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π and e

Bob Prielipp
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Two of the most famous numbers of mathematics are π and e, where π is the ratio of the circumference of a circle to its diameter and e is the base for the system of natural logarithms. Since even secondary school students are familiar with these numbers, it might be assumed that many years ago mathematicians determined answers to all of their questions about π and e. This, however, is not the case.

It may be of interest to note that Euler is largely responsible for our current use of the symbols π and e. The first appearance of the Greek letter π for the circle ratio seems to have occurred in 1706 in the Synopsis Palmariorum Matheseos. or A New Introduction to the Mathematics by William Jones. But it was Euler's adoption of the symbol π in 1737. and its use by him in his many popular textbooks, that made it widely known and employed. In a manuscript entitled "Meditation upon Experiments made recently on the firing of Canon" (Meditatio in Experimenta explosione tormentorum nuper instituta), probably written in 1727 or 1728, Euler used the letter e sixteen different times to represent $\lim_{n\to\infty}(1 + \frac{1}{n})^n$. The letter e was again employed by Euler to denote "that number whose hyperbolic logarithm = 1" in a letter to Goldbach which was written in 1731. This notation first appears in print in Euler's Mechanica, which was published in 1736. ("Meditation upon Experiments made recently on the firing of Canon" was first printed in 1862 in Euler's Opera postuma mathematics et physica edited by P. H. Fuss and N. Fuss). It has been suggested that perhaps the symbol e was derived from the initial letter of the word "exponential". Incidentally, the symbol i for $\sqrt{-1}$ is another notation introduced by Euler, although in this case the adoption came near the end of his life, in 1777.

Before presenting some open questions involving π and e, let's briefly review a portion of the historical background of these two numbers. We begin by recalling that an algebraic number is a complex number that satisfies an equation of the form $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$, where the a_i's, $i = 0, 1, \dots, n-1$, are all rational numbers and n is a positive integer. Some examples of algebraic numbers are $\frac{1}{2}$, $\sqrt{2}$, and i. ($\frac{1}{2}$ is a root of the equation $x - \frac{1}{2} = 0$, $\sqrt{2}$ is a root of $x^2 - 2 = 0$, and i is a root of $x^2 + 1 = 0$.) A polynomial having leading coefficient 1, such as the one indicated in the definition of an algebraic number, is called monic. Every algebraic number a satisfies a unique monic polynomial equation of least degree. This unique monic polynomial of least degree is called the minimal polynomial of a. The decree of the minimal polynomial of a is also the decree of a. The concept of algebraic number is a natural generalization of rational number.

Indeed, the rational numbers coincide with algebraic numbers of degree 1. A complex number that is not an algebraic number is called a transcendental number.

Every transcendental number is not a rational number. The preceding is generally stated in the form, "Every transcendental number is irrational." Our wording (following that of Niven) is an attempt to avoid the suggestion that a transcendental number must be a real number. Some algebraic numbers are rational numbers (2/3 and 5/8, for example) and some algebraic numbers are not rational numbers ($\sqrt{2}$ and i, for example). Thus knowing that a number is not a rational number is not sufficient to tell us whether the number is transcendental or algebraic.

It is not always a simple matter to determine if a particular real number is rational or irrational. However, using the standard infinite series expansion for e,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots,$$

where $n! = n(n-1)\dots 2\cdot 1$, a relatively simple proof that e is an irrational real number can be given. Suppose that e is a rational number. Then $e = \frac{m}{n}$ where m is an integer and n is a positive integer. Hence

$$n!\,(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \dots - \frac{1}{n!})$$

is an integer. Replacing e by its series expansion and simplifying, we have that

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

is an integer. But

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3}$$

$+ \dots = \frac{1}{n}$ using the formula for the sum of an infinite geometric series $\leq \frac{1}{1} = 1$. Thus

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

is a series of positive terms which converges to a positive value less than one and therefore is not an integer. From this contradiction we conclude that e is not a rational number. It can, in fact be established that if r is a rational number and $r \neq 0$ then e^r is not a rational number.

The irrationality of π was initially proved by Lambert in 1761 using continued fractions. Later it was also demonstrated that π^2 is irrational. This knowledge was not enough to settle the problem of "squaring the circle" which we will discuss in greater detail a little later.

In 1873. Hermite proved that e is transcendental. It has been reported that in that same year Hermite stated, "I shall risk nothing on an attempt to prove the transcendence of the number π. If others undertake this enterprise, no one will be happier than I at their success, but believe me, my dear friend, it will not fail to cost them some efforts." Nine years later (1882) Lindemann succeeded in demonstrating that π is a transcendental number. To produce his proof Lindemann developed an extension of the technique employed earlier by Hermite.

Before moving on, perhaps we should note that today we know that e^a is transcendental for any non-zero algebraic number a.

One of the most famous problems of antiquity was "squaring the circle", that is, constructing a square equal in area to a given circle using only the methods of the straight-edge and compass. The impossibility of this construction was established when Lindemann proved that π is transcendental. For, on the one hand, all line segments that can be constructed from a given unit length by a finite number of compass and straight-edge constructions have lengths that are algebraic numbers. On the other hand, given any circle, we may regard its radius as the unit of length, so that the circle has area π square units. Thus the problem of "squaring the circle" is equivalent to the problem of constructing a line segment of length $\sqrt{\pi}$ from a given unit length. Suppose this could be done. Then $\sqrt{\pi}$ would be an algebraic number, and from this it would follow that $\pi = \sqrt{\pi}\cdot\sqrt{\pi}$ would be an algebraic number since the algebraic numbers are closed under multiplication (indeed, the set of all numbers which are algebraic over any number field F is a field). Therefore it is impossible to "square the circle".

Another outstanding contribution to the theory of transcendental numbers was the proof of the Hilbert-Gelfond-Schneider theorem. This theorem provided a solution for the seventh of Hilbert's famous list of twenty-three outstanding unsolved problems. Although the list was announced in 1900, it was not until 1929 that Gelfond made the first real contribution to the solution of the seventh problem. Additional partial results were obtained by Kusmin, Siegel, and Boehle, and in 1934 a complete proof was given by Gelfond. Shortly thereafter an independent proof was supplied by Schneider. The Hilbert-Gelfond-Schneider theorem states that if a and β are algebraic numbers. β is not a rational number, and a is neither 0 or 1, then any value of α^β is transcendental. The hypothesis that "β is not a rational number" is usually stated in the form "β is irrational". Once again our wording is an attempt to avoid the suggestion that β must be a real number. In general, $a^\beta = e^{\beta \log_e \alpha}$ is multiple-valued. This is the reason for the phrase "any value of" in the statement of the theorem. One value of $i^{-2i} = {}^{-2i\,\log_e i}$ is e^n. Hence e^n is transcendental. The theorem also establishes the transcendence of such numbers as 5 and so-called Hilbert number $2^{\sqrt{2}}$.

Sometimes the Hilbert-Gelfond-Schneider theorem is stated in the following equivalent form: If a and β are algebraic numbers different from 0 and if $\beta \neq 1$, then $\log_e \alpha/\log_e \beta$ is either rational or transcendental. From this form of the theorem it follows that the logarithm of a positive rational number r to a positive rational base $b \neq 1$ is either a rational number or a transcendental number. This can be readily seen if one recalls the fact that $\log_b r = \log_e r/\log_e b$. Hence if r and $b \neq 1$ are positive rational numbers then $\log_b r$ is transcendental unless there exist integers m and n such that $r^m = b^n$.

Even though over the years much information has been gathered concerning π and e, it is not known if $\pi+e$, πe, e^e, π^π, or π^e are transcendental numbers. In fact, it is not even known if any of these numbers is irrational. Methods of attack which will tell us more about the character of the five numbers listed above do not appear to exist at the present time. The world is waiting for some clever mathematician to achieve another breakthrough.

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AN APOLOGY

In the Fall, 1970 issue, page 105, Dr. Francis Regan was incorrectly described as belonging to the Missouri Beta Chapter of Pi Mu Epsilon at St. Louis University. He has been associated with the Missouri Gamma Chapter.

PHI AM CURIOUS YELLOW: THE GOLDEN RATIO

Linda Riede
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All of you are undoubtedly familiar with my sister pi, the ratio of the circumference of a circle to its diameter. Being feminine, I am also irrational and have been accused of going on and on. In defense, I can only argue that I don't repeat myself like certain gentlemen I know. I feel somewhat slighted that pi gets so much attention, while I have as much to offer -- if not more. So few people these days recognize my inherent beauty and natural charm, missing the sheer delight that knowing me can brine them. I'd like to take this opportunity to barely introduce myself and to give you a somewhat meager taste of my curious talents.

Perhaps I can best describe myself geometrically.

A / B

$$\frac{A+B}{A} = \frac{A}{B}$$

If you let B = 1, you can easily compute my value from the following equations:

$$\frac{A+1}{A} = \frac{A}{1} \qquad A^2 = A + 1 \qquad A^2 - A - 1 = 0$$

By solving the quadratic equation above, one can see that I am exactly equal to $\frac{1+\sqrt{5}}{2}$. My decimal expansion is 1.61803390 · · ·. If instead, the length of A is taken as 1, then B will be my reciprocal; ie., 1/phi. Curiously, this value turns out to be .61803398 · · ·. I am the only positive number that becomes its own reciprocal by subtracting one.

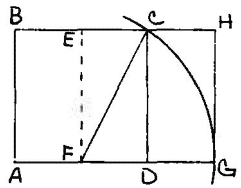
Like pi, I can be dressed in numerous outfits such as the sum of an infinite series. Since I have simple taste, the following show off my fundamental character:

$$\text{Phi} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

$$\text{Phi} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$$

When I'm in the mood, I enjoy making myself into lovely figures -- with my ratio intact, of course. Two of my favorites are the pentagram and the golden rectangle (rectangle with sides in golden ratio). I don't like to brag, but the Pythagorean brotherhood adopted the pentagram or five-pointed star as the symbol of their order because every segment in this figure is in golden ratio -- that's me -- to the next smallest segment.

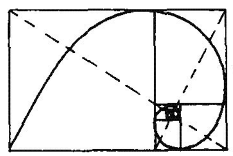
The golden rectangle is perhaps my most fascinating figure. If you would like to construct a golden rectangle, here's how you go about it.



The rectangle begins with a square, which is then divided into two equal parts by the dotted line E F. Point F now serves as the center of a circle whose radius is the diagonal F C. An arc of the circle is drawn (C G) and the base line A D is extended to intersect it. This becomes the base of the rectangle. The new side H G is now drawn at right angles to the new base. with the line B H brought out to meet it. That's all there is to it--or so it seems at first glance. Look!

If you cut off the original square, what remains will be a smaller golden rectangle. You can keep snipping off squares, leaving smaller and smaller golden rectangles. (This is an example of a perfect squared rectangle of order infinity.)

Successive points marking the division of sides into my golden ratio lie on a logarithmic spiral that coils inward to infinity, its pole being the intersection of the two dotted diagonals in the figure. Of course these "whirling squares," as they have been called, can also be whirled outward to infinity by drawing larger and larger squares.



If you place three golden rectangles so that they intersect each other symmetrically, each perpendicular to the other two, the corners of the rectangles-will mark the 12 corners of a regular icosahedron, as well as the centers of the 12 sides of a regular dodecahedron. You see. I do have the curious propensity for popping up when I am least expected. Many of you have probably seen me in art, architecture, nature, and so on; but, as has been my usual plight in life, I go unrecognized more than not. Leonardo Da Vinci used me in his figure of St. Jerome. If you draw a rectangle around the figure of St. Jerome, you will find a golden rectangle. Salvador Dali's "The Sacrament of the Last Supper" is painted inside a golden rectangle and other golden rectangles were used for positioning figures. Also, part of an enormous dodecahedron floats above the table. The Parthenon at Athens fits into a golden rectangle almost precisely. I could give you numerous elegant examples, but I fear my modesty would be endangered.

While I was still in my youth, I found that I was intimately involved with Mr. Fibonacci Series. Don't judge hastily -- we were married soon after.. Perhaps you have met him before. He looks like this: (1,1,2,3,5,8,13,21, 34, ...). Each term is the sum of the two preceding terms. As with many married couples, we get more alike as tine goes on. That is, if you take two successive terms and form a series of ratios as follows -- 5/3, 13/8, 21/13, . -- Fibonacci approximates my value more closely with each succeeding ratio in the series. Yes, he knows me well, but not perfectly -- one always needs that dash of mystery to be more interesting.

I will close my introduction with one last point of interest to you humans. A man named Frank A. Lonc had the audacity to measure the heights of 65 human women and compared these figures to the height of their navels. He happened to find me in that ratio. I was some what embarrassed and still blush whenever I think about it. but then -- I am curious.

AN APPLICATION OF BOOLEAN ALGEBRA TO FINITE TOPOLOGY

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A topology is essentially a set of subsets of a set, closed under the operations of union and intersection. So too is a Boolean algebra. Thus far, however, this relationship seems to have been ignored. It is the purpose of this paper to begin the development of this connection. This paper will restrict its attention to finite sets which will be denoted by X.

A glance at the list of postulates for a Boolean algebra, viewed as an algebra of sets, will show that a Boolean algebra satisfies all the requirements to he a topology. The only difference between the two types of structure is that a Boolean algebra contains the complements of all of its sets, or in topological terms, it contains all of its closed sets. This prompts an immediate definition:

Definition 1: A boolean topology is a topology containing its closed sets. The discrete and indiscrete topologies are Boolean; so is $\mathcal{T} = \{\emptyset, \{a\}, \{b,c\}, X\}$, where $X = \{a,b,c\}$.

Boolean topologies are a particularly simple type, which since they are Boolean algebras all contain a convenient number (2^m) of open sets. To convert an arbitrary finite topology to a Boolean one, it will be necessary either to add (redefine as open) closed sets that are not open, or to remove open sets that are not closed. In either case, the closed sets of the topology must receive some attention.

Lemma 1: The closed sets of a finite topology $\mathcal{T}$ form a topology.

Proof: $\emptyset$ and X are closed. Finite unions and intersections of closed sets are closed. Since $\mathcal{T}$ is finite, only these finite cases are relevant to the discussion.

Definition 2: The topology formed by the closed sets of the finite topology $\mathcal{T}$ is called the complementary topology to $\mathcal{T}$, and denoted $\mathcal{T}'$.

Since infinite unions of closed sets do not necessarily yield closed sets, the above lemma is not true for all infinite topologies. This is the main reason why this paper is restricted to the finite ones.

Several very easy results will help to link these new ideas to the more usual topological concepts.

Proposition 1: $\mathcal{T}$ is a Boolean topology iff $\mathcal{T} = \mathcal{T}'$.

Proposition 2: $\mathcal{T}$ is connected iff $\mathcal{T}$ and $\mathcal{T}'$ have only $\emptyset$ and X in common.

Proposition 3: $\mathcal{T}$ is a door space iff every subset of X is in $\mathcal{T}$ or $\mathcal{T}'$.

Propositions 1 and 2 together indicate how very disconnected a Boolean topology, in which every open set is closed, must be.

As is well known, the set of topologies on X forms a complete lattice under the operations $\vee$ and $\cap$.[1] An arbitrary topology can be made Boolean by combining it with its complementary topology under either of these operations:

Theorem 1: $\mathcal{T} \vee \mathcal{T}'$ is the coarsest Boolean topology finer than $\mathcal{T}$ (or $\mathcal{T}'$).

Proof: $\mathcal{T} \vee \mathcal{T}'$ is the topology having as subbasis the open sets of $\mathcal{T}$ and $\mathcal{T}'$, so it is certainly finer than either. Since all topologies under consideration

[1] Dugundji, Topology, p. 91.

are finite, an extension of the distributivity laws allows the usual **subbasis** characterization to be reversed to say that $\mathcal{T}\vee\mathcal{T}'$ has as open sets all finite intersections of all finite unions of open sets from $\mathcal{T}$ or $\mathcal{T}'$. Hence if G is an open set of $\mathcal{T}\vee\mathcal{T}'$, $G = \bigcap_n \bigcup_m G_{nm}$, where the G_{nm} are open in $\mathcal{T}$ or $\mathcal{T}'$. By DeMorgan's laws, $X \sim G = X - \bigcap_n \bigcup_m G_{nm} = \bigcup_n (X \sim \bigcup_m G_{nm}) = \bigcup_n \bigcap_m (X \sim G_{nm})$, which is in $\mathcal{T}\vee\mathcal{T}'$ since $X - G_{nm}$ is open in $\mathcal{T}$ or $\mathcal{T}'$. Therefore $\mathcal{T}\vee\mathcal{T}'$ is Boolean. Any Boolean topology finer than $\mathcal{T}$ contains all the open sets of $\mathcal{T}$, contains all open sets of $\mathcal{T}'$ since it is Boolean, contains all unions and intersections of such sets since it is a topology, and consequently will be finer than $\mathcal{T}\vee\mathcal{T}'$. The statement for $\mathcal{T}'$ may be proved similarly.

