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PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
Of THE HONORARY MATHEMATICAL FRATERNITY

David Ballew, Editor

Chapter reports, books for review, problems for solution and solutions to problems, should be mailed directly to the special editors found in this issue under the various sections. Editorial correspondence, including manuscripts (two copies) and news items should be mailed to THE EDITOR OF THE PI MU EPSILON JOURNAL. Dakota School of Mines and Technology, Rapid City, SD 57701. For manuscripts, authors are requested to identify themselves as to their class or year if they are undergraduates or graduates, and the college or university they are attending, and as to position if they are faculty members or in a non-academic profession.

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EDITORIAL POLICY

The Journal was founded in 1949 and dedicated to undergraduate students interested in mathematics. I believe the articles, features and departments of the Journal should be directed to this group. Undergraduates are strongly encouraged to submit their papers to the Journal for consideration and possible publication. Expository articles in all fields of mathematics are actively sought.

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David Ballew
Editor, Pi Mu Epsilon

Oh! My equation! Let us derive!
We can make beautiful slopes together.
I will range my sigma to the limit for yow epsilon.
A quantity squared cannot keep me from your tangent.
A delta is not as sweet, not near so tempting as your sine.
A new variable would not draw me away.
You satisfy my equation. Our relation has absolute value.
I think only in positive integers since our meeting.
You are the greatest integer in my domain.

David Haynes

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THE IMPOSSIBILITY OF TRISECTING ANGLES

by *Mark D. Meyerson*
University of Illinois

1. *Introduction*

One of the most intriguing geometrical problems of antiquity is to trisect an angle using a compass and straightedge. Although E. Galois proved (around 1830) that it is impossible, in general, to trisect an angle, much effort has since been wasted in futile constructions. Our goal is to give a brief and elementary proof of this nonconstructability. A few related theorems, such as the impossibility of duplicating the cube, are also included.

You might be surprised by all the algebra used in proving these geometric facts. The necessity of approaching these problems algebraically is the reason they were unsolved for so long. In fact, the most striking discoveries in mathematics often result from interplay between apparently unrelated fields, that is, the application of one branch of mathematics to another branch.

Here is a precise statement of the problem. Given an arbitrary angle, $\angle ABC$, one would like to construct a point D with the measure of $\angle DBC$ one-third the measure of $\angle ABC$. All construction must be done only with compass and straightedge. Given two points E and F, a compass may only be used to draw the circle through E with center F and straightedge may only be used to draw the line through E and F. Points are constructed by intersecting a line or circle with another line or circle. Although certain angles, such as a 90° angle, can be trisected in this manner, we will see that other angles, such as 60°, cannot be so trisected.

The sources for most of this paper are the two books, Elementary Geometry from an Advanced Standpoint, by Edwin E. Moise, and Topics in Modern Mathematics for Teachers, by Anthony L. Peressini and Donald R. Sherbert. These books are recommended if you desire to continue with the subject.

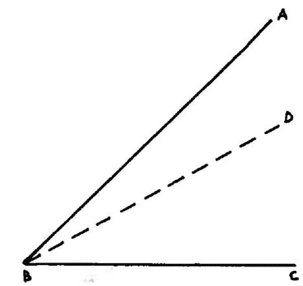

FIGURE 1
Angle DBC *has one third the measure of angle* ABC

2. *Subfields*

All our calculations will be done with real numbers. The set of real numbers is denoted by $\mathcal{R}$.

Definition 1. A subset, F, of $\mathcal{R}$, is called a ***subfield*** (of $\mathcal{R}$) if it contains 0 and 1, and if it is closed under division by non-zero elements of F and subtraction. For example, closed under subtraction means that if a and b are elements of F, so is $a - b$. Note that a subfield is closed under multiplication and addition, since $ab = a/(1/b)$ and $a + b = a - (0 - b)$. There is a technical definition of field which we will not need.

Examples. 1. $\mathcal{R}$ is a subfield.

2. A number is called ***rational*** if it can be written as p/q for p and q ($\neq 0$) integers. The set of rational numbers is denoted $\mathbb{Q}$. We show in the aside below that $\mathbb{Q} \neq \mathcal{R}$. But $\mathbb{Q}$ is a subfield, since $0 = 0/1$, $1 = 1/1$, $(p/q)/(r/s) = (ps)/rq)$ for $r/s \neq 0$ (hence $r \neq 0$), and $p/q - r/s = (ps - qr)/qs)$.

3. The set of integers is not a subfield, since $1/2$ is not an integer.

Aside. $\sqrt{2}$ is not rational.

Proof. Suppose $\sqrt{2}$ is rational. Then we could write it as p/q in reduced form. So $\sqrt{2}q = p$, and squaring, $2q^2 = p^2$.

Since p^2 is even, p must be even. So $p = 2m$ for some integer m. Substituting, we get $2q^2 = (2m)^2 = 4m^2$, or $q^2 = 2m^2$.

Since q^2 is even, q must be even. So p/q is ***not*** in reduced form, because p and q each have a factor of 2.

Hence $\sqrt{2}$ cannot be a rational number. Q.E.D.

We close this section with a theorem about the roots of an equation.

Theorem 1 (The *Rational Root Test*). If $a_n x^n + \ldots + a_1 x + a_0 = 0$ is a polynomial equation with integer coefficients and p/q is a rational root, in reduced form, then p divides a_0 and q divides a_n.

Proof. We have $a_n(p/q)^n + a_{n-1}(p/q)^{n-1} + \ldots + a_1(p/q) + a = 0$.

Multiplying by q^n we get $a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$. Since p and q each divide the right hand side of this equation, they each divide the left hand side. And since p divides each term on the left, except perhaps $a_0 q^n$, p must divide $a_0 q^n$ also. But p and q have no factors in common, so p divides a . Similarly, q divides $a_n p^n$, and so divides a_n. Q.E.D.

3. *Surds*

Definition 2. A number is called a surd if it can be calculated from 0 and 1 by a finite number of additions, subtractions, multiplications, divisions, and extractions of square roots.

Any rational number is a surd. $\sqrt{\sqrt{2}+1} - 3/2$ is a surd. There are many numbers which are not surds. We will see later that $\sqrt{2}$ and $\cos(20^{\circ})$ are not surds; also, π is not a surd.

The set of all surds forms a subfield. For 0 and 1 are surds, and if a, b and $c \neq 0$ are surds, so are a - b and a/c.

We now consider the Euclidean plane with a coordinate system.

Definition 3. A *surd-curve* is a circle or line with equation $A(x^2 + y^2) + Dx + Ey + P = 0$, such that all the coefficients are surds. We may assume that $A = 1$ for a circle and $A = 0$ for a line. A *surd-point* is a point (x,y) such that a; and y are surds.

Theorem 2. If $P = (x,y)$ lies on two distinct surd-curves, then P is a surd-point. ▪

Proof. This can be proven by solving for P, and showing that a; and y are surds. We prove only the hardest case, when both surd-curves are circles.

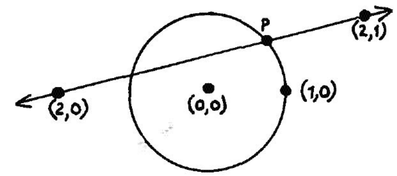

FIGURE 2
An *example of* two *surd-curves.*
$P = ((-2 + 4\sqrt{13})/17, (8 + \sqrt{13})/17)$ is a surd-point.

The two surd-curves have equations $x^2 + y^2 + Dx + Ey + F = 0$ and $x^2 + y^2 + Gx + Hy + I = 0$, with surd coefficients. Subtracting, we get $Jx + Ky + L = 0$, where J, K, and L are surds. J and K are not both zero. since if they were we would have distinct concentric circles meeting at P.

We now suppose $K \neq 0$. The proof is entirely analogous for the subcase $J \# 0$. So we can solve for y, $y = Mx + N$, where M and N are surds. Substituting into the very first equation, we get $ax^2 + bx + c = 0$, where a, b aid c are surds. Since $a = 1 + M^2$, $a \neq 0$. So $x = (-b \pm\sqrt{b^2 - 4ac})/(2a)$ and $y = Mx + N$, both surds. Q.E.D.

Theorem 3. Given a .collection of only surd-points, any point we can construct using compass and straightedge must be a surd-point.

Proof. Let $P = (a,b)$ and $Q = (c,d)$ be surd points.

It's easy to check directly that the line through P and Q has equation: $(d - b)x + (a - c)y + (bc - ad) = 0$, and that the circle with center P through Q has equation $x^2 + y^2 - 2ax - 2by + (2ac + 2bd - c^2 - d^2) = 0$. All coefficients are surds:

So only surd-curves can be constructed from surd-points. The only way to construct a new point is to consider the intersection of two of these surd-curves, which must be a surd-point by Theorem 2. We can continue constructing curves and points, but only surd-curves and surd-points. Q.E.D.

4. *Cubic Equations*

Definition 4. Let F be a subfield (of $\mathbb{R}$) and let k be a positive number in F such that $\sqrt{k}$ is not in F. Then $F(k)$ denotes the set of all numbers of the form $x + y\sqrt{k}$, where x and y are in F.

For example, if F = Q, k = 2, we get Q(2) which includes $3 + 2\sqrt{2}$, $(1/2) - \sqrt{2}$, and $3 = 3 + 0\cdot\sqrt{2}$. Or if F = Q(2), k = 3, we get $F(3) = (Q(2))(3)$. (It is easy to see that $\sqrt{3} \notin Q(2)$. Suppose $\sqrt{3} = r + s\sqrt{2}$ where r and s are rational. Squaring implies that $\frac{3 - r^2 - 2s^2}{2rs} = \sqrt{2}$, a contradiction.)

Each element of $F(k)$ can be written as $x + y\sqrt{k}$ in only one way. For if $a + b\sqrt{k} = c + d\sqrt{k}$, then $(a - c) = (d - b)\sqrt{k}$. If $b \neq d$, then $\sqrt{k} = (a - c)/(d - b)$ an element of F, contradicting the choice of k. So $b = d$, and hence $a = c$.

Also $F(k)$ is a subfield; let's check the definition of subfield. $0 = 0 + 0\cdot\sqrt{k}$ and $1 = 1 + 0\cdot\sqrt{k}$ are in $F(k)$, and $(a + b\sqrt{k}) - (c + b\sqrt{k}) = (a - c) + (b - d)\sqrt{k}$ an element of $F(k)$. So we only need check closure under division by non-zero elements. But

$$\frac{a+b\sqrt{k}}{c+d\sqrt{k}} = \frac{(a+b\sqrt{k})\ (c-d\sqrt{k})}{(c+d\sqrt{k})\ (c-d\sqrt{k})} = \frac{ac-bdk}{c^2-d^2k} + \frac{bc-ad}{c^2-d^2k}\sqrt{k}.$$

Note that $Q \subset Q(2) \subset$ the set of surds $\subset \mathbb{R}$.

Theorem 4. For $F(k)$ as above, suppose the coefficients of $x^3 + ax^2 + bx + c = 0$ are all in F and that $r + s\sqrt{k}$, an element of $F(k)$, is a root. Then some element of F is a root.

Proof. We may assume that $s \neq 0$, since otherwise we're done.

We have $0 = (r + s\sqrt{k})^3 + a(r + s\sqrt{k})^2 + b(r + s\sqrt{k}) + c = (r^3 + 3rs^2k + ar^2 + as^2k + br + c) + (3r^2s + s^3k + 2ars + bs)\sqrt{k}$. Write this as $A + B\sqrt{k} = 0$. So $A = B = 0$. Putting $r - s\sqrt{k}$ into the polynomial gives us $A - B\sqrt{k} = 0$, since only even powers of s occur in A and odd powers occur in every term of B. So $r - s\sqrt{k}$ is another root.

Now $x^3 + ax^2 + bx + c = (x - x_1)(x - x_2)(x - x_3) = x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3$, where x_1, x_2, and x_3 are the roots. So let's take $x_1 = r + s\sqrt{k}$, $x_2 = r - s\sqrt{k}$. Then $a = -(x_1 + x_2 + x_3) = -(r + s\sqrt{k} + r - s\sqrt{k} + x_3) = -(2r + x_3)$, so

$x_3 = -a - 2r$, an element of F. Q.E.D.

Theorem 5 (Main Theorem). Given cubic equation $x^3 + ax^2 + bx + c = 0$, where the coefficients are rational. If the equation has a surd as a root, then it has a rational root.

Proof. Suppose x_1 is a surd and a root. As a surd, x_1 is in some subfield $(\ldots(Q(k_1))(k_2)\ldots)(k_n)$. To see this, start to calculate x_1 from 0 and 1. (Recall that by definition, a surd can be calculated from 0 and 1 by additions, subtractions, multiplications, divisions, and extractions of square roots.) Let $\sqrt{k_1}$ be the first non-rational square root we extract. Continue, until we must extract a square root, $\sqrt{k_2}$, not in $Q(k_1)$. Continuing in this fashion, we get the above subfield.

By Theorem 4, the given cubic equation has a root in $(\ldots(Q(k_1))(k_2)\ldots)(k_{n-1})$. Applying Theorem 4 a total of n times, we see that the cubic equation has a root in Q. Q.E.D.

5. *Nonconstructability Proofs*

Theorem 6. The cube cannot be duplicated. In other words, given the edge of a unit cube (a unit segment), we cannot construct (with compass and straightedge) the edge of a cube of twice the volume. (The edge of such a cube would be $\sqrt[3]{2}$.)

Proof. We can think of this as being given surd-points (0,0) and (1,0) and being asked to construct $(\sqrt[3]{2}, 0)$. So it suffices to show that $\sqrt[3]{2}$ is not a surd.

Suppose it were. Then the cubic equation $x^3 - 2 = 0$ has a surd as a root. By the Main Theorem it has a rational root. But by the Rational Root Test, the only possible rational roots are ±1 and ±2 which are not roots. So $\sqrt[3]{2}$ is not a surd. Q.E.D.

Theorem 7. There are angles that cannot be trisected with compass and straightedge.

Proof. We actually show that no 60° angle can be trisected. Given a 60° angle, we can choose a coordinate system so that $A = (1, \sqrt{3})$, $B = (0,0)$, $C = (2,0)$ and the given angle is $\angle ABC$. (Note that A, B, and C are surd-points which form the vertices of an equilateral triangle.) We

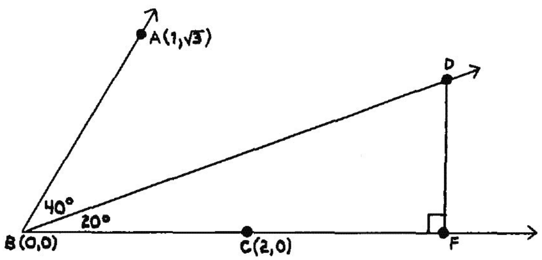

FIGURE 3

want to show that there is no surd-point D such that the measure of angle $DBC = 20^\circ$.

