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ELEMENTARY NUMBER THEORY
IN CERTAIN SUBSETS OF THE INTEGERS, II

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1. Introduction

In [2], the authors relativized the notion of divisibility to non-empty subsets A of the integers Z of the following type: (1) A # {0} and (2) if x ∈ A, then $-x$ ∈ A. For A = nZ, the multiples of an integer $n > 1$, the Fundamental Theorem of Arithmetic and unique factorization were discussed. Some interesting results and formulas for the arithmetic functions π_A, τ_A, and σ_A were also obtained. The development is continued in this paper, and the notions of greatest common divisor, relative primality, and the relative version of Euler's function, herein denoted by $, are discussed. The notations and results established in [2] will be used throughout.

2. Common Divisors, G.C.V., and Relative Primeness

We begin with a definition.

Definition 2.1. If a,b ∙ A, then X ∈ A is a *common divisor* of a and b (in A) if $x(A)a$ and $x(A)b$.

In the case A = Z, common divisors always exist for any two elements of Z. However, if A # Z, common divisors may not exist. For example, the set A of the primes in Z has no divisors in A and hence can have no common divisors in A. For A = nZ, common divisors do not always exist since divisors do not always exist. (This problem is easily remedied by adjoining the elements ±1 to A, however.)

At least two approaches are used to investigate "the notion of greatest common divisor (g.c.d.) of two integers x,y, both not zero. In [5], the greatest common divisor of x and y is defined as that common divisor which is greater than all other common divisors, while in [1] the g.c.d. is that (positive) common divisor d which has the property that any other common divisor f also divides d, that is, f | d. This second definition is not adequate for the types of sets we are considering

since we can take

$$A = \{\pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24\} .$$

Here 12 and 24 have the following positive common divisors in A: 2, 3, 4, and 6. 6 is the greatest of the common divisors by 4(/)6. For this reason we adopt the **following** definition.

Definition 2.2. Let $x, y \bullet$ A, not both zero, then d is the greatest common divisor of x, y in A, denoted $A(x, y)$, if

(1) d is a common divisor of x and y in A and

(2) d is greater than any other common divisor of x and y in A.

In the case of finite sets containing zero, even x = y = 0 have a **g.c.d.** in A, namely the largest integer in A. The definition produces the following **equalities**:

$$A(x, y) = A(y, x) =, A(x, -y) = A(-x, y) = A(-x, -y) .$$

Thus we assume x ≥ 0 and y ≥ 0. Also, if $x, y \bullet$ A have no common divisors (hence no **g.c.d.**) we **shall** write $A(x, y) = 0$.

Since **1 may** not be in A and since common divisors may not always exist in A, a natural way to define the notion of two integers being relatively prime in A **is**:

Definition 2.3. Let $x, y \bullet$ A. x and y are said to be relatively prime in A

(1) if $1 \in$ A, then $A(x, y) = 1$;

(2) if $1 \notin$ A, then

$$A(x, y) = \begin{cases} 0 \text{ (if } x \text{ and y have no common divisors)} \\ \\ \text{the least positive prime in A otherwise.} \end{cases}$$

If x and y are prime in A, then they are relatively prime in A. The converse is **false** as can be seen by taking A = Z. Also, if $x, p \in A$ where p is prime in A and $0 < x < p$, then x and p are relatively prime in A. However, **it** should be noted that $x \in A$ may have a factor y with x and y being relatively prime. For example, in **2Z,** 4 is a factor of 24, neither are prime, yet 4 and 24 are relatively prime in 22. This is not true in Z

Since the integer **1** plays an important role in the definition of relatively prime, we now investigate when two composite integers will

be relatively prime in the two sets nZ and $nZ \cup \{\pm 1\}$. For abbreviation, $nZ \cup \{\pm 1\}$ will be denoted nZ^{*}. We first state a proposition showing that the primes and composites in nZ^{*} are precisely those in nZ (the reader can easily supply a proof).

Proposition 2.1.

(1) x $\bullet$ nZ is prime if and only if **it** is prime in nZ^{*}.

(2) x $\bullet$ nZ is composite if and only if **it** is composite in nZ^{*}.

Theorem 2.2. Any two composites in nZ^{*} are not relatively prime in nZ^{*}.

Proof. He need consider only when x and y are composite in nZ. If x and y are composite in nZ, then $n(nZ^{*})x$ and $n(nZ^{*})y$ and so $nZ^{*}(x, y) \geq n$. Since $1 \in nZ^{*}$, x and y are not relatively prime in nZ^{*}.

Theorem 2.3. Two (positive) composites x = $k_1 n^2$ and y = $k_2 n^2$ are relatively prime in nZ if and only if k_1 and k_2 are relatively prime in Z.

Proof. We prove that x and y are not relatively prime in nZ if and only if k_1 and k_2 are not relatively prime in Z. Since x and y are positive, $k_1, k_2 > 0$. Since x and y are composite, they have divisors in nZ. Now, x and y are not relatively prime in nZ if and only if $nZ(x, y) = d$ where $d \in nZ$ and $d > n$. But $d \in nZ$ and d > n if and only if there is an $s \bullet$ Z such that $d = sn$ and $s > 1$.

Since $d(nZ)x$ and $d(nZ)y$, $\exists \, m_1, m_2 \in nZ$ such that $dm_1 = x = k_1 n^2$ and $dm_2 = y = k_2 n^2$. Since $m_1, m_2 \in nZ$, $\exists r_1, r_2 \bullet$ Z such that $m_1 = r_1 n$ and $m_2 = r_2 n$. Thus $(sn)(r_1 n) = k_1 n^2$ and $(sn)(r_2 n) = k_2 n^2$ or $s = k_1/r_1$ and $s = k_2/r_2$. Now $s > 1$ if and only if $k_1 > r_1 \geq 1$ and $k_2 > r_2 \geq 1$ if and only if $k_1, k_2 \geq 2$. Since $s = k_1/r_1 = k_2/r_2$, then

$$k_1 r_2 + (-k_2) r_1 = 0 . \tag{1}$$

That x and y are not relatively prime in nZ is equivalent to saying the linear Diophantine equation (1) above has solutions z_1, z_2 such that $1 < z_1 < k_1$ and $1 \leq z_2 < k_2$. Geometrically, this is equivalent to saying the line segment joining the lattice point (k_1, k_2) in the first quadrant to $(0, 0)$ contains at least one other lattice point, as shown in Fig. **1**.
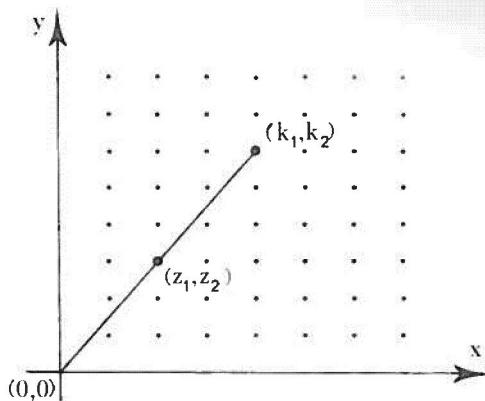
FIGURE 1

It is known (see [6, page 1101, or the reader can supply his own proof) that the line segment joining (k_1,k_2) to $(0,0)$ does not contain any lattice points if and only if k_1 and k_2 are relatively prime in Z, that is, $Z(k_1,k_2) = 1$. Hence, x and y are not relatively prime in nZ if and only if k_1 and k_2 are not relatively prime in Z.

3. Euler's Function

Let $\phi_A(x)$ denote *the number of positive integers in A which are less than or equal to x and relatively prime to x in A.*

For easy comparison, we state two theorems in connection with this function for the sets nZ and nZ^* and give the proofs later.

Theorem 3.1.
(1) If $x = kn$ is prime in nZ, then $\phi_{nZ}(kn) = k$.
(2) If $x = kn^2$ is composite in nZ, then $\phi_{nZ}(kn^2) = \pi nZ(kn^2) + \phi_Z(k)$.

Theorem 3.2.
(1) If $x = kn$ is prime in nZ^*, then $\phi_{nZ^*}(x) = k + 1 = \phi nZ(x) + 1$.
(2) If $x = kn^m$ is composite in nZ^*, then
   (a) if x is uniquely factorable,

$$\phi_{nZ^*}(x) = \left[\begin{array}{ll} \pi_{nZ^*}(x) & \text{if } k = 1 \\ \\ \pi_{nZ^*}(x) - 1 & \text{if } k \text{ is prime in } Z. \end{array} \right.$$

   (b) if x is not uniquely factorable,
$$\phi_{nZ^*}(x) = \phi_{nZ}(x) - \tau_Z(k) + 1 .$$

Proof of 3.1:
   (1) If x is prime in nZ, then x has no divisors in nZ and so every integer in nZ which is less than or equal to x is relatively prime to x in nZ. Thus $\phi_{nZ}(kn) = k$.
   (2) Now $\phi_{nZ}(x) = P + C$ where
      P = the number of primes in nZ which are less than X and are are relatively prime to x and
      C = the number of composites in nZ which are less than or equal to x and are relatively prime to x.
Since the primes in nZ have no divisors in nZ, every prime in nZ is relatively prime to x and the number of primes in nZ which are less than x is simply $\pi_{nZ}(kn^2)$. Thus, $P = \pi_{nZ}(kn^2)$.
   If $y \le x$ is also composite, then $y = k_1 n^2$. Since, by Theorem *2.3*, $nZ(x,y) = n$ if and only if $Z(k,k_1) = 1$, C equals the number of k_1 in Z which are less than or equal to k and are relatively prime to k in Z. Thus, $C = \phi_Z(k)$.

   Proof of 3.2:
   (1) If $x = kn$ is prime in nZ^*, then the only divisor (in addition to x) of x in nZ^* is 1 which is also relatively prime to x in nZ^*. Again, all other positive integers in nZ^* which are less than or equal to x are relatively prime to x in nZ^*. Hence, $\phi_{nZ^*}(x) = \phi_{nZ}(kn) + 1 = k + 1$.
   (2) Now $\phi_{nZ^*}(x) = P + C + 1$ where
      P = the number of primes in nZ^* which are less than x and relatively prime to x in nZ^*, and
      C = the number of composites in nZ^* which are less than or equal to x and relatively prime to x in nZ^*.
(The presence of 1 in the formula is due to the fact that 1 is also relatively prime to x in nZ^*.) But $C = 0$ since, by Theorem *2.2*, no two composites in nZ^* are relatively prime in nZ^*.
   Now P = the number of primes in nZ^* which are less than x minus the number of primes in nZ^* less than X which divide x in nZ^*.
(a) If x is uniquely factorable, then by [2, Theorem 2.3], $x = kn^m$ where; $k = 1$ or a prime in Z. If $k = 1$, then the only prime in nZ^* which divides x is n. Hence, $\phi_{nZ^*}(n^m) = \pi_{nZ}(n^m) - 1 + 1 = \pi_{nZ}(n^m)$. If k is a prime in Z, the only primes in nZ^* which divide x are n and kn. Hence, $\phi_{nZ^*}(kn^m) = \pi_{nZ}(kn^m) - 2 + 1 = \pi_{nZ}(kn^m) - 1$.

(b) If $x = kn^m$ is not uniquely factorable the only prime divisors in nZ^* of x are those obtained by splitting up k into its various divisors. Since there are $\tau_Z(k)$ divisors of k, there are $\tau_Z(k)$ prime divisors of x in nZ^*. Hence, $\phi_{nZ^*}(kn^m) = \tau_{nZ}(n^m) - \tau_Z(k) + 1$.

4. Conclusion

It is interesting to ask which results from elementary number theory relativize to nZ and nZ^*. For example, the following important basic theorem does not strictly generalize to these two sets: If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$. The proof of this theorem and many others in Z ultimately rest on Euclid's Division Algorithm, which does not appear to relativize either.

This line of inquiry has already turned out to be fruitful since we have been able to prove, based on the results in [2], that there are no odd perfect numbers, a result which will be presented elsewhere. It may be possible to approach other problems of number theory using the techniques presented here and in [2].