<u>Theorem 2</u>: $\mathcal{T}\cap\mathcal{T}'$ is the finest Boolean topology coarser than $\mathcal{T}$ (or $\mathcal{T}'$).

Proof: $\mathcal{T}\cap\mathcal{T}'$ is a topology coarser than $\mathcal{T}$ and $\mathcal{T}'$. If G is open in $\mathcal{T}\cap\mathcal{T}'$, it is open in $\mathcal{T}$ and $\mathcal{T}'$, so $X \sim G$ is open in $\mathcal{T}'$ and $\mathcal{T}$, so $X \sim G$ is open in $\mathcal{T}\cap\mathcal{T}'$. Consequently $\mathcal{T}\cap\mathcal{T}'$ contains all of its closed sets, and thus is Boolean. If G is any set in a Boolean topology coarser than $\mathcal{T}$, G and $X \sim G$ are in $\mathcal{T}$, or G is in both $\mathcal{T}$ and $\mathcal{T}'$, so G is open in $\mathcal{T}\cap\mathcal{T}'$. Hence $\mathcal{T}\cap\mathcal{T}'$ is the finest Boolean topology coarser than $\mathcal{T}$. Since $\mathcal{T}'' = \mathcal{T}$, the similar argument proves the statement for $\mathcal{T}'$.

Theorems 1 and 2 show that any topology $\mathcal{T}$ can be expanded or contracted to a Boolean topology by taking $\vee$ or $\cap$ with $\mathcal{T}'$. If $\mathcal{T}$ is already Boolean, then by Proposition 1, $\mathcal{T} = \mathcal{T}'$, and either procedure will leave $\mathcal{T}$ unchanged. More generally than this, $\vee$ or $\cap$ of any two Boolean topologies will be Boolean.

<u>Theorem 3</u>: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are Boolean, so are $\mathcal{T}_1\vee\mathcal{T}_2$ and $\mathcal{T}_1\cap\mathcal{T}_2$.

<u>Proof</u>: If G is open in $\mathcal{T}_1\vee\mathcal{T}_2$, then as in the proof of Theorem 1, $G = \bigcap_n \bigcup_m G_{nm}$, where G_{nm} will be open in $\mathcal{T}_1$ or $\mathcal{T}_2$, and $X \sim G = \bigcup_n \bigcap_m (X \sim G_{nm})$, open in $\mathcal{T}_1\vee\mathcal{T}_2$ since $\mathcal{T}_1$ and $\mathcal{T}_2$ are Boolean.

If G is in both $\mathcal{T}_1$ and $\mathcal{T}_2$, so is $X \sim G$.

The discrete and indiscrete topologies are Boolean, so the above theorem shows that the set of Boolean topologies of X forms an algebraically complete sublattice of the lattice of all topologies of X.

The collection of all topologies of X can be divided into the non-Boolean ones, which occur in complementary pairs, and the Boolean ones, which are self-complementary. The number of finite topologies of X will then have the form $2k + b$, where b is the number of Boolean topologies. b can be determined as follows:

<u>Theorem 4</u>: The number of Boolean topologies with n elements is equal to the number of partitions of a set of n elements.

<u>Proof</u>: It will be shown that each such partition determines, and is determined by a Boolean topology.

Let P be a partition of the set X, and let $\mathcal{T}$ be the topology having the sets of P as a basis. If G is an open set of $\mathcal{T}$, it is the (finite) union of some of the sets of P. Let H be the union of the sets of P not contained in G. Since the sets of P form a basis, H is open. Since they are exhaustive, $G \cup H = X$. Since they are pairwise disjoint, $G \cap H = \emptyset$. Hence $H = X \sim G$, $\mathcal{T}$ contains its closed sets, so $\mathcal{T}$ is Boolean.

Conversely, assume that $\mathcal{T}$ is Boolean. Since it is finite, its open sets can be ordered into pairs G_i and $X \sim G_i$. G_1 and $X \sim G_1$ form a partition of X. $G_1 \cap G_2$, $(X \sim G_1) \cap G_2$, $G_1 \cap (X \sim G_2)$, and $(X \sim G_1) \cap (X \sim G_2)$ (some may be empty and can be neglected) form a partition of X. If $A_1, \ldots, A_m$ is a partition of X, so is $A_1 \cap G_n, \ldots, A_m \cap G_n$, $A_1 \cap (X \sim G_n), \ldots, A_m \cap (X \sim G_n)$. Proceeding in this manner, $\mathcal{T}$ determines a partition of X.

Each partition of X determines a Boolean topology, and each Boolean topology determines a partition. It can be seen that each of these determinations is unique. Therefore the number of partitions and Boolean topologies is the same.

<u>Corollary 4–1</u>: There are as many Boolean topologies on X as there are equivalence relations among the elements of X^2.

<u>Proof</u>: Every partition determines an equivalence relation and conversely.

One reason that the Boolean topologies are useful is that they have a well-characterized number of elements—2^m, where $m \leq n$, the number of elements in X. Theorems 1 and 2 give uniquely the Boolean topologies "closest" to $\mathcal{T}$. It is of interest to study the number of open sets of $\mathcal{T}$ in relation to its associated Boolean topologies.

<u>Definition 3</u>: The number of open sets of finite topology $\mathcal{T}$ is denoted by $\#(\mathcal{T})$.

<u>Definition 4</u>: The coefficients of finite topology $\mathcal{T}$ are the numbers c and d given by $2^c = \#(\mathcal{T}\cap\mathcal{T}')$, $2 = \#(\mathcal{T}\vee\mathcal{T}')$. Some consequences of these definitions are contained in the following theorem.

<u>Theorem 5</u>: If c and d are the coefficients of $\mathcal{T}$, and X has n elements, then:

1) $1 \leq c \leq d \leq n$
2) $c = d$ iff $\mathcal{T}$ is Boolean
3) $c = 1$ iff $\mathcal{T}$ is connected
4) $c = n$ iff $\mathcal{T}$ is discrete
5) $d = 1$ iff $\mathcal{T}$ is indiscrete
6) $\mathcal{T}'$ has the same coefficients c and d.

<u>Proof</u>: 1) $\mathcal{T}\cap\mathcal{T}'$ is finer than the indiscrete topology, and obviously coarser than $\mathcal{T}\vee\mathcal{T}'$, which is coarser than the discrete topology.

2) If $\mathcal{T}$ is Boolean, $\mathcal{T} = \mathcal{T}'$ (Proposition 1), and $\mathcal{T}\cap\mathcal{T}' = \mathcal{T}\cap\mathcal{T} = \mathcal{T} = \mathcal{T}\vee\mathcal{T} = \mathcal{T}\vee\mathcal{T}'$. If $c = d$, then since $\mathcal{T}\cap\mathcal{T}'$ is in general coarser than $\mathcal{T}\vee\mathcal{T}'$, $\mathcal{T}\cap\mathcal{T}' = \mathcal{T}\vee\mathcal{T}'$. $\mathcal{T}$ is coarser than $\mathcal{T}\vee\mathcal{T}'$, so it is coarser than $\mathcal{T}\cap\mathcal{T}'$, which implies that every set open in $\mathcal{T}$ is open in both $\mathcal{T}$ and $\mathcal{T}'$, so $\mathcal{T}$ is Boolean.

3) This follows immediately from Proposition 2.

4) If $\mathcal{T}$ is discrete, so is $\mathcal{T}'$, and consequently so is $\mathcal{T}\cap\mathcal{T}'$. Hence $c = n$, since the discrete topology has 2 open sets. If $c = n$, $\mathcal{T}\cap\mathcal{T}'$ is discrete, so $\mathcal{T}$ which is finer than $\mathcal{T}\cap\mathcal{T}'$ is also discrete.

5) If $\mathcal{T}$ is indiscrete, so is $\mathcal{T}'$, hence so is $\mathcal{T}\vee\mathcal{T}'$, and $d = 1$. If $d = 1$, $\mathcal{T}\vee\mathcal{T}'$ is indiscrete, and $\mathcal{T}$ which is coarser than $\mathcal{T}\vee\mathcal{T}'$ is also indiscrete.

6) $\mathcal{T}\vee\mathcal{T}' = \mathcal{T}'\vee\mathcal{T}''$, and $\mathcal{T}\cap\mathcal{T}' = \mathcal{T}'\vee\mathcal{T}''$.

The effect of $\vee$ or $\cap$ on the $\#$ function is described in the next theorem.[3]

<u>Theorem 6</u>: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two finite topologies on X, then $\#(\mathcal{T}_1) + \#(\mathcal{T}_2) \leq \#(\mathcal{T}_1 \vee \mathcal{T}_2) + \#(\mathcal{T}_1 \cap \mathcal{T}_2)$.

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here are consequently 1,2,5,.5, and 87 Boolean topologies with 1, 2, 3, 4, and 5 elements respectively. It may be noted that in the first four cases, except for a minor correction factor of 0, 0, 1, and 5 (= (n-1)! - 1 ?) to be subtracted, that the <u>total</u> number of topologies with n elements equals $n!$ times the number of Boolean topologies with n elements.

similar formulas recur throughout mathematics when a measure is defined on the elements of a lattice. For instance, $P(A) + P(B) = P(A \cup B) + P(A \cap B)$ in probability theory, or $\dim V + \dim W = \dim (V + W) + \dim (V \cap W)$. The triangle inequalities of analysis also are of a related form.

Proof: All the open sets of $\mathcal{T}_1$ will be counted in $\#(\mathcal{T}_1 \vee \mathcal{T}_2)$. A set in $\mathcal{T}_2$ but not in $\mathcal{T}_1$ will also be counted in $\#(\mathcal{T}_1 \vee \mathcal{T}_2)$. Sets open in both $\mathcal{T}_1$ and $\mathcal{T}_2$, which are counted twice on the left side of the inequality are also counted twice on the right because they are in $\mathcal{T}_1 \cap \mathcal{T}_2$.

Full equality will occur in Theorem 6 if all the sets of $\mathcal{T}_1 \vee \mathcal{T}_2$ happened already to be open in either $\mathcal{T}_1$ or $\mathcal{T}_2$. Two sufficient conditions for this are: 1) one topology is finer than the other, so that $\vee$ simply gives the finer one; 2) one topology is a door space, and the other is its complement (see Proposition 3).

Letting $\mathcal{T}_1 = \mathcal{T}$ and $\mathcal{T}_2 = \mathcal{T}'$ in the theorem gives the following corollary;

Corollary 6–1: $2\#(\mathcal{T}) \leqq 2^c + 2$, where c and d are the coefficients of $\mathcal{T}$.

Proof: $\#(\mathcal{T}) = \#(\mathcal{T}')$, $2^c = \#(\mathcal{T} \cap \mathcal{T}')$, $2^d = \#(\mathcal{T} \vee \mathcal{T}')$.

This corollary is important because it gives a certain amount of structural information about a general finite topology. For example, if $\mathcal{T}$ is not discrete, then by parts 1 and 4 of Theorem 5, $c < n$, while $d \leqq n$, so $c \leqq n-1$, $d \leqq n$, and $2\#(\mathcal{T}) \leqq 2^{n-1} + 2^n$, or $\#(\mathcal{T}) \leqq 3 \cdot 2^{n-2}$. This means that only the discrete topology has more than 3/4 of the subsets of X. The result can be generalized:

Theorem 7: If $\mathcal{T}$ is not Boolean, then $\#(\mathcal{T}) \leqq 3/4\#(\mathcal{T} \vee \mathcal{T}') = 3/4 \cdot 2^d$.

Proof: By parts 1 and 2 of Theorem 5, $c < d$, or $c \leqq d-1$. From Corollary 6–1, $2\#(\mathcal{T}) \leqq 2^{d-1} + 2^d$, of $\#(\mathcal{T}) \leqq 3/4 \cdot 2^d = 3/4\ \#(\mathcal{T} \vee \mathcal{T}')$.

Various further conclusions can be obtained for specialized topologies. For example, if $\mathcal{T}$ is connected, Theorem 5 part 3 shows $c = 1$, and Corollary 6–1 becomes $\#(\mathcal{T}) \leqq 1 + 2^{d-1} < 1 + 2^{n-1}$. If $\mathcal{T}$ is a door space, then the discussion after Theorem 6 shows that $d = n$ and the inequality of Corollary 6–1 is an exact equality, so $\#(\mathcal{T}) = 2^{c-1} + 2^{n-1} \geqq 2^{n-1} + 1$. Combining these two results, all connected door spaces have $\#(\mathcal{T}) = 2^{n-1} + 1$.

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UNDERGRADUATE RESEARCH PROPOSAL

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It is not difficult to see that for each $A > \pi - 2$ there is an open, simply-connected subset of the closed unit disk whose closure contains a circle of radius r for each r, $0 < r \leq 1$, and whose area is less than A. Is there such a set with area equal to $\pi - 2$? Are there such sets with smaller area? What is the least such area?

NEWTON AND THE DEVELOPMENT OF THE CALCULUS

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This paper is intended to give a brief sketch of Sir Isaac Newton's role in the development of the calculus. In order to appreciate Newton's role somewhat better, a short description of earlier highlights in the development of the calculus will be given. Detail will be kept at a minimum because of the scope and complexity of the subject. A cursory glance at the celebrated priority controversy with Leibniz will also be offered.

D. E. Smith lists four steps in the development of the calculus: The Greek "method of exhaustion": the method of infinitesimals of Kepler and Cavalieri: fluxions, the method of Newton; and the method of limits, as used in the present day.[1]

The Greek method of exhaustion developed in the 5th century B.C. The four paradoxes of Zeno of Elea (495-435 B.C.) led to a consideration of infinitesimally small magnitudes.' The germ of the exhaustion method was introduced by the Sophist. Antiphon (c. 430 B.C.). Credit for developing this method to its most useful form is usually given to Eudoxus (408-355 B.C.).

According to W. W. Rouse Ball, Eudoxus' method "depends on the proposition that 'if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there will at length remain a magnitude less than the least of the proposed magnitude.'"[3] This method allowed the Greeks to sidestep the use of infinitesimals, which use was questioned by Zeno. The method was rigorous but awkward. Polygons whose boundary and area were successively less from the curve were inscribed and circumscribed about the curve to find the area bounded bv it.4

According to Smith. "It is to Archimedes himself (c. 225 B.C.) that we owe the nearest approach to actual integration to be found among the Greeks."[5] His method consisted roughly of drawing triangles under a curve such that the sum of the areas of two triangles equals 1/4 the area of an inscribed triangle. He then repeated this process with smaller triangles whose sum was $(1/4)^2$ of the original triangle, then $(1/4)^3$, etc. "He argued that by repeating this process indefinitely (in imagination) the parabolic segment would approach 'as near as one wished' to 'exhaustion'."[6] This process utilized the modern concept of a sum of an infinite series, a concept unknown to the Greeks.

Although there was some activity in this area of mathematics in the years following Archimedes,[7] the next great achievement was by Bonaventura Cavalieri (1598-1647), who (influenced by Johannes Kepler (1571-1630) and his problem of determining the volume of wine barrels. in which he used a "rude kind of integration"[8]) developed his "method of indivisibles". In this method, a solid is regarded as being made of surfaces, a surface made up of lines, a line made of points, in each case "these component parts being the ultimate possible elements in decomposition of the magnitude."[9] To find volumes, areas, or lengths, these "indivisibles" must be summed (an infinite sum of infinitesimals).

There was now a flurry of activity in this area, the most important steps being taken by Pierre Fermat (1601-1665), John Wallis (1616-1703), and Newton's teacher, Isaac Barrow (1630-1677).[10]

Fermat has been credited with the invention of calculus by the eminent mathematician, Joseph Lagrange, because "in his method Re maximis et minimis he equates the quantity of which one seeks the maximum or the minimum to the expression of the same quantity in which the unknown is increased by the in- determinate quantity."[11] He then causes radicals and fractions to disappear and takes this quantity to zero. While this is certainly part of the calculus and did influence Newton, it is no more "the calculus" than Cavalieri's summing of indivisibles is integration.

