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MORE ABOUT INTEGRAL TRIANGLES

by Jack Garfunkel
Forest Hills High School, Forest Hills, N. Y.

One of the many interesting properties of the arch-prime Pythagorean triangle (3, 4, 5) is that its sides are measured by consecutive integers. Furthermore it is the only right triangle having this property. On the other hand, an infinitude of acute triangles can be constructed whose sides are $a - 1$, a and $a + 1$, for any integer a exceeding 2.

Either leg of the 3-4-5 right triangle could be looked upon as an altitude, with the result that the area of the triangle is a whole number. A similar situation exists among the *Heronian* triangles, which are defined as those whose sides and whose areas are whole numbers. A familiar example is the triangle whose sides are 13, 14, 15 and whose altitude to the side of length 14 is 12. This is the only example of a triangle whose altitude h and sides a, b, c are consecutive integers.

The question now arises as to whether there are other Heronian triangles whose sides are consecutive integers. Such triangles would have at least one integral altitude. Furthermore, are there any triangles, not necessarily Heronian, with sides measuring consecutive integers and with an integral median or with an integral internal angle bisector? It is the purpose of this article to answer these questions.

Consider a triangle ABC whose sides are measured by three consecutive integers, a and a ti, while a certain other line segment drawn from vertex A to the opposite side a is also an integer. Suppose that this segment (or *Cevian*) is either the altitude h perpendicular to the side a, the angle-bisector t bisecting the angle A, or the median m bisecting the side a.

It is possible to express each of these Cevians in terms of the sides a, b, c of the triangle. We know that

$$h^2 = \frac{1}{4a^2}(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$$

$$t^2 = bc - \frac{a^2bc}{(b+c)^2}$$

$$m^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)\ .$$

Writing $a + 1$ for b and $a - 1$ for c, we find that

$$h^2 = \frac{3}{4}a^2 - 3$$

$$t^2 = \frac{3}{4}(a^2 - 1)$$

$$m^2 = \frac{3}{4}a^2 + 1\ .$$

The problem is thus reduced to solving in integers each of the equations

$$\left(\frac{a}{2}\right)^2 - 3\left(\frac{h}{3}\right)^2 = 1$$

$$a^2 - 3\left(\frac{2t}{3}\right)^2 = 1$$

$$m^2 - 3\left(\frac{a}{2}\right)^2 = 1$$

All three are variants of the Pell equation $x^2 - 3y^2 = 1$, the solution of which is given by $x = p$, $y = q$, where p and q are of different parity, with p greater than q and where p/q is one of the convergents of the simple continued fraction expansion of $\sqrt{3}$. Thus

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}\ \ddots}}}\ ,$$

the successive convergents of which are

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}, \cdots\ .$$

Selecting convergents having numerator and denominator of different parity, we have as solutions of $x^2 - 3y^2 = 1$

$$\begin{array}{ll} x_0 = 1 & y_0 = 0 \\ x_1 = 2 & y_1 = 1 \\ x_2 = 7 & y_2 = 4 \end{array}$$

$$\begin{array}{ll} x_3 = 26 & y_3 = 15 \\ x_4 = 97 & y_4 = 56 \\ x_5 = 362 & y_5 = 209 \\ x_6 = 1351 & y_6 = 780 \\ \vdots & \vdots \end{array}$$

Further solutions can be computed by means of the recurrence formulas

$$x_{n+1} = 2x_n + 3y_n\ , \qquad y_{n+1} = x_n + 2y_n\ .$$

By appropriate substitutions for h, t and m, we obtain triangles of the first kind with

