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Kenneth Loewen, Editor

Chapter reports, books for review, problems for solution and solutions to problems, and news items should be mailed directly to the special editors found in this issue under the various sections. Editorial correspondence, including manuscripts, should be mailed to THE EDITOR OF THE PI MU EPSILON JOURNAL, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73069.

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THE C. C. MACDUFFEE AWARD FOR DISTINGUISHED SERVICE

J.C. Eaves, President, Pi Mu Epsilon

Fellow members of Pi Mu Epsilon, honored guests: We have come together tonight for many reasons but none more important than the next event on our program.

Presidents of an organization such as ours come and go, but they live on, on the back of the printed copies of our Constitution and By-Laws. This is not the case with the veritable servants of our fraternity. Any interested individual can ascertain from a copy of our constitution that Professor E. D. Roe, Jr., was our organization's first president (then called Director General) but there is no reference thereon that Dr. John Steiner Gold, our honoree, served as Secretary-General with Professor Roe. It was during these early formative years of Pi Mu Epsilon that the size of an Institution seemed to be synonomous with interest in a strong Mathematics Department and thus, Dr. Gold was instructed to discourage smaller colleges when they sought information concerning the possibility of a chapter.

Our Constitution booklet carries the names Ingold, Owens, Evans, Milne, Fort, MacDuffee, but no mention that our honoree saw service with all of these. Dr. Gold was elected secretary in 1927 and served continuously until 1948, the longest term of any elected officer. In 1936 he worked arduously and with inborn devotion to revise the constitution, placing emphasis upon quality rather than size alone. He saw Pi Mu Epsilon Chapters swell from 14 when he took office in 1927 to almost 50 in 1948 when he chose not to seek reelection.

While we all find it difficult to accept the decision of a devoted servant to step out, we must, nevertheless, see some justification. As one of the main driving forces behind the society, Dr. Gold had seen it develop from one or two new chapters per year, each installing itself, preparing its own certificates, paying no dues, ordering seals from the Director General (at 1¢ per) to one which required an initiation ceremony usually attended by the Secretary, required membership certificates, individually written (in the fine Spencerian hand of Mrs. Gold), with ribbon and seal affixed by hand. He maintained detailed office records in the centralized headquarters of the fraternity from 1936 until his retirement. During much of Dr. Gold's period of service there were no national dues but the head office accepted donations of one or two dollars per year from each chapter

Professor and Mrs. John S. Gold

if the chapter took the hint and voluntarily contributed. In 1936 the seal of the Fraternity was placed in the hands of the Secretary Treasurer General, the new office created by the combination of the offices of Secretary, Treasurer, and Librarian. Each new member now was charged a Fee of **25¢** which barely covered Dr. Gold's cash outlay for certificate, seal, and ribbons. During the few minutes we had together prior to the beginning of our program tonight he also recalled some experiences such as an all night bus trip to clarify a point, and a trip on a bus which became snowbound and thus caused him to relay the initiation ceremony by phone thereby accomplishing the installation of a new Chapter.

Dr. Gold relates that money was scarce in the early 1930's and interest in mathematics also lagged. In case some of you think that a quarter wasn't much during those years, I must add that it was made of silver and I know some college men who survived on 25 to 30 cents per day for food and this included between meal snacks and night caps. These were also the years during which high school principals were replacing mathematics courses with manual training courses. This was truly, for some, an unimaginably rough period and it is probably no exaggeration to say that except for the stamina, the perserverance, the unforgettable belief in the value of recognition of a budding young mathematician, and the unregrettable devotion of many hours during 50 per cent of his active, productive, lifetime —we may not today be aware of Pi Mu Epsilon, it having long since slipped into oblivion. With leaders like Dr. Gold, the World of Mathematics scholars was not to lose conscious knowledge of this organization.

To me, Dr. Gold always exemplified aptness, unselfishness, and foresight, in his decisions for the fraternity. It was he who foresaw not only the possibility but also the necessity of a national publication, and this at a time barely following World War II.

Tonight we could honor others but none more deserving. Tonight we could read names from our roll of the many servants of Pi Mu Epsilon, but none more devoted. We could recount the accomplishments of president after president but scarcely a combination equals his indulgence. And for humility he has no equal.

Dr. Gold honors us tonight in accepting Pi Mu Epsilon's highest award, the C. C. **MacDuffee** Award. It was our intentions that this award say simply enough to anyone, "This is recognition of genuine, unselfish service." We intended that it be elegant enough to grace the finest wall and distinguished enough to become a cherished possession. We know, members of Pi Mu Epsilon, this is a tribute to a deserving member who did not labor that he saw a medal in sight but for the promotion of those true scholarly ideals he saw foremost in our organization.

In recognition of the contribution Mrs. Gold has made, both in **sharing** her husband's time with us and for the uncounted hours she spent in cutting ribbons, licking seals, and applying that long lost art of Spencerian penmanship, I am going to ask that she too stand and share in the presentation.

Dr. Gold, you honor us tonight and you both leave us forever in your debt.

August 29, 1967, On the occasion of the Annual Banquet, Toronto.

A CONVERGENT SERIES AND LOGARITHMS IN DIFFERENT BASES

Ali R. Amir-Moéz, Texas Technological College

In this note we first study a theorem of Euclidean geometry and use the result in obtaining logarithms in a positive base.

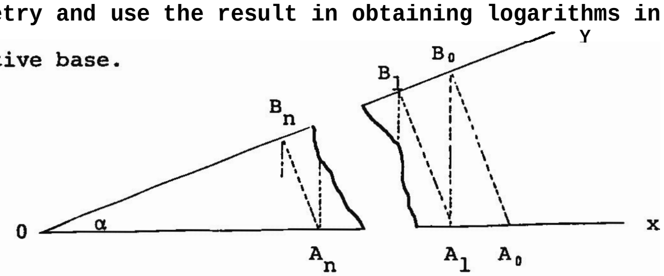

Fig. 1

1. Theorem: Let Ox and Oy be two half lines and a be the angle between them (Fig. 1). Let A_0 be a point on Ox. We consider points A_0, A , ... and B_0, B_1, ... such that B_n is the foot of the perpendicular through A to Oy, and A_{n+1} is the foot of the perpendicular through B_n to Ox. Let the segment $OA_n = a$, the segment $OB_n = b_n$, and the segment $A_nB_n = c_n$. Then

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} b_n, \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_n$$

are convergent for $0 < a < \frac{\pi}{2}$ and $0 < a_0 < \infty$.

proof: Let $a_0 = a$. Then $b_0 = a \cos a$ and $c_0 = a \sin a$. Thus

$$S = \sum_{n=0}^{\infty} a = a(1 + \cos^2 a + \dots + \cos a + \dots)$$

$$= \frac{a}{1 - \cos^2\alpha} = \frac{a}{\sin^2\alpha},$$

$$T = \sum_{n=0}^{\infty} b_n = a(\cos\alpha + \cos^3 a + \dots + \cos^{2n+1}\alpha + \dots)$$

$$= S \cos a = \frac{a \cos a}{\sin^2 a}$$

$$R = \sum_{n=0}^{\infty} c_n = a \sin a(1 + \cos a + \dots + \cos^n a + \dots)$$

$$= \frac{a \sin a}{1 - \cos a} .$$

2. Discussion: We observe that for $a = 0$ all three series are divergent. One easily sees that

$$S + T = \frac{a}{1 - \cos\alpha} .$$

This implies that

$$\frac{R}{S + T} = \sin\alpha .$$

We may study the case $a > \frac{\pi}{2}$. But it will not be very interesting.

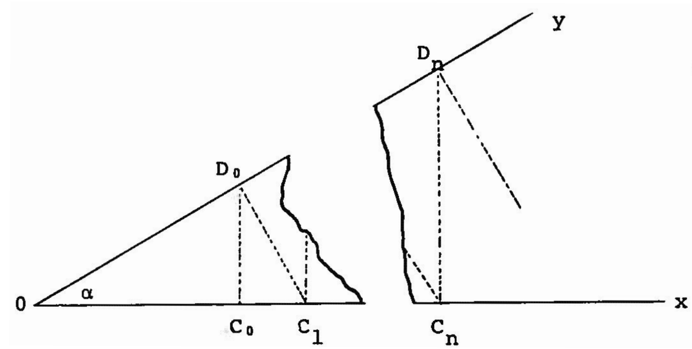

Fig. 2

3. Logarithms in a base: Let us look at a sequence related to the series S. Choose C_0 on Ox and let $OC_0 = d$. Then we draw the perpendicular to Ox through C_0 (Fig. 2). This perpendicular intersects Oy at D_0. Now we define C_1, ..., C_n on Ox and D_1, ..., D_n on Oy such that the

segment $C_k D_k$ is perpendicular to Ox and the segment $D_{k-1} C_k$ is perpendicular to Oy for $k = 0, 1, \ldots, n$. Let $OC_i = d_i$, $i = 0, 1, \ldots$. Then the sequence $d_0, \ldots, d_n, \ldots$ will will be

$$d_0 = d, d \sec^2 a, \ldots, d \sec^{2n} \alpha, \ldots \ .$$

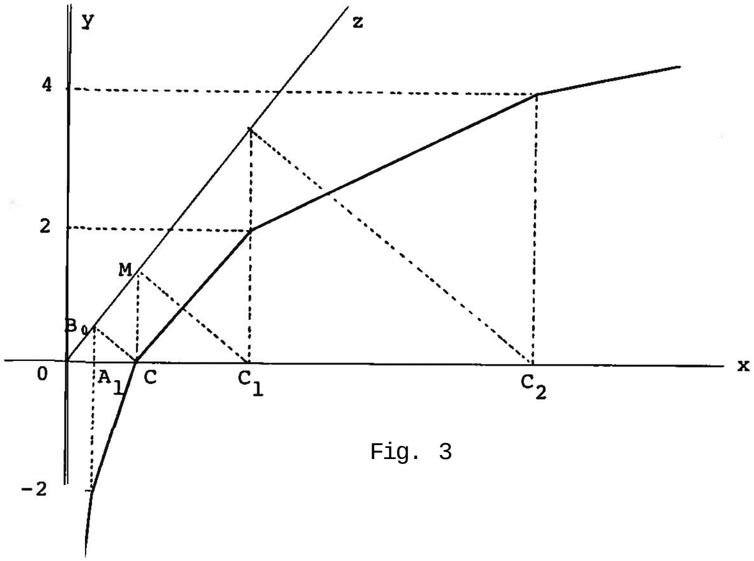

Fig. 3

In particular we are interested in the case that $d = 1$. Let h be a positive number and a base for logarithms. We consider a rectangular coordinate system (Fig. 3). We choose $C_0 = C$ to be $(1, 0)$. We construct the right triangle OCM such that C is the vertex of the right angle and $OM = h$. Let us denote the half line through 0 and M by Oz. It is clear that if a is the angle between Ox and Oz, then $\sec a = h$. This implies that C_n corresponds to (seca, 0), $n = 1, 2, \ldots$. Thus

$$\log_h d_n = 2n \ .$$

Here we consider points $(1, 0)$, $(d_1, 2)$, $(d_2, 4)$, $\ldots$, $(d_n, 2n)$. These points are on the graph of $y = \log_h x$.

