings are discussed in (2) pp. 38–39). This definition allows us to give a simple characterization of commutative hula hoops.

Theorem 8: If $R \in \Gamma$ and $M^{(\alpha)} \in \Omega(R)$, then $M^{(\alpha)}$ is commutative if and only if $\alpha - \bar{\alpha} (= 2\alpha - 1) \in A(M)$.

Proof: Necessity: $M^{(\alpha)}$ is commutative.

$a \circ 0 = 0 \circ a$ for all $a \in M$.
$\alpha \cdot a + \bar{\alpha} \cdot 0 = \alpha \cdot 0 + \bar{\alpha} \cdot a$
$\alpha a = \bar{\alpha} a$
$\alpha a - \bar{\alpha} a = 0$
$(\alpha - \bar{\alpha}) a = 0$ for all $a \in M$

Sufficiency: $\alpha - \bar{\alpha} \in A(M)$

$(\alpha - \bar{\alpha}) a = (\alpha - \bar{\alpha}) b$ (=0) for all $a, b \in M^{(\alpha)}$
$\alpha a + \bar{\alpha} b = \alpha b + \bar{\alpha} a$
$a \circ b = b \circ a$ for all $a, b \in M^{(\alpha)}$
$M^{(\alpha)}$ is commutative.

Corollary 9: If $R \in \Gamma$, $M^{(\alpha)} \in \Omega(R)$, and $A(M) = \{0\}$ then, $M^{(\alpha)}$ is commutative if and only if 2 is invertible and $\alpha = 2^{-1}$.

Proof: M^a is commutative if and only if $2a - 1 \in A(M) = \{0\}$. Moreover, $a = 2^{-1}$ if and only if $2\alpha - 1 = 0$.

Corollary 10: If $R \in \Gamma$ is simple, $M \neq \{0\}$, and $M^{(\alpha)} \in \Omega(R)$, then $M^{(\alpha)}$ is commutative if and only is 2 is invertible and $\alpha = 2^{-1}$.

Proof: $A(M)$ is an ideal in R. Since R is simple we must have either $A(M) = \{0\}$ or $A(M) = R$. Pick $a \in M$, $a \neq 0$. (Recall $M \neq \{0\}$).

$1 \cdot a = a \neq 0$

$\therefore 1 \notin A(M)$

$\therefore A(M) = \{0\}$ and in view of corollary 9 the proof is completed.

(vi) In the ring K of real numbers 2 has inverse 1/2. The commutative hoop $K^{(1/2)}$ appears in examples (i) and (iii).

(vii) Let the module N over the ring $K \oplus K$ be the abelian group K K on which scalar multiplication is defined by

$(\lambda, \mu)(a, a_2) = (\lambda a_1, \lambda a_2)$ for all $(\lambda, \mu) \in K \oplus K$ and all $(a_1, a_2) \in N$. $A(N) = \{(0, \mu) : \mu \in K\}$.

The reader may verify that $N^{((1/2,\, x))}$ is a commutative member of $\Omega(K \oplus K)$ for all $x \in K - \{0, 1\}$.

(viii) Z_9 is a non-simple ring since $3Z_9 = \{0, 3, 6\}$ is a non-trivial prober ideal. The group Z_9 is a module over the ring Z_9, scalar multiplication coinciding with ring multiplication in Z_9. $A(Z_9) = \{0\}$. The reader may verify that $Z_9^{(5)}$ is a commutative member of $\Omega(Z_9)$.

(ix) $F^{(a)}$ of example (iv) is non-commutative since $2(=0)$ is not invertible.

(x) $Z_5^{(2)}$ of example (v) is non-commutative since $2^{-1} = 3 \neq 2$. If $R \in \Gamma$ and $M^{(\alpha)}, N^{(\alpha)} \in \Omega(R)$ where M and N are isomorphic modules over R, then it is an easy consequence that $M^{(\alpha)}$ is isomorphic to $N^{(\alpha)}$. The converse of this statement is not true. $(1/2, 1/2)$ is medial in $K \oplus K$. $(1/2, 1/2) + (1/2, 1/2) = (1, 1)$; $(1/2, 1/2)^{-1} = (2, 2)$.

