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TWO ALGORITHMS FOR EXPRESSING $\sum_{i=1}^{n} i^k$ AS A POLYNOMIAL IN n

Louis D. Rodabaugh, Ph. D.
University of Akron

In this paper we shall illustrate and verify two algorithms for determining the coefficients in the polynomial representation of

$$1^k + 2^k + 3^k + \dots + n^k$$

for each non-negative integer, k, and each positive integer, n. These coefficients will be placed in an infinite triangle as follows:

$$\begin{array}{ccccc}
 & & & & R_{11} \\
 & & & R_{21} & R_{22} \\
 & & R_{31} & R_{32} & R_{33} \\
 & R_{41} & R_{42} & R_{43} & R_{44} \\
R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \\
\dots & \dots & \dots & \dots & \dots
\end{array}$$

so that if n is any positive integer, and k is any non-negative integer, then:

$$(1)\quad \sum_{i=1}^{n} i^k = R_{k+1,1}n^{k+1} + R_{k+1,2}n^k + \dots + R_{k+1,k+1}n.$$

Since $\sum_{i=1}^{1} i^k = 1^k = 1$, we know that

$$(2)\quad \sum_{i=1}^{r} R_{r,i} = 1 \qquad \text{for all positive integers } r.$$

From the well-known identity

$$(3)\quad n^k = \sum_{i=1}^{n} \left[i^k - (i-1)^k\right] \qquad \text{for all positive integers } k.$$

can be derived the equations

$$(4)\quad R_{r,1} = \frac{1}{r} \qquad \text{for all positive integers } r.$$

and

$$(5)\quad R_{r,s} = \frac{1}{r}\cdot\sum_{i=1}^{s-1}(-1)^{1+i}\,R_{r-i,s-i}\,P_{r+1,2+i}$$ for all integers r,s such that $1 < s \leq r$.

The $P_{m,n}$ appearing in equation (5) are the elements of Pascal's Triangle, placed and numbered as follows:

$$\begin{array}{rrrrr} & & & & P_{11}\\ & & & P_{21} & P_{22}\\ & & P_{31} & P_{32} & P_{33}\\ & P_{41} & P_{42} & P_{43} & P_{44}\\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55}\\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

so that

$$(6)\quad P_{m,n} = \binom{m-1}{n-1} = \frac{(m-1)!}{(n-1)!\,(m-n)!}\quad \text{for all positive integers } m,n.$$

Equations (2), (4), and (5) provide the basis for our FIRST ALGORITHM. We here illustrate this algorithm by using it in order to determine the fifth row of our coefficient triangle on the assumption that we have already determined the first four rows. We place a blank row beneath the fourth row, and place a "1" in the sixth row in the fifth column numbered from the right. We place as a column of multipliers the elements of the sixth row of Pascal's Triangle, beginning with a "1" in our sixth row and proceeding upward with alternating signs:

				1	-1
			$\frac{1}{2}$	$\frac{1}{2}$	5
		$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	-10
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	10
					-5
1					1
					MULTIPLIERS

				-1
			$\frac{5}{2}$	$\frac{5}{2}$
		$-\frac{10}{3}$	-5	$-\frac{5}{3}$
	$\frac{5}{2}$	5	$\frac{5}{2}$	0
1				
1	$\frac{5}{2}$	$\frac{5}{3}$	0	$-\frac{1}{6}$

COLUMN SUMS

If we now divide each column sum by 5 (the number of the row which we are **determining),** we shall have the fifth row of our coefficient triangle:

$$\frac{1}{5},\ \frac{1}{2},\ \frac{1}{3},\ 0,\ -\frac{1}{30}.$$

The appearance of the coefficient triangle, down to and including the fifth row, is therefore as follows:

				1
			$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$-\frac{1}{30}$

Our SECOND ALGORITHM is very largely based on

THEOREM 1: If r,s are integers and $1 < s < r$, then

$$(7)\quad R_{r,s} = \frac{r-1}{r-s+1}\cdot R_{r-1,s}.$$

Proof: Let m be any integer greater than 1. Then (7) holds for $r = m$ and $s = 1$, because

$$(8)\quad R_{m,1} = \frac{1}{m} = \frac{m-1}{m}\cdot\frac{1}{m-1} = \frac{m-1}{m-1+1}\cdot R_{m-1,1}\ .$$

Next, we assume that (7) holds not only for $R_{m,1}$ but also for the top q elements of the column headed by $R_{m,1}$; that is, that (7) holds for $R_{m,1}$, $R_{m+1,2}$, $R_{m+2,3}$, $\ldots$, $R_{m+q-2,q-1}$, and $R_{m+q-1,q}$ $(=R_{k,q})$.

Then:

$$(9)\quad R_{k+1,q+1} = \frac{1}{k+1}\sum_{i=1}^{q}(-1)^{1+i}\,R_{k+1-i,q+1-i}\,P_{k+2,2+i}$$

$$= \frac{1}{k+1}\ {}_{i=1}\ (-1)^{1+i}\,\frac{k-i}{k-q+1}\,R_{k-i,q+1-i}\,P_{k+2,2+i}$$

$$= \frac{1}{k+1}\ {}_{i=1}\ (-1)^{1+i}\,\frac{k-i}{k-q+1}\,R_{k-i,q+1-i}\,\frac{(k+1)!}{(1+i)!\,(k-i)!}$$

$$= \frac{k}{k-q+1}\cdot\frac{1}{k}\sum_{i=1}^{q}(-1)^{1+i}\,R_{k-i,q+1-i}\,\frac{k!}{(1+i)!\,(k-1-i)!}$$

COEFFICIENT TRIANGLE

for representing $\sum_{i=1}^{n} i^k$ as a polynomial in n. These are the first nineteen rows as constructed by an IBM computer in accordance with the SECOND ALGORITHM of Professor Louis D. Rodabaugh.

																		1
																	$\frac{1}{2}$	$\frac{1}{2}$
																$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
															$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
														$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{-1}{30}$
													$\frac{1}{6}$	$\frac{1}{2}$	$\frac{5}{12}$	0	$\frac{-1}{12}$	0
												$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{-1}{6}$	0	$\frac{1}{42}$
											$\frac{1}{8}$	$\frac{1}{2}$	$\frac{7}{12}$	0	$\frac{-7}{24}$	0	$\frac{1}{12}$	0
										$\frac{1}{9}$	$\frac{1}{2}$	$\frac{2}{3}$	0	$\frac{-7}{15}$	0	$\frac{2}{9}$	0	$\frac{-1}{30}$
									$\frac{1}{10}$	$\frac{1}{2}$	$\frac{3}{4}$	0	$\frac{-7}{10}$	0	$\frac{1}{2}$	0	$\frac{-3}{20}$	0
								$\frac{1}{11}$	$\frac{1}{2}$	$\frac{5}{6}$	0	-1	0	1	0	$\frac{-1}{2}$	0	$\frac{5}{66}$
							$\frac{1}{12}$	$\frac{1}{2}$	$\frac{11}{12}$	0	$\frac{-11}{8}$	0	$\frac{11}{6}$	0	$\frac{-11}{8}$	0	$\frac{5}{12}$	0
						$\frac{1}{13}$	$\frac{1}{2}$	1	0	$\frac{-11}{6}$	0	$\frac{22}{7}$	0	$\frac{-33}{10}$	0	$\frac{5}{3}$	0	$\frac{-691}{2730}$
					$\frac{1}{14}$	$\frac{1}{2}$	$\frac{13}{12}$	0	$\frac{-143}{60}$	0	$\frac{143}{28}$	0	$\frac{-143}{20}$	0	$\frac{65}{12}$	0	$\frac{-691}{420}$	0
				$\frac{1}{15}$	$\frac{1}{2}$	$\frac{7}{6}$	0	$\frac{-91}{30}$	0	$\frac{-143}{18}$	0	$\frac{-143}{10}$	0	$\frac{91}{6}$	0	$\frac{-691}{90}$	0	$\frac{7}{6}$
			$\frac{1}{16}$	$\frac{1}{2}$	$\frac{5}{4}$	0	$\frac{-91}{24}$	0	$\frac{143}{12}$	0	$\frac{-429}{16}$	0	$\frac{-455}{12}$	0	$\frac{-691}{24}$	0	$\frac{35}{4}$	0
		$\frac{1}{17}$	$\frac{1}{2}$	$\frac{4}{3}$	0	$\frac{-14}{3}$	0	$\frac{52}{3}$	0	$\frac{-143}{3}$	0	$\frac{260}{3}$	0	$\frac{-1382}{15}$	0	$\frac{140}{3}$	0	$\frac{-3617}{510}$
	$\frac{1}{18}$	$\frac{1}{2}$	$\frac{17}{12}$	0	$\frac{-17}{3}$	0	$\frac{221}{9}$	0	$\frac{-2431}{30}$	0	$\frac{1105}{6}$	0	$\frac{-11747}{45}$	0	$\frac{595}{3}$	0	$\frac{-3617}{60}$	0
$\frac{1}{19}$	$\frac{1}{2}$	$\frac{3}{2}$	0	$\frac{-34}{5}$	0	34	0	$\frac{-663}{5}$	0	$\frac{1105}{3}$	0	$\frac{-23494}{35}$	0	714	0	$\frac{-3617}{10}$	0	$\frac{43867}{798}$

Figure 2

$$= \frac{k}{k-q+1}\cdot\frac{1}{k}\sum_{i=1}^{(q+1)-1}(-1)^{1+i}\,R_{k-i,q+1-i}\,P_{k+1,2+i}$$

Thus

$$(10)\quad R_{k+1,q+1} = \frac{(k+1)-1}{(k+1)-(q+1)+1}\,R_{(k+1)-1,q+1}\ ;$$

That is, (7) holds also for the (q+1)th element (counting from the top down) of the column headed by $R_{m,1}$.

This concludes the Second-Principle Induction, and therefore also the proof, that (7) holds for every element of the column headed by $R_{m,1}$. Since m was an arbitrary integer greater than 1, we see that (7) holds for every element of the coefficient triangle not in the rightmost column. THEOREM 1 is therefore proved.

The validity of our SECOND ALGORITHM for the determination of the coefficient triangle is now established. We describe this algorithm:

I Use (4) to construct the first upper-right-to lower-left diagonal:

II Determine $R_{j,j}$ for each positive integer j, in either of the following two ways:

a) Apply the FIRST ALGORITHM to only the right-most column;

b) When the jth row is known except for the element $R_{j,j}$ use equation (2);

III Use equation (7) to determine $R_{r,s}$ for every r and s in I such that $2 \leq s < r$.

We next present the first nineteen rows of the coefficient triangle as determined by an electronic computer in accordance with the above program (specifically, instructions I, IIb, and III). See Figure 2.

In perusing the COEFFICIENT TRIANGLE, one observes the seeming alternation of upper-right-to-lower-left diagonals of zeros from the fourth on. We shall prove that these zero-filled diagonals do indeed alternate indefinitely.

Lemma 1: If k,n are positve integers and $(n, (k+1)!) = 1$. then

$$\sum_{i=1}^{n} i^k \text{ is divisible by } n.$$

Proof: We know from the established validity of the FIRST and SECOND ALGORITHMS that for any k,n positive integers there exist integers $a, a, a, \ldots, a_k, h$ such that

$$(11)\quad \sum_{i=1}^{n} i^k = \frac{a_0 n^{k+1} + a_1 n^k + \ldots + a_{k-1}n^2 + a_k n}{h}$$

and h divides $(k+1)!$. If k,n satisfy the hypothesis of Lemma 1, then $(n,h) = 1$. Since the left member of (11) is an integer, we see that $\frac{nq}{h}$ is also an integer, where

(12) $q = a_0 n^k + a_1 n^{k-1} + \cdots + a_{k-1} n + a$.

Therefore h divides nq. Since $(n,h) = 1$, this implies that h divides q. In other words, $\frac{q}{h}$ is an integer. Since

(13) $$\sum_{i=1}^{n} i^k = n \cdot \frac{q}{h},$$

we see that $\sum_{i=1}^{n} i^k$ is divisible by n, Q.E.D.

Lemma 1 and the fact that

(14) $$\sum_{i=1}^{n-1} i^k = \sum_{i=1}^{n} i^k - n^k$$

establish immediately

Corollary 1: If k,n are positive integers and $(n, (k+1)!) = 1$, then $\sum_{i=1}^{n-1} i^k$ is divisible by n.

Lemma 2: If h,n are positive integers. $(2n+1, (k+1)!) = 1$, and k is even, then $\sum_{i=1}^{n} i^k$ is divisible by $(2n+1)!$.

Proof: By Corollary 1,

(15) $$\sum_{i=1}^{2n} i^k \equiv 0 \pmod{(2n+1)} .$$

Since k is even, then for every $i \in \{1, 2, \ldots, n\}$ we have

(16) $$i^k \equiv [(2n+1) - i]^k \pmod{(2n+1)} .$$

Hence

(17) $$\sum_{i=1}^{2n} i^k \equiv 2 \sum_{i=1}^{n} i^k \pmod{(2n+1)} .$$

From (15) and (17) we have

(18) $$2 \sum_{i=1}^{n} i^k \equiv 0 \pmod{(2n+1)} .$$

From this, since $((2n+1),2) = 1$, we see that $(2n+1)$ divides $\sum_{i=1}^{n} i^k$,

THEOREM 2: If k,n are positive integers, $k \geq 3$, and k is odd, then, in the right-hand member of (11),

(19) $$a_k = 0 .$$

Proof: If, in (11), we select n as an odd integer, 2m+1, such that $(2m+1, (k+1)!) = 1$, then we can write

(20) $$\sum_{i=1}^{n} i^k = \sum_{i=1}^{2m+1} i^k$$

$$= (1^k + [(2m+1) - 1]^k) + (2^k + [(2m+1) - 2]^k)$$

$$+ (3^k + [(2m+1) - 3]^k) + \ldots + (m^k + [(2m+1) - m]^k) + (2m+1)^k.$$

From this we see that $\sum_{i=1}^{2m+1} i^k$ is divisible by $(2m+1)^2$ if and only if the expression

(21) $$k(2m+1)(1^{k-1} + 2^{k-1} + \ldots + m^{k-1})$$

is divisible by $(2m+1)^2$. This is seen to be the case whenever the expression

(22) $$1^{k-1} + 2^{k-1} + \ldots + m^{k-1}$$

is divisible by $(2m+1)$. The latter is the case, however, as we see from Lemma 2. We have, therefore, that if $n = 2m + 1$ and $(2m+1, (k+1)!) = 1$, then the right-hand member of (11) is equal to sn^2 for some integer s. From this it follows that

(23) $$a_0 n^{k-1} + a_1 n^k + \ldots + a_{k-1} n^2 + a_k n = hsn^2.$$

Thus n^2 divides $a_k n$ and hence n divides a_k. From the way in which n was selected we see that a_k is divisible by every positive integer which is relatively prime to $(k + 1)!$. It follows, of course, that $a_k = 0$,

Q.E.D.

MEETING ANNOUNCEMENT

Pi Mu Epsilon will meet in late August, 1970, at the University of Wyoming, **Laramie,** Wyoming, in conjunction with the Mathematical Association of America. Chapters should start planning NOW to send delegates or speakers to this meeting, and to attend as **many** of the lectures by other mathematical groups as possible.

The National Office of Pi Mu Epsilon will help with expenses of a speaker OR delegate (one per chapter) who is a member of Pi Mu Epsilon and who has not received a Master's Degree by April 15, 1970, as follows: SPEAKERS will receive 54 per mile or lowest cost, confirmed air travel fare; DELEGATES will receive 2 1/24 per mile or lowest cost, confirmed air travel fare.

Select the best talk of the year given at one of your meetings by a member of Pi Mu Epsilon who meets the above requirement and have him or her apply to the National Office. Nominations should be in our office by April 15, 1970. The following information should be included: Your Name; Chapter of Pi Mu Epsilon; school; topic of talk; what degree you are working on; of you are a delegate or a speaker; when you expect to receive your degree; current mailing address; summer mailing address; who recommended by; and a 50-75 word summary of talk, if you are a speaker. MAIL TO: Pi Mu Epsilon, 1000 Asp Ave., Room 215. Norman, Oklahoma 73069.