Now if we draw CB_0 perpendicular to Oz and B_0A_1 perpendicular to Ox and continue this way, we may obtain points with negative ordinates for $\log_h x$, for example $(a_1, -2)$ where $OA_1 = a_1$.

What has been studied in this section does not give a very accurate approximation of $\log_h x$. Thus we shall add a few ideas in order to get better approximations.

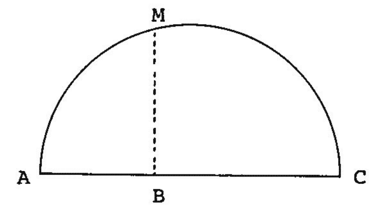

Fig. 4

4. <u>Geometric means</u>: Let a and b be two positive numbers denoted by the line segments AB and BC respectively (Fig. 4). We draw a half circle with diameter AC. Then we draw the line perpendicular to AC at B. This line intersects the half circle at M. It is quite easy to show that the length of BM is $\sqrt{ab}$, i.e., the geometric mean of a and b. One can easily show that

$$\log_h \sqrt{ab} = \frac{1}{2} [\log_h a + \log_h b].$$

5. <u>Refinement of the graph</u>: Using the geometric construction in 4, we can obtain more points for the graph of $y = \log_h x$. For example, let us consider C_1 and C_2 of figure 3. To C_1 and C_2 respectively correspond $d_1 = \sec^2 a$ and $d_2 = \sec^4 a$. We construct the geometric mean of d_1 and d_2. Let D correspond to this mean (Fig. 5). One can easily see that to D corresponds $(\sec^3 a, 0)$. We

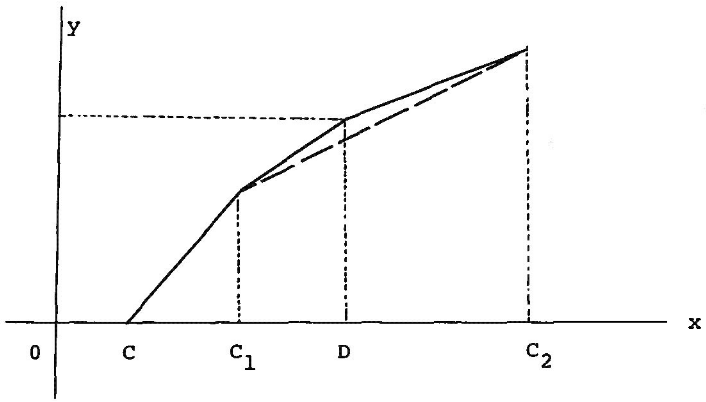

Fig. 5

observe that the arithmetic mean of $\log_h d_1$ and $\log_h d_2$ is 3. Thus the point $(\sec^3 a, 3)$ is on the graph of $y = \log_h x$. By obtaining geometric means on the x-axis and corresponding arithmetic means on the y-axis, one can obtain more points of the graph of $y = \log_h x$.

UNDERGRADUATE RESEARCH PROJECT

Proposed by Richard V. Andree, University of Oklahoma

Write the equation

1) $a_0 + a_1x + \ldots + a_nx^n = 0;$

in the form

2) $(a_0, a_1, \ldots, a_n)\cdot(1, x, \ldots, x^n) = 0.$

Then the problem of finding approximate solutions to 1) may be thought of as finding a vector

3) $(1, x, \ldots, x^n)$

which approximately solves 2).

A related (but not equivalent) problem is to find exact vectors for which 2) is approximately zero.

A very different problem would be. given x, which vectors $(a_0, a_1, \ldots, a_n)$ make 2) approximately zero.

Investigate these last two problems.

GENERALIZATIONS OF SEQUENCES

William L. Reynolds

Florida State University

INTRODUCTION: Since our first introduction to elementary calculus we have all been aware of the seeming lack of consistency in defining limits and convergence of sequences of real functions as compared with the definitions as applied to complex functions and functions of several variables. It is with the hope of wrapping all these definitions up into one neat package that we embark on this study of generalizations of sequences following the theory of E. H. Moore and H. L. Smith.

(1) DIRECTED SETS AND NETS: Let us examine briefly the concept of convergence of the real valued denumerable sequence $x_1, x_2, x_3, \ldots$ which converges to some k. Surely each of us is familiar with the classical "epsilon-delta" definition of convergence, and perhaps also with a more general topological definition. Let us put aside these definitions for the time being and develop a general definition involving the concept of a directed set, which we define as follows.

1.1) Definition: A directed set is a nonempty set N together with a binary relation $\geq$ (called the direction on N) such that
   a) $a \geq a$ for each a in N,
   b) $a \geq b$ and $b \geq c$ implies $a \geq c$ for all a, b, c in N, and
   c) if a and b are elements of N, then there is c in N such that $c \geq a$ and $c \geq b$.

We will denote the directed set by $(N, \geq)$ or by N if there is no confusion, and we say that N is directed by $\geq$. Obvious examples of directed sets are the set of positive integers and the reals with the usual $\geq$, and the collection of neighborhoods of a point in a topological space directed by inclusion.

1.2) Definition: A net in a space X is a function $S:N \to X$ from a directed set N into a space X.

We will denote the net simply by S, or in case the domain and its direction are not explicit by $(S, N, \geq)$. We say that S is in the space X if $S(n)$ belongs to X for each n in N; S is said to be frequently in a subset A of X if for each n_o in N, there is $n \geq n_o$ such that $S(n)$ is in A. S is eventually in the subset A if for some n_o in N, $S(n) \in A$ for each $n \geq n_o$.

1.3) Definition: A net S in a space X is said to converge to a point p in X if and only if S is eventually in each neighborhood of p.

We can now characterize open sets by the following

1.4) Theorem: A subset U (of a topological space X) is open if and only if no net in X-U can converge to a point of U. (In the interest of brevity the proofs of this theorem and of many of the following are left to the reader.)

1.5) Corollary: A subset V of a topological space X is closed if and only if no net in V converges to a point of X-V.

Student paper presented at the National Pi Mu Epsilon Meeting in Toronto, Canada, August, 1967.

One will note that, in general, limits of nets are by no means unique. The following theorem gives necessary and sufficient conditions for uniqueness.

1.6) Theorem: Let X be a topological space. Then each net in X converges to at most one point if and only if X is Hausdorff.

We extend the notion of accumulation points to nets by saying that a point s is an accumulation point of a net S if S is frequently in each neighborhood of s. It is interesting to compare this concept with the definition of convergence and ask under what conditions a net converges to its accumulation point(s). The following theorem establishes sufficient conditions for this convergence.

1.7) Theorem: Let S be a net in a space X with the property that, for each subspace A of X, S is eventually in A or eventually in X–A (S is universal). Then S converges to each of its accumulation points. (Proof is immediate upon establishment of the lemma: A universal net which is frequently in a set A is eventually in A.)

The study of subnets is appealing to one's intuition and supplements the theory of nets in much the same manner as subsequences supplement the theory of sequences. We will state the definition of subnet and a resulting theorem strictly as a point of interest, but, since we make no use of the concept in the sequel, we will not pursue the topic.

1.8) Definition: A net (T,D) is a subnet of a net (S,E) if and only if there is a function $N:D \to E$ such that

a) $T = S \cdot N$, or equivalently $T(i) = S(N(i))$ for each i in D, and

b) for each m in E there is n in D with the property that, if $p \geq n$, then $N(p) \geq m$.

1.9) Theorem: A point s in a space X is an accumulation point of a net S (in X) if and only if some subnet of S converges to s.

2) FILTERS: Our second generalization is somewhat less appealing to those indoctrinated with notions of sequences, but it nevertheless provides some interesting theory.

2.1) Definition: A filter in a set X is a nonempty family Φ of nonempty subsets of X such that

a) the intersection of any two sets in Φ contains an element of Φ, and

b) if $A \in \Phi$ and $A \subset B \subset X$, then $B \in \Phi$.

Obvious examples of filters are the set of all neighborhoods of a point in a topological space and the family of all sets with finite complements in an infinite set.

Convergence is of particular interest, and we make the following

2.2) Definition: A filter Φ is said to converge to a point x in a space X if and only if each neighborhood of x belongs to Φ.

This definition makes it possible to characterize open sets in topological spaces.

2.3) Theorem: A subset U of a set X is open if and only if U belongs to every filter in X which converges to a point of U.

It is also possible to characterize Hausdorff spaces by use of filters, as the following theorem indicates.

2.4) Theorem: A topological space X is Hausdorff if and only if each filter in X converges to at most one point.

We will say that the family B of subsets of a set X is a basis for a filter if and only if the collection of all subsets of X containing sets of B is a filter in X. Clearly if B' is a family such that each member of a filter Φ' contains a member of B*, then B' is a basis for Φ'.

2.5) Definition: A filter Φ is said to refine a filter Φ' if each member of Φ' is also a member of Φ. In such a case Φ is said to be a refinement of Φ'.

2.6) Lemma: Any refinement of a filter converging to a point also converges to that point.

Accumulation points of filters are of interest and are defined as follows: a point x is an accumulation point of a filter Φ if there is a refinement of 4 which converges to x.

2.7) Lemma: Given a filter Φ, the following are equivalent:

a) x is an accumulation point of Φ.

b) There is a filter Φ' which is a refinement of Φ and of N_x, the neighborhood system of x.

c) The intersection of sets (members) of Φ with the sets of N_x are nonempty.

3) RELATED PROPERTIES OF NETS AND FILTERS: In view of the close similarities of many of the notions of nets and filters, one's first reaction is probably to wonder if the two objects are equivalent. We will not Drove that they are equivalent. but as the following theorems indicate, each net determines a filter with similar properties, and conversely.

If $(S,D,\geq)$ is a net in a set X, then we will call the collection of sets $\{F_n\}$ where $F_n = \{S(m) \mid m \geq n;\ m,n \in D\}$ the filter basis associated with the net S. (It is left to the reader to show that $\{F_n\}$ is indeed a filter basis.) Recalling the definition of a filter basis, the net S thus defines a filter, called the filter associated with the net S.

3.1) Lemma: If the net $(S,D,\geq)$ is eventually in a set $E \subset X$, then E is an element of Φ, the filter associated with S.

3.2) Theorem: If the net S in the topological space X converges to a point x, then the filter Φ associated with S converges to x.

Proof: Recall that S converges to x if and only if S is eventually in each neighborhood of x. From the above lemma, each neighborhood of x belongs to Φ, so Φ converges to x.

Nets may be constructed from filters in the following manner. Let Φ be a filter in a space X, and let A be the collection of pairs (x,F) where $F \in \Phi$ and $x \in F$. We define a direction on A by requiring $(x,F) \geq (y,G)$ if and only if $F \subset G$. Then the function $S:A \to X$ defined by $S(x,F) = x$ is a net in X, called the net associated with the filter.

