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ELEMENTARY NUMBER THEORY IN CERTAIN SUBSETS OF THE INTEGERS, I

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The purpose of this paper is to study some aspects of elementary number theory in certain subsets of Z, the set of integers. The motivation for this study came about from discussions in number theory texts indicating that the fundamental theorem of arithmetic does not hold in some subsets of Z (see [1, p 28], [2, p 18], or [4, p 12]). The usual example given is $2Z$, the multiples of 2, where it is shown that 2, 6, and 18 are "prime" and that $36 = 6 \cdot 6 = 2 \cdot 18$.

In elementary textbooks on algebra, divisibility is usually studied in an integral domain (commutative ring with unity and no zero divisors) and, as is well known, the fundamental theorem does not always hold in this case either. The definitions are usually stated for a commutative ring and the main results usually hold in an integral domain. In this paper we make no such algebraic assumptions on the subsets considered. The assumptions which are made are more or less forced by the fact that we need to limit the kinds of subsets of Z we wish to consider but are weak enough to allow a wide variety of sets to be taken into consideration. However, our main results will be for the ring nZ, the multiples of some integer n, $n > 1$. These results, however, do not hold because of the algebraic properties of nZ but because the primes and composites (as we shall define them) are so nicely distributed in these sets.

The authors would like to thank the referee for his observations and comments.

1. *Preliminaries*

In looking at nZ one immediately sees that this subset of Z is closed under negation. To discuss divisibility in subsets of Z, this is precisely the restriction we wish to place on the types of subsets we are to consider (with the exception that we exclude the singleton set $\{0\}$). Thus in this paper we only consider non-empty subsets A of Z having the two properties: (a) $A \neq \{0\}$ and (b) if $x \in A$, then $-x \in A$.

To define divisibility in such subsets of Z we merely mimic the usual definition.

Definition: Let x, $y \in A$, $x \neq 0$. We say that x divides y in A (or x A-divides y) if there exists $n \in A$ such that $y = nx$.

To denote the fact that x A-divides y we write $x(A)y$; otherwise we write $x(\not A)y$. If $A = Z$, we adopt the usual notations $x \mid y$ and $x \nmid y$.

As a result of this definition the following can be easily verified:

(1) If $1 \in A$, then $1(A)x$ for all x in A.

(2) Divisibility in A is reflexive if and only if $1 \in A$.

(3) $x(A)(-x)$ if and only if $-x(A)x$ if and only if $-1 \in A$.

(4) If $0 \in A$, then $0(\not A)x$ for all x in A, but $x(A)0$ for all non-zero x in A.

(5) If $A \subset B$ and $x(A)y$, then $x(B)y$. Thus, if $x(A)y$, then $x \mid y$. On the other hand, if O is the set of odd integers, then $x(O)y$ if and only if $x \mid y$.

(6) Divisibility in A need not be transitive. For example, take $A = \{\pm 2, \pm 3, \pm 6, \pm 18\}$. Here $2(A)6$ and $6(A)18$, but $2(\not A)18$.

Although divisibility in A need not be transitive, about the best we can do in this regard is rhe following proposirion, which is easily verified.

Lemma 1: If A is closed under multiplication, then divisibility in A is transitive.

As a result, divisibility in nZ is transitive. The converse of this proposition is false as we see later.

To define primes in A we again mimic the usual definition.

Definition: Let $p \in A$. $p \neq 1$ is said to be *prime* in A, (or to be an *A-prime*),

(1) if $1 \notin A$, then $x(\not A)p$ for all x in A or

(2) if $1 \in A$ and $x(A)p$, then $x = \pm p$ or $x = \pm 1$.

x is composite in A if x is not 0 or ± 1 and is not prime in A.

Thus, for example, 2, 6, 10, 14,$\cdots$ are prime in $2Z$ and the primes in Z are prime in any set containing them. More generally, if $p \in A \subset B$ and p is prime in B, then p is prime in A. The converse is false--take $4 \in 4Z \subset 2Z$. Note also that p is prime in A if and only if $-p$ is prime in A, so as a result we shall omit discussing the negative primes.

There are only two subsets of Z which do not have primes, namely $\{\pm 1\}$ and $\{0, \pm 1\}$. For any other set A, the least positive integer in A which is greater than one is always prime in A. It is also easy to see that for any integer $n > 1$, there is a subset of Z with $-n$ and n as its only primes. Finally, if $p_1, p_2, \cdots p_n$ are any n distinct positive in Z, the set

$$A_n = \bigcup_{i=1}^{n} \{\pm(p_i^{\,m}) \mid m = 1,2,\cdots\}$$

has exactly $2n$ primes (n positive primes). Note that divisibility is transitive in A_n so that A_n serves as a counterexample to the converse of Lemma 1.

2. *The Fundamental Theorem of Arithmetic*

In this section we discuss the primes in nZ, the fundamental theorem of arithmetic in nZ and make some observations concerning primes in nZ and Z. For the remainder of the paper we use *FTA1* to mean there exists a factorization into a product of primes, *FTA2* to mean the factorization is unique, and FTA to mean both *FTA1* and *FTA2*.

The *FTA1* always holds in a subset A of Z of the type considered earlier (other than $A = \{\pm 1\}$ and $A = \{0, \pm 1\}$) and the proof can be patterned after that given in [4, p 11]. We shall show, however, that *FTA2* does not hold in nZ. There are proper subsets of Z for which the FTA does hold; for example take $A = \{\cdots, -p^2, -p, p, p^2, \cdots\}$ where p is any (positive) prime in Z.

There is a simple formula for the primes in nZ ($n > 1$) while, of course, no such formula is known to exist for the primes in Z.

Theorem 1: The positive primes in nZ ($n > 1$) are of the form $(kn + i)n$ for $i = 1,2,\cdots,n-1$ and $k = 0,1,2,\cdots$. All other posirive integers in nZ are composite and are of the form kn^2 for $k = 1,2,\cdots$.

Proof: Let $p \in nZ$ so that $p = mn$ for some $m \in Z$, and suppose that p is prime in nZ. Using the division algorithm (in Z) we can write $m = kn + i$, $0 \le i < n$. Now $i = 0$ iff $p = kn^2$ iff $n(nZ)p$ denying the primality of p. Thus if p is an nZ-prime, then $p = (kn + i)n$ for $i = 1,2,\cdots, n - 1$.

Theorem 2: For any integer $n > 2$, the FTA2 does not hold in nZ.

Proof: By Theorem 1, $n^2(n + 1)^2$ is composite in nZ and is not uniquely factorable since

$$n^2(n + 1)^2 = n \cdot [n(n + 1)^2] \quad \text{and}$$
$$n^2(n + 1)^2 = [n(n + 1)] \cdot [n(n + 1)]$$

are two differenr prime factorizations in nZ.

The following theorem identifies which composires in nZ are and which are not uniquely factorable in nZ. First we observe that if x is composite in nZ, we have $x = kn^2$ by Theorem 1. We then note that x can be written as $x = kn^2 = k_1 n^m$ where $m \geq 2$ and $n \nmid k_1$.

Theorem 3: Let x be an nZ-composite written $x = kn^m$ where $m \geq 2$ and $n \nmid k$. Then x is uniquely factorable if and only if $k = 1$ or k is prime in Z.

Proof: Let $k = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the unique prime factorization of k in Z. Observing that $k = 1$ is a trivial case, we go on to the cases when $k \neq 1$.

Case 1: $r = 1$.

(a) If $a_1 = 1$ then x is uniquely factorable as $x = n^{m-1}(np_1)$.

(b) If $a_1 \geq 2$, then x is not uniquely factorable since

$$\begin{aligned} x &= n^{m-1}(np_1^{a_1}) \\ &= n^{m-2}(np_1^{a_1 - 1})(np_1) \end{aligned}$$

are two different prime factorizations of x in nZ.

Case 2: $r \geq 2$.

Then x is not uniquely factorable since

$$\begin{aligned} x &= n^{m-2}(np_1^{a_1})(np_2^{a_2} \cdots p_r^{a_r}) \\ &= n^{m-1}(np_1^{a_1} \cdots p_r^{a_r}) \end{aligned}$$

are two different prime factorizations of x in nZ.

Theorem 1 allows us to make the following observations concerning primes in nZ and in Z; some of these will be discussed in more detail in the next section.

(1) There are infinitely many primes in each nZ, as there are in Z.

(2) There are not arbitrarily large gaps between primes in nZ as there are in Z. In fact, the gap between primes in nZ is either 0 or 1.

(3) There are infinitely many twin primes (primes separated by only one composite) in each nZ. Moreover, the twin primes in nZ have the form: $(kn + (n - 1)) \cdot n$ and $((k + 1)n + 1) \cdot n$ for $k = 0,1,2,\cdots$.

(4) Given any positive integer n, however large, nZ contains $n - 1$ consecutive primes. (In Z the largest sequence of consecutive primes is the sequence 2, 3.)

3. The Arithmetic Function $\pi(x)$

Let $\pi_A(x)$ denote rhe number of positive primes in A which are less than or equal to $x \in A$. Since the primes and composites in nZ are so nicely distributed, we are able to obtain an explicit formula for this function. In the following theorem, $[x]$ means the greatest integer (in Z) which is less than or equal to x.

Theorem 4: If $x = kn \in nZ$, then

$$\pi_{nZ}(kn) = k - [k/n] = \frac{n-1}{n} \cdot (k - r) + r \;,$$

where in the last equality $k = nq + r$ and $0 \leq r < n$, $q = [k/n]$.

Proof. The number of positive primes in nZ which are less than or equal to kn is equal to $T - C$ where

T = the total number of positive integers in nZ which are $\leq kn$, and

C = the number of positive composites in nZ which are $\leq kn$.

It is clear that $T = k$. As we saw in Theorem 1, the composites in nZ are of the form kn^2. Now the composites in nZ which are less than or equal to kn are: $n^2, 2n^2, \cdots, [k/n] \cdot n^2$. Thus, $C = [k/n]$.

In Z, the ratio $\pi(x)/x$ can be interpreted in two ways:

(1) as the number of positive primes which are $\leq x$ compared to x, or

(2) as the number of positive primes which are $\leq x$ compared to the number of positive integers which are $\leq x$.

Of course these two meanings are the same, and it is well-known that

$$\lim_{x\to\infty} \pi(x)/x = 0 .$$

However, in nZ the above meanings are different since the first gives the ratio $\pi_{nZ}(kn)/kn$ and the second gives the ratio $\pi_{nZ}(kn)/k$. In the next theorem we examine the two limits arising from these ratios.

Theorem 5: For a fixed $n > 1$

(1) $\lim_{k\to\infty} \frac{\pi_{nZ}(kn)}{k} = \frac{n-1}{n}$,

(2) $\lim_{k\to\infty} \frac{\pi_{nZ}(kn)}{kn} = \frac{n-1}{n^2}$.

Proof: Let $q = [k/n]$. From Theorem 4

$$\lim_{k\to\infty} \frac{\pi_{nZ}(kn)}{k} = \lim_{k\to\infty} \frac{k-q}{k} = \lim_{q\to\infty} \frac{nq+r-q}{nq+r} = \lim_{q\to\infty}\left(1 - \frac{q}{nq+r}\right) = \frac{n-1}{n} .$$

The proof of (2) follows from (1).

We observe that for a fixed n, $0 < \lim_{k\to\infty} \frac{\pi_{nZ}(kn)}{kn} < \lim_{k\to\infty} \frac{\pi_{nZ}(kn)}{k} < 1$ and that for large n, $\lim_{k\to\infty} \pi_{nZ}(kn)/kn$ is near zero while $\lim_{k\to\infty} \pi_{nZ}(kn)/k$ is near one.

4. *The Arithmetic Functions $\tau(x)$ and $\sigma(x)$*

In this section we discuss the following two functions:

$\tau_A(x)$ = the number of positive divisors of x in A,

$\sigma_A(x)$ = the sum of the positive divisors of x in A.

In Z, one normally establishes formulas for these functions by first taking x as prime, then x as a power of a prime, and then shows the above functions are multiplicative [$f(mn) = f(m) \cdot f(n)$ when m and n relatively

prime] so that one can rely on *FTA2* to obtain the formulas for composites. Since factorization is not unique in nZ, this process will not work. In the following approach we see that these functions are "nearly" multiplicative in nZ. We first establish the formulas for powers of n and then for composites. Clearly, if x is prime in nZ, then $\tau_{nZ}(x) = 0 = \sigma_{nZ}(x)$. We saw before that if x is composite in nZ, we can write $x = kn^m$ where $m \geq 2$ and $n \nmid k$.

Lemma 2: If $x = n^m$, then $\tau_{nZ}(n^m) = m - 1$ and $\sigma_{nZ}(n^m) = \sum_{r=1}^{m-1} n^r$.

Proof: The positive divisors of n^m in nZ are $n, n^2, n^3, \cdots, n^{m-1}$ and the formulas follow easily.

Theorem 6: If $x = kn^m$ where $m \geq 2$ and $n \nmid k$, then

$$\tau_{nZ}(kn^m) = (m-1)\cdot\tau_Z(k) = \tau_{nZ}(n^m)\cdot\tau_Z(k)$$

$$\sigma_{nZ}(kn^m) = \sum_{r=1}^{m-1} n^r \cdot \sigma_Z(k) = \sigma_{nZ}(n^m)\cdot\sigma_Z(k) .$$

Proof: Each divisor of k multiplied by n is a divisor of x. Each divisor of k multiplied by n^2 is a divisor of x. Inductively, each divisor of k multiplied by n^{m-1} is a divisor of x. Since k does not have n as a factor neither do the divisors of k and so the divisors named above are all distinct. Moreover, these are all the positive divisors of x. Since each row contributes $\tau_Z(k)$ divisors of x and since there are $m - 1$ rows, there are therefore $(m-1)\cdot\tau_Z(k)$ divisors of x in nZ. To find $\sigma_{nZ}(kn^m)$ we merely add up the divisors in each row. The sum of the jth row is $n^j \cdot \sigma_Z(k)$. Summing the $m - 1$ rows gives the middle equality. Lemma 2 gives the last equalities in the two formulas.