ELLIPTIC CURVES OVER LOCAL FIELDS

Bruce L. Rienzo
Rutgers University

Elliptic curves may be put into the standard form $y^2 = x^3 + Ax + B$, called the Weierstrass form. In this form, the points on the curve defined over a field k form an abelian group under an appropriate composition law. This group law also works for singular curves, provided we avoid the singular point.

Considering the curves over finite fields of p elements, we see that there can be only p^2 possible curves. We then may program the group law on a computer and run off all possible cases. Looking at these results, we can then make same conjectures as to the number of points on the curve mod p.

Having found the solutions mod p, we **procede** to develop a method for lifting these solutions to solutions mod p^N, for arbitrary N. This gives solutions in the p-adic fields.

Finally, we develop the **Nagell-Lutz** Theorem, for p-adic fields. By this theorem, points of finite order in the group must have integer coordinates.

1. Elliptic Curves and the Group Law.

51.1 Weierstrass Form

Rather than having to work directly with elliptic curves, we may first put them into a standard form. An elliptic curve, defined over a field k of characteristic not 2 or 3, is **birationally** equivalent to a plane cubic curve of the form $y^2 = x^3 + Ax + B$, provided the curve has a point defined over k.[1]

Thus, we will not need to consider general elliptic curves, only those of the form $y^2 = x^3 + Ax + B$. Curves of this form are said to be in the Weierstrass form. We will often denote the Weiestrass form by $y^2 = f(x)$, where $f(x)$ is a cubic.

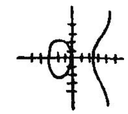
What do these curves look like? This question can be asked for various different fields. We will restrict our attention to points which are rational over the field. First, consider the field of real numbers.

51.2 The Real Ground Field

For the field of real numbers there are several cases depending on the roots of $f(x) = 0$.

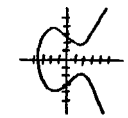
1) $f(x) = 0$ has three distinct real roots.

$y^2=x^3-3x$



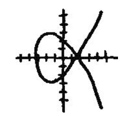
2) $f(x) = 0$ has only one real root.

$y^2=x^3-3x+3$



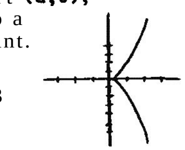
3) $f(x) = 0$ has a double root at $x=a$ and a distinct single root. The point $(a,0)$ is then a singular point of the curve.

$y^2=x^3-3x+2$



4) $f(x) = 0$ has a triple root at $x=a$. The curve then has a cusp at $(a,0)$; this is also a singular point.

$y^2=x^3$



51.3 Projective Space

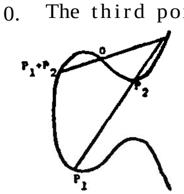
We will be considering these curves from the point of view of projective geometry. That is, we will be including points at infinity on the curve. Putting the equation into homogeneous form gives $Y^2Z = X^3 + AXZ^2 + BZ^3$. The points at infinity are the points with $Z = 0$. But this means that $X^3 = 0$. Thus there is only one infinite point on the curve (the point $(0, 1, 0)$ in projective coordinates); and the line at infinity intersects the curve at this point with multiplicity three.

51.4 The Group Law[2]

Consider a fixed elliptic curve defined over a field k. If we can devise a way of making the points of the curve into a group, we may then study the points by studying the structure of the group. We will see that we can in fact define such a group operation. It will turn out to be commutative, so we will call the operation addition and denote it "+".

Geometrically, the group operation for non-singular curves is based on the fact that, counting multiplicities, any line defined over k intersects the curve in exactly three points (over the algebraic closure of k). What this means is that if P_1 and P_2 are two points on the curve, we may draw the line through them. and this will give us a third point associated with P_1 and P_2.

Unfortunately, this easily defined composition is not a group operation. For one thing, it has no identity. However, we may remedy this situation by first fixing some point 0 on the curve to serve as the identity element of the group. Then when we get the third point of the curve on the line through P_1 and P_2, we simply draw the line through this point and the point 0. The third point on this line will be the desired point $P_1 + P_2$.



It is clear immediately that this addition law is commutative. (The line through P_1 and P_2 is certainly the same as the line through P_2 and P_1.) To show that the point 0 is indeed the identity, we let P be a point on the curve and find P + 0. The line through P and 0 intersects the curve in some third point Q. We then consider the line throught 0 and Q. But this must be the same line. Thus the third point must be P. That is, P + 0 = P as desired.

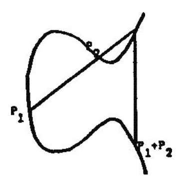
To get inverses, we draw the line through 0 twice (i.e. the line tangent to the curve at 0) and let S be the third point. Then if P is any point, the third point of the line through P and S is the point -P. (The third point of the line through P and -P is S.) Then the third point of the line through 0 and S is 0. So P + (-P) = 0.

The hard part is to show associativity. We omit this proof here, referring to Tate[3] for a proof. This difficulty may be avoided completely by using the definition of elliptic curves in Cassels[4].

We may choose any point on the curve to be the fixed point 0. If we choose the point at infinity, then the lines through 0 are just the vertical lines (and the line at infinity). That is, the line through 0 and P = (x,y,z) has the point (x,-y,z) as its third point of intersection with the curve.

Inverses are now simple to compute. The point S described above is now the point 0. (The line tangent to the curve at the point at infinity is the line at infinity, Z = 0. But we have seen that Z = 0 intersects the curve 3 times at the point 0. Thus, S = 0.) So, if P = (x,y,z) then -P is the third point of the line through P and 0 which is just (x,-y,z).

We may now restate the addition law. If P_1 and P_2 are two points on the curve, let the line through them have $P_3 = (x,y,z)$ as its third point. Then $P_1 + P_2 = (x, -y, z)$. That is, $P_1 + P_2 = -P_3$; or $P_1 + P_2 + P_3 = 0$, where P_1, P_2, and P_3 are collinear.



It will be useful to have an actual formula for the addition of two finite points. Let $P_1 = (x_1,y_1,1) = (x_1,y_1)$ and $P_2 = (x_2,y_2,1) = (x_2,y_2)$. If $x_1 = x_2$ and $y_1 = -y_2$, the points are inverses and $P_1 + P_2 = 0$. Otherwise, consider the line through P_1 and P_2. Say its equation is $y = \lambda x + \nu$. If $P_1 \neq P_2$, then the slope $\lambda = \frac{y_2-y_1}{x_2-x_1}$. If $P_1 = P_2$, the line through P_1 and P_2 is the tangent at that point. Then $y^2 = f(x)$ gives $\lambda = \frac{f'(x)}{2y}$.

In either case, $\nu = y_1 - \lambda x_1$ $(= y_2 - \lambda x_2)$. To get the third point $P_3 = (x_3,y_3)$, we plug $y = \lambda x + \nu$ into $y^2 = x^3 + Ax + B$:

$$(\lambda x + \nu)^2 = x^3 + Ax + B$$

$$\lambda^2x^2 + 2\lambda\nu x + \nu^2 = x^3 + Ax + B$$

$$0 = x^3 - \lambda^2x^2 + (A-2\lambda\nu)x + (B-\nu^2)$$

This is a cubic in x whose roots are just the x-coordinates of the three points of intersection of the curve with the line. The roots must equal the negative of the coefficient of the second order term. i.e. $x_1 + x_2 + x_3 = \lambda^2$. Thus the group law becomes:

$$x_3 = \lambda^2 - (x_1+x_2) \qquad -y_3 = \lambda x_3 + \nu$$

where $\lambda = \frac{y_1-y_2}{x_1-x_2}$ when $x_1 \neq x_2$, and $\lambda = \frac{f'(x)}{2y}$ when $P_1 = P_2$, and where $\nu = y_1 - \lambda x_1$.

<u>**Note:**</u> If the curve is given in the form $y^2 = x^3 + ax^2 + bx + c$, then $x_1 + x_2 + x_3 = \lambda^2 - a$. So the group law is $x_3 = \lambda^2 - a - (x_1+x_2)$.

These formulas could now be used to prove associativity.

51.5 <u>Singular curves</u>

We have described the group operation for non-singular curves. What can be said about the singular cases? We needed the fact that a line intersects the curve in exactly three points. This is still true provided the line does not pass through the singular point.

If P_1 and P_2 are two points on the curve, then the line through them does not pass through the singular point. (The singular point is in effect a double point, so any line through it can intersect in only one other point of the curve.)

Thus our group operation holds for points other than the singular point. That is, the complement of the singular point forms a group.

2. Local and Finite Fields

52.1 <u>P-adic fields</u>

Many of the most interesting results on elliptic curves come from looking at the curves over p-adic fields. We will not discuss the theory of p-adic numbers here. (For an explanation of p-adic numbers see a number theory text such as Borevich and Shafarevich[1].)

We will be using the exponential p-adic valuation, which is given by:

$$v_p(p^n u/v) = n \quad \text{where } p \nmid u \text{ and } p \nmid v.$$

If α, β are non-zero p-adic numbers, then

$$v_p(\alpha\beta) = v_p(\alpha) + v_p(\beta)$$

$$v_p(\alpha+\beta) \geq \min[v_p(\alpha), v_p(\beta)]$$

with equality if $v_p(\alpha) \neq v_p(\beta)$.

If $v_p(\alpha) \geq 0$, then α is a p-adic integer.

If $v_p(\alpha) = 0$, then α is a unit of the ring of p-adic integers. (Since we will in general be working with a fixed p, we will often write just $v(\alpha)$ to denote the valuation.)

Solution of $y^2 = x^3 + Ax + B$ in p-adic numbers is **equivalent** to solution of $y^2 \equiv x^3 + Ax + B \pmod{p^N}$ for arbitrarily high N. We consider first N = 1.

52.2 Solutions mod p -- Finite Fields

In this section we will be considering the group of points on the curve over the finite field of ρ elements (**i.e.** the field of numbers mod ρ where ρ is a prime. In general, we will avoid $\rho = 2$ and 3, since these cases present special problems with regard to singularities. (Note that fields of characteristic 2 and 3 were excluded from the discussion on page 1 of this paper.)

For a given ρ, there **are** only ρ^2 curves of the **form** $y^2 = x^3+Ax+B$. (A and B can each take only ρ values.) Thus it is possible to program the group law on a computer and run off all the possible cases.

Before seeing the actual results, how many points might we expect the curve to have for a given ρ? The **"Riemann hypothesis"**[2] gives the number of points as $N = \rho + 1 - a$ where $|a| \leq 2\sqrt{\rho}$. In other words, $\rho + 1 - 2\sqrt{\rho} \leq N \leq \rho + 1 + 2\sqrt{\rho}$.

Let's look first at $\rho = 5$. Then N should fall in the range $2 \leq N \leq 10$. The following chart gives the number of points in the group for each possible A and B.

B \ A	0	1	2	3	4
0	5^s	2×2	2	10	2×4
1	6	9	7	4^s	8
2	6	4	6^s	5	3
3	6	4	6^s	5	3
4	6	9	7	4	8

A single digit indicates that the group for that particular curve is cyclic of that order. For **example**, $y^2 = x^3 + 2x + 1$ has 7 solutions (including the **point** at **infinity**), and its **group is cyclic** of order 7.

When the entry is expressed as a product, the group is the direct product of cyclic groups. For example, $y^2 = x^3 + 4x$ has 8 solutions, and its group is the direct product of a cyclic group of order 2 and one of order 4.

An **"s"** indicates that the curve has a singular point. For example, $y^2 = x^3 + 2x + 3$ has 7 solutions, one of which is singular. Its group is cyclic of order 6. (The group does not include the singular point.)

Inspection of the chart shows many interesting features. First, all values are within the predicted $2 \leq N \leq 10$ range. In fact, all possible N within the range occur.

We know that the isomorphisms of the curves are given by $A \to c^4 A$, $B \to c^6 B$, where $c \in k$. (It is clear that these are isomorphisms; they take $x \to x/c^2$ and $y \to y/c^3$. For a proof that they are the only ones, see **Cassels**[3].)

In this case. all fourth powers $\equiv 1 \pmod 5$ and sixth powers $\equiv$ squares $\equiv \pm 1 \pmod 5$. So the isomorphisms are just the identity and $B \to -B$. This explains why the row B = 1 is the same as the row B = 4, and why the row B = 2 is the same as the row B = 3.

We also note (without explanation) that in any column (**i.e.** for any fixed A) the number of solutions are congruent mod p. Also, in any row (**i.e.** for any fixed **B**) no two numbers are congruent.

Before looking at $\rho = 7$, let's try to predict what we can. First, the number of solutions should be in the range $3 \leq N \leq 13$. As for the isomorphisms, fourth powers ≡1, 2, 4 (mod 7), and all sixth powers ≡±1 (mod 7). So the isomorphisms are the identity, $A \to 2A$, and $A \to 4A$. Thus we would expect the rows A = 1, A = 2, and A = 4 to be the same, and the rows A = 3, A = 5, and A = 6 to be the same.

Here's the chart:

B \ A	0	1	2	3	4	5	6
0	7	8	8	4×2	8	4×2	4×2
1	6×2	5	5	12	5	12	12
2	3×3	8^s	8^s	9	8^s	9	9
3	13	6	6	6	6	6	6
4	3	10	10	10	10	10	10
5	7	6	6^s	7	6^s	1	1
6	2	11	11	4	11	4	4

We see that our predictions are true, and that again all possible values in the range $3 \leq N \leq 13$ occur. Also, now in any row the number of solutions are congruent mod ρ. And, in any column, except A = 0, no two numbers are confluent.

When B = 0, N is always 8. (Note that for $\rho = 5$, when A = 0, N was always 6.) This $\rho+1$ phenomenon can be explained by looking at the automorphisms of the curves [see **Cassels**[4]]. The result is for $\rho \equiv 3 \pmod 4$, $N = p + 1$ for **B = 0; and for** $\rho \equiv 5 \pmod 6$, $N = \rho + 1$ **for A = 0.**

We make the following conjecture: If $\rho \equiv 3 \pmod 4$, then for B = 0, $N = p + 1$; and for any fixed $A \neq 0$, no two N are congruent mod p. If $\rho \not\equiv 3 \pmod 4$, then for any fixed A, all N are congruent. If $p \equiv 5 \pmod 6$, then for A = 0, $N = p + 1$; and for any fixed $B \neq 0$, no two N are congruent mod p. If $p \not\equiv 5 \pmod 6$, then for any fixed **B**, all N **are** congruent.

For $\rho = 11$, we predict a range of $6 \leq N \leq 18$. Also, the isomorphisms are the identity, A→5A **B→9B, A→4A B→3B, A→3A B→4B,** and A→9A **B→5B.** Thus the rows A = 1, A = 3, A = 4, A = 5, and A = 9 should be permutations of each other. The same should be true for the rows A = 2, A = 6, A = 7, A = 8, and A = 10; the columns B = 1, B = 3, B = 4, B = 5, **and B =** 9; **and** the columns B = 2, B = 6, **B =** 7, B = 8, and **B = 10.**

We have $11 \equiv 5 \pmod 6$ and $11 \equiv 3 \pmod 4$, so for either A = 0 or **B = 0,** we should have $N = \rho + 1$. Also, for any fixed A or for any fixed **B,** no two N should be confluent.

B \ A	0	1	2	3	4	5	6	7	8	9	10
0	11^s	12	6×2	12	12	12	6×2	6×2	6×2	12	6×2
1	12	14	16	18	9	11	12^s	15	17	4×2	10
2	12	8×2	9	13	6	10	14	7	10	15	8
3	12	18	12	4×2	14	9	15	10	16	11	17
4	12	9	17	14	11	4×2	16	12^s	10	18	15
5	12	11	10	9	4×2	18	17	16	15	14	12^s
6	12	13	14	15	8×2	6	7	8	9	10	10^s
7	12	15	7	10	13	8×2	8	10^s	14	6	9
8	12	6	10^s	8×2	10	15	9	14	8	13	7
9	12	4×2	15	11	18	14	10	17	12^s	9	16
10	12	10	8	6	15	13	10^s	9	7	8×2	14

§2.3 Solutions mod ρ^N

Now that we have the solutions mod ρ, we need a way of lifting them to solutions mod p^N. We do this one step at a time, i.e. from ρ to ρ^2, then from p^2 to p^3, etc. In general, we want to lift solutions mod ρ^n to solutions mod ρ^{n+1}.