3.3) <u>Lemma:</u> If Φ is a filter and $F \subset X$ is a member of Φ, then the net associated with Φ is eventually in F.

<u>Proof:</u> Let Φ be a filter and S the net associated with Φ as constructed above. Let $F \in \Phi$, and let (y,G) be any element of our directed set A (the domain of S) such that $(y,G) \geq (x,F)$. Then $G \subset F$ and $y \in G$, so $S(y,G) = y$ is in F. Thus S is eventually in F.

3.4) <u>Theorem</u>: Let Φ be a filter converging to a point x in a space X. Then the net S associated with Φ converges to x. (This is a direct consequence of the preceding lemma and the fact that each neighborhood of x belongs to Φ.)

The suspected relations involving accumulation points of associated nets and filters also hold, as the following theorems indicate.

3.5) <u>Theorem</u>: If x is an accumulation point of a net S, then x is also an accumulation point of the filter associated with S.

<u>Proof:</u> Suppose x is not an accumulation point of Φ, the filter associated with the net $(S,D,\geq)$. Then there is some neighborhood N of x and some $F \in \Phi$ such that $N \cap F = 0$ by 2.7. Recall that $\Phi = \{W | W \supset F_n\}$ where $F_n = \{S(k) \mid k \geq n\}$. So for some $n \in D$, $F_n \subset F$, hence $F_n \cap N = 0$, and therefore $S(j) \notin N$ for $j \geq n$; that is, S is not frequently in N, so x is not an accumulation point of S.

3.6) <u>Theorem</u>: Let Φ be a filter and S the associated net. If x is an accumulation point of Φ, then x is an accumulation point of S.

<u>Proof</u>: Let N be a neighborhood of x, and suppose $(a,F) \in A$, the domain of S. By 2.7 (c), $F \cap N \neq 0$, so for $b \in F \cap N$, $(b,F) \geq (a,F)$. Thus S is frequently in N.

So we see that for each net in a space X there is a filter in S with related properties, and conversely. This leads us to believe that any proposition whose proof requires the use of one of these objects could be justified by application of the other. Published research tends to justify this belief.

In the interest of brevity, only the basic definitions and theorems needed for a cursory comparison of nets and filters have been given. Many more results are known. While the independent study of nets and filters and their comparison is interesting and rewarding, the real challenge lies in developing, in accordance with a conjecture of E. H. Moore, a more general theory of sequential notions of which nets and filters are but special cases.

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SUMMATION OF GENERALIZED HARMONIC SERIES WITH PERIODIC SIGN DISTRIBUTIONS

B. J. Cerimele, Xavier University

In reinforcing the idea that a conditionally convergent series depends upon the presence of an infinite number of negative terms, one usually cites the example of the alternating harmonic series. A frequent question which arises in this consideration is whether other patterns, such as the negation of every third term, might render the harmonic series convergent. In this connection the author will discuss the convergence behavior of a generalized harmonic series having any periodic distribution of signs. In particular, it will be brought out that only those sign distributions in which a balance of plus and minus signs occur in the repeating sign block render the series convergent. Finally, a method of summing such convergent series by means of fundamental component subseries will be explained.

Consider the series, which shall be called the W-series, defined by

$$\sum_{i=0}^{\infty} 1/(j + ik),$$

where j and k are arbitrary positive integers. For $j=k=1$ this series reduces to the ordinary harmonic series, and for arbitrary j and k it diverges with the harmonic series. Patterns of sign distribution in the W-series which yield convergence are subject to the following theorem, due to Cesaro, which imposes a necessary condition on the relative frequency of plus and minus signs.

Cesaro's Theorem [3,p.318]. Let p_n and q_n denote the number of positive and the number of negative terms respectively in the first n terms of a series. If the series is conditionally convergent and its sequence of terms in absolute value is monotonically decreasing, then $\lim_{n\to\infty} p_n/q_n = 1$, where the limit is known to exist when the terms are of an order of magnitude not less than those of the harmonic series [2,p.17]. In a periodic pattern of sign distribution Cesaro's theorem leads to the requirement that there be a balance of plus and minus signs in the cyclic sign block.

In order to sum convergent W-series with a periodic sign distribution the **summability** of certain fundamental alternating series is needed. Consider the following alternating series:

$$W(j;\, k_1,\, k_2) = \sum_{i=0}^{\infty} (-1)^i \,/\, (j + qh + rk_1), \qquad (1)$$

where $h = k_1 + k_2$, $q = [i/2]$, $r = i$, mod 2.

Theorem: The series in (1) converges and has sum

$$W(j;\, k_1,\, k_2) = (1/h)\,[\psi\,((j + k_1)/h) - \psi\,(j/h)],$$

where $\psi\,(z)$ denotes the psi or **digamma** function [1, p.277].

Proof: That the series in (1) converges is an immediate consequence of the alternating series test. To generate the sum function the terms of the series are grouped in pairs yielding:

$$\sum_{i=0}^{\infty} 1/(j + ih) - 1/(j + k_1 + ih).$$

This grouped series can be expressed by:

$$\sum_{i=0}^{\infty} \int_j^{j+k_1} (x + ih)^{-2} dx.$$

Because the series

$$\sum_{i=0}^{\infty} (x + ih)^{-2}$$

is uniformly convergent for $x \geq 1$, the operations in the above grouped series may be permuted to give

$$\int_j^{j+k_1} dx \sum_{i=0}^{\infty} (x + ih)^{-2}.$$

Noting that [1, p.285]

$$\psi'(x/h) = \sum_{i=0}^{\infty} (i + x/h)^{-2},$$

one obtains the result:

$$W(j;k_1,k_2) = (1/h^2) \int_j^{j+k_1} \psi'\,(x/h)dx$$
$$= (1/h)\,[\psi\,((j + k_1)/h) - \psi\,(j/h)].$$

Utilization of an integral representation of the psi function [1, p.278] yields the following special case which is reducible to elementary functions.

Corollary. If $k_1 = k_2 = k$,

$$W(j;k,k,) = W(j;k) = (1/2k)\,[\psi\,((j/2k) + 1/2) - \psi\,(j/2k)]$$
$$= (1/2k) \int_0^1 \frac{t^{(j/2k)-1/2} - t^{(j/2k)-1}}{t - 1}\, dt$$

which by means of the transformation $t = x^{2k}$ becomes

$$= \int_0^1 x^{j-1}dx/(1 + x^k).$$

This last integral can be expressed in terms of elementary functions, viz.,

$$W(j;k) = (-1)^{j-1}(r/k)\,\ln\,(1 + x)$$
$$-(2/k) \sum_{i=0}^{q-1} [P_i(x)\,\cos\,((2i + 1)\,j\pi/k)$$
$$-Q_i\,(x)\,\sin\,((2i + 1)\,j\pi/k)]\,\Big|_0^1$$

where $q = [k/2]$, $r = k$, mod 2, $j \leq k$,

$$P_i(x) = (1/2)\,\ln\,[x^2 - 2x\cos((2i + 1)\pi/k) + 1]$$

$$Q_i(x) = \arctan[(x - \cos((2i + 1)\pi/k)/\sin\,(2i + 1)\pi/k].$$

Table 1 provides closed expressions for the sums of some of the fundamental alternating W-series.

Because of the cyclic pattern in the periodic sign distribution for a convergent W-series **it is** apparent that such series can be decomposed into component subseries of the form in (1); and moreover, the sum of the convergent W-series is given by the sum of the sums of the fundamental component subseries. Table 2 consists in a compilation of closed expressions and approximate numerical values for the harmonic series having convergent periodic sign patterns spanning **two, four, and six terms.**

The author acknowledges the computational support rendered by his student Timothy Luken.

Table 1

Closed Expressions for Some

Fundamental Alternating W-series

One parameter series: $W(j;k) = \sum_{i=0}^{\infty} (-1)^i / (j + ik)$

$W(j;k)$	$k=1$	2	3	4
$j=1$	$\ln 2$	$\pi/4$	$(1/9)[\pi\sqrt{3} + 3 \ln 2]$	$(\sqrt{2}/8)[\pi + 2 \ln(1 + \sqrt{2})]$
2		$(1/2) \ln 2$	$(1/9)[\pi\sqrt{3} - 3 \ln 2]$	$\pi/8$
3			$(1/3) \ln 2$	$(\sqrt{2}/8)[\pi - 2 \ln(1 + \sqrt{2})]$
4				$(1/4) \ln 2$

Two-parameter series: $W(j;k_1,k_2) = \sum_{i=0}^{\infty} (-1)^i / (j + qh + rk_1)$,

Where $h = k_1 + k_2$, $q = [i/2]$, $r = i$, mod 2

$h = 2$, $W(1;1,1) = W(1;1) = \ln 2$

$h = 3$, $W(1;1,2) = (\pi/9)\sqrt{3}$

$W(1;2,1) = (1/18)(\pi\sqrt{3} + 9 \ln 3)$

$h = 4$, $W(1;1,3) = (1/8)(\pi + 2 \ln 2)$

$W(1;2,2) = W(1;2) = \pi/4$

$W(1;3,1) = (1/8)(\pi + 6 \ln 2)$

Table 2

Summation of Harmonic Series

with Convergent Periodic Sign Distributions

Sign Pattern	Sum	Approximate value
+--+	$(1/4)(\pi - \ln 2)$	.61211
+-+-	$\ln 2$	.69315
++--	$(1/4)(\pi + 2 \ln 2)$	1.13197
+---++	$(1/3) \ln 2$	.23105
+--+-+	$(1/9)(\pi\sqrt{3} - 3 \ln 2)$	.37355
+--++-	$(1/18)(3 \ln 108 - \pi\sqrt{3})$	.47806
+-+--+	$(1/6)(\pi\sqrt{3} + \ln(4/27))$	.58864
+-++--	$(1/9)(\pi\sqrt{3} + 3 \ln 2)$	,83565
++---+	$(1/9)(2\pi\sqrt{3} - 3 \ln 2)$	.97815
++--+-	$(1/18)(\pi\sqrt{3} + 3 \ln 108)$	1.08266
++-+--	$(1/6)(\pi\sqrt{3} + \ln(27/4))$	1.22516
+++---	$(1/9)(2\pi\sqrt{3} + 3 \ln 2)$	1.44025

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A SHORTER AXIOMATIC SYSTEM FOR BOOLEAN ALGEBRA

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In volume 4, number 6 (this Journal), Lawrence J. Dickson lists seven axioms for Boolean Algebra. This list can be abbreviated.

I. The axioms:

A Boolean Algebra is a set X such that, for all a, b, c, $\ldots \varepsilon X$:

A. There is defined a (closed) binary operation $\cap$ such that:

Axiom 1: $a \cap (b \cap c) = (a \cap b) \cap c$

Axiom 2: $a \cap b = b \cap a$

Axiom 3: $a \cap a = a$

B. There exists an element $I \varepsilon X$ such that

Axiom 4: $a \cap I = a$ for all $a \varepsilon X$

C. There can be defined a function ' from X into itself such that:

Axiom 5: $(a')' = a$ for all $a \varepsilon X$

Axiom 6: $a \cap a' = I'$ for all $a \varepsilon X$

Axiom 7: $a \cap b = I' \rightarrow a \cap b' = a$.

