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In a calculus class a student noticed that $\frac{d}{dx}|x| = \frac{|x|}{x}$ and
$$\frac{d}{dx}|x^2 - u| = \frac{|x^2 - u|}{x^2 - u} (2x).$$

This led to the question: If $P(x)$ is a polynomial, is the following formula valid?
$$\frac{d}{dx}|P(x)| = \frac{|P(x)| \cdot P'(x)}{P(x)}$$

The class quickly found a counterexample. If $P(x) = x^3$, then $|P(x)|$ has a derivative everywhere, hence the formula, which would indicate that the derivative did not exist at the origin, is not valid in this case.

This brought up the question, what conditions can we impose on $P(x)$ so that the formula will hold? One student suggested the condition: if $P(a) = 0$, then $P'(a) \neq 0$. While another noticed that in all cases for which the formula held, the roots were distinct.

After considerable labor and prodding by the instructor, the following theorem emerged:

**Theorem:** If $P(x)$ is a polynomial, then the following three conditions are equivalent:

(i) If $a$ is real and $P(a) = 0$, then $P'(a) \neq 0$.

(ii) The real roots of $P(x)$ are distinct.

(iii) $\frac{d}{dx}|P(x)| = \frac{|P(x)| \cdot P'(x)}{P(x)}$.

(i) $\iff$ (ii) Suppose $P(x)$ has a repeated real root, $a$.

Then by the Factor Theorem, $P(x) = (x - a)^2 Q(x)$.

$P'(x) = (x - a)^2 Q'(x) + 2Q(x)(x - a)$ and $P'(a) = 0$.

(ii) $\iff$ (i) Suppose $P(a) = 0$, a real. Then $P(x) = (x - a)Q(x)$, and since $P(x)$ has no repeated real roots, $Q(a) \neq 0$. Then $P'(x) = (x - a)Q'(x) + Q(x)$ and $P'(a) = Q(a) \neq 0$.

(iii) $\iff$ (ii) Proof by induction on the number of real roots of $P(x)$. First we state two preliminary lemmas whose proofs are obvious.
Lemma 1. If \( Q(x) \) is a polynomial having no real roots, then
\[
D_x |Q(x)| \leq |Q'(x)|.
\]

Lemma 2. If \( P(x) = x - r, \ r \) real, then
\[
D_x |x - r| = \frac{|x - r|}{x - r}.
\]

Proof of \((iii) \Rightarrow (iii)\): First we show that \((iii)\) holds if \( P(x) \) has only one real root. Then \( |P(x)| = |(x - r_1)Q(x)| \) where \( Q(x) \) has no real roots.

\[
|P(x)| = |(x - r_1)| \cdot |Q(x)|
\]

\[
D_x|P(x)| = \left| \frac{x - r_1}{x - r_1} \cdot D_x|Q(x)| \right| \cdot |Q'(x)| \cdot D_x|\frac{x - r_1}{x - r_1}|
\]

and by the lemmas, this is
\[
\left| \frac{x - r_1}{x - r_1} \right| \cdot \left| \frac{Q(x)}{Q(x)} \right| \cdot \left| \frac{Q'(x)}{Q(x)} \right| \cdot \left| \frac{x - r_1}{x - r_1} \right| = |(x - r_1)Q(x)|
\]

Next we assume that \((iii)\) holds for a polynomial having exactly \( k \) distinct real roots. We want to show that \((iii)\) is valid if \( P(x) \) has exactly \( k + 1 \) distinct real roots.

\[
|P(x)| = |(x - r_1)(x - r_2) \ldots (x - r_{k+1})Q(x)|
\]

where \( Q(x) \) has no real roots.

\[
D_x|P(x)| = \left| \frac{x - r_1}{x - r_1} \right| \cdot \left| \frac{x - r_2}{x - r_2} \right| \ldots \left| \frac{x - r_{k+1}}{x - r_{k+1}} \right| \cdot D_x|Q(x)| \cdot |Q'(x)| \cdot D_x\left( \frac{x - r_1}{x - r_1} \right) \ldots \left( \frac{x - r_{k+1}}{x - r_{k+1}} \right)
\]

and by the induction hypothesis
\[
|Q(x)| \cdot |Q'(x)| \cdot D_x|Q(x)| = \frac{|Q(x)|}{x - r_1} \cdot \frac{|Q'(x)|}{x - r_1} \cdot \frac{|Q(x)|}{x - r_2} \ldots \frac{|Q'(x)|}{x - r_k} \ldots \frac{|Q(x)|}{x - r_{k+1}} \cdot \frac{|Q'(x)|}{x - r_{k+1}} \cdot \frac{|Q(x)|}{x - r_2} \ldots \frac{|Q'(x)|}{x - r_k} \ldots \frac{|Q(x)|}{x - r_{k+1}} \cdot \frac{|Q'(x)|}{x - r_{k+1}} \cdot \frac{|Q(x)|}{x - r_2} \ldots \frac{|Q'(x)|}{x - r_k} \ldots \frac{|Q(x)|}{x - r_{k+1}} \cdot \frac{|Q'(x)|}{x - r_{k+1}}
\]

Thus \( (iii) \Rightarrow (iii) \).

\[
|P(x)| = \left| \frac{(x - r_1)^2}{x - r_1} \right| Q(x)
\]

Thus the derivative exists at \( a \) and \((iii)\) does not hold.

\textbf{UNDERGRADUATE RESEARCH PROJECT}

Submitted by Dr. David Kay University of Oklahoma

This problem concerns a method of characterizing certain sets in the plane. Say a set is \((m,n)\)-convex if it has the property that among each \( m \) points of the set there are at least \( n \) pairs whose joins are in that set, \( m \geq 2 \) and \( n \geq 0 \). For example, the five-pointed star with its interior is a set which is \((2,0)\), \((3,1)\), \((4,3)\), \((5,5)\)-convex, etc. the set consisting of two intersecting lines is \((2,0)\), \((3,1)\), \((4,2)\), \((5,4)\)-convex, etc. The function \( c_m(m) \) is defined as the maximal number \( n \) such that the set \( S \) is \((m,n)\)-convex. Evidently \( c_m(m) \) reflects the character of \( S \). Thus it is easy to see that if \( S \) is a parabola \( c_m(m) = 0 \), if \( S \) is convex \( c_m(m) = \left( \frac{m}{2} \right) \leq [m(m - 1)]/2 \), and if \( S \) consists of the union of two convex sets \( c_m(m) \leq \left( \frac{[m/2]}{2} \right) \),

\[
\text{where} \ [m/2] \text{denotes the greatest integer in } m/2.
\]

Investigate certain sets in the plane to see if they may be characterized by the function \( c_m(m) \) defined above. Sample theorem: A set \( S \) consists of a convex set \( C \), and \( k \) isolated points not in \( C \), if and only if \( c_S(m) = \left( \frac{m}{2} \right) \). The Governing Council of Pi Mu Epsilon announces a contest for the best expository paper by a student (who has not yet received a masters degree) suitable for publication in the Pi Mu Epsilon Journal.

