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Comments on the Properties of Odd Perfect Numbers

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I. Regarding Background

Perfect numbers are positive integers with the unusual property that they are equal to twice the sum of their divisors, that is,

if $J(X) = \sum_{d/x} d$, then $J(N) = 2N$ for any perfect number N.

No odd perfect number has thus far been discovered, (and for goad reason as will soon become apparent), but restrictions on their existence have been demonstrated in that if in fact they do exist, they must have certain definite properties with regard to size, number of prime factors, general form and miscellaneous unusual bounds on the sums and products of the reciprocals of the primes which divide them.

Euclid proved all numbers of the form $N = (2^k-1)(2^{k-1})$ are perfect if (2^k-1) is prime. Euler demonstrated that in fact all even perfect numbers are of the same form, the first three perfect numbers being 6, 28, and 496.[1] It is curious to note that even perfect numbers all end in 6 or 28 (Novarese, 1887), but this is not an alternating sequence for the sixth perfect number (8,589,869,056) ends in '6' and not '8' as anticipated (Reid, p. 87). But so much for even perfect numbers.

The bulk of research regarding odd perfect numbers stretches backward several hundred years (including such names as Alcuin of York, Descartes, Fermat, Leibnitz, and Euler) while present day research primarily concerns itself with determination of bounds on the number of prime factors and size of the number through the use of computing machinery. Example: While Euler determined the necessary structural form of all odd perfect numbers to be p^aN^2 where $p \equiv a \equiv 1(4)$ and $(p,N) = 1$, Norton (1961) determined that if 17 is the smallest prime factor of an odd perfect number, then the number has at least 509 prime factors.

Let us examine only some of the necessary properties of the odd numbers which we deem perfect. The following will prove useful:

Lemma: (Bourlet, 1896)

If P^* is any perfect number, then $\sum_{d/P^*} (1/d) = 2$

Proof: Consider P^*/d_i where the d_i are the divisors of P^*; note that $P^*/d_1, P^*/d_2, \ldots P^*/d_{p^*}$ range over all the divisors of P^*

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[1]there are 23 known even perfect numbers and the largest of these is $2^{11,212}(2^{11,213}-1)$ which has 6751 digits. Editor.

and if $1 = d_1 d_2 d_3 \dots d_n = P^*$, then $P^*/d_1 = P^*$ and $P^*/d_{P^*} = 1$.

thus 1. $\sum_{d/P^*} \frac{P^*}{di} = d_1 + d_2 + \dots d_{P^*} = 2_{P^*}$

2. $P^* \sum_{d/P^*} \frac{1}{di} = 2P^*$

3. $\sum_{d/P^*} \frac{1}{d} = 2$

Lemma: If $n = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_k^{\alpha_3}$ where the Pi are distinct primes and the α_i are positive integers

then $J(N) = \prod_{i=1}^{K} \frac{P_i^{\alpha_i + 1} - 1}{P_i - 1}$

Proof of sketch: Note $J(N) = \prod_{i=1}^{K} J(P_i^{\alpha_i})$ since $J(X)$ is multiplicative and the Pi are relatively prime to each other. Consider the terms of $J(P_i^{\alpha_i})$ which are merely $1+P_i^1 + P_i^2 + \dots P_i^i$ whose sum is $(P_i^{\alpha_i + 1})/(P_i - 1)$. Continue for all k terms and the result follows.

II. Regarding the Form of an Odd Perfect Number

Euler was the first to prove that if N is any odd perfect number then N is of the form $P^a Q^2$ where P is a prime and a is a positive integer note: henceforth N shall be used to designate all odd perfects.

Proof 1. Let N be an odd perfect number with k prime factors such that $N = P_1^{\alpha_1} P_2^2 \dots P_k^{\alpha_k}$ where the P_i are distinct odd primes and the α_i are positive integers.

2. $J(N) = 2N$ Using the fact that $J(X)$ is multiplicative one obtains

3. $J(N) - J(P_1^{\alpha_1} \dots P_k^{\alpha_k}) = J(P_1^{\alpha_1}) \dots J(P_k^{\alpha_k})$

But 4. $J(P_1^{1}) \dots J(P_k^{\alpha_k}) = 2P_1^{\alpha_1} \dots P_k^{\alpha_k}$

5. This one of the $J(P_i^{\alpha_i})$ is the double of an odd number, let it be $J(P_1^{\alpha_1})$ and all the other $J(P_j^{\alpha_j})$ $i \neq j$ are odd.

6. $J(P_j^{\alpha_j})$ being odd implies α_j are even,

7. Thus $N = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_k^{\alpha_k} = P_1^{\alpha_1} P_2^{2b_2} \dots P_k^{2b_k}$

and thus $P^a Q^2$ where $Q = (P_2 b_2 P_3 b_3 \dots P_k b^k)$,

Q.E.D

Euler also proved that if m fact $N = P^a Q^2$ then $P \equiv a \equiv 1\ (4)$ and in particluar no N was of the form $4t + 3$.

Pepin (1897) proved no N was of the form $6t + 5$ and Touchard (1953) proved N is either of the form $12t + 1$ or $36t + 9$.

III. Regarding the Prime Factors of N

The question no doubt is raised that assuming the existence of N, might N be a prime or even an even power of a single prime. The answer to both questions is no as follows:

Case 1: N is a prime, If N is an odd prime and is perfect, then $J(N)=J(P)=2P$ by definition. But $J(P)=P+1$ since P has no divisors apart from itself and 1. Thus $P + 1 = J(P) = 2P$ implies $2P = P + 1$ implies $P = 1$ which is an absurd condition.

Case 2: N is an even power of a single prime, If N is an even power of an odd prime, then $J(P^k) = 2P^k$ if P^k is perfect. P^k is odd for $P > 2$ and $J(P^k)$ is even, Since all of the divisors of P^k are odd, P^k must have an even number of divisors; but P^k has exactly $(k+1)$ divisors $(1, P, P^2 \dots P^k)$ and thus k must be odd which is a contradiction.

How many prime factors may an odd perfect number contain? Or rather what is the minimum number of prime factors? The necessity for at least three distinct prime factors is attributed to Nocco (1863) by the following argument:

Let $a^m b^n$ be an odd perfect number where a,b are distinct odd primes and m,n are positive integers not necessarily distinct.

1) $2a^m = \frac{b^{n+1} - 1}{b - 1}$ $\quad b^n = \frac{a^{m+1} - 1}{a - 1}$ (proceeding lemma)

2) $a/2(b-1) = \frac{a^{m+1}}{2a^m(b-1)} = \frac{a^{m+1}}{b^{n+1}-1} = \frac{(a-1)b^n + 1}{b^{n+1} - 1}$

$= a+b(ab^n+2b^{n-1}+2) = 2+b(2b^n+2ab^{n-1})$

$a+b^{n+1}(a-2) = a+2b^m(a-1)-2b$

$b(a-2)b^n + a = 2(a-1)b^n + 2 - 2b$

$b(a-2)b^n > 2(a-1)b^n$

3) $ab-2b > 2a-2$ and $ab^n-2b^n = b^{n-1}$ $\quad b(a-2) > b^{n-1}(2a-2)$

$ab^n-2b^n > 2ab^{n-1} - 2b^{n-1}$

4) $ab^n + 2b^{n-1} > 2ab^{n-1} + 2b^n$

Reconsider now: $a+b(ab^n+2b^{n-1}+2) = 2+b(2b^n + 2ab^{n-1})$

Thus: $b(2b^n + 2ab^{n-1}) > b(ab^n + 2b^{n-1})$

5) $2b^n + 2ab^{n-1} > ab^n + 2b^{n-1}$ Contradiction to (4)

In 1908 Turcaninov proved that N has at least 4 distinct prime factors. In 1949 Kühnel increaded the minimum number of distinct prime factors to he 6 (implying that $N > 2\times10^6$).

In 1888 Servais proved a theorem to the effect that if N has k distinct prime factors then the smallestprime factor is less than or equal to k. The proof rests on the fact that if N = abc...z where a,b,z are distinct odd primes

then: $\frac{b}{b-1} < \frac{a+1}{}$ $\quad \frac{c}{c-1} < \frac{a+2}{a+1}$ etc.

and $2 < \frac{a}{a-1} \cdot \frac{b}{b-1} \ldots \frac{3}{} < \frac{a}{\cancel{a-1}} \cdot \frac{a+2}{\cancel{a+1}} \ldots \frac{a+n-1}{\cancel{a+n-2}}$

whence $2(a-1) < a+n-1$ where $a < n=1$

if L is the (m-1) st prime factor and a is the mth prime factor and if

$$\frac{a}{a-1} \cdot \frac{b}{b-1} \cdots \frac{\ell}{\ell-1} \quad L \quad 2$$

then L $\frac{s+1}{s} \ldots \frac{s+n-m}{s+n-m+1}$ by cancelling out adjacent numerators and denominators except for the first and last terms one obtains:

$$\frac{L(s+n-m)}{s-1} > 2$$

$$Ls - 2s \quad Lm - Ln - 2$$

$$s > \frac{Lm - Ln - 2}{L-2}$$

$$s \leq \frac{L(n-m) + 2}{2-L} \leq L(n-m) + 2 \leq 2(n-m) + 2 \quad n$$

When n is sufficiently large, the n "swamps" the values of m (and 2) such that $2(n-m)+2 < n$ and thus $s \leq n$.

Q.E.D.

This is a statement from the result of Cesano (1887) to the effect that $s < k\sqrt{2}$. Grun (1952) proved that the smallest prime factor of N was strictly less than (2/3) (k+3) where N has k distinct factors.

IV. Regarding: the Size of N

Muskat (1965) in his undergraduate thesis proved that any odd perfect number must be divisible by a prime power greater than 10^8, but later increased his lower bound to 10^{12} thru the use of the University of Pittsburgh's computing facilities in the following theorem:

Theorem (Muskat 1965)

Any odd perfect number must be divisible by a prime power greater than 10^{12}.

Proof Sketch: Assume each P^k that divides N is less than 10^{12} where P is an odd prime and k is a positive integer, Steurwald proved that if

$N = P^a Q_1^{2b_1} \ldots Q_i^4 \ldots Q_k^{2b_k}$ (Euler) then at least one of the b_i is greater than 1. Let it correspond to Q_i. Then,

$$10^{12} \geq N = P^a Q_1^{2b} Q_2^{2b_2} \ldots \geq P^a Q_1^{2b_1} \ldots Q_i^4 \ldots Q_k^{2b_k} \geq Q_i^4 .$$

Thus Q_i^4 10^{12} implies Q_i 1000

Now consider all odd primes Q_i such that Q_i 1000 and Q_i^4 10^{12}.

Computations courtesy of the University of Pittsburgh's IBM 7070/7090 reveals that of the 168 possible primes, each is succesively eliminated and thus an odd perfect number must have a prime power greater than 10^{12}.

Norton (1961) using unpublished results of Rosser and Schoenfeld as well as the computing facilities of the University of Illinois produced bounds on the number of distinct prime factors of N as well as the size of the least prime factor.

Theorem (Norton–1961)

Let N be an odd perfect number with smallest prime factor P and let b be any number less than 4/7.

Then N has at least a(n) distinct prime factors where $a(n) = \int_0^{P_n^2} \frac{dt}{\ln t} + O(n^2 e^{-\ln_b^b n})$. Also N has a prime factor at least as large as $P_n^2 + O(n^2 e^{-\ln n})$ ... and $\log N$ $2P_n^2 + O(n^2 e^{-\ln^b n})$.

Norton's theorem offers a relation between the least prime factor and the number of prime factors and is useful for generating estimates on the size of N. For example, if $3 \nmid N$, then N has at least 7 distinct prime factors while 541 is the least prime factor of N then N has at least 26,308 distinct prime factors and log N > 600,000!!!!! A sample of the Norton table is enclosed to demonstrate the rapidity at which the minimum number of prime factors of N increases.

Smallest Prime Factor P_n	Number of Prime Factors a(n)
3	3
5	7
7	15
11	27
13	41
17	62
19	85
23	115

(From Karl Norton, "Remarks on the Number of Factors of Odd Perfect Numbers", Acta Arith, 6(1960-61) pp. 365-74).

Norton's estimates on the size of N rest upon successive knowledge of the least prime factor of N. Kanold (1957) place a lower bound on all odd perfect numbers by proving that for all N, $N > 10^{20}$.

The evidence appears that odd perfect numbers are few and far between if in fact they are at all. It is not a surprise in the light of these theorems that none of the past mathematicians ever discovered any such beasts. Euler himself who elucidated the properties of all even perfect numbers could do no better than hypothesize.

V. Regarding $\sum_{P/N} 1/P$ and $\prod_{P/N} \frac{P}{P-1}$

Curiosity on the phenomena of odd perfect numbers has stimulated investigators into peculiar and rather unusual relationships between the sums and products of the primes which divide N. The most prolific of these investigators is Perisastri (1958) and Suryanarayana (1962,1966) who have come forth with the following inequalities:

N is an odd perfect number, p is a prime

i) $\frac{1}{2} < \sum_{P/N} \frac{1}{P} < 2 \log (\pi/2)$ Perisastri (1958)

(Ah, sweet mystery of π !!!)

ii) $\frac{\log^2}{5 \log 5/4} < \sum_{P/N} \frac{1}{P} < \log 2 + \frac{1}{338}$ if $N = 12t + 1$

iii) $\frac{1}{3} + \frac{\log \frac{4}{3}}{5 \log \frac{5}{4}} < \sum_{P/N} \frac{1}{P} \quad \log \frac{18}{13} + \frac{53}{150}$ if $N = 36t + 9$

(ii) and (iii) are both due to Suryanarayana (1962). Recently these bounds have been improved by the forenamed mathematician as follows (1966):

a) $\frac{1}{5} + \frac{1}{7} + \frac{\log 48/35}{11 \log 11/10} < \sum_{P/N} \frac{1}{P} < \frac{1}{5} + \frac{1}{2738}$ if $N = 12t + 1$ and $5/N$

b) $1 + \frac{\log 12/7}{11 \log 31/10} < \sum_{P/N} \frac{1}{P} < \log 2$ if $N = 12t + 1$ and $5 \nmid N$

c) $\frac{1}{3} + \frac{1}{5} + \frac{\log 16/15}{17 \log 17/16} < \sum_{P/N} \frac{1}{P} \quad \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log \frac{65}{61}$ if $N = 36t + 9$ and $5/N$

d) $\frac{1}{3} + \frac{\log 4/3}{7 \log 7/6} < \sum_{P/N} \frac{1}{P} \quad \frac{1}{3} + \frac{1}{338} + \log \frac{18}{13}$ if $N = 36t + 9$ and $5 \nmid N$

The prrofs of (a), (b), (c), (d) are rather long and the reader is referred to Suryanarayana, On Odd Perfect Numbers II, Proceedings American Mathematical Society 14(1963).