Wallis developed the limit concept and performed many useful integrations. He created the limit concept "by considering the successive values of a fraction formed in the study of certain ratios; these fractional values, steadily approach a limiting value, so that the difference becomes less than any assignable one and vanishes when the process is carried to infinity."[12]

Barrow was the first to realize that differentiation and integration were inverse operations.[13] His great achievement, at least as far as influencing Newton, was what we now call "Barrow's differential triangle". (See Appendix, Figure 1). This triangle has an important use in picturing the x-axis as being "in motion" or "in flux". Because of this, J. M. Childs says, "Isaac Barrow was the first inventor of the Infinitesimal Calculus: Newton got the main idea of it from Barrow by personal communication: and Leibniz was also in some measure indebted to Barrow's work." [14]

Newton was certainly influenced bv Barrow. However, neither Fermat nor Barrow can be credited with discovering the calculus, no matter how close they may have come. Barrow used geometrical notions and had no notations for first and higher derivatives.[15] Neither man had a complete system which would suffice for differentiation and interration of all curves and not just a number (however large) of special cases. It was the wide range of applications, along with a notation and general method, that constituted the discovery of the calculus. It was not an accident, even though earlier men had come more or less close to the discovery. It required a great deal of patience, thought, and insight to develop a method so general and useful as the calculus from a set of facts and methods relating only to specific cases. The only "accidental" feature involved is that Newton and Leibniz discovered the method independently within ten years of each other. The accomplishments of neither man is lessened because of the fact that certain specific parts had been in use before.

When Newton was at Cambridge in 1664, he had little mathematical background.[16] He later told the story of buying an astrology book. Because he could not understand the diagrams in it, he consulted Euclid's Elements for help. He regarded Greek geometry as self-evident and turned to Descartes' Geometric, not an easy book. However, "there can be no doubt that Newton's reading of Descartes''Geometrie'... was his kev to the reaches of higher mathematics." [17] He also studied Barrow and Wallis, being "particularly delighted with Wallis' Arithmetic of Infinities, a treatise frought with rich and varied suggestions." [18] Newton solved the problem of expanding $(1 - x^2)^{1/2}$, [19] which Wallis could not do and in the process developed his Binomial Theorem.

Newton also studied Fermat's method of drawing tangents to curves and admitted his indebtedness to Fermat.[20] As a student of Barrow. he learned to use Barrow's differential triangle. which became his starting point for developing his calculus.[21]

Another influence on Newton was Kepler's Law. for which he needed a powerful mathematical tool to find an exnlanation.[22]

During the plague years of 1665 and 1666, the university at Cambridge was forced to close. Newton went home to Woolsthorpe, where he spent much time on his researches in gravitation and optics. It is at this period that Newton first worked on developing his fluxionary calculus. There is a manuscript dated Mav 7R. 1665. where he stated some of his early results in drawing tangents.[23] The "Direct Method of Fluxions," what we now call the differential calculus. was set down in a manuscript dated November 13, 1665.[24] By May 1666. he was working on the Inverse Method of Fluxions.

Newton stated twelve problems which he proposed to solve through fluxions:

1. To draw tanpents to curve lines.
2. To find the quantity of the crookedness of lines.
3. To find the points distinguishing between the concave and convex portions of curved lines.
4. To find the point at which lines are most or least curved.
5. To find the nature of the curve line whose area is expressed bv any given equation.
6. The nature of any curve line being riven. to find other lines whose areas may be compared to the area of that riven line.
7. The nature of any curve line being riven. to find its area when it may be done; or two curved lines being riven, to find the relation of their area when it mav be.
8. To find such curved lines whose lengths mav be found, and also to find their lengths.
9. Anv curve line being given, to find other lines whose lengths may be compared to its lengths. or to its area. and to comnare them.
10. To find curve lines whose areas shall be equal, or have any riven relations to the length of anv given curve line drawn into a riven right line.
11. To find the length of any curve line when it mav he.
12. To find the nature of a curve line whose length is expressed by any given equation when it mav be done.[25]

Newton's first work revealing his fluxionary method is De Analvsi der Aequationes Numero Terminorum Infinitas. a tract he pave to Barrow in 1669.

> In this treatise the principle of fluxions, though distinctly pointed out, it is only partially developed and explained... The expression which was obtained for the fluxion (of a curve) he expanded into a finite or infinite series[26] of monomial terms, to which Wallis' rule was applicable.

Infinitely small quantities were "treated in the dynamic form...of the conatus of Hobbes rather than in the static form of Cavalieri's indivisible."[27] This is in keeping with the notation of a fluxion as a point in motion.

In his Method of Fluxions, Newton gave the most complete expose of his new calculus. He explains the expansion of fractional and irrational quantities into series. He then turns to the solution of the two problems

> which constitute the pillars, so to speak, of the abstract calculus:
> 'I. The length of the space described being continually (i.e. at all times) given; to find the velocity of the motion at any time proposed.
> II. The velocity of the motion is being continually given; to find

28

the length of the apace described at any time proposed.

He generalises, saying time does not necessarily have to be considered, "but I shall suppose soar of the quantities proposed, being of the same kind, to be increased by an equable fluxion, to which the rest may be referred, as it were to time, and therefore, by way of analogy, it may not improperly receive the name of time. He then makes his most important definition*:

> Those quantities which I consider as gradually and Indefinitely increasing, I shall hereafter call fluents, or flowing quantities, and shall repretent them by the final latter of the alphabet, v, x, y, and z;... and the velocity by which every fluent is increased by its generating motion (which I may call fluxions) or simply velocities, or celerities), I shall represent by the same letters pointed $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$.[30]

The fluxions themselves are not infinitely small, but the momenta of the fluxions, denoted $\dot{x}o$, $\dot{v}o$, etc are infinitesimally small. Theae are analogous to Leibnis' differentials, dx, dv, etc. Theae moments are important in that the fluenta x and y, when increased, after every indefinitely small interval of time, become $x + \dot{x}o$ and $y + \dot{y}o$. That is, $\dot{x}o$ and $\dot{y}o$ are the indefinitely small lengths the fluents increase in an indefinitely mall time.

For example, given $y = 3x - x^2$, substitute $x + \dot{x}o$ for x, $y + \dot{y}o$ for y in $3x - x^2 - y = 0$ and we obtain

$$3x + 3\dot{x}o - x^2 - 2x(\dot{x}o) - (\dot{x}o)^2 - y - \dot{y}o = 0$$

"Ignoring $(\dot{x}o)^2$ as negligible, and subtracting the original equation $3x - x^2 - y^2 = 0$, obtain

$$3\dot{x}o - 2x(\dot{x}o) - \dot{y}o = 0 \; ; \qquad \frac{\dot{y}o}{\dot{x}o} = 3 - 2x."[31]$$

This, of course, is the same reault as in modern procedures. Because to is infinitesimally small, $(\dot{x}o)^2$ is ignored. Newton became wary of this procedure after a while.

> In a portion of De quadratura (curvarum) which appeared in Wallis's Algebra ef 1693, Newton had said that terms multiplied by o be omitted as infinitely small, thus obtaining the result. In the 1704 publication of the work, on the other hand, he said clearly that 'errors are not to be disregarded in mathematics, no matter how small.'[32]

Instead, one oust find the "ultimate ratios" as theae terns become "evanescent", i.e. vanish. All traces of infinitesimally small terms were to be eliminated, although in practice they were not.[33] For example,

> Let the quantity x flow uniformly, and let it be proposed to find the fluxion of x^n.
>
> In the same time that the quantity x, by flowing, becomes x + 0, the quantity x^n will become $(x + 0)^n$, that is, by the method of infinite series,
>
> $$x^n + nox^{n-1} + \frac{n^2 - n}{2} oox^{n-2} + \&c.$$
>
> And the augment& and
>
> $$nox^{n-1} + \frac{n^2 - n}{2} oox^{n-2} + \&c.$$
>
> are to one another as 1 and
>
> $$nx^{n-1} + \frac{n^2 - n}{2} ox^{n-2} \&c.$$
>
> Now let these augments vanish, and their ultimate ratios will be 1 to nx^{n-1}.

This is again, the same result we obtain now.

Perhaps Newton's greatest problem was his system of notation. An excerpt from De quadratura curvarum will demonstrate this:

> In what follows I consider indeterminate quantities as increasing or decreasing by a continued motion, that is, as flowing forwards, or backwards, and I design them by the letters z, y, x, v, and their fluxions or celerities of increasing I denote by the same letters pointed $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$. There are likewise fluxions or mutations more or less swift of these fluxions, which may he called the second fluxions of the same quantities z, y, x, v, and may be thus designed $\ddot{z}$; $\ddot{y}$; $\ddot{x}$; $\ddot{v}$; and the first fluxions of these last, or the third fluxions of z, y, x, v, are thus denoted $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$ and the fourth fluxions thus $\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$, $\ddddot{v}$ and after the same means that $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$ are the fluxions of the quantities $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$, and these the fluxions of the quantities $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$; and these last the fluxions of the quantities z, y, x, v,: so the quantities z, y, x, v may be considered as the fluxions of others which I shall design thus $\acute{z}$, y, x, v; and these as the fluxions of others z, y, x, v; and these last still as the fluxions of others z, y, x, v. Therefore, $\acute{\acute{z}}, \acute{z}, z, \dot{z}, \ddot{z}, \dddot{z}, \ddot{z}, \ddot{z}$ &c. design a series of quantitieswhereof everyone that follows is the fluxions of the one immediately preceding, and everyone that goes before, is a flowing quantity having that which immediately succeeds, for its fluxion.[35]

It should be clear that this is a very tedious notation. It is hard to keep straight while writing. It is hard to read, especially for higher derivatives with an interesting number of dots over the variable. There is a possibility of confusing x for x (x-prime). Newton sometimes used $\boxed{x}$ for x. But "the rectangle was inconvenient in preparing a manuscript and well-nigh impossible for printing, when of frequent occurrence."[36]

It is no wonder that Leibniz' d-notation gained immediate acceptance in Europe. Not only was his work published before Newton's but his notation was much superior. Despite continued British use of Newton's notation, mainly due to nationalistic pride and honor in the priority controversy, Cajori shows that Leibniz' differential notation was used in England as early as 16?5. In fact, even John Keill, Newton's staunchest defender in the Leibniz dispute, used differential notation.[37] However, as Struik notes, the time derivative of x is to this day often denoted as $\dot{x}$.[38]

Leibniz originally used omn. (for omnia- "all") for his integrals, as in a manuscript from three days later, he wrote, "it will be useful to write ∫ for omn, as ∫l for omn. l, that is the sum of these l's.[39] The term ∫ is the long form of s, which is the first term of summa, or sum, which an integral is. Leibniz denoted the difference between "two proximate x's" as dx, or $\frac{x}{d}$. The differential of y was succesively denoted ω, l, $\frac{y}{d}$, and finally by the standard form dy. He also introduced the derivative $\frac{dx}{dy}$.[40] The connection between differentiation and integration as inverse operations, as denoted by Barrow, is accounted for by denoting an integral in the form ∫ p dy. Thus, we see that today's differential notation originated with Leibniz.

The celebrated priority controversy over who developed the calculus first, soon degenerated into a series of charges and countercharges over whether or not Leibniz plagiarized his discovery of the calculus from Newton's writings. Newton never published any of his writings until years after they were written. Thus, when Leibniz, who, it is safe to say, developed his differential and integral calculus independently of Newton's method, published his findings and Fatio de Duillier, a Swiss mathematician and adventurer, who held a grudge against Leibniz accused Leibniz of plagiarism (some 15 years after Leibniz' publication), Leibniz was accused of the lowest kind of plagiarism, stealing Newton's ideas "from personal letters solicited from him, and from private conversations with his friends." [41]

Both Brewster and More have written exhaustive accounts of the controversy,[42] which is beyond the scope of this paper. The controversy revolved mostly around a letter sent by Newton to the Secretary of the Royal Society, Henry Oldenburg, on October 24, 1676, which letter was to be forwarded to Leibniz. Known as the Epistola Posterior, the letter contains Newton's method of drawing tangents and certain maxima and minima problems. After these, Newton wrote to Leibniz, who has requested information on Newton's methods,

> The foundationof these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus:
>
> 6accdae13eff7i3l9n4o4qrr4s8t12vx
>
> On this foundation I have also tried to simplify the theories which concern the squaring of curves, and I have arrived at certain general theorems.[43]

Turnbull explains,

> The cipher is simply a transposition of the letters in the sentence: . Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; at vice versa ('given an equation involving any number of fluent quantities to find the fluxions, and conversely')... Such concealments were not unusual in the seventeenth century.[44]

Later in the same letter. Newton drops another "clue", which, when deciphered and translated into English reads:

> One method consists of extracting a fluent quality from an equation at the same time involving its fluxion; but another by assuming a series for any unknown quantity whatever, from which the rest could conveniently be derived, and in collecting homologous terms of the resulting equation in order to elicit the terms of the assumed series."

Oldenburg did not send this letter to Leibniz until May 2, 1677.[46] Leibniz answered it on June 11, 1677 and described some of his method. Regarding the scrambled jumble of letters, More writes:

> It is evident that no translation could by any possibility be made, and it was intended by the author that no one should be able to make any sense out of it till he chose to publish the key sentences. Futhermore, no mathematician could have obtained any help from such brief and obscure sentences if they had been written in plain English.[47]

Raphson, one of Newton's rabid supporters, claimed Leibniz solved the letters and found his calculus from these sentences. It should be clear that this could not be true. If the above was not convincing enough, More adds:

> The time between Newton's epistola posterior of 24 October, 1676, and the announcement to Oldenburg by Leibniz of his discovery of the Differential Calculus on 21 June, 1677, would have been absurdly short for him to have invented the calculus even if he had deciphered Newton's sentences. But, the fact is, the forwarding of Newton's letter was delayed for months. This is verified by incontestable evidence. 48

Thus, Leibniz had a very short time to decipher a jumble of letters which would yield only a vague hint of Newton's method, and develop a complete mathematical analysis from this. "For in his answer he frankly described his differential calculus, (and) pave its algorithm, or symbolic nomenclature, so perfectly that it is used today."[49]

Unfortunately, both sides, including the eminent mathematicians themselves, played the game very dirty. In fact, Newton even attacked Leibniz after the latter's death. This controversy is a blotch on the history of mathematicians. It is now unthinkable that Leibniz was a plagiarist in any sense.

Despite the detrimental effects of the controversy to Newton's reputation (not to mention Leibniz), it cannot be doubted that Newton's achievement in the development of the calculus was great indeed.

APPENDIX

Barrow's Differential Triangle50

In figure 1, part of a parabola is drawn. As x increases from A to B, y increases from P to Q. Triangle PQR is "Barrow's differential triangle."

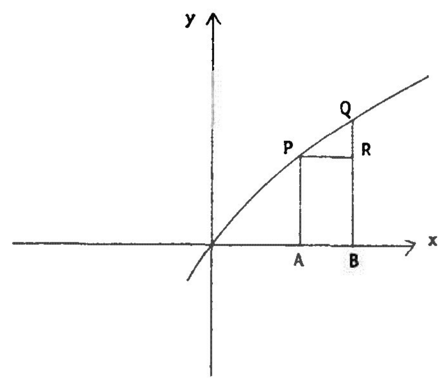

Figure 1

FOOTNOTES

[1]D. E. Smith, History of Mathematics (New York: Dover Publications, Inc., 1958), II, 676.

[2]Ibid., p. 677.

[3]W. W. Rouse Ball, A Short Account of the History of Mathematics (London: Macmillian and to., Ltd., 1927), p. 45.

[4]On the use of the name "exhaustion", see B. L. Van der Waerden, Science Awakening (New York: Oxford University Press, 1961), p. 184.

[5]Smith, p. 679

[6]Alfred Hooper, Makers of Mathematics (New York: Random House, 1948), pp. 241-244.

[7]The origin of the search for maxima and minima of curves is sometimes credited to Pappus (c. 300 B. C.). The process of integration was "Anticipated in some slight degree" by Iâbit ibn Qorra (c. 870), according to Smith, p. 685. The words fluxus and fluens were introduced in the second quarter of the fourteenth century by Richard Suiseth, known as Calculator. See Carl B. Boyer: The History of the Calculus and Its Conceptual Development (New York: Dover Publications, Inc., 1959, p. 79. Nicholas Oresme 323-82) considered the problem of motion that was not regular. He also discussed an instantaneous rate of change of velocity. See Boyer, p. 82 and H. D. Anthony, Sir Isaac Newton (London: Abelard-Schuman, 1960), p. 63. Blasius of Parma also wrote on infinitesimals, as did Nicholas of Cusa (1401–1464). Both Simon Stevin (1548-1620) and Luca Valerio (1552–1618) tried to substitute a sort of limit concept in their method of exhaustion.