REFERENCES

1. Andrews, G. E., Number Theory, W. B. Saunders Company, Philadelphia, Pennsylvania, 1971.

2. Artino, C. Q. and Kolod, J. R., "Elementary Number Theory in Certain Subsets of the Integers, I," this *Journal*, 5, No. 10 (1974), pp. 489-495.

3. Barnett, I. A., Elements of Number Theory, Prindle, Weber, & Schmidt Inc., Boston, Mass., 1969.

4. Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, 4th ed., Oxford University Press, New York, 1966.

5. Niven I. and Zuckerman, H. S., An Introduction to the *Theory* of Numbers, 2nd ed., John Wiley & Sons, Inc., New York, 1966.

6. Rademacher, H., Lectures on Elementary Number Theory, Blaisdell Publishing Company, New York, New York, 1964.

CORRECTION TO ELEMENTARY NUMBER THEORY IN SUBSETS OF Z, ▌

The authors of the preceding article wish to thank Professor Paul Schaefer for pointing out an error in the formulas of Lemma 2 and Theorem 6 on p. 495 of their previous article, Reference [2], above. The proofs of [2] apply as well to the following corrected versions: Lemma 2 should read: Let $x = n^m$. If n is prime in Z or if m $\leq$ 2, then $\tau_{nZ}(n^m) = $ m $-$ 1 and $\sigma_{nZ}(n^m) = \sum_{r=1}^{m-1} n^r = (n^m - n)/(n - 1)$. Theorem 6
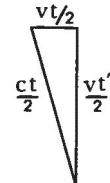
should read: Let $x = kn^m$, where m $\geq$ 2 and $n \nmid$ k. If n is prime in Z or if m $\leq$ 2 then $\tau_{nZ}(kn^m) = (m - 1)\tau_Z(k) = \tau_{nZ}(n^m)\tau_Z(k)$ and $\sigma_{nZ}(kn^m) = \sigma_Z(k)(n^m - n)/(n - 1) = \sigma_{nZ}(n^m)\sigma_Z(k)$.

PYTHAGORAS AND EINSTEIN

A reader, Bruce Bushman, Laguna Beach, California, has sent the following interesting observations and derivation of a formula in relativity.

The clock or time effect in special relativity is a simple application of the ancient theorem of Pythagoras. Consider a scientist riding a train at the given speed v. If he shines a flashlight down to a mirror laid at his feet, he can shine a pulse down and back up in a measured time t'. While he considers the path of the pulse to be two straight vertical lines of length ct', where c is the speed of light, an observer outside the train looking through a large window will consider the path of the pulse to be the shape of a "V", or along the hypotenuse of each of two right triangles with total horizontal length vt, where t is the time of the experiment as measured by the outside observer. The total length of the V-shaped path of light according to the outside observer is ct.

Now it is time to use the Pythagorean Theorem. Let us simplify by using only half the figure (one right triangle) as shown below. Doubling

and using the Pythagorean Theorem,

$$(ct)^2 = (vt)^2 + (ct')^2.$$

Our goal is to isolate t' to learn the relation between the moving clock and the stationary clock. Simple algebra and the fact that t' must be positive yields

$$t' = t\sqrt{1 - (v/c)^2} ,$$

which is Einstein's famous formula.

One immediately deduces from this the classical result that since $t' < t$ the moving clock is slower than the stationary clock.

ON ANALYTIC FUNCTIONS OF A QUATERNION VARIABLE

By **A. M. Buoncristiana**
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In a recent issue of this journal Joseph J. Buff [1] discussed a characterization of an analytic quaternion valued function of a quaternion variable. His development was based on a generalization of conventional complex variable theory and he deduced that a quaternion valued function, analytic according to his definition, had the general form of a "linear" function

$$f(Q) = cQ + B$$

where c is a real constant and B a quaternion constant. In this note we introduce a different definition of analytic function and examine it using the algebraic properties of quaternions. Our result, while similar in form to Buff's, allows a wider class of analytic functions. In fact, any complex analytic function can be extended directly to a quaternion analytic function.

A quaternion can be expressed as a linear combination of one real unit, e_0, and three imaginary units, e_1, e_2, e_3:

$$X = X^0 e_0 + X^1 e_1 + X^2 e_2 + X^3 e_3 \qquad (1)$$

where X^0, X^1, X^2, X^3 are real numbers. Upon multiplication of the units among themselves, e_0 acts like an identity, while the imaginary units satisfy

$$e_k e_k = -e_0 \qquad (2)$$

$$e_k e_\ell = e_m = -e_\ell e_k$$

with k, ℓ, m any cyclic permutation of $1, 2, 3$. We can readily see that the complex field is isomorphic to the subsystem of quaternions obtained by setting $X^2 = X^3 = 0$ and identifying

$$X^0 e_0 + X^1 e_1$$

with

$$X^0 + X^1 i.$$

For convenience we shall denote

$$X = X^0 e_0 + \vec{X}, \qquad (3)$$

X^0 and $\vec{X}$ being called the **real** and **imaginary** parts of X, respectively. The **conjugate** of the quaternion X, denoted by $\overline{X}$, is obtained by replacing the imaginary part by its negative:

$$\overline{X} = X^0 e_0 - \vec{X}; \qquad (4)$$

this operation $(X \to \overline{X})$ is an involution, that is, $\overline{\overline{X}} = X$ and $\overline{XY} = \overline{X}\,\overline{Y}$. Furthermore, it is easy to verify that

$$X + \overline{X} = T(X)e_0,$$
$$X\overline{X} = \overline{X}X = N(X)e_0, \qquad (5)$$

with $T(X)$ and $N(X)$ real valued functions of X^0, X^1, X^2, X^3 given by

$$T(X) = 2X^0,$$
$$N(X) = \sum_{\mu=0}^{3} (X^\mu)^2. \qquad (6)$$

From the equations (5) it is seen that every **quaternion** X satisfies the characteristic equation

$$\lambda^2 - T(X)\lambda + N(X)e_0 = 0; \qquad (7)$$

if we regard (7) as an equation over the complex field (with e_0 omitted) the roots of (7) are

$$\lambda_\pm = X^0 \pm i \left[\sum_{k=1}^{3} (X^k)^2 \right]^{1/2} \qquad (8)$$

$$= X^0 \pm i|\vec{X}|.$$

If $f(z)$ is an entire function of the complex variable z, so that for all finite z,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

we define an entire function of the quaternion **variable** X to be the corresponding representation by infinite series

$$f(X) = \sum_{n=0}^{\infty} c_n X^n, \qquad (9)$$

where infinite series of quaternions are formulated in terms of limits exactly as in complex variables. There is a straightforward extension

of this definition in case $f(z)$ is analytic in some restricted domain, but for simplicity we consider only entire functions here. To exploit this definition we need to examine the powers of X. Since X satisfies the quadratic equation (7) each power of X can be reduced to a linear function of X. Specifically, we have

$$X^n = A_n X + B_n e_0 \tag{10}$$

where A_n and B_n are real numbers satisfying the recursion formulas

$$A_{n+1} = T(X)A_n + B_n$$
$$B_{n+1} = -N(X)A_n$$

with initial conditions $A = 0$, $B_0 = 1$. The solution to this system is found by noting that over the complex field the equation corresponding to (10) also holds:

$$\lambda^n = A_n \lambda + B_n,$$

or

$$\lambda^n = A_n \lambda_+ + B_n, \qquad \lambda_-^n = A_n \lambda_- + B_n. \tag{10'}$$

Thus,

$$A_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \qquad B_n = \frac{\lambda_+ \lambda_-^n - \lambda_- \lambda_+^n}{\lambda_+ - \lambda_-}, \tag{11}$$

provided that $A_+ - A_-$ is non-zero. This latter case is easily handled separately, for we have $\lambda_+ = A$ if and only if $|\vec{X}| = 0$, or X real. Using equations (10) and (11) to eliminate X^n in equation (9) we obtain

$$f(X) = A(f,X)X + B(f,X)e_0 \tag{12}$$

where

$$A(f,X) = \frac{f(\lambda_+) - f(\lambda_-)}{\lambda_+ - \lambda_-}$$

and

$$B(f,X) = \frac{\lambda_+ f(\lambda_-) - \lambda_- f(\lambda_+)}{\lambda_+ - \lambda_-}$$

Thus while $f(X)$ has a linear form. the coefficients A and B are complex functions of the characteristic roots of X.

As an example consider a purely imaginary quaternion X (thus

$X + \overline{X} = 0$). The characteristic roots of X are given by $\lambda_\pm = \pm i|\vec{X}|$. We obtain directly, with $f(X) = e^X$,

$$A(f,X) = \frac{e^{i|\vec{X}|} - e^{-i|\vec{X}|}}{2i|\vec{X}|} = \frac{\sin|\vec{X}|}{|\vec{X}|},$$

$$B(f,X) = \frac{i|\vec{X}|e^{-i|\vec{X}|} + i|\vec{X}|e^{i|\vec{X}|}}{2i|\vec{X}|} = \cos|\vec{X}|,$$

or

$$e^X = \cos|\vec{X}| + \frac{X}{|\vec{X}|}\sin|\vec{X}|,$$

an obvious generalization of Euler's formula for complex numbers. From here it is easy to obtain a polar decomposition for quaternions.

Remark 1. Since all derivatives of a complex analytic function are analytic, we can define all derivatives of $f(X)$.

Remark 2. The definition given here can be extended to analytic functions of any power associative algebra. In particular it applies also to octonion valued functions of octonions. Furthermore, since the octonions also satisfy the quadratic identity (7) with T and N as given by equations (5), the general form of an analytic octonion function is given by (12) also.

REFERENCES[1]

1. Buff, J. J., "Characterization of an Analytic Function of a Quaternion Variable", this Journal, 5, No. 8 (1973), pp. 387-392.

[1]See also C. A. Deavours, "The Quaternion Calculus," American Mathematical Monthly, 80 (1973), pp. 995-1008, where yet another definition of quaternion analyticity appears and where a theory of quaternion integration is also undertaken. --Editor.

REFEREES FOR THIS ISSUE

A NEW PUZZLE

Victor Feser, St Louis, Missouri, has pointed out that the puzzle which appeared in the Fall 1973 issue cannot be solved without additional information, such as, for example, a comment by the native as to the identity of the other native. We therefore state below a corrected and slightly more involved version, and invite readers to find a solution.

On an island in the Pacific lived two tribes, the Blue men and the Green men. The Blue men always told the truth, and the Green men always lied, unless a Blue man was present, in which case they also told the truth. Once some men were shipwrecked on the island, and the natives agreed to release them if their captain could solve a puzzle. He was blindfolded, placed in a room with 10 natives, and was to guess their tribes within 5 minutes, using any clues he could get from the conversation. One of the natives spoke inaudibly, so the captain asked another native what he had said. The native answered "He said we are all Green men." The captain immediately identified all the natives and his men were released. What reasoning led the captain to his conclusion?

ESSAY ON A FIBONACCI-LIKE SEQUENCE

by Jeffrey Cohen
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Consider the sequence: 1, 1, 2, 3, 5, 8, 13, 21,·... This is the well-known non-repeating Fibonacci sequence. In the sequence below we consider a variation of the Fibonacci sequence: Whenever a two-digit number is reached, we add the sum of the individual digits to form another term which replaces the two-digit number, and continue the sequence in this manner. Thus:

$$1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8,$$
$$1, 9, 1, 1, 2, 3, 5, 8, \cdots \tag{1}$$

Notice, that after the 24th term, the series begins to repeat itself. This series when added has a total of 117.

Now take a similar series, beginning with 1, 3:

$$1, 3, 4, 7, 2, 9, 2, 2, 4, 6, 1, 7, 8, 6, 5, 2, 7, 9, 7, 7, 5, 3, 8, 2,$$
$$1, 3, \cdots.$$ Here again, after the 24th term, the series begins to repeat. Also the sum of the series is 117. Hereafter, a series like this will be termed a q series.