II.

Theorem 1: Axiom 3 is a consequence of axioms 5, 6, and 7.

Proof:

Let a be any element of X. By axiom 6, $a \cap a' = I'$. Hence by axiom 7, $a \cap (a')' = a$. But, by axiom 5, $(a')' = a$. Therefore $a \cap a = a$.

Theorem 2: Axioms 1 and 2 can be replaced by the axiom

Axiom 1': $a \cap (b \cap c) = (b \cap a) \cap c$,

as long as axiom 4 is retained.

Proof:

Clearly axiom 1' is a consequence of axioms 1 and 2. Suppose now that axiom 1' holds. First we show that axiom 2 holds.

$$\begin{aligned} a \cap b &= a \cap (b \cap I) && \text{by axiom 4} \\ &= (b \cap a) \cap I && \text{by axiom 1'} \\ &= b \cap a && \text{by axiom 4.} \end{aligned}$$

Then $a \cap (b \cap c) = (b \cap a) \cap c = (a \cap b) \cap c$, which shows that axiom 1 is satisfied.

Hence the original seven axiom system can be replaced by a system of five axioms, namely 1', 4, 5, 6, and 7.

DISTANCE PRESERVING FUNCTIONS ON THE REAL LINE

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In introducing a general theorem it is usually both interesting and instructive to examine its implication in restricted cases. It is important that the student be able to illustrate the theorem in a setting familiar to him. There are many theorems in mathematics which become quite elementary when applied to restricted cases. We would like to discuss isometrics on the real line. The notions of function, distance, composition of functions, and graphing will be illustrated.

Definition 1: The function $f = \{(x,y) \mid y=f(x)\}$ is an isometry if and only if for all real numbers a and b we have $|a-b| = |f(a) - f(b)|$.

Definition 2: Let f and u be functions and let F be the function $F = \{(x,y) \mid y = f[u(x)]\}$. Then F is called the composition (or composite) of f upon u; that is, F is the set of all ordered pairs of real numbers (x,y) such that $y = f[u(x)]$.

The reader is referred to reference [1] for further definitions.

The following theorem is basic to the study of isometries in Euclidean n-dimensional space (E^n).

Basic Theorem: Every isometry in Euclidean n-dimensional space can be represented by at most n+1 reflections through (n-1)-dimensional space. (Ref. [2])

Since use of the theorem here is restricted to 1-dimensional space, i.e., the real line, we will use a special case of the Basic Theorem: every isometry of Euclidean 1-dimensional space can be represented by at most two reflections through a point.

We will discuss a procedure whereby any number of reflections can be expressed as a combination of not more than two reflections. Note that the Basic Theorem merely guarantees the existence of such a representation, whereas this procedure will yield a method for computing the exact representation in terms of the information at hand.

Theorem 1: The graph of an isometry of a line into itself is a straight line having slope either +1 or -1.

Proof: Since every isometry can be represented as either a reflection through one point or a reflection through two points, we will consider two cases.

Case I: Reflection through one point.

Suppose the isometry is equivalent to a reflection through one point, say A. Then it follows that $f(A)=A$. For an arbitrary point x let $f(x)=x'$; then $|x-A| = |x'-A|$. It follows that either (1) $x-A = x'-A$, in which case $x=x'$ and thus $x=A$ (since A is the only fixed point) or (2) $x-A = -(x'-A)$ and $x' = 2A-X$. Therefore $f(x)=2A-x$. It will now be shown that the slope of the line segment joining (A,A) and $(x,2A-x)$ is -1. The slope of this line segment is given by

$$m = \frac{2A-x-A}{x-A} = \frac{A-x}{x-A} = -1.$$

Since x is an arbitrary point the result follows. See Figure 1.

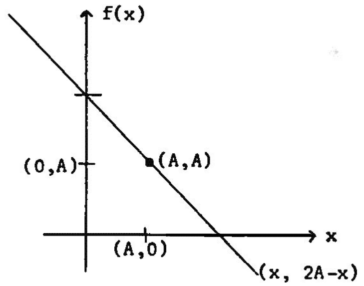

FIGURE 1: Graph of a reflection through points A and $(x,2A-x)$.

Case **II**: Reflection through two points.

Suppose the isometry is equivalent to reflections through two points A_1 and A_2 in that order. From the proof of Case I we know that given an arbitrary point x, the function $f(x)=2A_1-x$ reflects through the point A_1. Similarly the function $g(2A_1-x)=2A_2-(2A_1-x)$ reflects through the point A_2. Then $f[g(x)]=2A_2-2A_1+x$ is the composite function. Now $f[g(A_1)]=2A_2-2A_1+A_1=2A_2-A_1$. The slope of the line segment joining the two points $(x,2A_2-2A_1+x)$ and $(A_1,2A_2-A_1)$ is

$$m = \frac{(2A_2 - 2A_1 + x) - (2A_2 - A_1)}{x - A_1} = \frac{x - A_1}{x - A_1} = 1.$$

Since the points x, A_1 and A_2 were chosen arbitrarily, the proof is complete. See Figure 2.

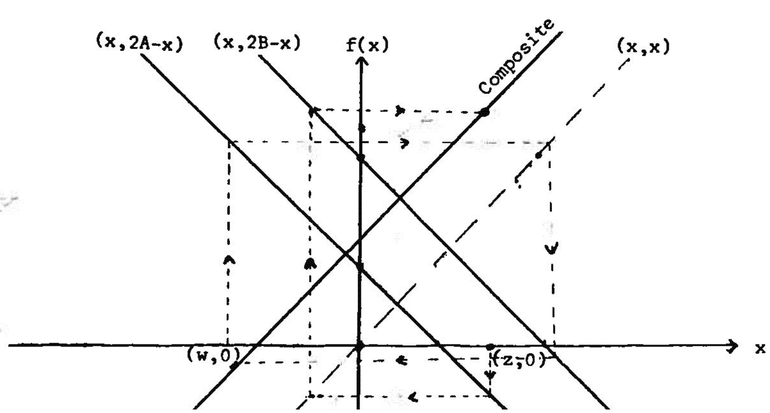

FIGURE 2: Composite of a reflection through B upon a reflection through A.

Theorem 2: The graph of an isometry F has slope $+1$ if and only if it can be decomposed into two isometries f and g whose graphs have slope -1.

Proof: Assume that F can be decomposed into two isometries f and g whose graphs have slope -1. Then $f = \{(x,y)|y=A-x$, for some fixed point $A\}$ and $g = \{(x,y)|y=B-x$, for some fixed point $B\}$. Therefore $F(x) = f(g(x)) = A - (B-x) = A - B + x$. Therefore F has slope +1.

Now suppose F has slope +1; then $F(x) = A+x$.

Let $f = \{(x,y)|y=A-x$, for some fixed point $A\}$ and $g = \{(x,y)|y= -x\}$. Then $f[g(x)]=A+x$. See Figure 3.

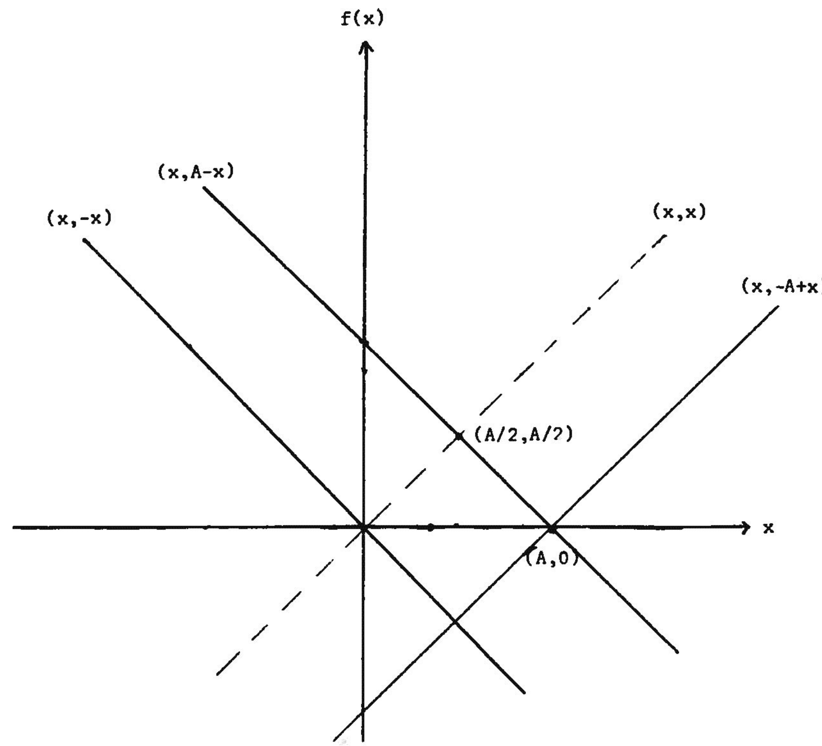

FIGURE 3: Decomposition of an isometry with slope $+1$ into two reflections.

Theorem 3: The composite of two isometries whose graphs have slope -1 and $+1$ is an isometry whose graph has slope -1.

Proof: Let $f = \{(x,y)|y=A-x$, for some fixed point $A\}$ and $g = \{(x,y)|y=B+x$, for some fixed point $B\}$. Then $f[g(x)]=A-(B+x)=A-B-x$ and has slope -1; also $g[f(x)]=B+A-x$ and has slope -1. See Figure 4.

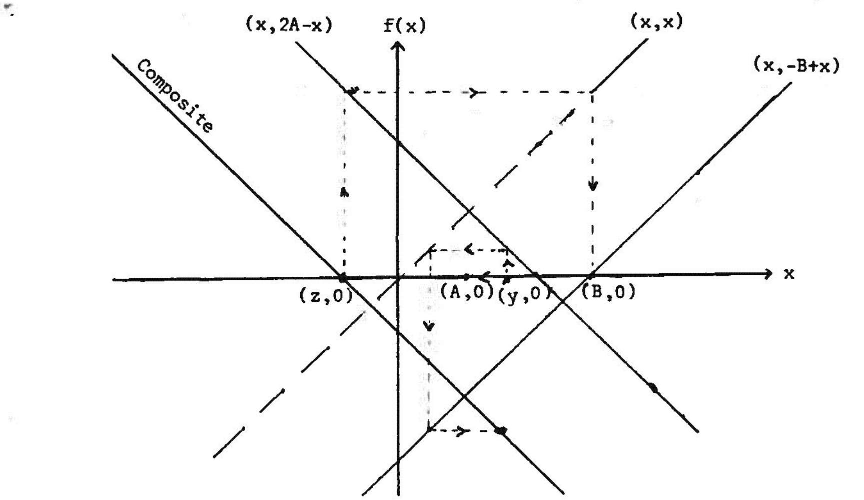

FIGURE 4: Composition of an isometry with slope -1 upon an isometry with slope $+1$.