The following prizes will be given

- $200. first prize
- $100. second prize
- $50. third prize

providing at least ten papers are received for the contest.

In addition there will be a $20. prize for the best paper from any one chapter, providing that chapter submits at least five papers.
AN EASIER CONDITION THAN TOTAL BOUNDEDNESS

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The condition of total boundedness is a useful one: for instance, in establishing compactness. Therefore it is worthwhile to find simpler properties equivalent to total boundedness. The condition I suggest is this:

Definition: A subset A of a metric space \((X,d)\) is Cauchy bounded, if for all \(\varepsilon > 0\) and all infinite subsets \(B\subseteq A\) there are two points \(x,y \in B\) with \(d(x,y) < \varepsilon\).

Compare this condition with that of total boundedness:

Definition: A subset A of a metric space \((X,d)\) is totally bounded, if for all \(\varepsilon > 0\) there is an \(\varepsilon\)-net consisting of a finite subset \(\{a_1, a_2, \ldots, a_n\}\), of A so that for any \(x \in A\) we have \(d(x, a_i) < \varepsilon\) for some i.

It is easy to see that if A were totally bounded and if \(B_\varepsilon(A)\) were infinite, then given \(\varepsilon > 0\), we could find \(x, y \in B\) with \(d(x, y) < \varepsilon\). The procedure is simple. We simply note the existence of a 1/2 \(\varepsilon\)-net and see that there are two points, \(x, y\), of the infinite set B clustered about one of the points, \(a_i\), in the 1/2 \(\varepsilon\)-net. From \(d(x, a_i) < \varepsilon/2\) and \(d(y, a_i) < \varepsilon/2\) and the triangle inequality, we see that \(d(x, y) < \varepsilon\). Thus total boundedness easily implies Cauchy boundedness.

That Cauchy boundedness implies total boundedness is not quite so easy to see but is still easy to prove.

Theorem: If a subset A of a metric space \((X,d)\) is Cauchy bounded, then it is totally bounded.

Proof: Suppose that A was not totally bounded. In that case we would have an \(\varepsilon > 0\) so that no \(\varepsilon\)-net existed. Choose a point \(a_1 \in A\). For every \(a_1 \in A\) there is a point \(a_2 \in A\) with \(d(a_1, a_2) > \varepsilon\). Choose a point \(a_3 \in A\) with \(d(a_2, a_3) > \varepsilon\) and \(d(a_3, a_1) > \varepsilon\). We note that \(\{a_1, a_2, a_3\}\) is not an \(\varepsilon\)-net, and this process continues in the same way. The result is a sequence of points, \(\{a_n\}\), of A, with the property that \(d(a_i, a_j) > \varepsilon\) if \(i \neq j\). The sequence \(\{a_n\}\) is an infinite subset of A that has no two points closer than \(\varepsilon\).

Thus we have a contradiction to the Cauchy boundedness of A, and the theorem is established.

Perhaps it is now easier to see why Cauchy boundedness is simpler to establish than total boundedness. The fact that Cauchy boundedness follows so easily from total boundedness implies that one might as well prove a set to be Cauchy bounded as totally bounded. Furthermore the fact that in order to establish total boundedness one must find an \(\varepsilon\)-net for each \(\varepsilon\) seems to indicate that total boundedness is harder to establish than Cauchy boundedness where one only needs to find two close points in an infinite subset.

As an illustration of what is involved, let me offer a new proof to an old result.

Theorem: If A is a subset of the space of continuous real-valued functions on [0,1] with the uniform metric, then A is compact if A is closed, equi-continuous, and uniformly bounded.

This is half of the Arzela-Ascoli theorem as stated in [1]. The procedure will be to show that A is complete and totally bounded. This is enough to prove that A is compact by a nice little theorem that says that a subset of a metric space is compact if, and only if, it is complete and totally bounded [2]. Since the metric space of continuous functions on [0,1] is complete and A is given to be closed, we know that A is complete. Thus we only need to prove that A is totally bounded. We will do this by route of Cauchy boundedness.

Let \(A_0\) be an infinite subset of A and let \(\varepsilon > 0\) be chosen. Since \(A_0\) is also equi-continuous, for all \(x \in [0,1]\) there exists \(N_x\) such that for all \(f \in A\) and all \(y \in N_x\), we have \(|f(y) - f(x)| < \varepsilon/3\), where \(N_x\) denotes an open set for which \(x \in N\). We note that the family of sets \(N_x\) for \(x \in [0,1]\) covers [0,1] and so we have a finite subcover of \([0,1]\), \(\{N_{x_1}, N_{x_2}, \ldots, N_{x_k}\}\).

Now consider the set \(\{f(x_i) / f \in A_0\}\). From the fact that \(A_0\) is both uniformly bounded and infinite, we know that there is an infinite subset \(A_0 \subseteq A\) such that for \(f, g \in A\), we have \(|f(x_i) - g(x_i)| < \varepsilon/3\). This follows from the well known Bolzano-Wierstrass theorem.

Let me put together what we have so far. Let \(y \in N_{x_1}\) and \(f, g \in A\), then:

\[|f(y) - g(y)| \leq |f(y) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(y)|.\]

We know that each of these quantities is less than \(\varepsilon/3\), so we have \(|f(y) - g(y)| < \varepsilon/3\). Thus, the functions of \(A_0\) uniformly approximate each other on \(N_{x_1}\).

The same trick works again and we find an infinite subset \(A_0 \subseteq A_0\) so that \(f, g \in A\), \(y \in N_{x_2}\) implies \(|f(y) - g(y)| < \varepsilon/3\). Of course, since
In the book One Hundred Problems in Elementary Mathematics, Hugo Steinhaus proposes the following problem: "Find n numbers in the unit interval such that the first two are in different halves, the first three in different thirds, the first four in different fourths, and so on, till the first n are in different n-ths. He gives a solution for n = 14 and in a footnote mentions that H. Warmus proved that n = 17 is the largest number for which the problem has a solution. A solution for n = 17 will be given by any set of numbers satisfying the following inequalities:

1. \[0 < x_1 \leq \frac{1}{17};\]
2. \[11/13 < x_1 < 6/7;\]
3. \[16/17 < x_2 < 1;\]
4. \[1/6 < x_3 < 3/17;\]
5. \[15/16 < x_3 < 13/14;\]
6. \[7/12 < x_5 < 5/11;\]
7. \[8/13 < x_5 < 5/8;\]
8. \[1/5 < x_7 < 4/17;\]
9. \[1/3 < x_{10} < 6/17;\]
10. \[11/17 < x_{11} < 11/16;\]
11. \[12/17 < x_5 < 5/7;\]
12. \[10/13 < x_11 < 11/14;\]
13. \[5/12 < x_6 < 3/7;\]
14. \[1/2 < x_{12} < 1/5;\]

We make a distinction between the ordered n-tuple \((a_1, a_2, \ldots, a_n)\) and the set \(\{a_1, a_2, \ldots, a_n\}\). The ordered n-tuple's "value" is changed if the elements are rearranged while the order of the elements has nothing to do with the "value" of the set. For example, \((1, 2, 3) \neq (3, 2, 1)\), but \([1, 2, 3] = [3, 2, 1]\).

