PI MU EPSILON JOURNAL

VOLUME 5 FALL 1973 NUMBER 9

CONTENTS

PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

David C. Kay, Editor

ASSOCIATE EDITORS

Roy B. Deal Leon Bankoff

OFFICERS OF THE FRATERNITY

President: H. T. Karnes, Louisiana State University

Vice-President: E. Allan Davis, University of Utah

Secretary-Treasurer: R. V. Andree, University of Oklahoma

Past-President: J. C. Eaves, University of West Virginia

COUNCILORS:

E. Maurice Beesley, University of Nevada

Gloria C. Hewitt, University of Montana

Dale W. Lick, Russell Sage College

Eileen L. Poiani, St. Peter's College

Chapter reports, books for review, problems for solution and solutions to problems, should be mailed directly to the special editors found in this issue under the various sections. Editorial correspondence, including manuscripts and news items should be mailed to THE EDITOR OF THE PI MU EPSILON JOURNAL, 601 Elm, Room 423, The University of Oklahoma, Norman, Oklahoma 73069.

For manuscripts, authors are requested to identify themselves as to their class or year if they are undergraduates or graduates, and the college or university they are attending, and as to position if they are faculty members or in a non-academic profession.

PI MU EPSILON JOURNAL is published semi-annually at The University of Oklahoma.

SUBSCRIPTION PRICE: To individual members, $4.00 for 2 years; to non-members and libraries, $6.00 for 2 years; all back numbers, $6.00 for 4 issues, $2.00 per single issue; Subscriptions, orders for back numbers and correspondence concerning subscriptions and advertising should be addressed to the PI MU EPSILON JOURNAL, 601 Elm Avenue, Room 423, The University of Oklahoma, Norman, Oklahoma 73069.

AN ANNOUNCEMENT FROM THE EDITOR

We think you will agree that the subscription rates for the *Journal* have been truly modest for many years. The present rate has been in effect ever since the publication was founded, in 1949. As a result of these low rates the *Journal* has required assistance from the Fraternity to meet its costs, as fully intended. It has always been the aim of the Councilors of Pi Mu Epsilon in establishing its rates to make the privileges of membership and the subscription to the *Journal* within reach of any college or university student.

The amount of assistance from the Fraternity has now reached in excess of $1000 per year, due primarily to the ever-increasing cost of paper. With great reluctance, the Council voted at its last meeting to increase the rates of the *Journal* in order to reduce the amount of drain on the general treasury. Thus the rates which are posted in the inside cover of this issue are effective on all new subscriptions and renewals, pending approval from the Internal Revenue Service. We anticipate that these new rates will be adequate for many years to come.

FORBIDDEN AREA

By John Vow. Iwaarden
Hope College, Holland, Michigan

In 1947, Norman Anning [1] proposed the following problem:

Consider the points on the median of a triangle. Through the centroid no lines can be dream, which will out off** 1/3 **of the area. Through a point,** 4/5 **of the distance from the vertex to the base, few such lines may be drawn. Find points on the median at which the number of possible lines change.

The problem remained unsolved for some time. In 1950 V. E. Hoggatt [2] published a solution which compounded his ideas with those of C. S. Ogilvy and F. Jamison. In this, Jamison pointed out that an interesting allied problem would be to determine through what part of the area of a triangle it is impossible to draw a line which will cut off one-third of the area of the triangle. This raised the issue of a "forbidden region" explained below.

In a circle C, every diameter D cuts the circle into two regions of equal area. A chord L of the circle C, not a diameter, cuts the circle into two regions of unequal areas whose areal ratio is $k < 1$. For each k, it 'is obvious that there is a chord L and a well-defined interior circle C_k such that:

(1) Through every point in the plane outside or on the boundary of C_k, there may be drawn at least one line dividing the area of circle C into two regions with areal ratio $k < 1$.

(2) No such lines may be drawn through any interior point of C_k; this is called the ***forbidden region*** of circle C, relative to areal ratio k.

(The reader should be able to determine the circle C_k for himself.)

Hoggatt raised the issue of forbidden area in [3] and listed some results for triangles,

This paper will list some additional findings for forbidden regions

of triangles. The proofs will use only elementary ideas from geometry and calculus.

For a given triangle of area 1 with a point P in its interior, draw any line ℓ passing through P. With an orientation given to ℓ, let 6

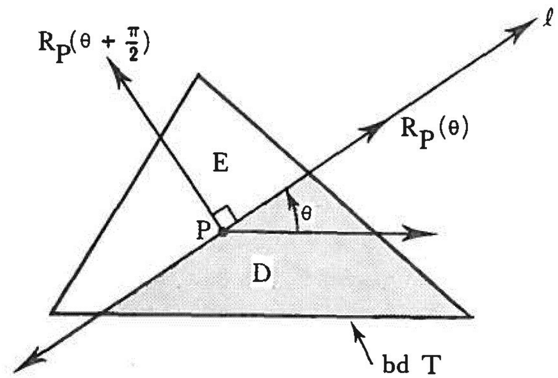


FIGURE 1

be the angle of inclination of ℓ with the horizontal, $0 \le 6 < 2\pi$. For each 6 there is a unique ray $R_P(\theta)$ on ℓ whose positive angle with the orienting ray is 6. The line ℓ then divides the triangle into two parts D and E as shown in Fig. 1, where E is the part which contains the ray $R_P(\theta + \pi/2)$. Let $A_P(\theta)$ = Area D, and $0 < k < 1/2$. Define

$$S_k = \textit{set of satisfactory points} = \{P \mid A_P(\theta) = k \text{ for some } \theta\}$$

and

$$F_k = \textit{forbidden region} = \textit{complementary set} = \{P \mid A_P(\theta) \ne k \text{ for all } \theta\}.$$

__*Theorem*__: The forbidden region is convex for $0 \le k \le 1/2$.

Some simple lemmas will be stated whose proofs should be obvious, which will then prove the theorem.

__*Lemma 1*__: $A_P(\theta + \pi) = 1 - A_P(\theta)$

__*Lemma 2*__: $A_P(\theta)$ is continuous as a function of θ and P. That is, if 6 and P are changed only slightly, then the corresponding value of

$A_P(\theta)$ *is changed only slightly.*

<u>Lemma 3</u>: *If $k = 0$ then only the boundary bd T of the triangle T lie; in S_k. Thus S_k = bd T and F_k is the interior of T. Since the interior of T is convex, the theorem is true for $k = 0$.*

<u>Lemma 4</u>: *If $k = 1/2$, $S_k = T$, F_k = empty set. Hence $F_{1/2}$ is convex*

Proof: Let $P \varepsilon T$. If $A_P(\theta) < 1/2$, look at

$$A_P(\theta + \pi) = 1 - A_P(\theta) > 1/2.$$

Thus there exists a value $\theta < \phi < \theta + \pi$ such that $A_P(\phi) = 1/2$.

Proof of Theorem: Let $0 < k < 1/2$. Let P_1 and $P_2 \ \varepsilon \ F_k$ and let point Q be on the segment joining P_1 and P_2. Suppose $Q \notin F_k$. Then there exists a line ℓ through Q intersecting segment P_1P_2 not passing through P_1 or P_2 such that $A_Q(\theta) = k$. Then P_1 or $P_2 \ \varepsilon \ D$, say P_1. Draw line ℓ' through P_1 parallel to ℓ. With the same orientation for D', $A_{P_1}(\theta') = A_{P_1}(\theta) < k$. Therefore $A_{P_1}(\theta + \pi) = 1 - A_{P_1}(\theta) > k$. But by continuity *there exists θ_k such that $A_{P_1}(\theta_k) = k$. This contradicts P_1 being a forbidden point.*

Theorem: *If the point P is the centroid of T and line ℓ is parallel to one of the sides, then $A_P(\theta) = 4/9$.*

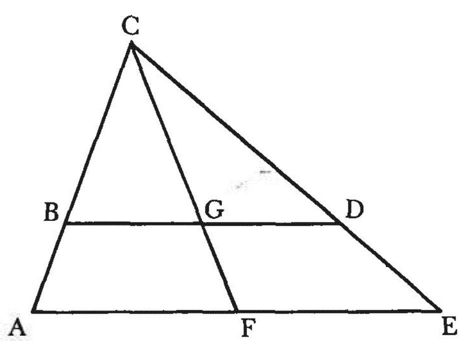


FIGURE 2

Proof: Let G be the centroid of T (see Fig. 2). According to a formula 61.1 in [4], if $x = CB/CA$, $y = CD/CE$ and the area of the triangle is 1, then Area BCD $= xy$. Since $CG/CF = 2/3 = x = y$, we have $A_G(\theta) = 4/9$. We note in general that since

$$\text{Area } BCG + \text{Area } GCD = \text{Area } BCD = xy,$$

we have $x/3 + y/3 = xy$. Thus the function $A_G(\theta)$ in x and y is subject to the restriction

$$x + y = 3xy.$$

It is easy to show by calculus that under these conditions the function

$$A_G(\theta) = xy$$

assumes its minimum precisely when $x = y$, or when ℓ is parallel to AE. Thus the value 4/9 is a minimum value for $A_G(\theta)$, and actually, $4/9 \le A_G(\theta) \le 1/2$.