Suppose there were such a D. Let $\overline{DF}$ be the perpendicular to the x-axis. F is a surd-point since it lies on the two surd-curves $y = 0$ and x = (x-coordinate of D). Since the distance between two surd-points is a surd, cos $20^\circ = BF/BD$ is a surd. Next we shall use the standard trigonometric identities:

$\cos(A+B) = \cos A \cos B - \sin A \ \sin B$

$\sin(24) = 2 \sin A \cos A$

$\cos(2A) = \cos^2 A - \sin^2 A$

$1 = \sin^2 A + \cos^2 A$

Now $\cos 3\theta = \cos(2\theta + 6) = \cos 26 \ \cos\theta - \sin 28 \sin 6$ $= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta = (2\cos^2\theta - 1 - 2(1 - \cos^2\theta))$ $\cos\theta = (4\cos^2\theta - 3)\cos\theta$. Since $\cos 60^\circ = 1/2$, we let $6 = 20^\circ$ to see that $\cos 20^\circ$ is a solution of $1/2 = 4y^3 - 3y$. Letting $y = x/2$, the surd $2\cos 20^\circ$ is a root of $x^3 - 3x - 1 = 0$. By the Main Theorem, $x^3 - 3x - 1 = 0$ has a rational root. But the only possibilities are ± 1, which are not roots. This contradiction implies the Theorem. Q.E.D.

Theorem 8. It is impossible to construct a regular seven sided polygon (heptagon) with compass and straightedge.

Proof. Suppose we could. Then we can construct the central angle, $8 = 360^\circ/7$. And so, as before, $x_0 = \cos 6$ is a surd.

Now $36 + 46 = 360^\circ$, so $\cos 3\theta = \cos(360^\circ - 4\theta) = \cos 4\theta$. So $4\cos^3\theta - 3\cos 6 = 2\cos^2 26 - 1 = 2(2\cos^2\theta - 1)^2 - 1$. Hence x_0 is a solution of $4y^3 - 3y = 2(2y^2 - 1)^2 - 1$, $4y^3 - 3y = 8y^4 - 8y^2 + 1$, and $16y^4 - 8y^3 - 16y^2 + 6y + 2 = 0$.

So $2x_0$ is a root of $x^4 - x^3 - 4x^2 + 3x + 2 = 0$. Since 2 is a root of this, we see that the left hand side equals $(x-2)(x^3 + x^2 - 2x - 1)$. But $x_0 = \cos 6 \neq 1$, so $2x_0 \neq 2$, and $2x_0$ is a surd and a root of $x^3 + x^2 - 2x - 1 = 0$. By the Main Theorem, there must be a rational root. But neither ± 1 are roots, so we have a contradiction. Q.E.D.

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Professor Milton D. Cox
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Miami University
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WEIGHTING INCONSISTENT JUDGMENTS

by Peter C. Morris
Shepherd College

In a "chautauqua-type short course" at the University of Maryland we studied an intriguing way to assign "weights" to items in a list, where "judgments" are made on pairs of them. The course director, T. L. Saaty who developed this method using "dominance matrices", has tested it on several projects with his family, such as choice of schools; he has used it when the judgments involved are about such things as psychological problem areas, power and world influence of nations, future energy needs, and transport planning (a major study to determine kinds of transportation needed in the Sudan [5]. (Several applications are discussed in [2], one of many publications on this method and its application.) As a group project in our short course, we used the method to digest our pooled opinions about the future of higher education.

Consider first the following ("consistent") judgment problem, taken from a civil war vintage arithmetic book.

> ***A person dying, worth $5,460, left a wife and too children, a son and daughter, absent in a foreign country. He directed that if his son returned, the mother should have one-third of the estate, and the son the remainder; but if the daughter returned, she should have one-third and the mother the remainder. Saw, it so happened that they both 'returned; how must the estate be divided to fulfill the father's intentions?*** [1]

That is, if the daughter (only) returns, divide the estate as 1:2 for the daughter's to the mother's portion; if the son (only) returns divide it as 1:2; but mother's portion to son's this time. Writing that last 1:2 ratio as 2:4, the combined ratios are 1:2:4 of daughter's to mother's to son's portion of the estate. This yields the desired division of $780, $1,560, and $3,120, respectively.

Consider this problem as a judgment of the mother (M) over the daughter (D) by a factor of 2, but of the son (S) over M by a factor of

2. Suppose also that the father would have preferred, had the mother died first and both the son and daughter returned, that the daughter should have had only one-fourth of the estate (say), the son the other three-quarters. That is, suppose that also he judged S over D by a factor of 3. The way to now settle the estate (with all three living and in the states) is not so simple.

Saaty's method would consider the following "dominance matrix" A to deal with such problems; always require a, b, and c to be positive numbers:

$$\begin{array}{c} \\ D \\ M \\ S \end{array}\begin{array}{c} \begin{array}{ccc} D & M & S \end{array} \\ \begin{bmatrix} 1 & 1/a & 1/b \\ a & 1 & 1/c \\ b & c & 1 \end{bmatrix} \end{array} = A.$$

The labeling of the rows and columns indicates that the ratio of the portions of M to D is $a:1$, whence (to explain the reciprocal entry) that of D to M is $1/a:1$ (that is, $1:a$). (a is 2 in both examples.) Likewise that of S to M is $c:1$ (c is 2 in both examples). Finally that of S to D is $b:1$. The judgments are called "consistent" in case $b = a \cdot c$. (Thus in the first example $b = 4$; in the second, b was defined to be 3.) Multiply the dominance matrix A_1 for the first example by its solution vector $X_1 = (1/7)\,[1,\ 2,\ 4\,]^t$:

$$A_1X_1 = \begin{bmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/2 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1/7 \\ 2/7 \\ 4/7 \end{bmatrix} = (1/7)\begin{bmatrix} 1+1+1 \\ 2+2+2 \\ 4+4+4 \end{bmatrix} = 3\begin{bmatrix} 1/7 \\ 2/7 \\ 4/7 \end{bmatrix} = 3\cdot X_1$$

That is, the solution vector X_1 is an "eigenvector" corresponding to 'eigenvalue" 3 for the (consistent dominance) matrix A_1.

That relationship (for the consistent judgment problem) motivates the following definition. Define the solution vector X (even for inconsistent judgments) by forming A as above; then X is the (normalized) eigenvector corresponding to (the real) eigenvalue λ of A. The entries of the solution vector X are called the relative weights. This paper derives a formula for that eigenvalue λ.

(Saaty's method in general considers pairwise comparisons of "n" items at a time, using hierarchies to avoid large n, and introduces a special scale. This paper considers only the special case of $n = 3$. In [3], Saaty has solved the corresponding eigenvalue problem for $n = 4$. His forthcoming book [4] contains further discussion of this method.)

Theorem. Let $d = \frac{ac}{b}$. (Require a, b, and c to be positive.) If $d \neq 1$ then A has exactly one real eigenvalue. A formula for it is $A = \sqrt[3]{d} + 1/\sqrt[3]{d} + 1$.

Proof. Writing $E = d + 1/d$, compute $\det(xI - A) = x^3 - 3x^2 + (2 - E)$; call this polynomial $f(x)$. If $d = 1$ then $E = 2$ yields $f(x) = x^2(x-3)$, whence the real eigenvalues are 0 and 3.

Consider the graph of f. From $f'(x) = 3x^2 - 6x = 3x(x-2)$, it follows that f has only one x-axis intercept in case its local maximum value at $x = 0$ is negative. But since $f(0) = 2 - E$, that is equivalent to $2 < d + 1/d$. After multiplying by $d > 0$, this can be written as $0 < (d-1)^2$. That is, if $d \neq 1$ then f has exactly one real root.

To find that root, substitute $x = y + 1/y + 1$ into $f(x) = 0$. After expanding and multiplying through by y^3, this yields $y^6 - Ey^3 + 1 = 0$. By the quadratic formula $y^3 = 1/2\,(E \pm \sqrt{\Delta})$, where $A = E^2 - 4 = (d - 1/d)^2$. If $d > 1$ then $\sqrt{\Delta} = d - 1/d$, whence $y^3 = 1/2\,[(d + 1/d) \pm (d - 1/d)]$ is d or $1/d$. On the other hand if $0 < d < 1$ then $\sqrt{\Delta} = 1/d - d$ yields again the pair of formulas d and $1/d$ for y^3. Since (in either case) the real solutions for y are reciprocals, there is only the single, desired, real solution A for x.

The theorem was discovered by using "Cardan's formula"; the substitution used in the paragraph above is a special case of that used [6; pp. 84–85] to derive the cubic formula.

Corollary 1. If λ_1 and λ_2 are the other eigenvalues, then $\lambda^2 - 3A = |\lambda_k|^2$.

Proof. Since λ, λ_1, and A_2 are roots of f, their product is the negative of its constant term. That is $\lambda\lambda_1\lambda_2 = E - 2$. But λ_1 and λ_2 are necessarily complex conjugates; thus $\lambda_1\lambda_2 = \overline{\lambda_2}\lambda_2 = \lambda_1\overline{\lambda_1} = |\lambda_k|^2$. Next substitute into the equation $f(x) = 0$ the product $\lambda|\lambda_k|^2$ for $E-2$ to obtain $\lambda^3 - 3\lambda^2 - \lambda|A_k|^2 = 0$. This equation yields the desired one after dividing by A.

Incidentally, letting $F = \sqrt[3]{d} - 1/\sqrt[3]{d}$ the formulas for those complex roots can be written as $A_k = \frac{1}{2}(3 - A) \pm \frac{1}{2}\sqrt{3}\,F\,i$. Notice

the eigenvalues are all continuous functions of d (whence of the matrix entries a, b, and c); in particular, the formulas for A give zero when $d = 1$.

Corollary 2. The eigenvector with entries the relative weights is $X = (1/\mu)\left[\sqrt[3]{d}/a,\ 1,\ c/\sqrt[3]{d}\right]^t$, where (to normalize) $\mu = \sqrt[3]{d}/a + 1 + c/\sqrt[3]{d}$.

Proof. Consider the matrix equation $(A - \lambda I)\,X = 0$. Using $\lambda - 1 = \sqrt[3]{d} + 1/\sqrt[3]{d}$, write it as the system of equations

$$(1)\quad (1/a)x_2 + (1/b)x_3 = (\sqrt[3]{d} + 1/\sqrt[3]{d})x_1$$

$$(2)\quad a\,x_1 + (1/c)x_3 = (\sqrt[3]{d} + 1/\sqrt[3]{d})x_2$$

$$(3)\quad b\,x_1 + c\,x_2 = (\sqrt[3]{d} + 1/\sqrt[3]{d})x_3$$

Selecting $x_2 = 1$ in equation (2) suggests trying $x_1 = \sqrt[3]{d}/a$ and $x_3 = c/\sqrt[3]{d}$. This trial solution reduces both equations (1) and (3) to the identity $ac = bd$.

Corollary 3. If A has eigenvector $\left[x_1, x_2, x_3\right]^t$ (for A) then $\left[1/x_1, 1/x_2, 1/x_3\right]^t$ is an eigenvector (for λ) of A^t.

Proof. Replacing each element of A by its reciprocal changes A into A^t. Notice that the formula for A is unchanged, as d is changed into $1/d$. The form for an eigenvector for A then follows from corollary 2.

Corollary 4. For fixed b and c, A is an increasing function of a in case $a > b/c$.

Proof. Write $\lambda = \lambda(x) = x^{1/3}\cdot\sqrt[3]{c/b} + x^{-1/3}\cdot\sqrt[3]{b/c} + 1$. compute $\lambda'(x) = (1/3)x^{-2/3}\cdot\sqrt[3]{c/b} - (1/3)x^{-4/3}\cdot\sqrt[3]{b/c} + 0$ $= (1/3)x^{-4/3}\cdot(x^{2/3}\cdot\sqrt[3]{c/b} - \sqrt[3]{b/c})$. Thus $\lambda'(x)$ is positive (whence A is increasing) precisely when $x > b/c$.

This corollary was suggested by one of Saaty's comments during our short course. (That comment is discussed, following the statement of the Perron-Frobenius theorem, in [2; p. 241].) ■

Corollary 5. $3 \leq \lambda \leq 2 + \max\{d, 1/d\}$.

Proof. If $d \geq 1$, since $\sqrt[3]{d} \leq d$ and $\sqrt[3]{1/d} \leq 1$, we have $\lambda = \sqrt[3]{d} + \sqrt[3]{1/d} + 1 \leq d + 1 + 1$. Similarly, if $d < 1$ then $\lambda \leq 1 + 1/d + 1$. Notice $3 \leq \lambda$ directly, for that assertation is equivalent to $2 \leq e + 1/e$, writing $e = \sqrt[3]{d}$. But that inequality can be restated as $0 \leq (e-1)^2$, after multiplying by $e > 0$.

In [2; theorem 1] it is shown that $\lambda = n$ (for the general $n \times n$ case) if and only if the matrix A is consistent (in the sense that all $a_{ij}a_{jk} = a_{ik}$).

Incidentally, $2 + \max\{\sqrt[3]{d}, \sqrt[3]{1/d}\}$ is clearly (when $d \neq 1$) a smaller upper bound for λ.

The bounds of corollary 5 for the "inconsistent" example above (with $a = 2$, $b = 3$, and $c = 2$) yield $3 \leq \lambda \leq 2 + 4/3$. (The better estimate, that is $2 + \sqrt[3]{d}$, gives $\lambda \leq 3.101$.) The formula (of the theorem) gives $\lambda = 3.009$. Relative weights (from corollary 2) are given by $X = (1/\mu)\left[1/\sqrt[3]{6},\ 1,\ \sqrt[3]{6}\right]^t \doteq \left[.16,\ .30,\ .54\right]^t$. Thus finally for this inconsistent example the estate division should be \$892.30, \$1,621.41, and \$2,946.29. (The consistent example (with $a = c = 2$ but $b = 4$) gave weights of $1/7 : 2/7 : 4/7$, or approximately $.14 : .29 : .57$.)

As a final example suppose that the judgment of M to D were still 2, but that of S to D is now 4 (as in the original consistent example), while that of S to M is 3. That is, suppose that $a = 2$, $b = 4$, and $c = 3$ to obtain matrix A, below. The eigenvalue $\lambda \doteq 3.018$; and the formula for the relative weights gives approximately $.14 : .24 : .63$. Using corollary 3, their reciprocals give an eigenvector for A^t, which normalizes to $\left[.56,\ .32,\ .12\right]^t$.

Notice that Saaty's method certainly is invariant of the ordering of the items to be compared; in particular, A^t is not the matrix to use to compare the items (which were judged to give the entries in A) in the reversed order (as transposition does not preserve all the pairwise comparisons). For this final example

$$A = \begin{array}{c} \\ D \\ M \\ S \end{array}\!\!\begin{array}{c} \begin{array}{ccc} D & M & S \end{array} \\ \begin{bmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/3 \\ 4 & 3 & 1 \end{bmatrix} \end{array} \text{ yields } A^t = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 3 \\ 1/4 & 1/3 & 1 \end{bmatrix}, \text{but } \begin{array}{c} \\ S \\ M \\ D \end{array}\!\!\begin{array}{c} \begin{array}{ccc} S & M & D \end{array} \\ \begin{bmatrix} 1 & 3 & 4 \\ 1/3 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix} \end{array} = B$$

would be the matrix to use to consider weights for S to M to D. (That is, an eigenvector for B can be obtained from an eigenvector for A by listing the entries in reversed order.)