The above remarks follow from the definition that \((a_1, a_2, \ldots, a_n) \neq (b_1, b_2, \ldots, b_n)\) if and only if \(a_1 \neq b_1\), \(a_2 \neq b_2\), \ldots, \(a_n \neq b_n\).

On the other hand, two sets are said to be equal if they contain the same elements. More precisely, sets A and B are equal if and only if every element of A is an element of B and every element of B is an element of A.

Example 1. If \([a, b] = [x, y]\) then either \(a < x\) and \(b > y\) or \(a > x\) and \(b < y\). Of course, both of these could occur, i.e., \(a \neq x\) and \(b = y\). We will usually omit this trivial case.

Example 2. If \([a] = [x, y]\) then \(a = x = y\) and \([a, x, y] = [a, y] = [x, y] = [a, x] = [a, y] = [x, y]\). More concretely, \([2, 3, 1, 3] = [2, 3]\). Example 3. \([a, \{a, b\}]\) is a set whose elements are themselves sets, namely \(\{a, b\}\). In addition, \([a], \{a, b\} \neq \{a, a, b\} = \{a, b\}\).

The object here is to define an ordered n-tuple in terms of sets. To dispose of the case where \(n = 1\), we shall define \((a) = \{a\}\). The case for \(n = 2\) is more interesting.

**Definition 1.** \((a, b) = \{[a, \{a, b\}]\}\) \((1)\)

The question now is whether or not the right hand side of (1) completely and uniquely determines the ordered pair \((a, b)\). The following theorem answers this question affirmatively.

**Theorem 1.** \((a, b) = ([x], [x, y])\), i.e., \(a = x\) and \(b = y\), if and only if \([a], [a, b] = [x], [x, y]\).

**Proof:** Suppose \((a, b) = ([x], [x, y]))\), i.e., \(a = x\) and \(b = y\). Then \([a] = [x]\) and \([a, b] = [x, y]\). Thus \([a], [a, b] = [x, y]\).

Suppose that \([a], [a, b] = [x, y]\). Two cases arise. We ignore the trivial case where \([a] = [a, b] = [x, y]\).

**Case 1.** \(\{a\} \neq [x]\) and \([a, b] = [x, y]\). (A)

**Case 2.** \(\{a\} = [x]\) and \([a, b] = [x, y]\). (B)
AN INTERESTING MAPPING OF TWO FIELDS

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In this article, I will prove an interesting theorem concerning a mapping of two fields, namely: If T is a one-to-one mapping of a field F onto a field F' such that \( T(a(b-1)) = T(a)T(b) - T(1) \), then T is an isomorphism.

In order to prove that T is an isomorphism, it is sufficient to show that T preserves addition and multiplication. To do this, I will characterize addition and multiplication in terms of the operation \( a(b-1) \) which will be denoted \( a^b \).

Theorem 1: \( ab = (a^0) \star ([1(0)\alpha b]) \)

Proof: \( (a^0) \star ([1(0)\alpha b]) = [a(0-1)] \star [1(0-1)\alpha b] \)

\[ = (-a)(-\alpha b) \]
\[ = (-a)(\alpha b) \]
\[ = (a)(-b-1) \]
\[ = (-a)(-b) = ab \]

Theorem 2: \( a+b = (a^0)\alpha ([ab^{-1}]\alpha b) \) if \( b \neq 0 \)

Proof: \( (b^0) \star ([ab^{-1}]\alpha b) = [b(0-1)] \star [ab^{-1}(0-1)] \)

\[ = (-b)(-\alpha b^{-1}) \]
\[ = (-a)(b^{-1}) \]
\[ = bab^{-1} + b \]
\[ = ab \]

If \( b \neq 0 \), we obviously have \( a+b = a+0b \).
We are given that \( T(a^0) = T(a) \star T(b) - T(1) \); thus in order to prove that \( T(a^0) = T(a) \star T(b) \), it is sufficient to prove that \( T(1) = 1 \) where 1 is defined to be the unity for \( F' \).

I will first prove that \( T(0) = 0' \) where 0' is the zero element of \( F' \).

Theorem 3: If T is a one-to-one mapping of a field F onto a field F' such that \( T(a(b-1)) = T(a)T(b) - T(1) \), then \( T(0) = 0' \).

Proof: \( T(0) = T(0) \star T(0) - T(1) \)
Assume \( T(0) \neq 0' \). Then we can cancel \( T(0) \) from both sides of (1). Therefore, \( 1' + T(a) - T(1) \), and \( T(0) = 0' \) for all a in F.

Therefore, T maps every element of F into one element of \( F' \), a contradiction since T is one-to-one.

Therefore \( T(0) = 0' \).
Theorem 4: If $T$ is as in the preceding theorem, then $T(1) = 1'$.

Proof: $1 = (-1)(0-1)$

$T(-1) = T(1) \left[ T(0) - T(1) \right]$
But $T(0) = 0'$ (Theorem 3)
Therefore, $T(-1) = -T(1)$

But $T(1) \neq 0'$ since $T$ is one-to-one.

Therefore, $T(-1) = -1'$.

It is now necessary to consider two cases: case I where $F$ is not of characteristic 2 and case II where $F$ is of characteristic 2.

CASE I: $F$ is not of characteristic 2 (i.e. $1 \neq -1$)

$-1 = 1(0-1)$

$T(-1) = T(1) \left[ T(0) - T(1) \right]$
But $T(0) = 0'$ and $T(-1) = -1'$
Therefore, $-1' = -T(1)^2$

Therefore, $T(1)^2 = 1'$.

Therefore, since $T$ is one-to-one and $-1 \neq 1$, $T(1) = 1'$.

CASE II: $F$ is of characteristic 2 (i.e. $1 = -1$)

Since $T$ is onto, every element of $F'$ has a preimage in $F$.

Let $a$ be the preimage of $1'$.

Then $T(a) = 1'$

$T(a-1) = T(1) \left[ T(a) - T(1) \right]$
But $T(1) = 1'$ (since $1 = -1$).

Therefore, $T(1) \left[ T(a) - T(1) \right] = (-1') \left[ 1' - (-1') \right]$

$T(a^2) = T(a(a-1)) = T(a) \left[ T(a) - T(1) \right]$

$= 1' \left[ (-1') + (-1') \right] = -1'$.

Therefore, $T(a^2) = T(1)$. But $T$ is one-to-one. Therefore, $a^2 = 1'$. Therefore, $a^2 - 1 = 0$.

But $a^2 - 1 = (a+1)(a-1)$, and this factorization is unique since a polynomial ring over a field is a unique factorization domain. (See, for example, G. Birkhoff and S. MacLane, A Survey of Modern Algebra, 3rd edition, page 72.)