The relationship between odd perfect numbers and the Rieman zeta funcation have been shown to involve the expression $\zeta(3)$ as follows (Suryanarayana, 1966):

1) $2 < \prod_{P/N} P/(P-1) < \frac{56791}{33612} \zeta(3)$ if $N = 12t + 1$ and $5/N$

2) $2 < \prod_{P/N} P/(P-1) < \frac{1760521}{1050375} \zeta(3)$ if $N = 12t + 1$ and $5 \nmid N$

3) $2 < \prod_{P/N} P/(P-1) < \frac{318897}{177023} \zeta(3)$ if $N = 36T + 9$ and $5/14$

4) $2 < \prod_{P/N} P/(P-1) < \frac{37061}{21252} \zeta(3)$ if $N = 36t + 9$ and $5 \nmid N$

The proofs of these results are equally long and the reader is referred to On Odd Perfect Numbers III, Proceedings American Mathematical Society, 18(1967).

In summary, the following conclusions can be drawn regarding the properties of odd perfect numbers:

1.	$N = P^a Q^2$ where $P \equiv a \equiv 1(4)$, P prime	Euler
2.	$N = 12t + 1$ or $36t + 9$	Touchard
3.	$N \neq 4t + 3$, $N \neq 6k + 5$	Euler/Pepin
4.	N has at least 6 (distinct) prime factors	Kuhnell
5.	$N > 10^{20}$	Kanold
6.	If P^k/N, then $P^k > 10^{12}$	Muskat
7.	It obeys the Survanaravana inequalities	Survanaravana

It appears unlikely that there are any odd perfect numbers, but until some mathematician proves these strange beasts out of existence, no real certainty can be achieved. Fata viam invenient.

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process allows us to continue choosing q_i's, even after enough have been chosen to make $\varepsilon_j < \varepsilon$. For every integer $n = \prod_{i=1}^{r} q_i$ where $r \geq s_t$, n satisfies the conclusion of the theorem. Consequently, there are infinitely many such n.

For R = 2, this theorem states that there are infintely many "almost unitary perfect" numbers. By choosing q_1 such that $q_1 \geq 3$. we can show that infinitely many odd "almost unitary perfect" numbers exist, although there are no odd unitary perfect numbers.

It is interesting that all of the n's produced by the method of the above theorem are of the form $\prod_{i=1}^{r} q_i$. Consider instead, integers of the form $\prod_{i=1}^{r} q_i^2$.

Unlike $U(\prod_{i=1}^{r} q_i)$, $U(\prod_{i=1}^{r} q_i^2)$ is bounded above. $U(\prod_{i=1}^{r} q_i^2) = \prod_{i=1}^{r} 1 + 1/q_i^2$.

$$\prod_{i=1}^{n} 1 + \frac{1}{p_i^2} = \sum \frac{1}{p_1^{a_1}, p_2^{a_2}, \ldots p_n^{a_n}}$$

where $a_i = 0$ or 1 and the sum is taken over all combinations of $a_1, a_2, \ldots, a_n$. Hence,

$$\prod_{i=1}^{\infty}\left(1 + p_i^{-2}\right) = \prod_{i=1}^{\infty}\left(1 - p_i^{-2}\right)^{-1} \Big/ \prod_{i=1}^{\infty}\left(1 - p_i^{-4}\right)^{-1} = \sum_{k=1}^{\infty}\left(k^{-2}\right) \Big/ \sum_{k=1}^{\infty}\left(k^{-4}\right) = \left(\pi^2/6\right) \Big/ \left(\pi^4/90\right) = 15/\pi^2 \; . < 2.$$

Thus we have proved the following theorem:

Theorem 2: There are no unitary perfect numbers of the form $\prod_{i=0}^{r} q_i^{a_i}$ where $a_i \geq 2$ for all i.

A similar proof can show that there are no unitary perfect numbers of the form $\prod_{i=1}^{n} q_i^{a_i}$ where $a_i \geq 2$ for $i \neq s$ and $a_s = 1$.

$$U\left(\prod_{i=1}^{r} q_i^{a_i}\right) = \left(1 + \frac{1}{q_s}\right) \cdot \left(\prod_{\substack{i=1 \\ i \neq s}}^{r} 1 + \frac{1}{q_i^2}\right) < \frac{q_s + 1}{q_s} \cdot \frac{q_s^2}{q_s^2 + 1} \cdot \frac{\pi^2}{6} = \frac{q_s^2 + q_s}{q_s^2 + 1} \cdot \frac{\pi^2}{6} .$$

If $g(x) = \frac{x^2 + x}{x^2 + 1}$, then $g'(x) = \frac{-x^2 + 1 + 2x}{(x^2 + 1)^2}$.

Thus $g'(x) = 0$ for $x = 1 + \sqrt{2}$, and $g'(x)$ 0 for all $x > 1 + \sqrt{2}$. Thus $g(x)$ is monotonically decreasing for $x > 1 + \sqrt{2} = 2.4+$. Hence, $\frac{q_s^2 + q_s}{q_s}$ is a maximum for either $q_s = 2$ or 3. Since $g(q_s) = 6/5$ for both $q_s = 2$ and 3, $\frac{q_s^2 + q_s}{q_s + 1} \cdot \frac{\pi^2}{6} \leq \frac{6}{5} \cdot \frac{\pi^2}{6} < 2$. Thus there are no unitary perfect numbers of the form $\prod_{i=1}^{r} q_i^{a_i}$ where $a_i \geq 2$ for $i \neq s$ and $a = 1$.

There are two known unitary perfect numbers of the form $\prod_{i=1}^{r} q_i^{a_i}$ where $a_i \geq 2$ for all $i \neq s$, t and $a_s = a_t = 1$: namely $60 = 2^2 \cdot 3 \cdot 5$ and $90 = 2 \cdot 3 \cdot 5$. Using $\pi^2/6$ as an upper bound for $\sum_{k \in S} 1/k^2$, an upper bound of $U(\prod_{i=1}^{r} q_i^{a_i})$ where $a_i \geq 2$ for $i \# s,t$ and $a = a = 1$ can be obtained. It is $(6/5) \cdot (6/5) \cdot \pi^2/6 \approx 2.37$.

UNDERGRADUATE RESEARCH PROPOSAL

Arthur Bernhart
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In the real number system there are three kinds of numbers: positive, zero, and negative. There are laws concerning these like the product of two positive numbers is positive, a positive times a negative is negative and so forth.

In a finite field we do not have the distinction between positive and negative, but there is another analogy which we can look at. In the reals, each non-zero number has a square which is positive. In a finite field, there are numbers which are sqaures and those which are not. Consider those which are squares (quadratic residues) in one class and those which are not squares (quadratic non-residues) in another, with zero in a separate class by itself. The product of two quadratic residues is a quadratic residue; the product of a non-residue and a residue is a quadratic non-residue. The product of two non-residues is a quadratic residue.

Here we have an analogy with the law of signs where the quadratic residues play the part of positive numbers and the non-residues play the part of negative numbers. How far can this analogy be pushed? You may want to consider vector spaces over the field and the resulting geometry. Try also to interpret distance relations and other parts of analytic geometry as well as purely algebraic results.

Binatural Numbers

D. R. Morrison
Sandia Laboratories

The natural numbers are, no doubt, the oldest, most fundamental, and most universally recognized and widely used mathematical system. The operation of counting, which gives rise to them, is probably the most elementary mathematical process. Kronecker is alleged to have said, "God gave us the integers; all the rest is due to man." Many civilizations, the Egyptians, the Romans, the Arabs, and the Mayans, to mention a few, invented systems of notation for natural numbers, all different in form and yet all representing the same abstract system.

Peano characterized the system N of natural numbers in terms of a first element 1 and a successor function S, under which N is closed, by three axioms.

- N1 S is one-to-one.
- N2 1 is not in the range of S.
- N3 The only subset of S which includes 1 and is closed under S is S.

The function S is, of course, the counting function, $S(n) = n + 1$. All the operations such as +, x, etcetera, which are traditionally defined in N, are defined inductively from 1 and S; and all their algebraic properties, such as commutativity, associativity, etcetera, are proved from N1, N2, and N3.

An obvious extension of the foregoing is a system B, of binatural numbers, characterized by a first member 1, and a pair L (left successor) and R (right successor) of successor functions, under each of which B is closed and which satisfies the three axioms:

- B1 L and R are both one-to-one.
- B2 The ranges of L and R are disjoint, and neither includes 1.
- B3 The only subset of B which includes 1 and is closed under both L and R is B.

While N has many representations, all of which are used essentially as counting systems, B has many representations which differ not only in form but also in use. Consider the different definitions of L and R (see following figure) and note the different forms of B to which they lead.

In example 2 the ranges of L, R, LL, LR, RL, RR, LLL, etcetera, lie in finer and finer partitions of the open interval (0,1) and have obvious application to the process of searching by halving, an important process in numerical analysis, measure theory, and other areas.

The natural vectors which arise in example 4 are a natural set of labels for things that may require extensive subdivision: **organizations**, subject matter categories, paragraph and section subdivision; of documents, etcetera. Example 5 is the free monoid with two generators, which plays and essential role in the coding of information for binary digital computers.

In each of these applications there are certain relations and operations that arise naturally in much the same way that + and x

Form Typical n	First n	Definition of L(n)	Range of L(n)	Definition of R(n)	Range of R(n)
1. Natural Number	1	2n	Even numbers	2n + 1	Odd numbers > 1
2. A number in the open interval (0,1)	$\frac{1}{2}$	$\frac{n}{2}$	A dense subset of the open interval $(0,\frac{1}{2})$	$\frac{n+1}{2}$	A dense subset of the open interval $(\frac{1}{2},1)$
3. A number in the Cantor set	$\frac{1}{3}$	$\frac{n}{3}$	A subset of the lower half of the Cantor set	$\frac{n+2}{3}$	A subset of the upper half of the Cantor set
4. A natural vector (finite sequence of natural numbers)	1 The natural vector of dimension 1, whose only term is 1	1,n Concatenation of 1 and n	Natural vectors whose first term is 1 and whose dimension is > 1.	1 + n The sum of the natural vectors 1 and n	Natural vectors whose first term is > 1.
5. A string (finite sequence) of zeros and ones	Ω The empty or zero-length string	n0 The extension of n by a zero	Strings whose last member is 0	n1 The extensions of n by a one	Strings whose last member is 1

EXAMPLES OF REPRESENTATIONS OF THE BINATURAL NUMBERS arise naturally in N. In each of them there is at least one "natural order," and in several of them there are one or more natural definitions of lenpth, magnitude, dimension, or some other measure that maps B into N which has useful properties relative to the operations that arise naturally. In several of the examples, concatenation, addition, and/or multiplication are defined for pairs. Involutions such as reversal or comnlementation are defined for individual elements, and these are in many cases automorphisms relative to the onerations.

It is not hard to show that the hinatural numhers, like the natural number?, are uniaue un to an isomorphism. It follows that anv operation, relation, or measure that arises in any representation of B has an analogous oneration, relation, or measure in each representation of B. This gives rise to a virtually inexhaustible list of interesting questions.

> What oneration on natural vectors corresnonds to + and x on natural numhers?
> What operation on natural numhers corresnonds to concatenation of natural vectors, or of strings of zeros and ones?
> What order relation on numhers in (0,1) corresponds to the well ordering by < among the natural numhers.
> There are twelve kinds of alphabetical order among the strings of zeros and 1's, corresnonding to the six permutations of 0,1 and Ω and the two directions (left and right) of concatenation. What are the corresnonding twelve order relations amonp natural numbers" Among natural vectors?

And so on. Some of these are easv to answer and some are hard. Some are interesting and some are not. I'll leave it to the reader to sort them out.

The on-to-one corresnondences amonp the various forms of binatural numbers are also interesting and useful. A function vhich assigns to each natural vector a unique natural integer makes it possible to process natural vector identifiers as single integers. This simplifies computer storage requirements, though it does generate rather large identifiers. Perhaps the readers can find other useful applications of these correspondence.

NEED MONEY?

The Governing Council of Pi Mu Epsilon announces a contest for the best expository paper by a student (who has not yet received a master's degree) suitable for publication in the Pi Mu Epsilon Journal.

The following prizes will be given:

$200.	first prize
$100.	second prize
$50.	third prize

providing at least ten papers are received for the contest.

In addition there will be a $20.00 prize for the best paper from any one chapter, providing that chapter submits at least five papers.

SOME COMMENTS ON "A CLASS OF FIVE BY FIVE MAGIC SQUARES"

Robert C. Strum

In the Fall, 1971 issue of the Pi Mu Epsilon Journal, Marcia Peterson presented a class of five-hy-five magic squares with a three-by-three magic center. The purpose of the current comments is to point out four errors in the magic square as it appeared in print, and to offer two correct magic squares.