$a = 4, 14, 52, 194, 724, 2702, \cdots,$

$h = 3, 12, 45, 168, 627, 2340, \cdots,$

triangles of the second kind (since $y = 2t/3$ must be even) with

$a = 7, 97, 1351, \cdots,$

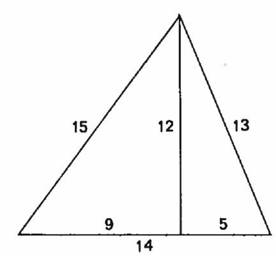
$t = 6, 84, 1170, \cdots,$

and those of the third kind with

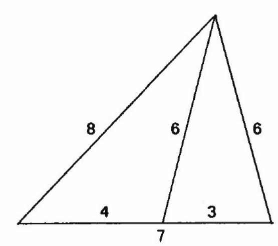
$a = 8, 30, 112, 418, 1560, \cdots,$

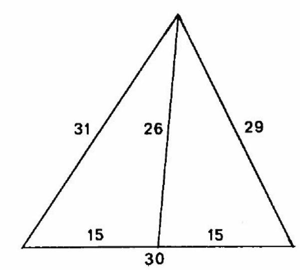
$m = 7, 26, 97, 362, 1351, \cdots.$

This is a nice example of solving three problems for the price of one. As a bonus, each triangle of the first kind has an integral area $ah/2 = 3xy$ and integral inradius y.



TRIANGLES WHOSE SIDES ARE CONSECUTIVE INTEGERS WITH INTEGER ALTITUDE, ANGLE BISECTOR AND MEDIAN





Up to this point, the only Cevians considered were those drawn to the side of middle length. The question arises: Are there any consecutive integral triangles with integral altitudes, medians or angle bisectors drawn to the shortest or to the longest side? This is a problem that is certainly worthy of further investigation.

REFEREES FOR THIS ISSUE

The *Journal* recognizes with appreciation the following persons who willingly devoted their time to evaluate papers submitted for publication prior to this issue: Robert Hemminger, Vanderbilt University; and members of the mathematics department at the University of Oklahoma, Arthur Bernhart, Bradford Crain, Stanley Eliason, John Green, and Bernard McDonald.

We also appreciate the competent work of the typist, Detia Roe.

UNUSUAL MAGIC SQUARES[1]

A Magic Square of Cards

Each row, column, and the two main diagonals contain each of the four values (Ace, King, Queen, Jack) in all four suits. Numerous other subsquares and symmetrical subsets of squares have the same property, such as the middle 2 x 2 square, and the four corner squares.

♠K	♣A	♦J	♥Q
♥J	♦Q	♣K	♠A
♣Q	♠J	♥A	♦K
♦A	♥K	♠Q	♣J

[1] From the *Recreational Mathematics Magazine*, No. 5 (1961), pp. 24-29.

Twin Magic Squares of Twin Primes

Corresponding entries in the two magic squares (with magic constants 1496 and 1504) are twin primes.

29	1061	179	227
269	137	1019	71
1049	101	239	107
149	197	59	1091

31	1063	181	229
271	139	1021	73
1051	103	241	109
151	199	61	1093

A Pythagorean Magic Square.

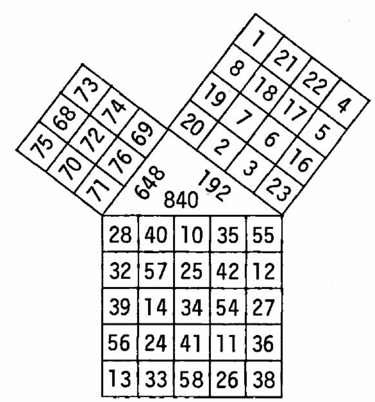
This arrangement is composed of three magic squares, a 3 x 3, 4 x 4, and a 5 x 5, with magic constants 216, 48, and 168, respectively. The total summations of the numbers in the squares shown on the three sides of the triangle have the property

$$648 + 192 = 840 .$$

The sums of the individual digits of the numbers in each of the squares are 99, 84, and 183, respectively, and

$$99 + 84 = 183 .$$

(One further relation is 216 - 48 = 168.)