Now, in a more general fashion, consider the following series:

$$f_1 = a, \; f_2 = b, \; f_3 = a + b, \; f_4 = a + 2b,$$

$$f_5 = 2a + 3b, \; f_6 = 3a + 5b, \; f_7 = 5a + 8b, \; f_8 = 8a + 13b,$$

and, in general

$$f_n = F_{n-2}a + F_{n-1}b, \tag{2}$$

where F_n is the nth Fibonacci number.

<u>Theorem 1</u>. $F_n = F_{n-2} + F_{n-1}$

Proof. Substitute $f_1 = a = 1$, and $f_2 = b = 1$ for equation two. Thus $f_n = F_{n-2}a + F_{n-1}b$, becomes $Fn = F_{n-2} + F_{n-1}$ and throughout.

<u>Definition 1</u>. If a and b are both integers such that:

$$0 < a < 10 \quad \text{and} \quad 0 < b < 10,$$

then $[a + b]^*$ means to add $(a + b)$, and if their sum exceeds 9, add the two separate digits of the sum to form one, one digit number.]

For example,

$$[5 + 7]^* = [12]^* = 3$$
$$[8 + 9]^* = [17]^* = 8$$

Note: For Definition 1, if a and b are positive integers not of the form $0 < a < 10$ or $0 < b < 10$, they may be made of that form by taking $[a]^*$ or $[b]^*$, and adding up the individual digits, so that a one-digit number is obtained (e.g. $[52]^* = 7$, $[2147]^* = 5$).

Theorem 2. $[a + 9]^* = a$.

Proof. Nine is the additive identity for this operation. This is because $[a + 9]^*$ may be written as $[[a] + 9]^*$. Here, $0 < [a]^* < 10$, and if $[a]^* = 1, 2, 3, \cdots, 9$, then:

$[1 + 9]^* = [10]^* = 1$	$[2 + 9]^* = [11]^* = 2$
$[3 + 9]^* = [12]^* = 3$	$[4 + 9]^* = [13]^* = 4$
$[5 + 9]^* = [14]^* = 5$	$[6 + 9]^* = [15]^* = 6$
$[7 + 9]^* = [16]^* = 7$	$[8 + 9]^* = [17]^* = 8$
$[9 + 9]^* = [18]^* = 9$	

Thus, all possible values for a have been verified.

Theorem 3. $[9 \cdot a]^* = 9$.

Proof. That nine acts like a zero in multiplication for this operation can be proved by again testing all possible values of a; thus:

$[1 \cdot 9]^* = [9]^* = 9$	$[2 \cdot 9]^* = [18]^* = 9$
$[3 \cdot 9]^* = [27]^* = 9$	$[4 \cdot 9]^* = [36]^* = 9$
$[5 \cdot 9]^* = [45]^* = 9$	$[6 \cdot 9]^* = [54]^* = 9$
$[7 \cdot 9]^* = [63]^* = 9$	$[8 \cdot 9]^* = [72]^* = 9$
$[9 \cdot 9]^* = [81]^* = 9$	

By applying Definition 1, to equation (2), any general term of the q series may be represented by the expression:

$$q_n = [F_{n-2}a + F_{n-1}b]^* \quad \text{where } n > 1 \text{ and } F_0 = 0.$$

---

[1]It may be easily shown that $[a + b]^*$ is the sum of a + b reduced modulo 9 -- Editor

Properties of the q Series

1. The q sequence repeats after every 24 digits,
2. The sum of the first 24 terms of the q sequence is $[117]^* = 9$.
3. The sum of two digits a and b, that are 12 tern's apart is $[a + b]^* = 9$.
4. The series contains two pairs of repeated numbers.

Property 1. The q sequence repeats after every 24 digits.

Proof. Let the first two terms be a, b. If the sequence repeats after every 24 digits, then the 1st and 2nd term should equal the 25th and 26th term, and so forth. Using $[F_{n-2}a + F_{n-1}b]^*$ for a general term of the sequence, then:

$$q_{25} = [F_{23}a + F_{24}b]^* \quad \text{and} \quad q_{26} = [F_{24}a + F_{25}b]^*$$

If the sequence repeats after 24 digits, then we should have:

$$a = [F_{23}a + F_{24}b]^* \quad \text{and} \quad b = [F_{24}a + F_{25}b]^*.$$

By equation (1) $[F_{23}]^* = 1$; $[F_{24}]^* = 9$; $[F_{25}]^* = 1$, so $q_{25} = [a + 9b]^*$ and $q_{26} = [9a + b]^*$. By Theorem 3, $[9a]^* = 9$ and $[9b]^* = 9$, so $q_{25} = [a + 9]^*$ and $q_{26} = [9 + b]^*$. By Theorem 2, $[a + 9]^* = a$ and $[b + 9]^* = b$, so $q_{25} = a$ and $q_{26} = b$.

Note: This only means the series repeats at least once after 24 terms. However, it may repeat in multiples of 24 (e.g., 1, 8, $\cdots$).

Property 2. The sum of the first 24 terms of the q sequence is $[117]^* = 9$.

Proof. The first few terms of the q sequence are:

$$a, b, a + b, a + 2b, 2a + 3b, \cdots, F_{n-2}a + F_{n-1}b,$$

so the sum of the first 24 terms is:

$$a + \sum_{n=2}^{24} [F_{n-2}a + F_{n-1}b]^* = [a + \sum_{n=2}^{24} F_{n-2}a + \sum_{n=2}^{24} F_{n-1}b]^*$$

But from equation (1),

$$\sum_{n=2}^{24} F_{n-2} = [117 - (1 + 9)]^* = [107]^* = 8,$$

and

$$\sum_{n=2}^{24} F_{n-1} = [117 - 9]^* = [108]^* = 9,$$

so

$$[a + 8a + 9b]^* = [9a + 9b]^* = [9(a + b)]^* = 9$$

by Theorem **3**.

Property **3**. In the q sequence the sum of two digits which are twelve terms apart is 9.

Proof. Let the first term equal $[F_{n-2}a + F_{n-1}b]^*$ and the twelfth term, $[F_{n+10}a + F_{n+11}b]^*$, so we must prove that

$$[F_{n-2}a + F_{n-1}b + F_{n+10}a + F_{n+11}b]^* = 9,$$

or

$$[(54F_{n+1} + 36F_n)a + (90F_{n+1} + 54F_n)b]^* = 9$$

by Theorem 1, as the following shows:

$$
\begin{aligned}
F_{n+10} &= F_{n+9} + F_{n+8} \\
&= 2F_{n+8} + F_{n+7} \\
&= 3F_{n+7} + 2F_{n+6} \\
&= 5F_{n+6} + 3F_{n+5} \\
&= 8F_{n+5} + 5F_{n+4} \\
&= 13F_{n+4} + 8F_{n+3} \\
&= 21F_{n+3} + 13F_{n+2} \\
&= 34F_{n+2} + 21F_{n+1} .
\end{aligned}
$$

Then,

$$
\begin{aligned}
F_{n+10} + F_{n-2} &= 55F_{n+1} + 34F_n + F_{n-2} \\
&= 54F_{n+1} + F_n + F_{n-1} + 34F_n + F_{n-2} \\
&= 54F_{n+1} + 36F_n .
\end{aligned}
$$

Similarly,

$$F_{n+11} + F_{n-1} = 90F_{n+1} + 54F_n .$$

But $[54]^* = [36]^* = [90]^* = 9$, so we obtain
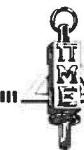
$$
\begin{aligned}
[(9F_{n+1} + 9F_n)a + (9F_{n+1} + 9F_n)b]^* &= [(9F_{n+1} + 9F_n)(a + b)]^* \\
&= [9(F_{n+1} + F_n)(a + b)]^* \\
&= 9 ,
\end{aligned}
$$

by Theorem **3**.

The q series totals 117, and repeats exactly 24 terms for most cases. Out of 81 possible cases for (a,b), the q series will total 117 and repeat after 24 terms, while for nine pairs the q series will **not**. These nine q series either repeat in multiples of 24 terms (when $(a,b) = (9,9)$), or repeat every 8 terms (when $(a,b) = (3,3),(3,6),(3,9)$, $(6,3),(6,6),(6,9),(9,3)$ or $(9,6)$). Thus, **after every** 24 terms, the series will repeat. These nine pairs of digits will add to $5(9,9)$ or $(45,45)$: $(3,3) + (3,6) + (3,9) + (6,3) + (6,6) + (6,9) + (9,3) + (9,6) = (45,45)$.

The q series actually consists of only 5 different series. There are 24 pairs (a,b) (e.g. $(1,1)$) which will generate the first series, 24 pairs (e.g. $(1,3)$) for the second series, 24 pairs (e.g. $(1,4)$) for the third series, eight pairs for the fourth series $((3,3),(3,6),(3,9)$, $(6,3),(6,6),(6,9),(9,3)$ and $(9,6))$, and one pair $(9,9)$ for the fifth series.

THE BINOMIAL AND POISSON DISTRIBUTION LIMIT THEOREM VIA MOMENT GENERATING FUNCTIONS

Professor Joseph M. Moser has indicated a very short and direct proof for Journal readers of the following well known theorem in Probability and statistics, which avoids the customary use of density functions.

Theorem. The Binomial Distribution approaches the Poisson Distribution as n approaches infinity.

Proof. The M.G.F. (moment generating function) of the Binomial Distribution is $(1 + p(e^t - 1))^n$. If one lets $\lambda = np$, one obtains $(1 + \lambda/n(e^t - 1))^n$. Now $\lim_{n \to \infty} (1 + \lambda/n(e^t - 1))^n = e^{(e^t - 1)}$, which is the M.G.F. of the Poisson Distribution.

EVEN ORDER MAGIC SQUARES WITHIN MAGIC SQUARES

by Steven J. Eberhard
Texas A&M University

In previous articles in this journal, magic squares containing magic squares within them have been considered. So far, all of these have been of odd order. Here we explore two such magic squares of even order. In Fig. 1 below, we exhibit a six by six magic square which contains a four by four magic square.

$n+(3c+1)b$	$n+(7c)b$	$n+(5c+1)b$	$n-(7c-2)b$	$n-(5c+2)b$	$n-(3c+2)b$
$n-(3c+3)b$	$n+(3c)b$	$n-(3c-1)b$	$n-(3c-2)b$	$n+(3c-3)b$	$n+(3c+3)b$
$n-(5c-1)b$	$n-(c+1)b$	$n+(c)b$	$n+(c-1)b$	$n-(c-2)b$	$n+(5c-1)b$
$n-(5c)b$	$n+(c-2)b$	$n-(c-1)b$	$n-(c)b$	$n+(c+1)b$	$n+(5c)b$
$n+(7c-1)b$	$n-(3c-3)b$	$n+(3c-2)b$	$n+(3c-1)b$	$n-(3c)b$	$n-(7c-1)b$
$n+(3c+2)b$	$n-(7c)b$	$n-(5c+1)b$	$n+(7c-2)b$	$n+(5c+2)b$	$n-(3c+1)b$

FIGURE 1

Each element of this magic square is of the form $(n + kb)$, where n and b are arbitrary whole numbers and k takes on thirty-six distinct values. In the notation of Strum in [2], these values of k are given by:

$$k = qc + p$$

where, with four exceptions,

$$q = -7, -5, -3, -1, 1, 3, 5, 7$$

and for each value of q except $q = -7$, $q = +7$,

$$p = -2, -1, 0, 1, 2.$$

For q = -7, $q = +7$,

$$p = -2, -1, 0.$$

The four exceptions to this pattern are the following: $k = -(c + 2)$, $k = +(c + 2)$, $k = -(5c - 2)$, $k = +(5c - 2)$ are replaced respectively, by $k = -(3c - 3)$, $k = +(3c - 3)$, $k = -(3c + 3)$, and $k = +(3c + 3)$. To insure that all thirty-six elements are distinct, it is sufficient that we have $c \geq 3$.