The magic square as published is shorn in Figure 1. Let each element of the five-by-five magic square be identified by $E(i,j)$ where:

$i = 1,2,3,4,5$, and indicates the row;
$j = 1,2,3,4,5$, and indicates the column.

Observe that each element is of the form $(n + kb)$ where k takes on twenty-five distinct values for the five-by-five magic square. The errors are as follows:

1) The set defining k in Figure 1 has only 22 elements. Given that 0 is also a member of that set, since it is used in $E(3,3)$, the set defining k is incomplete since 25 elements are required.

2) Because of 1), duplicate usage of two values of k ($k = +3c$ and $k = -3c$) is employed in $E(2,3)$, $E(3,1)$, and in $E(3,5)$, $E(4,3)$.

3) $\sum_{m=2}^{4} E(2,m) \neq 3n$ and $\sum_{m=2}^{4} E(4,m) \neq 3n$ but they should for a magic square three-by-three.

4) $\sum_{m=1}^{5} E(m,1) \neq 5n$ and $\sum_{m=1}^{5} E(m,5) \neq 5n$ but they should for a magic square five-hy-five.

To obtain a correct class of magic squares, one must add two members to the set defining k. Let these members be $k = +4c$ and $k = -4c$. Then a class of five-by-five magic squares with a three-by-three magic center is given in Figure 2.

It is interesting to note that, with two exceptions, the values of k are given by:

$$k = qc + p$$

where

$$q = -4,-3,-2,-1,0,1,2,3,4$$

and for each value of q except $q = -4$, $q = +4$,

$$p = -1,0,1$$

and for $q = -4$, $q = +4$,

$$p = -1,0$$

The exceptions to this symmetric pattern are $k = +(2c + 2)$ and $k = -(2c + 2)$ which are used instead of $k = +(3c - 1)$ and $k = -(3c - 1)$. A class of five-by-five magic squares with a three-by-three magic center using, for the values of k the set defined by $k = qc + p$ as described above is given in Figure 3.

n-(c-1)b	n-(2c+1)b	n-(3c+1)b	n+(2c+2)b	n+(4c-1)b
n-(2c-1)b	n-b	n+3cb	n+cb	n-2cb
n+3cb	n+(c+1)b	n	n-(c+1)b	n-3cb
n+2cb	n-cb	n-3cb	n+b	n+(2c+1)b
n-(4c-1)b	n+(2c+1)b	n+(3c+1)b	n-(2c+2)b	n+(c-1)b

In the above, b and n are arbitrary whole numbers. To be certain that all of the above entries are distinct we require only that the members of the set

$$\left\{\begin{matrix}-1,1,c,-c,c+1,c-1,2c,2c+1,2c-1,2c+2,3c,3c+1,4c-1,-(c+1),-(c-1),-2c,\\ -(2c+1),-(2c-1),-(2c+2),-3c,-(3c+1),-(4c-1)\end{matrix}\right\}$$

are all distinct. This will be true, for example, if $c \geq 3$.

Figure 1

n-(2c-1)b	n-(2c)b	n-(2c+2)b	n+(4c)b	n+(2c+1)b
n-(3c)b	n-b	n-(c-1)b	n+cb	n+(3c)b
n+(3c+1)b	n+(c+1)b	n	n-(c+1)b	n-(3c+1)b
n+(4c-1)b	n-cb	n+(c-1)b	n+b	n-(4c-1)b
n-(2c+1)b	n+(2c)b	n+(2c+2)b	n-(4c)b	n+(2c-1)b

Figure 2

n-(2c+1)b	n-(3c-1)b	n-(2c-1)b	n+(4c-1)b	n+(3c)b
n+(3c+1)b	n-b	n-(c-1)b	n+cb	n-(3c+1)b
n-(2c)b	n+(c+1)b	n	n-(c+1)b	n+(2c)b
n+(4c)b	n-cb	n+(c-1)b	n+b	n-(4c)b
n-(3c)b	n+(3c-1)b	n+(2c-1)b	n-(4c-1)b	n+(2c+1)b

Figure 3

Monte Carlo Estimate for Pi

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The purpose of this note is to provide a somewhat simpler experiment for caculating Pi than Buffon's needle experiment [2]. Let a region A be inscribed in a unit square. Assume that it is possible to select a point at random in the square. By "at random", we mean that every rectangular region R of area p has probability p of containing the point. Then the probability that the point will lie in the region A is equal to the area of A (see figure 1). This method of estimating the area of A is called a Monte Carlo Method [1] In particular, let a circle be inscribed in a unit square. If a point is selected at random in the square, then the probability that it will lie in the circle is $\frac{\pi}{4}$ (See figure 2).

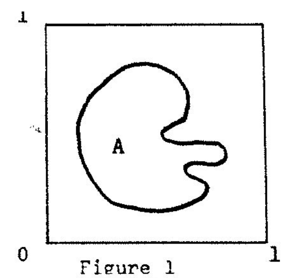

Figure 1

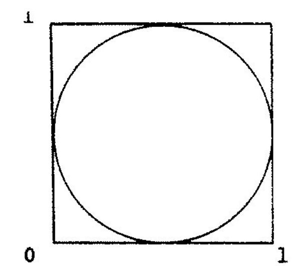

Figure 2

An experiment is constructed to calculate Pi as follows. A grid of perpendicular lines is drawn so that the distance between adjacent parallel lines is the diameter of a penny. A penny is thrown at random on the grid. The probability that the penny will cover an intersection of two grid lines is $\frac{\pi}{4}$. This may be verified by considering the center C of the penny as our random point. The center C of the penny will lie in the dotted circle, inscribed in a dotted square, if and only if the penny covers an intersection of two grid lines (see figure 3). In a classroom experiment 2500 pennies were tossed and 1961 hits were recorded. The approximate value for Pi obtained was 3.138. This particular experiment was somewhat better than expected. The experiment is a series of binomial trials, each of which has a probability $\frac{\pi}{4}$ of success. The standard deviation for such an experiment is known to be $\sqrt{n\frac{\pi}{4}\frac{4-\pi}{4}}$. The standard deviation for 2500 trials is approximately 20, which gives an accuracy of 0.8% for the estimation of Pi.

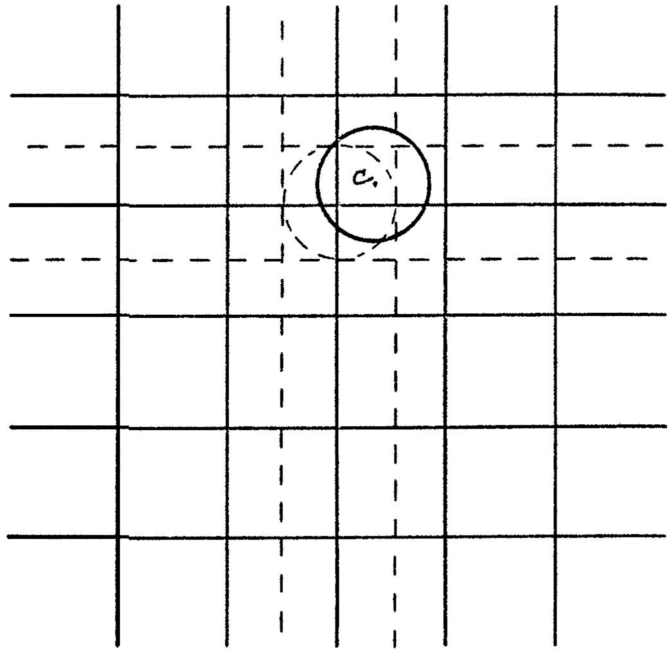

Figure 3

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A Note on the Integral and the Derivative of the Inverse Sine Function

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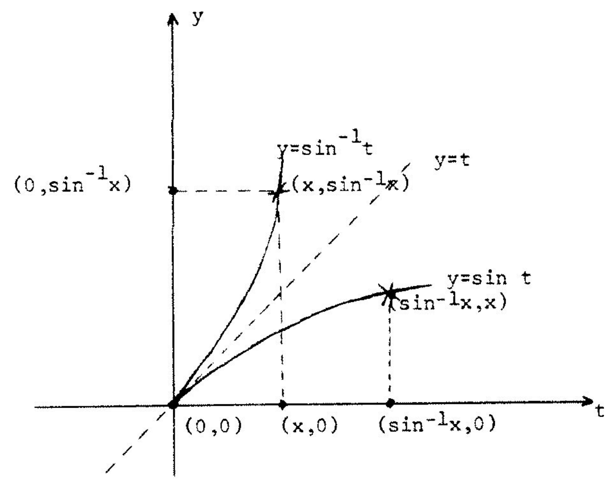
In a beginning calculus course a student encounters quite often the integral and the derivative of the inverse sine function, $\sin^{-1}$. The integral of $\sin^{-1}$ is usually obtained by integration by parts while the derivative can be obtained by applying the Inverse Function Theorem for Derivatives. This note shows how to handle both situations by means of a geometric argument.

On a single coordinate system, consider the graphs of $y = \sin^{-1}t$ where $0 \le t \le x$ (arbitrary x being positive for reasons of simplicity) and $y = \sin t$ where $0 \le t \le \sin^{-1}x$, as shown in the following figure.



The area of the region bounded by the curve $y = \sin t$, the t-axis, and the vertical line $t = \sin^{-1}x$ is given by $\int_0^{\sin^{-1}x} \sin t\,dt$. Rotating this region about the line $y = t$, the region will have the same area although the equations of its boundries become the curve $y = \sin^{-1}t$, the y-axis, and the horizontal line $y = \sin^{-1}x$. This area can be expressed as $x\cdot\sin^{-1}x - \int_0^x \sin^{-1}t\,dt$.

Hence, $\int_0^{\sin^{-1}x} \sin t\,dt = x\cdot\sin^{-1}x - \int_0 \sin^{-1}t\,dt$,

$$\text{or,}\quad \int_0^x \sin^{-1}t\,dt = x\cdot\sin^{-1}x - \int_0^{\sin^{-1}x} \sin t\,dt,$$

$$= x\cdot\sin^{-1}x + \cos t\Big]_0^{\sin^{-1}x},$$

$$= x\cdot\sin^{-1}x + \cos(\sin^{-1}x) - \cos 0,$$

$$= x\sin^{-1}x + \sqrt{1-\sin^2(\sin^{-1}x)} - 1,$$

$$\text{Thus,}\quad \int_0 \sin^{-1}t\,dt = x\sin^{-1}x + \sqrt{1-x^2} - 1. \qquad (A.)$$

As said before, the same result can be obtained by integration by parts although it is necessary to know that $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$; this derivative was not used to obtain (A.). To obtain such, one needs to differentiate both sides of (A.), assuming that the derivative of the $\sin^{-1}$ function exists.

$$\frac{d}{dx}\left(\int_0^x \sin^{-1}t\,dt\right) = \frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2} - 1\right),$$

$$\sin^{-1}x = 1\cdot\sin^{-1}x + x\cdot\frac{d}{dx}\sin^{-1}x - \frac{x}{\sqrt{1-x^2}},$$

$$\text{or,}\quad \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

MATCHING PRIZE FUND

The Governing Council of Pi Mu Epsilon has approved an increase in the maximum amount pep chapter allowed as a matching prize from \$25.00 to \$50.00. If your chapter presents awards for outstanding mathematical papers and students, you may apply to the National Office to match the amount spent by your chapter--i.e., \$30.00 of awards, the National Office will reimburse the chapter for \$15.00, etc.,--up to a maximum of \$50.00. Chapters are urged to submit their best student papers to the Editor of the Pi Mu Epsilon Journal for possible publication. These funds may also be used for the rental of mathematical films. Please indicate title, source and cost, as well as a very brief comment as to whether you would recommend this particular film for other Pi Mu Epsilon groups.

Spec (R) For A Particular R

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Let R be a commutative ring with identity. Define

$$\text{Spec}(R) = \{P \subset R:\ P \text{ is a prime ideal}\}.$$

We remark that the word "ideal" denotes "proper ideal." We shall write [P] for the element of Spec (R) given by the prime ideal P.

The Zariski topology on Spec (R) is given by: The closed sets are those of the form $\{[P]:\ P \supseteq A\}$ where A is a (possibly improper) ideal of R. We denote such a set by V(A). It is not hard to prove:

i) $V(\sum_a A_a) = \bigcap_\alpha V(A_\alpha)$
ii) $V(A \cap B) = V(A) \cup V(B)$

So the collection of closed subsets $\{V(A)\}$ does define a topology.

Let $\text{Spec}(R)_f = \{[P]:\ f \notin P\}$. Since $\text{Spec}(R)_f = \text{Spec}(R) - V((f))$, $\text{Spec}(R)_f$ is an open subset of Spec (R). These sets form a base for the open sets in Spec (R) since any open set $\text{Spec}(R) - V(A) = \bigcup_{f\in A}\text{Spec}(R)_f$.

Let k be a field, and consider the commutative ring with identity $R = \prod_{i=1}^{\infty} k$. We remark that $(u_1, u_2, \ldots)$ is a unit of R if and only if for all $i \in N$, $u_i \neq 0$.

In the preliminary version of his Introduction to Algebraic Geometry, (3, pp. 124-125, 140), David Mumford states that for $R = \prod_{i=1}^{\infty} k$, Spec (R) is the Stone-Cech compactification of N, the positive integers. This paper intends to present a proof of that statement, using the following characterization of the Stone-Cech compactification found in Gillman and Jerison's Rings of Continuous Functions, (1, page 86):

Theorem. Every completely regular space X has a Hausdorff compactification βX with the property that any two disjoint zero-sets in X have disjoint closures in βX. Futhermore, βX is unique: if a Hausdorff compactification T of X satisfies this property, then there exists a homeomorphism of βX onto T that leaves X pointwise fixed.

Proposition 1. The set $M_p = \{(k_1, k_2, \ldots) \in R:\ k_p = 0\}$ is a maximal ideal in R.