A NOTE ON THE SERIES FOR LOG 2

by N. Schaumberger
Bronx Community College, CUNY

R. Courant calls the series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \qquad (1)$$

a "remarkable formula ... whose discovery made a deep impression on the minds of the first pioneers of the differential and integral calculus" [1]. In a standard course the introduction of this series is postponed until Taylor's Theorem is reached. It is possible, however, to present it much earlier; namely, it may be presented right after integration and the introduction to the log function.

As our starting point we take the relation

$$\log(1 + t) = \int_0^t \frac{dx}{1 + x}$$

In particular,

$$\log 2 = \int_0^1 \frac{dx}{1 + x} . \qquad (2)$$

Using the definition of the definite integral to evaluate the right hand side of (2) we divide the interval [0,1] into n equal subintervals of length $1/n$. The lower sum for $\int_0^1 \frac{dx}{1 + x}$ is

$$\frac{1}{n}\left(\frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \cdots + \frac{1}{1 + \frac{n}{n}}\right)$$

$$= \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n}$$

and consequently,

$$\int_0^1 \frac{dx}{1 + x} = \lim_{n\to\infty}\left(\frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n}\right) . \qquad (3)$$

Now letting

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{(2n - 1)} - \frac{1}{(2n)} , \qquad (4)$$

we get

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n - 1} + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n - 1} + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$= \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n} .$$

Hence

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty}\left(\frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n}\right)$$

and by (3) we see that

$$\lim_{n\to\infty} S_{2n} = \int_0^1 \frac{dx}{1 + x} .$$

Whence, by use of (2) $\lim_{n\to\infty} S_{2n} = \log 2$. Furthermore $S_{2n+1} = S_{2n} + \frac{1}{2n + 1}$, and so $\lim S_{2n+1} = \lim_{n\to\infty} S_{2n}$. Thus, for n even or odd, we have $\lim_{n\to\infty} S_n = \log 2$. Finally, using (4), we obtain (1).

More generally, this approach can be used to show that for any positive integer r, log r can be expressed as the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r - 1} - \frac{r - 1}{r} + \frac{1}{r + 1} + \frac{1}{r + 2} + \cdots + \frac{1}{kr - 1}$$

$$- \frac{r - 1}{kr} + \frac{1}{kr + 1} + \cdots .$$

This series is obtained by replacing the term $1/rk$ $(k = 1,2,3,\cdots)$ in the harmonic series by the term $-(r - 1)/rk$ $(k = 1,2,3,\cdots)$. In particular,

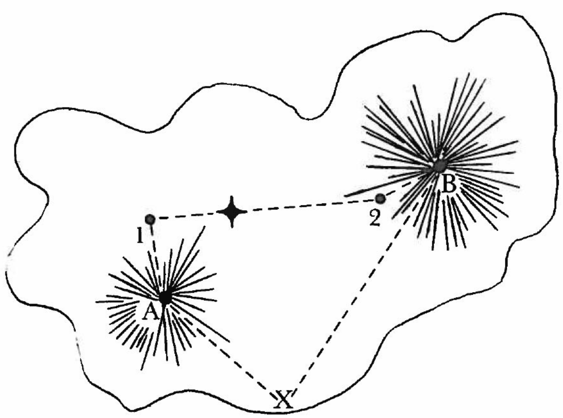
$$\log 3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots .$$

REFERENCE

1. R. Courant, *Differential end Integral Calculus*, Vol. 1, Interscience Publishers, Inc., Sew York, 1961, 317.

UNDERGRADUATE RESEARCH PROJECT

A map leading to a treasure chest of gold coins buried on an island which has only two trees and the remains of a sunken ship to serve as landmarks, bore these instructions: Proceed from the ship (X) to the smaller of the two trees (A), turn clockwise 30 degrees, pace off a distance equal to one-half XA, and drive a stake (1). Return to X, proceed to tree B, turn counterclockwise 150°, pace off a distance equal to one-fourth XB, and drive a second stake (2). The treasure is located on the line joining the two stakes at two-thirds the distance from stake 1 to stake 2. If a storm carried the ship to sea, thus destroying the reference point X in the map, the map would seemingly be worthless. Show, however, that the instructions in the map are still valid. Generalize.