In summary, consider the following restatement in words of corollary 2, about the weight vector $X = \left[w_1,\ w_2,\ w_3\right]^t$. The ratios of pairs of weights can be computed by dividing the corresponding comparisons by the "measure of consistency" $R = \sqrt[3]{ac/b} = \sqrt[3]{d}$. In the consistent case ($a c = b$) consider the eigenvectors $\left[1/a,\ 1,\ c\right]^t$ and $\left[1,\ a,\ b\right]^t$ for $\lambda_o = 3$ to see that the entries in the weight vector must satisfy these equations: $w_2/w_1 = a$, $w_3/w_2 = b/a = c$ and $w_3/w_1 = ac = b$. In fact also in the general case clearly (from the formula for X in corollary 2) $w_2/w_1 = a/R$, and $w_3/w_2 = c/R$. Finally, multiplying these last equations yields $w_3/w_1 = ac/R^2 = bd/R^2 = bR$. That is, even the non-adjacent weights satisfy this "dividing by R" property: the original comparison $1/b$ when divided by R yields w_1/w_3.

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THE APOLLONIUS TANGENCY PROBLEM

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To construct a circle tangent to three given circles is a problem proposed by the great geometer Apollonius of Perga [ca. 260-170 B.C.]. The Tactionibus, a text regarding tangencies, which contained the solution, is lost.

Francois Viete [1540-1603] reconstructs the Tactionibus under the title Apollonius Gallus. In his forword, Viete [12] refers to Pappus of Alexandria:

> *"Pappus Alexandrinus drew up to ten problems of Apollonius Perga, each of which I shall work out in the order that seems more convenient."*
>
> *"Apollonii Pergai Problemata ωθὶ ἐπαφων ad decem contraxit Pappus Alexandrinus, qua ideo singula persequar eo, qui convenientior videbitur, ordine."*

The following is a listing of the given information as used by Viete. The given information is in each of the ten cases followed by: construct a circle tangent to the given elements.

1. Three non-collinear points
2. Two points and one straight line
3. Three lines (not all three parallel)
4. Two lines and one point
5. One circle and two lines
6. One point, one line and one circle
7. Two circles and one line
8. Two points and one line
9. Two circles and one point
10. Three circles

Jean Etienne Montucla [1725-1799], in his text [7] *Histoire des Mathematiques,* refers to the tenth problem as the only difficult one.

The Belgianmathematician, Adrian van Roomen [1561-1615], who had challenged the French mathematicians with a 45th degree equation, was amazed that Viete had solved the problem. Viete in turn challenged van Roomen with the tenth problem: Given three circles, construct a circle

tangent to them. Van Roomen, according to Montucla [7], solved this problem by treating the "intersection of two hyperbolas as a center" of a desired tangent circle. Viete [12] rejects this proposition. Viete compares van Roomen's technique with Menechmus' method for duplicating the cube with parabolas, as well as with Nicomedes' efforts to duplicate the cube by means of conchoids. Viete is also convinced that van Roomen will work in vain, whenever the asymptotes of the associated hyperbolas are parallel.

> *"Therefore, my brilliant Adrian, and, if you please, Apollonius of Belga, because the problem that I have proposed is plane, you, however, explained it as solid; nor therefore have you made firm the meeting of hyperbolas, uhich you claim as your own doing; nor even now can you make it stick, because, as a matter of fact, if the asymptotes are parallel, you are working in vain."*
>
> *"Ergo clarissime Adriane, acsi placet Apolloni Belga, quoniam Problema quod proposui planum est, tu vero ceu solidum explicasti, neque ideo occursum hyperbolarum, quem ad factionem tuam adsumis, firmasti, neque etiamnum potes firmare, quoniam revera si asymptoti parallelae erit irritus labor."*

The following theorem is fundamental in generalizing van Roomen's proposition and involves addition or subtraction of radii of two circles.

Theorem. The loci of centers of circles tangent to two given circles are two distinct conic sections.

Figure 1 reflects three given circles as well as the common technique used to construct circles tangent to two circles. The explanatory notes which accompany the drawing lead to the important conclusion:

A LINE OF CENTERS CAN BE DETERMINED.

Each line of centers which can be determined as indicated contains centers of two circles tangent to the three given circles. It is the introduction of this line of centers which is believed to simplify the analytic solution of the problem. L. J. Ulman [11] seems to be the first mathematician to explicitly mention lines of centers as well as conic sections in connection with the solution of the Apollonius tangency problem. Ulman's discovery seems to have been ignored. The more difficult notions of radical center and axes of similitude seem to prevail.

R. F. Muirhead [8] states:

"It is well known that the contact circles occur in pairs such that each pair has for its radical axis one of the four axes of similitude of the given circles."

Julius Plücker, (1801-1868), [9] based on the general theorem quoted below, states that his sequence of development in obtaining an analytical solution of the Apollonius tangency problem is quite analogous to that of Viete, and is *"very elegant and compact."*

"The centers of the eight circles which simultaneously touch the same three given circles are distributed pairwise, on the perpendiculars dropped from the radical center of the three given circles onto their four axes of similitude."

'Les centres des huit cercles qui touchent a la fois les trois memes cercles donnes sont distribues, deux a deux, sur les perpendiculaires abaissees du centre radical de ces trois cercles sur leur quatre axes de similitude."

Definition. A hyperbola is the locus of a point that moves in a plane so that the difference between its undirected distances from two fixed points is a non-zero constant.

Observations. See Figure 1.

I. Given three non-intersecting, non-congruent circles with centers O , O_2 , and O , and radii R_1 , R_2 , and R_3 respectively,

II. Two auxiliary circles are constructed with centers O, and O , and radii $R_3 + R$ and $R_2 - R_1$.

III. Two arbitrary secants are drawn through O_1, and these scants intersect with the auxiliary circles at the points C and C' , and D and D'.

IV. Four isosceles trianges are constructed;

$A\ O_1AC$, $A\ O_1A'C'$, $A\ O_1BD$, and $A\ O_1B'D'$.

V. The difference between the specified line segments;

$$\overline{O_2B} - \overline{O_1B} = R - R_1 , \qquad \overline{O_1B'} - \overline{O_2B'} = R_2 \quad R_1 ,$$
$$\overline{O_3A} - \overline{O_1A} = R + R_1 , \qquad \overline{O_1A'} - \overline{O_3A'} = R_3 \quad R_1 .$$

VI. Points A and A' satisfy the definition of a hyperbola, and likewise points B and B'.

VII. Now, when points A and B' coincide, then that point is the

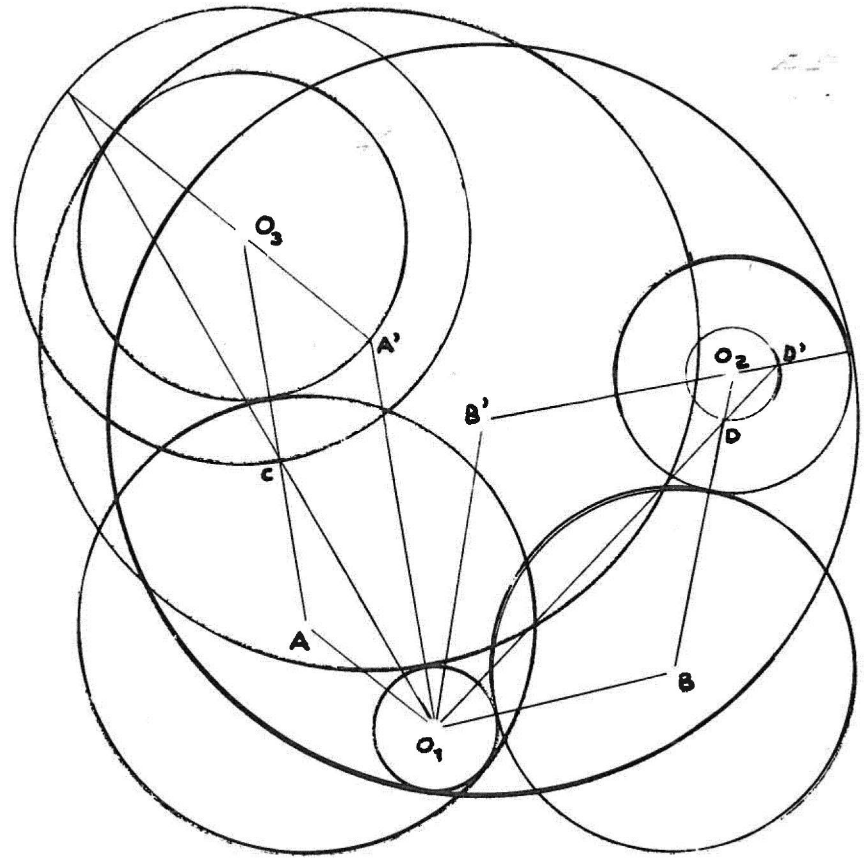

FIGURE 1

center of one of the circles tangent to the three given circles; and likewise, when A' and B coincide, then that point also will be the center of a desired tangent circle.

VIII. Since points A and A' are points on one hyperbola, and since points B and B' are points on another hyperbola, the conclusion is that when points A and B' coincide, as well as when points A' and B are coincident -- then ONE LINE IS DETERMINED.

Since two circles determine two conic sections, it is clear that three circles determine six conic sections. If the common intersections of these conic sections were to be determined by algebraic techniques, then indeed Montucla's comment [7] seems to take on credibility: *"analytic geometry is not suited for the Apollonius problem."* There is another similar comment by Benjamin Alvord [2], who in his article *"A geometrical solution of the ten problems in the tangencies of circles; and, also, of the fifteen problems in the tangencies of spheres,"* observes: *"if these problems had been solved algebraically -- the resulting equation would be one of the eighth degree."* However, Alvord does conjecture that quadratic equations should be associated with the tangency problem's solution; *"but I do not think it 'has ever- been obtained by mathematicians."*

Clearly the analytic geometry solution suggested by L. N. M. Carnot (1753-1823) [3] was not known to Alvord. Carnot states that a second degree equation in one variable is obtainable, but:

> *"I am not going to carry out the required calculations since the synthetic solutions by Viete, Newton, and Euler are much more elegant."*
>
> *"Je n'effectue pas ce calcul parce que ce probleme a ete resolu d'une maniere plus simple par des geometres de premier ordre, tels que Viete, Newton, Euler, et que la seule synthese en fournit plusieurs solution treselegantes."*

The comment by Montucla was made in connection with the two quadratic equations found by Descartes, and the one found by Princess Elisabeth. However, Montucla's comment may be contrasted with Descartes' [4] regarding the first solution.

> *"an equation is found where there is only* x *and* xx *unknown, so that the problem is in the plane, and there is no longer a need to go beyond. Because the rest is of no use in cultivating or entertaining the mind, but only in exercising the patience of some hard working mathematician."*
>
> *"on trouve une equation ou il n'y a que* x *et* xx; *de facon que le Probleme est plan, et il n'est plus besoin de passer outre. Car le reste ne sert point pow cultiver ou recreer l'esprit, mais settlement pour exercer la patience de quelque calculateur laborieux."*

Dr. A. Aeppli [1] has generalized Descartes' second proof to

n-dimensions.

J. M. Fitz-Gerald [5], while commenting on the solution by O. von Stoll, states: *"Some of his difficulties may no doubt be attributed to his extremely complex algebraic formulation of the problem."*

C. N. Mills [6] completed a *"general"* formula for the calculations of the length of the radii of the Apollonius contact circles. The complete calculations and proofrequired "18 *inch wall-paper 24 feet long."*

Von Stoll [10] in his article, *"Zum Problem des Apollonius,"* utilized analytic geometry effectively to determine solutions of desired tangent circles in pairs. According to R. F. Muirhead [8], von Stoll was the first author who considered the relative positions of the given circles in order to determine the number of possible solutions. Von Stoll correctly refers to some pitfalls when the centers of the given circles are collinear:

> *"In conclusion special consideration needs to be given to the case...where the centers of the given circles are collinear; this is the same situation where the Gergonne solution becomes illusory too."*
>
> *"Zum Schlusse verdient noch der Fall eine besondere Betrachtung, ...wo die Mittelpunkte der gegebenen Kreise auf einer geraden Linie liegen; es ist dies derselbe Fail, in welchem auch die Gergonne'sche Construction illusorisch wird."*

It should be mentioned that Viete [12] excludes from his first problem collinearity of the three given points. Von Stollrefers to another pitfall in the last paragraph when again no quadratic equation is obtainable:

> *"if the three circles are so located, that the external similarity point of the first and second is also the one of the second and third. ... Indeed one can then place two common external tangent lines on the three circles, which here are to be looked upon as circles with infinite long radii."*
>
> *"wenn die drei Kreise so liegen, class der äussere Aehnlichkeitspunkt des ersten and zweiten auch der des zweiten and dritten ist. ... In der That kann man dam an die drei Kreise zwei gemeinschaftliche geradlinige Tangenten legen, die hier als Kreise mit unendlich grossem Radius anzusehen sind."*

Von Stoll considers three additional cases with respect to internal and external similarity points and states in his last sentence:

"In all these cases two common tangent lines can be placed in contact with the three circles, and there exist besides these in general six solutions."

"In alien diesen Fällen Können an die drei Kreise zwei gemeinschaftliche Tangenten gelegt werden and existiren ausserdem im Allgemeinen sechs Lösungen."

The possibility of obtaining seven solution circles with finite radii and one solution with a center at infinity is overlooked by von Stoll. The attached numerical example reflects seven solution circles with finite radii and one common tangent line. This same numerical example can be used to show the limitation of the *"general"* solution by Mills.

Neither von Stoll nor Mills apparently heeded the objection by Viete [12] *"quoniam revera si asymptoti fuerint parallelae, erit irritus labor."* This objection to the van Roomen solution can now also be addressed. Whenever the asymptotes of the hyperbolas associated with the solution of the Apollonius tangency problem are parallel, then will the desired tangent circles be common tangent lines.

The task at hand then is to present a method for determining the centers of desired tangent circles in pairs, as well as provide an opportunity to analyze the problem and determine the number of possible solutions.

The proposed solution to determine pairs of centers:

SOLVE SIMULTANEOUSLY ONE LINEAR EQUATION IN TWO VARIABLES AND ONE SECOND DEGREE EQUATION IN TWO VARIABLES

$$ax + by + c = 0 \quad \text{and}$$
$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 .$$

This approach involves the use of a quadratic equation in one variable, and hence is constructable with straight-edge and compass.

The theorem which will be proven is:

Theorem. The centers of the eight desired tangent circles lie on two distinct conic sections, and these eight centers are distributed pair-wise on four lines of centers.

The development of the proof is intended to facilitate obtaining

numerical results through the use of electronic calculators as well as pictorial results by means of mechanical plotters.

Two theorems which can be proven by the reader:

Theorem. The four lines of centers are concurrent.

Theorem. The intersection of these lines of centers is the radical center.

PROBLEM

Determine the coordinates of centers of circles tangent to three given circles.

Analysis. The number of centers that can be obtained depends on the relative positions of the given circles.

Given.
$$(x - h_1)^2 + (y - g_1)^2 = R_1^2 ,$$
$$(x - h_2)^2 + (y - g_2)^2 = R_2^2 ,$$
$$(x - h_3)^2 + (y - g_3)^2 = R_3^2 .$$

Asked. Show that the intersections of lines and conic sections yield centers of desired tangent circles.

Solution. Let d_1, d_2, and d_3 be the respective distances from one of the desired centers to the centers of the given circles.

$$\text{I.} \quad d_1 = \sqrt{(x - h_1)^2 + (y - g_1)^2} ,$$
$$\text{II.} \quad d_2 = \sqrt{(x - h_2)^2 + (y - g_2)^2} ,$$
$$\text{III.} \quad d_3 = \sqrt{(x - h_2)^2 + (y - g_3)^2} .$$

Algebraic sums of pairs of equations shown yield possibilities:

$$d_3 \pm d_1 = \pm R_3 \pm R_1 ,$$
$$d_2 \pm d_1 = \pm R_2 \pm R_1 ,$$
$$d_3 \pm d_2 = \pm R_3 \pm R_2 .$$

Suggested sums or differences of known radii as shown below are useful in determining linear equations and second degree equations each in two variables.