(xi) Let M be the module $K \oplus K$ in which scalar multiplication coincides with ring multiplication and is denoted $(\lambda, \mu)\ (a_1, a_2)$. (xii) Let N be the module of example (vii), and let scalar multiplication be denoted by $(\lambda, \mu) \times (a_1, a)$. Clearly $M^{((1/2,\, 1/2))} = N^{((1/2,\, 1/2))}$

Lemma 11: $M \neq N$.

Proof: Let $h: M \to N$ be an arbitrary homomorphism, and let $h((1,1)) = (h_1, h_2)$.

$$\begin{aligned} h((2,1)) &= h((2,1)\cdot(1,1)) \\ &= (2,1) \times h((1,1)) \\ &= (2,1) \times (h_1, h_2) \\ &= (2h_1, 2h_2) \end{aligned}$$

1. $\therefore h((2,1)) = (2h_1, 2h_2)$

Now $h((2,2)) = h((2,2) \cdot (1,1))$

$= (2, 2 \times h((1,1))$

$= (2,2) \times (h_1, h_2)$

$= (2h_1, 2h_2)$

2. $\therefore h((2,2)) = (2h_1, 2h_2)$

3. $\therefore h((2,1)) = h((2,2))$

$\therefore$ h is not one-one.

But $h: M \to N$ was an arbitrary homorphism.

$\therefore M \neq N$.

Although we do not have a converse to our result, we do have a partial converse as a consequence of the next theorem. We shall need the following lemma.

Lemma 12: If $S \in \Gamma$, $N^{(\beta)} \in \Omega(S)$, H is a hoop, and $f: H \to N^{(\beta)}$ is an isomorphism, then $g: H \to N^{(\beta)}$ defined by $g(x) = f(x) - d$ is an isomorphism for all $d \in N$.

Proof: Let $y_o \in N^{(\beta)}$. Since f is onto, there exists $x_o \in H$ such that $f(x_o) = y_o + d$.

$\therefore g(x_o) = (y_o + d) - d = y_o$.

$\therefore$ g is onto.

Suppose $g(x_1) = g(x_2)$

$f(x_1) - d = f(x_2) - d$

$f(x_1) = f(x_2)$

$x_1 = x_2$ since f is one - one.

$\therefore$ g is one-one.

1. $g(a \circ b) = f(a \circ b) - d$
2. $g(a) \circ g(b) = (f(a) - d) \circ (f(b) - d)$

$= \beta(f(a) - d) + \bar{\beta}(f(b) - d)$

$= (\beta f(a) + \bar{\beta} f(b)) - (\beta + \bar{\beta})\, d$

$= f(a) \circ f(b) - d$

$= f(a \circ b) - d$ since f is a homomorphism.

$\therefore g(a \circ b) = g(a) \circ g(b)$ and g is indeed an ismorphism.

Theorem 13: If $R, S \in \Gamma$, $M^{(\alpha)} \in \Omega(R)$, $N^{(\beta)} \in \Omega(S)$, and $M^{(\alpha)} \cong N^{(\beta)}$;

then $M^+ \cong N$ where M^+ and N^+ are the additive groups of the modules M and N respectively.

Proof: Let $Q : M^{(\alpha)} \to N^{(\beta)}$ be an isomorphism. Then by lemma 12, $\Psi : M^{(\alpha)} \to N^{(\beta)}$ defined by $\Psi(x) = Q(x) - Q(0)$ is also an isomorphism, and $\Psi(0) = Q(0) - Q(0) = 0$. Moreover, since Y is a hoop ismorphism, it is a one-one mapping of the set M onto the set N.

1. $\Psi(\alpha^{-1}x \circ 0) = \Psi(\alpha\alpha^{-1}x + \bar{\alpha}0)$
$= \Psi(x)$.

2. $\Psi(\alpha^{-1}x \circ 0) = \Psi(\alpha^{-1}x) \circ \Psi(0)$
$= \Psi(\alpha^{-1}x) \circ 0$
$= \beta\Psi(\alpha^{-1}x) + \bar{\beta} \circ 0$
$= \beta\Psi(\alpha^{-1}x)$

3. $\beta\Psi(\alpha^{-1}x) = \Psi(x)$

4. $\Psi(\alpha^{-1}x) = \beta^{-1}\Psi(x)$ for all $x \in M$.

Similarly $\Psi(\bar{\alpha}^{-1}x) = \bar{\beta}^{-1}\Psi(x)$ for all $x \in M$.

Since Ψ is a one-one mapping of M onto N, it suffices to show that Ψ is a group homorphism.