In a future paper we shall discuss in nZ the notions of greatest common divisor, relative primeness, and Euler's function $\phi(x)$.

REFERENCES

1. Andrews, G. E., *Number Theory*, W. B. Saunders Company, Philadelphia, Pennsylvania, 1971.
2. Barnett, I. A., *Elements of Number Theory*, Prindle, Weber, and Schmidt, Inc., Boston, Massachusetts, 1969.
3. Hardy, G. H. and Wright, E. M., *An Introduction to the Theory of Numbers*, Oxford University Press, 4th ed., New York, 1960.
4. Niven, I. and Zuckerman, H. S., *An Introduction to the Theory of Numbers*, 2nd ed., John Wiley and Sons, Inc., New York, 1966.

A NESTED PRIME NUMBER MAGIC SQUARE

Reverend Victor Feser of St. Ambrose Church, St. Louis, has pointed out an example of a 13 × 13 nested magic square consisting entirely of *prime numbers*.[1] (See in this connection the article "Magic Squares Within Magic Squares" by Joseph Moser, this Journal, 5, No. 8 (Spring 1973), p. 430.) Each smaller square centered at 5437 is also a magic square,

1153	8923	1093	9127	1327	9277	1063	9133	9661	1693	991	8887	8353
9967	8161	3253	2857	6823	2143	4447	8821	8713	8317	3001	3271	907
1831	8167	4093	7561	3631	3457	7573	3907	7411	3967	7333	2707	9043
9907	7687	7237	6367	4597	4723	6577	4513	4831	6451	3637	3187	967
1723	7753	2347	4603	5527	4993	5641	6073	4951	6271	8527	3121	9151
9421	2293	6763	4663	4657	9007	1861	5443	6217	6211	4111	8581	1453
2011	2683	6871	6547	5227	1873	$\boxed{5437}$	9001	5647	4327	4003	8191	8863
9403	8761	3877	4783	5851	5431	9013	1867	5023	6091	6997	2113	1471
1531	2137	7177	6673	5923	5881	5233	4801	5347	4201	3697	8737	9343
9643	2251	7027	4423	6277	6151	4297	6361	6043	4507	3847	8623	1231
1783	2311	3541	3313	7243	7417	3301	6967	3463	6907	6781	8563	9091
9787	7603	7621	8017	4051	8731	6427	2053	2161	2557	7873	2713	1087
2521	1951	9781	1747	9547	1597	9811	1741	1213	9181	9883	1987	9721

[1] From the Recreational Mathematics Journal, No. 5 (October 1961), p. 28.

REFEREES FOR THIS ISSUE

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NICENESS OF THE SOCLE AND A CHARACTERIZATION OF GROUPS OF BOUNDED ORDER

by S. W. Talley[1]
Western Kentucky University

A classification result (that tells when or how two algebraic systems are of the same kind) is one of the most desired results in the study of any algebraic system. One of the most notable classification theorems is that of Ulm [10], which classifies countable reduced abelian primary groups in terms of a set of numerical invariants. This classification using the same invariants has been extended to a much larger class of abelian p-groups, the class of totally projective groups, in the work of Nunke [9], Hill [5] (see Griffith [3]), and Crawley and Hales [1].

Hill introduced the concept of "nice subgroup" and established the characterization of a totally projective group as a reduced p-group that contains "enough" nice subgroups. Moreover, Hill was able to show that the class of totally projective groups is the largest reasonable class of reduced primary abelian groups that can be classified by their Ulm invariants.

If G is an abelian p-group (p prime) and n is a non-negative integer, then $G[p^n]$ denotes the subgroup of G consisting of all elements having order less than or equal to p^n; $G[p]$ is called the *socle* of G. In this paper a necessary and sufficient condition is found for $G[p^n]$ to be a nice subgroup of the reduced abelian p-group G (see Theorem 1). This condition, together with the result of Hill [4], leads to the characterization of a reduced bounded p-group as a reduced p-group in which all subgroups are nice (see Theorem 4).

Preliminaries

All groups in this paper are assumed to be additively written reduced abelian p-groups. (A p-group is one in which all elements have order equal to a power of the prime p.) A group G is a divisible group

[1] The author wishes to acknowledge the help of K. D. Wallace in writing this paper.

if for each $x \in G$ and each positive integer n, there exists $y \in G$ such that $x = ny$. A reduced group is one which contains no (nonzero) divisible subgroups. (Clearly the element 0 is divisible by every positive integer n, since $0 = n0$.) Now let G be a p-group for the prime p. For each ordinal α, we define inductively a subgroup $p^\alpha G$ as follows: $pG = \{px \mid x \in G\}$; $p^{\gamma+1}G = p(p^\gamma G)$; if γ is a limit ordinal $p^\gamma G = \cap_{\beta<\gamma} p^\beta G$. We thus have a descending chain of subgroups

$$G \supseteq pG \supseteq \cdots \supseteq p^\alpha G \supseteq p^{\alpha+1}G \supseteq \cdots .$$

Note in particular if n is a positive integer and $x \in p^nG \backslash p^{n+1}G$ then x is divisible by p^n in G but x is not divisible by p^{n+1} in G. If x is divisible by p^n for every positive integer n, then $x \in p^\omega G$ where ω is the first infinite ordinal and x is said to have infinite height in G. In general, the element x is said to have p-height α in G, and we set $h_G(x) = \alpha$, if $x \in p^\alpha G \backslash p^{\alpha+1}G$. If $x \in p^\alpha G$ for each ordinal α then $h_G(x) = \infty$ where $\infty > \alpha$ for each ordinal α. Since $0 \in p^\alpha G$ for each ordinal α, $h_G(0) = \infty$.

The following fundamental properties are easily established. Let G be a p-group with $x, y \in G$. If $h_G(x) \neq h_G(y)$ then $h_G(x + y)$ is the smaller of the two heights. If $h_G(x) = h_G(y)$ then $h_G(x + y) \geq h_G(x)$. A homomorphism cannot decrease height; that is, if f is a homomorphism from G into G', then $h_G(x) \leq h_{G'}(f(x))$. If $p^\alpha G = p^{\alpha+1}G$, then $p^\beta G = p^\alpha G$ for $\beta > \alpha$ and $p^\alpha G$ is a divisible group. If G is a reduced p-group, there must be an ordinal γ such that $p^\gamma G = \{0\}$. The least such ordinal γ is called the length of G and is denoted by $\lambda(G) = \gamma$. A bounded group is a torsion group for which there exists a finite upper bound to the set of orders of all elements. That is, there exists a positive integer n such that $nx = 0$ for every x in the group. Alternatively, G is a bounded group if there exists a positive integer n such that $nG = 0$.

A subgroup A of the group G is a *nice* subgroup of G if and only if

$$p^\alpha\left(\frac{G}{A}\right) = \frac{p^\alpha G + A}{A}$$

for all α. We note that the subgroup A of G is nice if and only if for each $x \in G$ there exists some $a \in A$ such that $h_{G/A}(x + A) = h_G(x + a)$. The following characterization, given by Hill [5], will be utilized:

Theorem A: The subgroup A of a p-group G is a nice subgroup of G if and only if each coset $x + A$ contains an element $x + a$ that has p-height in G which is maximal among the elements of the coset $x + A$.

An immediate consequence of the above theorem is the fact that any finite subgroup is nice.

Characterization Theorems

Theorem 1: Let G be a reduced abelian p-group and n a positive integer. Then $G[p^n]$ is nice if and only if $\lambda(G) \leq \omega$.

Proof: Suppose n is an arbitrary but fixed positive integer, $G[p^n]$ is a nice subgroup of G, and that $\lambda(G) > \omega$. Consider the descending chain of subgroups

$$G \supset pG \supset \cdots \supset p^mG \supset \cdots \supset p^\omega G \supset p^{\omega+1}G \supset \cdots .$$

Since $\lambda(G) > \omega$, $p^\omega G \neq \{0\}$ and there exists $x \in p^\omega G - p^{\omega+1}G$, $x \neq 0$. Note the height of x is ω. Thus for the positive integer n there exists an element y in G such that $x = p^n y$. Moreover, for each positive integer m

$$x \in p^{n+m}G = p^n(p^mG)$$

and hence there exists $x_m \in p^mG$ such that $x = p^n x_m$. Now

$$p^n(x_m - y) = p^n x_m - p^n y = x - x = 0$$

so $x_m - y \in G[p^n]$ and thus

$$x_m + G[p^n] = y + G[p^n] .$$

Since $G[p^n]$ is nice, the coset $y + G[p^n]$ contains an element $y + a$ of maximal height. Hence $h_G(y + a) \geq h_G(y + g)$ for each $g \in G[p^n]$. Since $x_m \in y + G[p^n]$, $h_G(y + a) \geq h_G(x_m) \geq m$ for each positive integer m. Thus $h_G(y + a) \geq \omega$. Now observe that since $a \in G[p^n]$, $p^n a = 0$ and therefore $p^n y = p^n y + p^n a$. Thus

$$\omega = h_G(x) = h_G(p^n y) = h_G(p^n y + p^n a) = h_G(p^n(y + a)) \geq \omega + n,$$

which is absurd. (Indeed, $h_G(p^n x) \geq h_G(x) + n$ holds in general.) Thus if $G[p^n]$ is nice then $\lambda(G) \leq \omega$ for each positive integer n. If $\lambda(G) < \omega$, then the set consisting of all ordinals which serve as heights of elements of G is finite and consequently any set of elements of G must contain an element of maximal height. In particular, $G[p^n]$ is nice for each non-negative integer n. Now suppose $\lambda(G) = \omega$ and $G[p^n]$ is not nice. Then there exists an element $x \in G$ such that $x + G[p^n]$ does not contain an element of maximal height. (Note: $x \notin G[p^n]$, for otherwise $x + (-x) = 0 \in x + G[p^n]$ and 0 is a member of $x + G[p^n]$ of maximal height.) For each

$a \in G[p^n]$, there exists an element $\bar{a} \in G[p^n]$ such that $h_G(x + a) < h_G(x + \bar{a})$. Consequently for each positive integer N there exists $a \in G[p^n]$ such that $h_G(x + a) > N$. If $a \in G[p^n]$ then $p^n(x + a) = p^n x$ and $h_G(p^n x) = h_G(p^n(x + a)) \geq h_G(x + a)$. Thus $h_G(p^n x) \geq h_G(x + a)$ for each $a \in G[p^n]$ and consequently $h_G(p^n x) > N$ for each positive integer N. Therefore $p^n x \neq 0$ and $h_G(p^n x) > \omega$, contradictory to the assumption that $\lambda(G) = \omega$. Thus if $\lambda(G) \leq \omega$, $G[p^n]$ is a nice subgroup of the reduced p-group G.

A topology may be introduced on a reduced p-group G by letting the subgroups $\{p^n G\}_{n<\omega}$ serve as a basis for the neighborhoods of zero. This topology is called the *p-adic* topology. With this topology, G is Hausdorff if and only if G has no elements of infinite height and a subgroup H of G is closed in G if and only if G/H is without elements of infinite height. In [4], Hill established the following characterization for direct sums of cyclic groups.

<u>*Theorem* 2</u>: Let G be a primary group without elements of infinite height. Then G is a direct sum of cyclic groups if (and only if) there exists a collection $\mathcal{C}$ of subgroups of $G[p]$ such that:

(i) each member of $\mathcal{C}$ is closed in G,

(ii) 0 is a member of $\mathcal{C}$,

(iii) the group union in G of any collection of subgroups belonging to $\mathcal{C}$ again belongs to $\mathcal{C}$,

(iv) if $S \in \mathcal{C}$ and if $T \subseteq G[p]$ is such that $\frac{S + T}{S}$ is countable, then there exist.: $S' \in \mathcal{C}$ such that $S' \supseteq S + T$ and S'/S is countable.

Since $h_G(x + h) \leq h_{G/H}(x + H)$ for all $h \in H$, it follows from Theorem A that the closed subgroups of the *p-adic* topology are in fact nice subgroups of G if G is without elementsof infinite height. Moreover, if G is a p-group without elements of infinite height then a subgroup H of G is closed if and only if H is a nice subgroup of G. By utilizing Theorem 1, we may restate Theorem 2 as follows.

<u>*Theorem* 2'</u>: Let G be a primary group. Then G is a direct sum of cyclic groups if and only if there exists a collection $\mathcal{C}$ of subgroups of $G[p]$ such that:

(i) each member of $\mathcal{C}$ is a nice subgroup of G,

(ii) 0 is a member of $\mathcal{C}$,

(iii) the group union in G of any collection of subgroups belonging to $\mathcal{C}$ again belongs to $\mathcal{C}$,

(iv) if $S \in \mathcal{C}$ and if $T \subseteq G[p]$ is such that $\frac{S + T}{S}$ is countable, then there exists $S' \in \mathcal{C}$ such that $S' \supseteq S + T$ and S'/S is countable.