Proof: $M_p \neq \emptyset$ since $(0, 0, \ldots) \in M_p$. $M_p \neq R$ since $(1, 1, \ldots)$ does not belong to M_p. "Suppose $(k_1, k_2, \ldots)$ and $(k_1^1, k_2^1, \ldots)$ belong to M_p. Then $k_p = k_p^1 = 0$ and $k_p - k_p^1 = 0$ and $(k_1, k_2, \ldots) - (k_1^1, k_2^1, \ldots) \in M_p$. Suppose $(k_1, k_2, \ldots) \in M_p$ and $(r_1, r_2, \ldots) \in R$. Then $k_p \cdot r_p = 0$ and $(k_1, k_2, \ldots)\cdot(r_1, r_2, \ldots) \in M_p$. Hence M_p is an ideal in R.

Suppose M is an ideal of R containing M_p. Assume $(r_1, r_2, \ldots) \in M - M_p$.

Then $r_p \neq 0$. Nov, $(r_1, r_2, \ldots)(0, \ldots, 0, \overset{\text{pth place}}{\frac{1}{r_p}}, 0, \ldots) = (0,\ldots,0,\overset{\text{pth place}}{1},0\ldots) \in M$.

Since $(1,\ldots,1,\overset{\text{pth place}}{0},1\ldots) \in M_p$, we have $(0,\ldots,0,1,0\ldots) + (1,\ldots 1,0,1,\ldots) = (1,1,\ldots) \in M$ contradicting the fact that M is an ideal of R. Thus, $M = M_p$. Hence, M_p is maximal.

Corollary 1.1. M_p is principal; that is, $M_p = (f)$, where $f = (1,\ldots,1,\overset{\text{pth place}}{0},1\ldots)$.

Proof: Suppose $(k_1, k_2,\ldots) \in M_p$. Then $k_p = 0$. Hence $(k_1, k_2,\ldots) = (k_1, k_2,\ldots) \cdot f$. Thus, $M_p \subset (f)$. Hence, $(f) = M_p$.

Proposition 2. $\{[M_p]\}$ is open-and-closed.

Proof: $\{[M_p]\}$ is closed since $\{[M_p]\} = V(M_p)$. Let $g = (1,1,\ldots) - f$, vhere f is the generator of M_p. Suppose P is prime and $g \notin P$. We claim that $P = M_p$. $P \subset M_p$ since otherwise there would exist $(r_1, r_2,\ldots) \in P$ such that $r_p \neq 0$. In which case $(r_1, r_2,\ldots) \cdot (0,\ldots,0, \frac{1}{r_p}, 0,\ldots) = g$ belongs to P.

It suffices to show $f \in P$. Assume $f \notin P$. Let $(r_1,r_2,\ldots) \in P$. Then $(r_1, r_2, \ldots) = f \cdot (r_1, r_2, \ldots, r_{p-1}, 1, r_{p+1}, \ldots)$. But $(r_1, r_2, \ldots, r_{p-1}, 1, r_{p+1}, \ldots) \notin P$ since $P \subset M_p$ and we have assumed $f \notin P$. Contradiction of the fact P is prime. Hence $f \in P$ and $P = M_p$. Therefore, Spec $(R)_g = \{[M_p]\}$ and $\{[M_p]\}$ is open.

Corollary 2.1. N is homeomorphic with the subspace $\{[M_p]\}_{p\in N}$ of Spec (R).

Proof: N and $\{[M_p]\}_{p\in N}$ are in one-one correspondence by the map $p \leftrightarrow [M_p]$. Since both spaces are discrete, this map is a homeomorphism.

Proposition 3. Spec (R) is a T-space. In fact, P is the only prime ideal in R containing P; i.e., $\{[P]\} = V(P)$.

Proof! Suppose P_1 is prime and contains P. Assume $(k_1, k_2, \ldots) \in P_1 - P$. We may assume $k_i = 1$ or 0 for each $i \in N$, since $(k_1 a_1, k_2 a_2, \ldots) \in P_1 - P$ where $a_i = \{k_i$ if $k_i = 0$ $\frac{1}{k_i}$ otherwise$\}$

Let $(r_1, r_2, \ldots) = (1, 1, \ldots) - (k_1, k_2, \ldots)$. $(r_1, r_2, \ldots) \notin P_1$ since $(r_1, r_2, \ldots) + (k_1, k_2, \ldots) = (1, 1, \ldots)$. Hence $(r_1, r_2, \ldots) \notin P$. But $(r_1, r_2, \ldots) \cdot (k_1, k_2, \ldots) = (0, 0, \ldots) \in P$. Contradiction of the fact P is prime. Therefore, $P_1 = P$. Therefore, $\{[P]\} = V(P)$ and is closed.

Corollary 3.1. Every prime ideal in R is maximal.

Proof: Since R is a commutative ring with identity, every prime ideal P is contained in a maximal ideal M. Since M is prime, $M = P$.

Proposition 4. Spec (R) is Hausdorff.

Proof: Suppose $[P_1]$ and $[P_2]$ are distinct points in Spec (R). Then P_1 and P_2 are distinct prime ideals in R. Thus there exists $p^1 = (p_1^1, p_2^1,\ldots)$ such that $p^1 \in P_1 - P_2$ say. We may assume each p_i^1 is either 1 or 0 since $(p_1^1 a_1, p_2^1 a_2, \ldots) \in P_1 - P_2$ where $a_i = \{ 0$ if $p_i^1 = 0 \}$.

Let $p^2 = (p_1^2, p_2^2, \ldots) = (1, 1, \ldots) - (p_1^1, p_2^1, \ldots)$. Clearly $p^1 \cdot p^2 = (0, 0, \ldots)$. Thus $p^1 \cdot p^2 \in P$ for all prime ideals P in R. Hence, $p^2 \in P_2$. But $p^2 \notin P_1$ since $p^1 + p^2 = (1, 1, \ldots)$. Now, Spec $(R)_{p^1}$ and Spec $(R)_{p^2}$ are neighborhoods of $[P_2]$ and $[P_1]$ respectively. Futhermore, Spec $(R)_{p^1} \cap$ Spec $(R)_{p^2}$ is empty since $p^1 \cdot p^2 \in P$ for all $[P] \in$ Spec (R) and hence p^1 or p^2 must belong to P. Therefore, Spec (R) is Hausdorff.

Proposition 5. Spec (R) is compact.

Proof: Suppose $\{V(A_i)\}_{i\in I}$ is a family of closed sets in Spec (R) with the finite intersection property. Assume $\bigcap_{i\in I} V(A_i)$ is empty. Hence since $\bigcap_{i\in I} V(A_i) = V(\sum_{i\in I} A_i)$, $V(\sum_{i\in I} A_i) = \emptyset$. Thus no ideal in R contains A since R is a commutative ring with 1. (Hence, $\sum_i A_i = R$.) Thus, $1 = r_1 s_1 + \ldots + r_n s_n$ for certain $s_j \in A_{i_j}$. Since $(\bigcap_{j=1}^{n} V(A_{i_j})) \neq \emptyset$, there exists a maximal ideal M containing $\sum_{j=1}^{n} A_{i_j}$. Hence M contains $r_1 s_1 + \ldots + r_n s_n = 1$. Contradiction of the fact $1 \notin M$. Therefore, $\bigcap_{i\in I} V(A_i) \neq \emptyset$.

Proposition 6. The subspace $\{[M_p]\}_{p\in N}$ is dense in Spec (R).

Proof: Suppose $[P] \in$ Spec (R). Let Spec $(R)_f$ be a basic open neighborhood of [P]. It suffices to show that there exists $p \in N$ such that $f \notin M$. Assume $f \in M_p$ for all $p \in N$. Then $f = (0, 0, \ldots)$. Hence $(0, 0, \ldots) \notin P$. Contradiction of the fact P is an ideal. Hence there exists $p \in N$ such that $[M_p] \in$ Spec $(R)_f$. Thus $[P] \in \mathrm{cl}\ \{[M_p]\}_{p\in N}$. Therefore, Spec (R) $= \mathrm{cl}\ \{[M_p]\}_{p\in N}$.

Proposition 7. Any two disjoint subsets of $\{[M_p]\}_{p\in N}$ have disjoint closures in Spec (R).

Proof: Suppose n_1 and n_2 are disjoint subsets of $\{[M_p]\}_{p\in N}$. Assume $\mathrm{cl}_{\mathrm{spec}\,(R)} n_1 \cap \mathrm{cl}_{\mathrm{spec}\,(R)} n_2 \neq 0$. Let $[P] \in \mathrm{cl}\ n_1 \cap \mathrm{cl}\ n_2$. $[P] \notin \{[M_p]\}_{p\in N}$. Otherwise $[P] \in n_1 \cap n_2$ by Proposition 2. Futhermore, n_1 and n_2 must both be infinite; otherwise, n_1 say it is finite and hence closed in Spec (R). Thus $[P] \in n_1 \subset \{[M_p]\}_{p\in N}$.

So we mav write $n_1 = \{M_{p_1}, M_{p_2}, \ldots\}$ and $n_2 = \{M_{q_1}, M_{q_2}, \ldots\}$.

Define $n^1 = (n_1^1, n_2^1, \ldots)$ where n_i^1 is 0 if $i = p_n$ and 1 othervise.

Define $n^2 = (n_1^2, n_2^2, \ldots)$ where n_i^2 is 1 if $i = p$ and 0 otherwise.

Clearly $n^1 \cdot n^2 = (0,0,\ldots)$. Thus, n^1 say belongs to P. $n^0 \notin P$ since $n^1 + n^2 = (1,1,\ldots)$. Hence, Spec $(R)_{n^2}$ is a neighborhood of [P]. Thus, Spec $(R)_{n^2} \cap n_2 \neq 0$. Let $[M_{q_i}] \in n_2 \cap$ Spec $(R)_{n^2}$. Then $n^2 \notin M_{q_i}$. But since $n_1 \cap n_1 = \emptyset$, $n_{q_i}^2 = 0$. That is, $n^2 \in M_{q_i}$. Contradiction.

Therefore, $\mathrm{cl}_{\mathrm{Spec}(R)}\, n_1 \cap \mathrm{cl}_{\mathrm{Spec}(R)}\, n_2 = \emptyset$.

Theorem. Spec (R) is βN, the Stone-Cech compactification of N.

Proof: Spec (R) is a compact Hausdorff space, and N is homeomorphic to $\{[M_p]\}_{p\varepsilon N}$, a dense subspace of Spec (R). Since a zero-set is merely a special form of subset, any two disjoint zero-sets in N have disjoint closures in Spec (R). Therefore, by theorem Spec (R) is βN.

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(2) Kelley, John L., General Topology, D. Van Nostrand Company, Inc., 1955

(3) Mumford, David, Introduction to Algebraic Geometry, (preliminary version of first three chapters).

MEETING ANNOUNCEMENT

Pi Mu Epsilon will meet from August 28-30, 1972 on the Dartmouth campus, Hanover, New Hampshire, in conjuction with the Mathematical Association of America. Chapters should start planning NOW to send delegates or speakers to this meeting, and to attend as many of the lectures by other mathematical groups as possible.

The National Office of Pi Mu Epsilon will help with expenses of a speaker OR delegate (one per chapter) who is a member of Pi Mu Epsilon and who has not received a Master's Degree by April 15, 1972, as follows: SPEAKERS will receive lowest cost confirmed air fare (maximum of $300) from home or school, whichever is nearer, to Hanover, NH; or actual travel expenses, whichever is less; DELEGATES will receive 1/2 of the speaker's cost.

Select the best talk of the year given at one of your meetings by a member of Pi Mu Epsilon who meets the above requirements and have him or her apply to the National Office. Nominations should be in our office by April 15, 1972. The following information should be included: your name; Chapter of Pi Mu Epsilon; a school; topic of talk; what degree you are working on; if you are a delegate or a speaker; when you expect to receive your degree; current mailing address; summer mailing address; who you were recommended by; and a 50-75 word summary of talk, if you are a speaker. MAIL TO: Pi Mu Epsilon, 601 Elm Ave., Room 423, The University of Oklahoma, Norman, OK 73069.

A REGULAR NON-NORMAL TOPOLOGICAL SQUARE

William L. Quirin
Adelphi University

If $P = \{(x,y):\ x,y \in \mathbb{R},\ y > 0\}$ is the open upper half-plane with the Eulcidean topology, and if $L = \{(x,0):\ x \in \mathbb{R}\}$ is the real axis, we define a basis for a topology $\mathfrak{J}$ on $X = P \cup L$ as follows: for $(x,y) \in P$, the open disks with center (x,y) and radius $r \leq y$; for $x \in L$, all sets of the form $\{x\} \cup D$, where D is an open disk lying in 'the upper half-plane tangent to L at x. Such a set $\{x\} \cup D$ with radius r will be denoted $N_r(x)$. Note that if $r_1 > r_2$, then $N_{r_1}(x) \supset N_{r_2}(x)$.

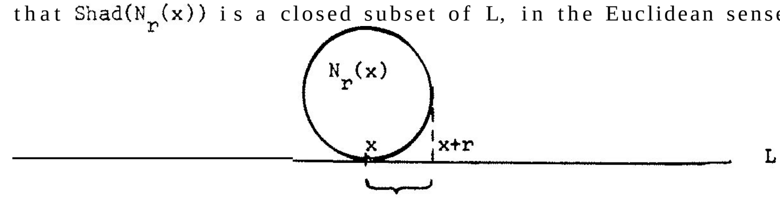
The Topological space $(X,\mathfrak{J})$, known as Niemytzki's Tangent Disk Space, is the classical example of a regular space which fails to be normal. However, an elementary proof that this space is not normal, which could be presented in an introductory undergraduate topology course, has to the author's knowledge, never appeared in print. In this article we present such a proof.

We begin with the following definition:

Definition: If $x \in L$ and $r > 0$, the shadow of the basic open set $N_r(x)$ is defined to be the set

$$\text{Shad}(N_r(x)) = \{y \in L:\ x \leq y \leq x + r\}.$$

Note that $\text{Shad}(N_r(x))$ is a closed subset of L, in the Euclidean sense.