[8]Smith, p. 686

[9]Ibid., pp. 686-687

Others include Gilles Personier (de) Boberval (1602-1675), who established certain integration formulas; Antonio de Monforte (1644-1717), who worked with maxima and minima, as did Rene Francois Walther de Sluze (Slusius) (1622-1685), Johann Hudde (1633–1704). Marin Mersenne (1588-1648), and Nicholas Mercator (1640-1687); Christiaan Huyens (1629-1695), whose Horologium oscillatorium "is a landmark on the path that led to the invention of the calculus," according to D. J. Struik, A Source Book in Mathematics: 1200–1800 (Cambridge, Massachusetts: Harvard University Press. 1969), p. 263; Evangelista Torricelli (1608-1647), who developed Cavalier's methods of indivisibles; Blaise Pascal (1623-1662), who was led to the "equivalent to our partial integration," (Struik, p. 241); Gregory St. Vincent (1584-1667) and his students Paul Guldin and Andreas Tacquit, all of whom worked on integrations and limit concepts; Thomas Hobbes (1588-1679), who invented the conatus (see J. W. N. Watkins, Hobbes's System of Ideas (London: Hutchinson University Library. 1965). p. 123); Galileo Galilei (1564-1642), who worked with infinitesimals and influenced his student, Cavalieri; and John Napier (1550-1617). Edward Wright, and James Gregory (1638-1675), who also worked with infinitesimals.

[11]Florian Cajori, "Who Was the First Inventor of the Calculus," American Mathematical Monthly, XXVI (January, 1919), 16–17. Also see (John Playfair), review of M. Le Comte Laplace, "Essai Philosophique sur les Probabilities," Edinburgh Review or Critical Journal, XXIII (September, 1814,) 324-325.

[12]Florian Cajori. A History of Mathematics (2d ed. rev.; New York: Macmillian, 1919), p. 192

[13]Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart. and Winston, 1961), p. 329.

[14]Quoted by Cajori, American Mathematical Monthly, XXVI, 16.

[15]Ibid., p. 17.

[16]Derek T. Whiteside (ed.), The Mathematical Works of Isaac Newton (New York: Johnson Reprint Corporation, 1964), I, ix.

[17]Ibid.

[18]Cajori, History of Mathematics, p. 192

[19]Hooper, p. 365

[20]Louis Trenchard More, Isaac Newton: A Biography (New York: Dover Publications, 1962), p. 185.

[21]Hooper, p. 310.

[22]Ibid., p. 305.

[23]Ball, p. 321

[24]Anthony, p. 64

Sir David Brewster, Memoirs of the Life, Writing!, and Discoveries of Sir Isaac Newton (New York: Johnson Reprint Corporation, 1965), II, 13–14.

[26]Cajori, History of Mathematics, p. 192

[27]Boyer, p. 195

[28]Cajori, History of Mathematics, p. 193

[29]Ibid, p. 194

[30]Ibid.

[31]Hooper, pp. 305-306.

[32]Boyer, p. 201

[33]Ibid.

[34]Struik, p. 306

[35]Ibid., pp. 306–307

[36]Florian Cajori. A History of Mathematical Notations. Vol. II: Notations Mainly in Higher Mathematics (Chicago: Open Court Publishing Company, 1929), p. 246.

[37]Ibid., pp. 244-245

[38] Struik, p. 270

[39] Cajori, Notations, p. 203

[40] Ibid.

[41] More, p. 188

[42] See Brewster, pp. 23–83, and More, pp. 565–607. For More's doubt as to Brewster's reliability, see More, p. vi.

[43] H. W. Turnbull, The Correspondence of Isaac Newton, Vol II! 1676–1687 (Cambridge, England: University Press, 1960), p. 134.

[44] Ibid., p. 153.

[45] Ibid., p. 159.

[46] Ibid., pp. 208, 219.

[47] More, p. 192.

[48] Ibid., pp. 192–193.

[49] Ibid., p. 193.

[50] Hooper, p. 289.

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POLYNOMIALS WHICH ASSUME INFINITELY MANY PRIME VALUES

E. F. Ecklund, Jr.

We will begin with Euclid, who stated and proved the following theorem: The number of primes is infinite. Without this theorem, our discussion would be moot.

Some time later, mathematicians began asking if there existed a function, p, for which $p(i)$ is the i-th prime, for each integer i. There is no known way to define such a function except by a pointwise definition. It was then asked if there is a function f such that $f(i)$ is a prime for each integer i. Note: f need neither assume values of all the primes, nor assume its values in any particular order. One attempt to produce such a function was Fermat's conjecture that $F(n) = 2^{2^n}+1$ was always prime. In 1732, Euler proved that $F(5)$ was composite. By the 1800's, it had been proven that no polynomial could assume only prime values. The next logical question would seem to be: Is there a function whose values are prime infinitely often? The answer was known to be yes, since it had been proven that $4x+3$ and $6x+5$ both produced sequences which are prime infinitely often. A search for conditions which would characterize such functions was now begun.

In 1837, G. L. Dirichlet proved that for $mx+n$ to represent infinitely many primes, it is necessary and sufficient that $(m,n) = 1$, where (m,n) denotes the greatest common divisor of m and n.

In 1854, V. Bouniakowsky conjectured that if $f(x)$ is a polynomial in x over the integers such that the coefficients of $f(x)$ have no common factors, and if N is the greatest common divisor of all integer values of $f(x)$, then if $f(x)$ is irreducible, $f(x)/N$ represents an infinitude of primes.

In 1904, L. E. Dickson introduced a new direction to these considerations. He asked if a_ix+b_i, $i = 1,..,n$ represented an infinitude of sets of n primes. We may ask what conditions are necessary for this to be true. First, for each i, the sequence a_ix+b_i must be prime infinitelv often: hence $(a_i,b_i) = 1$. Since each prime can occur only n times - once in each progression - we see that $\{a_ix+b_i;\ i = 1,..,n\} \cap \{a_iy+b_i;\ i = 1,..,n\} = 0$, for some integers x and y. If we let $P(x) = \prod_{i=1}^{n}(a_ix+b_i)$, we may restate this as $(P(x),P(y)) = 1$ for some integers x and y.

In 1958, A. Schinzel announced a conjecture which seems to combine the directions of thought of Bouniakowsky and Dickson.

First we introduce a necessary condition:

Condition S: Each of the polynomials $f_i(x)$, $i = 1,..,n$ is irreducible: its leading coefficient is positive; and there is no natural number $d > 1$ that is a divisor of each of the values of $P(x) = \prod_{i=1}^{n} f_i(x)$, $(x = 0,1,2,...)$.

Schinzel's conjecture is as follows:

Conjecture H: If n is a natural number and $f_i(x)$, $i = 1,..,n$, are polynomials with integral coefficients satisfying condition S, then there exist infinitely many natural numbers x for which each of the numbers $f_i(x)$ is prime, $i = 1,..,n$.

We now present some other conjectures which are corollaries to conjecture H.

Corollary 1. There occur infinitely often four consecutive primes p_1, p_2, p_3, and p_4 whose local distribution is such that $p_4-p_3 = p_2-p_1 = 2$ and $p_3-p_2 = 4$.

Proof: Let $f_1(x) = x^{2^n}+1$, $f_2(x) = x^{2^n}+3$, $f_3(x) = x^{2^n}+7$, and $f_4(x) = x^{2^n}+9$, for fixed n. Clearly for each i, $f_i(x)$ is irreducible and has positive leading coefficient. Let $P(x) = \prod_{i=1}^{4} f_i(x)$. $P(0) = 1\cdot3\cdot7\cdot9$, and $P(1) = 2\cdot4\cdot8\cdot10$. Thus condition S is satisfied, and by conjecture H, the $f_i(x)$'s are simultaneously prime infinitely often.

Corollary 1.1. There are infinitely many twin primes.

Proof: The pair $f_1(x)$ and $f_2(x)$ in Corollary 1 are twin primes when thev are simultaneously prime, which by Corollary 1 occurs infinitely often.

Corollary 1.2. There are infinitely many primes of the forms x^2+1 and x^4+1.

Proof: By Corollary 1, $x^{2^n}+1$ is prime infinitely often. Letting $v = x^{2^{n-1}}$, we have $x^{2^n}+1 = y^2+1$ is prime infinitely often. Similarly letting $v = x^{2^{n-2}}$, we have $x^{2^n}+1 = y^4+1$ is prime infinitely often.

Corollary 2. There exist infinitely many sets of three consecutive integers n, n+1, and n+2 such that each is the product of two distinct primes.

Proof: Let $p(x) = 10x+1$, $q(x) = 15x+2$, and $r(x) = 6x+1$. Clearly $p(x)$, $q(x)$ and $r(x)$ are irreducible and have positive leading coefficients. Let $P(x) = p(x)q(x)r(x)$; then $P(1) = 7\text{-}11\text{-}17$ and $P(3) = 19\text{-}31\text{-}47$. Hence $(P(1), P(3)) = 1$. Thus condition S is satisfied and by conjecture H, there exist infinitely many integers x such that $p(x)$, $q(x)$, and $r(x)$ are prime. For such an x, let $n = 3\cdot p(x)$, $n+1 = 2\cdot q(x)$, and $n+2 = 5\cdot r(x)$; ie $n = 30x+3$, $n+1 = 30x+4$, and $n+2 = 30x+5$. Then n, $n+1$, and n+2 are each the product of two distinct primes.

Note that we cannot find four such consecutive integers since one of them would be divisible by four. We may ask, however, how manv consecutive odd integers there are which are the product of two distinct primes. We see immediately that the maximum is eight since one of nine consecutive odd integers must be divisible by nine. It appears that the existence infinitelv often of eight such consecutive odd integers would be a corollary to conjecture H. In closing, we offer the following generalization of Corollarv 2:

Conjecture J. Given a natural number n, let m be the product of primes less than n. fhen there exist infinitely many sets of n^2-1 consecutive elements of arithmetic progressions with common difference m such that each is the product of two distinct primes.

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COLOR GROUPS

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An example of a group may be presented in many ways. One method is to define a set of generators and relations. Thus, for example,

$$\langle a\rangle,\ a^4 = e$$

describes the cyclic group of order four: there Is one generator, a, and on this generator we impose the restriction that it have order four. Similarly, the Klein group is given by

$$\langle a,b\rangle,\ a^2 = b^2 = e,\ ab = ba$$

The relations, of course, must be consistent; we do not concern ourselves here with the problem or independence.

Every formal expression which can be constructed by juxtaposition of generators, or integral powers (positive, negative, or zero) of generators, is a group element, called a word. The group operation is juxtaposition of these words. Two words are equivalent if one can be transformed into the other by means of finitely many applications of the group relations. An empty word is a word equivalent to the identity. A generator is free if it is unbound by any relation. A group is free if all its generators are free. We notice, then, that the free group on one generator is isomorphic to the group of integers under addition.

Consider, for example, the free group with three generators, $\langle a,b,c\rangle$. An example of a word in this group is

$$a^2b^{-5}a^3c^4ba^{31}$$

Since the group is free, this word is not equivalent to any other possible juxtaposition of powers of the generators. Now consider another group with the same generators: $\langle a,b,c\rangle$, $a = b^2 = c^3 = e$, $ba = a^6b$, $ac = ca$, $bc = cb$. In this group, the word $a^2b^{-5}a^3c^4ba^{31}$ is equivalent to a^2c. In this way, the elements of the group are the equivalence classes induced on the set of words by the (equivalence; relations..

These groups may be given a pictorial representation, called a graph or Cayley diagram: an array of points (vertices) and directed line segments (edges) of various colors. In print, the colors are usually represented by differently printed lines (solid, dashed, dotted, etc.). A Cayley diagram, then, is an array of vertices and edges which has the properties that

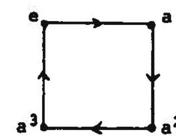
1) at each vertex there is an edge of each color directed toward the vertex and an edge of each color directed away from the vertex;

2) the figure is symmetric in each vertex.

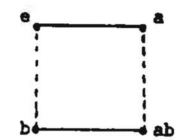
This second condition means that the structure of the diagram is constant, no matter from which vertex it Is "viewed". This does not necessarily mean a geometric symmetry in the plane, as several of the diagrams below illustrate, but rather an invariance under any permutation of the vertices.

The Cayley diagram corresponds to its group in this way

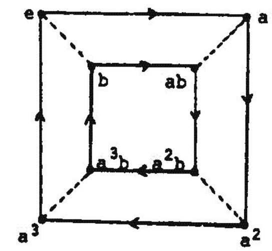
1) there is a one-to-one correspondence between the set of generators



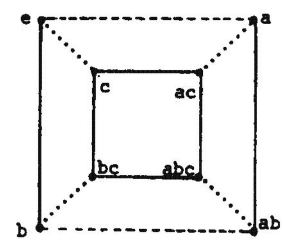
$\langle a\rangle$
$a^4 = e$



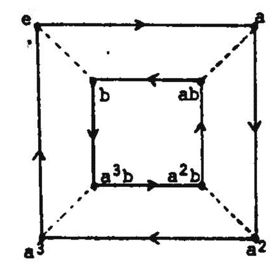
$\langle a,b\rangle$
$a^2 = b^2 = e$
$ab = ba$



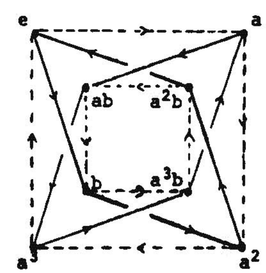
$\langle a,b\rangle$
$ab = ba$
$a^4 = b^2 = e$



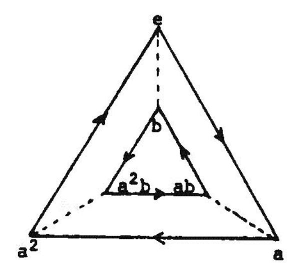
$\langle a,b,c\rangle$
$ab = ba$
$ac = ca$
$bc = cb$
$a^2 = b^2 = c^2 = e$



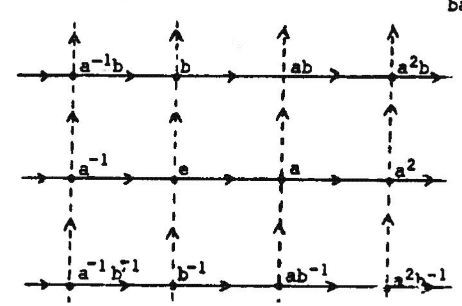
$\langle a,b\rangle$
$ba = a^3b$
$a^4= b^2 = e$



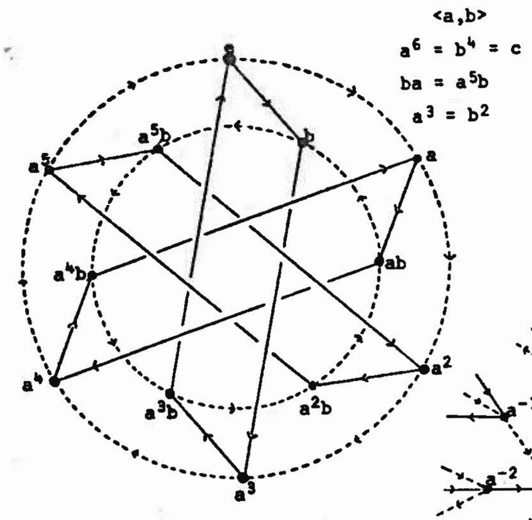
$\langle a,b\rangle$
$a^4= b^4 = e$
$ba = a^3b$
$a^2 = b^2$

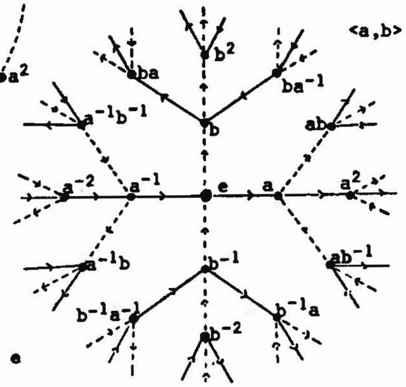


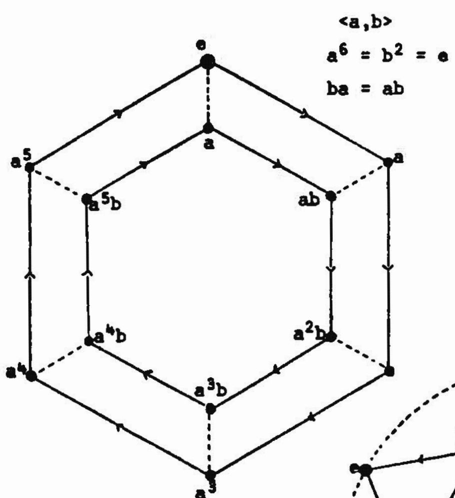
$\langle a,b\rangle$
$a^3 = b^2 = e$
$ba = a^2b$

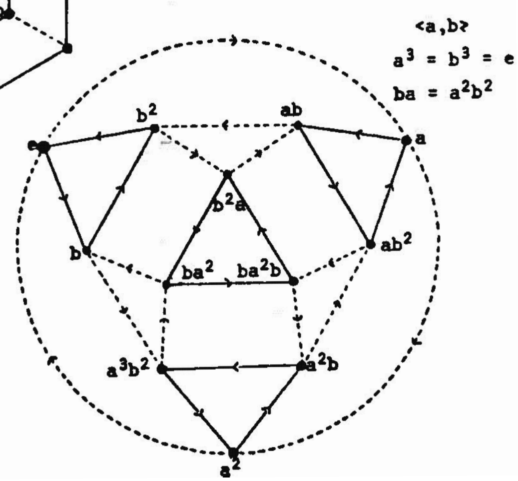


$\langle a,b\rangle$
$ab = ba$









and the set of colors used in the diagram;

2) there is a one-to-one correspondence between the set of group elements and the set of vertices;

3) the vertex corresponding to the identity may be arbitrarily chosen

4) if x and y are group elements corresponding to vertices X and Y respectively, and if a is a generator corresponding to color A, then xa = y if and only if there is an edge of color A leading from X to Y.