Any solution mod ρ^{n+1} must also be a solution mod p^n. Thus all solutions mod ρ^{n+1} are in the form $(x_0+s\rho^n, y_0+u\rho^n)$ where (x_0,y_0) is a solution mod p^n, and s and u are between 0 and p-1. Then,

$$(y_0+u\rho^n)^2 \equiv f(x_0+s\rho^n) \qquad (\text{mod } \rho^{n+1})$$
$$y_0^2+2y_0u\rho^n+u^2\rho^{2n} \equiv f(x_0+s\rho^n) \qquad (\text{mod } \rho^{n+1})$$
$$y_0^2+2y_0u\rho^n \equiv f(x_0+s\rho^n) \qquad (\text{mod } \rho^{n+1})$$
$$2y_0u\rho^n \equiv f(x_0+s\rho^n)-y_02 \qquad (\text{mod } \rho^{n+1})$$

The right side is divisible by p^n since $f(x_0)-y_0^2 \equiv 0$ (mod ρ^n).

$$2y_0u \equiv \frac{f(x_0+s\rho^n)-y_02}{\rho^n} \qquad (\text{mod } \rho)$$

Provided $y_0 \not\equiv 0$ (mod ρ), this is a simple linear congruence and so each value of s gives exactly one value of u. There are ρ such values, so there are p solutions

Suppose $y_0 \equiv 0$ (mod ρ). Then, as before

$$2y_0u\rho^n \equiv f(x_0+s\rho^n)-y_0^2 \qquad (\text{mod } \rho^{n+1})$$

But now, $y_0\rho^n \equiv 0$ (mod ρ^{n+1}), so

$$f(x_0+s\rho^n)-y_0^2 \equiv 0 \qquad (\text{mod } \rho^{n+1})$$

Using the expansion:

$$f(x_0+s\rho^n) = f(x_0) \text{ t } s\rho^n f'(x_0) + \frac{s^2\rho^{2n}}{2} f''(x_0) \text{ t } \ldots,$$

gives $\qquad f(x_0) \text{ t } s\rho^n f'(x_0) - y_0^2 \equiv 0 \qquad (\text{mod } \rho^{n+1})$

$$\frac{f(x_0)-y_0^2}{\rho^n} \text{ t } sf'(x_0) \text{ 5 } 0 \qquad (\text{mod } \rho)$$

Provided $f'(x_0) \not\equiv 0$ (mod ρ), there is exactly one s which solves this linear congruence. This value of s and any value of u gives a solution. there are ρ values of u, so again there are ρ solutions.

If both y_0 and $f'(x_0) \equiv 0$ (mod ρ), then the point (x_0,y_0) is a singular point. Otherwise, each point mod ρ^n lifts to ρ points mod ρ^{n+1}, and we need only solve a linear congruence to find then.

3. Nagell-Lutz Theorem

In this chapter, we give the major theorem on the structure of the group for curves over local fields. The proof given here generally follows the proof given by Lutz[1].

Let Γ be the group of points on the curve $y^2 = x^3 + Ax \text{ t } B$ over a p-adic field, where A and B are p-adic integers.

Lemma 3.1 Each rational point P = (x,y) in Γ has coordinates in the form $(\xi\rho^{-2n}, \delta\rho^{-3n})$, where $n \geq 0$ is an integer and ξ,δ are p-adic integers. ξ and 6 are units if $n > 0$.

proof: If x is a p-adic integer, then so is y; and then $\xi = x$, $\delta = y$, and $n=0$.

Otherwise, $v(x) < 0$, and we have $v(x^3) = 3v(x) < 0 \leq v(Ax)$. Also, $v(x^3) < 0 \leq v(B)$. Therefore, $v(y^2) = v(x^3+Ax+B) = v(x^3)$. $2v(y) = 3v(x)$ and so we must have $v(x) = -2n$ and $v(y) = -3n$ with $n > 0$. Thus, x and y are in the form $x = \xi\rho^{-2n}$ and $y = \delta\rho^{-3n}$ where ξ and 6 are units.

For any rational point P on the curve, let $n(P)$ be the integer n of the above lemma. Let Γ_m denote the set of points P with $n(P) \geq m$ (That is, with $v(x) \leq -2m$ and $v(y) \leq -3m$).

Theorem 3.2 Γ_m is a subgroup of Γ.

proof: Let $P_1 = (x_1,y_2)$ and $P_2 = (x_2,y_2)$ be two points in Γ_m. Let $P_3 = P_1 \text{ t } P_2$ and say $P_3 = (x_3,y_3)$.

Suppose $n(P_1) \neq n(P_2)$. We may assume $n(P_2) > n(P_1)$. The addition formula
$$x_3 = \frac{(y_2-y_1)^2 - (x_1 \text{ t } x_2)(x_2-x_1)^2}{(x_2-x_1)2} = \frac{x_2^2x_1 \text{ t } x_2x_1^2+A(x_1+x_2)+2B-2y_2y_1}{x_2^2-2x_2x_1+x_1^2}$$
gives $v(x_3) = v(x_2^2x_1+x_2x_1^2+A(x_1+x_2)+2B-2y_2y_1) - v(x_2^2-2x_2x_1 \text{ t } x_1^2)$.
$$= v(x_2^2x_1) - v(x_2^2) = v(x_1)$$

Thus for $n(P_1) \neq n(P_2)$, we have

$$n(P_3) = n(P_1 + P_2) = \min[n(P_1),n(P_2)]$$
$$n(P_1 - P_2) = \min[n(P_1),n(P_2)].$$

Suppose $n(P_1) = n(P_2)$. Then $P_1 = P_3 - P_2$, so if $n(P_3) \neq n(P_2)$ we have $n(P_1) = \min[n(P_3),n(P_2)]$. Thus, for $n(P_1) = n(P_2)$, we have $n(P_3) \geq n(P_1)$. In either case, we have $n(P_3) \geq \min[n(P_1),n(P_2)]$. Therefore, since

$n(P_1) \geq m$ and $n(P_2) \geq m$, we have $n(P_3) \geq m$, so P_3 is in Γ_m. Γ_m is therefore a subgroup of Γ.

Theorem 3.3 Γ_m has finite index in Γ, for integers $m > 0$.

Let $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ be two points in Γ. We need to consider conditions under which $(P_2-P_1) \in \Gamma_m$.

We may assume the $P_1,P_2 \notin \Gamma_m$, say $n(P_1) = n(P_2) = n < m$. Put $P_1 + (\xi_1\rho^{-2n},\delta_1\rho^{-3n})$ and $P_2 = (\xi_2\rho^{-2n},\delta_2\rho^{-3n})$ with $\delta_1^2 = \xi_1^3$ t $A\xi_1\rho^{4n}$ t $B\rho^{6n}$ and $\delta_2^2 = \xi_2^3 + A\xi_2\rho^{4n}$ t $B\rho^{6n}$, where ξ_1, ξ_2, δ_1, and δ_2 are units.

From the addition formula, $n(P_2-P_1) \geq m$ if and only if

$$\nu\left(\left(\frac{y_2+y_1}{x_2-x_1}\right)^2 - (x_2+x_1)\right) = \nu\left(\frac{y_2+y_1}{x_2-x_1}\right)^2 \leq -2m$$

$$\nu\left(\frac{y_2+y_1}{x_2-x_1}\right) \leq -m$$

But $$\left(\frac{y_2+y_1}{x_2-x_1}\right) = \frac{\delta_2+\delta_1}{\xi_2-\xi_1}\rho^{-n}$$

s o $n(P_2-P_1)$ m if and only if $\nu\left(\frac{\delta_2+\delta_1}{62-61}\right) = n+\nu\left(\frac{y_2+y_1}{x2-x1}\right) \leq n-m$.

Thus, $\nu(\xi_2-\xi_1) \geq m-n$ and $\nu(\delta_2-\delta_1) \geq m-n$ is clearly a necessary condition for $n(P_2-P_1) \geq m$.

If $\rho \neq 2$, $\delta_2+\delta_1$ is a unit, so $\nu(\delta_2+\delta_1) = 0$. Then, $(\xi_2-\xi_1) \geq m-n$ is a sufficient condition.

If $\rho = 2$, write

$$(\delta_2+\delta_1)(\delta_2-\delta_1) = \delta_2^2-\delta 1^2 = \xi_2^3 - \xi_1^3 + A(\xi_2-\xi_1)\rho^{4n}$$

$$= (\xi_2-\xi_1)(\xi_2^2+\xi_2\xi_1+\xi_1^2+A\rho^{4n}).$$

Since $\rho \neq 3$, $\xi_2^2+\xi_2\xi_1+\xi_1^2$ is a unit; and then so is $\xi_2^2+\xi_2\xi_1+\xi_1^2+A\rho^{4n}$. From the above, we know

$$\frac{\delta_2+\delta_1}{\xi_2-\xi_1} = \frac{\xi_2^2+\xi_2\xi_1+\xi_1^2+A\rho^{4n}}{\delta_2+\delta_1}$$

Therefore, $\nu(\delta_2-6_1)$ A m-n is a sufficient condition for $n(P_2-P_1) \geq m$, when $\rho = 2$.

Putting together the necessary and sufficient conditions gives that $(P_2-P_1) \in \Gamma_m$ if and only if $\nu(\xi_2-\xi_1) \geq m-n$ and $\nu(\delta_2-\delta_1) \geq m-n$.

In particular, when $n = m - 1$ this shows that Γ_{m-1}/Γ_m is finite for $m > 1$. [The lifting procedure described in 2. shows that the index of Γ_{m-1} in Γ_m is exactly ρ. Also, it should be noted that these arguments require that singularities be avoided.] This shows that Γ_m is of finite index in Γ_1.

We still must show that Γ/Γ_1 is finite. Let $n(P_1) = n(P_2) = 0$. We claim that $n(P2^-P1) \geq m$ when ξ_2 2 ξ_1 and $\delta_2 \equiv \delta_1$ modulo a sufficiently high power of p. Say $\xi_2 \equiv \xi_1$, $62 \equiv 61$ (mod ρ^{m+r}), for sufficiently large r.

If not, $n(P2-P1) < m$ implies $\nu(\delta_2+\delta_1) \geq r$ and $\nu(\xi_2^2+\xi_2\xi_1+\xi_1^2+A) \geq r$ by the above argument. Then, $\nu(2\delta_1) = \nu((\delta_1+\delta_2)-(\delta_2-\delta_1)) \geq r$ and $\nu(3\xi_1^2+A) = (\xi_2^2+\xi_2\xi_1+\xi_1^2+A) \geq r$. Thus we have $\nu(2\delta_1) \geq r$, and $\nu(3\xi_1^2+A) \geq r$.

However, we have the identity $-4A^3 - 27B^2 = (x^3+Ax+B)P(x) + (3x^2+A)Q(x)$ where $P(x) = 18Ax - 27B$, and $Q(x) = -6Ax^2 + 9Bx - 4A^2$. Putting $x = \xi_1$ and doubling, we get $2(-4A^3 - 27B^2) = 2\delta_1^2P(x)$ t $2(3\xi_1^2$ t $A)Q(x)$. But the right side $\equiv$Q (mod ρ^r) [since $2\delta_1^2$ 2 0, and $3\xi_1^2$ t $A \equiv 0$ (mod ρ^r)]. Thus, $2(-4A^3 - 27B^2)$ 5 0 (mod ρ^r), which cannot be for arbitrarily high r.

Thus Γ_m is of finite index in Γ.

In the previous theorem, we looked at groups Γ/Γ_m. These groups may be described in terms of what we did in Chapter 2 of this paper. The group Γ/Γ is just the group of points on the curve over the field of numbers mod ρ. In general. we have the following theorem for non-singular curves and $p \neq 2$. The singular cases are quite a bit more complicated.

Theorem 3.4 Γ/Γ_m is isomorphic to the group of points on the curve mod p^m.

proof: Let $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ be two finite points in with x_1 2 x2 and y_1 2 y_2 (mod ρ^m). Then we must show that $P_2-P_1 \in \Gamma_m$.

The addition formula gives the x-coordinate of the point P_2-P_1 as $\left(\frac{y_2+y_1}{x_2-x_1}\right)^2 - (x_2 + x_1)$. For $\rho \neq 2$, if $y_1 \equiv y_2 \not\equiv 0$ (mod ρ),

$$\nu\left(\left(\frac{y_2+y_1}{x_2-x_1}\right)^2 - (x_2+x_1)\right) = 2\nu\left(\frac{y_2+y_1}{x_2-x_1}\right) \leq -2m,$$ since y_2+y_1 is a unit and $x_2-x_1 \equiv 0$ (mod ρ^m).

So, $P_2-P_1 \in \Gamma_m$.

If $y_1 \equiv y_2 \equiv 0 \pmod{p}$, and the curve is non-singular mod p, we may write its equation in the form $y^2 = (x-x_0)g(x)$ where $x_1 \equiv x_2 \equiv x_0 \pmod{p}$ and $g(x_0)$ is a unit. Consider

$$\frac{y_2^2-y_1^2}{x_2-x_1} = g(x_2)\frac{x_2-x_0}{x_2-x_1} - g(x_1)\frac{x_1-x_0}{x_2-x_1}.$$ Here $g(x_1)$ and $g(x_2)$ are units.

We may assume that $v(x_1-x_0) \geq v(x_2-x_0)$. Suppose first $v(x_1-x_0) > v(x_2-x_0)$. Then

$$\begin{aligned} v(x_2-x_1) &= v((x_2-x_0)+(x_0-x_1)) \\ &= \min[v(x_2-x_0), v(x_0-x_1)] \\ &= v(x_2-x_0) \end{aligned}$$

This means that $\frac{x_2-x_0}{x_2-x_1}$ is a unit and $\frac{x_1-x_0}{x_2-x_1} \equiv 0 \pmod{p}$. Therefore, $\frac{y_2^2-y_1}{x_2-x_1}$ is a unit.

On the other hand, if $v(x_1-x_0) = v(x_2-x_0)$, we may write

$$\begin{aligned} \frac{y_2^2-y_1^2}{x_2-x_1} &= g(x_2)\frac{x_2-x_0}{x_2-x_1} - g(x_1)\left(\frac{x_1-x_2}{x_2-x_1} + \frac{x_2-x_0}{x_2-x_1}\right) \\ &= g(x_2)\frac{x_2-x_0}{x_2-x_1} - g(x_1)\frac{x_2-x_0}{x_2-x_1} + g(x_1) \\ &= \frac{g(x_2)-g(x_1)}{x_2-x_1}(x_2-x_0) + g(x_1). \end{aligned}$$

Since $g(x)$ is a polynomial, x_2-x_2 divides $g(x_2)-g(x_1)$. So $\frac{g(x_2)-g(x_1)}{x_2-x_2}$ is an integer. $x_2 \equiv x_0 \pmod{p}$, so $x_2-x_0 \equiv 0 \pmod{p}$. Thus the product $\equiv 0 \pmod{p}$. However $g(x_1)$ is a unit, so the sum is a unit. In either case, $\frac{y_2^2-y_1^2}{x_2-x_1}$ is a unit. We know

$$\frac{y_2+y_1}{x_2-x_1} = \frac{y_2^2-y_1^2}{x_2-x_1} \cdot \frac{1}{y_2-y_1}$$ where $y_1 \equiv y_2 \pmod{p}$. Therefore,

$$v\left(\frac{y_2+y_1}{x_2-x_1}\right) = v\left(\frac{1}{y_2-y_1}\right) \leq -m$$

$$v\left(\left(\frac{y_2+y_1}{x_2-x_1}\right)^2 - (x_2+x_1)\right) = 2v\left(\frac{y_2+y_1}{x_2-x_1}\right) \leq -2m.$$

Thus, $P_2-P_1 \in \Gamma_m$.

The curve may be parametrized as follows: $x = \frac{1}{t^2}$, $y = \frac{\varepsilon(t)}{t^3}$ where $\varepsilon(t) = (1 + At^4 + Bt^6)^{1/2}$. $\varepsilon(t)$ may be represented as a power series: $\varepsilon(t) = 1 + \sum_{i=2}^{\infty} \gamma_i t^{2i}$. [The series may be derived from the series $(1+u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 - \ldots$] This series converges p-adicly for $t \equiv 0 \pmod{p}$. From the formula for t, we see that t is the parameter of a point in Γ_m if and only if $v(t) \geq m$.