Let $a,b \in M^+$.

$$\begin{aligned}
\Psi(a + b) &= \Psi(\alpha\alpha^{-1}a + \bar{\alpha}\bar{\alpha}^{-1}b) \\
&= \Psi(\alpha^{-1}a \circ \bar{\alpha}^{-1}b) \\
&= \Psi(\alpha^{-1}a) \circ \Psi(\bar{\alpha}^{-1} b) \\
&= \bar{\beta}^{1}\,\Psi(a) \circ \bar{\beta}^{-1}\,\Psi(b) \\
&= \beta\beta^{-1}\,\Psi(a) + \bar{\beta}\bar{\beta}^{-1}\,\Psi(b) \\
&= \Psi(a) + \Psi(b)
\end{aligned}$$

$\Psi : M^+ \to N^+$ is an isomorphism.

References:

(1.) Hu, S.T., Elements of Modern Algebra, Holden-Day Inc., San Francisco, 1965.

(2.) McCoy, N. H., The Theory of Rings, The Macmillan Company, New York, 1964.

(3.) McCoy, N. H., Introduction to Modern Algegra, Allyn and Bacon, Inc. Boston, 1960.

I thank Dr. Hwa Tsang for her aid.

PROBLEM DEPARTMENT

Edited by

M. S. Klamkin, Ford Scientific Laboratory

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Michigan 48121.

PROBLEMS FOR SOLUTION

192.* Proposed by Oystein Ore, Yale University. Albrecht Durer's famous etching "Melancholia" includes the magic square

16	3	2	13
5	10	11	8
9	5	7	12
4	15	14	1

The boxed-in numbers 15-14 indicate the year in which the picture was drawn. How many other 4 x 4 magic squares are there which he could have used in the same way?

193. Proposed by William H. Pierce, General Dynamics, Electric Boat Division.

Two ships are steaming along at constant velocities (course and speed). If the motion of one ship is known completely, and if only the speed of the second ship is known, what is the minimum number of bearings necessary to be taken by the first ship in order to determine the course (constant) and range (time-dependent) of the second ship? Given this requisite number of bearings, show how to determine the second ship's course and range.

Editorial Note:

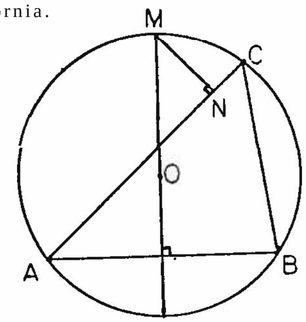
This problem was suggested by problem 186 which was given erroneously. See the comment on 196 (this issue).

194. Proposed by J. M. Gandhi, University of Alberta. Show that the equation

$$x^{x+y} = y^{y-x}$$

has no solutions in integers except the solutions (i) $x = \pm 1$, $y = \pm 1$,
(ii) $x = 3$, $y = 9$.

195. Proposed by Leon **Bankoff**, Los Angeles, California. Math. Mag. (Jan. 1963), p. 60, contains a short paper by Dov Avishdom, who asserts without proof that in the adjoining diagram, $AN = NC + CB$. Give a proof.



196. Proposed by R. C. Gebhart, Parsippany, N. J.

What is the remainder if x^{100} is divided by $x^2 - 3x + 2$?

197. Proposed by Joseph **Arkin**, Nanuet, N. Y.

A box contains $(1600\,u^2 + 3200)/3$ solid spherical metal bearings. Each bearing in the box has a cylindrical hole of length .25 centimeters drilled straight through its center. The bearings are then melted together with a loss of 4% during the melting process and formed into a sphere whose radius is an integral number of centimeters. How many bearings were there originally in the box?

198. Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn. A semi-regular solid is obtained by slicing off sections from the corners of a cube. It is a solid with 36 congruent edges, 24 vertices and 14 faces, 6 of which are regular octagons and 8 are equilateral triangles. If the length of an edge of this polytope is e, what is its volume.