A path in a Cayley diagram is any connected sequence of edges. Each path corresponds to a word; if the path P leads from vertex R to vertex S, and if P, R, and S correspond respectively to w, r, and s in the group, then rw = s, and conversely. Thus a closed path corresponds to an empty word. If two paths have the same initial point and the same terminal point, they correspond to equivalent words.

We present at the end of the paper the graphs of several groups. We remark that movement along an edge against its direction corresponds to multiplication by the inverse of that generator. We also remark that if a generator has order two, it is unnecessary to indicate direction along the corresponding edge - traffic can be considered to move in both directions along that street.

The graph of a group is not unique. The edges need not be straight lines, and variations are limited only by the artistic imagination of the individual. Some graphs are closed designs and other graphs fill the plane. There are seventeen groups whose graphs fill the plane with a continually repeated pattern. The group ⟨a,b⟩, ab = ba is one such group. The pursuit of this colorful topic can lead the mathematician into such useful occupations as designing tile floors, wallpaper, and Christmas cards.

Reference

Grossman and Magnus: Groups and their Graphs (Random House)

MATCHING PRIZE FUND

The Governing Council of Pi Mu Epsilon has approved an increase in the maximum amount per chapter allowed as a matching prize from \$25.00 to \$50.00. If your chapter presents awards for outstanding mathematical papers and students, you may apply to the National Office to match the amount spent by your chapter--i.e., \$30.00 of awards, the National Office will reimburse the chapter for \$15.00, etc.,--up to a maximum of \$50.00. Chapters are urged to submit their best student papers to the Editor of the PI Mu Epsilon Journal for possible publication. These funds may also be used for the rental of mathematical films. Please indicate title, source and cost, as well as a very brief comment as to whether you would recommend this particular film for other Pi Mu Epsilon groups.

A SEMI-NUMBER SYSTEM

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In the June–July, 1966 issue of The American Mathematical Monthly, Samuel Stern presented a paper entitled "The Semi-natural Numbers." In this article he outlined the development of a number system from four basic axioms in a manner similar to the development of the natural numbers from the Peano axioms. He also suggested a method of extending the semi-natural numbers to a larger set in which a certain kind of subtraction is possible. What follows is a description of the axioms for a semi-natural number system with a few consequences and examples and an extension of semi-natural numbers to a larger set of semi-integers. Then preceding beyond the limits of the original article, we shall construct ordered pairs of semi-integers to form semi-rational numbers and finally we shall define and briefly discuss semi-real numbers.

Definition. Let N be a set and $<$ be a binary relation on N. N is said to be system of semi-natural numbers if the following axioms are satisfied:

Axiom 1. N is simply ordered with respect to $<$.
Axiom 2. N is not the empty set.
Axiom 3. If x is an element of N, then there exists y in N such that $x < y$.
Axiom 4. If G is a subset of N such that, for all x in N, $I(x) \subseteq G$ ⇒ x is an element of G, then $G = N$. By $I(x)$ we mean $\{ y: y \text{ is in } N \text{ and } y < x\}$.

The article [St] by Stern demonstrates that this definition implies that a semi-natural system is well ordered by $<$, i.e. each non-empty subset has a least element.

Definition. For $x, y \in N$ such that $x < y$ and $x < z < y$ for no $z \in N$, then x is the immediate predecessor of y and y is the immediate successor of x. If an element p of N is such that p has no immediate predecessor, then p is said to be a primary semi-natural number.

Definition. The binary operation of addition is defined on N by:

(i). $x + p = x$, for all p, primary;
(ii). $x + Sy = S(x + y)$, where Sk denoted the successor of k.

The binary operation of multiplication is defined on N by:

(i). $xp = p$; (ii). $xSy = xy + x$.

Addition and multiplication can be shown to be well-defined and associative.

In seeking to find models for a semi-natural system, it is easy to verify that the natural numbers in union with zero, i.e. the non-negetive integers, in their usual order form such a system. As another, somewhat arbitrary example, consider $P = \{ 0, 1, 2, \ldots, 0', 1', 2', \ldots\}$, where the elements of P are ordered as they appear in the array. This set P demonstrates that addition and multiplication are not necessarily commutative in a semi-natural system. In P, $1 + 2' = 1 + S(1') = S(1 + 1') = S(1 + S(0')) = S(S(1 + 0')) = S(S(1)) = S(2) = 3$; while $2' + 1 = 2' + S(0) = S(2' + 0) = S(2') = 3'$. Also, $2' \cdot 1 = 2' \cdot S(0) = 2' \cdot 0 + 2' = 0 + 2' = 2$; while $1 \cdot 2' = 1 \cdot S(1') = 1 \cdot 1' + 1 = 1 \cdot S(0') + 1 = 1 \cdot 0' + 1 + 1 = 0' + 1 + 1 = 2'$.

Although a semi-natural system is not necessarily abelian, there are several properties which provide us with a weakened commutativity. Among these are: $x + y + z = x + z + y$ and $xyz = yxz$. Both of these statements are proven by using Axiom 4, the principle of transfinite induction, on z.

Also, the distributivity of multiplication over addition holds. There exist right additive identities, the primary elements, and left multiplicative identities, the successors of primary elements.

Let us consider the set $M = \{ (a,b) : a,b \in N \}$. The relation of equality is defined on M by: $(a,b) = (c,d)$ iff $a + d = c + b$. It follows from this definition that equality on M is an equivalence relation. Let us denote the equivalence class of (a,b) by $\overline{(a,b)}$ and set $I = \{ \overline{(a,b)}: a, b \in N \}$. The set I will be called a system of semi-integers.

Definition. The operation of addition is defined on I by:

$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a + c, b + d)}$.

The operation of multiplication is defined on I by:

$\overline{(a,b)} \odot \overline{(c,d)} = \overline{(ac + bd, ad + be)}$.

Again, these operations on I, just as the corresponding operations on N, are well-defined and associative, but not commutative. An important result regarding semi-natural numbers is that for any two numbers, a and $b \in N$, either $a = m + b$ or $b = n + a$, for some $m, n \in N$. Thus, the elements of I are of two forms: $\overline{(a, n + a)}$ or $\overline{(m + b, b)}$. Furthermore, a semi-integer of the form $\overline{(m + b, b)}$ equals $\overline{(m, p)}$, which equals $\overline{(m, 0)}$, where 0 is the least element of N. For $m \in N$ such that m is not primary, $m = \overline{(m, 0)}$ is said to be a positive semi-integer; for $p \in N$ such that p is primary, $\overline{(p,0)} = p$ is a primary semi-integer. Before defining a negative semi-integer, it should be noted that $\overline{(0,k)} \neq \overline{(p,k)}$ for all $k \in N$; and hence we must first find unique representations for elements of the form $\overline{(p,m)}$. To do this, we shall define the absolute value of semi-natural numbers: If $m \in N$, then the absolute value $|m|$ of m = m if $0 < m < p$, for all primary p such that $0 < p$;
= k if $m = p + k$ for some non-zero primary p such that, for all $p^* < m$ with p^* primary, $p^* \leq p$; $|k| = k$;
= 0 if m is primary

Then from the definition of equality. it follows that $\overline{(p, m)} = \overline{(p, |m|)}$. Thus, for non-primary m, $\overline{(p, |m|)} = -m$ is said to be negative semi-integer. Many familiar results regarding the integers also apply to the semi-integers; e.g., $-(-m) = m$, $m(-n) = (-m)n = -(mn)$, and $(-m)(-n) = mn$. A relation can be defined on I which preserves the relation of $<$ on the semi-integers of the form $\overline{(m,0)} = m$ which can also be regarded as semi-natural numbers.

Definition. The relation $\ll$ is defined on I by:

$m = \overline{(a, b)} \ll n = \overline{(c, d)}$ iff $a + d < c + b$.

If the non-negative integers are selected as the semi-natural system, the resulting set of semi-integers would be the familiar set of integers. If the set P, defined above, is chosen as the semi-natural system, then the set $\bar{P} = \{\ldots, -2, -1, \ldots, -2', -1', 0, 1, 2, \ldots, 0', 1', 2', \ldots\}$ is the corresponding system of semi-integers.

Consider the set $Q = \{ (a,b) : a, b \in I$ and b is not primary$\}$, with the relation of equality defined on Q by: $(a, b) = (c, d)$ iff $da = bc$. This relation is in fact an equivalence relation and partitions Q into mutually disjoint equivalence classes. Denote the equivalence class of (a, b) by $((a, b))$ and define $R = \{ ((a, b)): a, b \in I$ with $b \neq$ primary$\}$ to be a system of semi-rational numbers.

Definition. The operation of addition is defined on R by:

$((a, b)) \leftrightarrow ((c, d)) = ((da + bc, bd))$.

The operation of multiplication is defined on R by:

$((a, b)) \leftrightarrow ((c, d)) = ((ac, db))$.

Many properties of the semi-natural numbers and the semi-integers induce corresponding properties in the semi-rational numbers; e.g., left cancellation of multiplication by a non-primary element. Further, the semi-integers can be embedded in the semi-rational numbers by the mapping $\theta: I \longrightarrow R$ given by $m\theta = ((m, SO))$.

Among the new features of the semi-rational numbers, we can define the inverse of an element ((a, b)) where a is also non-primary, to be the semi-rational number $((b, |a|))$. This definition allows us to consider a new operation:

Definition. The operation of division is defined on R by:

$$\frac{((a, b))}{((c, d))} = ((c, d))^{-1}((a, b)), \text{ where } ((c, d))^{-1} = ((d, |c|)).$$

Another important result can be expressed as the following:

Theorem. Equations of the form $ax = b$, where $a,b \in R$ and $a \neq$ primary have a unique solution in R.

Proof: Clearly $x = a^{-1} \cdot b$ is a solution since $ax = a(a^{-1} \cdot b) = (SO)\ b = b$. Further, assume that x and y are both solutions to the equation. Then $ax = b = ay$, i.e., $ax = ay$. By left cancellation multiplicatively by a non-primary element, it follows that $x = y$; hence the solution is unique. It is interesting to note that while equations of the form $xa = b$ are soluble in R, these solutions are not unique; it is readily seen that the proof breaks down because there is no right cancellation multiplicatively by a non-primary element. As an example, choosing the set P defined above and denoting the resulting system of semi-rational numbers by $\bar{\bar{P}}$, the equation $x \cdot 2 = 6$ has two distinct solutions, viz., $x = 3$ and $x = 3'$. Another interesting property of semi-rational numbers is that for any two semi-rationals x and y such that $x < y$, there exists a semi-rational z such that $x < z < y$, i.e., the semi-rational numbers are dense.

In attempting to enlarge the semi-rational numbers into a more comprehensive type of semi-number, we are met with the same difficulty found in the extension of the rational numbers to form the real numbers; the method of constructing ordered pairs fails. In a manner completely analogous to Dedekind's method of "cuts" in the rational numbers, we proceed as follows:

Definition. Given a system, R, of semi-rational numbers, a cut is a partition of R, denoted (A, B), into two sets A and B such that

(i). $A \cup B = R$ $\quad A \cap B = \emptyset$;

(ii). $A \neq 0$ $\quad B \neq 0$;

(iii). For all $a \in A$ and $b \in B$, $a<b$.

(iv). For any $a \in A$, there exists $a' \in A$ such that $a<a'$.

Thus, there are two types of cuts: (a) A has no largest element and B has a smallest element, this type of cut is said to define a semi-rational number, i.e., the semi-rational number which is the least element of B; (b) A has no largest element and B has no smallest element, this type of cut is said to define a semi-irrational number. The set of all cuts will be termed the set of semi-real numbers.

The four standard arithmetic operations can be defined on R#, a system of semi-real numbers; in addition, extraction of roots is possible on R#. Many of the consequences regarding semi-real numbers follow from similar properties of semi-rational numbers, including the characteristic of density. Examples of semi-irrational numbers can also be given and proven to be semi-irrational, with the most obvious example being $\sqrt{S\ (SO)}$

REFERENCE

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Wronskian Identities

Martin Swiatkowski

Neither the statement nor the proof of the main result of this paper (Corollary 3.1) are original with this author. G. Polya [2, p.315] prefaces his use of the result with the words "...by the usual formula for a minor of the adjoint determinant." The proof given here is essentially that indicated by Philip Hartman [1,p.310]. .

The purpose of this paper is to provide an elementary and direct proof of Corollary 3.1 for those who lack the motivation and/or background to read Hartman's paper on differential equations and who are not familiar with "the usual formula for a minor of the adjoint determinant."

Definition. Let $f_1, \ldots, f_n$ be functions n-1 times differentiable over (a,b). The "Wronskian of $f_1, \ldots, f_n$," $W(f_1, \ldots, f_n)$ is the following determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots f_n^{(1)} \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots f_n^{(n-1)} \end{vmatrix}$$

One of the most basic identities involving Wronskians is the following fact:

$$(0.1) \qquad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} = \frac{\begin{vmatrix} v & u \\ v' & u' \end{vmatrix}}{v^2} = \frac{W(v, u)}{2}$$

Two identities will be proved which allow certain manipulations of Wronskians. Theorem 1 presents a sort of "factoring," theorem 2 permits an alteration in a Wronskian's size.

Theorem 1. If $Y(t) \neq 0$ for $t \in (a,b)$, then $W(YZ_1, \ldots, YZ_j) = Y^j W(Z_1, \ldots, Z_j)$ on (a,b).

Proof. $W(YZ_1, \ldots, YZ_j) =$

$$\begin{vmatrix} YZ_1 & \cdots & YZ_j \\ YZ_1' + Y'Z_1 & \cdots & YZ_j' + Y'Z_j \\ & & \vdots \\ \sum_{k=0}^{j-1} \binom{j-1}{k} Y^{(k)} Z_1^{(j-1-k)} & \cdots & \sum_{k=0}^{j-1} \binom{j-1}{k} Y^{(k)} Z_j^{(j-1-k)} \end{vmatrix}$$

Row operations eliminate all but the first terms of each element. Therefore $W(YZ_1, \ldots, YZ_j) =$

$$\begin{vmatrix} YZ_1 & \cdots YZ_j \\ \vdots & \vdots \\ YZ_1^{(j-1)} & \cdots YZ_j^{(j-1)} \end{vmatrix} = Y^j \begin{vmatrix} Z_1 & \cdots Z_j \\ \vdots & \vdots \\ Z_1^{(j-1)} & \cdots Z_j^{(j-1)} \end{vmatrix}$$

$$= Y^j W(Z_1, \ldots, Z_j).$$

Theorem 2. If $Y(t) \neq 0$ for $t \in (a,b)$, then $W(Y, Z_1, \ldots, Z_j) = Y^{j+1} W\left(\left(\frac{Z_1}{Y}\right)', \ldots, \left(\frac{Z_j}{Y}\right)'\right)$ on (a,b).

Proof. By theorem 1 $W(Y, Z_1, \ldots, Z_j) =$

$$Y^{j+1} W\left(1, \frac{Z_1}{Y}, \ldots, \frac{Z_j}{Y}\right) =$$

$$Y^{j+1} \begin{vmatrix} 1 & \frac{Z_1}{Y} & \cdots & \frac{Z_j}{Y} \\ 0 & \left(\frac{Z_1}{Y}\right)' & \cdots & \left(\frac{Z_j}{Y}\right)' \\ \vdots & \vdots & & \vdots \\ 0 & \left(\frac{Z_1}{Y}\right)^{(j)} & \cdots & \left(\frac{Z_j}{Y}\right)^{(j)} \end{vmatrix} = Y^{j+1} \begin{vmatrix} \left(\frac{Z_1}{Y}\right)' & \cdots & \left(\frac{Z_j}{Y}\right)' \\ \vdots & & \\ \left(\frac{Z_1}{Y}\right)^{(j)} & \cdots & \left(\frac{Z_j}{Y}\right)^{(j)} \end{vmatrix}$$

$$= Y^{j+1} W\left(\left(\frac{Z_1}{Y}\right)', \ldots, \left(\frac{Z_j}{Y}\right)'\right).$$

An ingenious identity indicated by Hartman can now be proved.