The following example of a divisible group shall play an important role in our remaining development. The reader is referred to Kaplansky [6] or Fuchs [2] for further properties and discussion of divisible groups. Let p be a fixed prime, and let P denote the additive group of those rationals whose denominators are powers of p. We denote the quotient group P/Z by $Z(p^\infty)$; it is understood that addition takes place modulo one. Since $Z(p^\infty)$ is a primary group, all of its elements are divisible by any integer prime to p. On the other hand, it is clear that every element of $Z(p^\infty)$ can be divided by arbitrary powers of p. Putting these two conditions together we have that $Z(p^\infty)$ is a divisible group and we further emphasize that addition is modulo one.

<u>Lemma 2</u>: $Z(p^\infty)$ is an epimorphic image of $\bigoplus_{k<\omega} Z(p^k)$.

Proof. For each positive integer n, the element $1/p^n$ generates a cyclic subgroup of $Z(p^\infty)$ having order p^n. Hence there exists a homomorphism (in fact isomorphism) of $Z(p^n)$ onto $\langle 1/p^n \rangle$. By the universal property of direct sums, it follows that there exists a homomorphism of $\bigoplus_{k<\omega} Z(p^k)$ onto $Z(p^\infty)$.

<u>Lemma 3</u>: Let G be an unbounded reduced p-group. If G is a direct sum of cyclic groups then $Z(p^\infty)$ is an epimorphic image of G.

Proof: Let $G = \bigoplus_{\alpha<\Gamma} Z(p^{n_\alpha})$ for some ordinal Γ where $\{n_\alpha\}_{\alpha<\Gamma}$ is an unbounded set of positive integers such that $n_\alpha \leq n_\beta$ if $\alpha < \beta$. Consider $\bigoplus_{k<\omega} Z(p^k)$. Since $\{n_\alpha\}$ is unbounded, for each k we may choose $\alpha_k < \Gamma$, $\alpha_k \notin \{\alpha_i\}_{i<k}$, such rhar $n_{\alpha_k} \geq k$. Hence for each k, rhere exists a homomorphism of $Z(p^{n_{\alpha_k}})$ onto $Z(p^k)$. Let $G' = \bigoplus_{k<\omega} Z(p^{n_{\alpha_k}})$. By the universal property of direct sums there exists a homomorphism Ψ of G' onto $\bigoplus_{k<\omega} Z(p^k)$. Since G' is a direct summand of G, there exists a homomorphism, ρ, of G onto G'. Thus $\rho\circ\Psi$ is a homomorphism of G onto $\bigoplus_{k<\omega} Z(p^k)$, and by Lemma 2 it follows that $Z(p^\infty)$ is an epimorphic image of G.

Theorem 4: Let G be a reduced *p*-group. Then G is a bounded group if and only if each subgroup of G is a nice subgroup of G.

Proof: If the group G is bounded then $\lambda(G) < \omega$. Hence any set of elements has an element of maximal height which implies that every subgroup is nice. Assume that every subgroup of G is nice. This implies, by Theorem 2' with C the class of all subgroups of $G[p]$, that the group G is a direct sum of cyclic groups. Then G is either bounded or unbounded. If G is bounded we are finished. Assume G is unbounded. Then by Lemmas 2 and 3 there exists an epimorphism from G onto $Z(p^\infty)$, say $\Psi : G \twoheadrightarrow Z(p^\infty)$. Note Ker $\Psi \neq \{0\}$ and $\frac{G}{\text{Ker } \Psi}$ is a divisible group (since $\frac{G}{\text{Ker } \Psi} \cong Z(p^\infty)$). Let $x \notin$ Ker Ψ; then $\infty = h_{\frac{G}{\text{Ker } \Psi}}(x + \text{Ker } \Psi) > h_G(x + k)$ for any $k \in$ Ker Ψ.

Thus Ker Ψ is not a nice subgroup of G which is a contradiction of our hypothesis. Hence if each subgroup of G is a nice subgroup of G then G is a bounded group.

REFERENCES

1. Crawley, P. and Hales, A. W., "The Structure of Abelian p-groups Given by Certain Presentation," *J.* of Alg. 12 (1969), pp. 10-23.
2. Fuchs, L., *Abelian* Groups, Publishing House of the Hungarian Academy of Sciences, Budapest, 1958.
3. Griffith, P., Infinite Abelian Group Theory, University of Chicago Press, 1970.
4. Hill, P., "Sufficient Conditions for a Group to be a Direct Sum of Cyclic Groups," Rocky *Mt. J.*, 1, No. 2, (1971), pp. 345-351.
5. Hill, P., "On the Classification of Abelian Groups," to appear.
6. Kaplansky, I., Infinite Abelian Groups, University of Michigan Press, Ann Arbor, 1954.
7. Kolettis, G., "Direct Sums of Countable Groups," Duke Math. J., 27 (1960), pp. 111-125.
8. Megibben, C.. "On Mixed Groups of Torsion-free Rank One," Illinois Math. *J.*, 11 (1967), pp. 134-144.
9. Nunke, R., "Homology and Direct Sums of Countable Groups," Math. Z., 101 (1967), pp. 182-212.
10. Ulm, H., "Zur Theorie der Abzahlbar-unendliched Albelschen Gruppen," Math. Ann., 107 (1933).
11. Wallace, K., "On Mixed Groups of Torsion-free Rank One with Totally Projective Primary Components," *J.* of *Alg.* 17 (1971), pp. 482-488.

USE OF MATRICES IN THE FOUR COLOR PROBLEM

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The four color problem has fascinated mathematicians and laymen for the pasr 100 years. Notwithstanding rhe massive research in this area and rhe rampant growth of graph theory, of which map coloring is a subfield, the four color problem remains unsolved to the present day. The reader may wish to consult Marshall [1] and May [5] for a historical account of the subject.

The four color problem may be stated with the aid of the following two definitions:

Definition 1: A finite collection $M = \{R_i\}$ of closed subsets of E^2 having the properties

1) each R_i has positive area,

2) $R_i \cap \text{int } R_j = \phi$ for all i and j, $i \neq j$,

and

3) each R_i is a homeomorphic image of the unit circle,

is called a map; the R_{ij}'s are called regions.

More refined definitions can be given using the concept of planar graphs; see Ore [3], Tutte's definition in Harary [4].

Definition 2: A *well-coloring* of a map M is a function assigning to each region $R_{i_0} \in M$ a color a_{i_0}, suah that, for all $j \neq i_0$, if $\partial R_{i_0} \cap \partial R_j$ is *a* set of positive measure then $c_{i_0} \neq a_j$.

The central question is: Given a map M *what* is the minimum number, x_M, of colors which is required to well-color the map? Obviously, if the map has n regions, $x_M \leq n$; however we can do much better than this. In fact, as seen in Ore [6], for any map M

$$x_M \leq 5 .$$

We may now ask: Is it possible that four colors are sufficient to well-color an arbitrary map M? An obviously related question is: Can we construct a counterexample which cannot be well-colored with four colors?

Amazingly, neither of these apparently simple questions has yet been answered, and this is what constitutes the so called Four Color Problem.

In this paper we definitely do not solve the four color problem, bur simply offer a new approach To the problem. To every map assign a region-region marrix $H = [a_{ij}]$ as defined as follows:

$a_{ij} = 1$ if $\partial R_i \cap \partial R_j$ is a set of positive measure,
$a_{ij} = 0$ if $\partial R_i \cap \partial R_j$ is a set of zero measure,

For a map with n regions the matrix H will be $n \times n$ and, as indicated later, it contains certain information concerning the map. (This notion is similar to that involved in a general graph when one constructs the so called adjacency matrix.) It follows directly from the definition that:

i) The matrix associated with any map is symmetric.
ii) The diagonal entries must be ones.
iii) A column cannot consist entirely of zeroes.
iv) In any row there arc at least *two* non-zero entries.

Property (i) implies that we need to studv only the upper triangular matrix. An example follows.

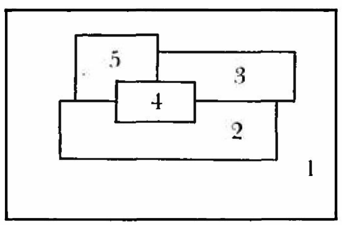


FIGURE 1

In the map illustrated in Fig. 1, we have

$\mu(\partial R_1 \cap \partial R_1) > 0, \quad a_{11} = 1,$
$\mu(\partial R_1 \cap \partial R_2) > 0, \quad a_{12} = 1,$
$\mu(\partial R_1 \cap \partial R_3) > 0, \quad a_{13} = 1,$
$\mu(\partial R_1 \cap \partial R_4) = 0, \quad a_{14} = 0,$ **and so forth.**

Hence the matrix associated with this map is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The upper triangle is

$$\begin{matrix} 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{matrix} .$$

The matrix can be read as follows: Region 1 touches regions 2, 3, 5 but not 4 (read row 1 across); region 3 touches regions 4, 5 (read row 3 across) and regions 1, 2 (read column 3).

Since the numbering of the regions in the collection M is arbitrary, we can use different numberings. This induces different matrices. There are in fact $n!$ "different" matrices for a given map. If H_0 is any particular matrix, the remaining $n! - 1$ matrices are called derivatives of H_0; symbol: ∂H_0 - any derivative of H_0.

We are now ready to define an equivalence relation between maps. The following is easily seen to be an equivalence.

<u>*Definition 3:*</u> $M_1 \sim M_2$ if the set of derivatives of a matrix H_1 of M_1 coincides with the set of derivatives of a matrix H_2 of M_2.

Obviously a necessary condition for two maps to be equivalent is that they have the same number of regions. On the other hand, two maps M_1 and M_2 being homeomorphic is sufficient to insure $M_1 \sim M_2$. For example, the two maps given in Fig. 2 are equivalent and consequently are considered to be indistinguishable.

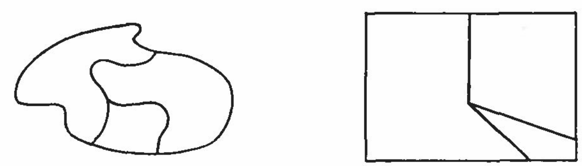


FIGURE 2

The converse, however, is not true, as Fig. 3 shows. These two naps are equivalent (as can be seen by writing out the $4! = 24$ matrices associated with each map) but are not homeomorphic, since in the second map $\partial R_2 \cap \partial R_4 \neq \phi$.

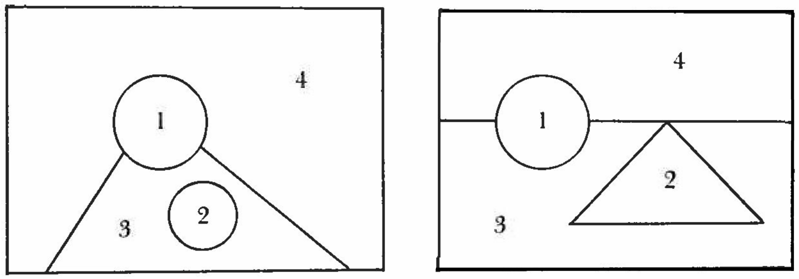


FIGURE 3

Given two equivalent maps, if we well-color one map and assign a numbering to the second map such that the matrix thus generated is equal to the matrix used to color the first map (we show below how this is done), then we automatically have well-colored the second map. Hence the chromatic characteristics of maps belonging to the same equivalence class are the same. For this reason we have only to study the equivalence classes of maps, which is equivalent to the study of the associated matrices, or in particular, any one of the associated matrices.

The equivalence relation introduced above reduces the number of maps in E^2 having n regions to a finite number of equivalence classes. The number of equivalence classes for maps with n regions is bounded by $2^{1/2n(n-1)}$ (use properties (ii), (iii), (iv) of the region-region matrix). The function f associating the region-region matrix to a map is a function into the set of symmetric matrices. It will be seen later that the matrix

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{array}$$

has no pre-image; for if it has a pre-image, it would require 7 colors to well-color it, but, as we have indicated, $a_M \leq 5$ for any map M.

Now we introduce a technique to well-color a map using the associated matrix without any reference to the actual map itself. We have already stated that we need investigate onlv the upper triangular matrix--one can

construct the triangular matrix by determining, for a particular region i, the regions with assigned integer ***greater*** or equal to i, which have a common border with region i. The well-coloring procedure is as follows: Suppose that we make the first region of color A. We then "multiply" the first row by A. For the preceding matrix map of Fig. 1 we would obtain

$$\begin{array}{ccccc} A & A & A & 0 & A \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{array} .$$

Now investigate the second (ith) region. We are allowed to make the second (ith) region of the same color A as the first region (the same color as the (i - 1)st, (i - 2)nd$\cdots$2nd, 1st regions) ONLY if the second (ith) *column* does not already contain color A (color of (i - 1)st, (i - 2)nd$\cdots$ etc.) We see we cannot use color A for the second region since the second column already has A in It; we must use a different color, say B, and hence "multiply" the row by B. We get

$$\begin{array}{ccccc} A & A & A & 0 & A \\ & B & B & B & B \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{array} .$$

Continuing this process we finally obtain

$$\begin{array}{ccccc} A & A & A & 0 & A \\ & B & B & B & B \\ & & C & C & C \\ & & & A & A \\ & & & & D \end{array} .$$

Note that in row four we can use color A again, since that column does not already contain A. This means that region 1 will be made color A; 2, B; 3, C; 4, A; 5, D. If we now go back to the actual map, we see that it is well-colored.

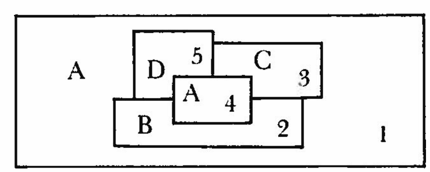


FIGURE 4

Definition 4: The *coloration* of a matrix is the assignment of a label $L(a_{ij})$ to every non-zero element a_{ij} of the matrix.