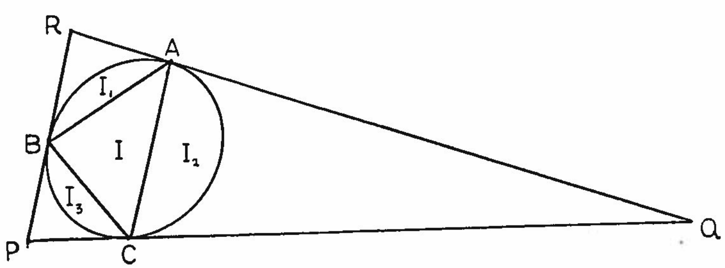
199. Proposed by Larry **Forman**, Brown University and M. S. Klamkin, Ford Scientific Laboratory.
Find all integral solutions of the equation $\sqrt[3]{x + \sqrt{y}} + \sqrt[3]{x - \sqrt{y}} = z$,

SOLUTIONS

181. Proposed by D. W. Crowe, University of Wisconsin and M. S. Klamkin, Ford Scientific Laboratory.
Determine a convex curve circumscribing a given triangle such that
1. The areas of the four regions (3 segments and a triangle) are equal and
2. The curve has minimum perimeter.

Solution by the proposers. Condition (2) is irrelevant. The only convex figure satisfying (1) is a circumscribed triangle whose sides are respectively parallel to the sides of the given triangle. Our proof depends on the known

Lemma* The area of any inscribed triangle in a given triangle cannot be smaller than each of the other three triangles formed. Consider a convex curve that is not a triangle and draw the three lines of support at the vertices.



* For a projective proof of this lemma and other references, see M. S. Klamkin and D. J. **Newman**, the Philosophy and Applications of Transform Theory, SIAM Review, Jan. 1961, p. 14.

Since $1 = I_1 = I_2 = I_3$ (in area), each of the triangles ABR, BCP, and CAQ would have greater area than $\triangle ABC$ contradicting the lemma. The case of two parallel support lines is also ruled out easily by area considerations.

182. Proposed by Gabriel Rosenberg, Temple University.
Prove that an odd perfect number cannot be a perfect square.

Editorial Note: Although **Euler** established the more general result that every odd perfect number must be of the form $p^{4a+1}k^2$ where p is a prime of the form $4n+1$, the above problem was given since it has an elegant solution in addition to the fact that there was a shortage of proposals.

Solution by H. **Kaye**, Brooklyn, N. Y.
If N is an odd perfect square then
1. The sum of its divisors = 2N,
2. The number of its divisors, each being odd, must also be odd since except for $\sqrt{N}$, the divisors occur in pairs d and N/d.

This gives a contradiction since the sum of an odd number of odd numbers is odd.

Also solved by Paul J. Campbell (University of Dayton), Paul Myers (Philadelphia, Pa.), Bob **Prielipp** (University of Wisconsin), Stanley Rabinowitz (Polytechnic Institute of Brooklyn), Dennis **Spellman** (Temple University), M. Wagner (New York, N. Y.), F. Zetto (Chicago, Illinois) and the proposer.

183. Proposed by R. Penney, Ford Scientific Laboratory.
If
$$r_0(r_1r_2 + r_2r_3 + r_3r_1) = 0\,,$$
$$r_1(r_0r_2 + r_0r_3 - r_2r_3) = 0\,,$$
$$r_2(r_0r_3 + r_0r_1 - r_3r_1) = 0\,,$$
$$r_3(r_0r_1 + r_0r_2 - r_1r_2) = 0\,,$$
Show that at least one of the quantities r_0, r_1, r_2, r_3 vanishes.

Solution by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.
If none of the r's vanish, then the expressions in the parenthesis must vanish. These conditions may then be written as

(1) $\quad 1/r_1 + 1/r_2 + 1/r_3 = 0\,,$

(2) $\quad 1/r_2 + 1/r_3 = 1/r_0\,,$

(3) $\quad 1/r_3 + 1/r_1 = 1/r_0\,,$

(4) $\quad 1/r1 + 1/r_2 = 1/r_0\,.$

Adding up (2), (3), and (4) and using (1) gives $3/r_0 = 0$ which is impossible. Hence, at least one of the r's must vanish.