Theorem 4: The composite of two **isometries** whose graphs have slope $+1$ is an isometry with slope $+1$.

Proof: Assume f and g are two functions whose slopes are $+1$. Let $f = \{(x,y)|y=A+x$, for some fixed point A} and $g = \{(x,y)|y=B+x$, for some fixed point $B\}$; then $F(x) = f[g(x)]=A+B+x$ whose slope is $+1$.

A method will now be given for finding the image of an arbitrary point x under a finite number of reflections. Let f be the function obtained by reflection through n points $A_1, A_2, \cdots, A_n$ in that order. Then $f_1 = \{(x,y)|y=2A_1-x\}$, $f_2 = \{(x,y)|y=2(A_2-A_1)+x\}$, $f_3 = \{(x,y)|y=2(A_3-A_2+A_1)-x\}$.

In general, for n-even, f_n is obtained by induction, first reflecting through the point whose coordinate is $A_1+A_3+A_5+\dots+A_{n-1}$ followed by a reflection through the point whose coordinate is $A_2+A_4+A_6+\dots+A$.

If n is odd, then f is obtained by a reflection through the point whose coordinate is given by $(A_1+A_3+A_5+\cdots+A\ ) - (A_2+A_4+A_6+\cdots+A_{n-1})$.

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A FUNCTIONAL APPROACH TO THE SINGULARITIES OF PLANE CURVES

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The problem of studying plane curves and their singularities has intrigued the men of mathematics for centuries. However, the necessity of working with poorly-behaved functions, the **parametrizations** of these curves, often quickly dampened the adventuresome spirits of many a would-be algebraic geometer. Many attempts have been made to avoid the use of such ill-behaved representations of plane curves. This paper will discuss a method employed by Hassler Whitney in a more general analysis of the singularities of maps for E^n into E^m (1).

Essentially this method reduces to several clear-cut steps. First, an investigation is made of a smooth map f from E into E , to determine its Jacobian $J(x,y)$. Second, determine the curve (or curves) C, called the general fold of f, in the pre-image space for which the Jacobian vanishes; i.e., $J(x,y)=0$, for all points (x,y) on C. Third, form the image $f(c)$, a curve in the image space, and examine it for possible singularities. Fourth, if the image of the general fold possesses singularities, analyze the general fold in the pre-image space near the inverse image of the singularity. It is this last step which will yield important information about the curve with singularities.

The crux of this procedure is: given a curve with singularities, does there exist a map whose points of vanishing Jacobian map into the given curve? This question is as yet unanswered, but the motivation of this paper lies in demonstrating how the above method can be effective in studying curves and their singularities.

I. INTRODUCTION

Let $f:E^2 \to E^2$ be a smooth mapping; i.e., possessing c ntinuous partials. Associated with each vector V in E and p in E , there is a unique vector $D_V f(p)$, given by

$$(1) \qquad D_V f(p) = \lim_{t\to 0} \frac{1}{t}[f(p + tV)-f(p)].$$

This represents the directional and magnitudinal change in the values of f as one travels through p in the direction of V.

Considering, for fixed p and f, $D_V f(p)$ as an operator on the set of vectors in E^2, it can easily be shown that the operator $D_V f(p)$ is actually a linear transformation from E^2 into E^2. If V is given by (a,b) with respect to some co-ordinate system (x,y), we have

(2) $$D_{(a,b)}f(p) = a \left.\frac{\partial f}{\partial x}\right|_p + b\left.\frac{\partial f}{\partial y}\right|_p .$$

Further, we can assume f to be given by component functions, u_1,u_2, where $u_1,u_2\colon E^2 \to E^1$. Then the following relationship between the D-operator for vector-valued functions and for real-valued functions holds.

(3) $$D_V f(p) = D_V(u_1(p),u_2(p)) = (D_V u_1(p),D_V u_2(p)).$$

Thus, the process of finding derivatives of vector-valued functions can be reduced to that of finding the derivatives of its corresponding component functions.

Combining equations (2) and (3) above, we arrive at the following evaluation of the D-operator of a function $f = (u_1,u_2)$:

(4) $$D_V f(p) = (D_{(a,b)}u_1(p),D_{(a,b)}u_2(p))$$
$$= (a \left.\frac{\partial u_1}{\partial x}\right|_p + b\left.\frac{\partial u_1}{\partial y}\right|_p , a\left.\frac{\partial u_2}{\partial x}\right|_p + b\left.\frac{\partial u_2}{\partial y}\right|_p).$$

Let $\phi(t)$ be a smooth parametrized curve in E^2. This maps the real line E^1 into some subset of the real plane E^2. For this curve, we define:

(5) $$d\phi/dt = D_{e_1}\phi(t),$$

where e_1 is the unit vector in E^1 and D is considered as an operator over a real valued function. The above definition of the tangent vector corresponds to the intuitive concept of tangent. It is a very specific example of the definition (1), where f is defined by $f(x,y) = \phi(x)$ for all y.

Lemma 1. Let f be a 2-smooth mapping, $f\colon E^2 \to E^2$, V(p) be a smooth vector function in E^2 and ϕ a 2-smooth curve in E^2 such that:

(6) $$d\phi(t)/dt = V(\phi(t)) \neq 0.$$

Then,

(7) $$\frac{d}{dt}(f\phi)(t) = D_V f(p),$$

(8) $$\frac{d^2}{dt^2}(f\phi)(t) = D_V D_V f(p),$$

where $p = \phi(t)$.

Proof: The restriction of ϕ in the hypothesis requires that its tangent vector for every t coincide with the vector function V(p), whenever $p = \phi(t)$.

By hypothesis, $D\phi = d\phi/dt$ maps e_1 into V(p): and Df maps V(p) into $D_V f(p)$. Thus the composite map $D(f\phi)(t) = \frac{d}{dt}(f\phi)(t)$ maps e_1 into $D_V f(p)$; i.e., equation (7) has been proven

Now, let us substitute the function $F(p) = D_V f(p)$ into the above results. Then,

$$\frac{d^2}{dt^2}(f\phi)(t) = \frac{d}{dt}(F\phi)(t) = D_V F(p) = D_V D_V f(p).$$

Thus, given the conditions of the hypothesis, the tangent and "acceleration" vectors can be simply described in terms of the D-operator, a fact which will play a major role in the coming developments.

II. SINGULARITIES

Definition: Let $f\colon R \subset E^2 \to E^2$ be a smooth mapping where R is open in E^2. Then we say f is regular at p if

(9) $$D_V f(p) \neq 0 \quad \text{for } V \neq 0.$$

What this means geometrically is that, in any direction, the directional derivative at p is non-zero, indicating that at f(p), there exists a non-zero image of V.

From this definition, we further define:

Definition: If f is smooth and f is not regular at p, then f is said to be singular at p.

Assume now that we have fixed co-ordinate systems (x,y), (u,v) in R and f(R) respectively. The Jacobian of a map f is given by

Definition: If $f\colon R \to E^2$ is a smooth mapping such that $f(x,y) = (u(x,y),v(x,y))$, then the Jacobian is

$$J = u_x v_y - u_y v_x .$$

Geometrically, J(p) represents the expansion (or contraction) factor of the mapping f at the point p. From the above definitions of regular and singular points, it can easily be shown that p is a regular or singular point of f according as $J(p) \neq 0$ or $J(p) = 0$.

Definition: Let f be a 2-smooth mapping. We say p in R is a good point if either $J(p) \neq 0$ or $DJ(p) \neq 0$.

Here, DJ(p) refers to the function $DJ(p)\colon E^2 \to E^1$ where $DJ(p)(V) = D_V J(p)$. Thus, p is good if either $J(p) \neq 0$ or, if $J(p) = 0$, either $J_x(p)$ or $J_y(p) \neq 0$.

The above definition merely asserts the condition that the singular points of f be well-behaved in the sense that their Jacobians vanish isolatedly.

We further define f to be good if every point of domain is good.

Example: Consider $f(x,y) = (x^2,y)$. Then, $J(x,y) = 2x$, $J_x(x,y) = 2$, $J_y(x,y) = 0$. Thus, $J \neq 0$ except where $x = 0$, in which case, $J_x \neq 0$.

As an example of a function which is not good, consider the map f given by $f(x,y) = ((x-4)^3,y)$. In this case, $J(x,y) = 3(x-4)^2$, $J_x(x,y) = 6(x-4)$, $J_y(x,y) = 0$. We see that $J \neq 0$ except where $x = 4$, in which case both J_x and J_y are both 0. If R is taken to be the real plane E^2, then f is not a good function as any point with the form $(4,y)$ is not good. However, if we restrict the domain R so as not to include any portion of the line $x = 4$, f will then be a good function.

We may now prove an important result for our development:

Lemma 2: Let f be good in R. Then the singular points of f form smooth curves in R.

Proof: If p is singular, $J(p) = 0$, and $DJ(p) \neq 0$. By the implicit function theorem, the solutions of $J(p) = 0$ near p lie on a smooth curve.

Definition: The smooth curves in R along which the Jacobian vanishes are called the general folds of f.

ample: 1. Let $f(x,y) = (x^3-12x,y)$. Then, the Jacobian $J(x,y) = 3x^2-12$, and the general folds are given by the smooth curves (lines) $x = \pm 2$.

2. Let $f(x,y) = (xy-x^3,y)$. In this case, $J(x,y) = y-3x^2$ and the general fold is as shown in Figure 1.

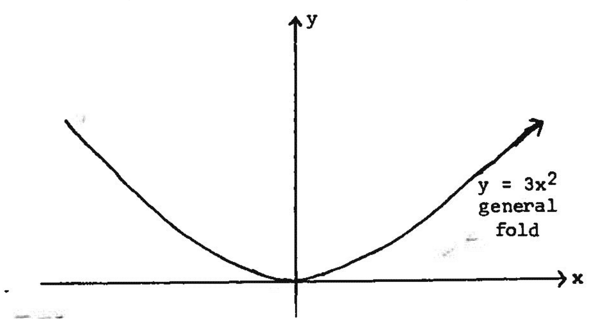


FIGURE 1

The primary fact to remember about the general fold is the Jacobian function vanishes at every point of the smooth curve. These points map into the "irregular" points in the image space $f(R)$. Now, let f be 3-smooth and good in R. Also, let p be a singular point of f and $\phi(t)$ any parametrization of the general fold through p such that $\phi(0) = p$. Using df/dt to mean $d/dt(f\phi)$, we define:

Definition: p is a fold point of f if

(10) $$df/dt \neq 0 \quad \text{at } p;$$

p is a cusp point if

(11) $$df/dt = 0, \quad d^2f/dt^2 \neq 0 \quad \text{at } p.$$

Thus, in example 1 above, all singular points of f are fold points, while in example 2, the point $(0,0)$ is a cusp point and all other points on the curve $y = 3x^2$ are fold points.

The condition of being fold or cusp points ties in intimately with the singularity of points of the image $f(C)$ of the general fold, as condition (11) is none other than the definition of a singular point for the curve $f\phi(t)$.