Let P_1, P_2 have parameters t_1, t_2 resp. Put $\varepsilon_1 = \varepsilon(t_1)$, $\varepsilon_2 = \varepsilon(t_2)$, $n_1 = n(P_1)$, and $n_2 = n(P_2)$. Let $P_3 = P_1 + P_2$ and have parameter t_3. We need to express the addition law in terms of the parameters; that is, we need t_3 in terms of t_1 and t_2.

We may assume $n_1 \leq n_2$.

$$\begin{aligned} \frac{y_2-y_1}{x_2-x_1} = \frac{\varepsilon_2 t_1^3 - \varepsilon_1 t_2^3}{t_1 t_2 (t_1^2-t_2^2)} &= \frac{t_1^3-t_2^3+\sum_2^{\infty} \gamma_i (t_2^{2i} t_1^3 - t_1^{2i} t_2^3)}{t_1 t_2 (t_1^2-t_2^2)} \\ &= \frac{t_1^3-t_2^3+t_1^3 t_2^3 \sum_2^{\infty} \gamma_i (t_2^{2i-3} - t_1^{2i-3})}{t_1 t_2 (t_1^2-t_2^2)} \end{aligned}$$

If we let $\theta = \sum_{i=2}^{\infty} \gamma_i \frac{t_2^{2i-3}-t_1^{2i-3}}{\quad}$, we get $\frac{y_2-y_1}{x_2-x_1} = \frac{t_1^2+t_1t_2+t_2^2-t_1^3t_2^3\theta}{t_1t_2(t_1+t_2)}$

By the addition formula,

$$\begin{aligned} \frac{1}{t_3^2} &= \left(\frac{t_1^2+t_1t_2+t_2^2-t_1^3t_2^3\theta}{t_1t_2(t_1+t_2)}\right)^2 - \left(\frac{1}{t_1^2} + \frac{1}{t_2^2}\right) \\ &= \frac{1}{(t_1+t_2)^2}\left(1-2t_1t_2(t_1^2+t_1t_2+t_2^2) + t_1^4t_2^4\theta^2\right) \end{aligned}$$

$$\frac{t_3}{t_1+t_2} = \left(1-2t_1t_2(t_1^2+t_1t_2+t_2^2)\theta + t_1^4t_2^4\theta^2\right)^{-1/2}$$

The right side of the above may be expressed as a power series using $(1-u)^{-1/2} = 1 + \frac{1}{2}u + \frac{3}{4}u^2 + \ldots$

We know $v(t_1) \geq n$, $v(t_2) \geq n$, and $v(t_1^2+t_1t_2+t_2^2) = v(t_1^2) \geq 2n_1$. **Therefore,** $v(t_1t_2(t_1^2+t_1t_2+t_2^2)) = v(t_1)+v(t_2)+v(t_1^2+t_1t_2+t_2^2) \geq 3n_1+n_2$.

If $\rho \neq 2$, θ is an integer (since the denominators of the γ_1 are powers of 2.) Then $v(t_1t_2(t_1^2+t_1t_2+t_2^2)\theta) \geq 3n_1+n_2$

$$\frac{t_3}{t_1+t_2} = v(t_1t_2t_1^2+t_1t_2+t_2^2\theta - \frac{1}{2}t_1^4t_2^4\theta^2+\dots)$$

$$= v(t_1t_2(t_1^2+t_1t_2+t_2^2)\theta) \geq 3n_1+n_2$$

If $P_1,P_2 \in \Gamma_m$, $v\left(\frac{t_3}{t_1+t_2} - 1\right) \geq 4m$

$$v(t_3-(t_1+t_2)) = v\left(\frac{t_3}{t_1+t_2} - 1\right) + v(t_1+t_2) \geq 4m+m = 5m.$$

Consider now multiples of a point P. By induction on ℓ in the above equation. we get (writing t for the parameter of P): $v(t(\ell P) - \ell t(P)) \geq 5m$. If $\ell = \rho$, $v(t(\rho P) - \rho t(P)) \geq 5n$, and so

$$v\left(\frac{t(\rho P)}{\rho t(P)} - 1\right) = v(t(\rho P) - \rho t(P)) - v(\rho t(P)) \geq 5n - (1+n) = 4n-1.$$

Then, by induction on s, $v\left(\frac{t(\rho^s P)}{\rho^s t(P)} - 1\right) \geq 4n-1$. Thus for all integers ℓ, $\left(\frac{t(\ell P)}{\ell t(P)} - 1\right) \geq 4n-1$. Therefore, for $P \in \Gamma_1$ (i.e. $n \geq 1$), if $\ell = r\rho^s$ where $\rho \nmid r$ then we have $v(t(\ell P) - \ell t(P)) = v(t(\ell P) - r\rho^s t(P))$

$$= v\left(\frac{t(\ell P)}{\ell t(P)} - 1\right) + v(r\rho^s t(P)) \geq (4n-1) + s > n+s.$$

That is, $n(\ell P) = n(P) + s$.

For $\rho = 2$, θ is not in general an integer. Therefore, we must make sure that it is not too bad---that is, that $v_2(\theta)$ is not too much below zero.

Going back to $(1+u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 - \dots = \sum_{i=0}^{\infty} \beta_i u^i$,

we see that $v_2(\beta_i) \geq -2i$. In the expansion for $\varepsilon(t)$, the u^i tern expands to terms in t^{4i} (and higher order terns). The coefficient of the t^i term is γ_{2i}; so $v_2(\gamma_{2i}) \geq -2i$, whereby $v_2(\gamma_i) \geq -i$.

We have $\theta = \sum_{i=2}^{\infty} \gamma_i \frac{t_2^{2i-3}-t_1^{2i-3}}{t_2-t_1}$.

$$v\left(\frac{t_2^{2i-3}-t_1^{2i-3}}{}\right) = v_2(t_2^{2i-4}+t_1t_2^{2i-5}+\dots+t_1^{2i-4})$$

$$\geq v_2(t_1^{2i-4}) \quad (2i-4)n_1.$$

Thus, v_2(ith term of θ) $\geq (2i-4)n_1 - i$.

For $n_1 \geq 1$ the series converges, and then $i - n_1(2i-4) \leq i-(2i-4) = 4-i \leq 2$, since the series starts at $i = 2$.

Again we have $v_2(t_1t_2(t_1^2+t_1t_2+t_2^2)) \geq 3n+n_2$. But now, $v_2(t_1t_2(t_1^2+t_1t_2+t_2^2)\theta) \geq 3n_1+n_2-2$. Now if $P_1,P_2 \in \Gamma_m$ with $m \geq 1$, $v_2(t_3-(t_1+t_2)) \geq 5m-2$. Proceding as before, $v_2(t(\ell P)-\ell t(P)) \geq 5m-2$,

$$v_2\left(\frac{t(2P)}{2t(P)} - 1\right) \geq 4n-3,$$

$$v_2\left(\frac{t(\ell P)}{\ell t(P)} - 1\right) \geq 4n-3.$$

So again for $P \in \Gamma$ and $\ell = r\rho^s$ $\rho \nmid r$, we have $n(\ell P) = n(P) + s$.

What this means is that n acts on the points of Γ_m in exactly the same way v acts on the p-adic integers. In fact, we have the following theorem:

Theorem 3.5 For $m \geq 1$, Γ_m is isomorphic to the additive group of p-adic integers.

proof: We need to show that there exists a $P_0 \in \Gamma_m$ such that any $P \in \Gamma_m$ can be uniquely expressed as $P = \zeta P_0$ where ζ is a p-adic integer.

From the values of $t \equiv 0 \pmod{p^m}$ choose a $t_0 \not\equiv 0 \pmod{\rho^{m+1}}$. Let P_0 be the point with t_0 as its parameter. Let $P_i = \rho^i P_0$ and t_i be the corresponding parameter. The preceding result gives $n(P_i) = n(P) + i$, i.e. $P \in \Gamma_{m+i}$. So,

$t_i \equiv 0 \pmod{\rho^{m+i}}$.

Let P be any point in Γ_m, and let $t \equiv 0 \pmod{\rho^m}$ be its parameter.

Let n_0 be the unique integer mod p such that $t \equiv n_0t_0 \pmod{\rho^{m+1}}$.

Let $P^{(1)} = P - n_0P_0$. Then $t^{(1)} = t(P^{(1)}) \equiv t - n_0t_0 \pmod{\rho^{5m}}$

$\equiv 0 \pmod{\rho^{m+1}}$.

So, $P^{(1)} \in \Gamma_{m+1}$.

Let n_1 be the unique integer mod p such that $t^{(1)} \equiv n_1t_1 \pmod{p^{m+2}}$.

Let $P^{(2)} = P^{(1)} - n_1P_1 \in \Gamma_{m+2}$ and let $P^{(2)}$ have parameter $t^{(2)}$.

Continue by induction:

$$P^{(k)} = P^{(k-1)} - n_{k-1}P_{k-1} = P - \sum_{i=0}^{k-1} n_iP_i \in \Gamma_{m+k}$$

$P = P^{(k)} + \sum_{i=0}^{k-1} n_iP_i$. As $k \to \infty$, $P^{(k)} \to 0$. So,

$$P = \sum_{i=0}^{\infty} n_i P_i = \sum n_i (\rho^i P_0) = P_0 (\sum_{i=0}^{\infty} n_i \rho^i).$$

Here $\sum n_i p^i$ is a p-adic integer, unique since the n_i are unique mod ρ.

Corollary 3.6 A point $P \varepsilon \Gamma$ of finite order is not in Γ_1. That is, it must have integer coordinates.

These results over p-adic fields have interesting consequences for the group of points on the curve over the field of rational numbers.

Theorem 3.7 **(Nagell-Lutz)** Let $y^2 = x^3 + Ax + B$ be non-singular and have integer coefficients. Then all rational points $P = (x,y)$ of finite order have integer coordinates such that $y = 0$ or $y^2 | -4A^3 - 27B^2$.

proof: If P is of finite order in the group of rational solutions, it is of finite order in the group of p-adic solutions for each p. Thus by the above corollary, x and y are integers in every p-adic field. But then they must be integers in the field of rationals.

If P is of order 2, then $y = 0$.

Otherwise, consider the point 2P. It is non-zero and of finite order. Thus it too has integer coordinates. The addition law gives the x-coordinate of 2P as $\left(\frac{f'(x)}{2y}\right)^2 - 2x$. For this to be an integer we must have $2y | f'(x)$ and then $y | f'(x)$. But we have the identity $-4A^3 - 27B^2 = f(x)P(x) + f'(x)Q(x)$ given in the proof of Theorem 3.3. $y^2 = f(x)$ so certainly $y | f(x)$. Now, $y | f(x)$ and $y | f'(x)$. Therefore, $y | f(x)P(x) + f'(x)Q(x)$. That is $y | -4A^3 - 27B^2$.

Footnoted References

Chapter 1

[1]Cassels, p. 211.
[2,3]Tate, Chapterl.
[4]Cassels, p. 210.

Chapter 2

[1]Borevich and Shafarevich, Chapter 1.
[2]Cassels, p. 242.
[3]Cassels, p. 211.
'tassels, p. 213

Chapter 3

[1]Lutz, pp. 239-244.

Bibliography

Borevich, Z.I., and I.R. Shafarevich, Number Theory, Academic Press, New York, 1966, 435 pp.

Cassels, J.W.S., "Diophantine Equations with Special Reference to Elliptic Curves", Journal of the London Mathematical Society, Vol. 41, London, 1966, pp. 193-291.

Lutz, Elizabeth. "Sur l'equation $y^2 = x^3-Ax-B$ dans les corps p-adiques", Journal fur die reine und angewandte Mathematik, Vol. 177, Walter de Gruyter & Co., Berlin, 1937, pp. 238-247.

Tate, John, Rational Points on Elliptic Curves, Haverford College, 1961, 107 pp.

Student paper presented at the meeting of Pi Mu Epsilon in Eugene, Oregon, August, 1969.

A DECIMAL APPROXIMATION TO π UTILIZING A POWER SERIES

Tim Golian and John Hanneken
Ohio University

CAN A CIRCLE BE SQUARED? This question has puzzled mankind since antiquity. Even before the 17th century mathematicians believed that the key to answering this question lie in a very special number - pi, the ratio of the circumference of a circle to its diameter. Since that time, mathematicians have tried to find a unique value for pi. Their attemps can be divided into three distinct periods.

In the first period, which was from the earliest times to the middle of the 17th century, the principle aims of mathematicians' studies were directed toward the approximation of pi by calculations of perimeters or areas of regular inscribed and circumscribed polygons.

From the middle of the 17th to the middle of the 18th century, the calculus of infinite series was utilized in the development of expressions for pi.

The last period, extending more then 150 years, pertained primarily to investigating and characterizing pi. In 1761. J.H. Lambert proved the irrationality of pi and in 1882 transcendence was established by F. Lindeman.

In the following development of π the specific objective relates to the second period, and thus the basic relations introduced in that era will be examined. Early expressions such as:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \ldots \text{ or } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots,$$

do not converge rapidly enough for practical use. For example, the latter expression, according to Newton, would require 5 BILLION terms to accurately calculate the value of pi to 20 decimal places. These relations were replaced by relations based upon the power series $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad (-1 \leq x \leq 1)$, which was discovered by James Gregory in 1671.

There are nine Important relations based on Gregory's series. These are:

#1. $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$ Charles Hutton - 1776

#2. $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$ John Machin - 1706

#3. $\frac{\pi}{4} = 8 \arctan \frac{1}{10} - 4 \arctan \frac{1}{515} - \arctan \frac{1}{239}$ S. **Klingenstierna** - 1730

#4. $\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}$ Euler - 1755

#5. $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{70} + \arctan \frac{1}{99}$ **Euler** - 1764

#6. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}$ L.K. **Schulz** von **Strassnitzky** - 1844

#7. $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$ Button - 1776

#8. $\frac{\pi}{4} = 3 \arctan \frac{1}{4} + \arctan \frac{1}{20} + \arctan \frac{1}{1985}$ S.L. **Loney** - 1893

#9. $\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$ Gauss

The **proofs** of these nine relations follows easily from the next example. The relations are found in "The Evolution of Extended Decimal Approximations to π," Wrench, Jr., **J.W.**, The Mathematics Teacher, December, 1960, pp. 644 - 650, which did not contain the proofs.

#1. SHOW: $2 \arctan \frac{1}{10} = \arctan \frac{1}{5} + \arctan \frac{1}{515}$

Let: $A = \arctan \frac{1}{10}$ $\quad 0 \leq A < \frac{\pi}{4}$

$B = \arctan \frac{1}{5}$ $\quad 0 \leq B < \frac{\pi}{4}$

$C = \arctan \frac{1}{515}$ $\quad 0 \leq C < \frac{\pi}{4}$

$$\tan (B + C) = \frac{\tan B + \tan C}{1 - (\tan B)(\tan C)}$$

$$\tan (B + C) = \frac{\frac{1}{5} + \frac{1}{515}}{1 - \left(\frac{1}{5}\right)\left(\frac{1}{515}\right)}$$

$$\tan (B + C) = \frac{20}{99}$$

$$\tan (2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\tan (2A) = \frac{2\left(\frac{1}{10}\right)}{1 - \left(\frac{1}{10}\right)^2}$$

$$\tan (2A) = \frac{20}{99}$$

$\tan (2A) = \tan (B + C)$

$2A = B + C$ Therefore: $2 \arctan \frac{1}{10} = \arctan \frac{1}{5} + \arctan \frac{1}{515}$

or: $\arctan \frac{1}{5} = 2 \arctan \frac{1}{10} - \arctan \frac{1}{515}$

#2. SHOW: $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$

Let: $A = \arctan \frac{1}{5}$ $\quad 0 \leq A < \frac{\pi}{4}$

$B = \arctan \frac{1}{239}$ $\quad 0 \leq B < \frac{\pi}{8}$

$$\tan (4A) = \frac{2 \tan (2A)}{1 - \tan^2 (2A)}$$

$$\tan (4A) = \frac{\frac{4 \tan A}{1 - \tan^2 A}}{1 - \frac{4 \tan^2 A}{(1 - \tan^2 A)^2}} = \frac{(4 \tan A)(1 - \tan^2 A)}{(1 - \tan^2 A)^2 - 4 \tan^2 A}$$

$$\tan (4A) = \frac{\left(\frac{4}{5}\right)\left(\frac{24}{25}\right)}{\left(\frac{24}{25}\right)^2 - \frac{4}{25}} = \frac{120}{119}$$

$$\tan (4A - B) = \frac{\tan (4A) - \tan B}{1 + \tan (4A) \tan B}$$

$$\tan (4A - B) = \frac{\frac{120}{119} - \frac{1}{239}}{1 + \left(\frac{120}{119}\right)\left(\frac{1}{239}\right)} =$$

$$\tan \frac{\pi}{4} = 1$$

$$\tan \frac{\pi}{4} = \tan (4A - B)$$

$\frac{\pi}{4} = 4A - B$ Therefore: $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$

#3. SHOW: $\frac{\pi}{4} = 8 \arctan \frac{1}{10} - 4 \arctan \frac{1}{515} - \arctan \frac{1}{239}$

Since: $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$

And; $\arctan \frac{1}{5} = 2 \arctan \frac{1}{10} - \arctan \frac{1}{515}$

Therefore substituting: $\frac{\pi}{4} = 8 \arctan \frac{1}{10} - 4 \arctan \frac{1}{515} - \arctan \frac{1}{239}$

This, therefore, completes the proof of relation number three.