First combination	Second Combination	Third Combination	Fourth Combination
$A = R_3 - R_1$	$A_3 = R_3 + R_1$	$A_3 = R_1 - R_3$	$A_3 = R_1 + R_3$
$A_2 = R_2 - R_1$	$A_2 = R_2 + R_1$	$A = R_1 + R_2$	$A_2 = R_1 - R_2$
$A_{,,} = R_3 - R_2$	$A_{,,} = R - R2$	$A_4 = R_2 + R_3$	$A_{,,} = R_2 + R_3$

Subtract equation I from equation III, then subtract I from II:

IV. $\sqrt{(x-h_3)^2+(y-g_3)^2} = \sqrt{(x-h_1)^2+(y-g_1)^2} + A_3$,

V. $\sqrt{(x-h_2)^2+(y-g_2)^2} = \sqrt{(x-h_1)^2+(y-g_1)^2} + A_2$.

Square both sides of IV and V, then collect like terms:

equations VI, and VII follow:

$$\text{VI.}\quad 2(h_1-h_2)x + 2(g_1-g_3)y + h_3^2 + g_3^2 - h_1^2 - g_1^2 - A_3^2 = 2A_3\sqrt{(x-h_1)^2+(y-g_1)^2},$$

$$\text{VII.}\quad 2(h_1-h_2)x + 2(g_1-g_2)y + h_2^2 + g_2^2 - h_1^2 - g_1^2 - A_2^2 = 2A_2\sqrt{(x-h_1)^2+(y-g_1)^2}.$$

Multiply both sides of VI by A , and multiply both sides of VII by A , then subtract. The result is a linear equation:

$$\text{VIII.}\quad A_1x + B_1y + C_1 = 0 .$$

The values of A , B_1, and C_1 are indicated below:

$A_1 = 2(A_2(h_1-h_3) - A_3(h_1-h_2))$,

$B_1 = 2(A_2(g_1-g_3) - A_3(g_1-g_2))$,

$C_1 = A^2\,(h_3^2+g_3^2-h_1^2-g_1^2-A_3^2) - A_3\,(h_2^2+g_2^2-h_1^2-g_1^2-A_2^2).$

By letting A and A_3 take on alternately the suggested values of the sum or difference of the radii, it is clear that four linear equations can be determined. Now note that a line may intersect with a conic in at most two points. So, if two second-degree equations each in two variables were determined, then each of these conic sections would contain at most four desired centers.

The derivation of a second degree equation is according to standard analytic geometry techniques:

Subtract equation II from equation III,

IX. $\sqrt{(x-h_3)^2+(y-g_3)^2} = \sqrt{(x-h_2)^2+(y-g_2)^2} + A_4$;

square both sides of equation IX, then collect like terms,

$$\text{X.}\quad 2(h_2-h_3)x + 2(g_2-g_3)y + h_3^2 + g_3^2 - h_2^2 - g_2^2 - A_4^2 = 2A_4\sqrt{(x-h_2)^2+(y-g_2)^2};$$

square both sides of X, then collect like terms in order to obtain an equation of the form

$$\text{XI.}\quad Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 .$$

The values of A, B, C, D, E, and F are indicated below:

$A = 4(h_2-h_3)^2 - A_4^2)$,

$B = 4(h_2-h_3)(g_2-g_3)$,

$C = 4(g_2-g_3)^2 - A_4^2)$,

$D = 2(h_2-h_3)(h_3^2+g_3^2-h_2^2-g_2^2-A_4^2) + 4h_2A_4^2$,

$E = 2(g_2-g_3)(h_3^2+g_3^2-h_2^2-g_2^2-A_4^2) + 4g_2A_4^2$,

$F = (h_3^2+g_3^2-h_2^2-g_2^2-A_4^2)^2 - 4A_4^2\,(h_2^2+g_2^2)$.

Observing that A_4 appears as a squared quantity only among the coefficients of equation XI, and using the two suggested values for A_4, i.e., $A_4 = R_3 - R_2$, or $A_4 = R_2 + R_3$, the conclusion is that in general two conic sections can be determined. The equation $Ax^2 + Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ represents one of nine curves; namely, an ellipse, an imaginary ellipse, a point, a hyperbola, a pair of intersecting lines, a parabola, a pair of parallel lines, a pair of imaginary parallel lines, or a pair of coincident lines. Since equation XI includes the hyperbola to which van Roomen refers, it is evident that analytic geometry provides a workable solution in a most general sense. The solution set of equations VIII and XI will reflect the coordinates of centers of circles tangent to the three given circles.

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REFEREES FOR THIS ISSUE

The following served as referees for papers considered since the last issue. The JOURNAL appreciates their help and contributions. Almost every article published is revised and improved by referees' comments and suggestions. *David Roselle,* Virginia Polytechnic Institute; *James Barksdale,* Western Kentucky University; *Bruce Peterson,* Middlebury College; *Robert Prielipp,* University of Wisconsin, Oshkosh; *L. Carlitz,* Duke University; *J. Sutherland Frame,* Michigan State; *Donald Bushnell,* Ft. Lewis College; *John Hodges,* University of Colorado; *Vern Nelson,* Metropolitan State College; *David Kay,* University of Oklahoma; *James Patterson, Roger Opp, C. A. Grimm, Dean Benson, Dale Rognlie, Ronald Weger, Sailes Sengupta, Sumedha Sengupta,* all from South Dakota School of Mines and Technology.

AN INEQUALITY FOR A MARKOV CHAIN WITH TRANSITION FUNCTION $p(i,j)$ AND STATE SPACE $J = \{1, 2, \ldots, n\}$.

by Michael Stephen Schwartz
Belfer Graduate School of Science

This paper originated from Problem Number 395 of the Spring, 1977 issue of this journal. The problem, as it appeared was:

'Assume that n independent Bernoulli experiments are made with $p = P[\text{success}]$, $1 - p = P[\text{failure}]$, and $0 < p < 1$. Intuitively it seems that P[success on the first trial|exactly one success] is always less than P[success on the first trial|at least one success . Verify directly that this is indeed the case."

Theorem. If $0 < p_i < 1$, for $i = 1, 2, \ldots, n$, then

$$(1)\qquad \frac{p_1 \prod_{j=2}^{n} (1 - p_j)}{\sum_{i=1}^{n} p_i \prod_{j\neq i}^{n} (1 - p_j)} < \frac{p_1}{1 - \prod_{j=1}^{n} (1 - p_j)} \qquad (n > 1).$$

Proof. We shall proceed by induction. For $n = 2$, we want to prove the following:

$$\frac{p_1(1 - p_2)}{p_1(1 - p_2) + p_2(1 - p_1)} \qquad \frac{p_1}{1 - (1 - p_1)(1 - p_2)}. \qquad (a)$$

After clearing the fractions and transposing, we have

$$(1 - p_2) - p_1(1 - p_2) < p_2(1 - p_1) + (1 - p_1)(1 - p_2)^2. \qquad (b)$$

Dividing both sides by $1 - p_1$ gives

$$(1 - p_2) < p_2 + (1 - p_2)^2. \qquad (c)$$

Since (c) reduces to $p_2^2 > 0$, reversing the steps from here establishes the truth for $n = 2$.

Next, suppose (1) holds for $n = k$. For $n = k + 1$, the left side of (1) can be written as

$$\frac{p_1 \prod_{j=2}^{k} (1 - p_j)}{\sum_{i=1}^{k} p_i \prod_{j\neq i}^{k} (1 - p_j)} \cdot \frac{(1 - p_{k+1}) \sum_{i=1}^{k} p_k \prod_{j\neq i}^{k} (1 - p_j)}{\sum_{i=1}^{k+1} p_i \prod_{j\neq i}^{k+1} (1 - p_j)},$$

which is less than

$$(1')\quad \frac{p_1}{1 - \prod_{j=1}^{k}(1 - p_j)} \cdot \frac{(1 - p_{k+1}) \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j)}{\sum_{i=1}^{k+1} p_i \prod_{j \neq i}^{k+1} (1 - p_j)}$$

by the induction hypothesis. The task which remains is to show that (1') is less than

$$\frac{p_1}{1 - \prod_{j=1} (1 - p_j)}.$$

Clearing

$$(2')\quad \sum_{i=1}^{k+1} p_i \prod_{j \neq i}^{k+1} (1 - p_j) - \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j)(1 - p_{k+1}) = p_{k+1} \prod_{j=1}^{k} (1 - p_j),$$

we readily obtain

$$(3')\quad -\prod_{j=1}^{k+1} (1 - p_j) \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j)(1 - p_{k+1}) < p_{k+1} \prod_{j=1}^{k} (1 - p_j) - \prod_{j=1}^{k} (1 - p_j) \sum_{i=1}^{k+1} p_i \prod_{j \neq i}^{k+1} (1 - p_j).$$

Transposing to isolate $p_{k+1} \prod_{j=1}^{k} (1 - p_j)$ on the right-hand side and dividing

$$(4')\quad \sum_{i=1}^{k+1} p_i \prod_{j \neq i}^{k+1} (1 - p_j) - \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j)(1 - p_{k+1}) - p_{k+1} \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j)(1 - p_{k+1}) < p_{k+1}.$$

which becomes

$$(5')\quad \prod_{j=1}^{k} (1 - p_j) + \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j)(1 - p_{k+1}) < 1$$

after (2') has been reapplied and p_{k+1} divided out. But

$$(1 - p_{k+1}) \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j) < \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j) \text{ and}$$

$$\prod_{j=1}^{k} (1 - p_j) + \sum_{i=1}^{k} p_i \prod_{j \neq i}^{k} (1 - p_j) < \prod_{i=1}^{k} (p_i + (1 - p_i)) = 1,$$

where the last inequality follows because $\prod_{i=1}^{k} (p_i + (1 - p_i))$ contains all of the terms on the left (plus others).

This establishes (1) for $n = k + 1$, so that inequality (1) holds for all integers $n > 1$. Q.E.D.

Comment. By the same argument we can obtain the following:

If there exists $j_1, j_2, \ldots, j_i, \ldots, j_n \in J$ such that $0 < p(i, j_1) < 1 \;\forall\, i \in J$, then

$$\frac{p(1, j_1) \prod_{k=2}^{n} (1 - p(k, j_k))}{\sum_{i=1}^{n} p(i, j_i) \prod_{k \neq i}^{n} (1 - p(k, j_k))} \qquad \frac{p(1, j_1)}{1 - \prod_{k=1}^{n} (1 - p(k, j_k))}.$$

I would like to thank the editor for helping me to write my paper in a better form.

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OBTAINING THE PROBABILITY DENSITY FUNCTION OF THE k-TH ORDER STATISTIC

by A. Banah, B. Crain, T. Mehle, B. Rahali, B. Showalter and P. Smith

Suppose that $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables (of absolutely continuous type), each with the same marginal probability density function $f(x)$ and cumulative distribution function $F(x)$. The joint pdf of $X_1, X_2, \ldots, X_n$ is then $f(x_1)f(x_2)\ldots f(x_n)$.

The order statistics $Y_1, Y_2, \ldots, Y_n$ are defined as usual (see Hogg and Craig [1965]) by

$$Y_1 = \text{smallest of } X_1, X_2, \ldots, X_n,$$
$$Y_2 = \text{next smallest of } X_1, X_2, \ldots, X_n,$$
$$Y_n = \text{largest of } X_1, X_2, \ldots, X_n.$$

Since we assume that the X's are of continuous type, the probability of any ties among the X's is zero; that is,

$$Pr(\bigcup_{i \neq j} [X_i = X_j]) = 0.$$

Thus we may assume that the Y's are well defined. By their definition the order statistics satisfy the chain of inequalities $-\infty < Y_1 \le Y_2 \le \ldots \le Y_n < \infty$. Order statistics are encountered in such places as the theory of reliability.

Let $g(y_1, y_2, \ldots, y_n)$ be the joint pdf of $Y_1, Y_2, \ldots, Y_n$, and let $g_k(y_k)$ represent the marginal pdf of the k-th order statistic Y_k. It is well-known (again, see Hogg and Craig) that $g(y_1, y_2, \ldots, y_n)$ and $g_k(y_k)$ may be written in terms of $f(x)$ and $F(x)$; and, in fact, the formulas are

$$(1)\quad g(y_1, y_2, \ldots, y_n) = \begin{cases} n! f(y_1)f(y_2)\ldots f(y_n), & -\infty < y_1 \le y_2 \le \ldots \le y_n < +\infty \\ 0 & \text{, otherwise} \end{cases}$$

and

$$(2)\quad g_k(y_k) = \frac{n!}{(k-1)(n-k)!} f(y_k)[F(y_k)]^{k-1}[1 - F(y_k)]^{n-k}, \; -\infty < y_k < \infty.$$

Formula (2) follows from formula (1) by repeated and laborious integration. In this article we give a short and direct way to obtain $g_k(y_k)$. Let $G_k(y_k)$ be the cdf for Y_k, so that

$$\frac{d}{dy_k} G_k(y_k) = g_k(y_k).$$

For $k = n$ the situation is easy:

$$G_n(y_n) = Pr(Y_n \le y_n) = Pr(X_1 \le y_n, X_2 \le y_n, \ldots, X_n \le y_n)$$

$= [F(y_n)]^n$ (by independence of the X's), so that

$$g_n(y_n) = (d/dy_n)[F(y_n)]^n = n[F(y_n)]^{n-1} f(y_n).$$

For k = 1 the situation is almost as easy:

$$G_1(y_1) = Pr(Y_1 \le y_1) = 1 - Pr(Y_1 > y_1)$$
$$= 1 - Pr(X_1 > y_1, X_2 > y_1, \ldots, X_n > y_1) = 1 - [1 - F(y_1)]^n$$

(by independence of the X's), so that

$$g_1(y_1) = (d/dy_1)\{1 - [1 - F(y_1)]^n\} = n[1 - F(y_1)]^{n-1} f(y_1).$$

For $1 < k < n$ the situation is more delicate:

$$G_k(y_k) = Pr(Y_k \le y_k) = Pr(k \text{ or more } X\text{'s} \le y_k).$$

Thus

$$(3)\quad G_k(y_k) = \sum_{j=k}^{n} \binom{n}{j} [F(y_k)]^j [1 - F(y_k)]^{n-j}$$

If we compute $g_k(y_k) = (d/dy_k)G_k(y_k)$ using (3) we obtain a horrible mess. Instead, write

$$(4)\quad G_k(y_k) = 1 - \sum_{j=0}^{k-1} \binom{n}{j} [F(y_k)]^j [1 - F(y_k)]^{n-j}$$
$$= 1 - H(k-1),$$

where $H(\cdot)$ is the cdf of a binomial random variable with parameters n and $p = F(y_k)$.