From the above definition, it can easily be proven that:

A. p is a fold point iff the image of C near p is smooth with a non-zero tangent at p.

B. p is a cusp point iff the tangent vector at p is zero, but becoming non-zero as we move away from p on C.

As a consequence of B, it is evident that the cusp points of f are isolated along C, implying that the corresponding singular points of $f(C)$ are also isolated. Note that these theorems must and can be proven to be independent of parametrization of the general fold C.

Definition: p is an excellent point of f (assumed good) if it is either regular, fold or cusp. f is excellent if all points of its domain are.

Now let us assume a 3-smooth co-ordinate system about p and also about $f(p)$. If we define a vector valued function $V(p)$ by

(12) $$V(p) = (-J_y(p),J_x(p)),$$

then,

$$D_{V(p)}J(p) = -J_y(p)J_x(p) + J_x(p)J_y(p) = 0.$$

Since $J(p)$ maps into E^1, then the above equation implies that the vector $V(p)$ is tangent to the general fold C at each singular point of the curve.

Using the above deductions with any fold of f, we can find a parametrization of C so that equation (1) holds. Then, invoking Lemma 1, with f assumed good, we have:

(13) $$p \text{ is a fold point iff } D_Vf(p) \neq 0,$$

(14) $$p \text{ is a cusp point iff } D_Vf(p) = 0 \text{ and } D_VD_Vf(p) \neq 0.$$

We have now reduced the determination for the singularities of the image curve $f(C)$ to calculations performed only in the image space. This enables us to determine whether the image curve possesses singularities without knowing its actual form. The key point of this theorem lies in the need to prove that the function under discussion is actually excellent.

III. STRUCTURE NEAR SINGULAR POINTS

Up to this point, attention has centered on the behavior of points actually on the general fold; i.e., the singular points. We now wish to relate these points to points in their two-dimensional neighborhoods.

Assume f to be an excellent map. For a given p in R, let $V' = D_V f(p)$ be a mapping from the set of vectors V into E^2 such that $|V| = 1$. Since the $Df(p)$ operator is linear (Section I), the above restriction is not prohibitive. Then, define R' to be the set of points p in R such that the set $\{V' = D_V f(p)\}$ has at least two non-equal elements.

Since the map f is excellent, for any p in R', there exists a unique pair of vectors V and $-V$ for which the quantity $|V'| = |D_V f(p)|$ is a minimum. Geometrically this means that for all points p in R', there exists a direction in which f varies the least. In this sense, we can assign a vector to each p in R' and this yields a system of smooth curves throughout R'.

Definition: The smooth curves defined above are called the curves of minimum Df. Figure 2 illustrates this concept.

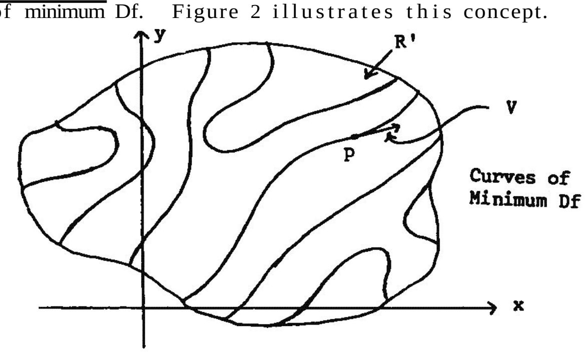


FIGURE 2

From the above definition, we have immediately that, if p is in R' and $V \neq 0$, then $D_V f(p) = 0$ iff p is a singular point and V is tangent to the curve of minimum Df at p. This follows since 0 is the minimum possible value for $|D_V f(p)|$. This is illustrated in Figure 3.

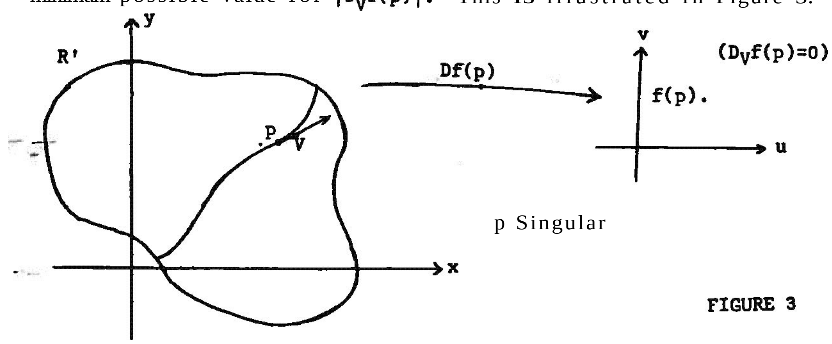


FIGURE 3

Now, let C be a general fold of f. If a curve of minimum Df cuts C at a positive angle, then for the tangent vector $V(p)$, $D_{V(p)} f(p) \neq 0$ and p is a fold point. However, let us assume that C is tangent to the curve of minimum Df at the point p. Since $V(p)$ may now be considered as the tangent vector to C at the point p, by equation (13), p is not a fold point. P is thus a cusp point, because f was assumed excellent. It is this last criterion which is extremely useful in determining the singular points of $f(C)$.

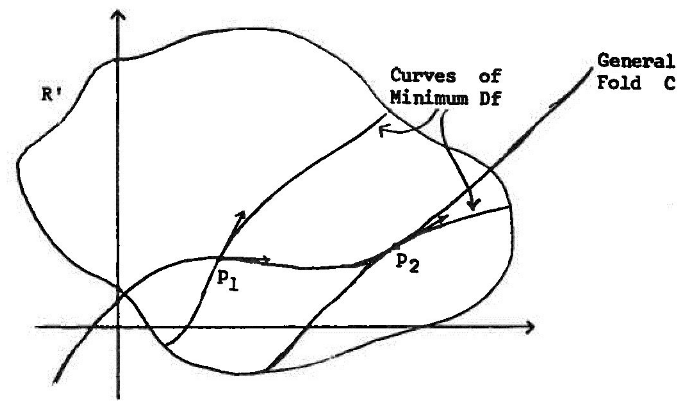


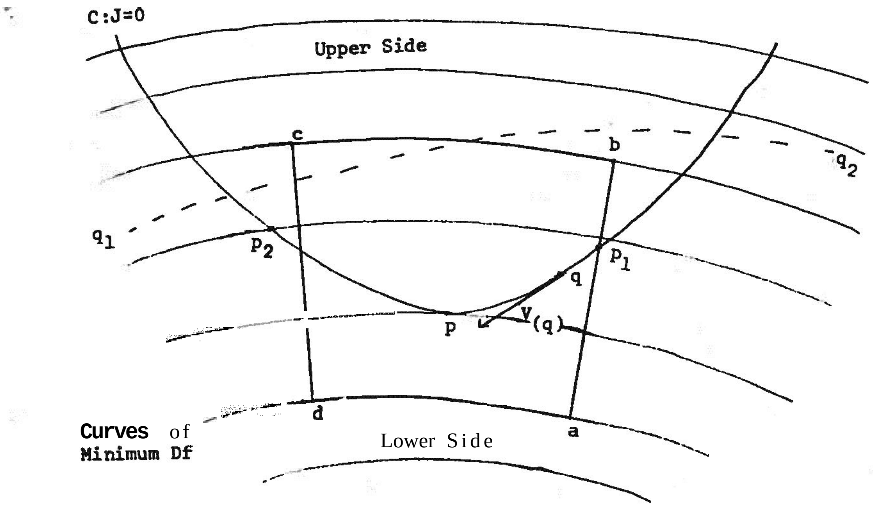
FIGURE 4

In Figure 4, p_1 is a fold point while the point p_2 is a cusp point of f.

Assume f to have a cusp point at p. Let

$$V^* = D_V D_V f(p).$$

Since f is excellent, $V^* \neq 0$. Since $D_V f(p) = 0$, $D_V f(p')$ is approximately in the direction of $\pm V^*$ for p' near p on C. It then follows that $D_W f(p)$ is a multiple of V^* for all W. As we move along C through the point p, $D_V f(p')$ changes from a negative to a positive multiple of V^* and thus $V(p')$ cuts the curves of minimum Df in opposite senses on each side of p. Therefore the curves of minimum Df lying on one side of C cut C on both sides of p. We call this side of C the upper side, and the other, the lower side. The figures 5 and 6 exemplify this concept, for both the pre-image space and the image space.



General Fold. Cusp at **p**.
FIGURE 5

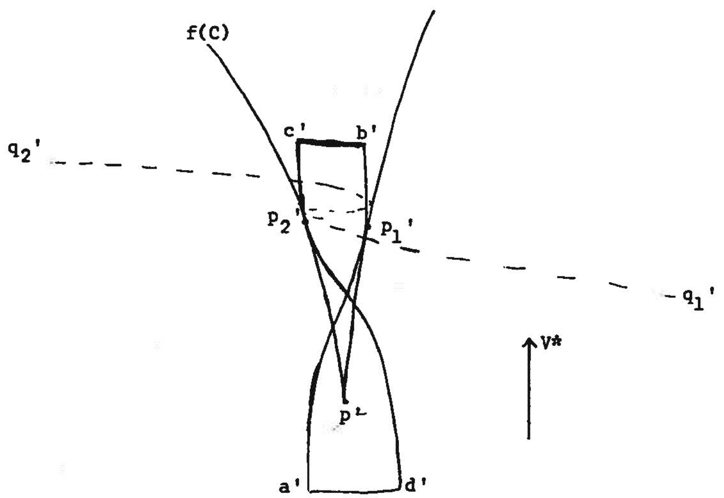


Image of general fold and of box about cusp point shown.
FIGURE 6

The image of C clearly has a cusp at $f(p)$, pointing, in the direction of $-V^*$. For any vector W not tangent to C at p, $D_W f(p)$ is a positive or negative multiple of V^* according as W points into the upper or lower side of C.

IV. SUMMARY

Considering the ultimate problem of this procedure, namely that of "reversing" the process involved, the author and his advisor, Professor V. O. McBrien, spent much time attempting to obtain an appropriate map which would yield the curve $y^2 = x^3 + x$. The analysis of this particular problem led us to a more complex one involving the solutions of partial differential equations. However, here again, devoted research might well open the doors to many problems herein encountered.

V. BIBLIOGRAPHY

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2. John W. Milnor, "Isolated Singularities of Hypersurfaces," Monthly, Vol. 74, No 4, April 1967, p. 467.

3. Egbert Brieskhorn, "Examples of Singular Normal Complex Spaces which are Topological Manifolds," Proceedings of the National Academy of Sciences, Vol. 55, No. 6, pp. 1395-1397, June, 1966.

The research in this paper was supported by a grant from the National Science Foundation. Student paper presented at the National Pi Mu Epsilon Meeting in Toronto, Canada, August, 1967.

NATIONAL MEETING

Pi Mu Epsilon will meet in conjunction with the Mathematical Association of America on August 27-28, 1968, at the University of Wisconsin in Madison. Members are urged to plan to attend these association meetings, as well as those of Pi Mu Epsilon, and take advantage of the opportunity to hear outstanding mathematicians.