Corollary: *If $k < 4/9$, then F # 4) and F is convex.*

In [3] Hoggatt noted that for $0 < k < 1/2$, the area of F is

$$A_F = \frac{1}{2}\left(3\sqrt{1-2k} - 1 - 3k \text{ loo } \left|\frac{1 - k + \sqrt{1-2k}}{2k}\right|\right)$$

It should be noted that for $k = 4/9$, $A_F = 0$ and that as $k \to 0$, $A \to 1$.

Theorem: *The forbidden region F_k is open.*

Proof: For point P in F_k, $A_P(\theta) \# k$ for all θ. But for some 8, $A_P(\theta) = 1/2$. Since $A_P(\theta)$ is continuous, there exists a neighborhood of P containing only points of the forbidden region, since a minimum must be met somewhere near P.

In the proofs of the lemmas and theorems, it should be noted that no use was made of the shape of the body except its convexity. (For a non-convex set, the convex hull satisfies the properties.) Thus, all the non-numerical *results above hold more generally for arbitrary convex sets with non-empty interiors.*

REFERENCES

1. *Anning, Norman, Elementary Problem Section, American Mathematical Monthly, 54, 1947, p. 281.*
2. *Hoggatt, Verner, Jr., Elementary Problem E-774 "Forbidden Area", American **Mathematical** Monthly, 69, 1962, pp. 98-104.*

3. Hoggatt, Verner, Jr., Solution to E-774, Elementary Problem Section, *American Mathematical Monthly*, 57, 1950, pp. 484-486.

4. Kay, D. C., *College Geometry*, Holt, Rinehart, and Winston, 1969, p. 205.

REFEREES FOR THIS ISSUE

The editorial staff sends a note of appreciation to the following persons who freely gave of their time to evaluate papers submitted for publication prior to this issue: Robert A. Stoltenberg, Sam Houston State University; Robert W. Prielipp, University of Wisconsin at Oshkosh; Bruce B. Peterson, Middlebury College; and A. Duane Porter, University of Wyoming.

The *Journal* also acknowledges with gratitude the typist for this issue, Theresa Killgore, who is a senior at the University of Oklahoma majoring in mathematics.

On an island in the Pacific lived two tribes, the Blue men and the Green men. The Blue men always told the truth and the Green men always lied. Once some men were shipwrecked on the island, and the natives agreed to release them if their captain could solve a puzzle. He was blindfolded, placed in a room with two natives, and was to guess their tribes within 5 minutes, using any clues he could get from the conversation. The first native spoke inaudibly, so the captain asked the second native what he had said. The native answered 'He said he was a Green man." The captain immediately identified the natives and his men were released. What reasoning led the captain to his conclusion?

A STEEP ASCENT OF THE FUNDAMENTAL THEOREM OF CALCULUS

by J. Michael Steele
Stanford University

In *What is Mathematics?* Courant and Robbins give a proof of Euclid's theorem on the infinity of primes in which they use the fact that the zeta function diverges at $z = 1$. Speaking of their proof they say, "Of course, this is much more involved and sophisticated than the proof given by Euclid. But it has the fascination of a difficult ascent of a mountain peak which could be reached from the other side by a comfortable road." If one borrows this attitude and the fact (which is also indicated in *What is Mathematics?*) that the integral of x^n can be calculated without using differentiation, it is possible to give a proof of the fundamental theorem of calculus which analysis classes may find amusing.

By giving $C'[a, b]$ the norm

$$\|f\| = \sup|f(x)| + \sup|f'(x)|$$

we can easily see that the linear functional defined by

$$F(f) = \int_a^b f'(t)dt - f(b) + f(a)$$

is continuous. Now, as mentioned above, we may assume the usual formulas for the integral and derivative of x^n. This allows us to check that $F(p) = 0$ if p is a polynomial. However, in view of Weierstrass's Theorem (applied to f' for a given function f in $C'[a, b]$) and the Mean Value Theorem the polynomials are dense in $C'[a, b]$. Hence we have that F must vanish identically, which is just what the fundamental theorem says.

THE POLE AND POLAR WITH RESPECT TO A QUADRIC

By ALI R. Amir-Moez
Texas Tech University

Many geometric problems can be simplified and generalized by the use of matrices. In this article we study pole and polar with respect to a quadric in an n-dimensional Euclidean space with techniques of matrices [1]. Thus whatever is done for an n-dimensional Euclidean space is not more difficult than similar ideas for the plane.

1. *Quadrics*

Let us consider the equation of a conic section with respect to an orthogonal Cartesian coordinate system (x,y). That is,

$$ax^2 + 2bxy + cy^2 + 2px + 2qy + r = 0. \tag{1}$$

We observe that (1) has the following matrix equation

$$[x \;\; y \;\; 1]\begin{bmatrix} a & b & p \\ b & c & q \\ p & q & r \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [0]. \tag{2}$$

We observe that $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is the matrix of the quadratic form of (1), namely,

$$[ax^2 + 2bxy + cy^2] \equiv [x \;\; y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The equation (2) can be written as

$$XAX' = 0, \tag{2'}$$

where $X = [x \;\; y \;\; 1]$ and A is the symmetric 3×3 matrix of (2). In general a quadric in Euclidean space of dimension n can be defined as the set of all points $(x_1, x_2, \cdots, x_n)$ such that $XAX' = 0$, where $X = [x_1 \;\; x_2 \;\; \cdots \;\; x_n \;\; 1]$ and A is any $(n+1) \times (n+1)$ symmetric matrix. If we omit the last row and last column of A, we obtain the matrix of the *quadratic* form of the quadratic. This matrix will be denoted by Q.

2. *Straight Lines*

It is clear that

$$\begin{cases} x = x_1 + pt \\ y = y_1 + qt \end{cases} \tag{3}$$

is a set of parametric equations of a straight line in the plane. A matrix equation for (3) is then

$$[x \;\; y \;\; 1] = [x_1 \;\; y_1 \;\; 1] + t[p \;\; q \;\; 0].$$

We observe that to each point (x, y) corresponds a matrix $[x \; y \; 1]$ and to each direction (p, q) corresponds a matrix $[p \; q \; 0]$.

Indeed in general the matrix equation of a straight line in a Euclidean space of dimension n can be taken as

$$X = Y + tD, \tag{3'}$$

where $X = [x_1 \cdots x_n \; 1]$, $Y = [y_1 \cdots y_n \; 1]$, and $D = [p_1 \cdots p_n \; 0]$. Here Y represents some fixed point and D represents a direction.

3. *Intersection* of a *Line* and a *Quadric*

Let $XAX' = 0$ be a quadric and $X = Y + tD$ a line in a Euclidean n-dimensional space. Then points of intersection of the line and the quadric are solutions of the system of matrix equations

$$\begin{cases} XAX' = 0 \\ X = Y + tD . \end{cases}$$

By substitution this set of equations gives us

$$(DAD')t^2 + (YAD' + DAY')t + YAY' = 0 \tag{4}$$

which is a second degree equation in t. One may easily show that $YAD' = DAY'$ and $DAD' = DQD'$, where Q is the matrix of the quadratic form of the quadric and in DQD' we have taken $D = [p_1 \cdots p_n]$. Thus (4) will become

$$(DQD')t^2 + 2(YQD')t + YAY' = 0. \tag{5}$$

(Note that, for example, DQD' is a matrix of one element which is considered as a scalar.)

A discussion of (5) about the existence and nature of roots should be given. Let it suffice for our purposes to mention only that if $DQD' \neq 0$, then the line intersects the quadric in two points which may be real and distinct, or real and double, or complex conjugates of one another. Indeed, this case (when $DQD' \neq 0$) is the one in which we are interested for studying pole and polar theory.

4. *Pole* and *Polar*

Let P be a point and $XAX' = 0$ be a quadric in an n-dimensional Euclidean space. Let any line through P intersect the quadric in two distinct points B and C, where we are considering only the case mentioned in the end of section 3. Let R be the harmonic conjugate of P with respect to B and C. Then the locus of R, as the line through P changes, is a hyperplane and is called the polar of P with respect to the quadric.

Proof: Let $(PRBC)$ be a harmonic set. Then

$$\frac{PB}{PC} = -\frac{RB}{RC}. \tag{6}$$

In order to make the idea clear we include a diagram in Euclidean three-

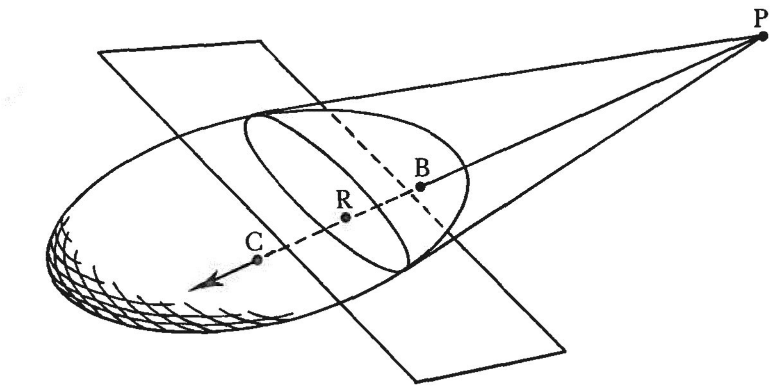


FIGURE 1

dimensional space (see Fig. 1). Let P denote the matrix $[y_1 \cdots y_n \; 1]$ which corresponds to the point P. Let a line through P have the equation

$$X = P + tD,$$

where D is a variable direction. Then points of intersection of the line and the quadric are obtained from

$$(DQD')t^2 + 2(PAD')t + PAP' = 0, \qquad DQD' \neq 0.$$

Here $t = 0$ corresponds to $X = P$. Let t_1 and t_2 correspond respectively to $X = B$ and $X = C$, and t to $X = R$. Then (6) becomes

$$\frac{0 - t_1}{0 - t_2} = -\frac{t - t_1}{t - t_2}$$

which implies that

$$t = \frac{2t_1t_2}{t_1 + t_2} = \frac{PAP'}{PAD'}$$

if $PAD' \neq 0$. Substituting this value for t in the equation of the line we obtain

$$(PAD')X = (PAD')P - (PAP')D.$$

Multiplying on the right by AP' we get

$$(PAD')(XAP') = (PAD')(PAP') - (PAP')(DAP') = 0.$$

Since $PAD' = DAP' \neq 0$ we obtain

$$XAP' = PAX' = 0,$$

which is a linear equation in $x_1, \cdots, x_n$, and thus the equation of a hyperplane.