But $a+1 = a^2 - 1 = (a - 1)^2$. Therefore, by the factor theorem, $a+1 = (a - 1)^2$. Therefore, $a^2 - 1 = (a - 1)^2$.

Therefore, in this case, $T(1) = 1'$.

Note that we have also shown that $F'$ is of characteristic 2.

Using the fact that $T(1) = 1'$, we can conclude that $T(ab) = T(a) \ast T(b)$. The proof of this obvious statement is left to the reader.

We are now ready to prove that $T$ preserves addition and multiplication.

Theorem 5: $T(ab) = a'b'$ where $a' = T(a)$ and $b' = T(b)$.

Proof: $T(ab) = T(ab0) \ast \left[ T(ab0) \ast b' \right]$
$= T(ab0) \ast \left[ T(ab0) \ast b' \right]$
$= T(ab0) \ast \left[ T(ab0) \ast b' \right]$
$= T(ab0) \ast \left[ T(ab0) \ast b' \right]$
$= T(ab0) \ast \left[ T(ab0) \ast b' \right]$

But $T(0) = 0', T(1) = 1, T(a) = a'$ and $T(b) = b'$. Therefore, $T(ab) = a'b'$ by Theorem 1.

Theorem 6: $T(ab) = a' + b'$

Proof: It is necessary to consider the cases $b = 0$ and $b \neq 0$ separately.

If $b = 0, T(a+0) = T(a) = a' = a' + 0' = T(a) + T(0)$
If $b \neq 0, T(ab) = T \left[ (b+0) \ast (ab') \right]$
$= T(b+0) \ast \left[ T(ab') \right]$

But since $T$ preserves multiplication, $T(ab') = a'b'$. Therefore, $T(ab) = (b' + 0') \ast (a'b')$. Therefore, $a' + b'$ by Theorem 2 is $b' \neq 0$. But since $b \neq 0$, and $T$ is one-to-one, we know that $b' \neq 0$.

Now that we have proven that $T$ is a homomorphism, it follows from the assumption that $T$ is one-to-one and onto that $T$ is an isomorphism.

$$\tan^3 x + \tan^3 y = 3 \tan x \tan y.$$
When \(1'\) holds replace \(a\) by \(b.c\) in it and use axioms \(1'\) and \(3\) to get

\[
(b.c)(b.c) = c((b.c).b) \\
b.c = b.(c.(b.c)) \\
= b.(b.(c.c)) \\
= b.(b.c) = c.(b.b) \\
= c.b
\]

Similarly when \(1''\) holds replace \(c\) by \(a.b\) in it and use axioms \(1''\) and \(3\) to get \(a.b = b.a\).

Thus \(\cdot\) is commutative in any case and this together with \(1'\) or \(1''\) implies \(1\). Axiom 4 is also satisfied in any case as follows:

\[
\begin{align*}
I' &= a.a' & \text{by axiom 6} \\
&= (a.a).a' & \text{by axiom 3} \\
&= a.(a.a') & \text{by axiom 1} \\
&= a.I' & \text{by axiom 6}
\end{align*}
\]

This implies

\[
\begin{align*}
(a.I')' &= a & \text{by axiom 7} \\
a.I &= a & \text{by axiom 5}
\end{align*}
\]

Independence. Let \(X = \{0, a, b, I\}\) with \(1.0 = a.b = 0, b.a = 1\) otherwise \(x.y = x;\) with \(x.y\) denoting any of the elements \(0, a, b, I\) and \(0' = I, I' = 0, a' = b, b' = a\). Then axioms 5, 6 and 7 are satisfied. But axiom \(1'\) is not satisfied since \(a.(a.I) = a\) and \(I.(a.a) = I\) and \(1''\) is also not satisfied since \((a.I).a = a\) and \(I.(a.a) = I\). The other three axioms are proved to be independent in [2].

**REFERENCES**

2. L. J. Dickson, ibid., V. 4, No. 6 (1967), pp. 253-257.
PROBLEM DEPARTMENT

Edited by
Leon Bankoff, Los Angeles, California

With this issue we introduce a new problem editor. To Murray Klamkin who served in this capacity for the past ten years we give our thanks for a job well done.

The new editor is Leon Bankoff who has contributed many problems and solutions to the problem department of the Journal in the past. He also served as joint editor of the problem department for one issue (Spring 1958). Dr. Bankoff practices dentistry in Los Angeles.

The new problem editor would like to publish solutions to all problems which have appeared in this Journal, but for which solutions have not yet been published. These problems (published prior to 1967) are listed below.

37. (April 1952) Proposed by Victor Thèbault, Tennie Sarthe, France. Find all pairs of three digit numbers M and N such that \( M \times N = P \) and \( M' \times N' = P' \), where \( M' \times N' \) and \( P' \) are the numbers \( M-N \) and \( P \) written backwards. For example,

\[
122 \times 213 = 25986
\]

and

\[
221 \times 312 = 68952
\]

48. (Nov. 1952) Proposed by Victor Thèbault, Tennie, Sarthe, France. Find bases \( B' \) and \( B'' \) such that the number \( 11, 111, 1111, 11111 \) consists of eleven digits in base \( B \) is equal to the number \( 111 \) consisting of twelve digits in base \( B'' \). (An incorrect solution by the proposer was given in the November 1953 issue. For further discussion see the Spring 1958 issue.)

65. (April 1954) Proposed by Martin Schechter, Brooklyn, N. Y. Prove that every simple polygon which is not a triangle has at least one of its diagonals lying entirely inside of it.

73. (April 1955) Proposed by Victor Thèbault, Tennie, Sarthe, France. Construct three circles with \( n \) centers such that the sum of the powers of the center of each circle with respect to the other two is the same.

83. (Spring 1956) Proposed by G. K. Horton, University of Alberta. Evaluate

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left( e^{(x-1)^2 + y^2} + e^{x^2 + (y-1)^2} \right) dx \, dy
\]

91. (Fall 1956) Proposed by Nathaniel Grossman, California Institute of Technology. Prove that

\[
\sum_{d|n} \phi(d) = n \tau(n)
\]

where \( \tau(n) \) denotes the number of divisors of \( n \), \( \sigma(n) \) is the sum of the divisors of \( n \) and \( \phi(n) \) is the Euler Totient function.

102. (Fall 1958) Proposed by Leo Moser, University of Alberta. Give a complete proof that two equilateral triangles of edge 1 cannot be placed, without overlap, into the interior of a square of edge 1.

120. (Spring 1960) Proposed by Michael Goldberg, Washington, D. C. All the orthogonal projections of a surface of constant width have the same perimeter. Does any other surface have this property? 2. A sphere may be turned through all orientations while remaining tangent to the three lateral faces of a regular triangular prism. Does any other surface have this property? Note that a solution to 2. is also a solution to 1.

128. (Spring 1961) Proposed by Robert P. Rudis and Christopher Sherman, AVCO. Given \( 2n \) unit resistors, show how they may be connected using \( n \) single pole single throw (SPST) and \( n \) single pole double throw (SPDT) (the latter with off position) switches to obtain, between a single fixed pair of terminals, the values of resistance of \( i \) and \( i^{-1} \) where \( i = 1, 2, 3, \ldots, 2n \).