To prove that $(X,\mathfrak{J})$ is not normal, we exhibit disjoint closed subsets of X which are not contained in disjoint open sets. Let

$$A = \{x \in L:\ x \text{ is rational}\}$$
$$B = \{x \in L:\ x \text{ is irrational}\}$$

Since L is a closed subset of X and since the relative topology on L is discrete, A and B are disjoint closed subsets of X. Suppose U and V are disjoint open sets such that $A \subset U$ and $B \subset V$.

Choose $x_1 \in A$. There is a basic open set $N_{\varepsilon_1}(x_1) \subset U$. Since there are irrational numbers arbitrarily close to any rational number, we can choose $x_2 \in B \cap \text{Shad}(N_{\varepsilon_1/2}(x_1))$, and we can find $N_{\varepsilon_2}(x_2) \subset V$ such that $\text{Shad}(N_{\varepsilon_2}(x_2)) \subset \text{Shad}(N_{\varepsilon_1}(x_1))$. In like manner, we can find $x_3 \in A \cap \text{Shad}(N_{\varepsilon_2/2}(x_2))$ and $N_{\varepsilon_3}(x_3) \subset U$ such that $\text{Shad}(N_{\varepsilon_3}(x_3)) \subset \text{Shad}(N_{\varepsilon_2}(x_2))$.

Continuing in this manner, we construct a sequence $\{x_n\}$ of points of L and a sequence of open sets $\{N_{\varepsilon_n}(x_n)\}$ such that:

(a) $x_{2n-1} \varepsilon A$ and $x_{2n} \varepsilon B$ for all $n \geq 1$,

(b) $x_{n+1} \varepsilon \text{Shad}(N_{\varepsilon_n/2}(x_n))$ for all $n \geq 1$,

(c) $N_{\varepsilon_{2n+1}}(x_{2n+1}) \subset U$ and $N_{\varepsilon_{2n}}(x_{2n}) \subset V$ for all $n \geq 1$,

(d) $\text{Shad}(N_{\varepsilon_n}(x_n)) \subset \text{Shad}(N_{\varepsilon_{n-1}}(x_{n-1}))$ for all $n \geq 2$.

Hence the sequence $\{x\}$ is a bounded increasing sequence of real numbers with the additional property that if $n \geq k$, then $x \varepsilon \text{Shad}(N_{\varepsilon_k}(x_k))$. In the usual topology on L, the sequence $\{x\}$ converges to some $x \varepsilon L$. Since $\text{Shad}(N_{\varepsilon_k}(x_k))$ is a closed set for each k, we have

$$x \varepsilon \bigcap_{k=1}^{\infty} \text{Shad}(N_{\varepsilon_k}(x_k)).$$

Now we must have either $x \varepsilon A$ or $x \varepsilon B$. However, if $x \varepsilon A$, then there exists $N_\varepsilon(x) \subset U$. Since $\{x\}$ converges to x, there exists x_{2k} such that $|x_{2k}-x| < \varepsilon$ Since $x \varepsilon \text{Shad}(N_{\varepsilon_{2k}}(x_{2k}))$, we see that

$$N_\varepsilon(x) \cap N_{\varepsilon_{2k}}(x_{2k})) \neq \phi,$$

and since $N(x) \subset U$ and $N_{\varepsilon_{2k}}(x_{2k})) \subset V$, we have $U \cap V \neq \phi$, contradicting our assumption that $U \cap V = \phi$. We arrive at a similar contradiction if we assume $x \varepsilon B$. Hence $x \notin A$ and $x \notin B$, and this final contradiction establishes the fact that $(X, \mathcal{J})$ is not normal.

We conclude by noting that the identical proof can be used to show that the following space, which is regular, is not normal. Let Y be the real line with topology generated by the intervals of the form $[a,b)$, and let $X = Y*Y$ with the product topology. The line $L = \{(x,y): x + y = 0\}$ is a closed subspace of X and the relative topology on L is discrete. If

$$A = \{(x,y) \varepsilon L: x,y \text{ are rational}\}$$
$$B = \{(x,y) \varepsilon L: x,y \text{ are irrational}\}$$

then A and B are disjoint closed subsets of X which are not contained in disjoint open sets.

For a higher level of proof, see Counterexamples in Topology by Steen 6 Seebach, Holt, Rinehart, & Winston, Inc., (1970), p. 100.

VECTOR GEOMETRY OF ANGLE-BISECTORS

Ali R. Amir-Moez
Texas Tech University

Problems involving anple-bisectors are usually very difficult; sometimes there is no peometric solution for constructing trianples for which some of the given parts are anple-bisectors. In this article we study the angle bisector of an anple through vectors and give some applications.

In what follows all vectors are in a Euclidean plane and will he denoted by Greek letters. The inner product of a and 6 will be denoted by (α,β) which is defined hy

$$(\alpha,\beta) = ||\alpha|| \; ||\beta|| \cos t,$$

where, for example, $||\alpha||$ means the norm of a and t is the angle between a and β. Other properties of vector algebra will he assumed and used [1, pp 1 - 74].

1. Bisectors: Let $\{\alpha,\beta\}$ be linearly independent (Fig. 1)** Let 6 he a non-zero vector on the bisector of an anple between a and β. Then the anple between a and δ is the same as the one between 5 and 6. This means

$$(1) \qquad \left(\frac{\delta}{||\delta||}, \frac{\alpha}{||\alpha||}\right) = \left(\frac{\delta}{||\delta||}, \frac{\beta}{||\beta||}\right).$$

This implies that

$$(2) \qquad \left(\delta, \frac{\alpha}{||\alpha||}\right) = \left(\delta, \frac{\beta}{||\beta||}\right)$$

which means that the algebraic projection of 6 on the axis $\left(\frac{\alpha}{||\alpha||}\right)$ is the same as its algebraic projection on the axis whose unit vector is $\frac{\beta}{||\beta||}$ (Fig. 2).

We may set

$$\lambda = \left(\delta, \frac{\alpha}{||\alpha||}\right) \frac{\alpha}{||\alpha||} \quad \text{and} \quad \mu = \left(\delta, \frac{\beta}{||\beta||}\right) \frac{\beta}{||\beta||}.$$

Then $||\lambda|| = ||\mu||$.

One observes that (1) implies that

$$(3) \qquad \frac{||\alpha||}{||\beta||} = \frac{(\alpha,\delta)}{(\beta,\delta)}.$$

We shall give a geometric interpretation. Let

$$\rho = \left(\alpha, \frac{\delta}{||\delta||}\right) \frac{\delta}{||\delta||}, \quad \theta = \left(\beta, \frac{\delta}{||\delta||}\right) \frac{\delta}{||\delta||}$$

**Figures are at the end of the article

which means that ρ and 8 are respectively projections of α and 6 on the axis $\left(\frac{\delta}{||\delta||}\right)$ (Fig. 3.)

Then (3) implies that:

(4) $$\frac{||\alpha||}{||\beta||} - \frac{||\rho||}{||\theta||} .$$

2. The convex hull of two vectors: Let {α,β} he linearly independent (Fig. 4.) and

$$\xi = a\alpha + b\beta, \quad a + b = 1, \; a > 0, \; b > 0.$$

This means that ξ ends on the open line segment connecting the end-points of a and 6. This line segment is called the convex hull of {α,β}. Then we observe that

$$\xi - a = aa + b\beta - a = b(\beta - a).$$

Similarly

$$\xi - 8 = a(\alpha - \beta).$$

Then

$$\frac{||\xi - \alpha||^2}{||\xi - \beta||^2} = \frac{b^2||\beta - \alpha||}{a^2||\alpha - \beta||} = \frac{b^2}{a^2}$$

This implies that

(5) $$\frac{||\xi - \alpha||}{||\xi - \beta||} = \frac{b}{a} .$$

3. The angle-bisector of a triangle: Let {α,β} be linearly independent am 6 be the angle bisector of the angle between a and 8 in the triangle formed by a and 8. (Figure 5). Then

$$\delta = a\alpha + b\beta, \quad a + b = 1, \; a > 0, \; b > 0,$$

and

$$\frac{(\delta,\alpha)}{||\alpha||} = \frac{(\delta,\beta)}{||\beta||} .$$

This implies that

$$\frac{(a\alpha + b\beta,\alpha)}{||\alpha||} = \frac{(a\alpha + b\beta,\beta)}{||\beta||}$$

or

$$\frac{a||\alpha||^2 + b(\alpha,\beta)}{||\alpha||} - \frac{a(\alpha,\beta) + h\,|\beta||^7}{||\beta||}$$

This equality implies that

(6) $$\frac{b}{a} = \frac{||\alpha||\,(\alpha,\beta) - ||\alpha||^2||\beta||}{||\beta||\,(\alpha,\beta) - ||\alpha||\,||\beta||^2} = \frac{||\alpha||}{||\beta||}$$

By (5) ve obtain

(7) $$\frac{||\delta - \alpha||}{||\delta - \beta||} = \frac{||\alpha||}{||\beta||}$$

4. The length of the bisector: Let 6 be the same as in 53, i.e.,

$$\delta = a\alpha + b\beta, \quad a + b = 1, \quad a > 0, \quad b > 0 \text{ and}$$

$$\frac{(\delta,\alpha)}{||\alpha||} = \frac{(\delta,\beta)}{||\beta||}$$

We note that

$$||\delta||^2 = a^2||\alpha||^2 + 2ab(\alpha,\beta) + b^2||\beta||^2.$$

He can write

$$\frac{1}{a^2}||\delta||^2 = ||\delta||^2 + 2\frac{b}{a}(\alpha,\beta) + \frac{b^2}{a^2}||\beta||^2 .$$

Using (6) we get

(8) $$\frac{1}{a^2}||\delta||^2 = 2||\alpha||^2 + 2\frac{||\alpha||}{||\beta||}(\alpha,\beta) .$$

Similarly we obtain

(9) $$\frac{1}{b^2}||\delta||^2 = 2||\beta||^2 + 2\frac{||\beta||}{||\alpha||}(\alpha,\beta) .$$

We compute a and b, we get

$$a = ||\delta||\sqrt{\frac{||\beta||}{2||\alpha||[||\alpha||\;||\beta|| + (\alpha,\beta)]}}$$

$$b = ||\delta||\sqrt{\frac{||\alpha||}{2||\beta||[||\alpha||\;||\beta|| + (\alpha,\beta)]}}$$

Thus

$$a + b = 1 = \frac{||\delta||}{\sqrt{2}}\sqrt{\frac{1}{||\alpha||\;||\beta|| + (\alpha,\beta)}}\left(\frac{||\alpha|| + ||\beta||}{\sqrt{||\beta||\;||\beta||}}\right)$$

Therefore we get

(10) $$||\delta||^2 = \frac{2||\alpha||\;||\beta||[||\alpha||\;||\beta|| + (\alpha,\beta)]}{(||\alpha|| + ||\beta||)^2}$$

Now we write this formula in terms of sides and angles. Let A and B respectively correspond to the end points of a and β. Thus C is the sane as the origin (Fig. 8). Then we observe that

$$||\delta|| = v_c, \quad ||\alpha|| = b, \quad 11811 = a,$$

and

$$(\alpha,\beta) = ab\cos C.$$

Thus

$$v_c^2 = \frac{2ab[ab(1 + \cos C)]}{(a+b)^2} = \frac{4a^2b^2\cos^2\frac{C}{2}}{(a+b)^2}$$

Since $C < \pi$, $\cos\frac{C}{2} > 0$. Therefore we obtain

$$v_c = \frac{2ab\cos\frac{C}{2}}{a+b} \tag{11}$$

This formula contains an angle. We shall obtain a formula in terms of sides of the triangle. It is clear that

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 - 2(\alpha,\beta) .$$

Thus we obtain

$$2(\alpha,\beta) = \|\alpha\|^2 + \|\beta\|^2 - \|\alpha - \beta\|^2$$

Substituting in (10) we get:

$$\|\delta\|^2 = \frac{\|\alpha\|\,\|\beta\|\left[(\|\alpha\| + \|\beta\|)^2 - \|\alpha - \beta\|^2\right]}{(\|\alpha\| + \|\beta\|)^2} \tag{12}$$

This can be written as

$$v_c^2 = \frac{ab[(a+b)^2 - c^2]}{(a+b)^2} .$$

5. Equal bisectors: The triangle for which two angle bisectors are equal is isosceles. To prove this we set, for example, $v_b^2 = v_c^2$. This amounts to

$$(b-c)\,\{[bc/(c+a)^2\,(a+b)^2]\,[a^2+2a(b+c)+(b^2+bc+c^2)+1]\} = 0.$$

Since the interior of braces is positive, we get

$$b - c = 0 \text{ or } b = c$$

We leave the details of the algebra to the reader.

REFERENCE

[1] A. R. Amir-Moez, Matrix Techniques, Trigonometry, and Analytic Geometry, Edwards Brother, Inc., Ann Arbor, Michigan (1964).