AN ALL-RULE AXIOMATIZATION OF THE PROPOSITIONAL CALCULUS AND THE EQUIVALENCE OF SOME WELL-KNOWN AXIOMATIZATIONS

by David Hoak
Occidental College

The propositional calculus is the branch of logic which deals with sentences or propositions built up from atomic sentences by connections such as $\neg$ (not) and $\rightarrow$ (implies). Thus, if A, B are atomic, we can form $\neg B \rightarrow A$, $A \rightarrow B$, $\neg(A \rightarrow B)$, and so forth.

The propositional calculus becomesinteresting if we consider as primitive propositions ones in whose truth we believe (based on some agreeable interpretation of the connectives), and use rules for building new propositions which preserve truth. For we can then generate new and sometimes surprising true propositions. The propositions so generated are valid propositions or tautologies. A tautology is a proposition true as a whole regardless of the truth or falsity of its components. Such propositions as $A \rightarrow A$ and $A \vee \neg A$ are tautologies under the standard interpretation of "implies" and "or" in two-valued propositional logic (the statement A is true or false).

An axiomatization of the propositional calculus is a set of fundamental propositions we may call axioms, and a set of rules by which we can, starting with the axioms alone, derive the tautologies of the calculus. A well-known result of the theory is that any axiomatization of the calculus by propositions must have at least one rule of inference. Apparently, this theorem was known to the nineteenth-century author and mathematician, Lewis Carroll, which places a large lower bound on its age. The result leads to the interesting question of axiomatizing the calculus by rules alone. The German, G. Gentzen, answered this question in 1934-35 with a list of fourteen rules from which he could derive the tautologies of the calculus (Gentzen [1]). The objective here is, from a more concise list of nine rules, to derive a standard axiomatization found in Kleene [3], which is also due essentially to Gentzen. Following this, will be a discussion of the equivalence of the axiomatizations found by Kleene, Rosser [6], Mendelson [4], and Hilbert, Ackerman [2].

The Kleene axiomatization is as follows, with "→," "¬," "∨," and "&" taken as primitives. Kleene also uses "~" (equivalence, iff) as a primitive, but we need not complicate matters with it here since $A \sim B$ *can be simply defined as* $(A \to B)\ \&\ (B \to A)$.

Axiom 1a: $\vdash A \to (B \to A)$

Axiom 1b: $\vdash (A \to B) \to ((A \to (B \to C)) \to (A \to C))$

Axiom 3: $\vdash A \to (B \to (A\ \&\ B)$

Axiom 4a: $\vdash (A\ \&\ B) \to A$

Axiom 4b: $\vdash (A\ \&\ B) \to B$

Axiom 5a: $\vdash A \to (A \vee B)$

Axiom 5b: $\vdash B \to (A \vee B)$

Axiom 6: $\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

Axiom 7: $\vdash (A \to B) \to ((A \to \neg B) \to \neg A)$

Axiom 8: $\vdash \neg\neg A \to A$

Rule. $A, A \to B \vdash B$ (modus ponens)

The lettering is Kleene's. The statement $\Gamma \vdash B$ *means the statement* B *can be deduced from the set of statements* Γ, *where* Γ *is a finite, possibly empty set of propositions. Thus, the axioms are statements that can be deduced, so to speak, from the empty set of propositons.*

We propose the following rule system:

R1: $A \vdash A$

R2: If $\Gamma \vdash C$, then $\Gamma, A \vdash C$

R3: If $\Gamma \vdash A$ and $\Gamma, A \vdash C$, then $\Gamma \vdash C$

R4: $\Gamma \vdash A \to B$ iff $\Gamma, A \vdash B$

R5: $\Gamma \vdash A\ \&\ B$ iff $\Gamma \vdash A$ and $\Gamma \vdash B$

R6: If $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$, then $\Gamma, A \vee B \vdash C$

R7: $\Gamma \vdash \neg A$ iff, for any proposition B, $(\Gamma, A) \vdash B$

R8: $\neg\neg A \vdash A$

R9: If $\Gamma, A, A \vdash A$, then $\Gamma, A, A \vdash A$ (R9 will not be cited in proofs.)