It is interesting to note, that while n and b may be chosen as arbitrary whole numbers, it is impossible to produce the "standard" numerical magic square of order six using whole numbers for n and b. (The "standard" numerical magic square of order six is the square which contains the consecutive whole numbers from 1 to 36. It has the magic sum of 111, and the four by four magic square contained in it has the magic sum of 74.) For obviously, in this case,

$$n = 111 \div 6 = 18\%.$$

Furthermore, we may generalize from this set of values as follows: n may be chosen as an arbitrary mixed number of the form $(j + \frac{1}{2})$, j an integer, provided that a is also of the form $(j + \frac{1}{2})$, $c \geq 2\frac{1}{2}$ (to insure thirty-six distinct values of k), and b is again an arbitrary whole number. It is this use of numbers of the form $(j + \frac{1}{2})$ that explains the absence of even numbers as values of q in the notation

$$k = qc + p$$

For clearly, if q were even, a of the form $(j + \frac{1}{2})$, and n of the form $(j + \frac{1}{2})$, we would have a fraction as an element of the magic square.

Fig. 2 exhibits an eight by eight magic square which contains a six by six magic square and a four by four magic square. The six by six magic square in the middle of this eight by eight magic square is exactly the same as the one which appears in Fig. 1. Therefore the same analysis applies, with the following exceptions:
The values of q are now

$$q = -13, -11, -9, -7, -5, -3, -1, 1, 3, 5, 7, 9, 11, 13$$

and for each value of q except q = -13, q = +13,

$$p = -2, -1, 0, 1, 2.$$

For q = -13, q = +13,

$$p = -2, -1.$$

There are two more exceptions to this pattern, in addition to those in the six by six magic square, as follows: k = -(9c - 2), and k = +(9c - 2) are replaced respectively, by k = -(7c + 3), k = +(7c + 3). The argument with respect to n of the form $(j + \frac{1}{2})$ still holds, so to insure that the sixty-four elements are distinct, it is sufficient that we have $c \geq 2\frac{1}{2}$.

In [1], Moser proposes that if one uses large enough values for q and p, any odd order magic square may be constructed which contains within it successive magic squares. Extending this proposition to even

n+(7c+1)b	n+(13c-1)b	n+(11c+1)b	n+(11c)b	n-(11c+2)b	n-(13c-2)b	n-(11c-1)b	n-(7c+2)b
n-(9c+1)b	n+(3c+1)b	n+(7c)b	n+(5c+1)b	n-(5c+2)b	n-(7c-2)b	n-(3c+2)b	n+(9c+1)b
n-(9c)b	n-(3c+3)b	n+(3c)b	n-(3c-1)b	n+(3c-3)b	n-(3c-2)b	n+(3c+3)b	n+(9c)b
n-(9c-1)b	n-(5c-1)b	n-(c+1)b	n+(c)b	n-(c-2)b	n+(c-1)b	n+(5c-1)b	n+(9c-1)b
n-(7c+3)b	n-(5c)b	n+(c-2)b	n-(c-1)b	n+(c+1)b	n-(c)b	n+(5c)b	n+(7c+3)b
n+(9c+2)b	n+(7c-1)b	n-(3c-3)b	n+(3c-2)b	n-(3c)b	n+(3c-1)b	n-(7c-1)b	n-(9c+2)b
n+(11c-2)b	n+(3c+2)b	n-(7c)b	n-(5c+1)b	n+(5c+2)b	n+(7c-2)b	n-(3c+1)b	n-(11c-2)b
n+(7c+2)b	n-(13c-1)b	n-(11c+1)b	n-(11c)b	n+(11c+2)b	n+(13c-2)b	n+(11c-1)b	n-(7c+1)b

FIGURE 2

order magic squares through the present paper, this author is of the opinion that a magic square of *any* order may be constructed which will contain within it successive magic squares.

REFERENCES

1. Moser, Joseph I-i., "Magic Squares Within Magic Squares," this *Journal,* 5, No. 8 (1973), p. 430.
2. Strum, Robert C., "Some Comments on 'A Class of Five by Five Magic Squares,'" this *Journal,* 5, No. 6 (1972), pp. 279-280.

1975 NATIONAL MEETING IN KALAMAZOO

It is not too early for local chapters to be making plans for the national meeting at Western Michigan University in Kalamazoo, Michigan in conjunction with the Mathematical Association of America. Plan now to send your best undergraduate speaker or delegate (or both) to that meeting. Travel money for approved speakers and delegates is available from National. Send requests and proposed papers to:

R. V. Andree
Secretary-Treasurer, Pi Mu Epsilon
601 Elm Avenue, Room 423
The University of Oklahoma
Norman, Oklahoma 73069

REGIONAL MEETINGS OF MM

Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such under-graduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send requests to:

R. V. Andree
Secretary-Treasurer, Pi Mu Epsilon
601 Elm Avenue, Room 423
The University of Oklahoma
Norman, Oklahoma 73069

1972-1973 MANUSCRIPT CONTEST WINNERS

The judging for the best expository papers submitted for the 1972-73 school year has now been completed. The winners" are:

FIRST PRIZE ($200): Sam W. Talley, Western Kentucky University, for his paper "Niceness of the Socle and a Characterization of Groups of Bounded Order" (this *Journal,* Vol. 5, No. 10, pp. 497-502).

SECOND PRIZE ($100): Daniel Minoli, Polytechnic Institute of New York, for his paper "Use of Matrices in the Four Color Problem" (this *Journal,* Vol. 5, No. 10, pp. 503-511).

THIRD PRIZE ($50): Roseann Moriello, Seton Hall University, for her paper "Partial Differentiation on a Metric Space" (this *Journal,* Vol. 5, No. 10, pp. 514-519).

1971-72 LOCAL CHAPTER CONTESTS

During 1971-72 two local chapters entered the manuscript contest for the first time, and the decision on winners has finally been reached.

MISSOURI GAMMA (St. Louis University). The winner of the $20 award is Dennis C. Swolarski for his paper "Generalizing Binary Operations." Honorable mention is awarded to Karin Stahara, Vasilios Alexiades, and Robert T. Griffin.

SOUTH DAKOTA BETA (South Dakota School of Mines). The winner of the $20 award is Clint R. Cole for his paper "A Poly-alphabetical Substitution Cipher with a Pseudo-Random One-Time Key." Honorable mention is awarded to Barbara J. Baskerville, Dale Koepp, and Jane Vande Bossche.

1974-75 CONTEST

Papers for the 1973-74 contest are now being judged, and we are receiving papers for this year's contest so be sure to send us your paper, or your chapter's papers. *In order to be eligible, authors must not have received a Master's degree at the time they submit their paper.*

GLEANINGS FROM CHAPTER REPORTS

CALIFORNIA ALPHA at the University of California at Los Angeles helped sponsor a new program of mathematics instruction for high school students called the *Innovative Mathematics Seminar*. The program, conducted through the summer of 1974, was directed by *Jeff Alpert* and *Patricia Yamamoto*. Its purpose is to (1) provide a course in the real mathematics of the mathematician to high school students, and (2) provide the experience of actual classroom teaching of mathematics to prospective mathematics teachers.

CALIFORNIA ETA at the University of Santa Clara heard *Professor Rafael Robinson* from the University of California at Berkeley speak on "Kronecker's Two Theorems About Equations with Integer Coefficients."

COLORADO DELTA at the University of Northern Colorado heard *Lawrence E. Gatterer* from the National Bureau of Standards, Boulder, Colorado, speak about the professional society in Thailand and his experiences with the Thailand Standards Team.

DELEWARE ALPHA at the University of Deleware heard *John Norton* of the Du Pont Company speak on "Three Levels of Mathematics Found in Today's Modern Businesses."

FLORIDA EPSILON at the University of Southern Florida sponsored a lecture by *Professor Garrett Birkhoff* from Harvard University on "A Role for Computing in Undergraduate Mathematics." Outstanding Senior for 1973, *Joseph Weintraub*, spoke on the topic "Continued Frctions."

FLORIDA ZETA at Florida Atlantic University heard *Peter DiPaola*, formerly Deputy Superintendent of Schools, New Rochelle, New York, speak on the topic "The Meaning of Standard Deviation: A Practical Approach."

GEORGIA GAMMA at Armstrong State College sponsored lectures by *Professor Anne L. Hudson* entitled "Lattice Points and the Greatest Common Divisor," and *Donald Braffit*, a senior, on Fibonacci Numbers.

ILLINOIS ZETA at Southern Illinois University at Edwardsville sponsored a lecture by *Professor Irving Kessler* on the topic "Partitions of

Integers," in addition to administering the *Mathematics* Field Day involving competition among approximately 400 high school students.

LOUISIANA EPSILON at McNeese State University sponsored a series of 6 film strips at two meetings on "Uses of Computers and History" by" *Sundown Swetharanyam*, director of the computer center.

MICHIGAN ALPHA at Michigan State University heard 2 students give talks on mathematics: *Dave Bowen*, on "Bells, Braids and Groups" and 'Conic Sections in a Finite Projective Geometry," and *Russel Caflisch* on Inversion of Power Series."

MICHIGAN DELTA at Hope College held 12 meetings during the year and heard several visiting lecturers, students and faculty members speak on a variety of topics, including *Professor Richard Schmidt* from SUNY at Buffalo on "A Career in Statistical Science" and *Professor Dean Sommers* on "Geometric Constructions." Student lecturers were *Sam Quiring, Sandra Brown, Barbara Watt, Kurt Avery, William Scrafford, Charmaine Mrazek, Richard Meyers, Ellenore Thompson,* and *Marvin Dietz*.

MINNESOTA ALPHA at Carleton College heard *Professor Thomas Hawkins* from Boston University speak on "Development of Linear Algebra from the Turn of this Century," and sponsored a lecture by *Franz E. Holm* from the University of Illinois on "What Is an Automaton" and "The Role of Mathematics in Modern Society." In addition, student talks were presented by *Richard Sprecher* and *William Lang*.

NEBRASKA BETA at Creighton University heard *Professor Henry Gale* (physiology and pharmacology) speak on mathematics from a biologist's point of view, and sponsored a *Mathematics* Field Day where approximately 700 high school students participated in three contests: Chalk talk derby (talks were given on Pascal's triangle and transfinite cardinal numbers), leap frog (a speed test) and marathon (a two member team speed test).

NEW JERSEY GAMMA at Rutgers University at Camden heard lectures by *Professor Francoise Shremmer* from Bryn Mawr College on "Is Calculus of Any Use in Mathematics?" and *Professor Tropper* on "The Perfect Group."

NEW JERSEY DELTA at Seton Hall University heard *Professor John Saccoman* speak on "Non-Standard Models." Members of the chapter, *Maureen Albers, Marilyn Koby,* and *Roseann Moriello*, and students, *Victor DeLorenzo, Sheila Paterson,* and *Richard Morgan*, presented papers at the Eastern

Colleges Science Conference held at Worcester Polytechnic Institute, April 18-20, 1974. The paper by Ro-ieann Moriello, "Partial Differentiation on a Metric Space" won First Prize at the conference.

NEW JERSEY EPSILON at St. Peter's College sponsored a lecture by H. O. Pollack, Director of Mathematical Research at Bell Labs, entitled "How to Embed an Arbitrary Graph in a Cube." Members of the mathematics faculty participated in a "Dissertation Memoirs" and presented a series of talks.

NEW YORK ETA at SUNY at Buffalo heard a lecture on "Probabilistic Potential Theory" by Professor Ann M. Piech of SUNY at Buffalo.

NEW YORK PI at State University College at Fredonia sponsored a bus trip to the Science Center in Toronto.

NEW YORK PSI at Iona College participated in the Fight Against Muscular Dystrophy Carnival by manning a NIM booth with the American Mathematics Society.

NORTH CAROLINA GAMMA at North Carolina State University heard Professor Herbert E. Speece talk on the subject "The Mathematics and Science Education Program at North Carolina State University," and also heard the following student speakers: Robert Bryant, Davis Blackwelder, Elizabeth Smith, and Maryo van der Vaart.