If we look at the case $PAD' = 0$, we note that P is at a center of the quadric and the polar is said to be at infinity. We shall leave this detail to the reader.

5. *Some properties*

There are many propositions concerning pole and polar in the plane. The reader may study them and give some generalizations to an n-dimensional Euclidean space. In fact, some of the ideas may be generalized to unitary spaces over the field of complex numbers. Here we would like to give some examples which the reader can try on his own.

(1) Let δ, γ be two lines through P intersecting the quadric at B and C and at D and E, respectively. Then the lines BD and EC intersect in a point which is on the polar of P with respect to the quadric. Also lines BE and CD intersect on the polar. (Special cases should be considered separately.)

(2) If a point A moves on the polar of P with respect to a quadric, then the polar of A with respect to the quadric passes through P.

(3) In Euclidean 3-space we may change the point P to a line and obtain a linear element in the space as the polar of the line. This idea may also be generalized.

REFERENCES

1. Amir-Moéz, A. R., *Matrix Techniques, Trigonometry, and Analytic Geometry*, Edwards Brothers, Inc., Ann Arbor, Michigan, 1964, pp. 122-132.

A NOTE ON THE EXISTENCE OF PERPETUAL MOTION MACHINES

by Edward J. Wegman

Historically, the role of a mathematician has been to determine the truth or falsity of propositions concerning "mathematical" objects (numbers, functions, sets, and so forth). In the 1930's the attention of a group of mathematicians shifted to the problem of the existence of algorithms or effective computational procedures for solving various problems. An algorithm is a set of instructions, requiring no creative thought, that provides procedures by which any one of a class of questions can be answered. In principle, it is always possible (though perhaps not practical) to construct a physical machine for carrying out such a set of instructions. Modern digital computers are examples of such machines and computer programs are examples of algorithms. The existence of simple algorithms can be proven simply by writing down the steps of the algorithm, but complex questions require more complex algorithms, the proof of whose existence requires much more sophisticated mathematical ideas. This body of ideas is known as *computability* theory or recursive function theory, and is frequently regarded as a branch of mathematical logic.

One tool in computability theory is a particularly simple mathematical model of machines known as a Turing machine, so named because of its inventor, A. M. Turing (see Turing [27]). A Turing machine may be thought of as a black box with a finite number of internal configurations which are called states. It also reads an infinite tape and based on what it reads on the tape and its present state it may move, or erase the tape and write something else, and also it may change states. This is an extremely simple model of a modern computer and its power lies in the fact that moving, erasing, writing and changing states are all actions which may be accomplished with no "creative" thought. We shall momentarily formalize these ideas with mathematical definitions. We note here that Turing machines exist in the same sense that many mathematical objects exist that is, it is a set whose existence is guaranteed by the

axioms of set theory. By appropriate construction of a Turing machine, we can demonstrate the existence of an amusing and interesting perpetual motion machine. (This is, of course, a real existence although not a physical existence.) To do this, we shall need the following sequence of formal definitions taken from Davis [1].

Definition 1: An expression is a finite sequence (possibly empty) of symbols taken from the list $q_0, q_1, q_2, \ldots; S_0, S_1, S_2, \ldots; R, L$.

Definition 2: A quadruple is an expression having one of the following forms:

(1) $q_i \; S_j \; S_k \; q_l$,

(2) $q_i \; S_j \; R \; q_l$,

(3) $q_i \; S_j \; L \; q_l$.

Definition 3: A *Turing* machine, Z, is a finite (nonempty) set of quadruples that contains no two quadruples whose first two symbols are the same. The q_i's and S_j's are called respectively its internal configurations (states) and its alphabet.

As we have already indicated, the Turing nachine may be thought of as a black box *scanning* an infinite tape. The tape contains a series of contiguous squares or positions. If the position scanned on the tape contains the symbol S_j and the machine is in state q_i, quadruple (1) in Definition 2 causes the machine to write symbol S_k and change to state q_l. Quadruple (2) causes the machine to scan one square or position to the right and change to state q_l. Similarly for quadruple (3).

Definition 4: An instantaneous description is an expression that contains exactly one q_i, neither L nor R, and is such that q_i is not the right-most symbol. An expression that consists entirely of the letters S_j is called a tape expression. The tape expression is obtained by removing the q_i from the instantaneous description.

p i : If P and Q are two expressions involving only letters of the alphabet, then the expression

$$P\, q_i\, S_j\, S_m\, Q$$

is an instantaneous description meaning the machine Z is in state q_i scanning the square containing S_j. If $q_i\, S_j\, S_k\, q_l$ is a quadruple, then the next instantaneous description is

$$P\, q_l\, S_k\, S_m\, Q.$$

If $q_i\, S_j\, R\, q_l$ had been a quadruple, the next instantaneous description would have been

$$P\, S_j\, q_l\, S_m\, Q.$$

Similarly for the third type of quadruple $q_i\, S_j\, L\, q_l$.

Consider a machine, Z, given in Table 1 with alphabet $\{0, 1, B\text{(blank)}\}$

$q_0\ B\ 0\ q_0$	$q_4\ 0\ R\ q_0$	$q_7\ 0\ R\ q_6$
$q_0\ 0\ L\ q_2$	$q_4\ 1\ R\ q_1$	$q_7\ 1\ R\ q_6$
$q_1\ B\ 1\ q_1$	$q_5\ B\ R\ q_6$	$q_8\ B\ L\ q_9$
$q_1\ 1\ L\ q_2$	$q_5\ 0\ R\ q_0$	$q_8\ 0\ R\ q_6$
$q_2\ 0\ B\ q_3$	$q_5\ 1\ R\ q_1$	$q_8\ 1\ R\ q_6$
$q_2\ 1\ B\ q_3$	$q_6\ B\ R\ q_7$	$q_9\ B\ L\ q_9$
$q_3\ B\ L\ q_4$	$q_6\ 0\ R\ q_6$	$q_9\ 0\ R\ q_0$
$q_4\ B\ L\ q_5$	$q_6\ 1\ R\ q_6$	$q_9\ 1\ R\ q_1$
	$q_7\ B\ R\ q_8$	

Table 1

and set of states $\{q_0, q_1, \ldots, q_9\}$. This machine is designed to operate with tapes containing finite strings of 0's, 1's, and B's with the restriction that within a string, 2 or more contiguous B's may not appear. Of course, on either side of the finite string, the tape will be filled with B's. For example, ...$BB01B1BB$... is allowed but ...$BB01BB1BB$... is not.

Let us consider the tape ...$BB01B1BB$... with initial instantaneous description $01Bq_4 1BB$. The computation is as shown in Table 2, where the list of instantaneous descriptions are obtained from Table 1.

In the above description, it is recognized that the tape contains blanks on either side of the expression, $01B1$. We have only included

Instantaneous Description	Tape Expression
$BB01Bq_4 1BB$	$BB01B1BB$
$BB01B1q_1 BB$	$BB01B1BB$
$BB01B1q_1 1B$	$BB01B11B$
$BB01Bq_2 11B$	$BB01B11B$
$BB01Bq_3 B1B$	$BB01BB1B$
$BB01q_4 BB1B$	$BB01BB1B$
$BB0q_5 1BB1B$	$BB01BB1B$
$BB01q_1 BB1B$	$BB01BB1B$
$BB01q_1 1B1B$	$BB011B1B$
$BB0q_2 11B1B$	$BB011B1B$
$BB0q_3 B1B1B$	$BB0B1B1B$
$BBq_4 0B1B1B$	$BB0B1B1B$
$BB0q_0 B1B1B$	$BB0B1B1B$
$BB0q_0 01B1B$	$BB001B1B$
$BBq_2 001B1B$	$BB001B1B$
$BBq_3 B01B1B$	$BBB01B1B$
$Bq_4 BB01B1B$	$BBB01B1B$
$q_5 BBB01B1B$	$BBB01B1B$
$Bq_6 BB01B1B$	$BBB01B1B$
$BBq_7 B01B1B$	$BBB01B1B$
$BBBq_8 01B1B$	$BBB01B1B$
$BBB0q_6 1B1B$	$BBB01B1B$
$BBB01q_6 B1B$	$BBB01B1B$
$BBB01Bq_7 1B$	$BBB01B1B$
$BBB01B1q_6 B$	$BBB01B1B$
$BBB01B1Bq_7 B$	$BBB01B1BB$
$BBB01B1BBq_8 B$	$BBB01B1BBB$
$BBB01B1Bq_9 B$	$BBB01B1BB$
$BBB01B1q_9 B$	$BBB01B1B$
$BBB01Bq_9 1B$	$BBB01B1B$
$BBB01B1q_1 B$	$BBB01B1B$

TABLE 1

these blanks as needed. Note that the last instantaneous description is the same as the second, except that the whole expression $01B1$ has been shifted one position to the right on the tape. Hence the machine will repeat this procedure of shifting, then recycling forever.[1] We have only to invent an interesting tape expression to complete our perpetual motion machine.