In this development relation number three will be used. The relation was originally discovered in 1730 by S. Klingenstierna and rediscovered by Schellbach in about 1830. In 1926 it was used by C.C. Camp to evaluate $\pi/4$ to 56 places. Pi was calculated to 10021 places on a **Pegasus** computer by G.E. **Felton** on March 31, 1957, at the Ferranti Computer Center in London. Thirty-three hours of computer time were required to accomplish this. A later check revealed that the computer erred and the result was only accurate to 7480 decimal places. This relation was later replaced by more efficient relations, such as relation number two.

After choosing which relation to use, the next logical step is to determine bounds on the error.

Theorem #1: The magnitude of the partial sums of a convergent continually decreasing alternating series is less than the magnitude of the first term.

Proof:

Consider the convergent alternating series,

$$\sum_{i=1}^{\infty} (-1)^{i+1} a_i = a_1 - a_2 + a_3 - \dots + (-1)^{n+1} a_n + \dots$$

with the two partial sums,

$$S_n = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \dots + (a_{n-1} - a_n)$$

$$S_{n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_n - a_{n+1}).$$

The quantities in parenthesis are positive because $0 < a_{n+1} < a$ for all positive integers (definition of convergence). Since all quantities are positive $S_n > 0$ and $S_{n+1} < a_1$ for all positive integers. Furthermore since $S_{n+1} = S_n + a_{n+1}$ then $S < S_{n+1}$ and $0 < S_n < S_{n+1} < a_1$.

Theorem #2: If $\sum_{i=1}^{\infty} (-1)^{i+1} a_i = a_1 - a_2 + a_3 - \dots + (-1)^{n+1} a_n + \dots$ is a convergent alternating series, then the error (R_n) in approximation the sum of the series after its first n terms is less than the absolute value of the first neglected term (a_{n+1}).

Proof:

Consider the convergent alternating series as:

$$\sum_{i=1}^{\infty} (-1)^{i+1} a_i = a_1 - a_2 + a_3 - \dots + (-1)^{n+1} a_n + R_n,$$

where R_n is the error in approximating the sum of the series after its first n terms. R_n may also be written as: $\sum_{i=n+1}^{\infty} (-1)^{i+1} a_i$, which would also be a convergent alternating series. As shown in Theorem #1, the magnitude of each of the partial sums of R_n is less than the magnitude of a_{n+1} and consequently $|R_n| < |a_{n+1}|$.

Theorem #3: If $|R_n(x_0)| < 5 \cdot 10^{-P}$, then a decimal approximation for $\arctan(x_0)$ correct to p-1 places can be obtained by using the first n terms of the power series (this follows directly from rules of round off).

<u>FIND THE NUMBER OF TERMS NECESSARY TO OBTAIN π CORRECT TO 16 DECIMAL PLACES.</u>

Since $\pi = 32 \arctan \frac{1}{10} - 16 \arctan \frac{1}{515} - 4 \arctan \frac{1}{239}$

or
$$\pi = 32\left[\sum_{n=1}^{m} \frac{(-1)^{n+1}\left(\frac{1}{10}\right)^{2n-1}}{2n-1} + S_m\right] - 16\left[\sum_{n=1}^{w} \frac{(-1)^{n+1}\left(\frac{1}{515}\right)^{2n-1}}{2n-1} + T_w\right] - 4\left[\sum_{n=1}^{v} \frac{(-1)^{n+1}\left(\frac{1}{239}\right)^{2n-1}}{2n-1} + Q_v\right]$$

where S_m, T_w, and Q_v are the respective remainders after the m^{th}, w^{th}, and v^{th} term. By theorem #3, the total remainder must be less than $(5)(10^{-17})$ for 16 place accuracy. Thus choose m, w, and v such that $|32 \cdot S_m - 16 \cdot T_w - 4 \cdot Q_v| < 5 \cdot 10^{-17}$. Now, $|32 \cdot S_m - 16 \cdot T_w - 4 \cdot Q_v| \leq 32|S_m| + 16|T_w| + 4|Q_v|$. According to theorem #2,

when: $m = 7$ $\quad |32 \cdot S_m| < 213.344 \times 10^{-17}$

$w = 2$ $\quad |16 \cdot T_w| < 8833.12 \times 10^{-17}$

$v = 2$ $\quad |4 \cdot Q_v| < 102588.924 \times 10^{-17}$.

Since each remainder taken separately is greater than $(5)(10^{-17})$ then at least one more term must be taken from each series.

when: $m = 8$ $\quad 32 \cdot S_m < 1.888 \times 10^{-17}$

$w = 3$ $\quad 16 \cdot T_w < 0.016 \times 10^{-17}$

$v = 3$ $\quad 4 \cdot Q_v < 1.284 \times 10^{-17}$.

Therefore, $32|S_m| + 16|T_w| + 4|Q_v| < 3.188 \times 10^{-17} < (5)(10^{-17})$.

Thus, π will be correct to 16 decimal places when 8 terms of **arctan** $\frac{1}{10}$ and 3 terms of **arctan** $\frac{1}{515}$ and 3 terms of **arctan** $\frac{1}{239}$ are calculated according to the **idenity** for π, which is:

$$\pi \approx 32\left[\frac{1}{10} - \frac{\left(\frac{1}{10}\right)^3}{3} + \frac{\left(\frac{1}{10}\right)^5}{5} - \frac{\left(\frac{1}{10}\right)^7}{7} + \frac{\left(\frac{1}{10}\right)^9}{9} - \frac{\left(\frac{1}{10}\right)^{11}}{11} + \frac{\left(\frac{1}{10}\right)^{13}}{13} - \frac{\left(\frac{1}{10}\right)^{15}}{15}\right] - 16\left[\frac{1}{515} - \frac{\left(\frac{1}{515}\right)^3}{3} + \frac{\left(\frac{1}{515}\right)^5}{5}\right] - 4\left[\frac{1}{239} - \frac{\left(\frac{1}{239}\right)^3}{3} + \frac{\left(\frac{1}{239}\right)^5}{5}\right].$$

BOUNDS ON ERROR DUB TO ROUNDING OFF.

$\pi = 32$ **arctan** $\frac{1}{10} - 16$ **arctan** $\frac{1}{515} - 4$ **arctan** $\frac{1}{239}$

Maximum error in **arctan** $\frac{1}{10}$: 8 terms, each with $0.5 error in the last digit used, or $(8)(0.5) = 4$; therefore, maximum error in 32 **arctan** $\frac{1}{10}$ is $(32)(4) = 128$, and

Maximum error in **arctan** $\frac{1}{515}$: 3 terms, each with $0.5 **error** in the last digit used, or $(3)(0.5) = 1.5$; therefore, maximum error in 16 **arctan** $\frac{1}{515}$ Is $(16)(1.5) = 24$, and

Maximum error in **arctan** $\frac{1}{239}$: 3 terms, each with $0.5 error in the last digit used, or $(3)(0.5) = 1.5$; therfore, maximum error in 4 **arctan** $\frac{1}{239}$ is $(4)(1.5) = 6$.

Total maximum error due to rounding off is $128 + 24 + 6 = 158$

Therefore,calculations oust be carried out to 20 decimal places to assure 16 place accuracy.

$\frac{1}{10^1} = 0.10000000000000000000$

$\frac{1}{3(10)^3} = 0.00033333333333333333$

$\frac{1}{5(10)^5} = 0.00000200000000000000$

$\frac{1}{7(10)^7} = 0.00000001428571428571$

$\frac{1}{9(10)^9} = 0.00000000011111111111$

$\frac{1}{11(10)^{11}} = 0.00000000000090909091$

$\frac{1}{13(10)^{13}} = 0.00000000000000769238$

$\frac{1}{15(10)^{15}} = 0.00000000000000006667$

$\frac{1}{17(10)^{17}} = 0.00000000000000000059$

$\frac{1}{515} = 0.00194174757281553398$

$\frac{1}{3(515)^3} = 0.00000000244037775828$

$\frac{1}{5(515)^5} = 0.00000000000000552070$

$\frac{1}{7(515)^7} = 0.00000000000000000001$

$\frac{1}{239} = 0.00418410041841004184$

$\frac{1}{3(239)^3} = 0.00000002441659178708$

$\frac{1}{5(239)^5} = 0.00000000000025647231$

$\frac{1}{7(239)^7} = 0.00000000000000000321$

arctan $\frac{1}{10}$:

term #	positive terms	negative terms
1	0.10000000000000000000	
2		0.00033333333333333333
3	0.00000200000000000000	
4		0.00000001428571428571
5	0.00000000011111111111	
6		0.00000000000090909091
7	0.00000000000000769238	
8		0.00000000000000006667
sum	0.10000200011111880349	0.00033334761995677662

$$\begin{array}{r} 0.10000200011111880349 \\ -0.00033334761995677662 \\ \hline 0.09966865249116202687 \end{array}$$

32 arctan $\frac{1}{10} \approx 3.18939687971718485984$

arctan $\frac{1}{515}$:

term #	positive terms	negative terms
1	0.00194174757281553398	
2		0.00000000244037775828
3	0.00000000000000552070	
sum	0.00194174757282105468	0.00000000244037775828

$$\begin{array}{r} 0.00194174757282105468 \\ -0.00000000244037775828 \\ \hline 0.00194174513244329640 \end{array}$$

16 **arctan** $\frac{1}{515} \approx 0.03106792211909274240$

arctan $\frac{1}{239}$:

term #	positive terms	negative terms
1	0.00418410041841004184	
2		0.00000002441659178708
3	0.00000000000025647231	
sum	0.00418410041866651415	0.00000002441659178708

$$\begin{array}{r} 0.00418410041866651415 \\ -0.00000002441659178708 \\ \hline 0.00418407600207472707 \end{array}$$

4 arctan $\frac{1}{239} \approx 0.01673630400829890828$

$$\pi = 32 \arctan \frac{1}{10} - 16 \arctan \frac{1}{515} - 4 \arctan \frac{1}{239}$$

$$\begin{array}{rr} & 3.18939687971718485984 \\ & -0.03106792211909274240 \\ \hline & 3.15832895759809211744 \\ & -0.01673630400829890828 \\ \hline \pi \approx & 3.14159265358979320916 \end{array}$$

According to theory the answer is only correct to 16 decimal places, therefore: π as 3.1415926535897932

Established value accurate to 16 decimal places: $\pi \approx 3.1415926535897932$ (from page A9 of Handbook of Chemistry and Physics, The Chemical Rubber Co., Cleveland, 47th Ed., 1966).

Student paper presented at the meeting of Pi Mu Epsilon in Eugene, Oregon, August, 1966.

NEED MONEY? AND MATCHING PRIZE FUND

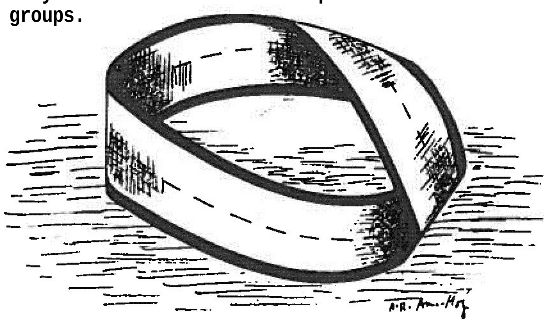
The Governing Council of Pi Mu Epsilon announces a contest for the best expository paper by a student (who has not yet received a masters degree) suitable for publication in the Pi Mu Epsilon Journal. The following prizes will be given

$200. first prize
$100. second prize
$ 50. third prize

providing at least ten papers are received for the contest.

In addition there will be a $20. prize for the best paper from any one chapter, providing that chapter submits at least five papers.

The Governing Council of Pi Mu Epsilon has approved an increase in the maximum amount per chapter allowed as a matching prize from $25.00 to $50.00. If your chapter presents awards for outstanding mathematical papers and students, you may apply to the National Office to match the amount spent by your chapter--i.e., $30.00 of awards, the National Office will reimburse the chapter for $15.00, etc.,--up to a maximum of $50.00. Chapters are urged to submit their best student papers to the Editor of the Pi Mu Epsilon Journal for possible publication. These funds may also be used for the rental of mathematical films. Please indicate title, source and cost, as well as a very brief comment as to whether you would recommend this particular film for other Pi Mu Epsilon groups.



SOME COMMENTS ON TERMINOLOGIES RELATED TO DENSENESS

R. Z. Yeh, University of Hawaii

The definitions of denseness, nowhere-denseness, and denseness-in-itself can be very confusing to the students learning about them for the first time. Perhaps more than anything the terminologies are at fault.

The familiar topological descriptions of sets, such as compactness, connectedness, openness and closedness, are either invariant or non-invariant with respect to subspace topologies. We recall that given a topological space X a subset A is said to be dense in a subset B if the closure of A contains B; in particular A is said to be dense in X (or dense everywhere) if the closure of A is X. A subset A is said to be nowhere-dense in B if the complement in B of the closure of A is dense in B; in particular A is said to be nowhere-dense in X if the complement of the closure of A is dense in X. Obviously, denseness and nowhere-denseness are non-invariant concepts. For example, the set of all rationals is dense in the real x-axis, nowhere-dense in the entire xy-plane, and neither in the union of the x-axis and the first quadrant. The often used phrase "dense everywhere", though convenient, is not really appropriate. It is almost as bad as if one were to say that A is "open everywhere" when one really means that A is open with respect to the topology of X. The word "nowhere" can be confusing also, especially when one has to consider "nowhere-dense? in the whole space X? or in some set B?" We also recall that a subset A of a topological space is said to be dense-in-itself if every point of A is a limit point of A. It is not difficult to show that if A is dense-in-itself with respect to the subspace topology of some set containing A, it is dense-in-itself with respect to the subspace topology of any set containing A. Denseness-in-itself is thus an invariant concept, and the term is suggestive of this. Only the word "dense" used here has nothing to do with the same word used earlier. One should keep this in mind or else substitute a new term for "dense-in-itself".

Hocking and Young [1] points out that the terminology "dense in itself" (meaning of course dense-in-itself, since every set is trivially dense in itself) is misleading. Dugundji [2] paranthetically calls a nowhere-dense set a sieve. The choice of a noun instead of an adjective, however, might obscure the fact that nowhere-denseness is only a non-invariant concept.

References

1. J. G. Hocking and G. S. Young, Topology, Addison-Wesley, Reading, Mass., (1961) p. 88.
2. J. Dugundji, Topology, Allyn and Bacon, Boston (1966) p. 250.

A NECESSARY AND SUFFICIENT CONDITION FOR CERTAIN TAUBERIAN THEOREMS

A. M. Fischer
West Virginia University

1. INTRODUCTION

This study is concerned with certain questions left open about Tauberian theorems by previous authors. Specifically, this paper demonstrates a necessary and sufficient condition for a generalization of a class of Tauberian theorems studied by Hardy and Littlewood [1] and more recently by A.E. Ingham [2]. For a brief discussion of the history of these theorems and eveolution of the methods employed in their proofs, the reader is referred to Ingham [2].