In Wilks [1962], p. 152, we find the following handy relation, where $q = 1 - p$:

$$\sum_{j=0}^{k} \binom{n}{j} p^j q^{n-j} = (n-k)\binom{n}{k}\int_0^q u^{n-k-1}(1-u)^k\,du .$$

This relation is proved using integration by parts, and applying it to our situation in (4) yields

$$(5)\quad G_k(y_k) = 1 - (n-k+1)\binom{n}{k-1}\int_0^{1-F(y_k)} u^{n-k}(1-u)^{k-1}\,du .$$

Using the Fundamental Theorem of Integral Calculus we differentiate (5) and obtain

$$g_k(y_k) = -(n-k+1)\,\frac{n!}{\overline{k-1}!\,\overline{n-k+1}!}\,[1-F(y_k)]^{n-k}[F(y_k)]^{k-1}\cdot -f(y_k)$$

$$= \frac{n!}{\overline{k-1}!\,\overline{n-k}!}\, f(y_k)\,[F(y_k)]^{k-1}\,[1-F(y_k)]^{n-k} .$$

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THE NATURAL DENSITY OF THE FIBONACCI NUMBERS

by I. P. Scalisi and G. Sethares
Bridgewater State College

In this paper, we compute the natural density of the Fibonacci Numbers by elementary techniques. Let N be the set of positive integers and $A \subseteq N$. We will think of the elements of A as being arranged according to size in the form of a sequence $\{a_k\}$. For example, if A denotes the set of positive even integers, then $a_k = 2k$. In this paper, we consider the set $A = \{F_k\}$, where F_k is the kth Fibonacci Number defined recursively by $F_1 = F_2 = 1$, and $F_{k+2} = F_{k+1} + F_k$, $k \in N$.

There are three types of density that one usually considers: asymptotic, natural, and Schnirelmann. They are defined in the following way:

Definition. Let $A(x)$ denote the number of positive integers in A that are less than or equal to x, where $x \in N$.

The <u>asymptotic density of A</u> is

$$\delta_1(A) = \liminf_{k\to\infty} \frac{A(k)}{k} .$$

The <u>natural density of A</u> is

$$\delta(A) = \lim_{k\to\infty} \frac{A(k)}{k}, \text{ if the limit exists.}$$

The <u>Schnirelmann density of A</u> is

$$d(A) = \inf_{k \in N} \left\{\frac{A(k)}{k}\right\} .$$

Clearly, $0 \le d(A) \le \delta_1(A) \le 1$, and if A has a natural density? then, $\delta(A) = \delta_1(A)$.

Following are a few examples illustrating these three densities:

a.) $A = N$. Then $d(A) = \delta(A) = \delta_1(A) = 1$.

b.) $A = \{2,4,6,\ldots\}$. Then, $A(k) = [k/2]$ and $\delta(A) = \delta_1(A) = 1/2$, but $d(A) = 0$.

c.) $A =$ the set of all positive primes. Let $\pi(x) =$ the number of positive primes $\le x$. The well-known Prime Number Theorem states that $\lim_{x\to\infty} \frac{\pi(x)}{x} \cdot \ln x = 1$.

Thus, using the notation, $A(k) = \pi(k)$, it follows that $\delta(A) = 0$.

Hence, $d(A) = \delta(A) = \delta_1(A) = 0$.

d.) $A = \bigcup_{n\in N} A_n$, where $A_n = \{x \mid (2n)! \le x < (2n+1)!\}$

Then $\delta(A)$ does not exist. (For a hint, see [3], p. 248.)

Note 1: The Schnirelmann density is sensitive to the first terms of the sequence $A = \{a_k\}$. For example, if $1 \notin A$, then $d(A) = 0$, and if $2 \notin A$, then $d(A) \le 1/2$, whereas δ_1 remains the same whether or not 1 or 2 are in A.

Note 2: The asymptotic density and the natural density, if it exists, are measures of the sparcity of the elements of A. For example, if A consists of the terms of an arithmetic sequence with difference D, then $\delta_1(A) = \delta(A) = \frac{1}{D}$.

Note 3: The first major result concerning the distribution of a set of positive integers was the Prime Number Theorem (see Example c.), proved independently by Hadamard and De La Vallée Poussin in 1896. The first serious study of the density of an arbitrary set of positive integers was made in 1930 by L. G. Schnirelmann (4). Schnirelmann conjectured the celebrated $\alpha\beta$ theorem which was first proved by H. B. Mann in 1942 [2].

Now, let $A = \{F_k\}$, the Fibonacci sequence. It is known that $d(A) = 0$. (Fibonacci Quarterly, vol. 4, #3, p. 284.) We will show that $\delta(A) = 0$, so that $0 = d(A) = \delta_1(A) = \delta(A)$. Note that

$$A(F_k) = A(n), \text{ for } F_k \le n < F_{k+1}$$

Hence,

$$\frac{A(F_k)}{F_k} \ge \frac{A(n)}{n}, \text{ for } F_k \le n < F_{k+1},$$

so that,

$$\limsup_{k\to\infty} \frac{A(k)}{k} \le \limsup_{F_k\to\infty} \frac{A(F_k)}{F_k} = \limsup_{k\to\infty} \frac{k}{F_k}.$$

We will show that

$$\lim_{k\to\infty} \frac{k}{F_k} = 0.$$

This will then imply that $\delta(A) = 0$.

(i) The sequence $\left\{\frac{k}{F_k}\right\}$ is monotonic decreasing and is bounded..

For $k \le 3$, we have $F_{k-2}/F_{k-1} \le 1$, so

$$1 + \frac{F_{k-2}}{F_{k-1}} \le 2 < k;$$

or

$$\frac{F_{k-1} + F_{k-2}}{F_{k-1}} < k.$$

Thus

$$\frac{F_{k-1}}{F_k} > \frac{1}{k}.$$

Upon adding 1 to this last inequality we get

$$\frac{F_{k+1}}{k+1} > \frac{F_k}{k},$$

or

$$\frac{k+1}{F_{k+1}} < \frac{k}{F_k}.$$

Thus, $\left\{\frac{k}{F_k}\right\}$ is monotone decreasing. Also, $2 \ge \frac{k}{F_k} > 0$, for $k \ge 2$.

Therefore, by the monotone convergence theorem, L exists, where $L = \lim_{k\to\infty} \frac{A(F_k)}{F_k} = \lim_{k\to\infty} \frac{k}{F_k}$.

(ii) Computation of the Limit ($L = 0$).

It is well known that $\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \frac{\sqrt{5}+1}{2}$;

call this value g (the "golden section"). Then,

$$L = \lim_{k\to\infty} \frac{k}{F_k} = \lim_{k\to\infty} \frac{k/F_{k+1}}{F_k/F_{k+1}}$$

$$= \lim_{k\to\infty}\left(\frac{F_{k+1}}{F_k}\right)\left[\frac{k+1}{F_{k+1}} - \frac{1}{F_{k+1}}\right]$$

$$= g \cdot L \; ,$$

or

$$L(1 - g) = 0.$$

But, $g \neq 1$, thus, $L = 0$.

The authors wish to thank the referee for several helpful suggestions.

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and *Mark Turella.*

1978-79 CONTEST

Papers for the 1977-78 contest have now been judged, and we are receiving papers for this year's contest, so be sure to send us your paper, or your chapter's papers (at least 5 entries must be received from the same chapter in order to qualify, with a $20 prize for the best paper in each chapter).

For all manuscript contests, in order for authors to be eligible, *they must not have received a Master's degree* at *the time they submit their paper.*

GLEANINGS FROM CHAPTER REPORTS

The *OREGON GAMMA CHAPTER* made over $150 selling used mathematics books in 1978.

The *OHIO DELTA CHAPTER* at Miami University hosted the Fifth Annual Pi Mu Epsilon Student Conference on September 30, 1978 (and will host the Sixth Annual Conference on September 28, 1979 this year -- see call for papers elsewhere in this issue). The contributed papers were: *Jim Snodgrass*, Western Kentucky University, "Toads, Frogs and Dominoes"; *Stan Mahaffey*, Appalachian State, "Scheduling Intramural Sports"; *Douglas W. Boeme*, Miami University, "Is It Possible to Lose the OL' Magic?"; *Margaret Shaw*, Appalachian State, "Best Seat"; *Carole Cook*, presented by *Todd O'Connell*, both of Miami University, "Exam Scheduling: An Example of Math Modeling"; *Patty Pagter*, Appalachian State, "Tomato Processing"; *Wayne Delia*, Clarkson College of Technology, "Meauring the Area of a Snowflake"; *Richard Griffin*, Lowell University, "Arithmetic a la Computer"; *Steve Ruberg*, Miami University, "Non-Consecutive Numbers in a Magic Square"; *Sheila Reaver*, Miami University, "Patterns and Happiness from Cubes"; *Barry Stoltz*, Miami University, "Mathematics in the Works of Lewis Carroll"; *Kathy Saunders*, Miami University, "Flatland".

(*Editorial Note* - Many sections of the Mathematical Association of America now have student paper sessions. Pi Mu Epsilon Chapter can and have been involved. Please send reports on these to the *Editor*).

The *TEXAS IOTA CHAPTER* at the University of Texas in Arlington heard the following papers by: *Dr. U. L. Tennison*, "Curves of Constant Width"; *Dr. Steve Bermfield*, "Mathematics"; and *Richard Bennett*, "Ground Resonance Instability of Helicopters".

PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinguish material which might well be the comain of the Problem Department, but the Editor reserves the right to make an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate, or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota 57701. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March 1, and that for each Sping issue, the following September 15.

Mathacrostic No. 7

O 1	V 2		H 3	S 4		W 5	G 6	S 7	O 8	I 9	B 10		E 11
U 12	N 13	W 14		A 15	H 16	G 17	M 18		S 19	X 20	M 21	R 22	E 23
Z 24	V 25	B 26		S 27	U 28	C 29	B 30	Y 31		U 32	C 33	J 34	O 35
F 36	L 37	K 38	S 39	D 40	T 41		W 42	C 43	B 44	A 45	R 46	Y 47	Z 48
	H 49	D 50		F 51	T 52	R 53		Y 54	T 55	E 56	V 57	U 58	X 59
K 60	G 61	J 62		S 63	U 64	C 65		N 66	G 67	I 68	L 69	O 70	X 71
P 72	Q 73	R 74	K 75	T 76	H 77		D 78	I 79	P 80	K 81	O 82	J 83	N 84
	P 85	B 86	Q 87		F 88	K 89		S 90		G 91	I 92	C 93	E 94
B 95	F 96	R 97	N 98		Q 99	C 100	I 101	U 102	A 103	G 104	V 105	T 106	P 107
Y 108	X 109		Z 110	L 111		S 112	H 113	Q 114		U 115	S 116	C 117	P 118
H 119	V 120	W 121		X 122	I 123	L 124		V 125	Z 126		F 127		Z 128
E 129	J 130	I 131	L 132	D 133	P 134	O 135	A 136	W 137	N 138		G 139	M 140	D 141
H 142		F 143	O 144	Q 145	X 146	E 147	V 148	K 149		D 150	M 151	J 152	A 153
	Z 154	R 155	A 156	F 157	U 158	V 159	N 160		T 161	F 162	G 163	J 164	U 165
N 166	K 167	Y 168	H 169		J 170	D 171	R 172	H 173	A 174	E 175	Y 176	Q 177	T 178
B 179	Z 180		E 181	K 182		T 183	E 184	P 185		B 186	F 187		X 188
Y 189	T 190	E 191	L 192	C 193									

Definitions and Key

A. formulated the four-color conjecture around 1852 — 156 45 15 153 103 136 174

B. composed of elements drawn from various sources — 179 44 95 86 30 26 186 10

C. early name for curve of constant width — 117 65 100 29 193 43 33 93

D. in 1837 he gave the first rigorous proof of the impossibility of trisecting the general angle — 150 141 50 78 133 171 40

E. a ratchet device which allows motion in one direction only in equal steps — 147 175 56 181 94 191 23 129 184 11

F. wink — 187 157 143 162 36 51 127 88 96

G. some call it the lazy dog curve — 61 163 104 91 139 6 17 67

H. punched-card pioneer (1860-1929) — 113 49 77 119 169 173 3 142 16

I. Paul Erdös' word for "child" — 79 68 101 9 131 92 123

J. complex analog of Euclidean space — 164 83 34 152 130 170 62

K. libratory motion of earth's axis — 149 167 81 38 75 60 89 182

L. author of WHOM THE GODS LOVE — 132 37 111 69 192 124

M. game described by J. L. Synge in SCIENCE - SENSE AND NONSENSE; short for "vicious circle" — 21 151 18 140

N. precise, neat and simple — 98 160 66 84 13 138 166

O. engaging in wanton behaviour — 70 1 144 135 82 8 35

P. bridge player's delight (2 wds.) — 80 85 134 72 118 107 185

Q. a group of nine — 73 177 145 114 99 87

R. an element of a topology (2 wds.) — 155 172 22 74 46 53 97

S. 1959 lithograph by M. C. Escher featuring tetrahedra and octahedra — 63 39 90 112 27 7 116 19 4

T. body of water which originally separated the supercontinents Gondwanaland and Laurasia (2 wds.) — 178 183 106 52 41 161 190 55 76

U. child's game played outdoors on an arrangement of rectangles — 28 64 115 58 165 32 102 158 12

V. between-the-acts music — 120 25 125 148 159 105 2 57

W. repeated design in a pattern — 121 137 14 5 42

X. game invented by William Rowan Hamilton; marketed in 1859 — 188 146 20 59 71 122 109

Y. female ring leader (1882-1935) — 54 108 176 189 31 47 168

Z. rage of the early 1950's; brainchild of Roger Price — 180 128 126 110 48 154 24

Mathacrostic No. 7

submitted by Joseph V. E. Konhauser
Macalester College, St. Paul, Minnesota

Like all other acrostics, this one is a keyed anagram. Identify the key words, matching their letters in order with the opposite sequence of numbers, and insert each letter of the key words in the square of the mathacrostic with the same number. Words end at blank squares, and some words may extend on to the next line.

When completed, the mathacrostic will be a 193-letter quotation and the 26 initial letters of the key words will spell the name of an author and the title of his book from which the quotation was derived.

Mathacrostic No. 8

The directions for this mathacrostic are similar to the one above. It has 29 key words, 202 letters, and the initial letters of the key words are an author and work.

Solutions

Mathacrostic No. 6 [Fall, 1978]

Definitions and key:

A. Hosel	H. Temporize	O. Near-beer	V. Raffles
B. Uncinate	I. Hadamard	P. Empty	W. Tete-beche
C. Nonplusses	J. Enprise	Q. Patterns	X. Iffy
D. Team up	K. Darboux	R. Redowa	Y. oddment
E. Lyceum	L. Ipse dixit	S. Offspeed	Z. Newcomen
F. Evolute	M. Versor	T. Poset	
G. YO-yo	N. Ischemia	U. Oxymoron	

First letters: HUNTLEY THE DIVINE PROPORTION

Quotation: *In mathematics the expedience of beauty may be compounded of surprise, wonder, awe, or of realized expectation, resolved perplexity, a sense of unplumbed depths and mystery; or of economy of the means to an impressive result.*

Solved by ROBERT PRIELIPP, *University of Wisconsin, Oshkosh;* VICTOR FESER, *Mary College, Bismarck;* GERALD PERHAM, *St. Joseph's University;* LOUIS CAIROLI, *Kansas State University;* JEANETTE BICKLEY, *Webster Groves H.S., Missouri; and the Proposer.*

The Bridge Game. [Fall, 1978] *submitted by Pier Square.*

Four men named Banker, Waiter, Baker and Farmer are playing bridge. Each man's name is another man's job. If the baker is Mr. Baker's partner, if Mr. Banker's partner is the farmer and if at Mr. Farmer's right is the waiter, who is sitting on the banker's left?