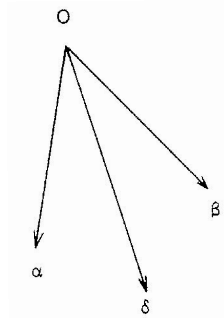


Fig. 1

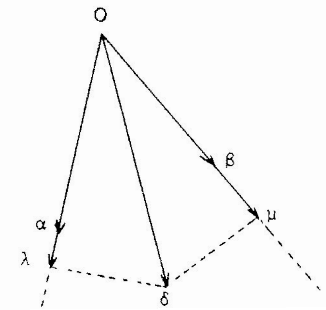


Fig. 2

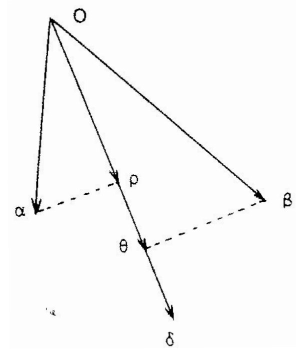


Fig. 3

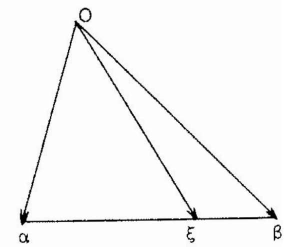


Fig. 4

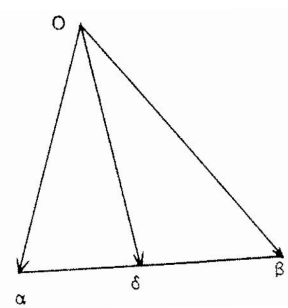


Fig. 5

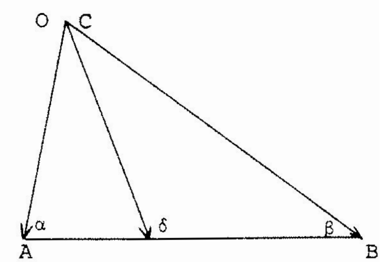


Fig. 6

PROBLEM DEPARTMENT

Edited by

Leon Bankoff, Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems characterized by novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if available, and by any information that will assist the editor. Contributors of proposals and solutions are requested to enclose self-addressed postcards to facilitate acknowledgements.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before November 1, 1972.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

270. Proposed by Leonard Carlitz, Puke University.

Let α, β, γ, denote the angles of a triangle. Show that

$$\cot\tfrac{1}{2}\alpha + \cot\tfrac{1}{2}\beta + \cot\tfrac{1}{2}\gamma \geq 3(\tan\tfrac{1}{2}\alpha + \tan\tfrac{1}{2}\beta + \tan\tfrac{1}{2}\gamma) \geq 2(\sin\alpha + \sin\beta + \sin\gamma).$$

271. Proposed by Solomon W. Golomb, California Institute of Technology and the University of Southern California.

Assume that birthdays are uniformly distributed throughout the year. In a group of n people selected at random, what is the probability that all have their birthdays within a half-year interval? (This half-year interval is allowed to start on any day of the year, in attempting to fit all n birthdays into such an interval.)

7 Proposed by Charles W. Trigg, San Diego, California.

A timely cryptarithm is the calendar verity

$$7(\text{DAY}) = \text{WEEK}$$

The letters in some order represent consecutive positive digits What are they?

273. Proposed by Charles W. Trigg, San Diego, California.

Twelve toothpicks can be arranged to form four congruent equilateral triangles, Rearrange the toothpicks to form ten triangles of the same size.

274. Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.

Find the value of $\displaystyle\sum_{i=1}^{\infty}\left(\frac{\sum_{j=1}^{k}\binom{k}{j} i^{k-j}}{(i^k)(i+1)^k}\right)$ for an arbitrary integer $k \geq 1$.

275. Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

If $t(n) = \frac{n}{2}(n+1)$, show that there are an infinite number of solutions in positive integers of

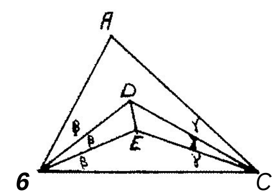
$$\sum_{i=0}^{r-1} t(a+i) = \sum_{i=0}^{s-1} t(a+r+i).$$

276. Proposed by R. S. Luthar, University of Wisconsin, Waukesha.

Find a such that the roots of $z^3 + (2+a)z^2 - az - 2a + 4 = 0$ lie along the line $y = x$.

277. Proposed (without solution) by Alfred E. Neuman, Mu Alpha Delta Fraternity. N.Y.

According to Morley's Theorem, the intersections of the adjacent internal angle trisectors of a triangle are the vertices of an equilateral triangle. If the configuration is modified so that the trisectors of one of the angles are omitted, as shown in the diagram, show that the connector DE of the two intersections bisects the angle BDC.



278. Proposed by Paul Erdős, University of Waterloo, Ontario, Canada.

Prove that every integer $\leq n!$ is the sum of $< n$ distinct divisors of $n!$. Try to improve the result for large n; for example, let $f(n)$ be the smallest integer so that every integer $\leq n!$ is the sum of $f(n)$ or fewer distinct divisors of n. We know $f(n) < n$. Prove $n - f(n) \to \infty$

279. Proposed by Stanley Robinowitz, Polytechnic Institute of Brooklyn.

Let $F_0, F_1, F_2, \ldots$ be a sequence such that for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{k=0}^{n}\binom{n}{k} F_k = F_{2n}.$$

280. Proposed by Kenneth Rosen, University of Michigan.

Find all solutions in integers of the Diophantine equation

$$x^3 + 17x^2y + 73xy^2 + 15y^3 + x^3y^3 = 10{,}000.$$

SOLUTIONS

248. [Spring 19711. Proposed by R. Luthar, University of Wisconsin, Waukesha.

For any positive integer n, prove that the following inequality holds:

$$n(n+1)^{n(n+1)} \geq \frac{2^{n(n+1)} \cdot (n!)^n}{\prod_{k=2}^{n-1} k!}$$

Solution by N.J. Kuenzi and Bob Prielipp, Wisconsin State University, Oshkosh.

The given inequality can be greatly improved. In fact, for each positive integer n.

$$[n(n+1)]^{n(n+1)} \geq (2n)^{n(n+1)}(n!)^{n+1}$$

with equality only when n = 1.

If n = 1, $[n(n+1)]^{n(n+1)} = (2n)^{n(n+1)}(n!)^{n+1}$. In what follows, we shall assume that $n \geq 2$. Using the arithmetic mean-geometric mean inequality (see Beckenbach and Bellman, An Introduction to Inequalities, New Mathematical Library, Random House, 1961, pp 54-59 for a detailed proof of this result), we have

$$(1+2+3+\dots+n)/n \geq (1 \cdot 2 \cdot 3 \dots n)^{1/n}.$$

Since $1+2+3+\dots+n = n(n+1)/2$, it follows that $[n(n+1)]^n \geq (2n)^n n!$. Thus $[n(n+1)]^{n(n+1)} \geq (2n)^{n(n+1)}(n!)^{n+1}$

Also solved by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.; C. B. A. Peck, State College, Pennsylvania; and the proposer.

249. [Spring 19711 Proposed by R. S. Luthar, University of Wisconsin Waukesha.

Prove that

$$p \mid (a+b) \Longleftrightarrow p^{m+1} \mid (a^{p^m} + b^{p^m}),$$

where p is an odd prime and m is a non-negative integer.

Solution by David Ballew, South Dakota School of Mines and Technology, Rapid City, South Dakota.

If $p^{m+1} \mid (a^{p^m} + b^{p^m})$ is true for? all non-negative m, then it is true for m = 1 and we have $p^2 \mid (a^p + b^p)$. By Format's Theorem, $a^p \equiv a(\text{mod } p)$ and $b^p \equiv b(\text{mod } p)$ so $a^p + b^p = a + b + xp$. Then we must have $p \mid (a+b)$.

Conversely assume that $p \mid (a+b)$. First we notice that $p^{m+1} \mid (a+b)^{p^{m+1}}$ and since

$$(a+b)^{p^{m+1}} = a^{p^{m+1}} + \binom{p^{m+1}}{1} a^{p^{m+1}-1} b + \dots + b^{p^{m+1}},$$

we have $p^{m+1} \mid (a^{p^{m+1}} + b^{p^{m+1}})$. Again by Fermat's Theorem $a^p = a + up$

and $b^p = b + vp$, so $a^{p^2} = a^p + \binom{p}{1} a^{p-1} up + \dots + (up)^p$ and $a^{p^2} \equiv a^p (\text{mod } p^2)$. By induction $a^{p^{m+1}} \equiv a^{p^m} (\text{mod } p^{m+1})$ and $b^{p^{m+1}} \equiv b^{p^m} (\text{mod } p^{m+1})$. Thus $a^{p^{m+1}} + b^{p^{m+1}} \equiv a^{p^m} + b^{p^m} (\text{mod } p^{m+1})$, so $p^{m+1} \mid (a^{p^m} + b^{p^m})$.

Also solved by Bob Prielipp, Wisconsin State University, Oshkosh, and the proposer.

250. [Spring 1971] Provosed by Charles W. Trigg, San Diego, Calif.

Identify the three mathematical terms represented by the following items:

(a) Bass made five yards over his own right tackle. Just as he was being tackled he tossed the ball back to Gabriel, who immediately flipped it back to Casey. After advancing ten yards, Casey threw the pigskin back to Mason, who lobbed it back to Bass, who continued on to a touchdown.

(b) As I was going up the stair
I met a man who wasn't there.
He wasn't there again today.
I wish, I wish he'd go away.

(c) Yukon Jake's tale was characteristically long, detailed, and profane: "At noon I found that a *** bear had discovered my cache and destroyed all the *** supplies. I was *** hungry and the nearest food was ten *** miles away, so I got the *** out of there fast. When I got to the *** cabin it was almost dark and I was *** tired. Them *** beans tasted *** good."

Solution by the Proposer.

(a) Complete quadrilateral, (b) imaginary number, (c) ellipses.

251. [Spring 1971] Proposed by Charles W. Trigg, San Diego, Calif.

If r_1, r_2, r are roots of $x^3 + px + q = 0$, show that

$$3 \Sigma r_i^2 \Sigma r_i^5 = 5 \Sigma r_i^3 \Sigma r_i^4$$

I. Solution by Sid Spital, California State College Hayward.

Since the given cubic is reduced (x^2 term missing), it easily follows that $\Sigma r_i = 0$ and $\Sigma r_i^2 = -2p$. Use of $r_i^3 = -pr_i - q$ then yields $\Sigma r_i^3 = -3q$, $\Sigma r_i^4 = 2p^2$, and $\Sigma r_i^5 = 3pq + 2pq = 5pq$. Hence $3 \Sigma r_i^2 \Sigma r_i^5 = -30p^2 q = 5 \Sigma r_i^3 \Sigma r_i^4$.

II. Solution by Proposer.

By Newton's Theorem,

$$\frac{x f'(x)}{f(x)} = n + \frac{\Sigma r_i}{x} + \frac{\Sigma r_i^2}{x^2} + \frac{\Sigma r_i^3}{x^3} + \dots$$

where n is the degree of the equation.

In the case under consideration,

$$\frac{x f'(x)}{f(x)} = \frac{3x^3 + px}{x^3 + px + q} = 3 + \frac{0}{x} - \frac{2p}{x^2} - \frac{3q}{x^3} + \frac{2p^2}{x^4} + \frac{5pq}{x^5} + \dots$$

ON "ALMOST UNITARY PERFECT" NUMBERS

Sidney Graham
The University of Oklahoma

A perfect number n is an integer n with the property that $\delta(n)=2n$ where $\delta(n)$ is the sum of the divisors of n. All known perfect numbers are even, but it has not been established that no odd perfect numbers exist.

Cramer [1] defined an "almost perfect" number to be an integer n with the property that $|2-(\delta(n)/n))| < \varepsilon$ for any preassigned $\varepsilon > 0$. He showed that for any ε, there exist infinitely many odd "almost perfect" numbers. Indeed, for any real $A > 1$, there exist infinitely many integers n with the property that $\delta(n)/n$ differs from A by less than ε.

Subbarao [4] defined a unitary divisor to be a divisor d of n with the property that $(d,n/d) = 1$. He also defined n to be unitary perfect if $\delta^*(n)=2n$ where $\delta^*(n)$ is the sum of the unitary divisors of n. It can easily be shown that no odd unitary perfect numbers exist. Subbarao and his associates have shown that 6, 60, 90, and 87,360 are the only unitary perfect numbers less than 10^{19}. Although a unitary perfect number greater than 10 has been discovered, Subbarao conjectures that only finitely many unitary perfect numbers exist.

Define an "almost unitary perfect" number to be a positive integer n such that $|2-(\delta^*(n)/n)| < \varepsilon$, for arbitrary fixed $\varepsilon > 0$. This paper will give a method for constructing infinitely many "almost unitary perfect" numbers.

First I wish to establish some notational conventions, p_i shall denote the ith prime; $p_1=2$, $p_2=3$, etc. q_i shall denote an arbitrary prime with the restriction that $q_i > q_j$ if an only if $i > j$.

Of primary importance is the formula for the sum of unitary divisors [5]. If $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$, then d is a unitary divisor if and only if $d = q_1^{e_1} q_2^{e_2} \cdots q_k^{e_k}$ where $e_i = a$ or 0. $\delta^*(q_i^{a_i}) = 1 + q_i^{a_i}$ and $\delta^*(n)$ is multiplicative. Thus $\delta^*(\prod_{i=1}^{k} q_i^{a_i}) = \prod_{i=1}^{k}(1 + q_i^{a_i})$.

Define $U(n) = \delta^*(n)/n$. n is unitary perfect if and only if $U(n) = 2$. $U(q_i^{a_i}) = (q_i^{a_i} + 1)/q_i^{a_i} = 1 + 1/q_i^{a_i}$, and $U(n)$ is multiplicative.

If q_i is fixed, $U(q_i^{a_i})$ is a maximum when $a_i = 1$. If a_i is fixed, $U(q_i^{a_i})$ is a maximum when q_i is 2. Also, $U(q_i) < U(q_j)$ if and only if $q_i > q_j$, and $\lim_{q_i\to\infty} U(q_i^{a_i}) = 1 = \lim_{a_i\to\infty} U(q_i^{a_i})$.

$U(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} 1 + 1/p_i > \sum_{i=1}^{n} 1/p_i$. It is well known (e.g., [2]) that $\lim_{n\to\infty} \sum_{i=1}^{n} 1/p_i = \infty$, thus $U(n)$ is unbounded above.

Theorem 1. Given any rational $R > 1$ and any real $\varepsilon > 0$, there exist infinitely many integers n such that $|R-U(n)| < \varepsilon$.