Editorial Note: Two more difficult related problems would be to obtain \( i \) and \( i^{-1} \) using the least number of only one of the above type of switches.

136. (Fall 1961) Proposed by Michael Goldberg, Washington, D. C. What is the smallest convex area which can be rotated continuously within a regular pentagon while keeping contact with all the sides of the pentagon? This problem is unsolved but has been solved for the square and equilateral triangle. For the square, it is the regular tri-arc made of circular arcs whose radii are equal to the side of the square. For the triangle, it is the two-arc made of equal 60° arcs whose radii are equal to the altitude of the triangle.

144. (Fall 1962) Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey. Find the shape of a curve of length \( L \) lying in a vertical plane and having its end points fixed in the plane, such that when it revolves about a fixed vertical line in the plane, generates a volume which when filled with water shall be emptied in a minimum of time through an orifice of given area \( A \) at the bottom. (Note: The proposer has only obtained the differential equation of the curve.)

166. (Fall 1964) Proposed by Leo Moser, University of Alberta. Show that 5 points in the interior of a 2x1 rectangle always determine at least one distance less than \( \sec 15° \).
This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Solutions of these problems should be submitted on separate signed sheets within four months after publication.

An asterisk (*) placed beside a problem number indicates that the problem was submitted without a solution.

Address all communications concerning problems to Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

205. Proposed by C. S. Venkataraman, Trichur, South India. ABC and PQR are two equilateral triangles with a common circumcenter but different circumcircles. PQR and ABC are in opposite senses. Prove that AP, BQ, CR are concurrent.

206. Proposed by Charles W. Trigg, San Diego, California. Identify the pair of consecutive three-digit numbers each of which is equal to the sum of the cubes of its digits.

207. Proposed by Charles W. Trigg, San Diego, California. Find a triangular number of the form abcdef in which def = 2 abc.

208. Proposed by Thomas Dobson, Hexham, England. Where must a man stand so as to hear simultaneously the report of a rifle and the impact of the bullet on the target?

209. Proposed by R. C. Gehhardt, Parsippany, New Jersey. At each play of a game, a gambler risks $1/x$ of his assets at the moment. What must be the odds so that, in the long run, he just breaks even?

210. Proposed by Leon Bankoff, Los Angeles, California. Three equal circles are inscribed in a semicircle as shown in the adjoining diagram. How is this figure related to one of the better-known properties of the sequence of Fibonacci numbers?

211. Proposed by Leonard Barr, Beverly Hills, California. It is known that the sum of the distances from the incenter I to the vertices of a triangle ABC cannot exceed the combined distances from the orthocenter H to the vertices. [Amer. Math. Monthly, 1960, 695; problem E 1397]. Show that the reverse inequality holds for their products, namely, that $AH \cdot BH \cdot CH \leq AI \cdot BI \cdot CI$.

212. Proposed by J. M. Gandhi, University of Manitoba, Winnipeg, Canada. If $M(n) = \sum_{s=0}^{n} \left( \binom{n}{s} \binom{nt}{s} \right)$ show that (a) $M(5m + 2) \equiv 0 \pmod{5}$. (b) $M(5m + 3) \equiv 0 \pmod{5}$.


SOLUTIONS

192. (Fall 1967). Proposed by Oystein Ore, Yale University. Albrecht Durer's famous etching "Melancholia" includes the magic square

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The boxed-in numbers 15-14 indicate the year in which the picture was drawn. How many other 4 x 4 magic squares are there which he could have used in the same way?

Solution by C. J. Bouwkamp, Phillips Research Laboratories, Eindhoven, Netherlands. There are exactly 32 solutions to this problem, including the Durer version shown above (with misprint from the Fall 1967 issue, page 295 corrected). Curiously, this number is contained in Durer's magic square in the middle of the top row. The construction is as follows. There exist four types according to the bottom row: 1, 15, 14, 4 (10 solutions), 2, 15, 14, 3 (6 solutions), 3, 15, 14, 2 (6 solutions), and 4, 15, 14, 1 (10 solutions). Further it is known that the sum of the four inner elements equals 34. Thus the sum of the two inner elements of the first row must be 5 (2+3 or 14+1). Similarly, the outer two elements of the first row can only be 13 and 16. All in all, there are then 16 types where the upper and lower rows are fixed. The remaining numbers 5 through 12 are to be distributed over the two middle rows. The inner four elements can be linearly expressed in terms of one parameter. Some easy manipulation then leads to all 32 possible Durer magic squares. The complete list of solutions is shown below.

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Solution by the Proposer. Designate the first ship, whose motion is known completely, as "own ship" and write its parametric equations of motion as

\[
x = x_0 + vt \cos \phi \quad y = y_0 + vt \sin \phi
\]

Similarly, designate the second ship as the "target ship" and write its parametric equations of motion as

\[
x = p_0 + at \cos \theta \quad y = q_0 + at \sin \theta
\]

In which only the speed \(v\) is known. The following picture will illustrate the situation.

![Diagram](image)

One Bearing Given

At time \(t_1\), bearing \(\beta_1\) is observed. Thus,

\[
\sin \beta_1 = \frac{Y_0 - y_0 + (v \sin \phi - a \sin \theta)t_1}{R_1}
\]

(1)

\[
\cos \beta_1 = \frac{X_0 - x_0 + (v \cos \phi - a \cos \theta)t_1}{R_1} \quad (R_1 \neq 0)
\]

which implies

\[
(\cos \beta_1)Y_0 + (t_1 \cos \beta_1)\frac{Y_0}{v} = \sin \beta_1X_0 + (t_1 \sin \beta_1)\frac{X_0}{v}
\]

(2)

It should be noted that (2) does not imply (1) as there are targets satisfying (2) which do not satisfy (1). Thus, targets whose bearing differs from \(\beta_1\) by 180°. Equation (2) is a linear equation in \(Y_0, v \sin \phi, X_0, v \cos \phi\) whose general solution is

\[
y_0 = q_0 + \frac{\lambda}{v} \sin \beta_1 + \frac{\mu}{v} \cos \beta_1
\]

(3)

\[
x_0 = p_0 + \frac{\lambda}{v} \cos \beta_1 + \frac{\mu}{v} \sin \beta_1
\]

where \(\lambda, \mu, \) and \(v\) are free parameters subject only to the restrictions

Editorial Note: This problem was suggested by problem 186 which was given erroneously. See the comment on 186 in the Fall 1967 issue.

Detailed discussions of magic squares may be found in the two well-known classics in mathematical recreations, by Kolatchik and by Ball and Coxeter. An additional bibliography appears in Martin Gardner's Second Scientific American Book of Mathematical Puzzles and Diversions in connection with a most refreshing chapter on this subject.