Solution:

MR FARMER
baker

MR BANKER
waiter

MR WAITER
farmer

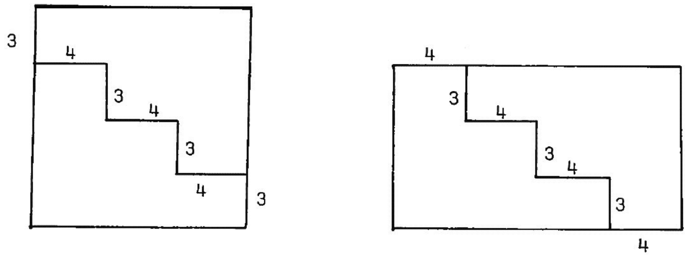
MR BAKER
banker

Solved by RANDOLPH ISTUANEK, *University of Wisconsin, Parkside;* VICTOR FESER, *Mary College, Bismarck;* JANDA COOK, *Lamar University;* MICHAEL YOUNG, *Portland State.;* AVI LOSIZE, *Torah Vodaath H.S., Brooklyn;* SAMUEL GUT, *Brooklyn;* RALPH KING, *St. Bonaventure;* MICHAEL GAIN, *Lament University;* RANDALL SCHEER, *Suny at Potsdam;* SUSAN HOFFMAN, *Iona College;* GEORGE RAINEY, *UCLA;* GERALD PERHAM, *St. Joseph's University;* SUSAN IWANSKI, *Greens Lawn, N.Y.;* CLAUDA EVANS, *LaMarque, Texas;* MARK EVANS, *LaMarque, Texas;* KENNETH LEMP, *Nassau Community College;* LOUIS CAIROLI, *Kansas State. University.*

A Single Cat [Fall, 1978] *submitted by Pier Square*

Is it possible to make a single cut in a 9 x 16 rectangle, rearrange the two parts and get a 12 x 12 rectangle?

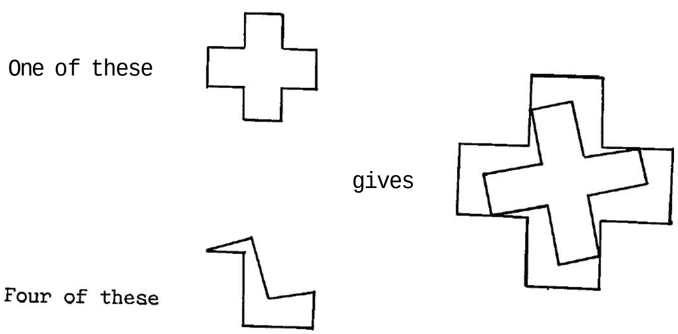
Solution:



Solved by VICTOR FESER, *Mary College, Bismarck;* RALPH KING, *St. Bonaventure;* MICHAEL GAIN, *Lamar University;* RANDALL SCHEER, *Suny at Potsdam;* GERALD PERHAM, *St. Joseph's University;* MARK EVANS, *LaMarque, Texas;* LOUIS CAIROLI, *Kansas State. University;* JEANETTE BICKLEY, *Webster Groves H.S., Missouri.*

Greek Crosses and Squares [Fall, 1977]

The last part of this five part problem was left open in the Fall, 1978 issue. It's solution is as follows:



Solved by VICTOR FESER, *Mary College, Bismarck;* MICHAEL GAIN, *Lamar University;* GERALD PERHAM, *St. Joseph's University;* KENNETH LEMP, *Nassau Community College.*

Mathacrostic No. 5 [Spring, 1978]

Also solved by ALLAN TUCKMAN *and* PATRICIA GROSS *of the. University of Illinois.* See Fall, 1978 issue for solution.

SUPERPROF

Professor	Associate Professor	Assistant Professor	Instructor	Graduate Assistant
Leaps tall buildings with a single bound.	Must take running start to leap over tall buildings.	Can leap over short buildings only.	Crashes into buildings when attempting to jump over them.	Cannot recognize buildings at all.
Is faster than a speeding bullet.	Is as fast as a speeding bullet.	Not quite as fast as a speeding bullet.	Slower than a slow bullet.	Wounds self with bullet when attempting to shoot.
Is stronger than a locomotive.	Is stronger than a bull elephant.	Is stronger than a bull.	Shoots the bull.	Smells like a bull.
Walks on water consistently.	Walks on water in emergencies.	Washes with water.	Drinks water.	Recognizes water.
Talks with God.	Talks with the angels.	Talks to himself.	Argues with himself.	Loses arguments with himself..
Creates consistent set of axioms.	Proves original theorems.	Accepts axioms.	Proves axioms	Disproves axioms.

Mathacrostic No. 8

E 1	D 2	B 3	J 4	d 5		H 6	G 7	W 8	A 9		C 10	S 11	I 12
O 13	X 14	a 15		G 16	R 17	L 18		F 19	D 20		b 21	S 22	V 23
C 24	N 25		K 26	K 27	W 28		d 29	U 30	Z 31	N 32	T 33	Y 34	R 35
	F 36	A 37		P 38	M 39	U 40	W 41	S 42		Q 43	K 44	N 45	T 46
K 47	E 48		T 49	b 50	D 51		M 52	R 53	N 54	X 55	K 56	F 57	
O 58	R 59	U 60		a 61	S 62		d 63	T 64	O 65		L 66	M 67	X 68
a 69	C 70	I 71	P 72	b 73	J 74	S 75	A 76		d 77	W 78	Q 79	d 80	R 81
	O 82	T 83	I 84	Y 85		D 86	H 87	Z 88	I 89	O 90	E 91		X 92
J 93	S 94		F 95	I 96	W 97	R 98	b 99	C 100	B 101		a 102	C 103	
F 104	N 105	Y 106	T 107		D 108	S 109		U 110	M 111	E 112	Z 113	Z 114	b 115
R 116	A 117	Y 118		T 119	U 120	V 121		L 122	W 123		d 124	K 125	D 126
P 127	X 128		B 129	S 130		I 131	P 132	D 133		G 134	S 135	d 136	R 137
Z 138	X 139	G 140	L 141	Y 142	J 143	R 144	N 145	I 146		Q 147	O 148	T 149	L 150
	Z 151	Y 152		d 153	H 154	B 155	N 156	E 157	F 158		a 159	S 160	
K 161	T 162	R 163	P 164	P 165	K 166	H 167		D 168	Y 169		S 170	H 171	
K 172	F 173		L 174	X 175	R 176	P 177	Z 178		P 179	P 180	a 181	T 182	B 183
J 184	L 185		A 186	Z 187	O 188	O 189		H 190	b 191		N 192	C 193	R 194
U 195	E 196	a 197	S 198	D 199	U 200		B 201	D 202					

Definitions and Key

A. A bingo-like game of Germanic origin — 76 186 9 117 37

B. Being — 183 3 129 201 101 155

C. Single-valued section of a multivalued function — 193 10 70 103 24 100

D. Famous criterion for irreducible polynomials over the rationals — 133 2 108 86 199 51 202 126 168 20

E. American sea captain who developed a method of determining longitude and latitude by observing heavenly bodies — 1 196 112 157 91 48

F. A chief city of the Philistines — 158 19 173 95 104 36 57

G. U.S. Mormon land — 134 140 16 7

H. Those function analytic in all of the complex plane — 171 87 6 190 154 167

I. Publication that should be read by all readers of this journal (two words; abbr.) — 71 89 131 96 12 84 146

J. A direct inference — 184 143 93 4 74

K. Profusely flowering dwarf chrysanthemum — 172 166 27 125 47 161 56 44 26

L. Provided with personnel — 122 150 141 174 66 18 185

M. A rounded elevation often accompanied by a corresponding depression on the opposite surface — 67 111 52 39

N. Victorious general in Shakespeare's Richard III — 32 156 54 25 45 192 145 105

O. The study of animal behavior — 65 82 148 58 188 13 90 189

P. Author of The Eumenides — 127 72 179 38 132 165 164 180 177

Q. Sequence of disjoint cells covering Euclidean n-space — 43 79 147

R. Plane of projective 2-space over a field — 35 137 81 163 176 17 98 53 194 144 116 59

S. In 1883, E. Lucas and DeParville presented this famous problem (3 words) — 42 160 170 11 94 130 62 22 75 135 109 198

T. Theorem extending functionals of linear subspaces to normed linear spaces (hypenated) — 49 119 64 33 46 149 162 107 182 83

U. Aiming at the improvement of race or breed — 60 40 200 30 120 110 195

V. Freud's pain/pleasure level of the psyche — 23 121

W. State of calmness or serenity — 28 123 41 8 78 97

X. Group having every element of finite order — 14 92 128 139 175 55 68

Y. An international understanding less binding than an alliance — 106 152 85 118 142 169 34

Z. Vector having direction and magnitude of the greatest rate of change — 88 138 31 114 151 113 187 178

a. A British soldier in the American Revolutionary War — 197 15 69 181 61 102 159

b. A rooflike cover over a door or window — 50 21 191 115 73 99

d. Structured a set so that each pair of elements has a supremum and infimum — 29 80 153 63 77 124 5 136

WILL YOUR CHAPTER BE REPRESENTED IN DULUTH?

It is time to be making plans to send an undergraduate delegate or speaker from your chapter to attend the annual meeting of Pi Mu Epsilon in Duluth, Minnesota during August, 1979. Each speaker who presents a paper will receive travel funds of up to $400, and each delegate, up to $200.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

At the suggestion of the Pi Mu Epsilon Council we have had a supply of 10 x 14-inch Fraternity crests printed. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock - - - - - $1.50/dozen,
(2) Purple and lavendar on goldenrod - $2.00/dozen.

LOCAL AWARDS

If your chapter has presented or will present awards this year to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal*.

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this Journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on separate sheets (one side only) properly identified with name and address and mailed before November 15, 1979.

Address all communications concerning this department to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048. A self-addressed postcard will expedite acknowledgements.

Problems for Solution

423. [Spring 19781. *Proposed by Richard S. Field, Santa Monica, California.*

Corrected version.

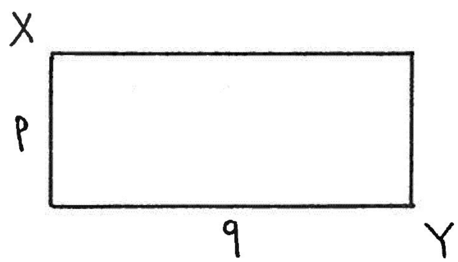
Find all solutions in positive integers of the equation $A - B^D = C^C$, where D is a prime number.

438. *Proposed by Ernst Straus, University of, California at Los Angeles.*

Prove that the sum of the lengths of alternate sides of a hexagon with concurrent major diagonals inscribed in the unit circle is less. than 4.

439. *Proposed by Richard 7. Hess, Palos Verdes, California.*

A bug starts at Monday noon at the upper left corner (X) of a p by q rectangle and crawls within the rectangle to the diagonally opposite corner (Y), arriving at 6 P.M. Exhausted, he sleeps till noon Tuesday. At that time he embarks for X, crawling along another path in the rectangle and arriving at X 6 P.M. Tuesday. Prove that at some time Tuesday the bug was at a point no farther than p from where he was at the same time Monday.



440. *Proposed by Charles W. Trigg, San Diego, California.*

Are there any prime values of $p < 10^5$ for which the equation

$$x^5 - y^5 = p$$

has a solution in positive integers?

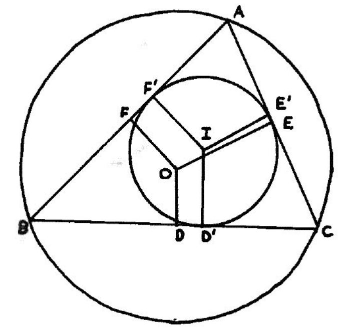
How about $x^5 + y^5 = p$?

441. *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Prove that a self-complementary graph with an even number of vertices has no more than $2i$ vertices of degree i, and that the number of them is even.

442. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, Nw York.*

Show that the sum of the perpendiculars from the circumcenter of a triangle to its sides is not less than the sum of the perpendiculars drawn from the incenter to the sides of the triangle.



443. *Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

If x and y are any real numbers, prove that

$$x^2 + 5y^2 \geq 4xy.$$

444. *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, Nw York.*

In terms of n, which is the first non-zero digit of

$$\prod_{i=1}^{n/2} (i)(n - i + 1) \quad \text{for even } n \geq 6?$$

445. *Proposed by Richard S. Field, Santa Monica, California.*

A "Tribonacci-like" integer sequence $\{A_n\}$ is defined in which $m_1A_i + m_2A_{i+1} + m_3A_{i+2} = A_{i+3}$, ($A_0 = A_1 = A_2 = 1$; m_1, m_2, m_3 are arbitrary integers).

A particular sequence of this kind is found ($m_1 = -1$, $m_2 = 5$, $m_3 = 5$) which appears to yield only perfect squares, viz.:

1, 1, 1, 9, 49, 289, 1681, ...

a) Prove that for this particular sequence, the successive terms continue to be perfect squares.

b) Can other values of m_1, m_2 and m_3 be found which result in the same property, namely, a sequence of perfect squares?

446. *Proposed by Clayton W. Dodge, University of Maine, Orono.*

A teacher showing the factorization of $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ emphasized that the second factor is not a square (not $(x + y)$ squared), and then chose $x = 5$ and $y = 3$ at random, obtaining

$$x^2 + xy + y^2 = 49,$$

which is a square.

a) Explain this apparent contradiction.

b) Show that the equation $x^2 + xy + y^2 = 49$ illustrates that a 3:5:7 triangle has a 120° angle.

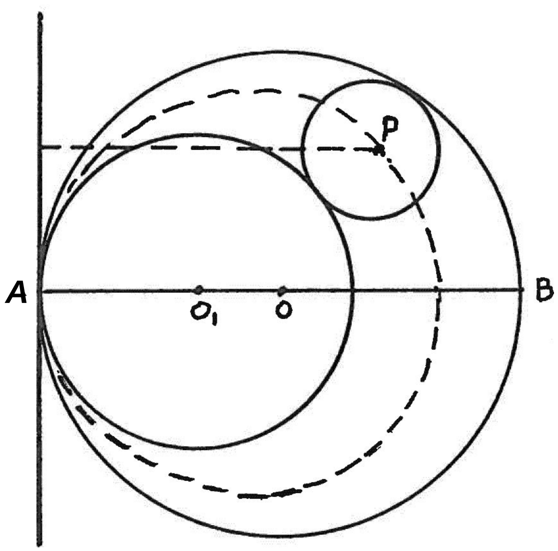
447. *Proposed by Zelda Katz, Beverly Hills, California.*

A variable circle touches the circumferences of two internally tangent circles, as shown in the figure.

a) Show that the center of the variable circle lies on an ellipse whose foci are the centers of the fixed circles.

b) Show that the radius of the variable circle bears a constant ratio to the distance from its center to the common tangent of the fixed circles.

c) Show that this constant ratio is equal to the eccentricity of the ellipse.