Let our tape expression be:

$$11B111B001B10B1B01B00B10B000.$$

This is the international Morse code for the word mountains. Hence, we have demonstrated not only the existence of a perpetual motion machine, but one that moves mountains!

REFERENCES

1. Davis, Martin, Computability and *Unsolvability*, McGraw-Hill, New York, 1958.
2. Turing, A. M., "On Computable Numbers With an Application to the Entscheidungsproblem," *Proc. Lond.* Math. *Soc.*, 42, 1936, pp. 230-265.

[1] That this will happen in general is a consequence of the following observations: The instantaneous description abq_4cBd will eventually become either aq_4bBcd (if $c \neq B$) or q_4aBbcd (if $c = B$); Bq_4BBab eventually becomes $BBBaq_6b$, and aq_6bcd becomes either abq_6cd (if b # B) or $abcq_6d$ (if b = B). Finally, $abcq_6BBB$ becomes abq_9cBBB which in turn becomes aq_4bBcBB, and the process repeats. One "turns the machine on" by inserting q_4 between the last two zeros of "mountains". --Editor.

SOME WORK ON AN UNSOLVED PALINDROMIC ALGORITHM

By Lee Ratzan
College of Medicine and Dentistry of New Jersey

Consider the following unsolved research problem posed by the editor in the Fall 1972 issue of the Pi Mu Epsilon Journal (Volume 5, Number 7, page 338): Let

$$k(n) = \frac{n}{2} \text{ if } n \text{ is even,}$$
$$= 3n + 1 \text{ if n is odd,}$$

and define

$$k^r(n) = k(k^{r-1}(n)) .$$

Is it true that $k(n)$ is a palindromic algorithm which acts to reduce the size of an integer and the sequence $\{k^1(n), k^2(n), \ldots\}$ resolves to 1 for any n? Analytically speaking, does there exist a positive integer s (s is a function of n) such that $k^s(n) = 1$ for any n?

The editor states that the problem has been partially investigated as to the activity of the function $k(n)$ and the truth of this conjecture (Conjecture I) has been verified by computer for all n less than 10,000. The present note expands the range for which the conjecture is true and introduces a new conjecture.

Through the use of the program below the author has increased the verification limit of Conjecture I. It can be shown that $k(n) \to 1$ for

PROGRAM TO VERIFY PROPERTIES OF K(N)

```
   SEED.. .INITIAL POINT (K = 1 IS TRIVIAL)
   K = 2
   LIMIT POINT.. .END OF INVESTIGATION
   L = 100000
 7 CONTINUE
   N = K
 1 TEST = N/2
   PARITY CHECK...ODD/EVEN
   IF (N-2* TEST) 4, 2, 4
 2 N = N/2
11 CONTINUE
   CHECKPOINT...N LT 1 = OVERFLOW;
        N EQ 1 = TERMINATION; N GT 1 = RECYCLE
   IF (N - 1) 12, 6, 1
```

```
 4 N = 3*N + 1
   GO TO 11
 6 K = K + 1
   IF (K - L) 7, 7, 8
12 WRITE (6, *) N, K
   GO TO 6
 8 WRITE (6, *) L
   END
```

all n less than 31,910, a tripling of the former range. It is interesting to note that for 30 values between 31,911 and 100,000 the conjecture is indeterminate due to the magnitude of the integers involved. At specific points in this open range the values assumed by $k(n)$ exceed $(2^{32} - 1)$ and the overflow of machine registers of the IBM 360/67 (Rutgers University) and the IBM 360/91 (Princeton University) indicated that the number being dealt with is out of range. Are these numbers exceptions to the *algorithm* or just large numbers? If these isolated points could be verified, Conjecture I would hold true for all n less than 100,000.

A study was made of the number of cycles necessary for the integers between 1 and 31,910 to resolve to 1. There is apparently no set pattern; however the mean value of s for the first 5,000 n was 78 cycles -- a surprisingly high value since "most" values fell in the range $12 < s < 25$. Analysis of histogram data reveals a wild distribution (verified by calculation of the variance and standard deviation); graphs of s versus n reveal no distinct functional or periodic relationship.

Let us consider a flow chart (Fig. 1) of the action of the $k(n)$ for each cycle for a given initial n. If at any point along the cycles $k(n)$ is odd, it undergoes the transformation $k(n) = 3n + 1$, making it even. At this point it is halved and if odd, recycled as before. If it is even,

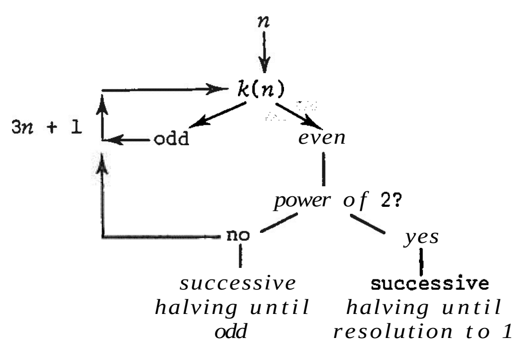


FIGURE 1

it will be halved, recycled, halved, and so forth, until it is odd and we have a perpetual circle. However, suppose $k(n)$ is a power of 2. Each successive halving diminishes the size of $k(n)$ and the cycle *will* terminate. The value of $k(n)$ will be $(2^L, 2^{L-1}, \ldots 4, 2, 1)$. We thus have the following analysis:

(1) Clearly if $n = 2^L$ for some integer L, $k^1(n), k^2(n), \ldots k^s(n)$ equals $n, n/2, n/2^2, \ldots n/2^L = 1$. Hence $s = L$.

(2) If n is odd, then a necessary condition for termination is that there exists an s such that $3k^s(n) + 1 = 2^L$ for some L, which in turn implies $k^s(n) = (2^L - 1)/3$ for some L. From this we have a test for termination of the sequence $k^1(n), k^2(n), \cdots$. If S is the set of values representable in the form $(2 - 1)/3$ (note that L must be even for this expression to be an integer) and if there exists an integer r such that $k^r(n)$ is an element of S for some r, then there is an s such that $k^s(n) = 1$.

(3) From this we can derive the general rule that if n is of the form $2^M(2^L - 1)/3$ for any nonnegative integer M and any even integer L, then $k^M(n) = (2^L - 1)/3$ and $k^{M+L+1}(n) = 1$. Hence, $s = M + L + 1$.

We can thus conclude that the conjecture is true for all n which are powers of 2 or representable in the form $2^M(2^L - 1)/3$ for appropriate choices of L, M. We can also monitor the values of $k(n)$ as it cycles and if at any point $k(n)$ assumes a value of the form $2^a(2^b - 1)$ for a, b integers, a nonnegative and b even, then a termination and resolution to 1 is certain in a finite number of steps.

If for values of n less than 31,910, the cycle length s is tabulated versus n, an interesting pattern can immediately be observed. It can be seen that, for example,

$$s(28) = s(29), \quad s(36) = s(37), \quad s(44) = s(45).$$

At specific intervals there are integers with the property that they have twin cycle lengths. This observation is more striking if the tabular form has 40 elements per row, for then every other line demonstrates this pairing. In analytic form we have:

Conjecture 11

Let $4k < n < 4(k + 1)$ for $k = 0, 1, 2, \cdots$ and define

$$f(n) = 8n + 8k + 20.$$

Then

$$s(f(n)) = s(f(n) + 1).$$

The question remains whether this conjecture be true for n outside the range of the present consideration.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

At the suggestion of the Pi Mu Epsilon Council we have had a supply of 10 × 14-inch Fraternity crests printed. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock - - - - - - $ 1.50/dozen,

(2) Purple and lavendar on goldenrod- - - $ 2.00/dozen.

MATCHING PRIZE FUND

If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30.00 of awards can result in the chapter receiving $15.00 reimbursement from the National Office. These funds may also be used for the rental of mathematical films. Write to the National Office for more details.

UNDERGRADUATE RESEARCH PROJECTS

1. *Proposed by John Van Iwaarden, Hope College, Holland, Mich.*

Let S be a closed bounded subset of E^2 with nonempty interior having unit area. The ***forbidden region*** of S relative to the real number k, $0 < k < 1/2$, is by definition the set F_k of points P through which no line may be passed cutting S in regions whose areas are in the ratio $k : (1 - k)$. (For more information, see the paper "Forbidden Area" by the same author in this issue of the *Journal*.)