Editorial Note: Bouwkamp verified his results by referring to two curious books in his possession, privately published by K. H. de Haas. 1) Albrecht Durer's Meetkundige Bouw van Reuter en Melencolia S 1, D. van Sijn en Zonen, Rotterdam, 1932. 2)Frenicle's 880 Basic Magic Squares of 4 x 4 Cells, Normalized, Indexed and Inventoried, by the same publisher in 1935. Gebhardt and Karst noted that a magic square remains magic if the same quantity is added to each element of the square, thus extending the number of solutions if the sequence 15-14 were permitted to appear in other rows.

192. (Fall 1967). Proposed by William H. Pierce, General Dynamics, Electric Boat Division, Groton, Connecticut. Two ships are steaming along at constant velocities (course and speed). If the motion of one ship is known completely, and if only the speed of the second ship is known, what is the minimum number of bearings necessary to be taken by the first ship in order to determine the course (constant) and range (time-dependent) of the second ship? Given this requisite number of bearings, show how to determine the second ship's course and range.

Also solved by R. C. Gebhardt, Parsippany, N. J.; Edgar Karst, University of Arizona; and Alfred E. Newman, New York, N. Y.
The first restriction eliminates from (3) those targets which do not satisfy (1) and the second restriction eliminates from (3) those targets which do not have the required target speed \( v \). There remains in (3), however, a multiplicity of targets with the required speed satisfying (1) so that we conclude that one bearing is insufficient to determine target motion when only target speed is given.

Two Bearings Given

A second bearing \( \beta_2 \) is observed at time \( t_2 > t_1 \), giving

\[
\begin{align*}
\sin \beta_2 &= y_2 - q_0 + (v \sin \phi - a \sin \theta) t_2 \\
\cos \beta_2 &= x_2 - p_0 + (v \cos \phi - a \cos \theta) t_2
\end{align*}
\]

(4)

which implies

\[
(\cos \beta_2) y_2 + (t_2 \cos \beta_2) v \sin \phi - (\sin \beta_2) x_0 - (t_2 \sin \beta_2) v \cos \phi = (\cos \beta_2) q_0 + (t_2 \cos \beta_2) a \sin \theta - (\sin \beta_2) p_0 - (t_2 \sin \beta_2) a \cos \theta
\]

(5)

The general solution of the linear equations (2) and (5) is

\[
\begin{align*}
y_0 &= q_0 - At \sin \beta_2 + \mu t^2 \sin \beta_1 \\
v \sin \phi &= a \sin \theta + \lambda \sin \beta_2 - \mu \sin \beta_1 \\
x_0 &= p_0 - \lambda t \cos \beta_2 + \mu t^2 \cos \beta_1 \\
v \cos \phi &= a \cos \theta + \lambda \cos \beta_2 - \mu \cos \beta_1
\end{align*}
\]

(6)

where \( \lambda \) and \( \mu \) are free parameters subject only to the restrictions

\[
\lambda > 0, \quad \mu > 0
\]

\[
v = \lambda^2 - 2 \lambda \mu \cos(\beta_2 - \beta_1) + \mu^2 + 2a [\lambda \cos(\beta_2 - \theta) - \mu \cos(\beta_1 - \theta)] + a^2
\]

The parameters here are not related to those used earlier, and the first restriction eliminates from (6) those targets that do not satisfy (1) and (4), while the second restriction eliminates from (6) those targets not having the required speed \( v \). Thus, there remains in (6) a one-parameter family of targets having the required bearings, and we conclude here also that two bearings are insufficient to determine target motion when only target speed is given. (If \( \sin(\beta_2 - \beta_1) = 0 \), then the two ships have either parallel motion or are on a collision course, and further information about target course can be obtained; discussion of this aspect is omitted.)

Three Bearings Given

A third bearing \( \beta_3 \) is observed at a time \( t_3 > t_2 > t_1 \), giving

\[
\begin{align*}
\sin \beta_3 &= y_3 - q_0 + (v \sin \phi - a \sin \theta) t_3 \\
\cos \beta_3 &= x_3 - p_0 + (v \cos \phi - a \cos \theta) t_3
\end{align*}
\]

(7)

which implies

\[
(\cos \beta_3) y_3 + (t_3 \cos \beta_3) v \sin \phi - (\sin \beta_3) x_0 - (t_3 \sin \beta_3) v \cos \phi = (\cos \beta_3) q_0 + (t_3 \cos \beta_3) a \sin \theta - (\sin \beta_3) p_0 - (t_3 \sin \beta_3) a \cos \theta
\]

(8)

The general solution of the linear equations (2), (5), and (8) is

\[
\begin{align*}
y_0 &= q_0 + H_\lambda t \sin \phi + a \sin \theta + H_2 A \\
x_0 &= p_0 + H_3 t \cos \phi - a \cos \theta + H_4 A
\end{align*}
\]

(9)

Where \( \lambda \) is an arbitrary non-zero parameter carrying the sign of \( H_\lambda \). (If \( H_\lambda \) = 0, \( H_2 \lambda \) = 0, \( H_3 \lambda \) = 0, \( H_4 \lambda \) = 0, then it can be proved that \( \beta_1 \) = \( \beta_2 \) = \( \beta_3 \) \( \frac{1}{2} \). It can further be shown that the three bearings must be such that \( \sin(\beta_3 - \beta_1) = 0 \), \( \sin(\beta_3 - \beta_2) = 0 \), and \( \sin(\beta_3 - \beta_2) = 0 \), all have the same sign which is opposite that of \( H_\lambda \).

The solution (9) is meaningful only when \( \sin(\beta_3 - \beta_1) \neq 0 \) in which case it can be proved that \( H_\lambda \) = \( H_2 \lambda \) = 0. It can further be shown that the three bearings must be such that \( \sin(\beta_3 - \beta_1) = \sin(\beta_3 - \beta_2) = \sin(\beta_3 - \beta_2) = 0 \), and \( \sin(\beta_3 - \beta_2) = 0 \), all have the same sign which is opposite that of \( H_\lambda \).

Solution (9) is a one-parameter family of targets in which the known target speed imposes a restriction on the parameter \( \lambda \). The restriction is
(10) \[(H_2^2 + H_4^2)k^2 + 2a(H_2 \sin \theta + H_4 \cos \theta) + a^2 - v^2 = 0\]

which is a quadratic in \(k\) that may have two, one or no roots, depending on the magnitude of the target speed \(v\). In any event, the parameter \(k\) is determinable (if it exists) from (10) and the associated target motion is determinable from (9). Target range and course are easily determined from the quantities \(\gamma_0\), \(v \sin \phi\), \(x_0\), and \(v \cos \phi\).

We therefore conclude that three bearings are needed to determine the target motion when target speed is known, and that this motion is obtained from (10) and then (9). Zero, one, or two targets may exist which satisfy the bearing and speed conditions. All additional bearings are determinable from the three given bearings, and no further bearings add any information.


Show that the equation
\[x^{\frac{1}{x+y}} = y^{-x}\]

has no solution in integers except the solutions:

(i) \(x = -1, y = 1\),
(ii) \(x = 3, y = 9\).