Before proceeding with the deduction, let us justify the rule modus ponens (MP).

$A, A \to B \vdash B.$

Proof. By R4, letting Γ be the set $\{A \to B\}$, we know that if $A \to B \vdash A \to B$, then $A, A \to B \vdash B$. But we always have $A \to B \vdash A \to B$ by R1. So modus ponens is justified. □

It is important to note here,-since the proceeding argument may seem circular, that it is an argument in the metalanguage, that is, the language we use to talk about the logical system and its symbols, based on what we all mean by "if" ... "then." We now derive Kleene's axioms:

<u>Axiom 1a.</u> $\vdash A \to (B \to A)$.

Proof.

1.	$A \vdash A$	R1
2.	$A, B \vdash A$	R2
3.	$A \vdash (B \to A)$	R4
4.	$\vdash A \to (B \to A)$. □	R4

<u>Axiom 1b.</u> $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$.

Proof.

1.	$A, A \to B \vdash B$	MP
2.	$A, A \to B, A \to (B \to C) \vdash B$	R2
3.	$A, A \to (B \to C) \vdash B \to C$	MP
4.	$A, A \to B, A \to (B \to C), B \vdash B \to C$	R2
5.	$B, B \to C \vdash C$	MP
6.	$A, A \to B, A \to (B \to C), B, B \to C \vdash C$	R2
7.	$A, A \to B, A \to (B \to C), B \vdash C$	R3(4,6)
8.	$A, A \to B, A \to (B \to C) \vdash C$	R3(2,7)
9.	$A \to B, A \to (B \to C) \vdash (A \to C)$	R4
10.	$A \to B \vdash (A \to (B \to C)) \to (A \to C)$	R4
11.	$\vdash (A \to B) \to (A \to (B \to C)) \to (A \to C)$. □	R4

<u>Axiom 3.</u> $\vdash A \to (B \to (A\ \&\ B))$.

Proof.

1.	$A \vdash A$	R1
2.	$A, B \vdash A$	R2
3.	$B \vdash B$	R1
4.	$A, B \vdash B$	R2
5.	$A, B \vdash A\ \&\ B$	R5
6.	$A \vdash B \to (A\ \&\ B)$	R4
7.	$\vdash A \to (B \to (A\ \&\ B))$. □	R4

Axiom 4a, 4b. $\vdash A \,\&\, B \to A,\ \vdash A \,\&\, B \to B.$

Proof:
1. $A \,\&\, B \vdash A \,\&\, B$ — $R1$
2. $A \,\&\, B \vdash A$ and $A \,\&\, B \vdash B$ — R5
3. $\vdash A \,\&\, B \to A$ and $\vdash A \,\&\, B \to B.$ □ — R4

Axiom 5a, 5b. $\vdash A \to (A \vee B),\ \vdash B \to (A \vee B)$

Proof: Let $\Gamma = 0$.
1. $A \vee B \vdash A \vee B$ — $R1$
2. $A \vdash A \vee B$ and $B \vdash A \vee B$ — $R6$
3. $\vdash A \to (A \vee B)$ and $\vdash B \to (A \vee B).$ □ — $R4$