Theorem 3. $(W(u_1, \ldots, u_n))^{k-1} W(u_1, \ldots, u_n, x_1, \ldots, x_k) = W(W(u_1, \ldots, u_n, x_1), \ldots, W(u_1, \ldots, u_n, x_k)).$

Proof. By induction on n.

Basis: $n=1$: $(W(u_1))^{k-1} W(u_1, x_1, \ldots, x_k) = W(W(u_1, x_k), \ldots, W(u_1, x_k)).$

$$(W(u_1))^{k-1} W(u_1, x_1, \ldots, x_k) = u_1^{k-1} W(u_1, x_1, \ldots, x_k)$$

$$= u_1^{k-1} u_1^{k+1} W\left(\left(\frac{x_1}{u_1}\right)', \ldots, \left(\frac{x_k}{u_1}\right)'\right) = u_1^{2k} W\left(\left(\frac{W(u_1, x_1)}{u_1^2}\right), \ldots, \left(\frac{W(u_1, x_k)}{u_1^2}\right)\right)$$

$$= u_1^{2k} u_1^{-2k} W(W(u_1, x_1), \ldots, W(u_1, x_k)) = W(W(u_1, x_1), \ldots, W(u_1, x_k)).$$

The first equality is by the value of a one element determinant, the second by theorem 2, the third by fact 0.1, the fourth by theorem 1, the last by adding exponents.

Induction hypothesis: Theorem 3.

Inductionstep: $n+1$: $(W(u_1, \ldots, u_{n+1}))^{k-1} W(u_1, \ldots, u_{n+1}, x_1, \ldots, x_k) = W(W(u_1, \ldots, u_{n+1}, x_1), \ldots, W(u_1, \ldots, u_{n+1}, x_k)).$

$$W(W(u_1, \ldots, u_{n+1}, x_1), \ldots, W(u_1, \ldots, u_{n+1}, x_k)) =$$

$$= W\left(u_1^{n+2} W\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_{n+1}}{u_1}\right)', \left(\frac{x_1}{u_1}\right)'\right), \ldots, u_1^{n+2} W\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_{n+1}}{u_1}\right)', \left(\frac{x_k}{u_1}\right)'\right)\right)$$

$$= u_1^{(n+2)k} W\left(W\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_{n+1}}{u_1}\right)', \left(\frac{x_1}{u_1}\right)'\right), \ldots, W\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_{n+1}}{u_1}\right)', \left(\frac{x_k}{u_1}\right)'\right)\right)$$

$$= u_1^{(n+2)k} \left(W\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_{n+1}}{u_1}\right)'\right)\right)^{k-1} W\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_{n+1}}{u_1}\right)', \left(\frac{x_1}{u_1}\right)', \ldots, \left(\frac{x_k}{u_1}\right)'\right)$$

$$= u_1^{(n+2)k} \left(u_1^{-(n+1)} W(u_1, \ldots, u_{n+1})\right)^{k-1} u_1^{-(n+k+1)} W(u_1, \ldots, u_{n+1}, x_1, \ldots, x_k)$$

$$= u_1^{(n+2)k - (n+1)(k-1) - (n+k+1)} \left(W(u_1, \ldots, u_{n+1})\right)^{k-1} W(u_1, \ldots, u_{n+1}, x_1, \ldots, x_k)$$

$$= \left(W(u_1, \ldots, u_{n+1})\right)^{k-1} W(u_1, \ldots, u_{n+1}, x_1, \ldots, x_k).$$

The first two equalities, by theorems 2 and 1, change a Wronskian of $n+2 \times n+2$ Wronskians to a product including a Wronskian of $n+1 \times n+1$ Wronskians, facilitating the use of the hypothesis. The fourth equality reinstates u_1 into the Wronskians by theorem 2, and the fifth gathers the exponents of u_1 which sum up to zero.

Polya's formula for a minor of the adjoint determinant is a specific case of theorem 3. Although probably proved by a tedious determinant method originally, it now becomes a simple corollary.

<u>Corollary</u> 3.1. $$\left(\frac{W(u_1,\ldots,u_n,Y)}{W(u_1,\ldots,u_n,X)}\right)' = \frac{W(u_1,\ldots,u_n)\,W(u_1,\ldots,u_n,X,Y)}{(W(u_1,\ldots,u_n,X))^2}$$

<u>Proof.</u> $$\left(\frac{W(u_1,\ldots,u_n,Y)}{W(u_1,\ldots,u_n,X)}\right)' = \frac{W(W(u_1,\ldots,u_n,X),\,W(u_1,\ldots,u_n,Y))}{(W(u_1,\ldots,u_n,X))^2}$$

$$= \frac{W(u_1,\ldots,u_n)\,W(u_1,\ldots,u_n,X,Y)}{(W(u_1,\ldots,u_n,Y))^2}$$

The proof uses fact .1 and theorem 3 with $k = 2$.

Hartman's article is suggested to the reader who is interested in further Wronskian identities and in their use in determining properties of solutions to linear differential equations.

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2. G. Polya, <u>On the mean value theorem corresponding to a given linear homogeneous differential equation.</u> Transactions of the American Mathematical Society vol. 24 (1922) pp. 312-324.

AN EXTENSION OF HERMITIAN MATRICES

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Hermitian matrices, defined to be identical with their transposed conjugates ($A^* = A$), possess so many interesting and useful properties that it is only natural to inquire what would happen if $A^* = A^n$, where n is an integer greater than 1. In this paper, such a type of matrix, and the analogously defined type of operator in Hilbert space will be considered.

A square matrix A satisfying the relation $A^* = A^n$, where n is an integer greater than 1 will be called a Hermitian matrix of degree n, or for short, an H matrix. Since $AA^* = AA^n = A^{n+1} = A^nA = A^*A$, any H matrix is normal. This implies that an H matrix is simple (diagonalizable--geometric = algebraic multiplicity for each eigenvalue), hence is unitarily similar to the diagonal matrix of its eigenvalues, and that eigenvectors associated with distinct eigenvalues are orthogonal.

A relationship can be obtained from $A^* = A^n$ not involving A^*. For since (as can easily be shown by induction) $(A^*)^n = (A^n)^*$, it follows that $A = A^{**} = (A^n)^* = (A^*)^n = (A^n)^n = A^{n^2}$, so $A = A^{n^2}$ for any H matrix A. If A^{-1} exists, this in turn implies that $A^{n^2-1} = I$. Conversely, if $A^{n^2-1} = I$, then since by assumption $n>1$, A^{n^2-2} exists and is A^{-1}. Hence if A is an H_n matrix, $A^{n^2-1} = I$ iff A is nonsingular. This can also he expressed by saying that if A is a nonsingular H matrix, its powers will form a cyclic group of order $n^2 - 1$, or some divisor of $n^2 - 1$.

Since any H_n matrix is unitarily similar to the diagonal matrix of its eigenvalues, it would he worthwhile to consider these eigenvalues themselves. The relations they satisfy arc similar in form to those obtained for the matrix: if μ is an eigenvalue, $\bar{\mu} = \mu^n$, and $\mu = \mu^{n^2}$. The first of these is proved in the same way that it is shown that the eigenvalues of a Hermitian matrix are real: $Ax = \mu x$ implies $x^*A^* = \bar{\mu}x^*$, so $x^*A^*x = \bar{\mu}x^*x$, but $x^*A^*x = x^*A^nx = \mu^n x^*x$, so $\bar{\mu}x^*x = \mu^n x^*x$. x is an eigenvector, $x \neq 0$, $x^*x \neq 0$, so $\bar{\mu} = \mu^n$. similarly, $\mu x = Ax = A^{n^2}x$

$= \mu^{n^2} x$, and $x \neq 0$ yields $\mu = \mu^{n^2}$. If μ is an eigenvalue of A, $\mu x = Ax$ implies $A^* x = A^n x = \mu^n x = \bar{\mu} x$, so $\bar{\mu}$ is an eigenvalue of A* with the same associated eigenvector x.

The scalar field will here be assumed to be the complex numbers, and so $\mu = \mu^{n^2}$ requires either that $\mu = 0$, or μ is an $n^2 - 1^{st}$ root of 1. It should be noted that while the eigenvalues of an H matrix, unlike those of a Hermitian matrix, are not necessarily real, they can be ordered since they all must lie on the unit circle (or at 0). The normality of an H matrix now implies that such a matrix will be unitarily similar to a diagonal matrix of 0's and $n^2 - 1^{st}$ roots of 1. If A is a nonsingular H matrix, it has no zero eigenvalues, and so this similarity description implies that $|\det A| = 1$.

The above requirement that μ be 0 or an $n^2 - 1^{st}$ root of 1 is only necessary, not sufficient, and so actually there is so far no proof that any H matrices really exist. This situation will now be remedied by showing that a matrix A is H iff it is unitarily similar to a diagonal matrix consisting of 0's and $n + 1^{st}$ roots of 1.

Sufficiency is clear, since if $A = U^* DU$, where U is unitary, and D is a diagonal matrix, then $A^n = U^* D^n U = (U^* (\bar{D})^n U)^* = A^*$ provided that $(\bar{D})^n = D$, or, taking conjugates, $D^n = \bar{D}$. If the eigenvalue μ, an element of D, is 0, $0^n = \bar{0}$. If $\mu^{n+1} = 1$, $\mu^n = \frac{1}{\mu}$, so expressing μ as $e^{i\theta}$, $\mu^n = \frac{1}{\mu} = \frac{1}{e^{i\theta}} = e^{-i\theta} = \bar{\mu}$.

Necessity is shown as follows. As was proved above, the nonzero eigenvalues of an H_n matrix must be $n^2 - 1$ roots of 1, i.e. must have the form $\mu = e^{i2k\pi/(n^2 - 1)}$, where k can range from 0 to $n^2 - 2$. Since $\mu^n = \bar{\mu}$, $e^{i2nk\pi/(n^2-1)} = e^{i\left[2m\pi - \frac{2k\pi}{n^2 - 1}\right]}$, where m is an integer. Solving, $nk = (n^2 - 1)m - k$, $k(n+1) = (n-1)(n+1)m$, or since $n > 1$, $\frac{k}{n-1} = m$, an integer. Resubstituting, $\mu = e^{i2m\pi/(n+1)}$, where m is an integer, and it can be seen that μ is indeed an $n + 1$ root of 1.

From this characterization, it can now be seen that all H matrices which are nonsingular are unitary, for if $A = U^* DU$ as above, then $AA^* = AA^n = A^{n+1} = U^* D^{n+1} U$. Since A is nonsingular, D is a diagonal matrix entirely of $n+1^{st}$ roots of 1, so $D^{n+1} = I$, and $AA^* = U^* IU = U^* U = I$.

There is an unlimited number of Hermitian matrices, since in the "unitarily similar to diagonal of eigenvalues" representation, any real number could serve as a possible eigenvalue. As was just shown, however, H_n matrices have only a finite number of choices available for eigenvalues, namely 0 and the $n + 1$ $(n + 1)^{st}$ roots of 1. Consequently only a finite number of H-matrices are distinct under unitary similarity transformations. This number

can easily be determined. Suppose all r x r H_n matrices are under consideration. A permutation matrix is unitary, so the r positions on the diagonal matrix of eigenvalues are indistinguishable under unitary similarity transformations. The number of different diagonal matrices is then the number of ways r indistinguishable positions can be assigned to n + 2 possible eigenvalues, which from occupancy theory is $\binom{n+r+1}{r}$.

The analysis up to this point has concentrated entirely on the original problem of $A^* = A^n$, where n is an integer > 1. However, in the process several related questions have also been solved. For instance, one might consider defining $H_{1/n}$ matrices, for n an integer > 1, by the relation $(A^*)^n = A$ suggesting $A^* = A^{1/n}$, if the root exists. However, taking conjugate transposes of both sides, this quickly becomes $A^* = (A^*)^{n*} = (A^n)^* = A^n$, which is back to the original problem $A^* = A^n$, so an $H_{1/n}$ matrix is simply H_n. Similarly, one might try to extend the concepts of both H and unitary matrices by considering matrices which satisfy $A^* A^n = I$ for n an integer > 1 (suggesting $A^* = A^{-n}$ if inverses exist), to be termed H_{-n} matrices. But if $A^* A^n = I$, $\det (A^* A^n) = 1$, $\det A \neq 0$, A^{-1} exists, and $A^* = A^{-n}$. An analysis of the eigenvalues exactly analogous to the one done for H_n matrices will show that H_{-n} matrices are unitarily similar to diagonal matrices of $n-1^{st}$ roots of 1 (0 is excluded since the matrices are nonsingular), and so for n > 3, the H_{-n} matrices correspond exactly with the nonsingular H_{n-2} matrices. (The H_{-2} and H_{-3} matrices are trivial cases—H_{-3} matrices are unitarily similar to diagonal matrices of ±1's, and only I is H_{-2}.)

Similar conclusions can also be obtained in the more general context of bounded linear operators in Hilbert space. For if T is such an operator which satisfies the relation $T^* = T^n$ (here, T^* is the adjoint of T, and T^n denotes the mapping T composed with itself n times), for n an integer >1, then again $T^* T = T^{n+1} = TT^*$, so T is normal. Also, if μ is an eigenvalue (element of the point spectrum) of T, then since $T = T^{**} = T^{n^2}$, for some $x \neq 0$, $\mu x = Tx = T^{n^2} x = \mu^{n^2} x$, so $\mu = \mu^{n^2}$, and if $\mu \neq 0$, $|\mu| = 1$. If T has no zero eigenvalue, then a finite-dimensional spectral theorem can be used to show that T is unitary in a finite-dimensional inner product space, and the same result can be obtained in the infinite case under the added assumption that T is completely continuous. Incidentally, since $T = T^{n^2}$, for all x, $||Tx|| = ||T^{n^2} x|| \leq ||T||^{n^2} ||x||$, so $||T|| \leq ||T||^{n^2}$, and if $T \neq 0$, $||T|| \geq 1$.

Returning now once again to the case of the H matrices, the representation in terms of the diagonal matrix of eigenvalues allows powers, roots, and finally, approximation of normal matrices by H matrices to be discussed. If A and B are two H matrices for which AB = BA, the AB is also H_n, since $(AB)^* = B^* A^* = B^n A^n = (AB)^n$. One sufficient condition for the required commutativity of A and B is for $A = U^* DU$ and $B = U^* EU$, where U is the same unitary matrix for both A and B, and D and E are the diagonal matrices of eigenvalues of A and B. A special case of this is of course when A = B, and so if A is H_n, so are all powers of A. The representation as $A = U^* DU$

gives slightly more information than this. If A is H_n, a nonzero element of D will be an $n+1^{st}$ root of 1. If $n+1 = ab$, where a, b are integers > 1, then $A^a = U^*D^aU$, and the elements of D^a will be b^{th} roots of 1 (or 0), so A^a will be H_{b-1} (if b = 2, A^a will be unitarily similar to a diagonal matrix of +1's or 0's, which if nonsingular will be idempotent). In a similar way, all rational roots of A can be defined (though not uniquely) as U^*RU, where R is a diagonal matrix of the corresponding roots of the elements of D. If A is H_n and r is an integer, then it can easily be verified that $A^{1/r}$ is H_{nr+r-1}.