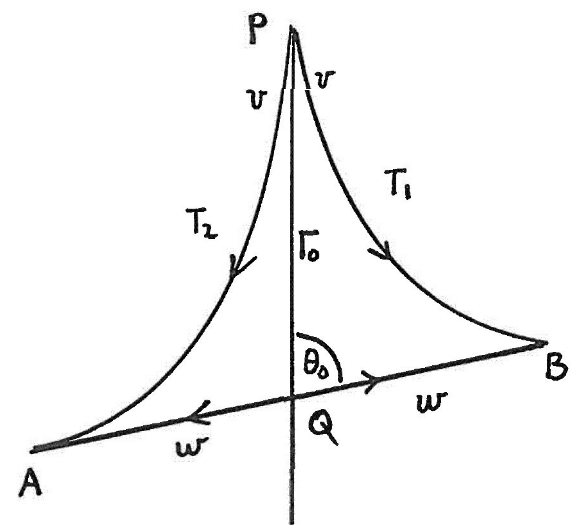


448. *Proposed by the late R. Robinson Rowe.*

Analogous to the median, call a line from a vertex of a triangle to a third point of the opposite side a "tredian". Then if both tredians are drawn from each vertex, the 6 lines will intersect at 12 interior points and divide the area into 19 subareas, each a rational part of the area of the triangle. Find two triangles for which each subarea is an integer, one being a Pythagorean right triangle and the other with consecutive integers for its three sides.

Solutions

401. [Fall 1977; Fall 1978]. The editor's comments following the solution of the Tom and the Pig pursuit problem mentioned a reference to Klamkin and Newman's article published in two parts in the American Mathematical Monthly, entitled Flying in a Wind Field. In addition to the treatment of related problems in the January and November 1969 issues, the article generalizes problem 401 to consider initial paths other than those at right angles



In the annexed figure, the Man starts at P with a speed of v, and the Pig starts at Q with a speed of w. If the Pig runs along QB, he is captured in time T_1. If, instead, he runs along QA, he is captured in time T_2. It is then shown that $T_1 + T_2$ is independent of Θ_0. Consequently if we take $\Theta = \pi/2$, as in Sam Loyd's Puzzle of Tom, the Piper's Son, then by symmetry,

$$T_1(90^\circ) = T_2(90^\circ).$$

If, on the other hand, $\Theta_0 = 0$, then

$$T_1(0^\circ) = \frac{\Gamma_0}{v + w} \qquad T_2(180^\circ) = \frac{\Gamma_0}{v - w}.$$

Finally, $T_1(90^\circ) + T_2(90^\circ) = 2T_1(90^\circ) = T_1(0^\circ) + T_1(180^\circ) =$

$$\frac{\Gamma_0}{v + w} + \frac{\Gamma_0}{v - w}.$$

This result justifies Sam Loyd's method of averaging the distances travelled by the pig if both ran forward on a straight line and if both ran directly toward each other.

412. [Spring 1978]. *Proposed by Solomon W. Golomb, University of Southern California, Lob Angeles, California.*

Are there examples of angles which are trisectibe but not constructible? That is, can you find an angle a which is not constructible with straightedge and compass, but such that when a is given, $\alpha/3$ can be constructed from it with straightedge and compass?

Solution by the Proposer.

Among an infinite number of such angles is $a = 3\pi/7 = 77\frac{1}{7}^\circ$. To trisect this angle, it suffices to double it, a trivial operation with straightedge and compass, to obtain $2\alpha = 6\pi/7$. The supplement of this angle is then $\pi - 2\alpha = \pi - 6\pi/7 = \pi/7 = \alpha/3$, the required trisection!

On the other hand, if α were constructible, then as we have seen, so too would be $\alpha/3$, from which $2\alpha/3 = 2\pi/7$ would be obtained by doubling. But $2\pi/7$ is the central angle corresponding to a side of the regular heptagon, whose constructibility is well known to be impossible.

Also solved by L. Carlitz, *Duke University*; Steven Izen, *Polytechnic Institute of New York, Brooklyn, N. Y.*; M. S. Klamkin and A. H. Rhemtulla, *(jointly), University of Alberta, Edmonton, Canada;* Stanley Rabinowitz, *Maynard, Mass.*; Léo Sauvé, *Algonquin College, Ottawa*; Dan Sokolowsky, *Antioch College, Yellow Springs, Ohio.*

Comment. The submitted solutions contained a wealth of related material which could be assembled into a most interesting expository article on Fermat primes, the cyclotomic equation and constructibility by straightedge and compass. For example, it was pointed out by Sauvé and by Sokolowsky that the angles

$$\alpha_k = \frac{6\pi}{3k-1}, \quad k = 1, 2, 3, \ldots$$

are all trisectible but not all constructible. Klamkin, Rhemtulla and Carlitz derived their solutions from the equation $\cos 3x = 4\cos^3 x - 3\cos x$. Klamkin and Rhemtulla offered a generalization showing that there exist non-constructible angles which when specified geometrically and arithmetically are n-sectible (for arbitrary integers n) by straightedge and compass. Solutions by Rabinowitz and Izen bore a strong resemblance to the Proposer's solution.

413. [Spring 1978]. *Proposed by the late R. Robinson Rowe.*

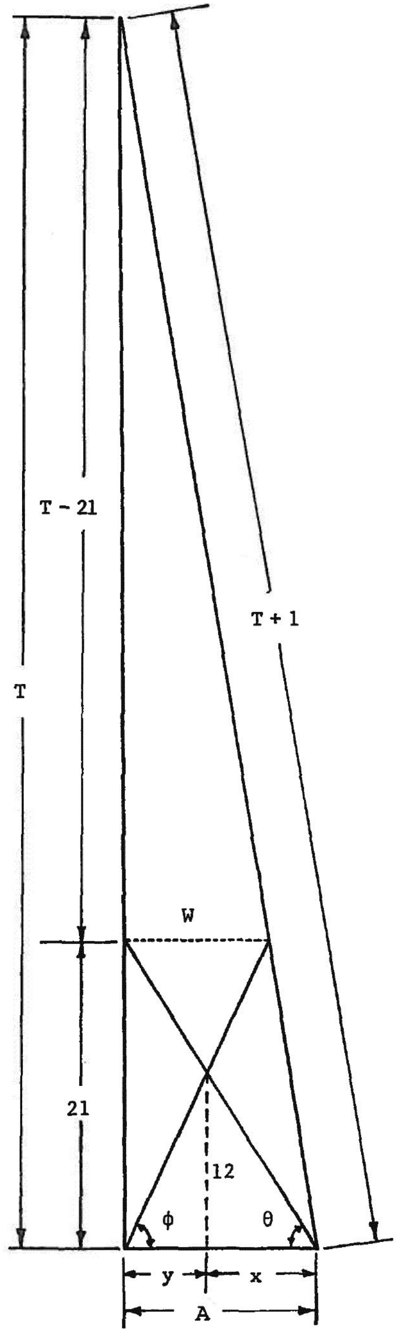
In a variation of the crossed-ladders-in-an-alley classic, the new tall building on one side of the alley was vertical, but on the other side the old low building, having settled, leaned toward the alley. Projected, its face would have met the top of the tall building and would have been one foot longer than the height of the tall building. The ladders, unequal in length, rested against the buildings 21 feet above the ground and crossed 12 feet above the ground. How high was the tall building and how wide was the alley?

Solution by David E. Penney, University of Georgia, Athens, Georgia.

Label the figure as shown: The height of the tall building is T; the width of the alley is A; the distance from the foot of the tall building to the point directly below the intersection of the ladders is y. Finally, $x = A - y$.

By similar triangles, $x = (4/7)A$, and thus $y = (3/7)A$. Also by similar triangles, $x = (4/7)W$. So $3A = 4W$, and thus $W = (3/4)W$. Since the dotted line marked W is therefore 1/4 of the way to the top of the tall building, T = 84 feet.

By the Pythagorean theorem, $A^2 = (T + 1)^2 - T^2 = 2T + 1 = 169$. So $a = 13$ feet.



Also solved, in a similar fashion by Rolan Christofferson, Charles R. Diminnie, *St. Bonaventure University, New York;* Mark A. Flood, *University of Toledo, Ohio;* Steve Leeland, *Phoenix, Arizona;* Charles H. Lincoln, *Goldsboro, N. C.;* Kenneth M. Wilke, *Topeka, Kansas;* and R. Robinson Rowe, *the Proposer,* who mentioned the simple but not well-known relation:

$$\frac{1}{H} + \frac{1}{f} = \frac{1}{d} + \frac{1}{e},$$

where H is the height of the tall building, d is the ordinate of the top of the ladder against the tall building, e the ordinate of the other ladder at its top, and f the ordinate of the intersection of the ladders.

Editor's Comment. Note the peculiar similarity between Rowe's formula for this problem and the classic crossed-ladder problem formula, $1/h + 1/k = 1/c$, where h and k are the heights of the tops of the ladders and c is the height of their intersection. An interesting treatment of the crossed-ladder problem may be found in William R. Ransom's One *Hundred Mathematical Curiosities*, still in print (fortunately) and available from J. Weston Walch, Portland, Maine.

414. [Spring 1978]. *Proposed by Steven S. Conrad, Benjamin N. Cardozo High School, Bayside, New York.*

In discussing the discriminant of a quadratic equation, a certain textbook says, "...if a, b and c are integers with $a \neq 0$ and if $b^2-4ac = 79$, the roots of $ax^2 + bx + c = 0$ will be real, irrational and unequal." Explain why this is incorrect.

Solution by Léo Sauvé, Algonquin College, Ottawa, Canada.

Apologies are due to the author of that textbook, for the statement quoted is perfectly correct. The statement is an implication in which the hypothesis is

H: a, b and c are integers with $a \neq 0$ and $b^2 - 4ac = 79$, and the conclusion is

C: the roots of $ax^2 + bx + c = 0$ will be real, irrational and unequal.

If H is true, then C is true by the theory of quadratics; while if H is false the implication is true by the laws of implication. So in either case the statement is true.

Whether H is true or false is entirely beside the point, but in fact it happens to be false. For if it is true, then b must be odd, say $2n + 1$, and then

$$b^2 - 79 = (2n + 1)^2 - 79 = 2(2n^2 + 2n - 39) \neq 4ac,$$

since $2n^2 + 2n - 39$ is odd and cannot equal $2ac$, so we have a contradiction.

Editor's Comment. Those who submitted solutions to this problem were divided into two camps. Sixteen solvers considered the problem incorrect because of the false hypothesis; seven solvers recognized the impossibility of the hypothesis, yet conceded the logical accuracy of the statement. It all boils down to the question: "What is the problem?" Is it: "Why is the statement incorrect?" or "Why is the hypothesis contradictory?" The difficulty, it seems, lies in the ambiguity in the enunciation of the proposal. Consider the analogous statement: If the moon is made of green cheese and if green cheese is delicious, then we may conclude that the moon is delicious. This is a true statement despite recent researches which suggest that the moon is not made of green cheese.

Solutions to this proposal were received from **Ronnie Adoubi, Steven R. Conrad** *(the Proposer)*, **Charles R. Diminnie, Michael W. Ecker, Victor G. Feser, Mark A. Flood, Robert A. Fuller, Andrew A. Galardi, Taghi Rezay Garacani, David Hammer, M. S. Klamkin, Peter A. Lindstrom, Charles A. Lincoln, Thomas E. Moore, James McKim** *[The University of Hartford Problem Group)*, **Sidney Penner, Bob Prielipp, George W. Rainey, Stanley Rabinowitz,** and **Kenneth M. Wilke.** *Also received were one unsigned solution and one with an illegible signature.*

415. [Spring 1978]. *Proposed by Charles W. Trigg, San Diego, California.*

A hexagonal number has the form $2n^2 - n$. In base 9, show that the hexagonal number corresponding to an n that ends in 7 terminates in 11.

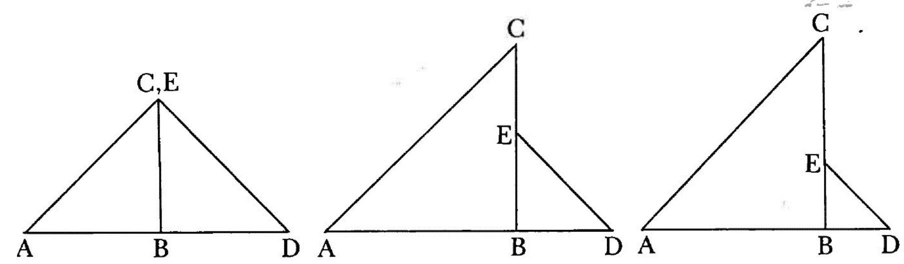
Practically all solutions submitted were almost identical, with slight variations in expression. The Editor's criteria for elegance, namely, accuracy, brevity and clarity, seem to have been best met by **Stanley Rabinowitz,** *Maynard, Mass.*, and by **Kenneth M. Wilke,** *Topeka, Kansas*, who say:

If $n = 9k + 7$, then $2n^2 - n = 2(9k + 7)^2 - (9k + 7) = 81(2k^2 + 3k + 1) + 10$. Thus $2n^2 - n \equiv 10 \pmod{11}$. But $10_{10} = 11_9$, so this number must end in 11.

Also solved by **Ronnie Aboudi, Rolan Christofferson, Victor G. Feser, Mark A. Flood, Taghi Rezay Garacani, Richard A. Gibbs, Samuel Gut, Howard Forman, John M. Howell, Charles H. Lincoln, Paul McGuire, Bob Prielipp,** and the *Proposer,* **Charles W. Trigg.**

416. [Spring 1978]. *Proposed by Scott Kim, Rolling Hills Estates, California.*

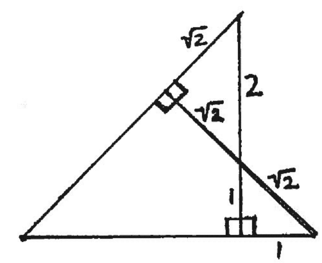
Each of the three figures shown below is composed of two isosceles right triangles, $\triangle ABC$ and $\triangle DBE$, where $\angle ABC$ and $\angle DBE$ are right angles, and B is between points A and D. Points C and E coincide in Figure (a), so that $CB/EB = 1$. In Figure (b), we are given that $CB/EB = 2$, and in Figure (c), we are given that $CB/EB = 3$. Consider each pair of triangles as a single shape and suppose that the areas of the three shapes are equal. (The figures are not drawn to scale.) Problem: For each pair of figures, find the minimum number of pieces into which the first figure must be cut so that the pieces may be reassembled to form the second figure. Pieces may not overlap, and all pieces must be used in each assembly.

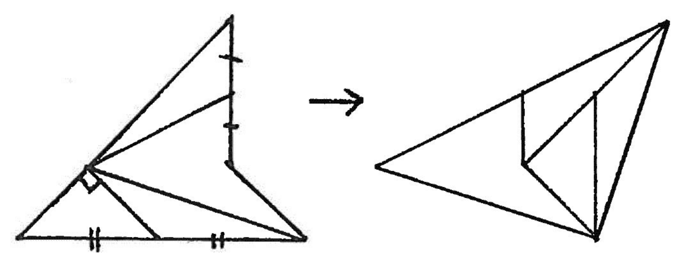


(a) (b) (c)

Solution by the Proposer.

Figures (b) and (c) are actually the same shape in two different orientations, so the dissection from Figure (b) to Figure (c) requires no cuts at all.





The minimal dissection from Figure (a) to Figure (b) or Figure (c) requires 4 pieces, as shown in the figure.

417. **[Spring 1978].** *Proposed by Clayton W. Dodge, University of Maine, Orono.*

1) Prove that the line joining the midpoints of the diagonals of a quadrilateral circumscribed about a circle passes through the center of the circle.

2) Let the incircle of triangle ABC touch side BC at X. Prove that the line joining the midpoints of AX and BC passes through the incenter I of the triangle.

Solution by the Proposer.