Axiom 6. $\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C)).$

Proof:
1. $A, A \to C \vdash C$ — MP
2. $A, A \to C, B \to C \vdash C$ — $R2$
3. $B, B \to C \vdash C$ — MP
4. $B, B \to C, A \to C \vdash C$ — $R2$
5. $A \to C, B \to C, A \vee B \vdash C$ — $R6(2,4)$
6. $A \to C, B \to C \vdash (A \vee B) \to C$ — $R4$
7. $A \to C \vdash (B \to C) \to ((A \vee B) \to C)$ — $R4$
8. $\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C)).$ □ — $R4$

Axiom 7. $\vdash (A \to B) \to ((A \to \neg B) \to \neg A).$

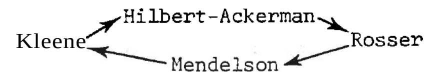
Proof:
1. $A, A \to \neg B \vdash \neg B$ — UP
2. $A, A \to \neg B, A \to B \vdash \neg B$ — $R2$
3. $A, A \to \neg B, A \to B, B \vdash C$, for any proposition C — $R7$ (from 2)
4. $A, A \to B \vdash B$ — MP
5. $A, A \to B, A \to \neg B \vdash B$ — R2
6. $A, A \to B, A \to \neg B \vdash C$, for any proposition C — $R3(5,3)$
7. $A \to B, A \to \neg B \vdash \neg A$ — $R7$
8. $A \to B \vdash (A \to \neg B) \to \neg A$ — R4
9. $\vdash (A \to B) \to ((A \to \neg B) \to \neg A).$ □ — R4

Axiom 8. $\vdash \neg\neg A \to A.$

Proof:
1. $\neg\neg A \vdash A$ — R8
2. $\vdash \neg\neg A \to A.$ □ — $R4$

Thus, we have shown that our system of rules is at least as strict an axiomatization as Kleene's. We could show the systems equivalent by deriving our rules from Kleene's axiomatization. We will not go into this in depth except to say that the proofs are not difficult and may be found in Kleene [3]. It should be pointed out that the system of rules proved in Kleene is Gentzen's; our list is a brief condensation of those rules in iff form with $R1$-$R3$ added.

To show the systems of Mendelson, Rosser, Kleene and Hilbert, Ackerman equivalent is a complicated by not impossible task employing the following plan of attack:



What complicates the proofs primarily is that each system is based on different primitive connectives. Consequently, to go from one to another we require definitions like $A \vee B \sim \neg(\neg A \,\&\, \neg B)$ and $A \,\&\, B \sim \neg(\neg A \vee \neg B)$, the familiar DeMorgan relations. The equivalences $\neg A \vee B \sim (A \to B)$ and $\sim(A \,\&\, \neg B) \sim (A \to B)$ must also be used. The axiomatizations are as follows:

L_1: (Hilbert-Ackerman [1950])

$\vee$ and $\neg$ are the primitives.

Def.:
1. $\vdash (A \to B) \sim (\neg A \vee B)$
2. $\vdash (A \,\&\, B) \sim \neg(\neg A \vee \neg B)$

Axioms:
1. $\vdash A \vee A \to A$
2. $\vdash A \to A \vee B$
3. $\vdash (B \to C) \to ((A \vee B) \to (A \vee C))$
4. $\vdash (A \vee B) \to (B \vee A)$

Rule: modus ponens

L_2: (Rosser [1953])

$\&$ and $\neg$ are the primitives.

Def.
1. $\vdash (A \to B) \sim \neg(A \,\&\, \neg B)$
2. $\vdash (A \vee B) \sim \neg(\neg A \,\&\, \neg B)$

Axioms:
1. $\vdash A \to A \,\&\, A$
2. $\vdash A \,\&\, B \to A$
3. $\vdash (A \to B) \to (\neg(B \,\&\, C) \to \neg(C \,\&\, A))$

Rule: modus ponens

L_3: (Mendelson [1964])

$\neg$ and $\rightarrow$ are the primitives.