Proof: The proof will be a method of constructing the required n's. All of the n's constructed here will be of the form:

$$\prod_{i=1}^{k} q_i,$$

but it is not necessary to restrict n in this manner.

Denote $Q_j = U(\prod_{i=1}^{1} q_i) = \prod_{i=1}^{1} 1 + 1/q_i$, and $\varepsilon_j = R - Q_j$. Since $\lim_{i\to\infty} U(p_i) = 1$, there exists some prime p_e such that $U(p_e) \le R$. Let $q_1 = p_e$. We could have have chosen q_1 to be any prime greater than p_e. The following method will be used to choose the remaining q_i's.

Case I. If p_k is the prime immediately following q_j, and if $(1 + \frac{1}{p_k}) \cdot Q_j \le R$, then let $q_{j+1} = p_k$. This process cannot be repeated indefinitely, however, for, as has already been pointed out, the infinite product $\prod_{i=1}^{\infty} 1 + \frac{1}{p_i}$ diverges. Since $Q_{j+1} > Q_j$, $\varepsilon_{j+1} < \varepsilon_j$ for every q_{j+1} chosen under this case.

Case II. If $(1 + \frac{1}{p_k}) \cdot Q_j > R$ for the prime p_k immediately following q_j, then let $Q_1 \cdot (1 + \frac{1}{b_j}) = R$. b_j need not necessarily be integral, and it can be determined by the formula $b_j = Q_j/(R - Q_j)$. One form of Bertrand's Postulate states that for any real $x > 1$, there exists a prime p such that $x < p < 2x$. [3] Let q_{j+1} be a prime satisfying the condition: $b_j \le q_{j+1}$ $2b_j$. Then $\varepsilon_j = R - Q_j = Q_j \cdot \frac{1}{b_j}$, and $\varepsilon_{j+1} = R - Q_{j+1} = R - Q_j(1 + \frac{1}{q_{j+1}}) = Q_j(\frac{1}{b_j} - \frac{1}{q_{j+1}})$.

Since $\frac{1}{q_{j+1}} > \frac{1}{2b_j}$, we have $Q_j(\frac{1}{b_j} - \frac{1}{q_{j+1}}) < Q_j(\frac{1}{2b_j})$, or $\varepsilon_{j+1} < \frac{1}{2}\varepsilon_j$.

Let q_{s_1} represent the first prime chosen by the method of Case II. $\varepsilon_{s_1} < \frac{1}{2}\varepsilon_{s_1-1} < \frac{1}{2}\varepsilon_0$. Because of the divergence of the product $\prod_{i=1}^{\infty} 1 + \frac{1}{p_i}$, Case II must be continually re-utilized. After t applications of Case II, $\varepsilon_{s_t} < \frac{1}{2^t}\varepsilon_0$, and for t sufficiently large, $\varepsilon_{s_t} < \varepsilon$, for any preassigned positive ε. Hence, $\varepsilon_{s_t} = R - Q_{s_t} = R - U(\prod_{i=1} q_i)$. Futhermore, the above

Now $3(-2p)(5pq) = -30p^2q = 5(-3q)(2p^2)$.

So $3\,\Sigma\, r_i^2\, \Sigma\, r_i^5 = 5\,\Sigma\, r_i^3\, \Sigma\, r_i^4$, as was to be proved.

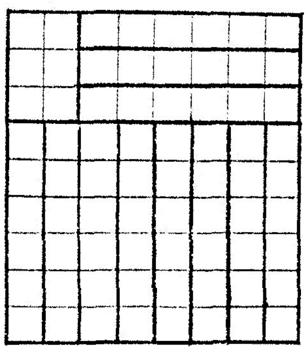
Also solved by Michael Mikolajczyk, New York Iota, Polytechnic Institute of Brooklyn; Joseph O'Rourke, Saint Joseph's College. Pennsylvania; Bob Prielipp, Wisconsin State University, Oshkosh; Kenneth Rosen and Jonathan Glauser (jointly) of the University of Michigan; and Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

252. [Spring 1971] Proposed by Solomon W. Golomb, University of Southern California.

There are 97 places where a 2 x 3 rectangle can be put on an 8 x 9 board. In how many of these cases can the rest of the board be covered with eleven 1 x 6 rectangles (straight hexominoes) and where are these locations?

I. Solution by the Proposer.

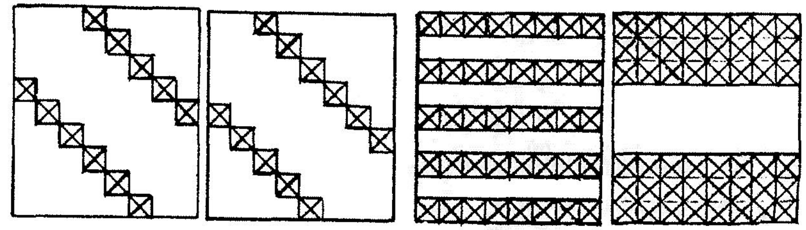
1	2	3	4	5	6	1	2
2	3	4	5	6	1	2	3
4	5	6	1	2	3	4	5
3	4	5	6	1	2	3	4
5	6	1	2	3	4	5	6
6	1	2	3	4	5	6	1
1	2	3	4	5	6	1	2
2	3	4	5	6	1	2	3
4	5	6	1	2	3	4	5



By Divine Inspiration, we introduce the coloring (numbering) of the 8 x 9 board as shown. We observe that a straight hexomino placed anywhere on the board must cover one equare of each color. Removing eleven squares of each color, we find that the left-over squares have the colors 1, 2, 3, 4, 5. Examing all 97 locations for the 2 x 3 rectangle, we discover that only the four corners, in the orientation indicated, are possible positionings. To verify that the corner location succeeds, we exhibit the "flag pattern" as shown in the second figure.

Also solved by Catherine Yee, Ohio State University. Miss Yee's solution is based on the observation that the 97 places available for the 2 x 3 rectangle can be reduced to 28 basic positions by taking into account reflections about the horizontal and the vertical axes of the board. Twenty-seven of these basic positions are then systematically eliminated from consideration by showing conflict with all possible placement of the eleven hexominoes.

The four patterns used in the elimination procedure are shown here. In each of the two diagonal patterns, a straight hexomino will cover only one black square. Since no black squares now remain for the 2 x 3 block, twenty-one of the twenty-eight basic positions are eliminated. In the striped patterns, the number of black squares covered by a straight hexomino is either 0, 3, or 6, with the result that the total number of squares covered by straight hexominoes is a multiple of 3. Thus five more of the 28 basic positions are eliminated in the narrow-striped pattern, while the wide-striped pattern eliminates still another position. The remaining corner placement, with the long edge of the 2 x 3 block parallel to the long edge of the board constitutes the only solution. If we add the three reflections we find that the four corner positions are the only ones to survive the elimination process.



253. [Spring 1971] Proposed by Erwin Just, Bronx Community College of the City University of New York.

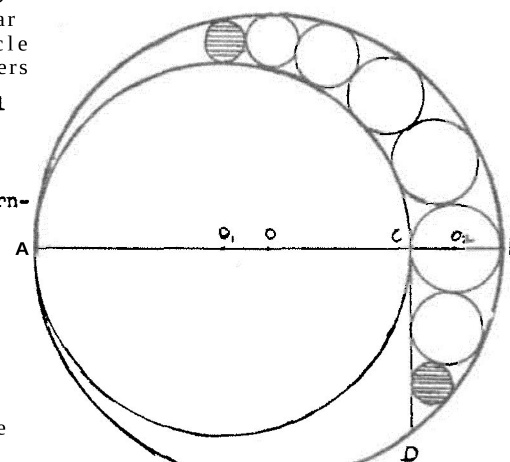
If $P(x)$ is an irreducible polynomial over the rationals and there exists a positive integer $k \neq 1$ such that r and r^k are both zeros of $P(x)$, prove that $P(x)$ is cyclotomic.

Solution by the Proposer.

Since $P(x^k)$ and $P(x)$ have a common zero, r, it must be the case that $P(x) \mid P(x^k)$, so that every zero of $P(x)$ is a zero of $P(x^k)$, from which it is easily found that $r, r^k, r^{k^2}, \ldots, r^{k^m}, \ldots$ are zeros of $P(x)$. Therefore, for some integers a and b, $r^{k^a} = r^{k^b}$ or $r^{k^a}(1 - r^{k^b - k^a}) = 0$, which implies that r is a root of unity. Since $P(x)$ is irreducible, it follows that $P(x)$ is cyclotomic.

254. [Spring 1971] Proposed by Alfred E. Neuman, Mu Alpha Delta Fraternity, New York.

In the adjoining diagram, CD is a half-chord perpendicular to the diameter AB of a circle (O). The circles on diameters AC and CB are centered on O_1 and O respectively. The rest of the figure consists of consecutively tangent circles inscribed in the horn-angle and in the segment as shown. If the two shaded circles are equal, what is the ratio of AC to AB?



Solution by the Problem Editor.

Let $AB = 2r$, $AC = 2r_1$, $CB = 2r_2$. Starting with the circle touching (O_2), the radii of the circles (ω_i) in the horn angle are denoted by ρ_i, and those of the circles (ω_i') in the half-segment by ρ_i', $(i = 1, 2, \ldots, n)$. The formula for the radii of the circles in the Pappus chain (i.e., in the horn angle) is $\rho_n = rr_1r_2/(rr_2 + n^2r_1^2)$, while the radii of the circles in the chain inscribed in the half-segment are given by

$$\rho'_n = \frac{4r_2r_1^{\,n}}{[(\sqrt{r} - \sqrt{r_2})^n + (\sqrt{r} + \sqrt{r_2})^n]^2}$$

For our purposes here, we use the simplified formulas,

$$\rho'_2 = r_2\left(\frac{r_1}{r + r_2}\right)^2 \quad \text{and} \quad \rho_5 = rr_1r_2/(rr_2 + 25r_1^{\,2})$$

Substituting $r_1 + r_2$ for r and equating ρ'_2 and ρ_5, we readily obtain $(r_1 + r_2)^2/r_1r_2 = 25/4$. Thus $25r_1r_2 = 4(r_1 + r_2)^2$. Let $r_1 = kr_2$. Then $25kr_2^{\,2} = 4r_2^{\,2}(k + 1)^2$ and $k = 4$. Hence $AC = 4(CB)$.

(Note: The solution $k = 1/4$ applies to the reflected figure, in which AC and CB are transposed. The formula for the radii of the circles in the half-segment was derived by a complicated inversion. Readers are invited to derive the expression for ρ'_2 by synthetic geometry.)

255. [Spring 1971] Proposed by C. Stanley Ogilvy, Hamilton College, Clinton New York.

Find a 3-digit number in base 9 which, when its digits are written in reverse order. yields the same number in base 7. Prove that the solution is unique.

I. Solution by Jeanette Bickley, Webster Groves Senior High School, Webster Groves, Missouri.

Below is a computer program and output from a XDS 940 computer. This program tests all possible digits (0, 1, 2, 3, 4, 5, 6 since base 7 is involved) and obtains the unique solution (other than the trivial solution): 305 in base 9 = 503 in base 7.

```
      INTEGER A,B,C
      DIMENSION A(7),B(7),C(7)
 90   FORMAT (I2,I1,I1," IN BASE 9 =",I2,I1,I1," IN BASE 7")
      DATA A/0,1,2,3,4,5,6/
      DATA B/0,1,2,3,4,5,6/
      DATA C/0,1,2,3,4,5,6/
      DO 20 I=1,7
      DO 20 J=1,7
      DO 20 K=1,7
      IF (40*A(I)+B(J)-24*C(K)) 20,30,20
 30   WRITE (1,90) A(I),B(J),C(K),C(K),B(J),A(I)
 20   CONTINUE
      STOP
      END
*XTRAN

 000 IN BASE 9 = 000 IN BASE 7
 305 IN BASE 9 = 503 IN BASE 7
*STOP*
```

II. Solution by Edward G. Gibson, Xavier University, Cincinnati.

Let the 3-digit number be ABC.

Thus $81A + 9B + C = 49C + 7B + A$
$2B = 48C - 80A$
$B = 8(3C - 5A)$.

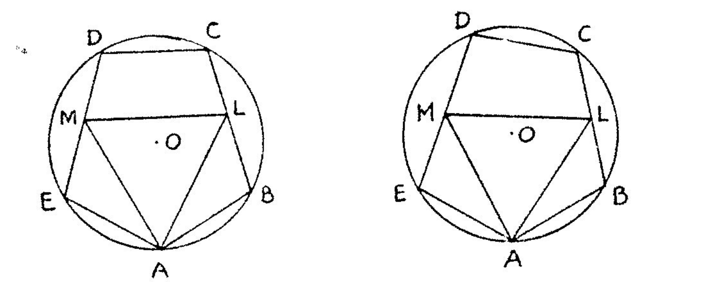
Since $B < 7$, $B = 0$. Hence $3C = 5A$.
Since $C < 7$, $C = 5$ and $A = 3$, a unique solution.
Hence the unique solution $(305)_9 = (503)_7$.

Note: This problem appears as Problem 93 on page 304 of Beiler's Recreations in the Theory of Numbers, Dover Publications, New York.

Also solved by Richard Ball, Portland State University, Portland, Oregon; S. Gendler, Clarion State College, Clarion, Pennsylvania; Marilyn Hoag, Lake-Sumter Community College, Leesburg, Florida; Carol Lancaster, St. Lawrence University, Canton, N.Y. ; Larry E Miller, Riverside, California; James R. Metz, St. Louis University; Bob Prielipp, Wisconsin State University, Oshkosh; Kenneth Rosen, University of Michigan; S. Swetharanyam, McNeese State University, Lake Charles, Louisiana; Charles W. Trigg, San Diego, California; and Gregory Wulczyn, Bucknell University.