The set of all rational roots of 1, the set of possible eigenvalues for H matrices, forms a dense subset of the unit circle. This would suggest that other matrices might be approximated in terms of H matrices. If N is an arbitrary normal matrix, then it is unitarily similar to the diagonal matrix of its eigenvalues, $N = V^*FV$. All rational roots can be taken (again not uniquely) by taking the corresponding roots of F. It is clear that there is an integer m, depending only on F (hence N), so that if r is an integer $\geq m$, all elements of $F^{1/r}$ will have modulus less than 2. If they are nonzero, they can be resolved uniquely into the sum of two points on the unit circle. (Using the familiar parallelogram law for addition, the points will be the intersection with the unit circle of the perpendicular bisector of the segment between the given point and the origin.) Thus for the 1-1 element of $F^{1/m}$, there is an integer n_1 such that the two points into which the element has been resolved are approximated to the desired closeness by the two n_1^{st} roots of 1, μ_1 and ν_1. (If the element is 0, simply let $\mu_1 = \nu_1 = 0$, and disregard n_1.) There are similar numbers $n_2, \ldots, n_p$ yielding the $n_2, \ldots, n_p^{st}$ roots of 1, $\mu_2, \ldots, \mu_p$ and $\nu_2, \ldots, \nu_p$, where F is a p x p matrix. Let n + 1 be the least common multiple of $n_1, \ldots, n_p$. Then $\mu_1, \ldots, \mu_p$ and $\nu_1, \ldots, \nu_p$ are all $n+1^{st}$ roots of 1, and $F^{1/m} \approx \mathrm{diag}(\mu_1 + \nu_1, \ldots, \mu_p + \nu_p)$. Let $A = \mathrm{diag}(\mu_1, \ldots, \mu_p)$ and $B = \mathrm{diag}(\nu_1, \ldots, \nu_p)$. Then V^*AV and V^*BV are H matrices, and $\left[V^*AV + V^*BV\right]^m = \left[V^*(A+B)V\right]^m = V^*(A+B)^mV \approx V^*FV = N$. On the other hand, $\left[V^*AV + V^*BV\right]^m$ may be expanded immediately to $\sum_{j=0}^{m} \binom{m}{j} V^*A^jB^{m-j}V$ ($A^0 = B^0 = I$), and each term $V^*A^jB^{m-j}V$ is also H_n. Consequently, any normal matrix can be approximated as the m^{th} power of the sum of two H matrices, or as the sum of 2^m H matrices (counting the sum as $\sum_{j=0}^{m} \binom{m}{j}$ terms), where m depends only on the normal matrix itself.

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Lancaster, Peter. Theory of Matrices. New York: Academic Press* Inc., 1969. Ch. 2.

BOOK REVIEWS

Edited by

Roy B. Deal, University of Oklahoma Medical Center

1. Elementary Number Theory By Ethan D. Bolker, W. A. Benjamin, Inc., New York, N. Y., 1970, xi + 180 pp., $8.50.

For the reader with the bare essentials of modem algebra, this introduction to number theory should prove interesting and should further enhance his insight into modern algebra, particularly if he works through a good portion of the exercises.

2. Computational Methods in Partial Differential Equations By A. R. Mitchell, John Wiley and Sons, Inc., New York, N. Y. 10016, 1969, vii + 255 pp., $11.00.

A reader with calculus, some matrix theory, and access to a high speed computer can learn a great deal about numerical analysis, particularly the use of finite difference techniques in solving partial differential equations, and gain a great deal of insight into the basic partial differential equation of mathematical physics from this very practical, but sound* introduction to the subject.

3. A Collection of Matrices for Testing Computational Algorithms By Robert T. Gregory and David L. Karney, John Wiley and Sons, Inc., New York, N. Y. 10016, Oct. 1969, ix + 154 pp., $9.95.

Although designed as a reference to assist in testing algorithms, anyone interested in computational methods involving matrices, and who has not already had a great deal of experience, will find the information on a wide variety of matrices to be of considerable use in itself,

4. Lie Algebras and Quantum Mechanics By Robert Hermann, W. A. Benjamin, Inc., New York, N. Y. 10016, 1970, xvi + 320 pp., $17.50 paperbound $7.95.

5. Vector Bundles in Mathematical Physics, Volume I By Robert Hermann, W. A. Benjamin, Inc., New York, N. Y. 10016, 1970, xiii + 441 pp. $17.50 paperbound $7.95.

6. Vector Bundles in Mathematical Physics, Volume II By Robert Hermann, W. A. Benjamin, Inc., New York, N. Y., 10016, 1970, ix + 400 pp. $17.50 paperbound $7.95.

These three books are probably too esoteric for most Pi Mu Epsilon Journal readers, but they represent an important contribution to the recent trend of bringing some of the modern mathematical concepts in Lie algebras and differential topology back to the modem physics from which it evolved. The reader must have some experience with these concepts and be seriously interested in this development before undertaking these volumes.

7. Beginner's Book of Geometry By Young and Young, Chelsea Publishing Company, Bronx, New York, 1970, xvi + 222 pp., $4.50.

8. Plane Trigonometry By Leonard E. Dickson, Chelsea Publishing Company, Bronx, New York, 1970, x + 176 pp., $3.95.

BOOK REVIEWS--Continued

9. Formulas and Theorems in Mathematics By George S. Carr, Chelsea Publishing Company, New York, N. Y., 1970, xxxvi + 935 pp., $12.50.

These are reprints of old books by the masters. Except for some interesting insights into elementary geometry in the book by the famous Youngs the first two books are basically of historic interest. The book by Carr is, however, unique. This amazing collection of about 6,000 propositions was written as a review book for the Tripos about 1880. The reviewer has found his original editions of two volumes from a second-hand bookstore a useful reference, even today, in such topics as infinite series, theory of equations, determinants, geometry of conics, elementary differential geometry, formulas of calculus, calculus variations, ordinary and partial differential equations, affine geometry, theory of plane curves, and solid analytical geometry. It is perhaps of both pedagogical and historical interest that this work was the basis of the education of the largely self-taught Indian mathematical genius, Srinivasa Ramanujan.

LISTED BOOKS

1. Table of Modified Bessel Functions By Henry E. Fettis and James C. Caslin, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, February 1969, iv + 232 pp.

2. Rational Approximations To A Class of G-Functions By Jerry L. Fields, Midwest Research Institute, Kansas City, Mo., Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio,

3. A Convergence Theorem For Noncommutative Continued Fractions By Wyman Fair, Midwest Research Institute, Kansas City, Missouri Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, March, 1970, iii + 5 pp.

4. Derivative-Free Iteration Processes of Higher Order, Jet Wine, Energetics Research Laboratory, Midwest Research Institute, Kansas City, Mo., Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, iv + 10 pp.

5. Inequalitites For Generalized Hypergeometric Functions By Yudell L. Luke, Midwest Research Institue, Kansas City, Mo. Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, March 1970, iv + 32 pp.

6. More Zeros of Bessel Function Cross Products By Henry E. Fettis and James C. Caslin, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, Dec. 1968, v + 56 pp.

LISTED BOOKS--Continued

7. A Table of the Complete Elliptic Integral of the First Kind For Complex Values of the Modulus, Part I By Henry E. Fettis and James C. Caslin, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force Base, Wright-Patterson Air Force Base, Ohio, Nov. 1969, iv + 298 pp.

8. A Table of the Complete Elliptic Integral of the First Kind For Complex Values of the Modulus, Part II By Henry E. Fettis and James C. Caslin, Applied Mathematics Research Laboratories, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, November 1969, iv + 250 pp.

9. Tables of Toroidal Harmonics, II: Orders 5-10, All Significant Degrees By Henry E. Fettis and James C. Caslin, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, December 1969, iv + 179 pp.

10. Tables of Toroidal Harmonics, I: Orders 0-5, All Significant Degrees By Henry E. Fettis and James C. Caslin, Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, February 1969, iv + 209 pp.

11. Calculus Supplement By Robert A. Kurtz, W. A. Benjamin, Inc., New York, N. Y., 1970, ix + 274 pp.

NEED MONEY?

The Governing Council of Pi Mu Epsilon announces a contest for the best expository paper by a student (who has not yet received a masters degree) suitable for publication in the Pi Mu Epsilon Journal.

The following prizes will be given

$200.	first prize
$100.	second prize
$50.	third prize

providing at least ten papers are received for the contest.

In addition there will be a $20.00 prize for the best paper from any one chapter, providing that chapter submits at least five papers.

PROBLEM DEPARTMENT

Edited by

Leon Bankoff, Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems characterized by novel and elegant methods of solution are also acceptable. Solutions should be submitted on separate, signed sheets and mailed before November 15, 1971.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

248. Proposed by R. S. Luthar, University of Wisconsin, Waukesha.

For any positive integer n, prove that the following inequality holds:

$$\{n(n+1)\}^{n(n+1)} \geq \frac{2^{n(n+1)}}{\prod_{k=2}^{n-1} k!} \cdot (n!)^n$$

249. Proposed hy R. S. Luthar, University of Wisconsin, Waukesha.

Prove that

$$p \mid (a+b) \Longleftrightarrow p^{m+1} \mid (a^{p^m} + b^{p^m}),$$

where p is an odd prime and m is any non-negative integer.

250. Proposed by Charles W. Trigg, San Diego, California.

Identify the three mathematical terms represented by the following items:

(a) Bass made five yards over his own right tackle. Just as he was being tackled he tossed the ball back to Gabriel, who immediately flipped it back to Casey. After advancing ten yards, Casey threw the pigskin back to Mason, who lobbed it back to Bass, who continued on to a touchdown.

(b) As I was going up the stair

I met a man who wasn't there.
He wasn't there again today
I wish, I wish he'd go away.

(c) Yukon Jake's tale was characteristically long, detailed, and profane: "At noon I found that a *** bear had discovered my cache and destroyed all the *** supplies. I was *** hungry and the nearest food was ten *** miles away, so I got the *** out of there fast. When I got to the *** cabin it was almost dark and I was *** tired. Them *** beans tasted good."

251. Proposed by Charles W. Trigg, San Diego, California.

If r_1, r_2, r_3 are roots of $x^3 + px + q = 0$, show that

$$3 \Sigma r_i^2 \Sigma r_i^5 = 5 \Sigma r_i^3 \Sigma r_i^4$$

252. Proposed by Solomon W. Golomb, University of Southern California.

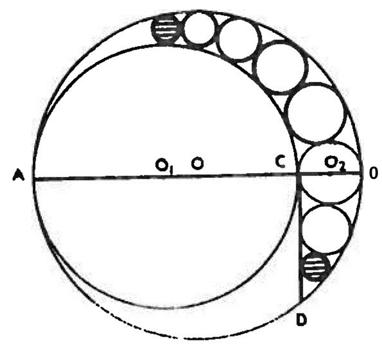
There are 97 places where a 2 X 3 rectangle can be put on an 8 X 9 board. In how many of these cases can the rest of the board be covered with eleven 1 X 6 rectangles (straight hexominoes) and where are these locations?

253. Proposed by Erwin Just, Bronx Community College of the City University of New York.

If P(x) is an irreducible polynomial over the rationals and there exists a positive integer $k \neq 1$, such that r and r^k are both zeros of P(x), prove that P(x) is cyclotomic.

254. Proposed by Alfred E. Neuman, Mu Alpha Delta Fraternity, New York.

In the adjoining diagram, CD is a half-chord perpendicular to the diameter AB of a circle (O). The circles on diameters AC and CB are centered on O_1 and O_2 respectively. The rest of the figure consists of consecutively tangent circles inscribed in the horn-angle and in the segment as shown. If the two shaded circles are equal, what is the ratio of AC to AB?

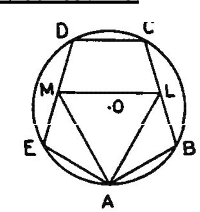


255. Proposed by C. Stanley Ogilvy, Hamilton College, Clinton, N. Y.

Find a 3-digit number in base 9 which when its digits are written in reverse order, yields the same number in base 7. Prove that the solution is unique.

256. Proposed by R. S. Luthar, University of Wisconsin, Janesville.

ABCDE is a pentagon inscribed in a circle (O) with sides AB, CD and EA equal to the radius of (O). The midpoints of BC and DE are denoted by L and M respectively. Prove that ALM is an equilateral triangle.



257. Proposed by Mike Louder and Richard Field, Los Angeles, California.

If x, y, z are the sides of a primitive Pythagorean triangle with $z > x > y$, can x and $(x - y)$ be the legs of another Pythagorean triangle?

SOLUTIONS

220. (Spring 1969 and Fall 1970) Proposed by Daniel Pedoe, University of Minnesota.

a) Show that there is no solution of the Apollonius problem of drawing circles to touch three given circles which has only seven solutions.

b) What specializations of the three circles will produce 0, 1, 2, 3, 4, 5 and 6 distinct solutions?

Solution I by the Proposer was published in the Fall 1970 issue,

II. Solution by Charles W. Trigg, San Diego, California.

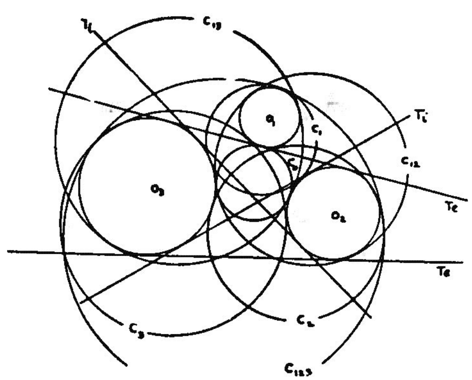
Since the proposer did not specifically state that straight lines would be considered circles of infinite radius, the following solution deals with circles of finite radius only.

Given three non-tangent circles, O_1, O_2, O_3, in the plane with $r_1 < r_2 < r_3$. Let T represent the common external tangents of O_2 and O_3, and T_i their internal tangents. This discussion will be based generally upon fixed O_2 and O_3 with a moving O_1.

~~Case I.~~ No circle lies between the common external tangents of the other two.

In general* there are eight circles tangent to the three O_i: C_0, including none of the three; C_1, C_2, C_3, covering one O_i only; C_{12}, C_{13}, C_{23}, surrounding two O_i only; and C_{123} encompassing all three O_i. The subscripts indicate the O_i's encompassed by the particular C_i. Neither a straight line (circle with infinite radius) nor one of the O_i can qualify as a C_i.

As O_1 approaches a T_e between the T_i, r_{23} increases. At tangency C_{23} merges with the T , generally leaving 7 solutions.



Case I

When O_1 rolls along the T toward O_2 until it becomes tangent to a T_i, C_{13} merges with the T_i, leaving 6 solutions.

When O_1 continues to tangency with O_2, C_1 merges with the T_e, leaving 5 solutions.

When O_1 continues to tangency again with the T_i, C_3 vanishes, leaving 4 solutions.

In the last situation, if O_2 and O_3 are tangent, the only 3 solutions are C_0, C_2 and C_{123}; and if O_1 is tangent simultaneously to the T_i and O_2, the only 2 solutions are C_0 and C_{123}. Indeed, this is also true when the three circles are tangent by two's.

Case II. O_1 is between T_e with O_2 between O_1 and O_3.

Here, situations may exist where two circles, C and C', include a particular circle and are tangent to the other two, and so on.

O_2 and O_3 not tangent, O_1 and O_2 not tangent

O_1 not tangent to either T_e, then we may have C_{123}, C'_{123}, C_{12}, C'_{12}, C_{23}, C'_{23}, C_2, and C'_2 - 8 solutions

O_1 tangent to one T_e, then only one C_{123} - 7 solutions

O_1 tangent to both T_e, hence no C_{123} - 6 solutions

O_2 and O_3 externally tangent, O_1 and O_2 not tangent

O_1 not tangent to either T_e, then we may have C_{123}, C'_{123}, C_{12}, C_{23}, C'_{23}, and C_2 - 6 solutions

O_1 tangent to one T_e - 5 solutions

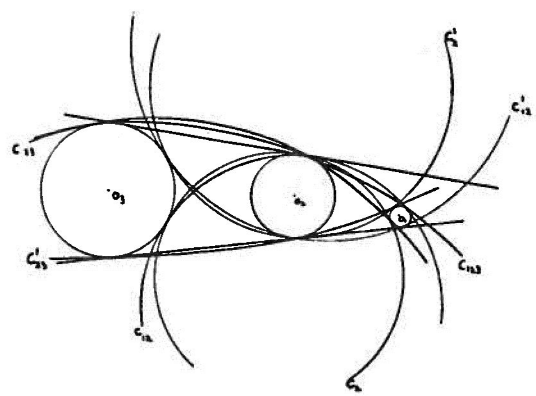
O_1 tangent to both T_e - 4 solutions

O_2 tangent to both O_3 and O_1

O_1 not tangent to either T_e - 4 solutions

O_1 tangent to one T_e - 3 solutions

O_1 tangent to both T_e - 2 solutions



Case II

Case III. O_1 is between the T and between O_2 and O_3.

In some of the situations in this category, the relative values of r_1, r_2, and r_3 are critical.