<u>*Questions*</u>:

(1) If S has a nonempty forbidden region, does it contain the centroid of S?

(2) Can the theory of forbidden regions be extended in any way to three-dimensional space or to unbounded regions?

2. *Proposed by David L. Drennan and Kirby C. Smith*
University of Oklahoma

A city engineer was asked to completely describe the traffic flow over a given time period through a T-type intersection where one two-lane road ended at another two-lane road. The description had to contain answers to the following questions. Of those cars which entered the intersection from a given direction, how many turned right? How many turned left? How many went straight?

The devices available to the engineer were traffic counters, each capable of counting the cars in one lane which passed it. He could also use people standing on the corners of the intersection but, unfortunately, each person could count only one thing (i.e. of the cars which entered the intersection nearest his corner he could count only one of the following: the right turners, the left turners, or the straight goers).

Since the use of people cost more than the machines, the engineer wanted to minimize the number of people used. Having accomplished this, for aesthetical reasons, he wished to minimize the number of counters he used, although plenty were available. What was the engineer's solution? If possible, generalize to more complicated intersections.

1971-72 MANUSCRIPT CONTEST WINNERS

The judging for the best expository papers submitted for the 1971-72 school year has now been completed. The evaluation by the judges resulted in a tie for third place, so the amount of that award will be shared equally among the two winners. We congratulate the following winners (all affiliations are those at the time the papers were written):

FIRST PRIZE ($200): Christopher Scussel, Michigan State University, for his paper "Goldbach's Conjecture" (this *Journal*, Vol. 5, No. 8, pp. 402-408.

SECOND PRIZE ($100): Joseph J. Buff, New York University, for his paper "Characterization of an Analytic Function of a Quaternion Variable" (this *Journal*, Vol. 5, No. 8, pp. 387-392.

THIRD PRIZE ($50): Frank L. Capobianco, College of the Holy Cross, for his paper "Spec(R) for a Particular R" (this *Journal*, Vol. 5, No. 6, pp. 285-2881, and

Lee Ratzan, Courant Institute of Mathematical Sciences, for his paper "Comments on the Properties of Odd Perfect Numbers" (this *Journal*, Vol. 5, No. 6, pp. 265-271.

1972-73 CONTEST

We are now receiving papers for this year's contest, so be sure to send us your paper, or your chapter's papers, if you want to participate. Papers submitted to the *Journal* for publication will automatically be'entered if the author is an undergraduate, but we must receive a total of at least ten papers during the year in order to conduct the contest. *In order to be eligible, authors must not have received a Master's degree at the time they submit their paper.* In addition to the prizes listed above, there is a $20.00 prize for the best paper from any one chapter, providing that chapter submits at least five papers.

BOOK REVIEWS

Edited by Roy B. Deal
University of Oklahoma Health Sciences Center

The Letters on Probability. By *Alfred Renyi*. Wayne State University Press, Detroit, Michigan, 48202. 1972. v + 86 pages.

The most fascinating book ever reviewed by this author for the *Pi Mu Epsilon Journal*. Any scholar who has heard of Fermat's last theorem and who has a modicum of interest in history will find the intrigue surrounding these letters from Pascal to Fermat and their discovery to be exciting. The story began with a discussion at the Pascal Memorial Symposium in Clermont-Ferrand, on June 9, 1962, between Renyi and Henri Trouversien, a professor of the history of mathematics at the University of Contebleu. Pascal wrote a letter to the Parisian Academy in 1654 in which he mentions a treatise on an entirely new theme, the mathematics of chance, never systematically dealt with by anybody thus far. It was listed with some of his nearly completed works shortly to be presented to the Academy and there had been a great deal of work searching for the missing documents. Renyi had suggested that perhaps the information was contained in letters to Fermat instead of in a manuscript. The interesting way that Trouversien came across these letters, his discovery of another document of Fermat, and his intriguing reason for asking Renyi to publish them, along with an essay, give this little book the enticement of a thrilling novel. The Appendix contains five interesting little notes of Renyi: A Short Biography of Pascal, Dating the Letters, History of the Probability Theory, On the Mathematical Concept of Probability, and A Further Letter to the Reader. In addition, there appears a brief note by the translator, Laszlo Vekerdi, about Renyi.

What is Mathematical Logic. By *J. N. Crossley, C. J. Ash, J. C. Stillwell and N. H. Williams*. Oxford University Press, N. y., N. y., 10016

December, 1972. 82 pages. $1.95.

This book is based on a series of lectures given by the authors at Monash University and the University of Melbourne in the autumn and winter of 1971. They make an enthusiastic attempt to make mathematical logic a "living and lively subject" to a wide audience. They discuss important ideas behind Godel's theorems, computability and recursive functions, and consistency and independence in axiomatic set theory.

Explorations in Number Theory. *By Jeanne Agnew.* Brooks-Cole Publishing Company, Monterey, California. 1972. xi + 308 pages. $10.95.

This book reflects the author's sincere effort to write for the student, The first three units leisurely cover the basic concepts and the last five offer a wide variety of fascinating diversions, some of which have not been given lucid elementary expositions. The units on number-theoretic functions and the p-adic integers provide some excellent motivation examples for understanding more general algebraic systems.

Inference and Decision. *By D. A. S. Fraser.* Halsted Press, Division of John Wiley and Sons, Inc., New York. 1973. vii + 82 pages. $13.75.

A set of invited papers presented at several seminars held at a number of Canadian universities during 1970-71. They cover a variety of interesting observations on statistical inference and decision theory by some outstanding people.

A Geometric Introduction to Topology. *By C. T. C. Wall.* Addison-Wesley Publishing Co., 1972. iv + 168 -pages.

An introduction which does not presume general topology and does quite well at the advanced undergraduate level those parts of homotopy theory which have a strong geometric intuition for many students.

Introduction to Semigroups. *By Mario Petrich.* Charles E. Merrill Publishing Co., 1973. viii + 198 pages.

The first introductory book devoted explicitly to the basic results in semigroups. It also serves as a reference,

Theories of Probability. *By Terrance L. Fine.* Academic Press, New York. 1973. xi + 263 pages.

In these days when axiomatic probability theory, and even bayesian inference seems thoroughly entrenched, it is refreshing to find a rather thorough discussion of a wide variety of aspects of probability and many of its ramifications in concrete applications. The author discusses such subjects as axiomatic comparative probability, relative frequency in probability, logical probability, and subjective or personal probability.

An Introduction to Confirmation Theory. *By Richard Swinburne.* Methuen and Co., Ltd., London. 1973. vi + 218 pages.

Like the previous book reviewed, this one will be of interest to those who like to see relationships between mathematics and our culture. This book relates to the field of philosophy. The author states in the preface that "Confirmation theory is the theory of when and how much different evidence renders different hypotheses probable. The aim of this book is to expound and criticize the views of philosophers on confirmation theory, and in the process to contribute towards the construction of a correct confirmation theory."

Computational Methods in Ordinary Differential Equations. *By J. D. Lambert.* John Wiley and Sons, Inc., New York, N.Y., 10016. 1973. xv + 278 pages. $15.50.

It is impressive to see how far the author is able to take advanced undergraduate students in the basic classical foundations of the numerical analysis and stability theory of ordinary differential equations,

Functional Analysis. *By Walter Rudin.* McGraw-Hill Book Company, New York. 1973. xi + 397 pages.

Another fine book by an established author in a field which seems to attract excellence in exposition. Perhaps this is because the subject is basic to so much mathematics and is amenable to good pedagogy. The subject has broadened in scope over the years, and while there is no attempt to be encyclopedic, the second part of the book covers fairly thoroughly distributions and Fourier transforms, with some applications to differential equations and Tauberian theory, and Part Three discusses Banach algebras and spectral theory, with chapters on bounded and unbounded operators in a Hilbert space.

LOCAL AWARDS

If your chapter has presented awards to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal*. The listing of new initiates had been discontinued.

MOVING??

BE SURE TO LET THE JOURNAL KNOW!

Send your name, old address with zip *code* and new address with zip *code* to:

Pi Mu Epsilon Journal
601 Elm Avenue, Room 423
The University of Oklahoma
Norman, Oklahoma 73069

GLEANINGS FROM CHAPTER REPORTS

FLORIDA ZETA CHAPTER at Florida Atlantic University sponsored a lecture by *Professor John Scheidell*, chairman of the Economics Department, on the topic "The Fecundity of Mathematics in Economics".

LOUISIANA EPSILON CHAPTER at McNeese State University presented a film series on space flight and held a lecture by *Professor Harlin Brewer* on "Mathematics -- Definition by Recursion".

MARYLAND ALPHA CHAPTER at the University of Maryland sponsored a series of highly diversified lectures, which included "Equations of Motion of the Planets" by *Professor Larry Goldstein*, and "Derivatives, Derivates, and Arbitrary Functions" by *Professor James A Hummel*. Another speaker was *Professor Oved Shisha*, Naval Research Laboratory.

MICHIGAN DELTA CHAPTER at Hope College held its first meeting on November 9, 1972. *Professor J. S. Frame*, Michigan State University, spoke on the topic "Continued Fractions", after which he installed the new chapter. Professor Frame is a former national president of Pi Mu Epsilon.