Editorial Note: This problem has arisen in the proposer's paper, "Geometrization of a Complex Scalar Field. I. Algebra," Jour. of Math. Physics, March 1966, pp. 479-481. It also follows that at least 2 r's must vanish. The problem can be extended to any number of variables, e. g., for 5 variables we would have:

$$r_0(r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2) = 0\,,$$
$$r_1(r_0r_2r_3 + r_0r_3r_4 + r_0r_4r_2 - r_2r_3r_4) = 0\,,$$
$$r_2(r_0r_3r_4 + r_0r_4r_1 + r_0r_1r_3 - r_3r_4r_1) = 0\,,$$
$$r_3(r_0r_4r_1 + r_0r_1r_2 + r_0r_2r_4 - r_4r_1r_2) = 0\,,$$
$$r_4(r_0r_1r_2 + r_0r_2r_3 + r_0r_3r_1 - r_1r_2r_3) = 0\,.$$

It follows in a similar fashion as before that at least two of the r's must vanish. Also solved by Paul J. Campbell (University of Dayton), Mario E. Ettrick (Brooklyn, N. Y.), R. C. Gebhart (Parsippany, N. J.), Theodore Jungreis (New York University), Bruce W. King (Burnt Hills, N. Y.), Edward Lacher (Rutgers university), Paul Myers (Philadelphia, Pa.), Charles W. Trigg (San Diego, Calif.), F. Zetto (Chicago, 111.) and the proposer.

184. Proposed by Alan Lambert, University of Miami.
Find a parametric representation for transcendental curve given by the equation

$$x^y = y^x .$$

Solution by Bruce W. King (Burnt Hills, N. Y.) and Peter Lindstrom (Union College).
Let $y = xt$. Then since $y\log x = x\log y$
we have, $t\log x = \log xt$
or $x = t^{1/(t-1)}$ and $y = t^{t/(t-1)}$

Editorial Note: This parametric representation was already known to Euler. It can also be used to determine all rational solutions of the above equation.

Also solved by R. C. Gebhart (Parsippany, N. J.) H. Kaye (Brooklyn, N. Y.), Stanley Rabinowitz (Polytechnic Institute of Brooklyn, F. Zetto (Chicago, Ill.) and the proposer.

185. Proposed by A. E. Livingston and L. Moser, University of Alberta.

Given: $$f(n) = f(n+3) \quad , \quad n = 1,2, \ldots \quad ,$$

$$f(1) = 1 \quad , \quad \sum_1^\infty f(n)/n = 0 \quad .$$

Find $f(2)$.

Solution by R. C. Gebhart (Parsippany, N. J.).

Starting from the known summations,

$$\sum_1^\infty \frac{\cos nx}{n} = -\ln 2\sin\frac{x}{2} \quad , \quad \sum_1^\infty \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad ,$$

we can obtain the particular sums

$$(1) \qquad 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \ldots = \frac{\pi}{3\sqrt{3}} \quad ,$$

$$(2) \qquad 1 - \frac{1}{3} + \frac{1}{4} - \frac{1}{6} + \frac{1}{7} - \frac{1}{9} + \ldots = \frac{\ln 3}{2} + \frac{\pi}{6\sqrt{3}} \quad .$$

Nor multiply (1) by -a and (2) by -b.

This will give the initial series if

$$a = f(2), \quad b = f(3) \quad ,$$

$$a + b + 1 = 0 \quad ,$$

$$\frac{a\pi}{3\sqrt{3}} + b\left(\frac{\ln 3}{2} + \frac{\pi}{6\sqrt{3}}\right) = 0 \quad .$$

Whence,

$$f(2) = \frac{\pi + 3\sqrt{3}\,\ln 3}{\pi - 3\sqrt{3}\,\ln 3}$$

Editorial Note: A more direct way of summing these type of series would be to use generating functions which lead to the evaluation of definite integrals.

If $$G(x) = \frac{x}{1} + \frac{x^4}{4} + \ldots + \frac{x^{3m-2}}{3m-2}$$

then $$G'(x) = 1 + x^3 + \ldots + x^{3m-3} = \frac{1-x^{3m}}{1-x^3} \quad ,$$

and $$G(1) = \int_0^1 \frac{1-x^{3m}}{1-x^3}\,dx \quad .$$

Similarly, $$H(x) = \frac{x^2}{2} + \frac{x^5}{5} + \ldots + \frac{x^{3m-1}}{3m-1} \quad ,$$

$$I(x) = \frac{x^3}{3} + \frac{x^6}{6} + \ldots + \frac{x^{3m}}{3m} \quad ,$$

$$H(1) = \int_0^1 \frac{x(1-x^{3m})}{1-x}\,dx \quad , \quad I(1) = \int_0^1 \frac{x^2(1-x^{3m})}{1-x}\,dx \quad .$$