OHIO LAMBDA at John Carroll University heard a lecture by Professor John Baker from Kent State University on "Cardinal and Ordinal Numbers" and sponsored a High School Math Day.

OHIO NU at the University of Akron heard Professor Neal C. Raber speak on "The Marriage Problem", and participated in the Ohio Section of the American Mathematical Association at Muskingham College, May 3, 1974.

OHIO ZETA at the University of Dayton sponsored talks by students, Steve Stoner on "Computer Art" and Brad Plohr on "The Delta Function and Other Such Nonsense."

PENNSYLVANIA NU at Edinboro State College was installed as a new chapter with 33 charter members on May 4, 1974 by Councilor Eileen Poiani, who talked on "New Directions in Mathematics." Following the installation banquet, Bill Means gave an entertaining demonstration on the Tower of Hanoi.

PENNSYLVANIA THETA at Drexel University sponsored a lecture "Some Mathematical Aspects of Long Distance Running" by Professor Thomas J. Osler from Glassboro State College.

RHODE ISLAND BETA at Rhode Island College sponsored a regional conference in cooperation with Providence College in April, 1974. Talks were presented by Kirk House, Charlie Huot, John Andreozzi, Joseph A. Capalbo, David J. Del Sesto, Eddy Jutras, Cathy A. Green, and Stephen M. Raymond.

TENNESSEE BETA at the University of Tennessee at Chattanooga heard Professor Clinton Smullen speak on the topic "The Friendship Theorem" and also sponsored the Freshman Mathematics Award.

TEXAS EPSILON at Sam Houston State University heard a lecture on "The Electrifying Matrix" by Professor Joe Obrien.

VIRGINIA GAMMA at Madison College heard two speakers from the University of Virginia: Professor Stephen Hedetniemi on "Theory and Application of Trees" and Lucille Whyburn on "The Calculating Scotchman: John Napier." Students who presented their own work in mathematics included Audrey Stoat and Nancy Ballard.

VIRGINIA DELTA at Roanoke College heard Professor R. E. Cape from the University of Virginia speak on the topic "What is an Operating System?"

WEST VIRGINIA BETA at Marshall University, in addition to scheduling several lectures by professional mathematicians, sponsored a College Bowl Format which pitted two teams of four mathematics students and four faculty members against each other, and a second interdepartmental college bowl in which two students and two faculty members from each of the Physics and Mathematics Departments opposed each other (Physics team won).

WISCONSIN ALPHA at Marquette University heard James Grotelueschen on "Circuital Approach to Tic-Tac-Toe," Professor Douglass Harris on "Reluctant Functions," and viewed films entitled "Inversion" and "Maurits Escher, Painter of Fantasies."

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if available, and by any information that will assist the editor.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of May 1975.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

326. *Proposed by Zazou Katz, Beverly Hills, California.*

Find solutions of the equation $x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + d^2$, where each of the sets x, y, z and a, b, c, d consists of consecutive integers.

327. *Proposed by Charles W. Trigg, San Diego, California.*

On a remnant counter there are six rolls of ribbons containing 31, 19, 17, 15, 13 and 8 yards. There are two widths of ribbons, some rolls being twice as wide as the others. There are no price marks, but all the ribbons sell for the same price per square inch. If you wish to buy $14.00 worth of each width, buying every roll but one, which roll would you leave on the counter?

328. *Proposed by Joe Van Austin, Emory University, Atlanta, Georgia.*

A group of *366* people are sequentially asked their date of birth. Assuming birthdates are independent and all days are equally likely,

find P_k, the probability that the first match is obtained when the kth person is asked. As 366 people must have at least one match,

$$\sum_{k=1}^{366} P_k = 1.$$

Show this directly.

329. *Proposed by Bernard C. Anderson, Henry Fohd Community College, Dearborn, Michigan.*

Show that $f(x) = 2x + \sin x$ is a strictly increasing function on $(-\infty, +\infty)$ by using only pre-calculus methods.

330. *Proposed by R. Robinson Rowe, Sacramento, California.*

Starting at zero-zero latitude and longitude at 12:00 noon on Monday, Rumline Crowe flew his plane at a constant 180 knots loxodromically North 45° West. Where was he on Tuesday at 12:00 noon, local standard time?

331. *Proposed by Jack Garfunkel, Forest Hills High School, New York.*

In a right triangle ABC, A = 60° and B = 30°, with D, E, F the points of trisection nearest A, B, C on the sides AB, BC and CA respectively. Extend CD, AE and BF to intersect the circumcircle (O) at points P, Q, R. Show that triangle PQR is equilateral.

332. *Proposed by Richard Field, Santa Monica, California.*

Several years ago I was spending the evening at the home of a friend who is a musicologist. While there, I received a call from the president of my company, who apologetically told me that he had traced me to ask a question he had to answer at the next morning's board meeting. Specifically, was our monthly average rate of sales growth *(6%)* compatible with his forecast that our business would double in the next year? I promised to call him back as quickly as possible with an answer. At first I thought I would have to dash home to consult my slide rule, log tables, etc. -- but then in a flash it occurred to me that my musicologist's library should provide the answer. And indeed it did! I called back in 5 minutes with the answer and proceeded without further disturbance to my social evening. What do you suppose gave me the answer?

333. *Proposed by Charles W. Trigg, San Diego, California.*

Find integers in the scale of 8 whose 6-digit squares are permutations of sets of consecutive digits.

334. *Proposed by Richard Field, Santa Monica, California.*

What is the 37th digit in the decimal fraction

$$\sum_{n=1}^{\infty} \frac{1}{10^n - 1} = .122324\cdots ?$$

After how many digits does the first zero occur?

335. *Proposed by Victor G. Feser, St. Louis University, St. Louis, Missouri.*

Problem 65 in this *Journal* (first presented in April 1954; re-presented in Fall 1968; solved in Fall 1969) showed that every simple non-triangular polygon has at least one interior diagonal, i.e., a diagonal lying entirely inside the polygon.

a) Show that every simple polygon of n sides, $n \geq 3$, has at least $(n - 3)$ interior diagonals.

b) Show that for every $n \geq 3$, there exists a simple polygon having exactly $(n - 3)$ interior diagonals.

336. *Proposed by Zazou Katz, Beverly Hills, California.*

On the diameter AB of a semicircle (O) perpendiculars are erected at arbitrary points C and D cutting the semi-circumference at points E and **F** respectively. A circle (P) touches the arc of the semicircle and each of the two half-chords. Show that PQ, the distance from P to the diameter AB, is equal to the geometric mean of AC and DB. (See Fig.1).
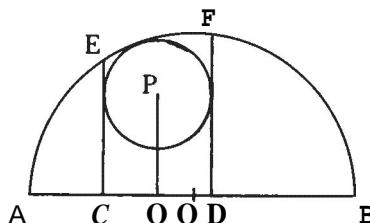
FIGURE 1

337. *Proposed by the Problem Editor.*

If R, **r** and p denote the circumradius, the **inradius** and the orthic triangle **inradius** respectively of an acute triangle ABC, show that $r^2 \geq pR$. (The orthic triangle is determined by the feet of the altitudes of the parent triangle).

Solutions

284. **[Fall 1972]** *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

A polygonal number can be defines:

$$\overset{m}{P}(n) = \frac{n}{2}[(m - 2)n - (m - 4)].$$

An r-digit automorph integer base b can be defined:

$$(n_1, n_2, \cdots, n_r)_b^2 = (\cdots, n_{r+1}, n_1, n_2, \cdots, n_r)_b.$$

If $b = 2m = 2(2r + 1)$, show that the last two digits of $\overset{m}{P}(b + 1)_b$ is a two-digit automorph.

Solution by N. J. Kuenzi and Bob Prielipp, The University of Wisconsin-Oshkosh.

We shall assume that b = 2m where m is odd, $m \geq 3$. Then

$$\overset{m}{P}(b + 1) = (b + 1)\frac{(b - 2)^2}{4} = (b + 1)(m - 1)^2$$

$$= (b + 1)\left[\frac{m - 3}{2} \cdot 2m + (m + 1)\right] = (b + 1)\left[\frac{m - 3}{2} \cdot b + (m + 1)\right]$$

$$= \frac{m - 3}{2} \cdot b^2 + \left[\frac{m - 3}{2} + (m + 1)\right]b + (m + 1)$$

$$= \frac{m - 3}{2} \cdot b^2 + \frac{3m - 1}{2} \cdot b + (m + 1).$$

Thus the last two digits of $\overset{m}{P}(b + 1)_b$ are $\frac{3m - 1}{2}$ and $m + 1$ respectively.

Also

$$\overset{m}{P}(b + 1)^2 = Ab^2 + (3m - 1)(m + 1)b + (m + 1)^2$$

$$= Ab^2 + (3m - 1)(m + 1)b + (\frac{m + 1}{2} \cdot 2m + m + 1)$$

$$= Ab^2 + (3m - 1)(m + 1)b + (\frac{m + 1}{2} \cdot b + m + 1)$$

$$= Ab^2 + \left[(3m - 1)(m + 1) + \frac{m + 1}{2}\right]b + (m + 1)$$

$$= Ab^2 + \left(\frac{3m + 1}{2} \cdot 2 + \frac{3m - 1}{2}\right)b + (m + 1)$$

$$= (A + \frac{3m + 1}{2})b^2 + \frac{3m - 1}{2} \cdot b + (m + 1)$$

Thus the last two digits of $\overset{m}{P}(b + 1)_b^2$ are $\frac{3m - 1}{2}$ and $m + 1$ respectively.

Therefore $\overset{m}{P}(b + 1)_b$ is a two-digit automorph.

Also solved by the Proposer.

297.   *[Spring* 1973] *Proposed by Roger E. Kuehl, Kansas City, Miss-ouri.*

A traffic engineer is confronted with the problem of connecting two non-parallel straight roads by an S-shaped curve formed by arcs of two equal tangent circles, one tangent to the first road at a selected point and the other touching the second road at a given point. (Fig. 2)
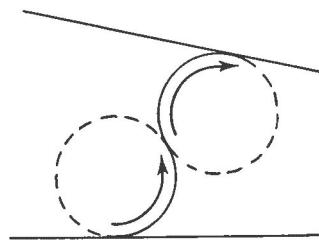
FIGURE 2

1) Determine the radius of the equal circles synthetically, trig-onometrically or analytically.

2) If the figure lends itself to an Euclidean construction, how would one go about it?

Solution by R. Robinson Rowe, Sacramento, California.