Part 1. Let the circle be the unit circle centered at the origin of the complex plane, $|z| = 1$. Let the points of tangency on sides AB, BC, CD, DA be P, Q, R, S. Now lines AB and BC have the parametric equations, with real parameters t and u,

$$z = p + ipt \qquad \text{and} \qquad z = q + iqu.$$

Since $p\bar{p} = 1$, we multiply the first equation by $\bar{p}$ to get

$$z\bar{p} = 1 + it, \qquad \text{and} \qquad \bar{z}p = 1 - it$$

by taking conjugates. Adding these two equations, we get

$$z\bar{q} + \bar{z}p = 2$$

as an equation for side AB. Similarly, for side BC, we have

$$z\bar{q} + \bar{z}q = 2.$$

Now solve these two equations simultaneously to get their point B of intersection:

$$b = 2/(\bar{p} + \bar{q}).$$

Since similar expressions hold for C, D, and A, we find the midpoints M and N of the diagonals are given by

$$m = \frac{1}{\bar{p} + \bar{q}} + \frac{1}{\bar{r} + \bar{s}} \qquad \text{and} \qquad n = \frac{1}{\bar{q} + \bar{r}} + \frac{1}{\bar{s} + \bar{p}}$$

To prove that M, N, and O are collinear, it suffices to show that $m\bar{n} - \bar{m}n = 0$. To that end, we have

$$m\bar{n} = \left(\frac{1}{\bar{p} + \bar{q}} + \frac{1}{\bar{r} + \bar{s}}\right)\left(\frac{1}{q + r} + \frac{1}{s + p}\right)$$

$$= \frac{\bar{p} + \bar{q} + \bar{r} + \bar{s}}{(\bar{p} + \bar{q})(\bar{r} + \bar{s})} \cdot \frac{p + q + r + s}{(q + r)(s + p)} \cdot \frac{1}{pqrs} \cdot \frac{1}{\bar{p}\bar{q}\bar{r}\bar{s}}, \text{ since } p\bar{p} = 1, \text{etc.}$$

$$= \frac{p + q + r + s}{(q + p)(s + r)} \cdot \frac{\bar{p} + \bar{q} + \bar{r} + \bar{s}}{(\bar{r} + \bar{q})(\bar{p} + \bar{s})}$$

$$= \left(\frac{1}{p + q} + \frac{1}{r + s}\right) + \left(\frac{1}{\bar{q} + \bar{r}} + \frac{1}{\bar{s} + \bar{p}}\right)$$

$$= \bar{m}n.$$

The proof of Part 1 is complete.

Part 2. Call ABCD a quadrilateral and apply Part 1.

Also solved by **Sister Stephanie Sloyan**, *Georgian Court College, Lakewood, N. J.*

Klamkin called attention to a related problem involving the quadrilateral <u>inscribed</u> in a circle. The version proposed by Dodge concerns the quadrilateral <u>circumscribed</u> about the circle and is offered here as an example of the application of complex numbers in the solution of a geometrical problem. The problem was first proposed and solved by Isaac Newton (*Book* I, *Lemma* XXV, COR. 3) and applies to a quadrilateral circumscribing a conic.

418. **[Spring 1978].** *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find all angles θ other than zero such that $\tan 11\theta = \tan 111\theta = \tan 1111\theta = \tan 11111\theta = \ldots.$

Solution by Léo Sauvé, Algonquin College, Ottawa, **Canada.**

Let α be one of the angles 11θ, 111θ, 1111θ, ..., say

$$\alpha = \frac{10^n - 1}{9}, \qquad n \geqslant 2;$$

then α has the desired property if and only if

$$\tan\alpha = \tan(10\alpha + \theta). \tag{1}$$

Suppose (1) holds, then

$$\sin(10\alpha + \theta)\cos\alpha - \cos(10\alpha + \theta)\sin\alpha = 0,$$

that is, $\sin(9\alpha + \theta) = 0$. Thus

$$9\alpha + \theta = k\pi, \quad k \in Z \qquad \text{and} \qquad 10^n\theta = k\pi,$$

from which

$$\theta = \frac{k\pi}{10^n}, \qquad k, n \in Z, \quad n \geqslant 2. \tag{2}$$

Conversely, if $\alpha = \left(\frac{10^n - 1}{9}\right)\theta$, where θ is given by (2), it is easy to verify that $10\alpha + \theta = k\pi + \alpha$, and so (1) holds. We conclude that all the answers are given by (2).

Also solved by Rolan Christofferson, Michael W. Ecker, Victor G. Feser, Mark A. Flood, Howard Forman, Taghi Rezay Garacani, Charles H. Lincoln, Bob Prielipp, Kenneth M. Wilke and the Proposer, who indicated some special interest in the case $\theta = 9^\circ$, which yields the result $\tan 99^\circ = \tan 999^\circ = \tan 9999^\circ = \tan 99999^\circ = \ldots$.

419 [Spring 1978]. *Proposed by Michael W. Ecker, City University of New York.*

Seventy-five balls are numbered 1 to 75 and are partitioned into sets of 15 elements each, as follows: $B = \{1, \ldots, 15\}$, $I = \{16,\ldots,30\}$, $N = \{31,\ldots,45\}$, $G = \{46,\ldots,60\}$, and $O = \{61,\ldots,75\}$, as in Bingo.

Balls are chosen at random, one at a time, until one of the following occurs: At least one from each of the sets B, I, G, O has been chosen, or four of the chosen numbers are from the set N, or five of the numbers are from one of the sets B, I, G, O.

Problem: Find the probability that, of these possible results, four N's are chosen first.

(Comment: The result will be approximated by the situation of a very crowded bingo hall and will give the likelihood of what bingo players call "an N game", that is, bingo won with the winning line being the middle column N.)

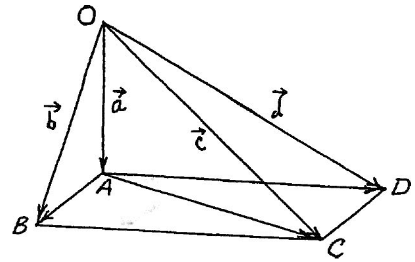
No solution has been received.

420. [Spring 1978]. *Proposed by Herbert Taylor, South Pasadena, California.*

Given four lines through a point in 3-space, no three of the lines in a plane, find four points, one on each line, forming the vertices of a parallelogram. (This is a variation of problem B-2 on the December 1977 William Lowell Putnam Mathematical competition.)

I. *Solution by David C. Kay, University of Oklahoma.*

A vector solution is obtained by letting the given lines pass through the origin O and allowing the lines to be represented by nonzero position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ (the points on those lines being given by $\lambda\vec{a}$, $\lambda\vec{b}$, $\lambda\vec{c}$, and $\lambda\vec{d}$ for real λ). If A, B, C, D are the desired vertices of the parallelogram, we see from the figure that the condition



demanded is $\vec{AC} = \vec{AB} + \vec{AD}$ or

$$\vec{a} - \vec{b} + \vec{c} - \vec{d} = \vec{0}. \qquad (1)$$

Hence we must find vectors representing the four lines which satisfy condition (1). But if $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are taken as a basis on lines OB, OC, OD and A is any point on the first line, then

$$\vec{a} = \beta\vec{e}_1 + \gamma\vec{e}_2 + \delta\vec{e}_3$$

for unique real β, γ, δ. To choose $\vec{b}$, $\vec{c}$, and $\vec{d}$ on the other three lines satisfying (1), merely set

$$\vec{b} = \beta\vec{e}_1, \qquad \vec{c} = -\gamma\vec{e}_2, \qquad \vec{d} = \delta\vec{e}_3.$$

If $\vec{b} = \lambda\vec{e}_1$, $\vec{c} = \mu\vec{e}_2$, and $\vec{d} = \nu\vec{e}_3$ were any other choice of vectors satisfying (1) then

$$\beta\vec{e}_1 + \gamma\vec{e}_2 + \delta\vec{e}_3 = \vec{a} = \lambda\vec{e}_1 - \mu\vec{e}_2 + \nu\vec{e}_3$$

and by linear independence of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, we have $\lambda = \beta$, $\mu = -\gamma$, $\nu = \delta$. Thus there is a unique parallelogram solution $ABCD$ corresponding to each point $A \neq O$ on the first line, and since the terminal points of $\lambda\vec{a}$, $\lambda\vec{b}$, $\lambda\vec{c}$, $\lambda\vec{d}$ for real $\lambda \neq 0$ lie in a plane parallel to that of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, the totality of solutions as λ varies with $\lambda\vec{a}$ occur in a uniquely determined family of parallel planes.

II. *Solution by Herb Taylor and Dennis Johnson, JPL, Pasadena.*

According to the Putnam Examination problem, if $ABCD$ is a convex quadrilateral and O is a point not in the same plane, then there exist points A' on OA, B' on OB, C' on OC, and D' on OD forming a parallelogram. First find the point X where AC meets BD. In the plane OAC we can find A' on OA, C' on OC, having X the midpoint of the segment $A'C'$. Likewise we can find B' on OB, D' on OD, having X the midpoint of the segment $B'D'$. Thus A', B', C', D' are found to be the vertices of a quadrilateral whose diagonals bisect each other, that is, a parallelogram.

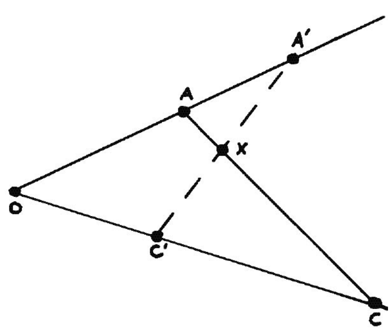
To return to the present problem, let a, b, c, d be the given four lines through a point O in 3-space. Let λ be the line formed by the intersection of the plane (ab) with the plane (cd). λ cannot be equal to a because a, c, d are not all in a plane. Similarly λ cannot be equal to b, nor c, nor d.

Now in the plane (ab) choose a ray a^+ of a and a ray b^+ of b so that a^+ and b^+ are on the same side of λ. Likewise in the plane (cd) put c^+ and d^+ on the same side of λ.

Now choose any point A on a^+, and any point D on d^+. The plane P which is parallel to λ and contains A and D will intersect b^+ in a point B and will intersect c^+ in a point C.

Since the line AB is parallel to the line CD, the four points A, B, C, D must be the vertices of a convex quadrilateral.

The solution to our problem follows by reduction to the Putnam problem. (All solutions are faces of parallelopipeds with O at the center.)



Also solved by M. S. Klamkin, *University of Alberta, Edmonton, Alberta, Canada.*

421. [Spring 1978]. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

If $F(x, y, z)$ is a symmetric increasing function of x, y, z, prove that for any triangle, in which w_a, w_b, w_c are the internal angle bisectors and m_a, m_b, m_c, the medians, we have

$$F(w_a, w_b, w_c) \le F(m_a, m_b, m_c)$$

with equality if and only if the triangle is equilateral.

Solution by the Proposer.

There are a number of special cases of this inequality in the literature. However, (1) follows simply from

$w_a \le m_a$, $w_b \le m_b$, $w_c \le m_c$, which apparently has been overlooked. No doubt it is buried somewhere in the literature.

$$\text{Proof: } 4w_a^2 = \frac{16\,bcs(s-a)}{(b+c)^2} \le 4s(s-a) = (b+c)^2 - a^2.$$

$$(b+c)^2 - a^2 \le 2b^2 + 2c^2 - a^2 = 4m_a^2, \text{ etc.}$$

The following are four known inequalities for the medians and the angle bisectors of a triangle [1], [2]:

(1) $w_a^2 + w_b^2 + w_c^2 \le m_a^2 + m_b^2 + m_c^2$,

(2) $w_a + w_b + w_c \le m_a + m_b + m_c$,

(3) $w_a w_b w_c \le m_a m_b m_c$,

(4) $w_a^6 + w_b^6 + w_c^6 \le m_a^6 + m_b^6 + m_c^6$,

with equality iff the triangle is equilateral.

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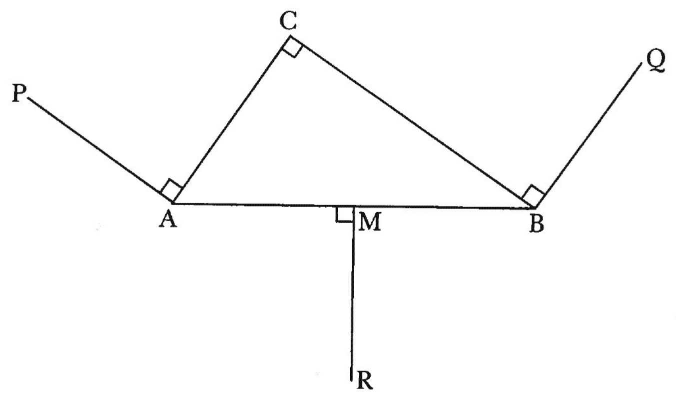
2. Leuenberger, F., Problem E 2060, Amer. Math. Monthly 76, 1969, 197, 198.

422. [Spring 1978]. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

If perpendiculars are erected outwardly at A, B, of a right triangle ABC ($C = 90^\circ$), and at M, the midpoint of AB, and extended to points P, Q, R such that $AP = BQ = MR = AB/2$, show that triangle PQR is perspective with triangle ABC.



$MR = BM = BQ = PA$

Solution by Stanley Rabinowitz, Digital Equipment Corporation, Maynard, Massachusetts.

Draw AQ, BP, CR meeting the sides of the triangle, BC, CA, AB respectively at Q', P', and R'. Let $BC=a$, $CA=b$, $AP=BQ=m$. since $\Delta ACQ' \sim \Delta QBQ'$, we have

$$\frac{CQ'}{Q'B} = \frac{b}{m}. \qquad (1)$$

Similarly,

$$\frac{CP'}{P'A} = \frac{a}{m}. \qquad (2)$$

Consider the circle with center M and radius m. It passes through points A, B, C, and R. Since MR is the perpendicular bisector of AB, R is the midpoint of minor arc $\widehat{AB}$. Thus CR bisects angle C. Consequently,

$$\frac{AR'}{R'B} = \frac{b}{a} \qquad (3)$$

Combining equations (1), (2), and (3), we get

$$\frac{AR'\cdot BQ'\cdot CP'}{R'B\cdot Q'C\cdot P'A} = \frac{bma}{abm} = 1$$

Thus, by Ceva's Theorem AQ, BP, CR concur.
Consequently, ΔPQR is perspective with MBC.

Also solved by Charles tl. Lincoln, Goldsboro, N. C. and the Proposer.

424. *[Spring]*. *Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Prove that

$$\left(x^{1/n} + y^{1/n}\right)^n > \left(\frac{x \quad y}{\ln x \ - \ln y}\right) \qquad (2n+2)$$

where n is an odd integer ≥ 3 and $0<y<x$

Consider the function

$$f(t) = \frac{n}{2n+2} \ln t - \frac{t^n - 1}{(t+1)}$$

$$f'(t) = \frac{n}{2n+2} \cdot \frac{(t-1)^{n+1} + g(t)}{t(t+1)^{n+1}}$$

where $g(t)$ is non-negative for $t \geq 0$.
Thus $f'(t) > 0$, $t > 1$.
Since $f(1) = 0$, $f(t) > 0$, $t > 1$.
Apply this fact to $t = \frac{x^{1/n}}{y^{1/n}}$. We get the required inequality.

Also solved by Robert A. Fuller, Armstrong State University, Savannah, Georgia.

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