Def.: 1. $\vdash (A \vee B) \sim (\neg A \rightarrow B)$

2. $\vdash (A \,\&\, B) \sim \neg(A \rightarrow \neg B)$

Axioms: 1. $\vdash A \rightarrow (B \rightarrow A)$

2. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

3. $\vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$

Rule: modus ponens

(We shall refer to Kleene's system, which we have already seen, as L_4.)

Because each system contains few elementary propositions and employs different primitives, we cannot prove the sentences of any one from one of the others directly. Rather, the strategy is to discover and prove as many theorems in each system as possible, and then attempt the needed proofs using these stronger tools. Since the problems involved in each $L_i \rightarrow L_j$ proof are of similar difficulty, we will consider only one in detail here, $L_2 \rightarrow L_3$. We include, with two proofs, the following results of L_2 which appear in Mendelson [4] lettered as below:

(*a*) $A \rightarrow B,\ B \rightarrow C \vdash \neg(\neg C \,\&\, A)$

(*b*) $\vdash \neg(\neg A \,\&\, A)$

(*c*) $\vdash \neg\neg A \rightarrow A$

(*d*) $\vdash \neg(A \,\&\, B) \rightarrow (B \rightarrow \neg A)$

(*e*) $\vdash A \rightarrow \neg\neg A$

(*f*) $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

(*g*) $\neg A \rightarrow \neg B \vdash B \rightarrow A$

(*h*) $A \rightarrow B \vdash (C \,\&\, A) \rightarrow (B \,\&\, C)$

(*i*) $A \rightarrow B,\ B \rightarrow C,\ C \rightarrow D \vdash A \rightarrow D$

(*j*) $\vdash A \rightarrow A$

(*k*) $\vdash A \,\&\, B \rightarrow B \,\&\, A$

(*l*) $A \rightarrow B,\ B \rightarrow C \vdash A \rightarrow C$

(*m*) $A \rightarrow B,\ C \rightarrow D \vdash (A \,\&\, C) \rightarrow (B \,\&\, D)$

(*n*) $B \rightarrow C \vdash (A \,\&\, B) \rightarrow (A \,\&\, C)$

(*o*) $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \,\&\, B) \rightarrow C)$

(*p*) $\vdash ((A \,\&\, B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$

(*q*) $A \rightarrow B,\ A \rightarrow (B \rightarrow C) \vdash A \rightarrow C$

(*r*) $\vdash A \rightarrow (B \rightarrow (A \,\&\, B))$

(*s*) $\vdash A \rightarrow (B \rightarrow A)$

(*t*) If $\Gamma, A \vdash B$, then $\Gamma \vdash A \rightarrow B$ (Deduction Theorem)

(*u*) $\vdash (\neg A \rightarrow A) \rightarrow A$

(*v*) $A \rightarrow B,\ \neg A \rightarrow B \vdash B$

We include the proofs of (*c*) and (*m*) since they involve manipulations typical in a system based on "&" and "$\neg$;" "*c*" is an especially important result since it assures us that the axioms we have laid down imply a two-valued logical system, and "*m*" is an extremely useful rule for involving "&" from two implications.

Proposition (*c*). $\vdash \neg\neg A \rightarrow A$.

Proof: 1.	$(\neg A \rightarrow (\neg A \,\&\, \neg A))$ $\rightarrow (\neg((\neg A \,\&\, \neg A) \,\&\, \neg\neg A) \rightarrow \neg(\neg\neg A \,\&\, \neg A))$	Ax. 3
2.	$\neg A \rightarrow (\neg A \,\&\, \neg A)$	Ax. 1
3.	$(\neg A \,\&\, \neg A) \rightarrow \neg A$	Ax. 2
4.	$((\neg A \,\&\, \neg A) \rightarrow \neg A) \rightarrow \neg((\neg A \,\&\, \neg A) \,\&\, \neg\neg A)$	Def. 1
5.	$\neg((\neg A \,\&\, \neg A) \,\&\, \neg\neg A)$	MP(3,4)
6.	$\neg(\neg\neg A \,\&\, \neg A)$	MP(2,1)MP(5,MP(2,1))
7.	$\neg\neg A \rightarrow A$. □	Def. 1, from 6

Proposition (*m*). $A \rightarrow B,\ C \rightarrow B \vdash A \,\&\, C \rightarrow B \,\&\, D$.