1) Analytically, take the origin (0,0) at point B, the coordinates of point A at (j,k), and the angle between tangents at C. (Fig. 3). Then $2R^2$ vers $C + 2R(j \sin C + k \cos C + k) - j^2 - k^2 = 0$, a quadratic easily solved for R.
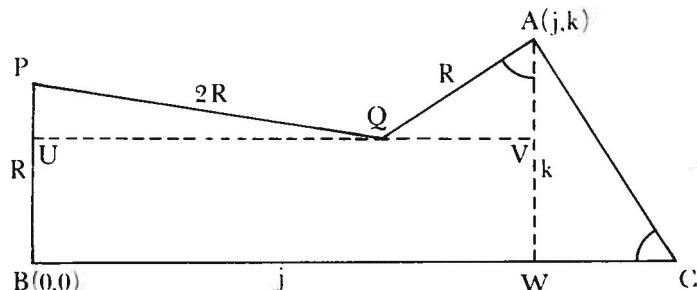
FIGURE 3

The derivation of this equation stems from the following procedure. Draw Ah' perpendicular to BC. Then **AW** = k and angle QAW = angle C. Draw UV through Q parallel to BC intersecting BP at U and AW at V. Then $UV = j$. Let angle $PQU = \phi$. Then $PU = 2R \sin \phi$ and $UQ = 2R \cos \phi$. In triangle AQV, QV = R sin C and $AV = R$ cos C. By algebraic addition of lines parallel to the axes:

$$UQ = UV - QV, \text{ whence } 2R \cos \phi = j - R \sin C, \qquad (1)$$

$$PU = BP + AV - AW, \text{ whence } 2R \sin \phi = R + R \cos C - k. \qquad (2)$$

Squaring (1) and (2) and adding:

$$4R^2 \cos^2 \phi + 4R^2 \sin^2 \phi = j^2 - 2jR \sin C + R^2 \sin^2 C + R^2 + R^2 \cos^2 C$$
$$+ k^2 + 2R^2 \cos C - 2kR - 2kR \cos C. \qquad (3)$$

Collecting terms and noting that $\sin^2 \mathbf{x} + \cos^2 \mathbf{x} = \mathbf{1}$, with $\mathbf{x} = \phi$ or C,

$$4R^2 = 2R^2 + 2R^2 \cos C = 2R(j \sin C + k \cos C + k) - j^2 - k^2. \qquad (4)$$

Then putting (4) in quadratic form and noting that $\mathbf{1} - \cos C = $ vers C,

$$2R^2 \text{ vers } C + 2R(j \sin C + k \cos C + k) - \mathbf{j}^2 - k^2 = \mathbf{0}. \qquad (5)$$

As an example of the application of this equation, suppose $(j,k) = (5,3)$ and C = sin-'0.6. Substitution in (5) reduces to $R^2 + 42R - 85 = 0$, whence $R = -21 \pm \sqrt{526}$, with the two solutions approximately +1.9 and -43.9, only one of which is meaningful.

Equation (5) can be solved to express R explicitly in terms of j, k and C, but the expression is unwieldy -- with 8 terms under the radical.

2) <u>Construction</u>. (See Fig. *4*). Erect **AD** ⊥ AC and BE ⊥ BC. On **AD** lay off any distance **AF,** and from **F** lay off **FG** = AF and parallel to BE. From A as a center and radius **AG** strike an arc, and from B as a center and 2AF as a radius strike another arc intersecting the first at H and **I.** Do likewise with another distance AJ, making JK = A J and par-allel to BE, then striking the arc $A(AK)$ and arc $B(2AJ)$ intersecting at L and M. Construct a circular arc through L, H, **I** and 14. (Any 3 of the points will suffice, and the center is on the extension of BA). Draw line **AGK** intersecting arc $LHIM$ at N. Draw line BN and bisect it at 0. Then BO is the required radius R.

Proof: Complete the construction of the reversed curve with cen-ters at P and Q and reversed tangency at T. Instead of solving for a 3-line linkage BP - PQ - QA, we substitute the 2-line linkage BN - NA, in which **NA** is the vectorial sum of NQ and QA. The locus of all points

with a ratio of distances from **B** and **A** equal to BN/NA is a circular arc, defined by location of **3** or more points on the arc. This was done with vectorial addition through F and J to G and K.
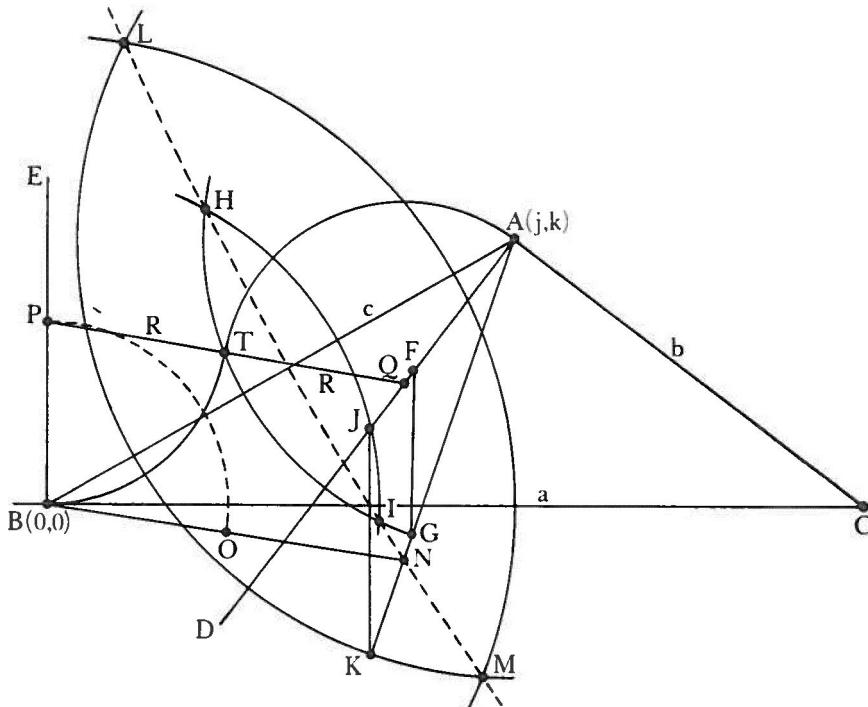

FIGURE 4

Solution of part (2) by Clayton W. Dodge, University of Maine at Orono. We shall show how to construct the desired circles at given points P and Q on roads m and n respectively. Let the perpendiculars to m at P and to n at Q meet at point **Q** (see Fig. *5).* The problem is then equivalent to locating points **A** on line **PR** and **B** on line **QR** so that $2PA = AB$ = $2BQ$. Then **A** and **B** are the centers of the desired circles. The Euclidean construction is by the "rule of false position". Choose any convenient length **s** and mark **A** on **PR** and B' on **QR** so that $s = PA + QB'$. Swing an arc of radius **2s** with **A** as center to cut at B_1 the parallel to PQ through B'. How PB_1 cuts **RQ** at **B,** one of the desired centers. On **PR** mark **PA = QB.**

To prove this construction, draw through B_1 the parallel to **QR** to, cut **PR** at R_1 and **PQ** at Q_1. In triangle PR_1Q_1 we have $2PA_1 = A_1B_1 = 2B_1Q_1$ by construction. Now a homothety with **P** as center maps Q_1 to Q, R_1 to R, and B_1 to **B**. (See Dodge, **Euclidean Geometry and Transformations,** Problem **28.10** for the similar problem requiring **PA = AB = BQ**.)
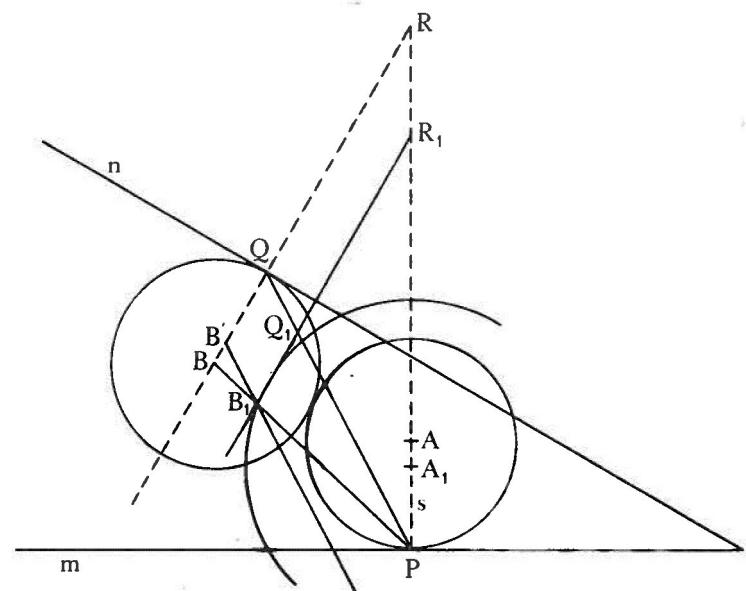

FIGURE 5

The first part of this problem was also solved by JOHN TOM HURT, *Texas A & M University and by* **C.** STANLEY OGILVY, Hamilton College, Clinton. New York. Dr. Ogilvy provided a general analytic solution for the radius of any desired number of equal circles connecting the two roads. The proposer, Mr. Roger Kuehl, derived a quadratic equation for the solution of this problem using algebraic methods. It is interesting to note that this problem happened to be a practical one confronted by Traffic Engineer Kuehl.

303. *[Fall* 1973] *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, New York.*

By means of an ϵ, δ proof only, show that a polynomial function is

continuous at any real number.

Solution by Clayton U. Dodge., University of Maine at Ohono.

Let $f(x) = \sum_{k=0}^{n} a_k x^k$. We show that $\lim_{x \to b} f(x) = f(b)$. If $f(x)$ is constant, choose $\delta = 1$ and the proof is easy. Thus assume that $f(x)$ is not a constant. Now choose $\epsilon > 0$ and let

$$\delta = \min (1, \epsilon/2 \sum_{k=1}^{n} k|a_k| (|b| + 1)^k).$$

Whenever $|b - x| < \delta$, we have

$$|f(b) - f(x)| = |\sum_{k=0}^{n} a_k b^k - \sum_{k=0}^{n} a_k x^k|$$

$$= |\sum_{k=1}^{n} a_k (b - x)(b^{k-1} + b^{k-2}x + b^{k-3}x^2 + \cdots + x^{k-1})|$$

$$\leq \sum_{k=1}^{n} |a_k| \cdot |b - x| \cdot |k(|b| + 1)^k|$$

$$= |b - x| \sum_{k=1}^{n} k|a_k|(|b| + 1)^k$$

$$< \epsilon.$$

Also solved by **VICTOR G. FESER**, *St. Louis, Missouri, and by the Proposer.*

304. **[Fall 1973]** *Proposed by Charles U. Trigg, San Diego, California.*

(A) One of the four digits **1**, 2, 3, 4 is placed at the midpoint of each edge of a cube in such a manner that four different digits are on the perimeter of each square face.

(B) The digits are placed on the vertices of the cube so that again there are four different digits on the perimeter of each face.

Show that in each case the clockwise cyclic order of the digits is different on each face.

Solution by Clayton W. Dodge., University of Maine at Orono.

(A) We may choose **1**, 2, 3, 4 for the four consecutive edges of one face. Then edge **a** (see Fig. *6*) is common to both the right and lower faces, so **a** # **1** and **a** $\neq$ 2. By symmetry **it** is immaterial whether **a** = 3 or **a** = 4, so let **a** = 3. Then we must have b = 2, c = **1**, and d = 4. Also e = 4, f = 3, g = 2, and h = **1** are forced. **It** is easily seen that all six possible cyclic arrangements are represented on the six faces. We also observe that each given number labels three mutually skew edges.
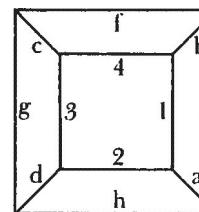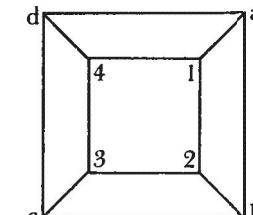

FIGURE 6   **FIGURE 7**

(B) We may choose **1**, 2, 3, 4 for the four consecutive vertices of one face without loss of generality. Then vertex **a** (see Fig. **7**) is common to both the upper and right faces. Hence we must have a = 3. Now b = 4, c = **1**, and d = 2 follow. Again all six arrangements are represented on the faces. Notice here that each number must label two opposite vertices.

Also solved by **VICTOR** G. **FESER**, *St. Louis, Missouri;* **CHARLES H. LINCOLN**, *Raleigh,* **N. C.;** **BRUCE LOVETT**, *Rutgers College, New Brunswick,* N. J.; **JIM METZ**, *Springfield, Illinois;* **PAOLO RANALDI**, *Akron, Ohio;* **TERRY SMETANKA**, *University of Toledo, Toledo, Ohio; and the Proposer.*

305. **[Fall 1973]** *Proposed by Jack Garfunkel, Forest Hills High Scliooi, New York.*

In ar acute triangle *ABC*, AF is an altitude and F is a point on AF such that AP = $2r$; where r is the inradius of triangle ABC. If D and E are the projections of F upon *AB* and AC respectively, show that the perimeter of triangle ADE is equal to that of the triangle of least perimeter that can be inscribed in triangle ABC.