Ingham's theorem [2, Th. A, p. 160], in fact a generalization of Hardy and Littlewood's, states:

"Suppose that

$$F(s) = \int_0^\infty A(u)e^{-su}du \qquad (s > 0),$$

where $A(u)$ is positive and for $u > 0$. Let $L(u)$ be a (strictly) positive function such that $L(cu) \sim L(u)$ as $u \to \infty$ for each fixed $c > 0$; and suppose $a > -1$. Suppose, further, that

$$F(s) \sim A\frac{\Gamma(\alpha+1)}{s^{\alpha+1}} L(1/s) \qquad \text{as } s \to 0_+.$$

Then

$$A(u) \sim Au^\alpha L(u) \qquad \text{as } u \to \infty."$$

What this paper demonstrates is the following generalization:

THEOREM 1. Let

$$g(1/x) = xF(x) = x\int_0^\infty A(t)e^{-xt}dt \qquad (x > 0),$$

where $A(t)$ is non-negative and monotonic (in the wide sense) but not identically zero, for $t > 0$. Then if $\xi > 0$, the following statements are equivalent:

(1) $\xi A(x) \sim g(x)$ as $x \to \infty$

(2) $\xi g(x) \sim \int_0^\infty g(xt)e^{-t}dt$ as $x \to \infty$

Although the proof is omitted, it is interesting to note that Ingham's theorem can be deduced directly from Theorem 1.

Theorem 1 is a conclusion of Lemmas 2 and 4: Lemma 2 establishes that (1) $\Rightarrow$ (2); Lemma 4 shows that (2) implies

$$\limsup_{x\to\infty} A(x)/g(x) \le 1/\xi \le \liminf_{x\to\infty} A(x)/g(x),$$

which completes the proof. Lemmas 1 and 3 pertain to the behavior of g. Lemma 1 is interesting in its own right insofar as it demonstrates a necessary restriction on the rate at which g can decrease.

2. NOTATION AND ASSUMPTIONS

The notation employed herein should be construed as follows: $\uparrow$ and $\nearrow$ respectively signify 'strictly increasing' and 'non-decreasing' just as $\downarrow$ and $\searrow$ respectively signify 'strictly decreasing' and 'non-increasing'. In addition $\uparrow\infty$ indicates 'increasing and unbounded'.

Since Theorem 1 is the goal of this paper, its hypotheses are assumed without further mention. Furthermore, the convergence of the integrals $\int_0^\infty A(t)e^{-xt}dt$ and $\int_0^\infty g(xt)e^{-t}dt$ for $x > 0$ is also assumed.

3. PROOF

Before starting any proofs, it is wise to note two important facts: first, from the definition of g, it follows that $g(x) > 0$ $(x > 0)$, and that if $A(x) \nearrow$ or $\searrow$, then $g(x)$ behaves in the same respective manner; and second, that

$$g(x) = \int_0^\infty A(xt)e^{-t}dt \ge \begin{cases} e^{-1}A(x) & \text{for } A\nearrow \\ (1-e^{-1})A(x) & \text{for } A\searrow \end{cases}$$

so that lim inf A/g and lim sup A/g actually exist.

LEMMA 1. If either (1) or (2) is true, then $xg(x)\uparrow\infty$.

Proof: Under the hypotheses of Theorem 1, $xg(x) = \int_0^\infty A(t)e^{-t/x}dt$ so that $xg(x)\uparrow$. Assume that $xg(x)$ is bounded, then $\lim_{x\to\infty} xg(x) = C > 0$.

CASE i. If (1) is true, then $\exists x' \ni \forall x > x'$: $\xi xA(x) > C/2$

$$xg(x) = \int_0^\infty A(t)e^{-t/x}dt > \int_{x'}^\infty A(t)e^{-t/x}dt > C/(2\xi)\int_{x'}^\infty t^{-1}e^{-t/x}dt,$$

which indicates $xg(x)$ is unbounded. This is a contradiction; consequently $xg(x)$ is unbounded.

CASE ii. If (2) is true, then $\exists x' \ni \forall x > x'$ we have both $xg(x) > C/2$ and $2\xi xg(x) > \int_0^\infty g(t)e^{-t/x}dt$. Hence

$$4\xi xg(x) > 2\int_{x'}^\infty g(t)e^{-t/x}dt > C\int_{x'}^\infty t^{-1}e^{-t/x}dt,$$

which also indicates $xg(x)$ is unbounded, a contradiction.

LEMMA 2. $(1) \Rightarrow (2)$.

Proof: $(1) \Rightarrow \forall\, \varepsilon > 0,\ \exists x' \ni \forall x > x'$: $|g(x)/A(x)-\xi| < \varepsilon$. Hence

$$|g(x)^{-1}\int_0^\infty g(xt)e^{-t}dt - \xi| \leq g(x)^{-1}\int_0^\infty |g(xt)-\xi A(xt)|e^{-t}dt$$

$$< g(x)^{-1}\int_{x'/x}^\infty \varepsilon A(xt)e^{-t}dt + x^{-1}g(x)^{-1}\int_0^{x'} |g(t)-\xi A(t)|e^{-t/x}dt$$

$$< \varepsilon\ t\ [\int_0^{x'} g(t)+\xi A(t)dt]/[xg(x)].$$

As a consequence of Lemma 1, and since ε is arbitrary, it follows that $(1) \Rightarrow (2)$, which was to be shown.

Lemmas 3 and 4 are devoted to showing that $(2) \Rightarrow (1)$.

In Lemmas 3 and 4 we shall take $\nu=+1$ if $A\nearrow$; $\nu=-1$ if $A\searrow$ (if A is constant, arbitrarily take $\nu=+1$).

LEMMA 3. Define $B(q) = \limsup_{x\to\infty} g(xq)/g(x)$. If (2), then $B(q)$ exists for every q and $\lim_{q\to 1_+} B(q^\nu) = 1$.

CASE i. If A^* then $g\downarrow$ and $\nu=-1$. $B(q)$ exists for $q \geq 1$ simply because g is non-increasing. If $q > 1$, then by Lemma 1 we have

$$(x/q)g(x) \leq (x/q)g(x/q) \leq xg(x)$$

from which we infer that $B(q^{-1})$ exists and that $1 \leq B(q^{-1}) \leq q$.

CASE ii. If $A\nearrow$, from (2) we see: $\forall\, \varepsilon > 0,\ \exists x' \ni \forall q > 0,\ \forall x > x'$:

$$(3)\qquad (\xi+\varepsilon)g(x) > \int_q^\infty g(xt)e^{-t}dt \geq \int_q^\infty g(xq)e^{-t}dt = e^{-q}g(xq)$$

and also

$$(\xi+\varepsilon)g(x/\log 2) > \log 2\int_0^\infty g(xt)2^{-t}dt \geq \log 2\int_q^\infty g(xq)2^{-t}dt$$

$$= 2^{-q}g(xq),$$

consequently $B(q)$ exists. Put $q = 1/\log 2$ in (3) and obtain

$$(4)\qquad g(xq) < (\xi+\varepsilon)^2\exp(1/\log 2)2^q g(x).$$

Now consider $1 < q < 1/\log 2$. For $\forall \varepsilon > 0$ and for x sufficiently large

$$\frac{1}{\xi+\varepsilon}g(xq) - \frac{1+\varepsilon}{\xi}g(x) < \int_0^\infty [g(xqt)-g(xt)]e^{-t}dt$$

$$< \int_1^\infty g(xt)e^{-t/q}[q^{-1}-\exp(-t(1-q^{-1}))]dt$$

$$< \int_1^\infty g(xt)e^{-t/q}(q^{-1}-q^{-t})dt < c_0 g(x)\int_1^\infty 2^{-t}e^{-t/q}(q^{-1}-q^{-t})dt,$$

by (4), where $c_0=(\xi+1)^2\exp(1/\log 2)$. Divide this by $g(x)$, take the lim sup as $x\to\infty$, the limit as $\varepsilon\to 0$, and note that $g\nearrow$. This results in

$$1 \leq B(q) \leq 1 + 2c_0 q(1-q\log 2)^{-2}e^{-1/q}\log q.$$

The lemma follows immediately.

LEMMA 4. Define ζ_L and ζ_H respectively as $\liminf_{x\to\infty}$ and $\limsup_{x\to\infty}$ of $\frac{A(x)}{g(x)}$; then $(2) \Rightarrow \zeta_L \geq 1/\xi \geq \zeta_H$. In the interest of brevity only the first inequality will be shown in full detail, the proof of the second is conceptually the same.

Proof: Consider an arbitrary q $(1 < q < 1/\log 2)$. Define $p(t) = 4(2^{-t}-2^{-2t})$ and select an $N \geq 2$, then set $H(t) = p^N(t)$ and $h(t) = H(t)-\theta^{N-1}p(t)$ where $\theta=\theta(q) = \max\{p(q^{-1}),p(q)\}$. Since $p(t)\uparrow$ $(0\leq t\leq 1)$ and $p(t)\downarrow$ $(t\geq 1)$, obviously $\theta < 1$. Furthermore

$$(5)\qquad H(t) \text{ and } h(t) \text{ are both of the form } \sum_{r=1}^R v_r e^{-t\rho_r}\quad (\rho_r > 0),$$

$$(6)\qquad H(t) \geq 0\ (t\geq 0) \text{ and } h(t) \leq \begin{cases} H(t) & \\ 0 & t\notin(q^{-1},q)\end{cases}.$$

For an additional ease of notation also define

$$J_q = \int_{q^{-1}}^q H(t)dt,\quad \begin{matrix} J(x) \\ j(x)\end{matrix}\Big\} = \int_0^\infty A(xt)\begin{Bmatrix} H(t) \\ h(t)\end{Bmatrix}dt, \text{ and } \begin{matrix} J_q(x) \\ j_q(x)\end{matrix}\Big\} = \int_{q^{-1}}^q A(xt)\begin{Bmatrix} H(t) \\ h(t)\end{Bmatrix}dt.$$

It follows from (6) that

$$(7)\qquad 0 < J(x) \geq J_q(x) \geq j_q(x) \geq j(x).$$

It will be clear that (8) through (13) hold for any particular $\varepsilon > 0$ if x is sufficiently large. Now observe that

$$j(x)-q^{-1}J(x) = \int_0^\infty A(xt)[H(t)-\theta^{N-1}p(t)-q^{-1}H(t)]dt$$

$$> -4\theta^{N-1}\int_0^\infty A(xt)2^{-t}dt = -\frac{4}{\log 2}\theta^{N-1}g(x/\log 2)$$

$$(8)\qquad > -c_1\theta^{N-1}g(xq^\nu)$$

where $c_1 = 4[B(q^{-\nu}/\log 2)+1]/\log 2 > 0$. In view of (7), this leads to

$$J(x) \leq q[J_q(x)-(j(x)-q^{-1}J(x))] < q\int_{q^{-1}}^q A(xt)H(t)dt + c_1 q\theta^{N-1}g(xq^\nu)$$

$$(9)\qquad \leq qA(xq^\nu)J_q + c_1 q\theta^{N-1}g(xq^\nu).$$

Clearly $\xi \int_0^\infty A(xt)v_r e^{-t\rho_r}dt \sim \int_0^\infty g(xt)v_r e^{-t\rho_r}dt$ $(\rho_r > 0)$; hence, in light of (5) [since x is sufficiently large]

$$J(x) = \int_0^\infty A(xt)H(t)dt \geq (\xi+\epsilon)^{-1}\int_0^\infty g(xt)H(t)dt > (\xi+\epsilon)^{-1}\int_{q^{-1}}^{q} g(xt)H(t)dt$$

$$(10) \quad \geq (\xi+\epsilon)^{-1}g(x/q^{\nu})J_q > (\xi+\epsilon)^{-1}(B(q^{2\nu})+\epsilon)^{-1}g(xq^{\nu})J_q,$$

where the last step is a result of Lemma 4. Combine (9) and (10) and divide by $g(xq^{\nu})qJ_q$ to obtain

$$(11) \qquad \frac{1}{(\xi+\epsilon)(B(q^{2\nu})+\epsilon)q} < \frac{A(xq^{\nu})}{g(xq^{\nu})} + \frac{c_1}{J_q}\theta^{N-1}.$$

Now let us momentarily consider J_q. Since $p''(t) \leq 0$ $(0 \leq t \leq)$, $p(0) = 0$ and $p(1) = 1$ it is evident that $p(t) \geq t$ $(0 \leq t \leq 1)$. Thus

$$J_q = \int_{q^{-1}}^{q} H(t)dt > \int_{q^{-1}}^{1} t^N dt = \frac{1}{N+1}(1-q^{-N-1}).$$

However, $q^{-1} \leq p(q^{-1}) \leq \max\{p(q^{-1}),p(q)\} = \theta < 1$, so

$$(12) \qquad J_q > \frac{1}{N+1}(1-\theta^{N+1}).$$

Combine (11) and (12) and then take limits as $x\to\infty$ and $\epsilon\to 0$; we readily obtain

$$(13) \qquad \zeta_L \geq [\xi B(q^{2\nu})q]^{-1} - c_1(N+1)(1-\theta^{N+1})^{-1}\theta^{N-1}.$$

Because N was chosen as any integer ≥ 2, it can be taken large enough so that the last term in (13) is arbitrarily close to zero (recall that $\theta < 1$). Finally take the limit as $q\to 1_+$ and apply Lemma 3. This proves the first inequality of Lemma 4.

To prove the second inequality, alter the definitions of $J(x)$, $j(x)$, $J_q(x)$ and $j_q(x)$ by replacing A with g. Then in parallel to (8), (9) and (10), we obtain

$$j(x)-q^{-1}J(x) > -4\theta^{N-1}(\xi+\epsilon)g(x/\log 2) > -c_2\theta^{N-1}g(xq^{\nu}),$$

$$J(x) < qg(xq^{\nu})(J_q+c_2\theta^{N-1}) < qg(x/q^{\nu})(B(q^{2\nu})+\epsilon)(Jq+c_2\theta^{N-1}),$$

and

$$J(x) > (1+\epsilon)^{-1}\xi\int_{q^{-1}}^{q} A(xt)H(t)dt \geq (1+\epsilon)^{-1}\xi A(x/q^{\nu})J_q,$$

from which $\zeta_H \leq 1/\xi$ is a simple deduction.

REFERENCES

1. G.H. Hardy, "Tauberian Theorems Concerning Power Series and Dirichlet's Series Whose Coefficients Are Positive", Proc. London Math. Soc. (2) 13 (1913) 174–91.

2. A.E. Ingham, "On Tauberian Theorems", Proc. London Math. Soc. (3) 14A (1965) 157–73.

The author is indebted to editors of the Jour. Am. Math. Soc. for these references:

3. A. Edrei and W.H.J. Fuchs, Tauberian Theorems for a class of meromorphic functions, Proc. Int. Conf. in Function Theory, Erevan, Armenian S. S. R. (1966). 339-58.

4. D. Drasin, Tauberian Theorems and slowly varying functions, Trans. Amer. Math. Soc. 138(1968), 333–56.

5. D. Shea, On a complement to Valiron's tauberian theorem for the Stieltjes transform, Proc. Amer. Math. Soc. (1969), 1–9.

Student paper presented at the meeting of Pi Mu Epsilon in Eugene, Oregon, August, 1969.

UNDERGRADUATE RESEARCH PROPOSALS

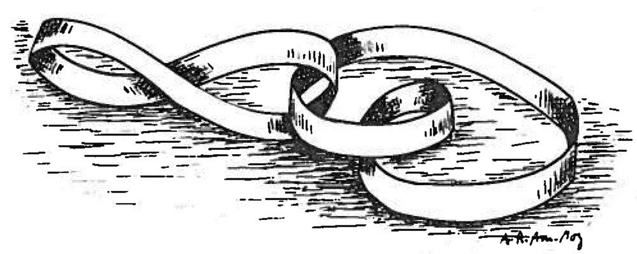
Bernard McDonald
University of Oklahoma

1) Develop a theory for the n x n matrices over a field having the property such that every submatrix has non-zero determinant. Is the Vandermond matrix of this form?

2) Two matrices over a finite field $GF(p^n)$ are to be considered equivalent if they differ by a row or column permutation. Count the number of equivalence classes and number of matrices in each class.

3) Determine a cannonical matrix for the ring $M_n(R)$ of n x n matrices over a principal ideal domain R under operation on the left by unimodular matrices and on the right by permutation matrices.