Definition 5: An $n \times n$ matrix M with a coloration is *normal* if

1) for all i, $1 \le i \le n$, $L(a_{it}) = L(a_{ii})$ for all non-zero elements a_{it},
2) for all j, $1 \le j \le n$, $L(a_{jj}) \ne L(a_{tj})$ for all non-zero elements a_{tj}, $t < j$.

Theorem 1: Let H be some matrix associated with a map M. Then M is well-colored if and only if H is normal.

Proof: Let the matrix be normal; the color on the diagonal is different from any color in that column. Consider a_{jj}. The fact that a_{j1}, a_{j2}, $a_{j3}, \cdots, a_{jj}$ are, say, 1's means that the (jth) region touches the 1st, 2nd,$\cdots$, $(j-1)$st region; but by hypothesis the color of the jth region is different from that of the 1st, 2nd,$\cdots$, $(j-1)$st region, contiguous to it. This being true for all columns implies that the map is well-colored.

Conversely, let the map be well-colored. Then any region j is surrounded by regions with colors different from that of the region j. Give an arbitrary numbering to the map and also write down a letter symbol for the color in the appropriate region. Now construct the associated matrix, but instead of 0's and 1's, put 0's and the letter of the color. Say that the (uncolored) matrix is of the form

$$\begin{array}{cccccc} 1 & 1 & 1 & 0\cdots 1 & 1 \\ & 1 & 1 & 1\cdots 1 & 1 \\ & & 1 & 1\cdots 1 & 1 \\ & & & \cdots\cdots & . \end{array}$$

By hypothesis all regions around region j, say $j = 3$, have different colors. This means that the entries in column j, have different colors (by construction of the matrix). So the matrix will be of the form

$$\begin{array}{cccccc} 1 & 1 & A & 0\cdots 1 & 1 \\ & 1 & B & 1\cdots 1 & 1 \\ & & C & 1\cdots 1 & 1 \end{array}$$

and hence by the multiplication principle

$$\begin{array}{cccccc} A & A & A & 0\cdots A & A \\ & B & B & B\cdots B & B \\ & & C & C\cdots C & C \end{array}$$

and it is normal (the argument above being true for all columns).

A question of considerable importance is if the map is well-colored with the minimum possible number of colors. Obviously, normality does not imply that the map is well-colored with the minimum number of colors; the converse is however true. In order to well-color with minimum number of colors, every time we get to a new row we must investigate to see if we can use again colors already used.. Sometimes we will have a choice of two colors (already used) leading to a different number of total colors used. In any case the matrix will be normal. Example:

$$\begin{array}{ccccc} A & A & 0 & 0 & A \\ & B & 0 & B & B \\ & & A & A & A \\ & & & C & C \\ & & & & P \end{array} \qquad\qquad \begin{array}{ccccc} A & A & 0 & 0 & A \\ & B & 0 & B & B \\ & & B & B & B \\ & & & A & A \\ & & & & C \end{array}$$

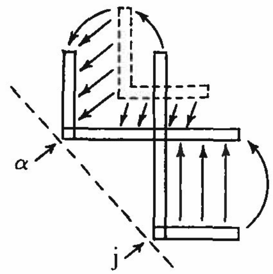
(4 colors needed) (3 colors needed)

This does not present a problem, however, since In application of this procedure to the four color problem we suppose there exists a matrix with $n + 1$ columns, representing the smallest map M in E^2 requiring five colors. If we can fuse together two regions in such a way that the resulting $n \times n$ matrix (map) still requires 5 colors, we achieve a contradiction since M was the smallest such map; hence no such map could exist. Of course, we already know that n has to exceed 40. See Ore and Stemple [7]. To that end we present (without proof) the following theorem.

Theorem 2: Let M be a map, H its matrix. Let R_α, R_j be two contiguous regions (i.e. $a_{\alpha j} = 1$) and assume $j > \alpha$. Let M' be a map identical with M except that the regions R_j and R_α are combined. Let H' be a a matrix obtained as follows:

1) "Add" the corresponding entries of row j In H to row α in H as follows: $1 + 0 = 0 + 1 = 1$, $0 + 0 = 0$, $1 + 1 = 1$.
2) "Fold" the jth column at entry α. Then:
   a) "Add" (as above) the corresponding elements of the unfolded part to the entries of the α column

b) "Add" (as above) the corresponding elements of the folded part to the a row.



3) All other rows and columns of H remain unchanged.

Then there exists a numbering of the regions of M' such that the matrix of M' is exactly H'.

Example: Fuse region 2 and region 4 in map M below:

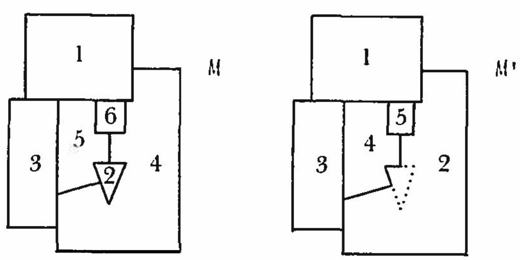
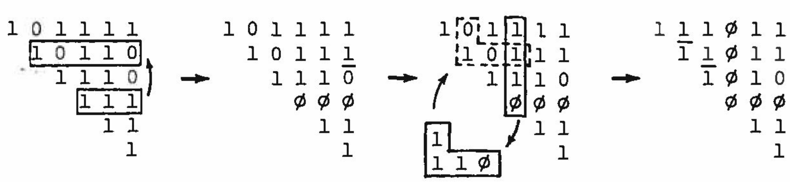


FIGURE 5

$$H = \begin{array}{rrrrrr} 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & \boxed{1} & 1 & 0 \\ & & 1 & 1 & 1 & 0 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{array} \qquad H' = \begin{array}{rrrrr} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{array}$$

The matrix H' is obtained from H by the following steps:



and finally

$$\begin{array}{rrrrr} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{array}$$

It should be noted that even though one row and one column disappeared, this fusion "swallows up" zeroes in the remaining rows and columns. The presence of more one's tends to increase the number of colors needed to normalize the matrix. Hence the proof of the following conjecture, as indicated earlier, would solve the four color problem.

Conjecture: Given an $n + 1$ matrix which requires a minimum of 5 colors to be normalized, then there exists one fusion of regions such that the resulting $n \times n$ matrix requires 5 colors to be normalized.

The above, and this we believe is the merit of this paper, is void of any geometric concept and constraint: Given a matrix which requires 5 letters to be normalized, fuse two rows (in the sense of the preceding theorem -- without reference to geometry) such that the resulting matrix still requires 5 letters. The efforts of the author toward settling the above conjecture have not yet produced satisfactory results.

REFERENCES

1. Harary, F. (editor), Graph Theory and Theoretical Physics, Academic Press, New York, 1967.
2. Harary, F., Graph Theory, Addison Wesley, Reading, Mass., 1969.
3. Liu, C. L., Introduction to *Combinatorial* Mathematics, McGraw-Hill, New York, 1968.
4. Marshall, C. W., Applied Graph Theory, Wiley-Interscience, New York, 1971.
5. May, K. O., "The Origin of the Four Color Conjecture," Iris, 56, (1965), pp. 346-348.
6. Ore, O., The Four Color Problem, Academic Press, New York, 1967.
7. Ore, O. and Stemple, Joel, "Numerical Methods in the Four Color Problem," Journal of Combinatorial Theory, 8, (1970), pp. 65-78.

EULER'S CONSTANT AND EULER'S e

By Peter A. Lindstrom
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The sequence $\{a_n\}$ defined by

$$a_n = \sum_{i=1}^{n} \frac{1}{i} - \ln n$$

can be shown to converge by demonstrating that the sequence is (1) decreasing and (2) bounded below by 0. The value of this limit is often referred to as *Euler's* constant. Another constant associated with Euler is e. Euler introduced e to represent the base of the natural logarithm. One of the familiar properties of e is the following inequality:

$$\text{(A)} \qquad (1 + \tfrac{1}{i})^{i} < e < (1 + \tfrac{1}{i})^{i+1}, \qquad i = 1, 2, 3, \cdots.$$

The purpose of this note is to show that by using (A), one can show that the above sequence $\{a_n\}$ is convergent.

Since by (A)

$$e < (1 + \tfrac{1}{n})^{n+1},$$

then:

$$(1 + \tfrac{1}{n}) > e^{1/(n+1)}$$

$$\ln\left(\frac{1 + \frac{1}{n}}{e^{1/(n+1)}}\right) > 0,$$

$$\ln\left(\frac{n+1}{n} \cdot \frac{1}{e^{1/(n+1)}}\right) > 0,$$

$$\ln\left[\frac{\prod_{i=1}^{n} e^{1/i} / n}{\prod_{i=1}^{n+1} e^{1/i} / (n+1)}\right] > 0,$$

$$\ln\left(\frac{\prod_{i=1}^{n} e^{1/i}}{n}\right) > \ln\left(\frac{\prod_{i=1}^{n+1} e^{1/i}}{n+1}\right),$$

$$\left(\sum_{i=1}^{n} \ln e^{1/i} - \ln n\right) > \left(\sum_{i=1}^{n+1} \ln e^{1/i} - \ln (n+1)\right),$$

$$\left(\sum_{i=1}^{n} \frac{1}{i} - \ln n\right) > \left(\sum_{i=1}^{n+1} \frac{1}{i} - \ln (n+1)\right),$$

so that the sequence $\{a_n\}$ is decreasing for all n.

Since also by (A)

$$(1 + \tfrac{1}{i})^{i} < e,$$

then:

$$e^{1/i} > (1 + \tfrac{1}{i}) = \left(\frac{i+1}{i}\right),$$

$$\prod_{i=1}^{n} e^{1/i} > \prod_{i=1}^{n} \left(\frac{i+1}{i}\right) = n + 1.$$

Thus, the above is greater than n, so we have

$$\ln\left(\prod_{i=1}^{n} e^{1/i}\right) > \ln n,$$

$$\sum_{i=1}^{n} \ln e^{1/i} > \ln n,$$

$$\left(\sum_{i=1}^{n} \frac{1}{i} - \ln n\right) > 0,$$

and the sequence $\{a_n\}$ is bounded below by 0 for all n.

PARTIAL DIFFERENTIATION ON A METRIC SPACE[1]

by Roseann Moriello
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The "metric differentiability" of functions between abstract metric spaces is a relatively new and undeveloped field of mathematics. The metric derivative of a function with real domain was first defined in 1935 by W. A. Wilson (See [2]). In 1971, E. Braude arrived at the same definition after having researched the topic independently. Unlike his predecessor, Braude sought to develop his concepts of metric differentiability for functions with abstract metric domains, In this paper we explore the idea of partial differentiation for functions defined on the cartesian product of two metric spaces.

Partial differentiation on a metric space

Let f be a function from a metric space (X,s) into a metric space (Y,t) and let $b \bullet X$. f is said to be *metrically differentiable* at b if b is not discrete and if a real number $f^I(b)$ exists with the property that for every $\varepsilon > 0$, a positive number 5 can be found such that if $s(x', b) < \delta$, $s(x'', b) < \delta$ and $x' \neq x''$, then

$$\left| \frac{t(f(x'), f(x''))}{s(x', x'')} - f^I(b) \right| < \varepsilon$$

The assertion "f: $X \to Y$ is metrically differentiable" means that f is metrically differentiable at every point of X. (See [2].)

Proceeding almost directly from the definition of the metric derivative is the definition of the metric partial derivative. Let f be a function from a metric space $(X{\times}Y,p)$ into a metric space (Z,p') and let (x,y) be an element of $X{\times}Y$ for which every open ball containing (x,y) contains an element (x',y) with $x' \neq x$; f has a *metric partial derivat e* with respect to x at (x,y) if a real number $f_x^I(x,y)$ exists with the property that for every $\varepsilon > 0$, a positive number 5 can be found such that if $p(x', x) < 6$, $p(x'', x) < 6$ and $x' \neq x''$, then

$$\left| \frac{p'(f(x', y), f(x'', y)}{p((x', y), (x'', y))} - f_x^I(x, y) \right| < \varepsilon ,$$

and f has a metric partial derivative with respect to y at (x,y) if every open ball containing (x,y) contains an element (x,y') with $y \# y'$ and a real number $f_y^I(x,y)$ exists with the property that for every $\varepsilon > 0$, a positive number δ can be found such that if $p(y', y) < \delta$, $p(y'', y) < \delta$ and $y' \# y''$, then

$$\left| \frac{p'(f(x, y'), f(x, y'')}{p((x, y'), (x, y''))} - f_y^I(x, y) \right| < \varepsilon .$$

It follows from the work of Braude in [1] that the derivative of f with respect to x in the case $X = Y = Z = R$, where $p((x, y), (x', y')) = [(x - x')^2 + (y - y')^2]^{1/2}$, and $p'(z, a') = |z - z'|$, is equal to the absolute value of the ordinary partial derivative of f with respect to x. Similarly, the metric partial derivative of f with respect to y in this case is equal to the absolute value of the ordinary partial derivative with respect to y.

The Cauchy-Riemann equations are an important criterion for the differentiablility of a function from R^2 into R^2. A necessary and sufficient condition for $f(x + iy) = U(x, y) + iV(x, y)$ to be differentiable is that $U_x(x, y) = V_Y(x, y)$ and $U_Y(x, y) = -V_x(x, y)$ together with the continuity of U_x, U_y, V_x, and V_Y where U_x and V_X are the partial derivatives of U and V with respect to x, and U_Y and V_y are the partial derivatives of U and V with respect to y. The following theorems give conditions analogous to the Cauchy-Riemann equations which are necessary but not sufficient for a function f to be metrically differentiable.