Solution by Charles W. Trigg, San Diego, California. The given equation may be written in the form:
\[(xy)^x = (y/x)^y\]

The left hand member is an integer, so the right hand member must be an integer also. This requires that \(y = kx\), \(k\) an integer.

Thus
\[(kx)^x = (k^{kx})^k\]

or \(x = k(k-1)/2\).

Consequently, \(k = \text{odd}\) and has the form \(2m + 1, m\) an integer; or \(k\) has the form \(n\), \(n\) a non-zero integer.

The complete solution is \(x = -1, y = -1; x = (2m + 1)^{m-1}, y = (2m + 1)^{m-1}, m = 0, 1, 2, \ldots; x = n^{n-1}, y = n^{n-1}, n\) a non-zero integer. The proposition as stated is false.

Also solved by R. C. Gehhardt, Parsippany, N. J.; Erwin Just, Bronx Community College; Bruce W. King, Burnt Hills-Ballston Lake High School; Bob Nemec; Bob Priellip, Wisconsin State University; Phillip Singer, Michigan State University; and Gregory Walczyn, Bucknell University.

195. (Fall 1967). Proposed by Leon Bankoff, Los Angeles, California. (Jan. 1967), p. 60, contains a short paper by Dov Avishalom, who asserts without proof that in the adjoining diagram \(AN = NC + CB\).

Give a proof.

Solution I by Josef Konhauser, University of Minnesotta.

AMCB is a cyclic quadrilateral, so \(AM + CB + MC + AB = AC + MB\).

Let \(F\) be the foot of the perpendicular from \(M\) to \(AB\). Then, using \(AM = MB\), we have \(AM + CB + 2 MC + AF = AC + MA\). Triangles \(M AF\) and \(MNC\) are similar, so \(MC + AF = AN + MC\). Therefore, since \(AC + AN + NC\), it follows that \(AM + CB + 2 MC + NC = (AN + NC)AM\).

Simplifying gives the desired result.

Solution II by Charles W. Trigg, San Diego, California. From \(M\) drop a perpendicular to \(BC\) extended, meeting it at \(D\). Draw \(MA\) and \(MB\). Since \(MA = MB\) and since angle \(MAC\) and angle \(MBD\), the right triangles \(MAC\) and \(MBD\) are congruent, and \(AN = DC + CB\).

Also, \(MN = MD\) and \(MC = NC\).

Therefore right triangles \(MNC\) and \(MDC\) are congruent, and \(MC = DC\). Finally, \(AN = NC + CB\).

Solution III by Leon Bankoff, Los Angeles, California.

Extend \(AC\) to \(D\) so that \(CD = BC\). If \(F\) is diametrically opposite \(M\), we find that \(CP\), the bisector of angle \(ACB\) is perpendicular to \(MC\). Therefore \(MC\) bisects angle \(CBD\), and we have angle \(MCB = angle MDC\). Since \(MBC\) and \(MCD\) are congruent, and \(MD = MB = MA\). It follows that the right triangles \(MAB\) and \(DMD\) are congruent. Hence \(AN = MD = MC + CD = MC + CB\).

Also solved by Dan Deignan, Miami University (by trigonometry); William Tally, University of Southwestern Louisiana; and Gregory Walczyn, Bucknell University (using polar coordinates).

196. (Fall 1967). Proposed by R. C. Gehhardt, Parsippany, N. J.

What is the remainder if
\[x^{100} \text{ is divided by } x^2 - 3x + 27\]

Amalgam of almost identical solutions submitted by E. A. Franz, Illinois College, Jacksonville, Illinois; Erwin Just, Bronx Community College; and Charles W. Trigg, San Diego, California.

Division gives rise to the identity
\[x^{100} = f(x) = (x - 1)(x - 2) q(x) + Ax + B,\]

where \(Ax + B\) is the remainder sought. The term \((x - 1)(x - 2) q(x)\) can be eliminated by the substitution of either 1 or 2 for \(x\).

Thus, we have
\[f(1) = A + B\]
\[f(2) = 2A + B\]

Then
\[A = f(2) - f(1) = 2^{100} - 1\]
\[B = 2f(1) - f(2) = 2 - 2^{100} .\]

Therefore the remainder, \(Ax + B\), is equal to \((2^{100} - 1)x + 2 - 2^{100}\) or \(2^{100}(x - 1) - (x - 2)\).

A box contains \((1600 \ u^2 + 3200) / 3\) solid spherical metal bearings. Each bearing in the box has a cylindrical hole of length .25 centimeters drilled straight through its center. The bearings are then melted together with a loss of 4% during the melting process and formed into a sphere whose radius is an integral number of centimeters. How many bearings were there originally in the box?

Solution by Charles W. Trigg, San Diego, California. The volume of the "wedding ring" left after a cylindrical hole with axis along a diameter is drilled through a sphere is the same as that of a sphere with diameter equal to the length of the hole. [Cf., e.g., Charles W. Trigg, Mathematical Quickies, McGraw-Hill (1967), page 179.] Hence we have

\[ 4\pi k^3 / 3 = (1600(u^2 + 2)/3)(\pi/6)(1/4)^3(96/100) \] or \( k^3 = u^2 + 2. \)

The only value of \( u \) satisfying this last expression is \( u = 5 \), whereupon \( R = 3 \) cm. and the original number of bearings was 14, 400.


198. (Fall 1967). Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn. A semiregular solid is obtained by slicing off sections from the corners of a cube. It is a solid with 36 congruent edges, 24 vertices and 14 faces, 6 of which are regular octagons and 8 are equilateral triangles. If the length of an edge of this polytope is \( e \), what is its volume?

Solution by Leon Bankoff, Los Angeles, California.

The eight sliced-off pyramids can be assembled to form a regular octahedron of edge \( e \), whose volume is known to be \( \frac{3}{2} \sqrt{2} e^3 / 3 \). Subtracting this quantity from \( e^3 (\sqrt{2} + 1)^3 \), the volume of the cube, we find that the volume of the truncated cube is \( \frac{7}{3} e^3 (3 + 2 \sqrt{2}) / 3 \).

Also solved by Charles W. Trigg, San Diego, California, and the proposer.

Editorial Note. The term "truncated cube" is more descriptive of the residual polyhedron than is the word "polytope", which is general enough to apply to points, segments, polygons, polyhedra and hyper dimensional solids.

199. (Fall 1967). Proposed by Larry Forman, Brown University, and M. S. Klamkin, Ford Scientific Laboratory. Find all integral solutions of the equation

\[ \sqrt{x + \sqrt{y}} + \sqrt{x - \sqrt{y}} = z. \]

Solution by Charles W. Trigg, San Diego, California.

It is evident upon inspection that if \( x = 0 \), then \( z = 0 \) (and conversely), and \( y \) is indeterminate. Also, if \( y = 0 \), then \( x = (z/2)^2 \), so \( z \) is even.