O_2 and O_3 not tangent

O_1 not tangent to the T nor to O_2 or O_3, then we may have C_0, C'_0, C_1, C'_1, two of C_2, C'_2, C_{13}, C'_{13}, and two of C_3, C'_3, C_{12}, C'_{12} - 8 solutions

O_1 tangent to one T but not the circles, there can be only one C_0 - 7 solutions

O_1 tangent to one circle but no T - 6 solutions

O_1 tangent to a circle and one T_e - 5 solutions

O_1 tangent to both circles but no T - 4 solutions

O_1 tangent to both circles and a T - 3 solutions

O_2 and O_3 externally tangent

O_1 not tangent to a T or to O_2 or O_3, we may have C_0, C'_0, C_1, C'_1, one of C_2, C_{13}, and one of C_3, C_{12} - 6 solutions

O_1 tangent to T but no circle - 5 solutions

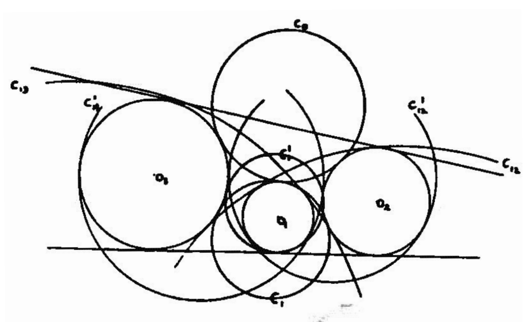
O_1 tangent to one circle but no T - 4 solutions

O_1 tangent to one circle and T - 3 solutions

O_1 tangent to both circles but no T - 2 solutions

O_1 tangent to one circle, to T_e, and to the internal tangent of O_2 and O_3 - 2 solutions

O_1 tangent to T_e, O_2 and O_3 - 1 solution



Case III

~~Case IV.~~ One or two circles inside the third

O_1 outside O_2, both inside O_3

No tangencies -	8 solutions
One tangency -	6 solutions
Two tangencies	4 solutions

Three Tangencies	- 2 solutions
In general	- (8 - 2t) solutions

O_1 inside O_2, both inside O_3

No tangencies	- 0 solutions
One or two distinct tangencies	- 2 solutions
All tangent at a point	- an infinity of solutions

O_1 inside O_2, both outside O_3

No tangencies	- 0 solutions
One tangency	- 2 solutions
Two distinct tangencies	- 1 solution
All tangent at a point	- an infinity of solutions

~~Case V.~~ Intersecting circles, none completely including another.

Each circle intersects every other with no triple point

No common tangent	- 8 solutions
One common tangent	- 7 solutions
Two common tangents	- 6 solutions

Two tangent circles, each intersected by the third, no triple point

No common tangent	- 6 solutions
One common tangent	- 5 solutions
Two common tangents	- 4 solutions

One circle intersected by two others, no tangencies or triple

No common tangent	- 4 solutions
One common tangent	- 3 solutions
Two common tangents	- 2 solutions

Three circles having one common point

No tangencies	- 4 solutions
Two circles tangent at the point	- 2 solutions

Three circles having two common points - 0 solutions

Only two circles intersecting

No common tangent	- 2 solutions
One common tangent	- 1 solutions
Two common tangents	- 0 solutions

Clearly, this is not a complete census, either of configurations or of special cases, such as those where O_1 intersects a T, the three radii are equal, when a particular placement of the circles modifies the announced number of solutions, etc.

EDITOR'S NOTE: Diagrams for Case IV and Case V are left as an exercise for the reader.

232. (Spring 1970) Proposed by Solomon W. Golomb, University of Southern California, Los Angeles.

Find a direct combinatorial interpretation of this identity:

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$

Solution by Murray S. Klamkin, Ford Motor Company

If we have n points A, B, C,... then the left hand side can be interpreted as the number of pairs of segments formed by the n points. Now add an extra point O and consider the number of combinations four at a time. The combination O, A, B, C gives rise to three pairs of segments, i. e., AH. AC; AB, BC; BC, AC. The combination A, B, C, D also gives rise to three pairs of segments, i. c., AB, CD; AC, BD; AD, BC. And the number of these is then the right hand side of the identity.

Also solved by Kenneth Rosen, University of Michigan and by the proposer.

233. (Spring 1970) Proposed by Charles W. Trigg, San Diego, California.

The director of a variety show wanted to give the female impersonator a job, but questioned his ability to dance with the high-kicking Folies Bergere chorus. In reply to the director's query, the impersonator's Spanish agent said:

"SI/HE = . CAN CANCAN...,

but CAN be less than one-fourth effective in his demonstration today."

If each letter of the cryptarithm uniquely represents a digit in the scale of eleven, what is the sole solution?

Solution by the Proposer

Let F = .CANCANCAN ...
Then 1000 F = CAN .CANCANCAN ...
whereupon (1000 − 1)F = CAN, in the scale of eleven.

Hence SI/HE = CAN/XXX = CAN/(13) (37)
= CAN/ (32) (35) = CAN/ (18) (64).

The denominators contain the only two-digit factors of XXX. Consequently HE equals one of them, and its associate times SI equals CAN. But .CAN < 1/4 = .2828 ···. Now 282/12 ≐ 24, so the associate of HE is 13 or IS. The three-digit multiples of these two numbers < 282 are listed and those with duplicate digits or digits in common with SI or HE are discarded, leaving the unique solution: 19/87 = .235235235 ···.

The ratio of the first two digits of the repetend happens to be the ratio of the sums of the digits of the numerator and the denominator.

Also solved by Wesley Johnston, Springfield, Illinois; Donald E. Marshall, U. C. L. A.; and Kenneth Rosen, University of Michigan in Ann Arbor.

234. (Spring 1970) Proposed by Charles W. Trigg, San Diego, California.

Show that when the nine positive digits are distributed in a square array so that no column, row, or unbroken diagonal has its digits in order of magnitude, the central digit must always be odd.

2	8	7
6	1	4
5	3	9

Solution by D. J. Deignan, Indiana University.

Consider the eight outside digits as four diametrically opposite pairs. If an even digit occupies the central position, there remain an odd number of lower digits and an odd number of higher digits. Thus at least one of the four pairs must consist of one lower and one higher digit, thus contradicting the hypothesis.

Also solved by Al Davis, Albany, N. Y.; Joel Feingold, Sheepshead Bay High School, Brooklyn, N. Y.; Wesley Johnston, Springfield, Illinois; Donald E. Marshall, Pasadena, California; Kenneth Rosen, Madison, Wisconsin; Donald R. Steele, Elizaville, N. Y.; and the proposer.

235. (Spring 1970) Proposed by James E. Desmond, Florida State University.

Prove that a^{n+1} divides $(ab + c)^{(ad)^n} - c^{(ad)^n}$ for integers a > 0, b, c, d > 0 and $n \geq 0$.

I. Solution by the Proposer.

Using induction on n, the case n = 0 is clear. Suppose a^{k+1} divides $(ab + c)^{(ad)^k} - c^{(ad)^k}$ for $k \geq 0$. Then for some integer N we have

$$(ab + c)^{(ad)^{k+1}} - c^{(ad)^{k+1}} = (ab + c)^{(ad)^k ad} - c^{(ad)^{k+1}}$$
$$= \left[(ab + c)^{(ad)^k} - c^{(ad)^k} + c^{(ad)^k}\right]^{ad} - c^{(ad)^{k+1}}$$
$$= (a^{k+1}N + c^{(ad)^k})^{ad} - c^{(ad)^{k+1}}$$
$$= \sum_{i=1}^{ad} \binom{ad}{i} (a^{k+1}N)^i (c^{(ad)^k})^{ad-i}$$

which is divisible by a^{k+2}.

We note that the result also follows from E 2058, Amer. Math. Monthly, 75(1968)189; 76(1969)196.

II. Solution by Murray S. Klamkin, Ford Motor Company

Expanding by the binomial theorem, the term of lowest degree in a is

$$ab(ad)^n c^{(ad)^n - 1},$$ which is divisible by a^{n+1}.

Also solved by Genevieve Lento, Philadelphia; D. E. Marshall, Pasadena, California; and Larry E. Miller, University of California, Riverside.

236. (Spring 1970) Proposed by Erwin Just, Bronx Community College.

If k is a positive integer, prove that $(6^{16k+2}/2) - 1$ is not a prime.

I. Solution by Bob Prielipp, Wisconsin State University. Oshkosh.

We shall show that 17 divides $(6^{16k+2}/2) - 1$ for each positive integer k. Since $6 \equiv 2 \pmod{17}$. $6^{16k+2} \equiv 2^{8k+1} \pmod{17}$ where k is an arbitrary positive integer. Thus $6^{16k+2}/2 \equiv 2^{8k} \pmod{17}$. But $2^8 \equiv 1 \pmod{17}$. Hence $2^{8k} \equiv 1 \pmod{17}$. Therefore $6^{16k+2}/2 \equiv 1 \pmod{17}$, or 17 divides $(6^{16k+2}/2) - 1$.

II. Solution by the Proposer

Since $(6^{16k+2}/2) - 1 = (6^2/2)(6^{16k} - 1) + (6^2/2) - 1$

$$= 18\,[(6^k)^{16} - 1] + 17,$$

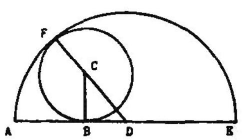
and since Fermat's theorem guarantees that $(6^k)^{16} - 1$ is divisible by 17, it follows that $(6^{16k+2}/2) - 1$ is divisible by 17.

Also solved by Walter Wesley Johnston, Springfield, Illinois; Murray S. Klamkin, Ford Motor Company; Donald E. Marshall, U. C. L. A.; Kenneth Rosen, University of Michigan, Ann Arbor; Donald R. Steele. Elizaville, N. Y.; C. L. Sabharwal, Saint Louis University; and Richard Zanghi, Deer Park, N. Y.

237. (Spring 1970) Proposed by Leonard Barr, Beverly Hills, California.

The diameter of a semi-circle is divided into two segments, a and b, by its point of contact with an inscribed circle. Show that the diameter of the inscribed circle is equal to the harmonic mean of a and b.

Solution by Robert J. Herbold, Cincinnati. Ohio.



Let C be the center of the inscribed circle and D be the center of the larger circle. If B is the point where the inscribed circle is tangent to the diameter of the semi-circle, then angle CBD = 90°. If we let r be the radius of the inscribed circle then FC = CB = r. Also if AB = a and BE = b then FD = (a +b)/2, BD = (a +b)/2 − a, and CD = (a + b)/2 − r. Since angle CBD is a right angle,

$$(CB)^2 + (BD)^2 = (CD)^2$$

and $\quad r^2 + [(a+b)/2 - a]^2 = [(a+b)/2 - r]^2$

Solving for r, we obtain $r = ab/(a + b)$. So the diameter of the inscribed circle is equal to the harmonic mean of a and b.

Also solved by Robert C. Gebhardt. Parsippany, N. J.; Theodore Jungreis, Brooklyn, N. Y.; Bruce King, Adirondack Community College, Glen Falls, N. Y.; Donald E. Marshall, U. C. L. A.; Kenneth Rosen, University Michigan, Ann Arbor; and Donald R. Steele, Elizaville, N. Y.

Rosen located the problem in Eve's A Survey of Geometry, Volume One, page 103, problem 2.6–14.

238. (Spring 1970) Proposed by David L. Silverman, Beverly Hills, California.

A necessary and sufficient condition that a triangle exist is that its sides, a, b, and c satisfy the inequalities (1) a < b + c, (2) b < a + c, (3) c < a + b. Express (1), (2) and (3) in a single inequality.

I. Solution by Joseph D. E. Konhauser, Macalester College.

Applying Hero's formula, a necessary and sufficient condition that a, b, c be the sides of a non-degenerate triangle is that $(a + b + c)(a + b - c)(b + c - a)(c + a - b) > 0$ or

$$2(a^2b^2 + b^2c^2 + c^2a^2) > a^4 + b^4 + c^4$$

II. Solution by Charles W. Trigg, San Diego, California.

This is problem 3188, School Science and Mathematics, 69 (April, 1969), p. 350.

Method I. The given inequalities may be written in the forms $a - c < b$, $b - c < a$, $c - b < a$, so $|b - c| < a < b + c$.

Method II. Represent the three quantities as a_i, a_j, a_k with i, j, k = 1, 2, 3; $i \neq j \neq k$; then $a_i < a_j + a_k$.

III. Solution by Sid Spital, Hayward, California.

Clearly the three inequalities are equivalent to

$$\max(a, b, c) < a + b + c - \max(a, b, c).$$

with $\max(a, b, c) = \max(\max(a,b), c)$ and $\max(x,y) = (x + y + |x - y|)/2$, this becomes

$$|a - b| + |a + b - 2c + |a - b|| < a + b.$$

Also solved by Steven Blumenthal, Bayside, N. Y.; Steven R. Conrad, Bayside, N. Y.; Robert J. Herbold, Cincinnati, Ohio; Wesley Johnston, Springfield, Illinois; Murray S. Klamkin, Ford Motor Company; C. B. A. Peck, State College, Pennsylvania; Kenneth Rosen, University of Michigan, Ann Arbor; and the proposer.

EDITOR'S NOTE: Herbold, Peck and Rosen offered the solution $\max\{a, b, c\} < \min\{b+c, a+c, a+b\}$, following the assumption that $a \leq b \leq c$. Johnston, using the methods of Solution II, remarked that Method I is a single sentence but is actually a double inequality while method II represents the conditions in a single inequality form but not as an actual inequality in a single sentence. Klamkin gave the references Q 269, "Math. Magazine, Sept. − Oct., 1960, p. 58; D. S. Mitrinovic, Elementary Inequalities, Noordhoff, Netherlands, 1964, P. 113, (6.7). Trigg called attention to Problem 423. Mathematics Magazine, Sept.-Oct. 1960, p. 50 and Sept. − Oct., 1961, pp. 364–365, as well as Silverman's comments on problem 423 on page 62 of the Jan.–Feb. 1962 issue of the Mathematics Magazine.

ERRATA: The following errors and misprints were called to the attention of the Editor by Alfred E. Neuman of the Mu Alpha Delta Fraternity:

Page 132, line 3: "Angles" should read "Angeles".
Page 132: The line connecting C and Q should not have terminated on the circumference of the circle.
Page 132, bottom line: The minus sign on the right side should be a plus sign.
Page 132: the word "number" in problem 240 should read "numbers".
Page 131, line 19: "propser" should read "proposer".
Page 142, 6th line from bottom: ${}_iC_i$ should read $\bigcup_i C_i$.
Page 144, line 2: "a " should read "A_n".

The last proposal in the Spring 1170 issue and the first proposal in the Fall 1970 issue were both Inadvertently numbered "230". Henceforth the former problem will be referred to as 239-a.

Mr. Neuman was also disturbed about the use of the neologism "Planidrome" for the more conventional, tried and true word "Palindrome" (page 148).

There was a misprint on the cover of the Fall 1970 Journal. Volume 6 was printed on it, but it should have read Volume 5.

MEETING ANNOUNCEMENT

Pi Mu Epsilon will meet on August 31 and September 1, 1971, at Pennsylvania State University, University Park, Pennsylvania, in conjuction with the Mathematical Association of America. Chapters should start planning MOW to send delegates or speakers to this meeting, and to attend as many of the lectures by other mathematical groups as possible.

The National Office of Pi Mu Epsilon will help with expenses of a speaker OR delegate (one per chapter) who is a member of Pi Mu Epsilon and who has not received a Master's Degree by April 15, 1971, as follows: SPEAKERS will receive 5¢ per mile or lowest cost, confirmed air travel fare; DELEGATES will receive 2-1/2¢ per mile or lowest cost, confirmed air travel fare.

Select the best talk of the year given at one of your meetings by a member of Pi Mu Epsilon who meets the above requirements and have him or her apply to the National Office. Nominations should be in our office by May 15, 1971. The following information should be included: your name; Chapter of Pi Mu Epsilon; school; topic of talk; what degree you are working on; if you are a delegate or a speaker; when you expect to receive your degree; current mailing address; summer mailing address; who you were recommended by; and a 50-75 word summary of talk, if you are a speaker. MAIL TO: Pi Mu Epsilon, 1000 Asp Ave., Room 215, Norman, Oklahoma 73069.

NEW INITIATES

ARIZONA ALPHA, University of Arizona

Stanley D. Carlson
Richard V. Heinisch
Bobbie Jo Rench
Robert J. Rose
Alan B. Sternstein
Thomas M. Stoker

ARIZONA BETA, Arizona State University

Gordon Anderson
Randy Anderson
Lucas H. B. Bunt
Merrill Fastcott, Jr.
Edward Hargens
Albert Heitzmann
Lewis Kopera
Janet McIver
Robert Hecht-Neilsen
Daniel Phillips
Robert Rowley, II
Teddy N Wang

COLORADO ALPHA, University of Colorado

William R. Arney
Suzanne M. Barker
James D. Clark
Virgil B. Fenn
Steven Jacobs
Joyce Lewis
Robert P. Pellett
David W. Smith
Daniel M. Swirsky
Ray E. Watkins

FLORIDA GAMMA. Florida Presbyterian College

Margie A. Hale
Evalyn E. Hall
Kirk A. Jackson
Carolyn J. Thomas

FLORIDA EPSILON, University of South Florida

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