Theorem 1: Let X be a metric space with metric p and let g be the metric on $X{\times}X$ defined by

$$g((x', y'), (x'', y'')) = [p(x', x'')^2 + p(y', y'')^2]^{1/2}.$$

Furthermore, let f: $(X{\times}X,g) \to (X{\times}X,g)$ with $f(x, y) = (U(x, y), V(x, y))$ where U, V: $X{\times}X \to X$. If f is metrically differentiable at (x, y) and U_x, U_y, V_x, and V_Y exist, then $U_x^I(x, y)^2 + V_x^I(x, y)^2 = U_y^I(x, y)^2 + V_y^I(x, y)^2$.

Proof: Since the quantity

$$\frac{g(f(x', y'), f(x'', y''))}{g((x', y'), (x'', y''))}$$

gets arbitrarily close to $f^I(x, y)$ for (x', y') and (x'', y'') sufficiently

[1]One of several papers produced as a result of a research project funded by the Research Corporation under the Cottrell College Science Program, grant number C-205/308, directed by Professor E. J. Braude, Seton Hall University.

close to (x, y), it can be said that the quantity

$$\frac{g(f(x', y), f(x'', y))}{g((x', y), (x'', y))}$$

gets arbitrarily close to $f^I(x, y)$ for x' and x'' sufficiently close to x. Thus

$$f^I(x, y) = \lim_{\substack{(x', y') \to (x, y) \\ (x'', y'') \to (x, y)}} \frac{g(f(x', y'), f(x'', y''))}{g((x', y'), (x'', y''))}$$

$$= \lim_{\substack{x' \to x \\ x'' \to x}} \frac{g(f(x', y), f(x'', y))}{g((x', y), (x'', y))}$$

$$= \lim_{\substack{x' \to x \\ x'' \to x}} \frac{g((U(x', y), V(x', y)), U(x'', y), V(x'', y)))}{g((x', y), (x'', y))}$$

$$= \lim_{\substack{x' \to x \\ x'' \to x}} \frac{\sqrt{p(U(x', y), U(x'', y))^2 + p(V(x', y), V(x'', y))^2}}{\sqrt{p(x', x'')^2 + p(y, y)^2}}$$

$$= \sqrt{\left[\lim_{\substack{x' \to x \\ x'' \to x}} \frac{p(U(x', y), U(x'', y))}{p(x', x'')}\right]^2 + \left[\lim_{\substack{x' \to x \\ x'' \to x}} \frac{p(V(x', y), V(x'', y))}{p(x', x'')}\right]^2}$$

$$= [U_x^I(x, y)^2 + V_x^I(x, y)^2]^{1/2}.$$

Similarly, $f^I(x, y) = [U_y^I(x, y)^2 + V_y^I(x, y)^2]^{1/2}$. Hence,

$$U_x^I(x, y)^2 + V_x^I(x, y)^2 = U_y^I(x, y)^2 + V_y^I(x, y)^2.$$

In order to extend the ideas of Theorem 1 to abstract metric spaces, it is necessary to introduce the following lemma.

Lemma 1: Let (X, p) be a metric space and let h be such that (1) $h: R^+ \times R^+ \to R^+$, (2) $h(s, t) = 0$ if and only if $s = t = 0$, (3) $h(b+c, e+f) \leq h(b, e) + h(c, f)$, and (4) if $a \leq b$ and $c \leq d$, then $h(a, c) \leq h(b, d)$. Furthermore, let g be a mapping from $X \times X$ into R^+ defined by

$$g((x', y'), (x'', y'')) = h(p(x', x''), p(y', y'')).$$

Then g is a metric on $X \times X$.

Proof: We make the following observations:

(1) The domain of g is $(X \times X) \times (X \times X)$.

(2) The range of g is a subset of $[0, \infty)$.

(3) The following is true:

(a) $g((x', y'), (x', y')) = h(p(x', x'), p(y', y')) = h(0, 0) = 0$,

(b) $g((x', y'), (x'', y'')) = h(p(x', x''), p(y', y'') > 0$, if $(x', y') \neq (x'', y'')$,

(c) $g((x', y'), (x'', y'')) = h(p(x', x''), p(y', y'')) = h(p(x'', x'), p(y'', y')) = g((x'', y''), (x', y'))$,

(d) $g((x', y'), (x'', y'')) = h(p(x', x''), p(y', y'')) \leq h((p(x', x) + p(x, x'')), (p(y', y) + p(y, y''))) \leq h(p(x', x), p(y', y)) + h(p(x, x''), p(y, y'')) = g((x', y'), (x, y)) + g((x, y), (x'', y''))$; therefore $g((x', y'), (x'', y'') \leq g((x', y'), (x, y)) + g((x, y), (x'', y''))$.

Hence g is a metric by definition.

The restrictions placed on g and h by the preceding lemma allow for a more general statement of Theorem 1.

Theorem 2: Let (X, p) be a metric space, and let h be a continuous function of $R^+ \times R^+$ into R^+ satisfying the conditions on h of Lemma 1, as well as the conditions $h(a, 0) = a$, $h(0, b) = b$, and $ch(a, b) = h(ca, cb)$ for every c, a, and b in R^+. Let g be the metric on $X \times X$ defined in Lemma 1 by $g((x', y'), (x'', y'') = h(p(x', x''), p(y', y''))$. Then for every metrically differentiable function $f: (X \times X, g) \to (X \times X, g)$ given by $f(x, y) = (U(x, y), V(x, y))$ for which U_x^I, V_x^I, U_y^I, and V_u^I exist, $h(U_x^I(x, y), V_x^I(x, y)) = h(U_y^I(x, y), V_y^I(x, y))$.

Proof:

$$f^I(x, y) = \lim_{\substack{(x', y') \to (x, y) \\ (x'', y'') \to (x, y)}} \frac{g(f(x', y'), f(x'', y''))}{g((x', y'), (x'', y''))}$$

$$= \lim_{\substack{x' \to x \\ x'' \to x}} \frac{g(f(x', y), f(x'', y))}{g((x', y), (x'', y))}$$

$$= \lim_{\substack{x' \to x \\ x'' \to x}} \frac{g((U(x', y), V(x', y)), (U(x'', y), V(x'', y)))}{g((x', y), (x'', y))}$$

$$= \lim_{\substack{x' \to x \\ x'' \to x}} \frac{h(p(U(x', y), U(x'', y)), p(V(x', y), V(x'', y)))}{h(p(x', x''), p(y, y))}$$

$$= \lim_{\substack{x'\to x\\ x''\to x}} \frac{h(p(U(x', y), U(x'', y)), p(V(x', y), V(x'', y)))}{h(p(x', x''), 0)}$$

$$= \lim_{\substack{x'\to x\\ x''\to x}} \frac{h(p(U(x', y), U(x'', y)), p(V(x', y), V(x'', y)))}{p(x', x'')}$$

$$= \lim_{\substack{x'\to x\\ x''\to x}} h\left[\frac{p(U(x', y), U(x'', y))}{p(x', x'')}, \frac{p(V(x', y), V(x'', y))}{p(x', x'')}\right]$$

$$= h\left[\lim_{\substack{x'\to x\\ x''\to x}} \frac{p(U(x', y), U(x'', y))}{p(x', x'')}, \lim_{\substack{x'\to x\\ x''\to x}} \frac{p(V(x', y), V(x'', y))}{p(x', x'')}\right]$$

$$= h(U_x^I(x, y), V_x^I(x, y)),$$

since h is continuous and the two limits exist by hypothesis. Similarly,

$$f^I(x, y) = h(U_y^I(x, y), V_y^I(x, y)).$$

Therefore,

$$h(U_x^I(x, y), V_x^I(x, y)) = h(U_y^I(x, y), V_y^I(x, y)).$$

The above theorem gives necessary but not sufficient conditions for determining the metric differentiability of a function.

Remark 1: *Theorem 1 is a special case of Theorem 2 in which* $h(s, t) = \sqrt{s^2 + t^2}$. The *function* h *satisfies the conditions of Lemma 1:*

(1) $h\colon R^+ \times R^+ \to R^+$

(2) $h(s, t) = \sqrt{s^2 + t^2} = 0$ *if and only if* $s = t = 0$

(3) *The following statements are equivalent for any* b, c, e *and* f in R^+

$(bf - ec)^2 \geq 0$; $2(bf)(ec) \leq (bf)^2 + (ec)^2$; $bc + ef \leq \sqrt{b^2c^2 + e^2f^2 + e^2c^2 + b^2f^2}$; $(b + c)^2 + (e + f)^2 \leq [\sqrt{b^2 + e^2} + \sqrt{c^2 + f^2}]^2$; $\sqrt{(b + c)^2 + (e + f)^2} \leq \sqrt{b^2 + e^2} + \sqrt{c^2 + f^2}$;

$h(b + c, e + f) \leq h(b, e) + h(c, f)$.

(4) *If* $a \leq b$ *and* $c \leq d$, *then* $a^2 + c^2 \leq b^2 + d^2$, *and so* $\sqrt{a^2 + c^2} \leq \sqrt{b^2 + d^2}$ (i.e. $h(a, c) \leq h(b, d)$.)

It is an elementary fact that h is continuous. The equations $h(a, 0) = a$ and $h(0, b) = b$ *are clear.* Finally, *if* $c \in R^+$, *then for any* (a, b) *in* $R^+ \times R^+$, $ch(a, b) = c\sqrt{a^2 + b^2} = \sqrt{(ca)^2 + (cb)^2} = h(ca, cb)$.

Remark 2: *Now consider the function* $h\colon R^+ \times R^+ \to R^+$ *defined by* $h(s, t) = \max(s, t)$.

(a) *Since* $h^{-1}(a, b) = (a, b) \times (a, b)$ *for every open interval* (a, b) in R^+, *and* $h^{-1}[0, b) = [0, b) \times [0, b)$ *for every* $b > 0$, *h is continuous.*

(b) *h satisfies the conditions of Lemma 1:*

(1) $h\colon R^+ \times R^+ \to R^+$

(2) $\max(s, t) = 0$ *if and only if* $s = t = 0$

(3) $h(b, e) + h(c, f) = \max(b, e) + \max(c, f) \geq b + c$.

Similarly, $h(b, e) + h(c, f) \geq e + f$.

Thus, $h(b, e) + h(c, f) \geq \max(b + c, e + f) = h(b + c, e + f)$.

(4) $a \leq b$ *and* $c \leq d$ *imply* $\max(a, c) \leq \max(b, d)$. Now, $c(\max(a, b)) = \max(ca, cb)$ *since* "$d \geq k$" *and* "$cd > ck$" *are equivalent for* $c \geq 0$. The *equations* $h(a, 0) = a$ *and* $h(0, b) = b$ *are clear.*

Applying Theorem 2 to the function h of Remark 2 gives the following result.

Corollary: *Let X be a metric space with* metric p *and let* g *be the metric on* $X\times X$ *given by* $g((x', y'), (x'', y'')) = \max[p(x', x''), p(y', y'')]$. Furthermore, *let* $f\colon(X\times X, g) \to (X\times X, g)$ *with* $f(x, y) = (U(x, y), V(x, y))$ *where* $U, V\colon X\times X \to X$ *and* U_x^I, V_x^I, U_y^I, *and* V_Y^I *exist. If* f *is metrically* differentiable *at* (x, y), *then*

$$\max [U_x^I(x, y), V_x^I(x, y)] = \max [U_y^I(x, y), V_y^I(x, y)].$$

REFERENCES

1. Braude, *E., "A Metric Space* Derivative," *American Mathematical Society Notices* 20, (1973), p. A608.

2. Dugundji, *J. Topology*, Allyn *and Bacon, Boston, 1965.*

A SUGGESTION FOR ENJOYABLE READING

Professor Robert W. Prielipp of the University of Wisconsin, Oshkosh, highly recommends three relatively little known books to *Journal* readers.

Fantasia Mathematica, assembled and edited by Clifton Fadiman, Simon and Schuster, Inc., New York, 1958 (Paperback). Noteworthy selections in this collection of short stories and poems are:

(a) *Young Archimedes*, by Aldous Huxley -- the difficulties faced by a young Italian peasant boy who seems to be a mathematical genius.

(b) *The Devil and Simon Flagg*, by Arthur Porges -- a mathematical version of Stephen Vincent Benet's *The Devil and Daniel Webster*.

(c) *--And He Built a Crooked House*, by Robert A. Heinlein -- unusual phenomena in a house having dimension higher than three.

(d) *A Subway Named Moebius*, by A. J. Deutsch -- strange things happen to the Boston subway when the Boylston shuttle is installed.

The Mathematical Magpie, assembled and edited by Clifton Fadiman, Simon and Schuster, Inc., New York, 1962 (Paperback). Choice selections in this anthology are:

(a) *The Law*, by Robert M. Coates -- unlikely events become the rule.

(b) *The Appendix and the Spectacles*, by Miles J. Brown, M.D. -- a mathematician uses the fourth dimension to get even with a bank president.

(c) *The Nine Billion Names of God*, by Arthur C. Clarke -- Tibeton monks try to use a computer to set down all the possible names of God.

(d) *Milo and the Mathemagician*, by Norton Juster -- the adventures of a little boy and a dog (who ticks) in Digitopolis.

Whom the Gods Love (The Story of Evariste Galois), by Leopold Infeld, McGraw-Hill Book Company, Inc., New York, 1948 -- the moving story of a young mathematical genius who is caught up in the struggle for a new French republic in the days after Napoleon's defeat during the reign of the Bourbons, His father commits suicide, his mathematical accomplishments go unrecognized during his lifetime, and at age twenty he is killed in a duel resulting from a love affair plotted by his political enemies.