Bun, $$\sum_1^\infty \frac{f(n)}{n} = \lim_{m \to \infty} \{f(1)G(1) + f(2)H(1) + f(3)I(1)\}$$

In oilier to have convergence,

$$f(1) + f(2) + f(3) = 0 \quad ,$$

and then $$\int_0^1 \frac{f(1)+f(2)x+f(3)x^2}{1-x^3}\,dx = 0 \quad .$$

Since $f(1) = 1$, we have two linear equations in $f(2)$ and $f(3)$, leading to the previous results. Another way of solving the problem is via the function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad .$$

Here, we consider the more general problem of summing

$$S = \sum_{n=1}^\infty \left\{\frac{A_1}{rn+1} + \frac{A_2}{rn+2} + \cdots + \frac{A_s}{rn+s}\right\}$$

where r,s are rational and $r > 0$, $1 < s \le r$.

For convergence, it is necessary and sufficient that

$$A_1 + A_2 + \cdots + A_s = 0 \quad .$$

Since

$$\frac{1}{a+b} + \frac{1}{a+2b} + \cdots + \frac{1}{a+nb} = \frac{1}{b}\left\{\psi\left(\frac{a}{b} + n+1\right) - \psi\left(\frac{a}{b} + 1\right)\right\}$$

and

$$\lim_{n \to \infty} \{\psi(a+n) - \psi(n)\} = 0 \quad ,$$

we obtain

$$S = -\sum_{j=1}^{s} A_j\psi\left(\frac{j}{r} + 1\right)/r \quad .$$

Since $\psi(x)$ is known explicity for rational x, the series can also be explicitly evaluated.

In particular,

$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}} ,$$

$$\psi\left(\frac{2}{3}\right) = -\gamma - \frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}} , \quad \psi(1) = -\gamma .$$

Using these results on our original problem, we find that

$$f(2) = \frac{\psi(1)-\psi(1/3)}{\psi(1)-\psi(2/3)}$$

Also solved by the proposers.

186. Proposed by Lt. Jones-Bateman, U.S.C.G. and M. S. Klamkin, Ford Scientific Laboratory.

Two ships are steaming along with constant velocities. What is the minimum number of bearings necessary to be taken by one ship in order to determine the velocity of the other ship? Given this requisite number of bearings, show how to determine the other ship's velocity. (It is assumed that range-finding equipment is either non-existent or non-operative).

Solution by William H. Pierce, General Dynamics, Electric Boat Division.

Under the assumption that both ships are proceeding at constant velocities (courses and speeds), then no number of bearings taken by one ship can determine the velocity of the other ship. To fee this, let the parametric coordinates of the two ships be denoted by

$$x_1 = a_1 + v_1 t\cos\theta_1 , \quad y_1 = b_1 + v_1 t\sin\theta_1 ,$$

$$x_2 = a_2 + v_2 t\cos\theta_2 , \quad y_2 = b_2 + v_2 t\sin\theta_2 .$$

Then it follows easily that the following family of target ships will generate the same bearings as the target ship above with respect to the first ship:

$$x_2' = a_2' + v_2' t\cos\theta_2' , \quad y_2' = b_2' + v_2' t\sin\theta_2'$$

where

$$a_2' = \lambda(a_2-a_1) + a_1 , \quad b_2' = \lambda(b_2-b_1) + b_1 ,$$

$$v_2'\cos\theta_2' = \lambda(v_2\cos\theta_2 - v_1\cos\theta_1) + v_1\cos\theta_1 ,$$

$$v_2'\sin\theta_2' = \lambda(v_2\sin\theta_2 - v_1\sin\theta_1) + v_1\sin\theta_1 .$$

Editorial **Note:** See problem 193, this issue for a corrected version of this problem.

MOVING??,

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BOOK REVIEWS

Edited by
Roy B. Deal, Oklahoma University Medical Center

1. Elements of Real Analysis. By Sze-Tsen Hu, Holden-Day, Inc., San Francisco, California, 1967.

   A comprehensive, but readable, beginning graduate level introduction to Real Analysis, following the CUM recommendations with some optional material. It treats the fundamentals of topological spaces, metric spaces, topological linear spaces, including Hilbert spaces "studied leisurely in a single section," measures and integrals, including Daniell integrals and. Haar measure, and brief, but modern, introductions to differentiation and distributions.