4) Let $m > 0$ be square free. Take $a = a + b\sqrt{m}$, where a and b are from Q, the set of rationals. Let G be the multiplicative group of $Q[\sqrt{m}]$ and H be $Q - \{0\}$. Is the quotient G/H finitely generated?



A CHARACTERIZATION OF HOMEOMORPHIC T1 SPACES

W. M. Priestley

Beginning students of topology appreciate the following theorem, whereas writers of elementary textbooks apparently do not.

THEOREM: Let $(X, \mathcal{S})$, $(Y, \mathcal{J})$ be T1 spaces. $(X, \mathcal{S})$ and $(Y, \mathcal{J})$ are homeomorphic $\Longleftrightarrow$ $(\mathcal{S}, \subseteq)$ and $(\mathcal{J}, \subseteq)$ are isomorphic as partially ordered sets.

PROOF: $(\Longrightarrow)$ If $f: X \to Y$ is a homeomorphism, then $I: \mathcal{S} \to \mathcal{J}$ defined by $I(G) = \{f(x) \mid x \in G\}$ for $G \in \mathcal{S}$ is an order isomorphism.

$(\Longleftarrow)$ If $I: \mathcal{S} \to \mathcal{J}$ is an order isomorphism, consider the complementary lattices $\mathcal{S}'$ and $\mathcal{J}'$ of closed subsets and the induced order isomorphism $I': \mathcal{S}' \to \mathcal{J}'$ defined for $F \in \mathcal{S}'$ by $I'(F) = I(F')'$, where S' denotes the complement of the set S. In T1 spaces singleton sets are closed. They are also minimal in the sense that each is preceded by exactly one other set (the empty set ϕ) in the ordering $\subseteq$. An order isomorphism sends minimal elements into minimal elements, and it therefore makes sense to define a function $f: X \to Y$ by $\{f(x)\} = I'(\{x\})$ for $x \in X$. f is one-one and onto since I' is, by an elementary argument similar to that given in [1]. It is a simple exercise to show that for each $F \in \mathcal{S}'$, $f(F) = I'(F)$, from which it follows that both f and f^{-1} are continuous.

The example of $X = \{1\}$, $Y = \{1,2\}$, $\mathcal{S} = \mathcal{J} = \{\phi, \phi'\}$ shows the T1 hypothesis to be essential.

Compare Kelley's final remark on p. 130 of [2].

References

1. Chan Kai-Meng, An alternative formulation of an unsolved problem of set theory, Amer. Math. Monthly, 76(1969) 53.

2. John L. Kelley, General topology, van Nostrand, Princeton, 1955.

PROBLEM DEPARTMENT

Edited by
Leon Bankoff, Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Solutions should be submitted on separate, signed sheets and mailed before August 1, 1970

Address all communications concerning problems to Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

232. Proposed by Solomon W. Golomb, University of So. Calif., Los Angeles.

Find a direct combinatorial interpretation of this identity:

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}.$$

233. Proposed by Charles W. Trigg, San Diego, California.

The director of a variety show wanted to give the female impersonator a job, but questioned his ability to dance with the high-kicking Folies Bergere chorus. In reply to the director's query, the impersonator's Spanish agent said:

"SI/HE = .CAN CANCAN...,

but .CAN be less than one-fourth effective in his demonstration today."

If each letter of the cryptarithm uniquely represents a digit in the scale of eleven, what is the sole solution?

234. Proposed by Charles W. Trigg, San Diego, California.

Show that when the nine positive digits are distributed in a square array so that no column, row, or unbroken diagonal has its digits in order of magnitude, the central digit must always be odd.

2	8	7
6	1	4
5	3	9

235. Proposed by James E. Desmond, Florida State University.

Prove that a^{n+1} divides $(ab + c)^{(ad)^n} - c^{(ad)^n}$ for integers $a > 0$, b, c, $d > 0$ and $n \geq 0$.

236. Proposed by Erwin Just, Bronx Community College.

If k is a positive integer, prove that $(6^{16k+2}/2) - 1$ is not a prime.

237. Proposed by Leonard **Barr,** Beverly Hills, California.

The diameter of a semi-circle is divided into two segments, a and b, by its point of contact with an inscribed circle. Show that the diameter of the inscribed circle is equal to the harmonic mean of a and b.

238. Proposed by David L. Silverman, Beverly Hills, California.

A necessary and sufficient condition that a triangle exist is that its sides, a, b, and c satisfy the inequalities (1) $a < b + c$, (2) $b < a + c$, (3) $c < a + b$. Express (1), (2), and (3) in a single inequality.

239. Proposed by David L. Silverman, Beverly Hills, California.

A pair of **toruses** having hole-radius = tube-radius = 1 are linked. a) What is the smallest cube into which the **toruses** can be packed? **b)** What convex surface enclosing the linked **toruses** has the smallest volume? **c)** What convex surface enclosing the linked **toruses** has the smallest area? **d)** What is the locus of points in space equidistant from the two links?

SOLUTIONS

213. (Spring 1969) Proposed by Gregory Wulczyn, Bucknell University.

Prove that a triangle is isosceles if and only if it has a pair of equal ex-symmedians. (Editorial note: See Mathematics Magazine, Problem 637, November 1966, May 1967 and January 1968, for the corresponding problem involving **symmedians.)**

Solution by the Proposer.

Let a, **b,** c denote the sides opposite vertices A, **B,** C of the triangle and let x_a and x_b denote the lengths of the ex-symmedians issuing from A and B and terminated by the opposite sides.

I. If a = b, we have

$$x_a = \frac{b \sin C}{\sin(B-C)} \qquad x_b = \frac{c \sin A}{\sin(A-C)}$$

[Davis, "Modern College Geometry", p. 1711

Then, since b sin C = c sin **B,**

$$x_a = \frac{b \sin C}{\sin(B-C)} = \frac{c \sin B}{\sin(B-C)} = \frac{c \sin A}{\sin(A-C)} = x_b$$

II. If $x_a = x_b$, then

$$\frac{b \sin C}{\sin(B-C)} = \frac{c \sin A}{\sin(A-C)} = \frac{c \sin B}{\sin(B-C)}$$

It follows that sin A sin (B-C) = sin B sin (A-C), which simplifies to sin (A-B) = 0. Hence A = **B,** and the triangle is isosceles. The proposer also supplied a geometric solution.

214. (Spring 1969) Proposed by Charles W. **Trigg,** San **Diego,** California.

Find the unique nine-digit triangular number A which has distinct digits and for which n has the form abbbb.

Solution by the Proposer.

In A $= n(n+1)/2$, the last three digits of $n(n+1)$ determine the last two digits of A . Thus we find that for b = 0, 3, 6, 9, duplicate digits terminate A . Now

$$n^2 < n(n+1) < (n+1)^2, \text{ so } n = [\sqrt{2\Delta_n}].$$

Therefore, since A has nine digits, $n \leq [\sqrt{2(987654321)}] = 44444$, and $a \leq 4$. Furthermore, $n \geq [\sqrt{2(102345678)}] = 14307$. Consequently, there are only seventeen possible values of n, all of which yield a A having duplicate digits except $\Delta_{25555} = 326541790$.

Answers (without solutions) were also supplied by Carl A. **Argila,** TRW Inc., Houston, and by Kenneth A. Leone, Michigan State University.

215. (Spring 1969) Proposed by Leon **Bankoff,** Los Angeles, California.

In an acute triangle ABC whose circumcenter is 0, let **D,** E, F denote the midpoints of sides BC, CA, **AB,** and let P, Q R denote the midpoints of the minor arcs BC, CA, AB of the circumcircle. Show that

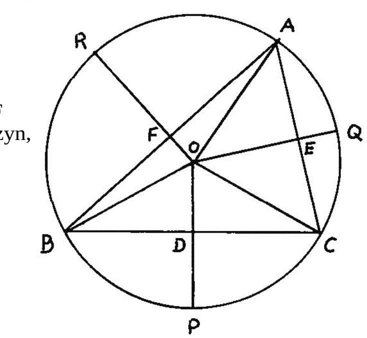
$$\frac{DP+EQ+FR}{OB+OD+OC+OE+OA+OF} = \frac{\sin^2(A/2)+\sin^2(B/2)+\sin^2(C/2)}{\cos^2(A/2)+\cos^2(B/2)+\cos^2(C/2)}.$$

Solution by Alfred E. Neumann, New York City.

It is known that OD+OE+OF = R + r. Since $\sum \cos^2(A/2) = 2 + r/2R$ and $\sum \sin^2(A/2) = 1 - r/2R$, we have

$$\frac{\sum\sin^2(A/2)}{\sum\cos^2(A/2)} = \frac{2R-r}{4R+r} = \frac{3R-(R+r)}{3R+(R+r)} = \frac{OP+OQ+OR-(OD+OE+OF)}{OB+OC+OA+(OD+OE+OF)} = \frac{DP+EQ+FR}{OB+OC+OA+OD+OE+OF}$$

Also solved by Guy Gardner, USAF Academy, Colorado; Gregory Wulczyn, Bucknell University; and the proposer.



216. (Spring 1969) Proposed by Erwin Just, Bronx Community College.

Prove that the Diophantine equation

$$x^9 + 2y^9 + 3z^9 + 4w^9 = k$$

has no solution if $k \varepsilon \{11, 12, 13, 14, 15, 16\}$.

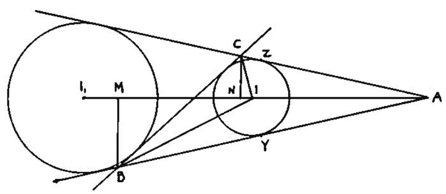
Solution by the Proposer.

Since $\phi(27) = 18$, $x^{18} \equiv 1 \pmod{27}$ when $(x, 27) = 1$. This implies $(x^9 - 1)(x^9 + 1) \equiv 0 \pmod{27}$, from which it may readily be concluded that $x^9 \equiv \pm 1 \pmod{27}$. On the other hand, if $(s,27) \neq 1$, then it must follow that $x^9 \equiv 0 \pmod{27}$. Thus, in all cases either $x^9 \equiv 0 \pmod{27}$, $x^9 \equiv 1 \pmod{27}$, or $x^9 \equiv -1 \pmod{27}$.

As a result, when the given Diophantine equation is viewed as a relation among the integers (modulo 27), it is apparent that none of the permitted values of k will enable the equation to be true. Since there can be no solutions (modulo 27), it follows that the given equation has no solutions in integers.

217. (Spring 1969) Proposed by C.S. Venkataram, Sree Kerala Varma College, Trichur, South India.

A transverse common tangent of two circles meets the two direct common tangents in B and C. Prove that the feet of the perpendiculars from B and C on the line of centers are a pair of common inverse points of both the circles.



Solution by the Proposer.

Let the direct common tangents meet in A. Then the two circles are plainly the incircle and excircle opposite to A of triangle ABC. Therefore let us denote their centers by I, I_1, respectively..

Let M, N be the feet of the perpendiculars from B, C on II_1, the line of centers, and let Y, Z be the points of contact of the incircle with AB, AC respectively. Join BI, CI.

Adopting the usual notation for a triangle ABC, we obtain readily that:

$$IN = CI \cos NIC = CI \cos (A/2 + C/2) = CI \sin (B/2)$$
$$IM = BI \cos BIM = BI \cos (A/2 + B/2) = BI \sin (C/2).$$

Therefore $IN \cdot IM = (BI \sin\frac{B}{2})(CI \sin\frac{C}{2}) = IY \cdot IZ = r^2$

So N,M are inverse points with respect to the circle (I). Similarly, they are inverse points with respect to the circle (I_1).

Also solved by Alfred E. Neumann, New York City, who found the problem stated but not solved in Forder's "Higher Course Geometry", page 182, problem 48.

218. (Spring 1969) Proposed by Charles W. Trigg, San Diego, California.

Find the three 3-digit numbers each of which is equal to the product of the sum of its digits by the sum of the squares of its digits.

Solution by the Proposer.

If three digits, a, b, c, have a fixed sum, the minimum value of $a^2 + b^2 + c^2$ is attained when $a = b = c$. Since $3(5)[3(5^2)] > 1000$, then $a + b + c < 15$.

$N = (a + b + c)(a^2 + b^2 + c^2) \equiv (a + b + c) \pmod 9$, so

$(a + b + c)(a^2 + b^2 + c^2 - 1) \equiv 0 \pmod 9$.

We need consider only those digit sets whose sum $\equiv 0$, and those the sum of whose squares $\equiv 1 \pmod 9$. In the latter case, one square must be $\cong 1$ and each of the other two squares $\equiv 0 \pmod 9$. It is necessary to examine only the twenty-four sets, 009, 018, 027, 036, 045, 117, 126, 135, 144, 225, 234, 333, 001, 008, 031, 038, 061, 068, 091, 331, 338, 361, 391, 661, to see if the product of the sum of the digits by the sum of the squares of the digits in any of these sets is equal to one of the six permutations of the set.

The three solutions are: $133 = 7(19)$; $315 = 9(35)$; and $803 = 11(73)$.

Also solved by Carl A. Argila, TRW Systems, Houston and by Kenneth Leone, Michigan State University.

219. (Spring 1969) Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.

Consider the following method of solving $x^3 - 11x^2 + 36x - 36 = 0$. Since $(x^3 - 11x^2 + 36x)/36 = 1$, we may substitute this value for 1 back in the original equation to obtain

$$x^3 - 11x^2 + 36x(x^3 - 11x^2 + 36x)/36 - 36 = 0,$$

or $x^4 - 10x^3 + 25x^2 - 36 = 0$, with roots -1, 2, 3, and 6. We find that $x = -1$ is an extraneous root.

Generalize the method and determine what extraneous roots are generated.

Solution by Charles W. Trigg, San Diego, California.

The polynomial equation $f(x) = 0$ has a constant term a_n. When "the method" is applied to this equation by multiplying the term $a_{n-k}x^k$ by 1, that is, by $[f(x) - a_n]/(-a_n)$, we have

$$f(x) - a_{n-k}x^k + a_{n-k}x^k[f(x) - a_n]/(-a_n) = 0.$$

This simplifies to

$$(a_{n-k}x^k - a_n) f(x) = 0.$$

Consequently, the extraneous roots introduced by "the method" are the roots of $a_{n-k}x^k = a_n$.

Also solved by the Proposer.

220. (Spring 1969) Proposed by Daniel Pedoe, University of Minnesota.

a) Show that there is no solution of the Apollonius problem of drawing circles to touch three given circles which has only seven solutions. b) What specializations of the three circles will produce 0, 1, 2, 3, 4, 5, and 6 distinct solutions?

The solution to problem 220 will appear in the next issue.

221. (Spring 1969) Proposed by Murray S. Klamkin, Ford Scientific Laboratory.

Determine 8 vertices of an inscribed rectangular parallelepiped in the sphere

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

Solution by Charles W. Trigg, San Diego, California.

Obviously, the following eight points fall on the surface of the sphere:

$A(x_1, y_1, z_1)$, $B(x_1, y_1, z_2)$, $C(x_2, y_1, z_2)$, $D(x_2, y_1, z_1)$,
$A'(x_1, y_2, z_1)$, $B'(x_1, y_2, z_2)$, $C'(x_2, y_2, z_2)$, $D'(x_2, y_2, z_1)$.

Clearly, $AA' = |y_1 - y_2| = BB' = CC' = DD'$,

$AB = |z_1 - z_2| = CD = A'B' = C'D'$,

$AD = |x_1 - x_2| = BC = A'D' = B'C'$,

so ABCD-A'B'C'D' is a parallelepiped. Also,

$$(A'B')^2 = (x_1 - x_1)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = (AA')^2 + (AB)^2,$$

$$(DB)^2 = (x_1 - x_2)^2 + (y_1 - y_1)^2 + (z_1 - z_2)^2 = (AD)^2 + (AB)^2,$$

$$(A'D)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_1)^2 = (AD)^2 + (AA')^2,$$

and the tree face angles at A are right angles. Therefore, ABCD-A'B'C'D' is an inscribed rectangular parallelepiped.

Also solved by the Proposer.

37. (April 1952) Proposed by Victor Thebault, Tennie, Sarthe, France.