The Annual Banquet for Pi Mu Epsilon members and their guests will be held at 6 p.m. on Tuesday, August 27, in the East Dining Room of the Wisconsin Center. Tickets may be obtained for this dinner at the registration desk; price is $2.90 per plate.

There will be the usual breakfast on Wednesday, August 28, and members will go through the cafeteria line between 8:00 and 8:30 a.m. and take their trays across the hall to the Plaza Room for an informal meeting and get-together.

All chapters are urged to submit applications for speakers as soon as possible. Applicants must be members of Pi Mu Epsilon who will not have received the Master's Degree by April 15th. Travel reimbursement will be paid by the National Office to a maximum of $150. Delegates' travel will be paid to a maximum of $75. Every chapter should plan to be represented at Madison this summer. Will your chapter be there?

A PRELIMINARY EXAMINATION OF ROUND-ROBIN TOURNAMENT THEORY

Charles A. Bryan

John Carroll University

1. We begin round-robin tournament theory with two essential concepts: a finite set of points $V = \{0, 1, \ldots n-1\}$ and a set of ordered pairs $D \subset V\times V$. With these two concepts we formulate a number of definitions.

Directed Graph: a finite set of points V and a set of ordered pairs $D \subset V\times V$.

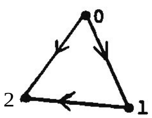
Complete, Asymmetric Diagraph: for every pair of points i and j in V, $(i,j)\varepsilon$ D iff $(j,i)\not\varepsilon D$.

i is adjacent from j iff $(j,i)\varepsilon D$.

i is adjacent to j iff $(i,j)\varepsilon D$.

The **score** of i is the cardinality of $\{j|(i,j)\varepsilon D\}$.

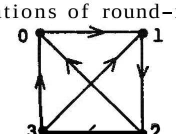
Example:



$V = \{0,1,2\}$
$D = \{(0,1), (1,2), (0,2)\}$

In this case 1 is adjacent from 0 and 0 is adjacent to 1. The score of 0 is 2.

2. We give three applications of round-robin tournament theory.



Game Interpretation: Every player is represented by a point and each game is represented by a line. Each player plays every other player once. In this tournament we have four players. Player 1 plays player 2 and defeats him. The arrow on the line always points in the direction of the defeated player.

Preference Interpretation: This interpretation is used in psychology and the social sciences. There is a subject who is asked to choose between objects in a method of paired comparisons. Thus the points now represent objects. Thus in the example we have objects 2 and 3 and the subject has chosen object 2. The arrow always points in the direction of the object not chosen.

Dominance Interpretation: This interpretation is used in biology. Each point represents a particular trait, characteristic, or species. The arrow always points in the direction of the trait which is overcome.

3. Now we move into the theory itself.

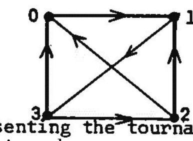
Definition: A tournament $T = (V,D)$ is a complete, asymmetric diagraph.

Example: .

A possible tournament would be the following:

T: $V = \{0,1,2,3\}$
$D = \{(0,1), (2,0), (3,0), (2,1), (3,2), (1,3)\}$

There is another way of representing the same tournament. So far we have used a graphic method. For the tournament above the graph would be:



A second method of representing the tournament is by a matrix. Let T be the matrix representing the tournament T. Then $t_{ij} = 0$ or 1 and $t_{ij} = 1$ iff $(i,j)\varepsilon D$.

	0	1	2	3
0	0	1	0	0
1	0	0	0	1
2	1	1	0	0
3	1	0	1	0

(Here the score of i is the sum of elements in row i of the matrix. Ed.)

4. Let us now proceed to examine the score sequences for tournaments.

Theorem 1: A sequence of n non-negative integers--$s_0, \ldots, s_{n-1}$-- may be considered as the score sequence of a tournament iff the following two conditions are fulfilled:

1. $\sum_{0}^{n-1} s_i = \frac{1}{2} n(n-1)$

2. If k is any positive integer less than n, then $\sum_{0}^{k-1} s_i \geq \frac{1}{2} k(k-1)$

Example:

$V = \{0,1,2,3\}$

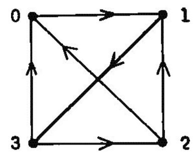
$D = \{(0,1), (1,3), (2,1), (3,2), (2,0), (3,0)\}$

$s_0 = 1$

$s_1 = 1$

$s_2 = 2$

$s_3 = 2$



We check to make sure this score sequence satisfies our two conditions.

$$\sum_{0}^{3} s_i = 1+1+2+2 = \tfrac{1}{2}(4)(3) = 6$$

Suppose k = 3.

$$\sum_{0}^{2} s_i = 1+1+2 = 4 > \tfrac{1}{2}(3)(2) = 3$$

Thus, this score sequence satisfies our two conditions.

5. To further examine the nature of score sequences a program was written to feed back the possible score sequences for n equal to the integral values up to 30. However, the number of score sequences possible for even relatively small n turned out to be so large that the General Electric 215 computer's work was terminated after n = 10 had been computed.

The following theorem was used in writing the program.

Theorem: Let T be a tournament with score sequence $\langle s_0, \ldots, s_{n-1}\rangle$ such that $s_0 \le s_1 \le \ldots \le s_{n-1}$. Then every score satisfies the inequalities $\tfrac{1}{2}(k-1) \le s_k \le \tfrac{1}{2}(n+k-2)$.

Some of the results of the program are tabulated below.
Score Sequences Obtained From the Program:

n	Sequences
2	<0,1>
3	<0,1,2> <1,1,1>
4	<0,1,2,3> <0,2,2,2> <1,1,1,3> <1,1,2,2>
5	<0,1,2,3,4> <0,1,3,3,3> <0,2,2,2,4> <0,2,2,3,3>
	<1,1,1,3,4> <1,1,2,2,4> <1,1,2,3,3> <1,2,2,2,3> <2,2,2,2,2>

As is immediately seen, every n has a perfect **hierarchial** score **sequence** for one of its score sequences. In this perfect hierarchy, the worst player defeats no one, the second worst player defeats only the worst player and so on. This is indicated in the score sequence by a sequence such as <0,1,2>.

A tabulation of the number of possible score sequences for values of n from 1 to 9 is given below.

n	Number of score sequences
2	1
3	2
4	4
5	9
6	22
7	59
8	164
9	496

References (supplied by editor)

Harary, Frank and Leo Moser, "The theory of Round Robin Tournaments," American Mathematical Monthly, 73 (1966) 231-246.

Kemeny, J.G., J.L. Snell, G.L. Thompson, Introduction to Finite Mathematics, Chapter VII, Englewood Cliffs, New Jersey, 1957.

Student paper presented at the National Pi Mu Epsilon Meeting in Toronto, Canada, August, 1967.

NEED MONEY?

The Governing Council of Pi Mu Epsilon announces a contest for the best expository paper by a student (who has not yet received a masters degree) suitable for publication in the Pi Mu Epsilon Journal.

The following prizes will be qiven

$200. first prize

$100. second prize

$ 50. third prize

providing **at** least ten papers are received for the contest.

In addition there will be a $20. prize for the best paper from any one chapter, providing that chapter submits at least five papers.

PROBLEM DEPARTMENT

Edited by

M. S. Klamkin, Ford Scientific Laboratory

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate and/or candidate for the Master's Degree. Solutions of these problems should be submitted on separate signed sheets within four months after publication.

An asterisk(*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to M. S. Klamkin, Ford Scientific Laboratory, P. O. Box 2053, Dearborn, Michigan 48121.

PROBLEMS FOR SOLUTION

200*. Proposed by Helen M. Marston, Douglas College.
The arithmetic identities

$$6 + (7\times 36) = 6 \times (7+36),$$

$$10 + (15\times 28) = 10 \times (15+28),$$

$$12 + (15\times 56) = 12 \times (15+56),$$

suggest the problem of finding the general solution, in positive integers, to the equation

$$a + (b\times c) = a \times (b+c).$$

In particular, how many pairs of positive integers (b,c) with $b < c$ satisfy the latter equation if $a = 21$?

201. Proposed by R. C. Gebhardt, Parsippany, N. J.
Out of the nine digits 1,2,3,...,9, one can construct 9! different numbers, each of nine digits. What is the sum of these 9! numbers?

202. Proposed by Leon Bankoff, Los Angeles, California.
Let I, O, H, denote the incenter, circumcenter, and orthocenter, respectively, of a right triangle. Find angle HIO given that ΔHIO is isosceles.

203. Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.
Let P denote any point on the median AD of ΔABC. If BP meets AC at E and CP meets AB at F, prove that AB = AC, if and only if, BE = CF.

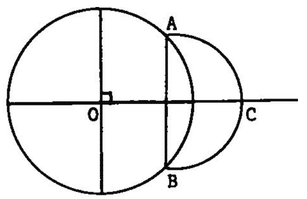
204. Proposed by M. S. Klamkin, Ford Scientific Laboratory.
If $a_{n+1} = \sqrt{2+a_n}$, n = 0,1,2,... , $a_0 = \sqrt{x}$, find

$$\lim_{x \to 4} \frac{a_r - 2}{x - 4}$$

Editorial note: Special cases of this problem occur in R. E. Johnson, F. L. Kiokemeister, Calculus with Analytic Geometry, 3rd Edition, Allyn and Bacon, Boston, p. 74.

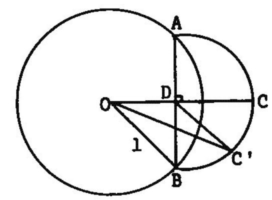
SOLUTIONS

187. Proposed by R. C. Gebhardt, Parsippany, N. J. A semicircle ACB is constructed, as shown, on a chord AB of a unit circle. Determine the chord AB such that the distance OC is a maximum.



Solution by M. S. Klamkin, Ford Scientific Laboratory. For OC to be a maximum for a given length chord $\overline{AB}$, $\overline{OC}$ will have to be perpendicular to $\overline{AB}$. This follows immediately from the triangle inequality:

OC = OD+DC = OD+DC' > OC'.



Now if $OD = \sqrt{a}$,
$BD = DC = \sqrt{1-a}$. Thus,

$$OC^2 = 1 + 2\sqrt{a(1-a)} = 1 + 2\left\{(1/4)-(a-(1/2))^2\right\}^{1/2}.$$

Whence, $OC_{max} = \sqrt{2}$, occurring for $a = 1/2$.

Alternatively, if $\angle DOB = \theta$, then

$OC = \sin\theta + \cos\theta = \sqrt{2}\sin(\theta+\pi/4)$. For a maximum, $\theta = \pi/4$.

Also solved by David W. Erbach, University of Nebraska; Bruce W. King, Burnt Hills-Ballston Lake High School; Paul Myers, Philadelphia, Pennsylvania; Stanley Rabinowitz, Polytechnic Institute of Brooklyn; Dennis Spellman, New York University; M. Wagner, New York City; and F. Zetto, Chicago, Illinois.

Editorial note: All the solutions except Spellman's used calculus to obtain the maximum. King also noted that at the maximum, the chord $\overline{AB}$ subtends a right angle at O.