II. Put \( x = k^3 / m^3 \) and \( x - \sqrt{y} = n^3 \), whereupon

\[ \sqrt{y} = \left( m^3 - n^3 \right) / 2, \]

\[ x = \left( m^3 + n^3 \right) / 2, \]

and \( y = \left( (m^3 - n^3) / 4, z = m + n, \right. \)

where \( m \) and \( n \) are integers with the same parity. This two-parameter solution includes (I), for \( m = n \).

III. Solution II is based upon the restricted assumption that \( m \) and \( n \) are integers. Cubing both sides of the given equation, we have

\[ \left. \sqrt{x} = \left( \left( m^3 + n^3 \right) / 2 \right)^3 \right/ \left( m^3 - n^3 \right)^2 / 4, z = m + n, \right. \)

Solutions not given by (II), for example, when \( z = 3 \) and \( k = 1 \), are given by this two-parameter solution in \( z \) and \( k \), with \( z \) even or with \( z = 0 \) and both odd, and \( z^2 \neq 3k \), \( z^2 \neq k \). These two restrictions are necessary for consistency with (I). The penultimate restriction is necessary because if \( x < 0 \), then \( x = 0 \); and the last one because if \( y > 0 \), then \( x = z^3 / 8 \), whereas if \( z^2 = k \), then \( x = -z^3 \).

Editorial Note. Klamkin cubed the given equation to obtain

\[ 2x + 3z \sqrt{3} - y = z^3. \]

Thus \( 3\sqrt{3} = y + (m \text{ an integer}), \) and the desired solution is

\[ x = \left( z^3 - 3zm \right) / 2, y = \left( \left( z^3 - 3zm \right) / 2 \right) - m^3, \]

where \( z \) and \( m \) are arbitrary integers, provided either \( z \) is even or both \( z \) and \( m \) odd. For values of \( z \) and \( m \) that result in \( y > 0 \), the "proper" cube roots must be extracted to satisfy the original equation. For example, suppose \( z = 3 \) and \( m = 1 \). Then \( x = 0, y = -27, \) and \( z = 3 \). Substitution of these values in the given equation yields

\[ \sqrt{3} \sqrt[3]{1}, \sqrt[3]{11} \sqrt[3]{3}. \]

Recasting this in the form \( \sqrt[3]{3} \sqrt[3]{11} - \sqrt[3]{3} \sqrt[3]{11} = 3, \) an obvious impossibility, we are confronted by the intrusion of an extraneous root. On the other hand, the solution becomes acceptable by the following procedure:

\[ \sqrt[3]{27} + \frac{1}{\sqrt[3]{27}} = 27^{(1/3)} + (-1)^{1/3} \]

\[ = \sqrt[3]{27} \left( e^{\pi i / 3} + e^{-\pi i / 3} \right) \]

\[ = 2\sqrt[3]{3} \cos(\pi / 3) \]

\[ = 3. \]

Also solved by Edgar Karst, University of Arizona, and Gregory Walczyn, Bucknell University, both of whom submitted partial solutions for integral values of \( x + \sqrt{y} \).
BOOK REVIEWS

Edited by
Roy B. Deal, Oklahoma University Medical Center


A series of seventeen chapters, written by well-known biologists, which gives an excellent survey of a variety of the important areas in biology where rather extensive and interesting mathematical models promise to play a big role.


A third and revised edition of the now famous classic in modern mathematical writings. Many proofs and developments have been modernized. In particular the chapter on fluctuations in coin tossing and random walks has been extensively rewritten and expanded to incorporate modern probabilistic arguments. Sections have been added on branching processes, on Markov chains, and on the De Moivre-Laplace theorem. These changes, along with other clarifications and rearrangements, and the established importance of the earlier editions make this also a valuable book.


Whereas the first volume was basically a study of discrete probabilities and was a pioneer in its mathematical treatment of applied problems, the second volume covers a larger spectrum, utilizes Lebesque measure, has many theorems and applications on more general multidimensional distributions, on more general Markov processes, random walks, renewal theory and other aspects of stochastic processes, and many interesting uses of such things as semi-groups, Tauberian theorems, Laplace transforms, and harmonic analysis. This volume may not have as much of the pioneering aspect but it reflects the same organizational talent of a master who can bring difficult subjects to within the grasp of one with a minimal background, say elementary real analysis and volume one.


Although this is an advanced and extensive book on integration, and perhaps beyond the level of many Pi Mu Epsilon readers, it is such an excellent book that it should be brought to the attention of most members. It is a completely revised and enlarged edition of his well-written earlier book "An Introduction to the Theory of Integration."

A comprehensive, coordinated collection of combinatorial identities including "The most extensive array of inverse relations available," and a survey of number-theoretical aspects of partition polynomials.


Although there are many fine books on Quantum Mechanics at the first-year graduate level, this little book which grew out of a course for final year honors students of mathematical physics is, because of the spirit in which it is written, perhaps the best introduction to mathematical quantum mechanics for mathematics students at the senior-first year graduate level.


The North-Holland Series in Applied Mathematics and Mechanics is attempting to foster a continuing close relationship between applied mathematics and mechanics by publishing authoritative monographs on well-defined topics. This reasonably self-contained book presents the main problems considered in the theory of dynamic deformation of plastic bodies. It gives many details regarding mechanical models, computing methods, and programs for the integration with computers. Although it is written so that no previous knowledge of plasticity is required, the solutions to many problems are given with such detail and modern methods that they may be used directly by the practicing engineer.


This little book meets quite well its stated objective of giving a brief, modern introduction to the subject of ordinary differential equations with an emphasis on stability theory to the reader with only a "modicum of knowledge beyond the calculus".


This brief but excellent account of practical numerical methods for solving very general two-point boundary-value problems would follow quite well the above book by Sanchez. "Three techniques are studied in detail: initial-value or "shooting" methods, finite-difference methods, and integral-equation methods. Each method is applied to non-linear second-order problems and eigenvalue problems; the first two methods are applied also to first-order systems of non-linear equations."


While this book should be very valuable for its stated purpose as a nearly encyclopedic single reference source for scientific programmers with a bachelors degree and a mathematics major, it might also serve to provide the undergraduate mathematics major with a feeling for this important aspect of real world problems, as well as delineate the essential features for many of today's more sophisticated users.

BOOKS RECEIVED FOR REVIEW


Note: All correspondence concerning reviews and all books for review should be sent to PROFESSOR ROY B. DEAL, UNIVERSITY OF OKLAHOMA MEDICAL CENTER, 800 NE 13th STREET, OKLAHOMA CITY, OKLAHOMA 73104.

MATCHING PRIZE FUND

The Governing Council of Pi Mu Epsilon has approved an increase in the maximum amount per chapter allowed as a matching prize from $20.00 to $25.00. If your chapter presents awards for outstanding mathematical papers and students, you may apply to the National Office to match the amount spent by your chapter--i.e., $30.00 of awards, the National Office will reimburse the chapter for $15.00, etc.,--up to a maximum of $25.00. Chapters are urged to submit their best student papers to the Editor of the Pi Mu Epsilon Journal for possible publication.
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