WELCOME TO NEW CHAPTERS

The *Journal* extends its welcome to the following new chapters of Pi Mu Epsilon which were recently installed:

ALABAMA EPSILON at Tuskegee Institute, installed by Houston T. Karnes, Council President.

CALIFORNIA IOTA at the University of Southern California, installed by J.C. Eaves, past Council President.

GEORGIA GAMMA at Armstrong State College, installed April 2, 1974 by Houston T. Karnes.

ILLINOIS EPSILON at Northern Illinois University, installed by J. Sutherland Frame, past Council President.

ILLINOIS ZETA at Southern Illinois University, installed June 1, 1973 by Houston T. Karnes.

KENTUCKY BETA at Western Kentucky University, installed by J.C. Eaves.

LOUISIANA KAPPA at Louisiana Tech, installed May 9, 1973 by Houston T. Karnes.

MICHIGAN DELTA at Hope College, installed by J. Sutherland Frame.

NEBRASKA BETA at Creighton University, installed April 25, 1973 by R.V. Andree, Council Secretary-Treasurer.

PENNSYLVANIA MU at the University of Scranton, installed by Eileen L. Poiani, Councilor.

SOUTH CAROLINA BETA at Clemson University, installed March 27, 1973 by Houston T. Karnes.

LOCAL CHAPTER AWARDS WINNERS

The *Journal* inadvertently omitted the CALIFORNIA ETA (University of Santa Clara) report of their annual awards in the last issue. The *Award for Excellence* for 1973 was presented to

Karen A. Moneta,

and the *George W. Evans II Memorial Prize* for highest placing in the William Lowell Putnam Mathematical Competition was received by

Paul N. Ilacqua
Kathleen M. Daly.

The winner of the *Freshman Mathematics Prize* for 1973 was

James L. Hafner.

A FRIENDLY REMINDER

We are no longer printing the long initiate lists in the *Journal*, but we do print the names of those who have distinguished themselves in mathematics in their local chapters. *PLEASE SEND US THE NAMES OF YOUR AWARDS WINNERS* and let your exceptional chapter members (or local students) will receive the recognition they deserve.

In connection with this, remember that the National Office can supply you with impressive award certificates. Write to:

R. V. Andree
Secretary-Treasurer, Pi Mu Epsilon
601 Elm Avenue, Room 423
The University of Oklahoma
Norman, Oklahoma 73069

NO ANNUAL MEETING THIS SUMMER

Because the annual summer meeting of the Mathematical Association of America will not be held this year, Pi Mu Epsilon also will not have its meeting. The Council urges as many local chapters as possible to make plans to send delegates to regional meetings of the MAA in their area instead. Many of these regionals include sessions for undergraduate papers, so be aware of this opportunity for your members.

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if mailable, and by any information that will assist the editor.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of November 1974.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

314. *Proposed by J. A. H. Hunter, Toronto, Canada.*

Show that

$$\frac{\sin^2 45^\circ - \sin^2 15^\circ}{\sin^2 30^\circ - \sin^2 10^\circ} = \frac{\sin 80^\circ}{\sin 30^\circ}$$

315. *Proposed by Charles W. Trigg, San Diego, California.*

One type of perpetual calendar consists of two white plastic cubes resting on a tilt-back base. On each face of each cube is a single digit. The digits are so distributed that the cubes can exhibit any date from 01 to 31 on their front faces.

Could this type of calendar be constructed if a base of numeration smaller than ten were employed?

316. *Proposed by Zazou Katz, Beverly Hills, California.*

If you were marooned on a desert island without a calculator or tables of trigonometric functions, how would you go about determining which is greater:

$$2\tan^{-1}(\sqrt{2}-1) \quad \text{or} \quad 3\tan^{-1}(1/4)+\tan^{-1}(5/99)\ ?$$

317. *Proposed by the Editor of the Problem Department.*

A rectangle $ADEB$ is constructed externally on the hypotenuse AB of a right triangle ABC (Fig. 1). The lines CD and CE intersect the line AB in the points F and G respectively. a) If $DE = AD\sqrt{2}$, show that $AG^2 + FB^2 = AB^2$. b) If $AD = DE$, show that $FG^2 = AF \cdot GB$.

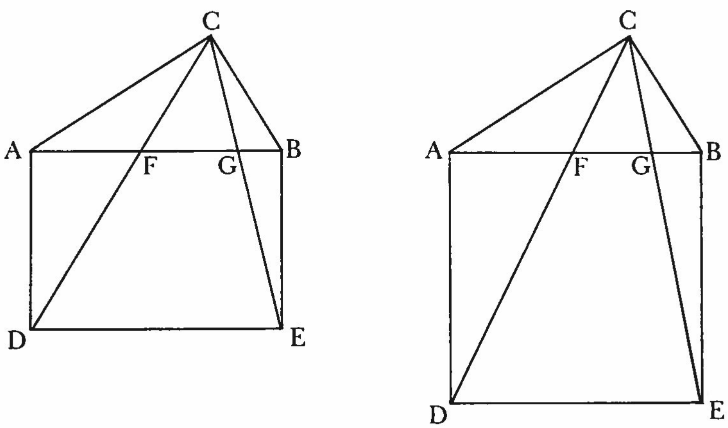


FIGURE 1

318. *Proposed by R. Robinson Rowe, Sacramento, California.*

Two equal cylindrical tanks, Tank A above Tank B, have equal orifices in their floors, capable of discharging water at the rate of $13\sqrt{h}$ gallons per minute, where h is the depth of water in feet. At 10:20 a.m. Tank B is empty and water is 10 feet deep in Tank A, as discharge begins. At noon Tank A is just emptied. What was the maximum depth in Tank B, and when? How deep is the water in Tank B at noon, and when will it be empty?

319. *Proposed by Professor M. S. Longuet-Higgins, Cambridge, England.*

Let A', B', C' be the images of an arbitrary point in the sides BC, CA, AB of a triangle ABC. Prove that the 4 circles $AB'C'$, $BC'A'$, $CA'B'$, ABC are all concurrent.

320. *Proposed by H. S. M. Coxeter, Toronto, Canada.*

Prove that the projectivity $ABC \barwedge BCD$ (for 4 collinear points) is of the period 4 if and only if $H(AC, BD)$.

321. *Proposed by Nosmo King, Raleigh, North Carolina. (Dedicated to the memory of Leo Moser).*

According to Merten's Theorem

$$\prod_{p\le n}\left(1-\frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\log n}$$

where γ denotes Euler's constant, (0.57721...) and where the product on the left is taken over all primes not exceeding n. (See Hardy and Wight, *The Theory of Numbers*, p. 351 or Trygve Nagell's *Introduction to Number Theory*, p. 298). Can you estimate

$$\prod_{p\le n}\left(1-\frac{2}{p}\right)\ ?$$

322. *Proposed by Jack Garfunkel, Forest Hills High School, New York.*

It is known that the ratio of the perimeter of a triangle to the sum of its altitudes is greater than or equal to $2/\sqrt{3}$. (See *American Mathematical Monthly*, Problem E 1427, 1961, pp. 296-297). Prove the stronger inequality for the internal angle bisectors t_a, t_b and t_c:

$$2(t_a + t_b + t_c) \le \sqrt{3}(a + b + c)$$

equality holding if and only if the triangle is equilateral.

323. *Proposed by David L. Silverman, Los Angeles, California.*

Call plane curves such as the circle of radius 2, the square of side 4, or the 6 × 3 rectangle in Fig. 2 *isometric* if their perimeter is numerically equal to the area they enclose. What is the maximum area that can be enclosed by an isometric curve?

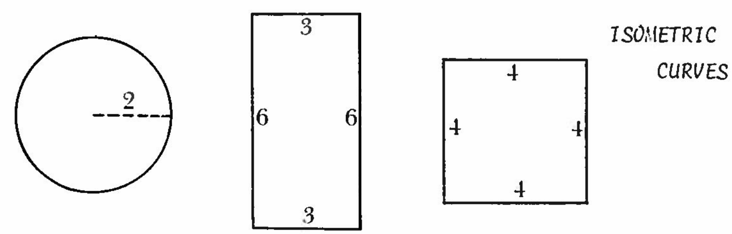


FIGURE 2

324. *Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Evaluate

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \frac{j}{j+1}}{n}$$

325. *Proposed by Charles W. Trigg, San Diego, California.*

Show that there is only one third-order magic square with positive prime elements and a magic constant of 267.

Solutions

292. [Spring 1973] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

If perpendiculars are constructed at the points of tangency of the incircle of a triangle and extended outward to equal length:, then the ioins of their endpoints form a triangle perspective with the given triangle.

Solution by Clayton W. Dodge, University of Maine at Orono.

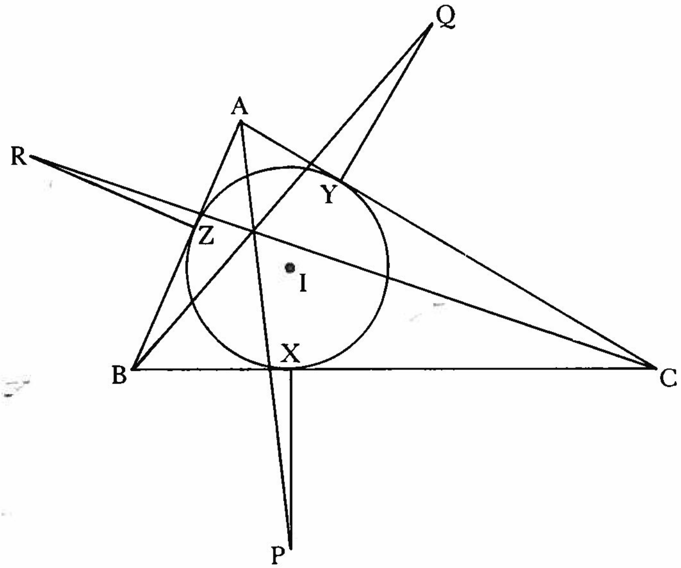


FIGURE 3

Let the incircle touch sides BC, CA, AB of triangle ABC at points X, Y, Z, (see Fig. 3) and let the other ends of the constructed equal perpendiculars be P, Q, R, respectively. It is well known that points A, B, C and X, Y, Z are in perspective at the Gergonne point for triangle ABC. But also triangles XYZ and PQR are in perspective at point I, the incenter for triangle-ABC. It is also known that perspective is transitive; that since triangles ABC and XYZ are in perspective and triangles XYZ and PQR are in perspective, then triangles ABC and PQR are in perspective also.

Also solved by ZAZOU KATZ, *Beverly Hills , California, and the Proposer.*

Problem Editor's Note

Two recent texts describing the Gergonne point and other notable points of the triangle are: (1) David C. Kay, *College* Geometry, Holt, Rinehart and Winston, Inc., 1969; (2) Clayton W. Dodge, *Euclidean* Geometry and Transformations, Addison-Wesley Publishing Company, Inc., 1972.

293. [Spring 1973] *Proposed by Lew Kowarski, Morgan State College, Baltimore, Maryland.*

Prove that $N = 53^{103} + 103^{53}$ is divisible by 78.

I. *Solution by R. Robinson Rowe, Sacramento, California.*

Since $53 \equiv -1 \pmod 6$ and $103 \equiv 1 \pmod 6$, we have $53^{103} \equiv (-1)^{103} \pmod 6$ $\equiv -1 \pmod 6$ and $103^{53} \equiv 1 \pmod 6$.

Also, since $53 \equiv 1 \pmod{13}$ and $103 \equiv -1 \pmod{13}$, we have $53^{103} \equiv 1 \pmod{13}$ and $103^{53} \equiv -1 \pmod{13}$.

Hence N is divisible by both 6 and 13, which are relatively prime. Therefore N is divisible by $6 \cdot 13 = 78$.

Similar solutions were offered by MERYL J. ALTABET, *Bronx, N. Y.;* RAYMOND E. ANDERSON? *Montana State University, Bozeman, Montana;* DONALD CASCI, *Rhode Island College, Providence, Rhode Island;* RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* DONNELLY JOHNSON, *Major, USAF, Air Force Institute of Technology, Fairborn, Ohio;* BRUCE LOVETT, *Rutgers College N u Brunswick, N. J.;* T. E. MOORE, *Bridgewater State College, Bridgewater, Mass.;* THERESA PRATT, *N. Easton, Mass.;* PAOLO RANALDI, *Akron, Ohio;* MICHAEL SCHWARZSCHILD, *Polytechnic Institute of Brooklyn.*

II. *Solution by Charles W. Trigg, San Diego, California.*

Examining the congruences of 53^{103} and 103^{53} for the moduli 3, 4, 7; 13 and 25, it is seen that N is divisible not only by 78 = 2(3)(13) but also by any combination of the factors 2^2, 3, 5^2, 7, and 13.

Solutions recognizing this extension of the proposed problem were offered by CLAYTON W. DODGE, *University of Maine and by* DAVID L. SILVERMAN, *Los Angeles, California.*

111. *Solution by Frank Massimo and Mark Yankovich, Juniors at Drexel University, Philadelphia, Pennsylvania.*

Expansion of $(78 - 25)^{103}$ and $(78 + 25)^{53}$ by the binomial theorem shows that each term contains a factor of 78 except $(-25)^{103}$ and 25^{53}. Therefore, if $25^{53} - 25^{103}$ is divisible by 78 then so is N. Rewrite $25^{53} - 25^{103}$ as $-(25)^{53}(25^{50} - 1)$, which equals $-(25)^{53}(625^{25} - 1)$. It is known that $(625^{25} - 1)$ is divisible by $(625 - 1) = 8 \times 78$. Therefore 78 divides $25^{53} - 25^{103}$ and also $53^{103} + 103^{53}$.