256. [Spring 1971] Proposed by R. S. Luthar, university of Wisconsin, Janesville.

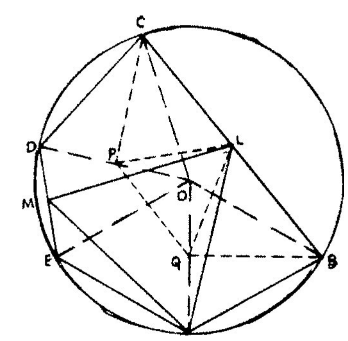
ABCDE is a pentagon inscribed in a circle (O) with sides AB, CD and EA equal to the radius of (O). The midpoints of BC and DE are denoted by L and M respectively. Prove that ALM is an equilateral triangle.



Solution by Charles W. Trigg, San Diego, California.

Let P and Q be the midpoints of the radii OD and OA, respectively. Then CP and BQ are equal altitutes of the congruent equilateral triangles COD and BOA.

OL is the perpendicular bisector of the base of the isosceles triangle COB. Consequently, CL = BL and $\angle OCL = \angle OBL$. Then since $\angle PCO = 30° = \angle QBP$, tirangles PCL and QBL are congruent, and PL = QL.



Two opposite angles in each of the quadrilaterals OLBQ and OLPC are right angles, so the quadrilaterals are inscriptible. Hence
$\angle OLP = \angle OCP = 30°$ and
$\angle OLQ = \angle OBQ = 30°$.
It follows that $\angle PLQ = 60°$ and triangle PLQ is equilateral.

MP is parallel to EO, and so makes an angle of 60° with AQ. PL makes an angle of 60° with QL. Hence $\angle MPL = \angle AQL$. Then MP = EO/2 = AO/2 = AQ and PL = QL, so triangles MPL and AQL are congruent.
Hence, ML = AL and $\angle PLM = \angle QLA$. Thus, $60° = \angle PLQ = \angle PLQ - \angle PIM + \angle QLA = \angle MLA$. Therefore, triangle ALM is equilateral.

Editor's Note:
The stated problem does not require DC to be parallel to EB although the diagram inadvertently creates the impression that it is. Consequently, it was necessary to reject several solutions stemming from this misleading hypothesis. If DC and EB are parallel, the problem is considerably simplified and lends itself to an easy synthetic solution. One such solution, offered by Alfred E. Neuman, Mu Alpha Delta Fraternity, New York, notes that the sum of the angles DOC, AOE and BOA is 180° with the result that EOD and COB are isosceles right triangles. It follows that OL = LB = OM = ME and that triangles OMA, OLA, EMA, and BLA are congruent. Since MA and LA are bisectors of the angles EAO and OAB, the equal segments MA and AL have a mutual inclination of 60°, thus making triangle HAL equilateral.

Samuel L. Greitzer, Rutgers University offered a synthetic solution for ED = CB and called attention to the fact that this problem is a special case of Problem B-1 of the William Lowell Putnam Mathematical Competition held on December 2, 1967. (See The American Mathematical Monthly, Aug.-Sept. 1968 pp 732-739. The more general problem reads as follows;

Let (ABCDEF) be a hexagon inscribed in a circle of radius r. Show that if AB = CD = EF = r, then the midpoints of BC, DE, FA are the vertices of an equlateral triangle. (This problem and its solution also appear in A Survey of Geometry, Howard Eves, p.184, Vol. 2., Allyn and Bacon, 1965.)

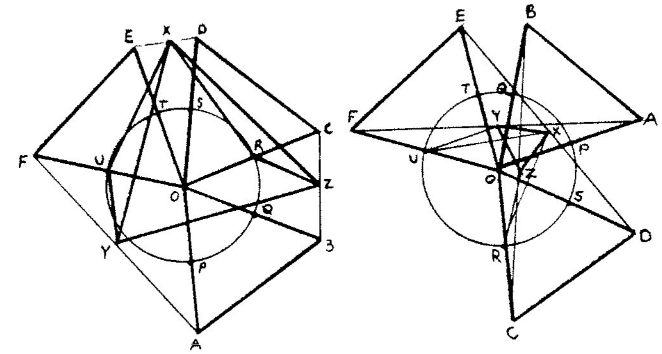
In the special case of Problem 256, the vertices F and A coincide with the "midpoint" of FA. The various methods of solution of the general version of the problem are, of course, applicable here. Despite the elegance of the solution by the use of complex numbers, a solution by synthetic, Euclidean, high-school geometry may be of interest.

Let X, Y, Z, P, Q, R, S, T, U denote the midpoints of DE, FA, BC, OA, OB, OC, OD, OE, and OF, respectively.

In the congruent triangles EUX, DRX, we have UX = RX. Since XORD is a cyclic quadrilateral, $\angle RXO = \angle RDO = 30°$. So $\angle RXU = 60°$ and triangle URX is euqilateral.

Since UY is parallel to OA and equal to OA/2, and since RZ is parallel to OB and equal to OB/2, we have UY = RZ and $\angle(UY,RZ) = 60°$. By a rotation of 60° about X, triangle XUY may be made to coincide with triangle XRZ. So XY = XZ and $\angle YXA = 60°$. Hence triangle XYZ is equilateral.

Also solved (analytically) by Lew Kowarski, Morgan State College, Baltimore, Maryland and by the proposer. Both solvers used a rectangular coordinate system with O as origin and with A lying on the Y-axis. Letting the radius of the circle equal unity, the coordinates of the points are: A(0,-1); B($\sqrt{3}/2$, -1/2); C(cos a, sin a); D (cos (60°+ a), sin (60° + a)); E ($-\sqrt{3}/2$, -1/2). The coordinates of M and L are now easily found and the distance formula yields the solution $AL = LM = MA = \frac{1}{4}\sqrt{16 + 12 \sin \alpha + 4\sqrt{3} \cos a}$.



257. [Spring 1971] Proposed by Mike Louder and Richard Field, Los Angeles, California.

If x, y, z are the sides of a primitive Pythaporean triangle with z > x > y, can x ana (x - y) be the lees of another Pythagorean triangle?

Solution by Charles W. Trigg, San Diego, California.
The two legs of every primitive Pythaporean triangle have the forms $m^2 - n^2$ and 2mn, where m and n are relatively prime and have different parities. The hypotenuse, $z = m^2 + n^2$. Hence one leg is even and the other two sides are odd.
In non-primitive triangles, both legs may be even, but both may not be odd.

First case. $x = m^2 - n^2$, $y = 2mn$, $x - y = m^2 - n^2 - 2mn$.
Since x and x - y are both odd, they cannot be the legs of another Pythagorean triangle. This is confirmed by the identity

$$(m^2 - n^2)^2 + (m^2 - n^2 - 2mn)^2 = 2(m^4 - 2m^3n - 2mn^3 + n^4).$$

The quantity in the parentheses on the right is odd, so the entire expression cannot be the square of a hypotenuse.

Second case. $x = 2mn$, $y = m^2 - n^2$, $x - y = 2mn - m^2 + n^2$.
If x and x - y, which are relatively prime, are to be legs of a Pythagorean trianfle, it must be primitive. Then the odd x - y will have to have the form $p^2 - q^2$, and the factors of x must be regroupable into 2pq, with p and q relatively prime and of opposite parity.
Furthermore, $p^2 + q^2 < m^2 + n^2$. Also, $2mn - m^2 + n^2 > 0$.
That is, $(m + n)^2 > 2m^2$, so $m > n > m(\sqrt{2} - 1)$.
If m = ab and n = cd, the factors of mn may be regrouped in four basic wavs:

A. p = mn, q = 1.

Now $(m^2 - 1)(n^2 - 1) \geq 0$,

so $m^2n^2 + 1 \geq m^2 + n^2$. Hence, this regrouping is impossible.

B. p = mc, q = d > 1, c > 1. Thus m > n > d, so

$(m^2 - d^2)(c^2 - 1) > 0$, whereupon

$m^2c^2 + d^2 > m^2 + c^2d^2 = m^2 + n^2$. Therefore, this regrouping is impossible.

C. p = a > q = bn, b > 1. Then

$a^2 - b^2n^2 = 2abn - a^2b^2 + n^2$

$n^2(b^2 + 1) + 2abn - a^2(b^2 + 1) = 0.$

$n = a(-b \pm \sqrt{(b^2 + 1)^2 + b^2})/(b^2 + 1).$

Now $n > ab(\sqrt{2} - 1)$, so

$-b \pm \sqrt{(b^2 + 1)^2 + b^2} > (b^2 + 1)b(\sqrt{2} - 1) = (b^3 + b)\sqrt{2} - b^3 - b$

$b^4 + 3b^2 + 1 > b^6(3 - 2\sqrt{2}) + 2b^4(a - \sqrt{2}) + 2b^2$

$b^2 + 1 > (b^6 + b^4)(3 - 26)$

$1 > b^4(3 - 2/51 = b^4(0.1716)$

This inequality clearly does not hold for integer values of $b > 1$.

Otherwise. Since n is an integer, $(b^2 + 1)^2 + b^2 = x^2$. Then $4b^4 + 12b^2 + 9 = 4x^2 + 5$.

Let $2b^2 + 3 = z$ and $2x = y$.
Then $(z - y)(z + y) = 5$.
Solving $z + y = 5$ and $z - y = 1$ simultaneously, $y = 2$ and $z = 3 = 2b^2 + 3$. Whereupon $b = 0$, contrary to the hypothesis.

D. $p = ac$, $q = bd = bn/c$, where a, b, c, d are relatively prime integers, and each is greater than 1. Then

$a^2c^2 - b^2n^2/c^2 = 2abn - a^2b^2 + n^2$

$(b^2 + c^2)n^2 + 2abc^2n - a^2c^2(b^2 + c^2) = 0$

$n = ac[-bc \pm \sqrt{b^2c^2 + (b^2 + c^2)}\,]/(b^2 + c^2).$

Necessarily, $b^2c^2 + (b^2 + c^2)^2 = x^2$

$4b^4 + 12b^2c^2 + 9c^4 = 4x^2 + 5c^4$

Let $2b^2 + 3c^2 = z$ and $2x = y$, then

$(z + y)(z - y) = 5c^4.$

The factors on opposite sides of this equation may be matched in six ways;

I. $z + y = 5c^4$ and $z - y = 1$. Simultaneous solution gives

$z = (5c^4 + 1)/2 = 2b^2 + 3c^2$, so

$b^2 = (5c^4 - 6c^2 + 1)/4 = (5c^2 - 1)(c^2 - 1)/4.$

If b^2 is to be an integer, $c = 4k + 1$ or $c = 4k + 3$.
For $c = 4k + 1$,

$b^2 = 80k^2(k + 1)^2$ and $b = 4k(k + 1)\sqrt{5}$, which is not an integer.
For $c = 4k + 3$,

$b^2 = 8(20k^2 + 30k + 11)(k^2 + 3k + 1) = 8$(an odd integer), which is not the square of an integer.

II. $z + y = 5c^3$ and $z - y = c$, whereupon

$z = (5c^3 + c)/2 = 2b^2 + 3c^2$. Hence,

$b^2 = c(5c - 1)(c - 1)/4.$

If $c = 4$, $b^2 = 57$, which is not a square. Otherwise, c and b have a common factor, contrary to the hypothesis.

III. $z + y = 5c^2$ and $z - y = c^2$. Then

$z = 3c^2 = 2b^2 + 3c^2$. Hence, $b = 0$, contrary to the hypothesis.

IV. $z + y = c^4$ and $z - y = 5$. Then

$z = (c^4 + 5)/2 = 2b^2 + 3c^2$, so

$b^2 = (c^2 - 1)(c^2 - 5)/4$. If b^2 is to be an integer then $c = 4k + 1$ or $4k + 3$.

If $c = 4k + 1$, $b^2 = 4(2k)(2k + 1)(4k^2 + 2k - 1)$, but the three quantities in parentheses are relatively prime, so the product cannot be a square integer.

If $c = 4k + 3$, $b^2 = 8(2k^2 + 3k + 1)(4k^2 + 6k + 1) = 8$ (an odd number), which cannot be a square integer.

V. $z + y = c^3$ and $z - y = 5c$, $c > 2$. Hence

$z = c(c^2 + 5)/2 = 2b^2 + 3c^2$. Consequently, $b^2 = c(c - 5)(c - 1)/4$.

If $c = 4$, $b^2 = -3$. Otherwise, b and c have a common factor, contrary to the hypothesis.

VI. $z + y = 5c$ and $z - y = c^3$, $c = 2$. Hence $2x = y = 1$ and $x = 1/2$, which is not an integer.

Therefore, in no case can x and $x - y$ be legs of a Pythagorean triangle.

Editor's Note: Mr. Charles W. Trigg was kind enough to point out the following errata in the problem department of the Fall 1971 issue of the Pi Mu Epsilon Journal.

Page 241 - In proposal 258, "verticle" should read "vertical".

Page 243 - The symbol "h", representing the segment ON, has been left out of Figure 2, which should be rotated counterclockwise so that GE becomes the X-axis and HH becomes the Y-axis.

Page 244 - Twenty-first line from the bottom - "plaindrome" should read 'palindrome".
- Seventh line from the bottom should read $[\sqrt{2N}][\sqrt{2N} + 1] = 2N$.

Page 246 - Line 11 - "figure 1" should read "figure 3".
- Line 16 should read $\angle AEC = \angle DAC$ (1).
- Fourth line from the bottom - "synthetic" should not have been capitalized.
- In Figure 4, the "I" should read "1".

Page 247 - Line 7 - The "t" of "triangles" should not have been capitalized.
- Line 8 - "porportional" should read "proportional".
- Line 8 - $\angle MPR = \angle QMP$ should read $\angle MRP = \angle QMP$.

Page 248 - Line 10 - This and the following line are editorial comments and are not part of the submitted solution,

Page 249 - Line 3 - "Proposer" has been misspelled.
Line 4 - "Proposed" was misspelled.

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