188. Proposed by Waldemar Carl Weber, University of Illinois.
For any two real numbers x and y with $0 < x \le y$, verify the following procedure for adding on a slide rule using the A, S, and T scales.

First setting of the slide:

A	opposite y	opposite x
T	set right index	read angle θ, $0 < \theta < \pi/4$

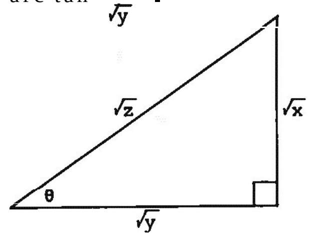
Second setting of the slide:

A	opposite x	read x + y
S	set angle 6	opposite right index

Solution by R. C. Gebhardt, Parsippany, N. J.
The **S(sine)** and **T(tangent)** scales on a slide rule are normally intended for use with the C and D scales. Using them with the A scale involves the square roots of the numbers employed, as illustrated below.
In the first step of the procedure, $6 = \arc\tan \frac{\sqrt{x}}{\sqrt{y}}$.

In the second step, $\theta = \arcsin \frac{\sqrt{x}}{\sqrt{z}}$.

Therefore, $\arctan \frac{\sqrt{x}}{\sqrt{y}} = \arcsin \frac{\sqrt{x}}{\sqrt{z}}$.

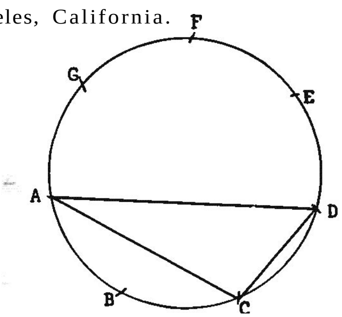


From the right triangle in the figure, it now follows that $z = x + y$.

Also solved by Albert Good, University of California, San **Diego**; H. Kaye, Brooklyn, N. Y., Paul Myers, Philadelphia, Pa., Stanley Rabinowitz, Polytechnic Institute of Brooklyn; and the proposer.

Editorial note: Good notes in his solution that it is necessary to use the proper half of the A scale and that the method will fail if the number of digits of y exceeds the number of digits of x by more than one.

189. Proposed by Leon **Bankoff**, Los Angeles, California. If **A,B,C,D,E,F,** and G denote the consecutive vertices of a regular heptagon, show that CD is equal to half the harmonic mean of AC and AD.



Solution by Stanley Rabinowitz, Polytechnic Institute of Brooklyn. By the law of sines, CD, AC, and AD are proportional to $\sin \pi/7$, $\sin 2\pi/7$, and $\sin 3\pi/7$. Since

$$\frac{1}{\sin 2\pi/7} + \frac{1}{\sin 3\pi/7} = \frac{\sin 2\pi/7 + \sin 3\pi/7}{\sin 2\pi/7 \sin 3\pi/7} = \frac{\sin 2\pi/7 + \sin 3\pi/7}{2 \sin\pi/7 \cos\pi/7 \sin 3\pi/7}$$

$$= \frac{\sin 2\pi/7 + \sin 3\pi/7}{\sin 2\pi/7 + \sin 4\pi/7} \cdot \frac{1}{\sin \pi/7} = \frac{1}{\sin \pi/7},$$

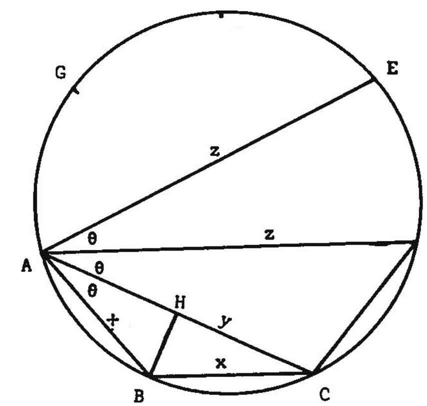
$\frac{1}{CD} = \frac{1}{AC} + \frac{1}{AD}$ which was to be shown.

Solution by C. W. **Trigg,** San **Diego,** California.

In the figure $\angle EAD = \angle DAC = \angle CAB = \theta$, $AB = BC = CD = DE = x$, $AC = y$, $AD = AE = z$, and BH is the $\perp$ bisector of AC. In ΔABC, $\cos \theta = y/2x$. In ΔACD and ΔADE,

$y^2 + z^2 - 2yz \cos \theta =$
$x^2 = z^2 + z^2 - 2z^2 \cos\theta$.
Whereupon,

$y^2 - z^2 = 2z(y-z)\cos\theta = (y-z)(zy/x)$. Then since $y \neq z$, $x = yz/(y+z)$.



Also solved by Steven Ferry, Michigan State University; Edgar Karst, University of Arizona; H. Kaye, Brooklyn, N. Y.; Gregory Wulczyn, Bucknell University; **F.** Zetto, Chicago, Illinois; and the proposer.

191. Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn. Let P and **P'** denote points inside the rectangles ABCD and **A'B'C'D'**, respectively. If $PA = a+b$, $PB = a+c$, $PC = c+d$, $PD = b+d$, $P'A' = ab$, $P'B' = ac$, $P'C' = cd$, prove that $P'D' = bd$.

Solution by Helen **Marston,** Douglas College.

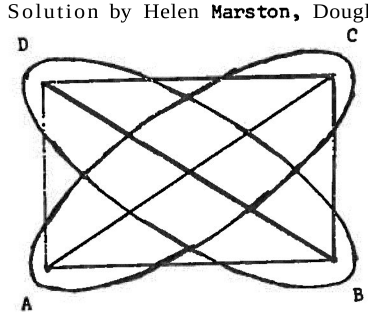


FIGURE 1

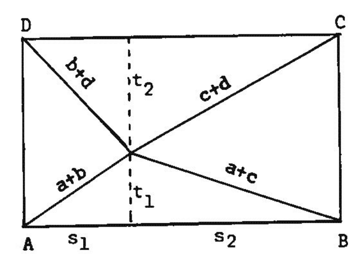


FIGURE 2

Since PA + PC = a+b+c+d = PB + PD, P must lie on each of two ellipses having major axes of length a+b+c+d and foci at A and C, B and D, respectively. Hence there are four possible positions for P (or one, should a+b+c+d = AC) and either a = d or b = c, from which it follows that P'D' = bd.

Less Interestingly, we can use the Pythagorean theorem on Figure 2:

$$(a+b)^2 - s_1^2 = t_1^2 = (a+c)^2 - s_2^2 ,$$
$$(b+d)^2 - s_1^2 = t_2^2 = (c+d)^2 - s_2^2 .$$

Subtracting, we obtain $(b-c)(a-d) = 0$ and either a = d or b = c as before.

Also solved by David W. Erbach, University of Nebraska; Albert Good, University of California, San Diego; H. Kaye, Brooklyn, N. Y.; Charles W. Trigg, San Diego, California; Cornelia Yoder, Michigan State University; and the proposer. There was also one unsigned solution.

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BOOK REVIEWS

Edited by
Roy B. Deal, Oklahoma University Medical Center

1. Convex Polytopes. By Branko Grunbaum with the cooperation of Victor Klee, M. A. Perles, and G. C. Shepard, Interstate Publishers, A Division of John Wiley and Sons, New York, New York, 1967, xiv + 456 pp., $18.75.

A comprehensive account of the recent history and results in this important aspect of modern geometry which could, seve either as a reference for someone working in this field, as a text for courses at many levels from the advanced undergraduate level up, or as an excellent source for an advanced undergraduate student with adequate dilligence to bring himself to the current frontiers in an interesting subject.

2. Inequalities. By Oved Shisha, Academic Press, Inc., New York, New York, 1967, xiv + 360 pp., $18.00.

This is the preceding~of a symposium held at Wright-Patterson Air Force Base, Ohio, August 19-27, 1965, and consists, essentially, of the one-hour lectures presented there by an Imposing collection of outstanding mathematicians. *Many* branches of algebra, geometry, and applied mathematics are covered and the levels of difficulty and prerequisites vary considerably.

3. Difference Methods for Initial-Value Problems. Second Edition By Robert D. Richtmyer and K. D. Morton, Interscience Publishers, a Mvision of John Wiley and Sons, New York, New York, 1967, xiv + 405 pp., $14.95.

In this revision of a book copywrited in 1957, there is even a greater attempt to bridge the gap between the highly theoretical analysis of partial differential equa tions and the solutions obtained by users with modern machine algorithms. Although most Pi Mu Epsilon readers will find parts of the book very difficult those who are really interested in the application of partial differential equations and/or their numerical solutions will find a rich source of fundamental ideas.

4. Probability Measures on Meric Spaces. By K. R. Parthasarathy, Academic press, Inc., New York, New York, 1967, xi + 276 pp., $12.00.

Although the subjects covered seem somewhat sophisticated, a reader with a minimum knowledge of measure theory and general topology will find an accessible interesting collection of results on probability distributions and limit theorems in metric spaces, as well as convolutions in locally compact Abelin groups and Hilbert spaces.

5. Complex Numbers in Geometry. By I. M. Yaglom, Academic Press, Inc., New York, New York, 1967, xii + 243 pp., Clothbound, $7.50, Paperbound, $4.25.

This book is divided into sections with the intent to make various parts of the subject of interest to top high school students, undergraduate mathematics majors, as well as junior high and high school teachers. It is a good. source for the reader who would, like to know some of the classical foundations of Euclidean and. non-Euclidean geometries from a readable point of view with the use of ordinary and generalized complex numbers.

BOOKS RECEIVED FOR REVIEW

1. Characters of Finite Groups. By Walter Feit, W. A. Benjamin, Inc., New York, New York, 1967, vi - 186 pp., Clothbound $9.50, Paperbound $4.95.

2. Random Matrices and the Statistical Theory of Energy Levels. By M. L. Mehta, Academic Press, Inc., New York, New York, 1967, x + 259 pp., $12.00.

3. Renewal Theory. By D. R. Cox, Barnes and Noble, Inc., New York, New York, 1967, ix + 142 pp., Clothbound $3.75, Paperbound $2.35.

4. The Theory of Games and Linear Programming. By S. Vajda, Barnes and Noble, Inc., New York, New York, 1967, 106 pp., Clothbound $3.25, Paperbound $1.75.

5. Mathematics for the Liberal Arts Student. By Richarman, Walker, and Wisner, Brooks/Cole Publishing Company, Belmont, California, 1967, viii + 190 pp.

6. Introduction To Mathematical Structures. By Charles K. Gordon, Jr., Dickenson Publishing Company, Inc., Belmont, California, 1967, viii + 164 pp.

7. Calculus and Linear Algebra. By Burrowes Hunt, W. H. Freeman and Company, San Francisco, California, 1967, xii + 401, $9.50.

8. Ordinary Differential Equations. By Fred Brauer and John A. Nohel, W. A. Benjamin, Inc., New York, New York, 1967, xvi + 457 pp., $10.75.

9. Intrcduction To Number Systems. By Spooner and Mentzer, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968, x + 339 pp., $7.95.

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