Other solutions involving the binomial theorem were offered by HYMAN CHANSKY, *University of Maryland;* STANN CHONOFSKY, *Worcester Polytechnic Institute, Worcester, Mass.;* JACK GIAMMERSE, *Louisiana State University, Baton Rouge, Louisiana;* PETER A. LINDSTROM, *Genesee Community College, Batavia, New York;* T. PAUL TURIEL, *SUNY at Binghamton, New York;* CHARLES W. TRIGG, *San Diego, California; and the Proposer.*

Comment

Generalizations of the problem were submitted by Theodore Jungreis, Brooklyn, New York and by Bob Prielipp, University of Wisconsin at Oshkosh: If $p = a_1m + 1 = a_2n - 1$ and $q = a_3m - 1 = a_4n + 1$, and p and q are odd, it is found that $p^q + q^p$ is divisible by $2mn$.

294. [Spring 1973] *Proposed by Charles W. Trigg, San Diego, California.*

In Fig. 4 show that *ABCD* is a square.

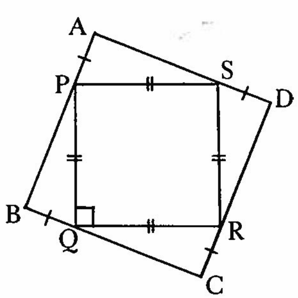


FIGURE 4

Comment by Clayton W. Dodge, University of Maine at Orono.

The solution of the analogous problem for nested equilateral triangles appeared in the November 1970 issue of the *Mathematics Magazine*, Problem 754, pages 280-281. Comment by K. R. S. Sastry, Makele, Ethiopia, in the November 1971 issue indicates that the method of the solution by Michael Goldberg holds for any regular polygon. Hence it holds for a square, proving problem 294.

Solutions were also offered by ZAZOU KATZ, *Beverly Hills, California;* ALFRED E. NEUMAN, *Mu Alpha Delta Fraternity;* R. ROBINSON ROWE, *Sacramento, California; and the Proposer.*

295. [Spring 1973] *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Determine an equation of a regular dodecagon (the extended sides are not to be included).

Solution by the Proposer.

The equation

$$\sum_{n=1}^{N} \{|a_nx - b_n| + |c_ny_n - d_n|\} = \lambda$$

represents a $4N$-gon. By symmetry we try

$$|x - 1| + |x + 1| + |y - 1| + |y + 1| + a(|x| + |y|) = \lambda .$$

After some elementary calculations,

$$a = \sqrt{3} - 1 \qquad \lambda = 2(2 + \sqrt{3}) .$$

296. [Spring 1973] *Proposed by Solomon W. Golomb, University of Southern California, Department of Electrical Engineering.*

1) Combine 2, 5, and 6 to make four 2's.

2) Combine 2, 5, and 6 to make four 4's.

3) Combine 2, 5, and 6 to make four 5's.

4) Combine 2, 5, and 6 to make four 7's.

5) Combine 1, 5, and 6 to make four 7's.

Solution by Charles W. Trigg, San Diego, California.

In general,

$6 + 2 - 5 = 3 = (a + a + a)/a;$

$2(6 - 5) = 2 = a/a + a/a = 6 + 1 - 5;$

$2 + 5 - 6 = 1 = (a/a)(a/a) = aa/aa = a/a + a - a = a^a/a^a = 1(6 - 5);$

$6 - 5 - 1^2 = 0 = a/a - a/a = a + a - a - a = aa - aa = a^a - a^a =$

$6 - 5 = 1;$

where a is any positive digit, including 2, 4, 5 and 7. (The symbol $!x$ indicates *sub-factorial* x. For example: $!1 = 0$, $!2 = 1$, and $!5 = 44$.)

Attention is called to the solution of Problem E861, American Mathematical Monthly, January 1950, page 35, where Vern Hoggatt and Len Moser show that every integer may be expressed by using p a's and a finite number of operator symbols (including log) used in high school texts; $p > 3$ and $a > 1$ are integers.

For other representations where a has a specific value, space is conserved by first listing certain non-negative integers with their representations by 2, <, and 6. Namely:

$0 = (6 + 5)[!(!2)]$
$1 = (6/2)! - 5$
$2 = 5 - 6/2$
$3 = 5 - \sqrt{6 - 2}$
$4 = 2(5) - 6$
$5 = (.5)(6) + 2$
$6 = 2(.5)(6)$
$7 = 2(6) - 5$
$8 = 5 + 6/2$
$9 = 6 + 5 - 2$
$10 = 6 + 2/(.5)$
$11 = (6/2)! + 5$
$12 = 2(6!)/(5!)$

$13 = 2 + 5 + 6$
$15 = 5(6)/2$
$16 = 2(5) + 6$
$18 = 6(5 - 2)$
$20 = 5(6 - 2)$
$21 = 26 - 5$
$22 = 2(6 + 5)$
$24 = 2(6)/(.5)$
$25 = 5(6 - !2)$
$26 = 2^5 - 6$
$28 = 6(5) - 2$
$30 = 5(6)(!2)$
$32 = 2 + 5(6)$

$35 = 5(6 + !2)$
$36 = (6!/5!)^2$
$40 = 5(2 + 6)$
$42 = 6(2 + 5)$
$44 = (!5)(!\sqrt{6 - 2})$
$46 = !5 + \sqrt{6 - 2}$
$48 = !5 + 6 - 2$
$49 = !5 + 6 - !2$
$50 = (!5)(!2) + 6$
$64 = 2^{(6!)/(5!)}$
$88 = (!5)\sqrt{6 - 2}$
$625 = 5^{(6 - 2)}$

There are many alternate simple representations of these integers by 2, 5, and 6, as well as representations of many other integers.

1) The following representations using four 2's can be matched against the previous representations using 2, 5, and 6:

$0 = (2 + 2)/2 - 2$
$1 = 2^2/(2)(2)$
$2 = 2 + 2 - 2(!2)$
$3 = 2 + 2 - 2/2$
$4 = 2(2)(2)/2$
$5 = 2 + 2 + 2/2$
$8 = 2 + 2 + 2 + 2$
$10 = 2(2)(2) + 2$
*$11 = (22/2)(!2)$
$12 = (2 + 2 + 2)(2)$

$16 = (2 + 2)(2 + 2)$
*$18 = 22 - 2 - 2$
$20 = (2/.2)\sqrt{(2)(2)}$
*$22 = 22(2)/2$
$36 = (2 + 2 + 2)^2$
*$44 = 22 + 22$
*$46 = 2 + 2(22)$
$64 = [(2)(2)(2)]^2$
*$88 = 2(2)(22)$

2) All of the integers in 1), except those marked by an asterisk (*), can be represented properly by replacing each 2 with $\sqrt{4}$. Other representations by four 4's are:

$7 = 4 + 4 - 4/4$
$8 = (4 + 4)(4/4)$
$9 = 4 + 4 + 4/4$
$12 = 4 + 4 + \sqrt{4} + \sqrt{4}$
$15 = 4(4) - 4/4$
$16 = 4 + 4 + 4 + 4$

$24 = 4(4) + 4 + 4$
$32 = 4(4)(4)/\sqrt{4}$
$40 = 4\sqrt{(4)(4)}/(.4)$
$42 = 4(4)/(.4) + \sqrt{4}$
$44 = 4(4)/(.4) + 4$
$64 = (4 + 4)(4 + 4)$

3) Some representations of certain of the integers in the first list given above but using four 5's are:

$3 = 5 - (5 + 5)/5$
$6 = 5/5 + \sqrt{5(5)}$
$7 = 5 + \sqrt{5 - 5/5}$
$9 = 5 + 5 - 5/5$
$10 = (5 + 5)(5/5)$
$11 = 5 + 5 + 5/5$
$15 = (5)(5) - 5 - 5$

$20 = 5 + 5 + 5 + 5$
$24 = (5)(5) - 5/5$
$25 = 5(5)(5/5)$
$26 = (5)(5) + 5/5$
$30 = 5(5 + 5/51$
$35 = 5 + 5 + 5(5)$
$625 = 5(5)(5)(5)$

4) Some representations of certain of the integers in the first list given above but using four 7's are:

$4 = 77/7 - 7$
$7 = 7 + 7 - \sqrt{7(7)}$
$9 = (7 + 7)/7 + 7$
$13 = 7 + 7 - 7/7$
$15 = 7 + 7 + 7/7$
$18 = 77/7 + 7$

$21 = 7 + 7 + \sqrt{7(7)}$
$28 = 7 + 7 + 7 + 7$
$48 = 7(7) - 7/7$
$49 = 7(7)(7/7)$
$50 = 7(7) + 7/7$

Using match sticks or toothpicks and rearranging:

II + V + VI → 7 − 7 − 7 − 7

5) Combining 1, 5, and 6 to make four 7's:

$6/\sqrt{5 - 1} = 3 = (7 + 7 + 7)/7$
$(6!)/(5!) - 1 = 5 = 7 - (7 + 7)/7$
$\sqrt{6(5 + 1)} = 6 = \sqrt{7(7)} - 7/7$
$(6!)/(5!) + 1 = 7 = (7/7)\sqrt{7(7)}$
$6 + \sqrt{5 - 1} = 8 = 7/7 + \sqrt{7(7)}$

$(6 + 1)/(.5) = 14 = (7 + 7)(7/7)$
$(5!)(.1) + 6 = 18 = 77/7 + 7$
$16 + 5 = 21 = 7 + 7 + \sqrt{7(7)}$
$!5 - 16 = 28 = 7 + 7 + 7 + 7$
$61 - 5 = 56 = 7(7 + 7/7)$

Using toothpicks and rearranging:

I + V + VI → 77 − 7 − 7

Ncte: In each of the five cases, the three unlike digits have been combined so as to equal the *sum* of the four like digits.

Problem Editor's Comment

The multitude of solutions listed below, in some cases duplicating representations already listed in Trigg's solution, display the ingenious variety accomplished by the following solvers: CLAYTON W. DODGE, *University of Maine at Orono;* JOAN INNES, *Creighton University, Omaha, Nebraska;* RICK JOHNSON, *Burroughs Corporation, Wilmington, N. C.;* BRUCE LOVETT, *Rutgers College, New Brunswick, N. J.;* JIM METZ, *Springfield, Illinois;* T. E. MOORE, *Bridgewater State College, Bridgewater, Massachusetts;* BOB PRIELIPP, *The University of Wisconsin, Oshkosh;* R. ROBINSON ROWE, *Sacramento, California; and the Proposer.*

The potpourri of solutions submitted by the above-named solvers is as follows:

1) $2 \cdot 5 + 6 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^2 \cdot 2^2$

$6/2 + 5 = 2 + 2 + 2 + 2$

$256 = [(2^2)^2]^2$; $2^{(.5 \times 6)} = 2 + 2 + 2 + 2 = 2^2 + 2^2$

$[52/6] = -[-6^2/5] = 2 + 2 + 2 + 2$,

where brackets Indicate the greatest integer function.

2) $\sqrt{256} = 2 \cdot 5 + 6 = 4 + 4 + 4 + 4$

$256 = 4 \cdot 4 \cdot 4 \cdot 4$; $2 \cdot 5 \cdot 6 = 4 \cdot 4 \cdot 4 - 4$;

$5 \cdot 6^2 = 4 \cdot 44 + 4$.

3) $5(6 - 2) = 5 + 5 + 5 + 5$; $5^{6-2} = 5 \cdot 5 \cdot 5 \cdot 5$;

$625 = 5 \cdot 5 \cdot 5 \cdot 5$.

4) $56/2 = 7 + 7 + 7 + 7$; $65 - 2 = 7 \cdot 7 + 7 + 7$;

$5 \cdot 6 - 2 = 7 + 7 + 7 + 7$; $5 + 6 - 2 = (7 + 7)/7 + 7$;

$2 \cdot 5 - 6 = 77/7 - 7$.

5) $1(6 - 5) = 77/77$; $(1 + 5)/6 = 77/7$; $1 + 5 + 6 = (77 + 7)/7$;

$5(6 + 1) = 7 \cdot 7 - (7 + 7)$; $6^5 + 1 = 7777$.

297. [Spring 1973] *Proposed by Roger E. Kuel, Kansas City, Missouri.*

A traffic engineer is confronted with the problem of connecting two non-parallel straight roads by an S-shaped curve formed by arcs of two equal tangent circles, one tangent to the first road at a selected point and the other touching the second road at a given point.

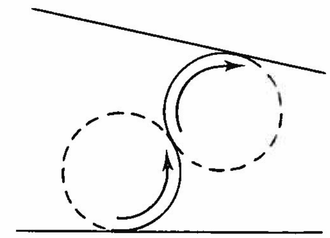


FIGURE 5

1) Determine the radius of the equal circles synthetically, trigonometrically or analytically.

2) If the figure lends itself to an Euclidean construction, how would one go about it?

Problem Editor's Note

Because of the complexity and difficulty of this problem the deadline for consideration of solutions is hereby extended to November 30, 1974. A synthetic solution would be particularly welcome.

298. [Spring 1973] *Proposed by Paul Erdös, Budapest, , and Jan Mycielski, University of Colorado, Boulder, Colorado.*

Prove that

$$1)\quad \lim_{n\to\infty} \frac{1}{n}(\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n}) = 1,$$

$$2)\quad \lim_{n\to\infty} \frac{1}{n}(n^{1/\log 2} + n^{1/\log 3} + \dots + n^{1/\log n}) = e.$$

Solution by Donnelly J. Johnson, Major, USAF, Wright-Patterson Air Force Base, Ohio.