NEW JERSEY DELTA CHAPTER at Seton Hall University had several of its members present papers at the annual *Eastern Colleges Science Conference* held at Pennsylvania State University. Those presenting papers were *Michael Martin*, *Roseann Moriello* (whose paper "Partial Differentiation on a Metric Space" won first prize), and *Karen Pukatch*.

NEW JERSEY EPSILON CHAPTER at St. Peter's College sponsored a lecture presented by *Professor B. Melvin Kiernan* entitled "What Would You Say to a Hypercube if You Met One on the Street?" and hosted a regional meeting of the Mathematical Association of America.

NEW YORK ETA CHAPTER at S. U. N. Y., Buffalo, recently honored its faculty correspondent, *Professor Harriet Montague*, for her great efforts and outstanding success in leading the chapter through many difficult years.

NEW YORK PHI CHAPTER at the State University College at Potsdam participated in a joint meeting of the Pi Mu Epsilon chapters of St. Lawrence University, Clarkson College and Potsdam State University, at which a representative from each chapter gave a short talk.

NORTH CAROLINA GAMMA CHAPTER at North Carolina State University sponsored a series during the year, one of which was "Non-Standard Analysis" by *Professor Robert T. Ramsay.*

OHIO NU CHAPTER at the University of Akron sponsored a tour of the Goodyear Computer Center.

OHIO ZETA CHAPTER at the University of Dayton participated in an *Alumni Seminar* where graduates in mathematics described to undergraduates what they were doing and answered questions from the audience.

OKLAHOMA ALPHA CHAPTER at the University of Oklahoma heard *Professor W. T. Reid* speak at its annual awards banquet on noted mathematicians of this century who owed their education either directly or indirectly to the faculty at the University of Chicago during the early 1900's. In particular, members of the mathematics department heard their individual "R. L. Moore numbers" (the number of mathematical "generations" from advisor to advisor beginning with Professor Moore) traced.

TENNESSEE BETA CHAPTER at the University of Tennessee recently honored *Winston Massey*, Guerry Professor of Mathematics, for forty years of service to the university. A *Winston Massey Mathematics Award* has been established to be given to an outstanding junior in mathematics, in addition to the award presently given to an outstanding freshman each year.

VIRGINIA GAMMA CHAPTER at Madison College heard several stimulating lectures during the year, including "The Supposed Glut of Educated Minds" by *Professor William L. Duren, Jr.* (Applied Mathematics and Computer Science Department), and "A Unified Theory of Integration" by *Professor E. J. McShane*, both from the University of Virginia. At the initiation banquet *Professor Gordon Fisher* spoke on "Some Interesting Personalities in Mathematics'.

WISCONSIN ALPHA CHAPTER at Marquette University sponsored a trip to the-Chicago Museum of Science and Industry, and heard several lectures including "Pythagorean Proofs" by *William Wepfer*, "Chemistry of Lines and Planes" by *Gary Schaefer* (both undergraduates), "Non-Standard Reals" by *Jeffrey Posluszny*, a graduate student, and "Alphametrics, a Mathematical Diversion" by *Professor Raymond Honerlah.*

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems characterized by novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if available, and by any information that will assist the editor. Contributors of proposals and solutions are requested to enclose a self-addressed postcard to expedite acknowledgement.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before May 31, 1974.

Address all communications concerning problems to Dr. Lean Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

303. *Proposed by Peter A. Lindstrom, Genessee Community College, Batavia, New York.*

By means of an ε, 6 proof only, show that a polynomial function is continuous at any real number.

304. *Proposed by Charles W. Trigg, Sun Diego, California.*

(A) One of the four digits 1, 2, 3, 4 is placed at the midpoint of each edge of a cube in such a manner that four different digits are on the perimeter of each square face.

(B) The digits are placed on the vertices of the cube so that again there are four different digits on the perimeter of each face.

Show that in each case the clockwise cyclic order of the digits is different on each face.

305. *Proposed by Jack Garfunkel, Forest Hills High School, New York.*

In an acute triangle ABC, AF is an altitude and P is a point on AF such that $AP = 2r$, where r is the inradius of triangle ABC. If D and E are the projections of P upon AB and AC respectively, show that the perimeter of triangle ADE is equal to that of the triangle of least perimeter that can be inscribed in triangle ABC.

306. *Proposed by David L. Silverman, Los Angeles, California.*

On alternate days A and B play games that are similar except with respect to the question of which player does the paying. In both versions A selects one number from the set (1, 2, 3) and B selects two numbers from the same set. If the two selections are disjoint, no payment is made. If the two selections have a number in common, the "payer" pays that number of dollars to the "receiver". They alternate daily in assuming the roles of payer and receiver. Does the arrangement favor either player?

307. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

Let $\tau(n)$ denote the number of divisors of n. For square-free n greater than 1, prove that $\tau(n^2) = n$ if and only if $n = 3$.

308. *Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur 4, South India.*

Defining a *proper number* as one which is equal to the product of all its proper divisors, show that an integer is a *proper number* if and only if it is the cube of a prime or the product of two different primes.

309. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the volume of the solid formed by the elliptic papaboloids $2h - z = ax^2 + by^2$ and $z = ox^2 + dy^2$, where a, b, c, d and h are all positive.

310. *Proposed by Sidney Penner, Bronx Community College, Bronx, New York.*

If x and y are integers and $x < y$ then let $[x, y] = \{z : x \le z \le y$, and z is an integer$\}$. Also, for any set S, let $N(S)$ be the cardinal number of S.

Let n and k be positive integers with $k > 1$ and let $G = [2, (2n)^k - 1]$. If V is a subset of G such that $N(V) = (2n)^k - 2n$ and V # $[2n, (2n)^k - 1]$ then there are at least two distinct members of V each of which is the product of k members (not necessarily distinct) of V.

311. *Proposed by Charles W. Trigg, San Diego, California.*

On opposite sides of a diameter of a circle with radius $a + b$ two semicircles with radii a and b form a continuous curve that divides the circle into two tadpole-shaped parts.

(I) Find the angle that the join of the centroids of the two component parts makes with the given diameter of the circle.

(II) For what ratios $a : b$ does the continuous curve pass through one of the centroids?

(III) When $a = b$, find the moment of inertia of one of the component areas about an axis through its centroid and perpendicular to its plane.

312. *Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Let $\{a_n\}$ be a sequence such that $a_1 = 1$ and for $n > 1$

$$a = a_{n-1} + 1 + (-1)^n + \frac{3}{2}[1 + (-1)^{n+1}].$$

Show that the sequence $\{a_n\}$ has infinitely many primes.

313. *Proposed by Murray S. Klamkin, Fohd Motor Company, Dearborn, Michigan.*

Give an elementary proof that

$$(1 + \cos^2 A)(1 + \cos^2 B)(1 + \cos^2 C) \geq 64 \sin^2 A \sin^2 B \sin^2 C,$$

where A, B, C are the angles of an acute triangle ABC.

<u>*Remark*</u>

J. Gillis gave a proof using calculus techniques in Problem E 2119, *American Mathematical Monthly*, 1969, p. 831.

Solutions

281. [Fall **1972**] *Proposed by Solomon W. Golomb, University of Southern California.*

We define an "average" number to be a real number for which the average of the digits in its decimal expansion is $(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)/10 = 4.5$. Prove that the number $1/p$, for p prime, is an "average" number if and only if the period of its decimal expansion has an even number of digits.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

We shall show that the proper fraction a/p, for p prime, is an 'average" number if and only if the period of its decimal expansion has an even number of digits.

Let the fraction a/p, for p prime, be such that the period of its decimal expansion has an even number of digits. (Since it is clear that p must be different from 2 and also different from 5, the period will begin immediately after the decimal point.) We begin by considering some examples. The period of 5/7 consists of the six digits 714285. We split them in half and add the numbers so formed: $714 + 285 = 999$. The period of 1/17 is 0588235294117647 which when split and added gives $05882352 + 94117647 = 99999999$. For the period of $1/11$, which is 09, we have $0 + 9 = 9$.

It can be established that the sum of the two halves of the period will always turn out this way when the period belongs to the fraction a/p whose denominator p is a prime, provided the period has an even number of digits. For a proof of this fact, see Rademacher and **Toeplitz**, The Enjoyment of Mathematics, Princeton University Press, 1957, pp. 158-160. Another proof may be found in W. G. Leavitt, "A Theorem on Repeating Decimals," American Mathematical Monthly, June-July, 1967, pp. 669-673.

Let the period of a/p be $b_1 b_2 \cdots b_k b_{k+1} \cdots b_{2k}$. Then $b_1 + b_2 + \cdots + b_k + b_{k+1} + \cdots + b_{2k} = 9 + 9 + \cdots + 9$ [k addends of $91 = 9k$, so that a/p is an "average" number.

Suppose the fraction a/p for p prime is an "average" number. Then clearly p must be different from 2 and also different from 5, so the period will begin immediately after the decimal point. Let the period of a/p be $c_1 c_2 \cdots c_j$. Because a/p is an "average" number, $c_1 + c_2 +$ — $+ c_3 = (9/2)j$. But $c_1 + c_2 + \cdots + c_3$ is a positive integer. Hence,

$(9/2)j$ is a positive integer, which implies that j is an even positive integer. Thus the period of the decimal expansion of a/p has an even number of digits.