Find all pairs of three-digit numbers, M and N, such that (M)(N) = P and (M')(N') = P', where M', N', and P' are the numbers M, N, and P written backwards. For example:

$$(122)(213) = 25986$$
$$(221)(312) = 68952$$

I. Solution by Charles W. Trigg, San Diego, California.

A) If M = abc, N = def, P = vwxyz, (M)(N) = P, M' = cba, N' = fed, P' = zyxwv, and (M')(N') = P', clearly no columnar sum can exceed 9 in the multiplication


```
      d e f
      a b c
      * * *
    * * *
  * * *
  v w x y z
```

No one of a, c, d, f can be zero. To avoid duplication of pairs, Keep $M \leq N$.

If M = 101, then e may be any one of the ten digits, and $d + f \leq 9$. Thus there are $10(8 + 7 + \dots + 1)$ or 360 accompanying values of N.

If M = 102, then $2d + f \leq 9$ and $d, e, f < 5$. Hence, there are $5(4 + 4 + 3 + 1) - 1$ or 59 accompanying values of $N \geq M$.

For other possible values of $M \leq N$, either the restrictions on the digits of N or the values of N accompanying that M are tabulated below together with the frequency of the N's for that M.

M	N	Frequency
103	$3d + f \leq 9$; $d, e, f < 4$	22
104	111, 112, 121, 122, 201, 211, 221	7
105 - 108	In each case, 111 only	4
111	$d + e + f \leq 9$; $d, e, f < 8$	112
112	$2e + f \leq 9$; $2d + e + f \leq 9$; $d, e, f < 5$	32
113	113, 121, 122, 123, 201, 202, 203, 211, 212, 221	10
114	120, 121	2
121	$2d + e \leq 9$; $d + 2e + f \leq 9$; $e + 2f \leq 9$	34
122	$2d + e \leq 9$; $2d + 2e + f \leq 9$; $e + f < 5$	15
123	201, 202, 203, 211	4
124, 134, 144	201	3
131	201, 202, 203, 211, 212, 221, 301, 302, 303	9
132	201, 202, 203, 211, 212, 301, 302, 303	8
133	201, 202, 203	3
141	201, 202, 211, 212	4
142	201, 202, 211	3
143	201, 202	2
201	$d + 2f \leq 9$; $d, e, f < 5$	40
202	$d + f < 5$, $d, e, f < 5$	14
203	211, 221, 231	3
211	$d + 2e \leq 9$; $d + e + 2f \leq 9$; $d, e, f < 5$	20
212	212, 221, 231, 301, 311	5
221	221, 301, 302, 303, 311, 312, 401, 402	8
222, 232,	301	2

231	301, 302, 303	3
301	301, 302, 311, 312, 321, 322, 331, 332	8
302	311, 321, 331	3
311	311, 321	2
	Total for all 36 values of M	801

For each M, N the corresponding M', N' necessarily also appears in the tabulation.

B) If (abc)(def) = uvwxyz and (cba)(fed) = zyxwvu, then (c)(f) = pn, where $n = p \pm 1$. The only possible terminal duos are 2(6) = 12, 3(4) = 12, 5(9) = 45, and 7(8) = 56.

Now in P', 399(499) = 199101, so 3, 4 may not be a terminal duo. Also, 299(699) = 209001, but (b2)(e6) = 100be + 20(e + 3b) + 12, so in P the penultimate digit is not zero, which rules out 2, 6 as a terminal duo.

If (5b5)(9e9) or (7b7)(8e8) provide a solution, the P = P', so the product must be palindromic and therefore divisible by 11. Hence, any solutions must come from (5b5)(979), (7b7)(858), or (737)(8e8). There are only four such solutions:

(555)(979) = 543345, (737)(888) = 654456,
(707)(858) = 606606, (858)(777) = 666666.

No other solutions appear when the products (5b9)(9e5), (7b8)(8e7), (5b7)(9e8), and (5b8)(9e7) are exhausted.

II. Solution by Carl A. Argila, TRW Systems, Houston, Texas.

Given any three digit integer, I, we define the function β as follows:

$$\beta(I) = 100\left(I - 100\left[\frac{I}{100}\right] - 10\left[\frac{I - 100\left[\frac{I}{100}\right]}{10}\right]\right) + 10\left[\frac{I - 100\left[\frac{I}{100}\right]}{10}\right] + \left[\frac{I}{100}\right]$$

where [A] is the greatest integer in A. Note that $\beta(I)$ is just I written backwards. We wish to find all pairs of three digit integers, M and N, for with $\beta(M)$ and $\beta(N)$ are also three digit integers and for which

$$\beta(M\times N) = \beta(M)\times\beta(N).$$

By means of a simple computer program we determine that there are 805 distinct pairs of three digit numbers which satisfy this condition.

83. (Spring 1956) Proposed by G.K. Horton, University of Alberta.

Evaluate

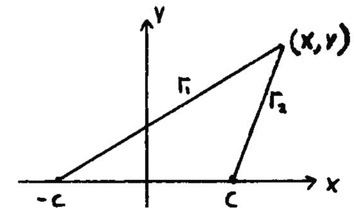
$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp - \left\{\sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + (y-1)^2}\right\} dxdy.$$

Solution by Murray S. Klamkin, Ford Scientific Laboratory.

It follows by symmetry that

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(r_1+r_2)}dxdy$$

where $c = \sqrt{2}$.



We first transform the rectangular coordinates (x,y) into elliptic coordinates (ξ,η) (see Stratton, Electromagnetic Theory, McGraw-Hill, N.Y., 1941, pp. 52–54.) Here

$$\xi = \frac{r_1+r_2}{2c}, \quad \eta = \frac{r_1-r_2}{2c}$$

and the region of integration is $\xi \geq 1$, $-1 \leq \eta \leq 1$.

Also

$$dxdy = c^2\left\{\frac{\xi^2-\eta^2}{\xi^2-1}\cdot\frac{\xi^2-\eta^2}{1-\eta^2}\right\}^{1/2}d\xi d\eta.$$

Thus,

$$I = 2c^2\int_1^{\infty}d\xi\int_0^1\frac{(\xi^2-\eta^2)e^{-2c\xi}}{\sqrt{(\xi^2-1)(1-\eta^2)}}d\eta \quad \text{or}$$

$$I = 2c^2\int_1^{\infty}\frac{e^{-2c\xi}d\xi}{\sqrt{\xi^2-1}}\int_0^1\left\{\frac{\xi^2-1}{\sqrt{1-\eta^2}} + \frac{1-\eta^2}{\sqrt{1-\eta^2}}\right\}d\eta.$$

Integrating with respect to η;

$$\frac{2I}{\pi c^2} = 2\int_0^{\infty}\sqrt{\xi^2-1}\,e^{-2c\xi}d\xi + \int_0^{\infty}\frac{e^{-2c\xi}d\xi}{\sqrt{\xi^2-1}}.$$

Now let $\xi = \cosh\theta$ giving

$$\frac{2I}{\pi c^2} = 2\int_0^{\infty}\sinh^2\theta\, e^{-2c\cosh\theta}d\theta + \int_0^{\infty}e^{-2c\cosh\theta}d\theta.$$

Differentiating the known integral

$$K_0(a) = \int_0^{\infty}e^{-a\cosh\theta}d\theta \quad (K_0 - \text{modified Bessel function})$$

twice with respect to a, we obtain

$$K_0''(a) = K_2(a) - \frac{1}{a}K_1(a) = \int_0^{\infty}\cosh^2\theta\, e^{-a\cosh\theta}d\theta.$$

Whence,

$$\int_0^{\infty}\sinh^2\theta\, e^{-a\cosh\theta}d\theta = K_2(a) - \frac{1}{a}K_1(a) - K_0(a).$$

and

$$\frac{2I}{\pi c^2} = 2\{ K_2(2c) - \frac{1}{2c}K_1(2c) - K_0(2c) \} + K_0(2c)$$

or

$$I = \frac{\pi c^2}{2} \{ 2K_2(2c) - \frac{1}{c}K_1(2c) - K_0(2c) \}.$$

Now just replace c by $\sqrt{2}$.

91. (Fall 1956) Proposed by Nathaniel Grossman, California Institute of Technology.

Prove that

$$\sum_{d/n} a \frac{n}{d} \phi(d) = n \cdot T(n)$$

where $T(n)$ denotes the number of divisors of n, $\sigma(n)$ is the sum of the divisors of n, and $\phi(n)$ is the Euler Totient function.

I. Solution by James E. Desmond, Florida State University.

It is well known that a, ϕ and T are multiplicative number-theoretic functions. As shown in (Calvin T. Long, Number Theory, D.C. Heath and Co., Boston, 1965, p. 103),

$$F(n) = \sum_{d/n} \sigma(n/d)\phi(d)$$

is multiplicative. We note that

$$\sigma(p^{r-s})\phi(p^s) = p^r - p^{s-1}$$

for any prime p and integers $r \geq s > 0$. Therefore $F(p^r) = p^r \cdot T(p^r)$.

Write n in standard form, $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then

$$\sum_{d/n} \sigma(n/d)\phi(d) = F(n) = F(p_1^{a_1})F(p_2^{a_2})\ldots F(p_k^{a_k}) = n \cdot T(n).$$

We note that the result appears without proof in History of the Theory of Numbers by Leonard E. Dickson, P. 285, and is generalized

to $\sigma_t(n) = \sum_{d/n} d^t$ on p. 286.

II. Solution by Solomon W. Golomb, University of Southern California.

For $R^{\ell}(s) > 2$, the following identities hold:

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$ Titchmarsh (1.2.12) page 6

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$ Titchmarsh (1.3.1) page 8

$$\sum_{n=1}^{\infty} \frac{T(n)}{n^s} = \zeta^2(s)$$ Titchmarsh (1.2.1) page 4

Therefore, since both

$$\sum_{n=1}^{\infty} \frac{nT(n)}{n^s} = \sum_{n=1}^{\infty} \frac{T(n)}{n^{s-1}} = \zeta^2(s-1)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d/n} \phi(d)\sigma\left(\frac{n}{d}\right) = \sum_{a=1}^{\infty} \frac{\phi(a)}{a^s} \sum_{b=1}^{\infty} \frac{\sigma(b)}{b^s} = \frac{\zeta(s-1)}{\zeta(s)} \cdot \zeta(s)\zeta(s-1) = \zeta^2(s-1)$$

the corresponding coefficients of n^{-s} must be equal:

$$nT(n) = \sum_{d/n} \phi(d)\sigma\left(\frac{n}{d}\right).$$

Reference: E.C. Titchmarsh, The Theory of the Riemann Zeta Function, Oxford, Clarendon Press, 1951.

Also solved by Marco A. Ettrick, Brooklyn, N.Y.; Murray S. Klamkin, Ford Scientific Laboratory; Bob Prielipp, Wisconsin State University; Cary C. Todd, Buies Creek, North Carolina, and Alfred E. Neumann, New York City.

111. (Spring 1960) Proposed by M.S. Klamkin, AVCO RAD, and D.J. Newman, Brown, University.

It is conjectured by at most N - 2 super-queens can be placed on an NxN ($N > 2$) chessboard so that none can take each other. A super-queen can move like an ordinary queen or a knight. (It should have been stipulated that N is even. For N = 5, Michael J. Pascual has shown that one can place 4 super-queens.)

Comment by Martin Gardner, Hasting-on-Hudson, N.Y.

"In 1965 a reader of Scientific American Column, Hilario Fernandez Long, (of Fernandez Long y Reggini, Esmeralda 356, Buenos Aires) sent me the following counter-example to the conjecture---10 super-queens on the 10 x 10.

		1							
					2				
								3	
4									
			5						
						6			
									7
	8								
				9					
							10		

He said a computer program had shown this to be a unique solution for 10 super-queens on the 10 x 10."

Comment by the Editor.

Solomon W. Golomb notes that if $n \geq 10$ is either a prime or one less than a prime, there is a construction which places n mutually non-attacking super-queens on the n x n board. Furthermore, for n prime, the board may even be regarded as a torus! In the example shown above, if a row is added above the board and a column to the left, a super-queen can be placed in the upper left corner thus rendering the solution applicable for a torus.

Also solved by George S. Cunningham, University of Maine; Richard E. Sot, University of Toledo; and Stanley Rabinowitz, Far Rockaway, N.Y.

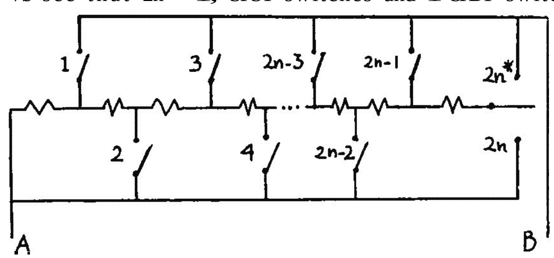
128. (Spring 1961) Proposed by Robert P. Rudis and Christopher Sherman, AVCO RAD.

Given 2n unit resistors, show how they may be connected using n single throw (SPST) and n single pole double throw (SPDT) (the latter with off position) switches to obtain, between a single fixed pair of terminals, the values of resistance of i and i^{-1} where $i = 1, 2, 3, \ldots, 2n$.

Editorial Note: Two more difficult related problems would be to obtain i and i^{-1} using the least number of only one of the above type of switches.

Solution by C.W. Dodge, University of Maine, Orono.

The accompanying circuit is minimal since, for the series resistance 2n connection, switch 2n* must be closed with all others open, and for the parallel resistance 1/2n connection, all other switches must be closed. Thus the number of permanent connections is a maximum. We see that $2n - 1$, SPST switches and 1 SPDT switch **are used.**



The series resistances are obtained by closing switch 2n* and also switches $2n - 2$ and $2n - 1$, $2n - 2$, $2n - 4$ and $2n - 1$, $2n - 4$, ..., 2, $2n - 1$, none, for 1, 2, 3, 4, ..., $2n - 2$, $2n - 1$, 2n ohms resistance, respectively. The parallel resistances require closing switches 1 and 2n*, 1 and 2 and 2n, 1 and 2 and 3 and 2n*, ..., 1 through 2n, for 1, 1/2, 1/3, ..., 1/2n ohms resistance, respectively.

Finally, observe that the lone SPDT switch does not need to have an off position.

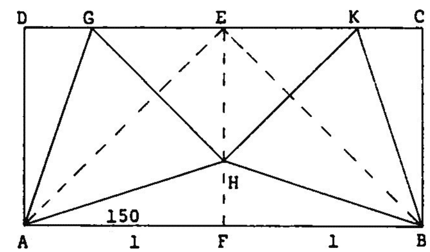
166. (Fall 1964) Proposed by Leo Moser, University of Alberta.

Show that 5 points in the interior of a 2-by-1 rectangle always determine at least one distance less than sec 15°.

Solution by Charles W. Trigg, San Diego, California.

In the 2-by-1 rectangle ABCD connect the midpoint E of a long side DC to the extremities and midpoint F of the opposite side. From A draw lines making angles of 30° with AE, meeting DC in G and EF in H. Also, from B draw lines making 30° angles with BE, meeting DC in K and (by symmetry) EF in H. Thus the triangles AGH and BKH are isosceles, and consequently are equilateral triangles inscribed in unit squares. As may be seen from right triangle AHF, each side of the triangles is sec 15°. The five points A, G, H, K, B are as widely separated as possible in or on the boundary of the 2-by-1 rectangle. Clearly, any movement of one of these points will reduce the distance between it and at least one of the other points. Since the boundary is excluded in this problem, it follows that at least one distance between two of the points is less than sec 15°.

This method follows that of Dewey Duncon in dealing with substantially the same problem in Mathematics Magazine, 23 (March, 1950), page 206.



Solution II by C.W. Dodge, University of Maine, Orono, Maine.

First we show that 3 points on a unit square determine at least one distance not exceeding sec 15°. The maximum distance between the points occurs when one point A is at a vertex of the square and the other two points X and Y lie on sides BC and CD of the square to form an equilateral triangle. By symmetry it follows that angle BAX = 15° = angle YAD. Then AX = XY = YA = sec 15°. Since now angle YXC = 45°, then CY > (sec 15°)/2. Reflecting this figure in side BC produces a 1 by 2 rectangle with 5 points thereon and inside determining distances of at least sec 15°. By symmetry, sec 15° is the largest value this minimum distance can have. It follows that 5 points all strictly interior to the rectangle cannot obtain this minimum value.

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