Let $\varepsilon > 0$ be given. Using l'Hospital's rule we see that $\lim_{n\to\infty} \frac{\log n}{n} = 0$. Thus, there is a function $M(\varepsilon)$ so that $|\sqrt[n]{n} - 1| < \varepsilon$ if $n \ge M(\varepsilon)$. Now we claim $\left|\frac{1}{n}\sum_{k=2}^{n} n^{1/k} - 1\right| < \varepsilon$ if

$$n \ge \max\left\{\frac{3}{\varepsilon}, [M(\frac{\varepsilon}{6}\log(1 + \frac{\varepsilon}{3}))]^2\right\}.$$

Observe that $n^{1/k} = \exp\left(\frac{\log n}{k}\right)$, so if

$$k > \frac{\log n}{\log(1 + \frac{\varepsilon}{3})}$$

then

$$\frac{\log n}{k} < \log(1 + \tfrac{\varepsilon}{3}) \quad \text{and} \quad n^{1/k} < 1 + \frac{\varepsilon}{3} .$$

Also observe that

$$\frac{\log n}{n^{1/2}} = \frac{2 \log n^{1/2}}{1/2} ,$$

so if

$$n^{1/2} > M\left(\frac{\varepsilon}{6} \log(1 + \tfrac{\varepsilon}{3})\right)$$

then

$$\frac{\log n}{n^{1/2}} < \frac{\varepsilon}{3} \log(1 + \tfrac{\varepsilon}{3}) .$$

Thus if $n \ge \frac{3}{\varepsilon}$, $n^{1/2} \ge M(\frac{\varepsilon}{6} \log(1 + \frac{\varepsilon}{3}))$, and $m = \left[\frac{\log n}{\log(1 + \varepsilon/3)}\right]$, then

$$\left| \frac{1}{n} \sum_{k=2}^{n} n^{1/k} - 1 \right| = \left| \frac{1}{n} \left(-1 + \sum_{k=2}^{n} (n^{1/k} - 1) \right) \right|$$

$$\le \frac{1}{n} + \sum_{k=2}^{m} \frac{n^{1/k} - 1}{n} + \frac{1}{n} \left[\sum_{k=1+m}^{n} (n^{1/k} - 1) \right]$$

$$< \frac{1}{n} + \frac{\log n}{\log(1 + \frac{\varepsilon}{3})} \left(n^{-1/2} \right) + \frac{1}{n} \sum_{k=1}^{n} \frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{as required.}$$

Also solved by the Proposers.

299. [Spring 1973] *Proposed by David L. Silverman, West Los Angeles, California.*

On the back of an envelope you see the results of an interrupted game by two players whom you know to be tic-tac-toe experts. It is generally recognized that the expert never puts himself into a potentially losing position and always wins if his opponent gives him the opportunity. There are 2 X's and 2 O's on the diagram. It is impossible to deduce whose move it is. Neglecting symmetry, what is the position?

Solution by Clayton W. Dodge, University of Maine at Orono.

Let us number the boxes in the tic-tac-toe array along each row from left to right using 1 to 9. Thus the first column is 1-4-7. Of the 2 X's and 2 O's, it is clear that :

1. No mark (say *X*) can occupy box 5, since then a like mark is adjacent to it, so the opposite mark (*O*) must complete that row or column or diagonal. The last O cannot now be placed so that a win cannot be forced in some way.

2. No two like marks are symmetric to box 5, since then a like mark in box 5 would win or observation 1 would be violated.

3. The same mark (say *X*) cannot appear in two adjacent corners (say 1 and 3), since then O must occupy box 2 and some other box that would enable O to win if he plays next.

4. The same mark (say *X*) cannot appear in two middle-of-an-edge squares (such as 2 and 8 or 2 and 4) without violating observation 2 (as in boxes 2 and 8) or, if we assume X's in boxes 2 and 4, then O must occupy the corner (box 1) between the X's and another box (either 6 or 8) to prevent X from winning with box 5. But then O can win with box 9.

5. The same mark (say *X*) cannot occupy two adjacent squares (say 1 and 2), since then the opposite mark would have to complete that row or column (*O* in box 3) and another box, enabling O to win if he moves first.

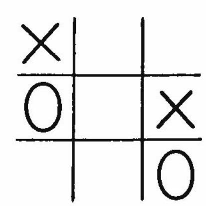
6. Thus each mark occupies one corner and one middle-of-an-edge box (such as 1 and 6) that is opposite. There are four cases.

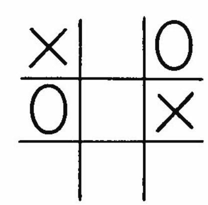
If X lies in 1 and 6 and *O* in 2 and 9, then X in 4 forces a win for X.

If X lies in 1 and 6 and O in 2 and 7, then X in 5 forces a win for *X*.

If X is in boxes 1 and 6 and if O is in 4 and either 3 or 9, then expert play will produce a tie in all cases.

Hence we have two distinct solutions to the problem, either of which could have been on the envelope:





Also solved by RICK JOHNSON, *Wilmington*, N. C.; ZAZOU KATZ, *Beverly Hills, California; and the Proposer.*

300. [Spring 1973] *Proposed by the Problem Editor.*

It can be shown with difficulty that if the opposite angles of a skew quadrilateral are equal in pairs, the opposite sides are also eaual in pairs. (The reward of instant immortality is offered the solver who can prove this without difficulty). If two opposite sides of a skew quadrilateral are equal and the other two unequal, is it possible to have one pair of opposite angles equal?

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

The answer is "yes." We make use of the "ambiguous" case *SSA* for congruence of triangles. Let *ABC* be anv obtuse triangle with obtuse angle at *B* and with side *AB* greater than side *BC* (Fig. 6). With center *B* swing an arc of radius *BC* to cut side AC again at *E*. Then triangles *ABC* and *ABE* are not congruent but satisfy the *SSA* condition. Construct triangle *BCD* (on side *BC*) congruent to triangle *EBA*. Now fold the paper along line *BC*. Then the skew quadrilateral *ACDB* has angles *A* and *D* equal, sides *AB* and *CD* equal, and sides *AC* and *BD* unequal.

II. *Solution by Zazou Katz, Beverly Hills, California.*

Consider the annexed figure, in which chords *CD* and *AB* are parallel (Fig. 7). It is clear that angles *DCB* and *DAB* are equal and that *CB* = *AD*. if triangle *CDB* is hinges about *DB* and *C* is lifted from the plane of the figure, we obtain a skew quadrilateral in which the angles at *C* and *A* remain equal, the opposite sides *BC* and *AD* are eaual, and the opposite sides *CD* and *AB* are unequal.

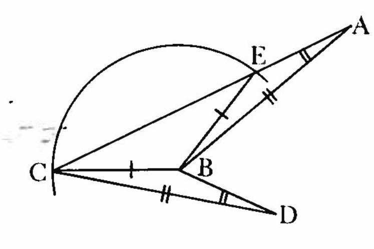


FIGURE 6

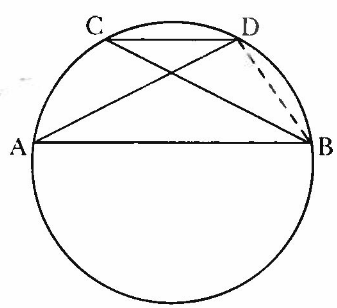


FIGURE 7

Also solved by ALFRED E. NEUMAN, *Mu Alpha Delta Fraternity, New York, and by* NOSMO KING, *Raleigh, N.* C.

Problem Editor's Note.

Readers interested in skew quadrilaterals will find solutions of the introductory theorem on pages 1026 and 1027 of the November 1973 issue of the *American Mathematical Monthly* (solutions to Question B-5 of the William Lowell Putnam Mathematical Competition held on December [illegible], 1972).

301. [Spring 1973] *Proposed by Neal Jacobs, Bronx, N. Y.*

(Corrected). One-fifteenth can be expressed in "decimals" in manv ways, for example, as $.\overline{0421}$ in base eight, or as $.\overline{013}$ in base five. Show that in any base n, the "decimal" for one-fifteenth will have no more than four recurring digits.

Solution by Donnelly J. Johnson, Major, USAF, Wright-Patterson Air Force Base, Ohio.

The representation of 1/15 as a "decimal" expression to base n, (n = 2, 3, 4, $\cdots$,) either terminates or has a repeating block of length 1, 2, or 4. To prove this, it is sufficient to show that there is a repeating block of length 4. Also it is sufficient to show that [illegible] 2, 3, 4, $\cdots$, then 15 divides $n(n^4 - 1)$ since if $n(n^4 - 1)/15$ = [illegible]

$K = \sum_{i=1}^{4} a_i/n^i$ with $0 \le a_i < n$, then

$$\sum_{r=0}^{\infty} \frac{K}{n^5}\left(\frac{1}{n^4}\right)^r = \frac{K}{n^5} + \frac{K}{n^9} + \frac{K}{n^{13}} + \cdots$$

is seen to represent a number in "decimal" base n with repeating block of length 4 digits; and

$$\sum_{r=0}^{\infty} \frac{K}{n^5}\left(\frac{1}{n^4}\right)^r = \frac{K/n^5}{(n^4 - 1)/n^4} = \frac{K}{n(n^4 - 1)} = \frac{1}{15}.$$

To complete the proof that 15 divides [illegible] show that for p = 3 and for p = 5, p divides either [illegible] this it is sufficient to show for p = 3, 5 that if a [illegible]), then $a^4 - 1$ is a multiple of p: thus, $1^4 - 1$ = [illegible] - 1 = 80; 4^4 - 1 = 215.

Also solved by RICHARD A. GIBBS, [illegible], Durango, Colorado; BOB PRIELIPP, *The University* [illegible]; and the Proposer. *All solvers noted the misprint* [illegible] problem.

Professor Prielipp cited the Teacher's Manual for Excursions *Into Mathematics by Beck, Bleicher and Crowe, pp. 271-272. Gibbs noted the general result found in B. M. Stewart, Macmillan 1952,* p. 234.

302. [Spring 1973] *Proposed by David L. Silverman, West Los Angeles, California and Alfred E. Newman, Mu Alpha Delta Fraternity, New York.*

A tapestry is hung on a wall so that its upper edge is a units and its lower edge b units above the observer's eye-level. Show that in order to obtain the most favorable view the observer should stand at a distance $\sqrt{ab}$ from the wall.

I. *Solution by R. Robinson Rowe, Sacramento, California.*

In Fig. 8, AB is the tapestry on wall ABC and the observer's eye is at I at a distance of IC = d from the wall. For the most favorable view, angle AIB = θ should be a maximum.

$$\theta = AIC - BIC = \tan^{-1}a/d - \tan^{-1}b/d$$
$$= \tan^{-1}\frac{d(a + b)}{d^2 + ab}$$

Since θ is less than 90°, it can be maximized by maximizing its tangent, and hence by maximizing its tangent divided by $(a - b)$. Letting $F(d) = d/(d^2 + ab)$, we obtain $F'(d) = (ab - d^2)/(d^2 + ab)^2 = 0$, whence $ab - d^2 = 0$ and $d = \sqrt{ab}$.

Comment by Problem Editor

While this solution is correct in theory, there are practical limitations. As C approaches B, b and d approach zero and θ approaches 90° Thus, if a = 4 and b = .01, we find that d = .2 and even a myopic observer would get a jaundiced view.

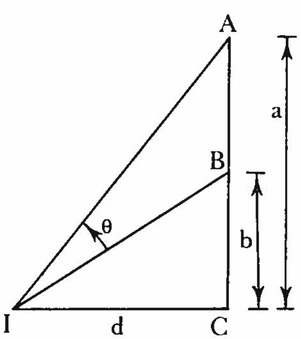


FIGURE 8

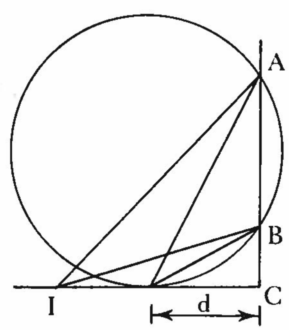


FIGURE 9

II. *Amalgam of geometric solutions by Scott H. Brown, West Virginia University, Morgantown, West Virginia and Clayton W. Dodge, University of Maine at Orono.*

Construct the circle through the top and the bottom of the tapestry and tangent to the eye-level line of the observer (Fig. 9). The point of contact of the circle with the line is the optimum position. For all points on the line outside the circle, the tapestry subtends a smaller angle than at the point of tangency. Hence the distance from the wall is $\sqrt{ab}$.

Other almost identical calculus solutions were offered by RAYMOND E. ANDERSON, *Montana State University;* ROBERT C. GEBHARDT, *Hopatcong, N. J.;* JOHN M. HOWELL, *Littlerock, California;* DONNELLY JOHNSON, *Wright-Patterson Air Force Base, Ohio;* G. MAVRIGIAM, *Youngstown State University;* BOB PRIELIPP, *The University of Wisconsin, Oshkosh;* DAN SCHOLTEN, *Wesleyan University. Also solved by the Proposers using synthetic geometry.*

Problem Editor's Comment

It is hard to find an elementary calculus text that has not used some variant of this problem. Howell gave the references to W. L. Hart's Analytic Geometry and *Calculus*, p. 265, problem #47 and to Smail's Analytic Geometry and *Calculus*, p. 301, problem #3. The geometric version can be found in an article by J. H. Butchart and Leo Moser, published in the September-December 1952 issue of *Scripta Mathematica*. An extension of the geometric treatment may be found in problem E1128, pages 184-185 of the *American* Mathematical Monthly, March 1955. Bob Prielipp called attention to the article "How to Break a Window Without Calculus", by John H. Hughes in the January 1974 issue of The Mathematics Teacher.

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