2. Exercises in Mathematics. By Jean Bass, Academic Press, Inc., New York, 1966. xii + 457 pp., $14.75.

   A valuable collection of carefully worked out problems in real and complex analysis, ordinary and partial differential equations, and various aspects of applied mathematics.

3. A Hilbert Space Problem Book. By Paul R. Halmos, Van Nostrand Co., Inc., Princeton, New Jersey, 1967.

   A unique collection of problems and solutions of varying levels of difficulty designed to illustrate the "central techniques and ideas of the subject. The user of this book should have some knowledge of general topology, real and complex analysis, linear topological vector spaces, including Hilbert spaces, and real and. complex analysis. The first one hundred and forty pages list the problems with brief historical and mathematical comments and a few definitions. The next 22 pages give hints to the solutions and the last 185 pages give fairly complete solutions.

4. Differential Geometry. By Louis Auslander, Harper and Row, New York, 1967.

   A fine modern treatment of classical differential and global geometry for the advanced undergraduate or first year graduate with two years of calculus and one year of modern algebra. Prerequisite of algebraic topology is skillfully avoided. An interesting feature is the treatment of matrix Lie groups instead of abstract Lie groups.

5. Boundary Value Problems of Mathematical Physics. By Ivar Stakgold, MacMillan Co., Riverside, New Jersey, 1967.

   For the graduate students in applied mathematics, engineering, and physical sciences, with advanced calculus and elementary complex variables, this first of two volumes on linear boundary value problems is an excellent introduction to the study of Green's Functions, including the theory Of distribution, and eigenfunction expansions in linear integral equations and. the spectral theory of second-order differential operators.

6. Addition Theorems. By Henry B. Mann, John Wiley, Inc., New York, 1965, xi + 113 pp., $8.75.

This little book on additive theory of sets in number theory and more general Abelian groups or semi groups affords one of the best opportunities for the interested reader with a limited background, such as a course in modern algebra and a minimal introduction to number theory, to follow some recent, important research results in mathematics.

7. Information A Scientific American Book. By W. H. Freeman and Co., San Francisco, California, 1966, 218 p., 52 illustrations, 15 plates, Clothbound $5.00, Paperbound $2.50.

A collection of the faclnating series of articles appearing in the September, 1966 Scientific American on the state of the computer art and its' impact on society.

8. ~~Numbers and~~ Ideals. By Abraham Robinson, Holden-Day, Inc., San Francisco, California, 1965.

A very elementary introduction to algebraic number theory with its own introduction to modern algebra in a small book that could be studied, for example, by freshman honor students in mathematics.

9. Applied Combinatorial Mathematics. Edited by Edwin F. Beckenback, John Wiley, Inc., New York, 1964, xxi + 591 pp., $13.50.

A collection of eighteen chapters by outstanding mathematicians, covering a wide variety of aspects of this much recently emphasized aspect of mathematics. Varying levels, generally from advanced calculus on are required to seriously study the articles.

10. Mathematical Logic. By Stephen Cole Kleene, John Wiley, Inc., New York, 1967.

A junior (also written for juniors) addition of the author's famous "Introduction to Metamathematics" which should, prone to be one of the most popular books on the subject for Mathematics majors.

11. ~~Evolution of~~ Mathematical Thought. By Meschkowski, Holden-Day, Inc., San Franciso, California, 1965, 155 pp., $5.95.

This book, translated from the German, is an interesting discussion "about" the foundations of mathematics for a wide range of readers in both interest and level of background.

12. Geometric Programming. By Duffin, Peterson and Zener, John Wiley, Inc., New York, 1967, xi + 278 pp., $12.50.

Although the material was collected for use in solving engineering design problems, and many examples are given, many readers, interested in optimization problems such as generalization of linear programming problems, would, find this a profitable book to study.

13. The Theory of Gambling and Statistical Logic. By Richard A. Epstein, Academic Press, Inc., New York, 1967, xiii + 485 pp., $10.00.

A scholarly treatise on gambling with many thought-provoking ideas on a wide variety of subjects as well as some interesting mathematical results in probability theory and game theory for readers at about the advanced calculus level of mathematical maturity.

14. ~~A Second Course~~ In Complex Analysis. By William A Veech, W. A. Benjamin, Inc., New York, 1967, ix + 245 pp., $8.75.

A very good book for those with an elementary course in complex varables and general topology who would like fundamental grounding in some important aspects of the subject including the prime numbers theorem.