Comment by Sid Spital, Hayward, California. This problem appears on page 24 of the USSR Olympiad Problem Book, W. H. Freeman and Company, 1962.

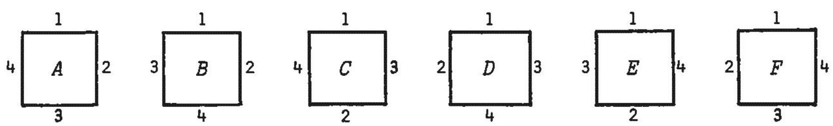
Also solved by STUART MARGOLIS, *Rutgers College;* GREGORY WULCZYN, *Bucknell University, Lewisburg, Pennsylvania, and the Proposer.*

282. [Fall 1972] *Proposed by Charles W. Trigg, San Diego, California.*

Four differently colored isosceles right triangles can be assembled to form a square in six essentially different ways (not counting rotations). By joining these tetrachrome squares domino-like with like-colored sides meeting, a variety of configurations can be formed. Show that (a) they can be so assembled into a 2×3 rectangle with solid colors along each side and that (b) they can not be so assembled into a 2×3 rectangle with its four sides differently colored.

Solution by the Proposer.

The color-numbers along the edges of the six squares are shown below:



a) A typical assemblage, as shown in the 2×3 figure, leads to three more arrangements by adding 1 to each color-number (5 becomes 1), and continuing the process, reducing each sum modulo 4, as shown in Fig. 1.

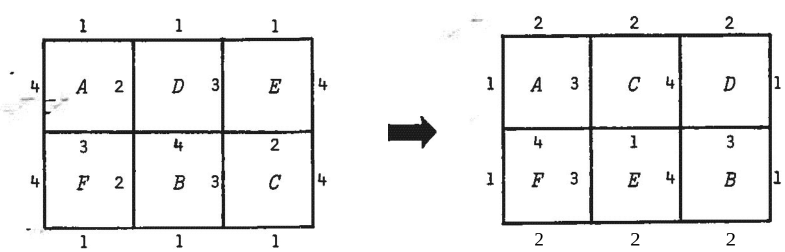


FIGURE 1

Note that this merely rotates A and F and cyclically permutes the others along a crossed loop. The columns in each arrangement may also be cyclically permuted to give a total of 12 such rectangles.

b) Internal color segments occur in matched pairs, so if three like colors appear on one side of the rectangle, the sixth segment of that color must appear on another side thus spoiling the four-colored arrangement.

Also solved by MERYL J. ALTABEL, *Herbert H. Lehman College, New York;* ERIC HARTSE, *Missoula, Montana;* CHARLES H. LINCOLN, *Terry Sanford Senior High School, Fayetteville, North Carolina; and* NOSMO KING, *Raleigh, North Carolina.*

283. [Fall 1973] *Proposed by David L. Silverman, Lot, Angela, California.*

Let $a_1 = \sin 1$ and for every positive integer n, let $a_{n+1} = \sin a_n$ Does Σa_n $(n = 1, 2, 2, \ldots, \infty)$ converge?

Solution by the Proposer.

By examining the first two terms of the Taylor expansion of the sine function, it is evident that $1/(n + 1) < \sin (1/n)$ for $n = 1,2,3, \ldots$. Then, since $1/2 < \sin 1$, it follows that $1/3 < \sin (1/2) < \sin \sin 1$; also, $1/4 < \sin (1/3) < \sin \sin \sin 1$, etc. Consequently, since it dominates the truncated harmonic series, the series Σa_n diverges.

Also solved by BENJAMIN L. SCHWARTZ, *McLean, Virginia. Four incorrect solutions were received, one of them asserting convergence.*

285. [Fall 1972] *Proposed by Murray S. Klamkin, Ford Motor Company Scientific Laboratory, Dearborn, Michigan.*

Solve the equations:

$$a(x^2 - y^2) - 2bxy + cx - dy + e = 0$$
$$b(x^2 - y^2) + 2axy + dx + cy + f = 0$$

Solution by the Proposer.

Both equations can be written as the complex equation

$$(a + ib)(x + iy)^2 + (c + id)(x + iy) + e + if = 0$$

which can be solved by the quadratic formula. The square root can be extracted by converting to polar form.

286. [Fall 1972] *Proposed by A. M. Gustafson, Salt Lake City, Utah.*

Given decimals $x = .x_1x_2x_3\cdots$, $y = .y_1y_2y_3\cdots$ in $[0,1]$, define $x \star y = .x_1y_1x_2y_2x_3y_3\cdots$, and let $P_x = \{x \star y: y \in [0,1]\}$. (The only decimal ending in 9's is 1.)

1) Use the sets P_x, $0 \le x \le 1$, to write $[0,1]$ as the union of c pairwise disjoint perfect sets.

2) There are many ways to write $[0,1]$ as the union of c pairwise disjoint perfect sets P_x, $0 \le x \le 1$. Let T be any family of such decompositions $\{P_x : 0 \le x \le 1\}$ such that no two decompositions in T have a set in common. Prove that the cardinal number of T cannot exceed c.

3) Modify (1) to obtain a family T of the kind considered in (2) with the cardinal number c.

For references see E. Hewett, Real and Abstract *Analysis*, Springer, New York, 1965, and I. P. Natanson, Theory of Functions of a Real Variable, Frederick Ungar Publishing Co., New York, 1961. (In particular, see Ex. 5, p. 54.)

Solution by the Proposer.

1) P_x is closed: Consider a Cauchy sequence $\{f_n\} \subset P_x$, $f_n = x * y_n$. For fixed k, it is possible to show that the first k digits of f_n are the same for all large n. Let g_k be the eventual value of the kth digit of f_n, and put $g = .g_1g_2g_3\cdots$. A short computation shows $g = x * y$ with $y = .g_2g_4g_6\cdots$, and $f_n \to g$. Therefore, P_x is closed.

P_x is perfect: Given $x * y \in P_x$, select a sequence $\{y_n\}$ of distinct points with limit y. Then $x * y_n \to x * y$, so P_x is perfect.

2) Let R be the family of all perfect sets from the decompositions in T. Then R has cardinal number c, because there are c closed sets in $[0,1]$. Therefore, T has cardinality at most c.

3) Let $k = \{k_n\} \in N^N$ be a sequence of natural numbers. Define $T_k = \{P_x(k) : 0 \le x \le 1\}$ as follows: each $P_x(k)$ is the set of all decimals in $[0,1]$ with x_n in position $k_1 + \cdots + k_n + n$. Each T_k is a collection of pairwise disjoint perfect sets with union $[0,1]$. Since N^N has cardinal number c, the family $T = \{T_k : k \in N^N\}$ has the desired property.

287. [Fall 1972] *Proposed by Erwin Just, Bronx Community College.*

For each real number x, prove that

$$-x^n + \sum_{k=0}^{2n} (-1)^k x^k \ge 0 .$$

Solution by Sid Spital, Hayward, California

Evaluation of the geometric sum and a little algebra converts the problem to a proof of

$$(x^n - 1)(x^{n+1} - 1) \gtrless 0 \text{ for } x \gtrless -1.$$

This is true since both factors are of like sign for $x > -1$ and of opposite sign for $x < -1$ ($x = -1$ is trivial).

Also solved by K. BURKE, *Seton Hall University, South Orange., N. J.;* ERIC HARTSE, *Missoula, Montana;* CHARLES H. LINCOLN, *Terry Sanford Senior High School, Fayetteville, N. C.;* PETER A. LINDSTROM, *Genesee Community College, Batavia, N. Y.;* SIDNEY PENNER, *Bronx Community College;* BOB PRIELIPP, *University of Wisconsin, Oshkosh;* GREGORY WULCZYN, *Bucknell University, Lewisburg, Pa.; and the. Proposer.*

288. [Fall 1972] *Proposed by Leon Bankoff and Alfred E. Neuman, Mu Alpha Delta Fraternity.*

If $\alpha + \beta + \gamma = \pi$ show that

(1) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le \sin \alpha + \sin \beta + \sin \gamma$

(2) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le$
$\sin \alpha + \sin \beta + \sin \gamma + \sin 3\alpha + \sin 3\beta + \sin 3\gamma$

equality holding if and only if $\alpha = \beta = \gamma$.

I. *Solution by L. Carlitz, Duke University, Durham, North Carolina.*

The inequality

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le \sin \alpha + \sin \beta + \sin \gamma$$

is well known (see for example O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović and P. M. Vasić, *Geometric Inequalities*, Groningen, 1969, p. 18, No. 2.4).

As for the second inequality, we have

$$\sum \sin \alpha = \frac{s}{R} \qquad \sum \sin 2\alpha = \frac{2rs}{R^2}$$

$$\sum \sin 3\alpha = \sum (3 \sin \alpha - 4 \sin^3 \alpha)$$
$$= \frac{3}{2R} \sum a - \frac{1}{2R^3} \sum a^3$$

$$= \frac{3s}{R} - \frac{1}{2R^3}\sum a^3$$
$$= \frac{3s}{R} - \frac{1}{2R^3}(2s^3 - 12rsR - 6r^2s).$$

Thus

$$\sum \sin a + \sum \sin 3a - \sum \sin 2a = \frac{4s}{R} - \frac{1}{2R^3}(2s^3 - 12rsR - 6r^2s) - \frac{2rs}{R^2}$$
$$= \frac{s}{R^3}(4R^3 + 4rR + 3r^2 - s^2).$$

Now it is known (Geometric *Inequalities*, p. 51, No. 5.9) that

$$s^2 \le 4R^2 + 4rR + 3r^2$$

with equality if and only if $a = b = c$. Thus the stated result follows at once.