Solution by Zazou Katz, Beverly Hills, California.

It is known that the triangle of minimum perimeter inscribed in an acute triangle ABC is its orthic triangle, determined by the feet of the altitudes from A, B and C. Furthermore, the perimeter p of the orthic triangle is equal to twice the area of triangle ABC divided by its circumradius R. (See N. **A.** Court, College Geometry, Barnes and Noble, 1952, p. 100). Hence:

$$p = \frac{r(a + b + c)}{R} = 2r(\sin A + \sin B + \sin C).$$

The points A, D, P, E are concyclic, lying on the circumference

of a circle whose diameter AP is equal to $2r$. So

$$AD = 2r \sin APD = 2r \sin B,$$

$$AE = 2r \sin APE = 2r \sin C,$$

and

$$DE = 2r \sin EAD = 2r \sin A.$$

It follows that the perimeter of triangle ADE is equal to

$$2r(\sin A + \sin B + \sin C)$$

and is therefore equal to the perimeter of the orthic triangle.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* DAVE LOGOTHETTI, *University of Santa Clara, California;* R. ROBINSON ROWE, *Sacramento, California; and the Proposer.*

306. **[Fall 1973]** *Proposed by David L. Silverman, Los Angeles, California.*

On alternate days A and B play games that are similar except with respect to the question of which player does the paying. In both versions A selects one number from the set (1, 2, 3) and B selects two numbers from the same set. If the two selections are disjoint, no payment is made. If the two selections have a number in common, the "payer" pays that number of dollars to the "receiver". They alternate daily in assuming the roles of payer and receiver. Does the arrangement favor either player?

Solution by the Santa Clara University Gamesters: M. Chamberlain, T. Pennello, M. Fay, J. Moore, D. Wong, J. Dechene, *and* K. Daly.

This problem may be analyzed with elementary game theory. On the day when B pays A, we set up the following payoff matrix for the game:

	B		
	(1,2)	(1,3)	(2,3)
A 1	1	1	0
A 2	2	0	2
A 3	0	3	3

Solving this game by the simplex algorithm, we find that A's optimal stratgy is (0, .6, .4), that is, he should avoid choosing 1, pick 2 60% of the time, and 3 40% of the time; B's optimal strategy is (.6, .4, 0), that is, he should select the pair (1,2) 60% of the time and (1,3) 40% of the time. With these optimal strategies, the value of the game is 1.2, so that B expects to pay $1.20 on the average when playing this version of the game.

On the day when A pays B, we set up a similar payoff matrix:

	B		
	(1,2)	(1,3)	(2,3)
A 1	-1	-1	0
A 2	-2	0	-2
A 3	0	-3	-3

The entries in the matrix are now negative, since A is paying B. The optimal strategies in this case are (1, 0, 0) for A and any convex combination of (2/3, 1/3, 0) and (1/2, 1/2, 0) for B. With these optimal strategies, the value of the game is -1, so that A expects to pay B $1.00 on the average.

In the long run, then, A will reap an average gain of 20¢ every two days. Thus the game favors A.

Also solved by R. ROBINSON ROWE, *Sacramento, California;* ZAZOU KATZ, *Beverly Hills, California; and the Proposer.* *Three incorrect solutions were received.*

307. **[Fall 1973]** *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

Let $\tau(n)$ denote the number of divisors of n. For square-free n greater than 1, prove that $\tau(n^2) = n$ if and only if $n = 3$.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

Since n is square-free and greater than 1, $n = p_1 p_2 \cdots p_k$ where $p_1, p_2, \cdots, p_k$ are distinct prime numbers. If $n = \tau(n^2)$, then $n = (p_1^2 p_2^2 \cdots p_k^2) = 3 \cdot 3 \cdots 3 [k \text{ factors of } 3] = 3^k$. Thus $n = 3$ because n is square-free. .

The fact that if $n = 3$ then $\tau(n^2) = n$ is immediate.

Similar solutions were offered by CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *St. Louis, Missouri;* RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* CHARLES H. LINCOLN, *Raleigh, N. C.;*

PETER A. LINDSTROM, *Genesee Community College, Batavia;* BRUCE LOVETT, *Rutgers College, N w Brunswick, N. J.;* T. E. MOORE, *Bridgewater State College, Bridgewater, Mass.;* DAVID A. ROSEN, *Cornell University, Ithaca, N w York; the University of Santa Clara Problem Solving Seminar; and the Proposer.*

308. **[Fall 1973]** *Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur 4, South India.*

Defining a proper number as one which is equal to the product of all its proper divisors, show that an integer is a proper number if and only if it is the cube of a prime or the product of two different primes. *Solution by Clayton W. Dodge, University of Maine at Orono.*

A proper number n has exactly three distinct proper factors, namely 1, a, and b, with $ab = n$. If a and b are distinct primes or if $a = p$ and $b = p^2$ for any prime p, then the definition is satisfied for n to be proper. Any other factorization for a number n leads to less than or more than three proper divisors, so such n cannot be proper.

Also solved by VICTOR G. FESER, *St. Louis, Missouri;* RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* CHARLES H. LINCOLN, *Raleigh, N. C.;* BRUCE LOVETT, *Rutgers College, New Brunswick, N. J.;* T. E. MOORE, *Bridgewater State College, Bridgewater, Mass.;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* PAOLO RANALDI, *Akron, Ohio;* WILLIAM J. RICKERT, *Toms River, N. 3.;* DAVID A. ROSEN, *Cornell University, Ithaca, N. Y.; and the Proposer.*

309. **[Fall 1973]** *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the volume of the solid formed by the elliptic paraboloids $2h - z = ax^2 + by^2$ and $z = cx^2 + dy^2$, where a, b, c, d and h are all positive. *Solution by the Proposer.*

The region E in the XY plane is the ellipse $(a + c)x^2 + (b + d)y^2 = 2h$:

$$V = \iint_E \left(\int_{cx^2+dy^2}^{2h-ax^2-by^2} dzdA \right)$$

$$= \iint_E [2h - (a + c)x^2 - (b + d)y^2]dA$$

$$= 2h \iint_E \left[1 - \frac{x^2}{\frac{2h}{a + c}} - \frac{y^2}{\frac{2h}{b + d}} \right] dA.$$

Let $x = u\sqrt{\frac{2h}{a + c}}$, $y = v\sqrt{\frac{2h}{b + d}}$, and C be the unit circle. Then

$$V = \frac{4h^2}{\sqrt{(a + c)(b + d)}} \iint_C [1 - u^2 - v^2] \, dA'$$

$$= \frac{4h^2}{\sqrt{(a + c)(b + d)}} \int_0^{2\pi} \int_0^1 (1 - r^2)r \, dr \, d\theta$$

$$= \frac{2\pi h^2}{\sqrt{(a + c)(b + d)}}.$$

Also solved by GUS MAVRIGIAN, *Youngstown State University, Youngstown, Ohio and by* R. ROBINSON ROWE, *Sacramento, California.*

310. **[Fall 1973]** *Proposed by Sidney Penner, Bronx Community College, Bronx, New York.*

If x ana y are integers and x < y then let $[x,y] = \{z: x \le z \le y$, and z is an integer$\}$. Also, for any set S, let $N(S)$ be the cardinal number of S.

Let n and k be positive integers with k > **1** and let $G = [2,(2n)^k-1]$. If V is a subset of G such that $N(V) = (2n)^k - 2n$ and V # $[2n, (2n)^k-1]$ then there are at least two distinct members of V each of which is the product of k members (not necessarily distinct) of V. *Solution by the Proposer.*

The key to our proof (as we shall show) is that a desired product will exist in which (at least) k - **1** of the multiplicands will be the smallest element of V. The case $n = 1$ is vacuously true; assume n > **1**. Let V' be the complement of V (relative to G); clearly $NV' = 2n - 2$. Let m be the smallest element of V; clearly $2 \le m \le 2n - 1$.

Case **I.** $2 \le m \le n$. Let $c = 2^k n - 1$.

$$2m^{k-1} = g_2$$
$$3m^{k-1} = g_3$$
$$(1) \qquad\qquad \vdots \qquad \vdots$$
$$cm^{k-1} = g_c$$

Since $g_c = (2^k n - 1)m^{k-1} \leq (2^k n - 1)n^{k-1} \leq (2n)^k - 1$, we see that g_2, $g_3, \cdots, g_c$ are in G. Since $NV' = 2n - 2$ there are at least $2^k n - 2 - (4n - 4) = (2^k - 4)n + 2$ equations of (1) all of whose elements are in V.

Case 11.

Lemma *If n and j are positive integers then*

$$(2n - 1)^j \leq (2n)^j - 1.$$

Proof. *Easily done by induction on j; we omit the details.*

$$(n + 1)m^{k-1} = h_{n+1}$$

$$(n + 2)m^{k-1} = h_{n+2}$$

(2)

$$(2n)m^{k-1} = h_{2n}$$

$$(2n + 1)m^{k-1} = h_{2n+1}$$

If $k = 2$, then $h_{2n+1} = (2n + 1)m \leq (2n + 1)(2n - 1) = 4n^2 - 1$.

If $k > 2$, then $h_{2n+1} \leq (2n + 1)(2n - 1)^{k-1} = (2n + 1)(2n - 1)(2n - 1)^{k-2}$

$\leq (4n^2 - 1)[(2n)^{k-2} - 1] = (2n)^k - (2n)^{k-2} - 4n^2 + 1 < (2n)^k - 1.$

Hence we see that the h_i of (2) are in G. Since $NV = 2n - 2$ and $[2,n]$ is a subset of V', there are $n - 1$ unknown elements of V'.

Since $2n + 1 < n^2 + 2n + 1 = (n + 1)^2 \leq (n + 1)(n + 1)^{k-1} \leq (n + 1)m^{k-1} = h_{n+1}$, we see that the set $\{n + 1, n + 2, \cdots, 2n + 1, h_{n+1}, h_{n+2}, \cdots, h_{2n+1}\}$ has $2(n + 1)$ elements. Hence there are at least two equations of (2) all of whose elements are in V.

Remark. *The case $G = [2,99]$ (with $n = 5$, $k = 2$) and $V = \{9\} \cup [11,99]$ shows that our result is best possible if it does not include n or k.*

311. [Fall 1973] Proposed by Charles W. Trigg, San Diego, California.

On opposite sides of a diameter of a circle with radius $a + b$ two semicircles with radii a and b form a continuous curve that divides the circle into two tadpole-shaped parts.

(i) Find the angle that the join of the centroids of the two component parts makes with the given diameter of the circle.

(ii) For what ratios $a:b$ does the continuous curve pass through one of the centroids?

(iii) When $a = b$, find the moment of inertia of one of the component areas about an axis through its centroid and perpendicular to its-plane.
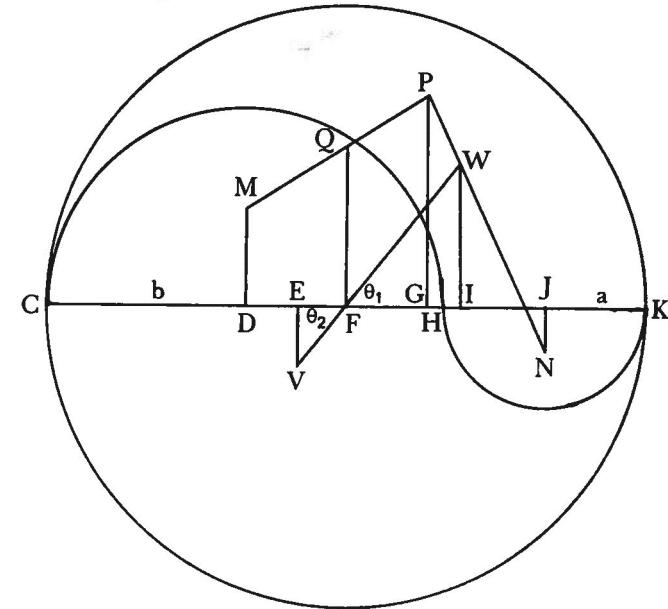

FIGURE 8

Solution by the Proposer.