15. Plane Geomtry and Its Groups. By Guggenheimer, Holden-Day, Inc., San Francisco, California, 1967.

Although the generation of motions by reflections is classic, this treatment and selection of topics has such a modern flavor that many students of today will find it interesting and profitable to pursue.

16. Foundations of Mechanics. By Ralph Abraham, W. A. Benjamin, Inc., New York, 1967, ix + 290 pp., $14.75.

This is truly a foundations text. It is based on a series of lectures directed primarily at physicists at the graduate level, but the mathematical background is so well presented that it might also serve as an excellent introduction for mathematics majors to a modern treatment of differential manifolds, vector and tensor bundles, and integration on manifolds. The study of celectial mechanics contains very recent basic results.

NOTE: All correspondence concerning reviews and all books for review should be sent to PROFESSOR ROY B. DEAL, UNIVERSITY OF OKLAHOMA MEDICAL CENTER, 800 NE 13th STREET, OKLAHOMA CITY, OKLAHOMA 73104.

BOOKS RECEIVED FOR REVIEW

1. Topological Vector Spaces and Distributions By John Horvath, Addison-Wesley Publishing Co., Inc., Reading, Mass., Vol. I., 1966.

2. Computer Programming and Related Mathematics By R. V. Andree, John Wiley, Inc., New York, 1967, 281 pp., $6.50.

3. Abstract Theory of Groups By 0. U. Schmidt, Translated from the Russian by Fred Holling and J. B. Roberts, Edited by J. B. Roberts, W. H. Freeman and Co., San Francisco, California, 1966, 174 pp., $5.00.

4. Value Distribution Theory By Leo Sario and Kiyoshi Noshiro, Van Nostrand Co., Inc., Princeton, N. J., 1967, xi + 235 pp., $11.50.

5. Linear Geometry By K. W. Gruenberg and A. J. Weir, Van Nostrand and Co., Inc., Princeton, N. J., 1967, viii + 181 pp., $6.00.

6. Set Theory and the Number Systems By May Risch Kinsolving, International Textbook Co., Scranton, Penn.

7. Vector Analytic Geometry By Paul A. White, Dickenson Publishing Co., Inc., Belmont, California, 1965.

8. Introduction to Modern Abstract Algebra By David M. Burton, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1967.

9. Abstract Algebra By Chih-Hah Sah, Academic Press, Inc., New York, 1966, xiii + 335 pp., $9.75.

10. Theory of Functions of a Complex Variable Vol. 11 By A. I. Markushevich, Translated from the Russian by Richard A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, N J., xii + 329 pp., $12.00.

11. Theory of Functions of a Complex Variable Vol. III By A. I. Markushevich, Translated from the Russian by Richard A. Silverman, Printice-Hall, Inc., Englewood Cliffs, N. J., 1967, xi + 355 pp., $12.95.

12. Introduction to Real Analysis By Casper Goffman, Harper and Row, New York, 1966.

13. Vectors, ~~Tensors and~~ Groups By Thor A. Bale and Jonas Lichtenburg, W. A. Benjamin, Inc., New York, 1967.

14. Functions of One and Several Variables By Thor A. Bak and. Jonas Lichtenburg, W. A. Benjamin, Inc., New York, 1967.

15. Series, Differential Equations and Complex Functions W. A. Benjamin, Inc., New York, 1967.

16. Introduction to Probability and Statistics By William Mendenhall, Wadsworth Publishing Co., Belmont, California, 1967, xiii + 389 pp., $8.50 text, $11.35 trade.

17. Guidelines For Teaching Mathematics By Donovan A. Johnson and Gerald R. Rising, Wadsworth Publishing Co., Inc., Belmont, California, 1967, viii + 439 pp., $6.95.

18. Introductory Geometry: An Informal Approach Brooks/Cole Publishing Co., Belmont, California, 1967, xii + 238 pp., $7.50.

19. Stationary and Related Stochastic Processes By Harold Cramer and M. R. Leadbetter, John Wiley, Inc., New York, 1967, xii + 345 pp., $12.50.

20. Combinatorial Methods In the Theory of Stochastic Processes By Lajos Takac, John Wiley, Inc., New York, 1967, xi + 255 pp., $12.75.

21. First Order Mathematical Logic By Angelo Margaris, Blaisdell Publishing Co., Waltham, Mass., 1967.

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