11. *Solution by the Proposers.*

(1) If Δ denotes the area of a triangle ABC with inradius r and circumradius R, we have

$$2\Delta = R\sum a \cos A = r\sum a$$

from which we obtain

$$\frac{r}{R} = \frac{\sum a \cos A}{\sum a} = \frac{\sum \sin 2A}{2\sum \sin A} \le \frac{1}{2} \qquad (*)$$

or $\sum \sin A \ge \sum \sin 2A$.

(2) Since $\sum \sin 3A = 3\sum \sin A - 4\sum \sin^3 A$, it follows that

$$\sum \sin A + \sum \sin 3A = 4\sum \sin A - 4\sum \sin A \sin^2 A$$
$$= 4\sum \sin A(1 - \sin^2 A) = 4\sum \sin A \cos^2 A$$
$$= 2\sum \sin 2A \cos A \ge \frac{2\Delta}{R^2} = \frac{2r}{R}\sum \sin A$$

Then, by (*), we have $\sum \sin A + \sum \sin 3A \ge \sum \sin 2A$.

289. [Fall 1973] *Proposed by R. S. Luthar, University of Wisconsin, Waukesha.*

If $p_1, p_2, \cdots p_n$ are the first n primes, prove that for $n > 2$,

$$p_n < p_1 + p_2 + \cdots + p_{n-1}$$

and hence show that between p_n and $p_1 + p_2 + \cdots + p_n$, there always lies a prime number.

Solution by Charles H. Lincoln, Terry Sanford Senior High School, Fayetteville, North Carolina.

The problem as stated is incorrect. It should read "for $n > 3$", since $p_1 + p_2 = p_3$. The proof is by induction. For $n = 4$, $p_1 + p_2 + p_3 > p_4$, or $2 + 3 + 5 > 7$. Assume that for some $k > 3$

$$p_1 + p_2 + p_3 + \cdots + p_{k-1} > p_k.$$

Then

$$p_1 + p_2 + \cdots + p_{k-1} + p_k > 2p_k > p_k.$$

Since by Bertrand's Postulate for every positive integer m there is a prime p such that $m < p \le 2m$, we now have:

$$p_1 + p_2 + \cdots + p_{k-1} + p_k > 2p_k > p_{k+1} > p_k$$

or

$$p_1 + p_2 + \cdots + p_k > p_{k+1},$$

thus completing the proof. Since $p_1 + p_2 + \cdots + p_n > 2p_n > p_n$, by what has been said above, there exists a prime with the desired property.

Similar solutions were offered by ERIC HARTSE, *Missoula, Montana;* MATTHEW [illegible]. KOCH, *Buffalo, N. Y.;* THOMAS MOORE *and* DONALD SIMPSON, *Bridgewater State College, Bridgewater, Mass.;* BOB PRIELIPP, *University of Wisconsin, Waukesha;* PETER A. LINDSTROM, *Genesee Community College, Batavia, N. Y.;* SID SPITAL, *Hayward, California; and the Proposer. Most solvers noted the inaccuracy in the statement of the problem and some sought to make the correction by adding the equality sign to the strict inequality.*

290. [Fall 1972] *Proposed by Solomon W. Golomb, University of Southern California.*

Let M be an $a \times b$ matrix of ab distinct real numbers, with $ab > 1$. Show that there exists a real number μ such that either every row of M or every column of M (or possibly both) has an entry less than μ and an element greater than p.

1. *Solution by Sid Spital, Hayward, California.*

Let $[s_i, t_i]$ be the smallest closed interval containing the elements of the ith row of M. The intersection of these intervals is either a closed interval or empty. If it is a closed interval, by choosing μ anywhere in the corresponding open interval, the requested result becomes clear for rows. If the intersection is empty, then there are two disjoint row intervals--that is for some j and k, the open interval (t_j, s_k) is non-empty. By choosing M anywhere in this interval the requested result becomes shown for columns by an inspection of the jth and kth entry in any column:

jth entry $\leq t_j < \mu < s_k \leq$ kth entry.

11. Solution by the Proposer.

If either a = 1 or b = 1 the result is trivial. For the general case, find the minimum element in each column, and let a (in column C) be the largest of these. Either there is an entire column C' of elements smaller than a or there is not. If such a column C' exists, then each row has an element in C which is $\geq$ a and an element in C' which is $<$ a. We then pick μ less than a but larger than any element of C'. If no such column C' exists, then each column has an element $>$ a, as well as its minimum element $\leq$ a. For this case, we may pick μ greater than a but smaller than any element of M which exceeds a.

The matrix $\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}$ with $\mu = 2\frac{1}{2}$ shows it is possible (in some matrices) for each row and each column to have an element $> \mu$ and an element $< \mu$. The matrix $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ shows that this does not happen in all matrices.

Also solved by MERYL J. ALTABET, *Bronx, N. Y.;* ERIC HARTSE, *Missoula, Montana;* N. J. KUENZI *and* BOB PRIELIPP, *The University of Wisconsin, Oshkosh; and* CHARLES H. LINCOLN, *Terry Sanford Senior High School, Fayetteville, N. C.*

291. [Fall 1972] Proposed by Charles W. Trigg, San Diego, California

How may a square card be folded into a tetrahedron? What is the volume of the tetrahedron in terms of the side of the square?

All of the solvers listed here submitted practically identical solutions: ROBERT C. GEBHARDT, *Hopatcong, N. J.;* ERIC HARTSE, *Missoula, Montana;* JOHN M. HOWELL, *Littlerock, Calif.;* CHARLES H. LINCOLN, *Fayetteville, N. C.; and the Proposer.*

Solution:

Join the midpoints of two adjacent sides of the square and then join these points to the opposite vertex. Crease along the joins and fold up into a tetrahedron. In the trirectangular tetrahedron so formed, the volume is $(1/3)(1/2)(s/2)^2 s$, or $s^3/24$, where s is the side of the square.

LOCAL CHAPTER AWARDS WINNERS

ALABAMA BETA (Auburn University). Recognition for outstanding work in mathematics was given to

Diane Jordan,
Keith Lane.

FLORIDA ZETA (Florida Atlantic University). An award of \$33.33 to be used for the purchase of books was presented to

Zachary Cabany,
Scott Demsky,
John Leach.

GEORGIA BETA (Georgia Institute of Technology). For achieving at least a 3.7 average (4.0 perfect) in all mathematics courses a book on mathematics of the recipient's choice was awarded to

Equine. Hubbahd,
John H. Nading,
Mark V. Peavy,
Keith R. Propst.

NEW YORK PHI (State University College at Potsdam). Undergraduate senior awards for excellence in mathematics based on cumulative grade point average in mathematics were presented to

Richard Gustafson,
Colleen Guhin.

OKLAHOMA ALPHA (University of Oklahoma). The Nathan *Altshiller* Court Award of \$50 for the best freshman man and woman in mathematics was given to

Larry F. Knight
Karen L. Boyd

The *Samuel Watson Reaves Scholarship* given each year to a senior for graduate work in mathematics was presented to

David Young

OHIO NU (University of Akron). A *Samuel Selby Scholarship* was presented to

Michael Margreta,
Mary Ann Schuerger,
Allan Wilcox.

PENNSYLVANIA BETA (Bucknell University). The first annual *John Steiner Gold Mathematical Competition* involved 72 individuals from 24 high schools and resulted in the winners listed below. The individual winners each received the four-volume set of *The World of Mathematics,* and the first, second and third place team winners received respective prizes of $100, $50, and $25 for their school mathematics library.

INDIVIDUAL WINNERS:

Andrew Boyer (First Place), *LEWISBURG AREA SR. HIGH SCHOOL*
Lee. Klinger (Second Place Tie), *SELINSGROVE AREA JR./SR. HIGH SCHOOL*
Larry Smith (Second Place Tie), *CENTRAL COLUMBIA HIGH SCHOOL*
William Bachman (Fourth Place Tie), *WARRIOR RUN SR. HIGH SCHOOL*
Thomas Smith (Fourth Place Tie), *MILTON AREA SR. HIGH SCHOOL*
Philip G. Staubs (Fourth Place Tie), *BENTON AREA JR./SR. HIGH SCHOOL*

TEAM WINNERS:

LEWISBURG AREA SENIOR HIGH SCHOOL (First Place)
(Andrew Boyer, Walter T. Bromfield, Cynthia Walter)

CENTRAL COLUMBIA HIGH SCHOOL (Second Place)
(Diana Schell, Larry Smith, Michael Winseck)

WILLIAMSPORT AREA HIGH SCHOOL (Third Place)
(William Carpenter, Nora Landale, David L. Plankenhorn)

WEST VIRGINIA ALPHA (West Virginia University). For presenting outstanding papers in mathematics, membership in the Mathematical Association of America was awarded to

Stephen Summers,
Bonnie White.,
Barry Dooley,
Michael Mays.