In the conventional manner designate the semicircle with center F by (F) (see Fig. 8). The radii and areas of (D), (J), and (F) are b, a, $a + b$ and $\pi b^2/2$, $\pi a^2/2$, $\pi(a + b)^2/2$, respectively. For the tadpole areas, $A = (J) + (F) - (D)$ and $B = (D) + (F) - (J)$.

Pappus' second theorem states that the volume of a solid of revolution, formed by revolving a plane area about a line in its plane not cutting the area, is equal to the product of the generating area and the circumference of the circle described by the centroid of the area.

The center of gravity of a semicircle falls on the radius r perpendicular to the diameter at a distance d from the diameter. Rotating the-area about the diameter generates a spherical volume. Thus

$$4\pi r^3/3 = (\pi r^2/2)(2\pi d), \text{ so } d = 4r/3\pi.$$

(i) If M, Q, N, P, W, and V are the centroids of (D), (F), (J), $(F) - (D)$, A, and B respectively, then DM = $4b/3\pi$, EG = $4(a + b)/3\pi$, and NJ = $4a/3\pi$. Furthermore, DF = HJ = JX = a, and CD = DG = FJ = b. Using horizontal moment arms--

Taking moments about Q:

$$a \cdot \frac{\pi b^2}{2} = FG \left[\frac{\pi(a + b)^2}{2} - \frac{\pi b^2}{2}\right] .$$

Taking moments about W:

$$(b - FI) \cdot \frac{\pi a^2}{2} = (FI - FG)\left[\frac{\pi(a + b)^2}{2} - \frac{\pi b^2}{2}\right] .$$

Eliminating FG and solving,

$$FI = \frac{b}{2} .$$

Using vertical moment arms--

Taking moments about Q:

$$\left[\frac{4(a + b)}{3\pi} - \frac{4b}{3\pi}\right]\left(\frac{\pi b^2}{2}\right) = GP - \frac{4(a + b)}{3\pi}\left[\frac{\pi(a + b)^2}{2} - \frac{\pi b^2}{2}\right] .$$

Taking moments about W:

$$(P - IW)\left[\frac{\pi(a + b)^2}{2} - \frac{\pi b^2}{2}\right] = (IW + \frac{4a}{3\pi})(\frac{\pi a^2}{2}) .$$

Eliminating GP and solving,

$$IW = \frac{2b}{\pi} .$$

Hence, θ_1 = arctan $(\frac{IW}{FI})$ = arctan $\frac{4}{\pi}$ = 51.85° (approximately).

Similarly, or by symmetry, VE = $2a/\pi$, EF = $a/2$, and θ_2 = arctan $(\frac{VE}{EF})$ = arctan $\frac{4}{\pi}$..

Consequently, VW is the straight line join of V and W and makes the angle arctan $4/\pi$ with the diameter. Furthermore,

$$VW = (a + b) \frac{\sqrt{\pi^2 + 16}}{2\pi} ,$$

so is constant in length and inclination as a and b vary within the constant a + b.

(ii) With CX the X-axis and F the origin, the equation of the circle (D) is $(x + a)^2 + y^2 = b^2$. If this circle passes through $W(b/2, 2b/\pi)$, then

$$(\frac{b}{2} + a)^2 + \frac{4b^2}{\pi^2} = b^2$$

so

$$a = -\frac{b}{2} \pm \sqrt{b^2 - 4b^2/\pi^2}$$

and

$$a/b = -1/2 + \sqrt{1 - 4/\pi^2} \doteq 0.271178 .$$

Clearly, the other centroid will fall on the dividing curve when b/a has the same value, so the corresponding value of a/b is the reciprocal of its other value or approximately 3.68761.

(iii) If a = b, the figure becomes the familiar Yin and Yang [see, e.g., C. W. Trigg, "Bisection of Yin and of Yang," *Mathematics Magazine*, 34 (November 1960), pp. 107-1081 and the component parts are congruent.

The moment of inertia of the circle about an axis perpendicular to its plane at F is $\pi(2a)^4/2$, so I_F of Yin is $\pi(2a)^4/4$. Then

$$I_W = \frac{\pi(2a)^4}{4} - [\frac{\pi(2a)^2}{2}][\frac{a\sqrt{\pi^2 + 16}}{2\pi}]^2 = \frac{a^4(7\pi^2 - 16)}{2\pi}$$

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* ZAZOU KATZ, *Beverly Hills, California; and by* R. ROBINSON ROWE, *Sacramento, California.*

312. [Fall 1973] *Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Let $\{a_n\}$ be a sequence such that a_1 = 1 and for $n > 1$

$$a_n = a_{n-1} + 1 + (-1)^n + \frac{3}{2}[1 + (-1)^{n+1}] .$$

Show that the sequence $\{a_n\}$ has infinitely many primes.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

From the formula given for a_n, it follows that

$$a_{2n+1} = a_{2n} + 3, \qquad n = 1, 2, 3, \cdots$$

and

$$a_{2n} = a_{2n-1} + 2, \qquad n = 1, 2, 3, \cdots .$$

Thus a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 8, a_5 = 11, a_6 = 13, a_7 = 16, a_8 = 18, $\cdots$. Hence

$$a_{2n+1} = 5n + 1, \qquad n = 0, 1, 2, \cdots$$

and

$$a_{2n+2} = 5n + 3, \qquad n = 0, 1, 2, \cdots .$$

It is known that if s is a positive integer then there are infinitely many primes p such that p is congruent to 1 modulo s. [For a proof of this result, see Theorem 2 on pp. 250-251 of Goldstein, Abstract Algebra: A First Course, Prentice-Hall, Inc. 1973. The proof given uses the

theory of cyclotomic polynomials and does not invoke Dirichlet's theorem.] Hence the subsequence generated by $a_{2n+1} = 5n + 1$, $n = 0$, 1, 2,$\cdots$ contains infinitely many prime numbers, and thus automatically the given sequence $\{a\}$ contains infinitely many prime numbers.

Alternately, we may apply Dirichlet's theorem to guarantee that both the subsequence generated by $a_{2n+1} = 5n + 1$, $n = 0$, 1, 2,$\cdots$ and the subsequence generated by $a_{2n+2} = 5n + 3$, $n = 0$, 1, 2,$\cdots$ contain infinitely many prime numbers. [Dirichlet's theorem: Let k and j be relatively prime positive integers. Then there exist infinitely many primes in the arithmetic progression k, $j + k$, $2j + k$,$\cdots$.]

It follows immediately that the given sequence $\{a_n\}$ contains infinitely many prime numbers.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, St. *Louis, Missouri;* RICHARD A. GIBBS, *Ft. Lewis College, Durango, Colorado;* CHARLES H. LINCOLN, *Raleigh,* N. C.; PETER A. LINDSTROM, *Genesee Community College, Batavia, New York;* BRUCE LOVETT, *Rutgers College, New Brunswick,* N. J.; SIDNEY PENNER, *Bronx Community College, Bronx, New York;* DAVID A. ROSEN, *Cornell University, Ithaca, New York; and the Proposer.*

313. **[Fall 1973]** *Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.*

Comment by the Problem Editor.

Victor G. Feser of St. Louis, Missouri and R. Robinson Rowe of Sacramento, California called attention to a misprint in the published proposal. The corrected version follows and we invite solutions from readers who were frustrated by the published version.

Give an elementary proof that

$(1 + 8\cos^2 A)(1 + 8\cos^2 B)(1 + 8\cos^2 C) > 64 \sin^2 A \sin^2 B \sin^2 C$,

where A, B, C are the angles of an acute triangle ABC.

Remark

J. Gillis gave a proof using calculus techniques in Problem E 2119, *American Mathematical Monthly*, 1969, p. 831.

Corrections

Spring 1973, p. 436, line 7: $r^2 + 2s^2 + 4t^2$ should read $r^2 + 2s^2 = 4t^2$.

Fall 1973, p. 477, 7th line from bottom: **'papaboloids'** should read **'paraboloids'**.

Spring 1974, p. 533, line 1: **'Kuel'** should read **'Kuehl'**.

The *Journal* Editor apologizes to R. Robinson Rowe for **incorrectly** attributing his comment which followed the solution to Problem 302 in the Spring 1974 issue, p. 538, to the Problem Editor. The comment was due to R. Robinson Rowe and not the Problem Editor.

LOCAL CHAPTER AWARDS WINNERS

CALIFORNIA ETA (University of Santa Clara). Recognition for outstanding achievement in upper division mathematics classes went to

Kathy Daly,
Michael Fay,
Jeffrey Moon.

COLORADO DELTA (University of Northern Colorado). The winner of the *Outstanding Freshman* award based on the performance on an examination was

Brian Peterson,

who received a $20 bookstore gift certificate. The *Outstanding Senior* award based on achievement, ability, potential, interest, and enthusiasm went to

Bruce Lewis

who received a $40 bookstore gift certificate.

GEORGIA GAMMA (Armstrong State College). Four mathematics majors, chosen on the basis of scholarship, were presented awards at the President's Awards Banquet on May 15, 1974.

Donald Braffitt

received a one-year membership in the Mathematical Association of America, and

Marshall Hinds,
Patricia Spence,
Mary Stalnaker

each received a one-year membership in the American Mathematical Society.

IOWA ALPHA (Iowa State University). An award of $50 each was presented to the two second year mathematics majors who have completed the calculus sequence and have attained the highest scholarship in all their course work:

Marcia Mason,
Evan G. Person.

SEW JERSEY BETA (Douglass College). The recipients of the *New Jersey Beta Book Award* were

Helen Amunds,
Mary Campanale,
Kathleen Gossard,
Deborah Silliman.

NEW YORK PHI (State University College at Potsdam). The outstanding graduating senior mathematics major for 1973-74 was

Harris Schlesinger.

NEW YORK PSI (Iona College). Awards winners this year were as follows:

Van Bonacorsi (winner of the *Sullivan Award*),
Gregory Hubertus (winner of the *Joseph E. Power Award*),
Brother Jonathan Paolicelli (winner of the *Julia Friedman Award*).

OHIO EPSILON (Kent State University). The award for *Excellence in Mathematics* consisting of a plaque and $25 for books went to

Richard James Nelson.

OHIO LAMBDA (John Carroll University). The winner of the annual essay contest for 1974 was

Beverly Bruss

for her paper "A Discriminitive Study of Conic Sections." Although not awarded by the local chapter,

Mitchell Spector

was judged the best participant from John Carroll in the annual Northeastern Ohio Intercollegiate Mathematical Competition.

OHIO NU (University of Akron). The annual *Selby Scholarship* award went to

Paolo Ranaldi,

and a one-year student membership in the American Mathematical Association for scholastic achievement in mathematics went to (in addition to the student named above)

Mike Margreta,
Judy Park,
Mary Anne Schuerger,
Linda Talkington,
Allan Wilcox.

In addition to these awards the following Junior High School Science

Fair winners were recognized:

Robert Braun,
Joel Godard.

OHIO ZETA (University of Dayton). The Sophomore Award for Excellence in Mathematics was presented to

Richard Grote,

who received $25 for the purchase of mathematics books.

OKLAHOMA ALPHA (University of Oklahoma). The Nathan Altshiller Court Award of $50 each for the best freshman man and woman in mathematics was given to

Margaret R. Barrett,
William C. Wright.

The Samuel Watson Reaves Scholarship given each year to a senior for graduate work in mathematics was awarded to

Rex Allen McCaskill.

RHODE ISLAND BETA (Rhode Island College). An undergraduate senior award for excellence in mathematics based on the highest cumulative grade point average in mathematics was presented to

Christina Marcoccio.

VIRGINIA GAMMA (Madison College). A certificate and $50 award for mathematical materials were presented on Honors Day, April 11, 1974, to the outstanding senior mathematics major,

Nancy L. Ballard.

MATCHING PRIZE FUND

If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of mathematical films. Write to the National Office for more details.