Proof: 1.	$A \rightarrow B$	premise
2.	$\neg(B \,\&\, C) \rightarrow \neg(C \,\&\, A)$	Ax. 3 and MP
3.	$C \,\&\, A \rightarrow B \,\&\, C$	(*g*)
4.	$A \,\&\, C \rightarrow C \,\&\, A$	(*k*)
5.	$A \,\&\, C \rightarrow B \,\&\, C$	(*l*)
6.	$C \rightarrow D$	premise
7.	$\neg(D \,\&\, B) \rightarrow \neg(B \,\&\, C)$	Ax. 3 and MP
8.	$B \,\&\, C \rightarrow D \,\&\, B$	(*g*)
9.	$D \,\&\, B \rightarrow B \,\&\, D$	(*k*)
10.	$B \,\&\, C \rightarrow B \,\&\, D$	(*l*)
11.	$A \,\&\, C \rightarrow B \,\&\, D$. □	(*l*)(5,10)

Incidentally, the proof of the important result (*t*) is the standard proof by induction on the length of the proposition using (*j*), (*q*) and (*s*).

We now examine the proof $L_2 \rightarrow L_3$. We shall derive the axioms and definitions of Mendelson's system using the derived theorems, axioms and definitions of Rosser's system.

Axiom 1. $\vdash A \to (B \to A)$.

Proof. Immediate from (s).

Axiom 2. $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

Proof:		
1.	$A \to B,\ A \to (B \to C) \vdash A \to C$	(q)
2.	$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$. □	(t) twice

Axiom 3. $\vdash ((\neg B \to \neg A) \to ((\neg B \to A) \to B))$.

Proof:		
1.	$\neg B \to \neg A$	*premise*
2.	$A \to B$	(g)
3.	$(\neg B \to A)$	*premise*
4.	$(\neg B \to A) \to (\neg A \to \neg\neg B)$	(f)
5.	$(\neg A \to \neg\neg B)$	$MP(3,4)$
6.	$\neg\neg B \to B$	(c)
7.	$\neg A \to B$	$(l)(5,6)$
8.	B	$(v)(2,7)$
9.	$(\neg B \to \neg A),\ (\neg B \to A) \vdash B$	from 1-8
10.	$\vdash (\neg B \to \neg A) \to ((\neg B \to A) \to B)$. □	(t) twice

Definition 1. $\vdash A \vee B \sim (\neg A \to B)$.

Proof:		
1.	$\neg(\neg A \;\&\; \neg B) \to \neg A \to B$	*Def.* 1
2.	$A \vee B \to \neg(\neg A \;\&\; \neg B)$	*Def.* 2
3.	$\vdash A \vee B \to (\neg A \to B)$.	(l)

The converse follows by the reverse implications of the two definitions. □

Definition 2. $\vdash A \;\&\; B \sim \neg(A \to \neg B)$.

Proof:		
1.	$\vdash (A \to \neg B) \to \neg(A \;\&\; \neg\neg B)$	*Def.* 1
2.	$\neg\neg(A \;\&\; \neg\neg B) \to \neg(A \to \neg B)$	(f) and *MP*
3.	$(A \;\&\; \neg\neg B) \to \neg\neg(A \;\&\; \neg\neg B)$	(e)
4.	$(A \;\&\; \neg\neg B) \to \neg(A \;\&\; \neg B)$	$(l)(3,2)$
5.	$A \to A$	(j)
6.	$B \to \neg\neg B$	(e)
7.	$(A \;\&\; B) \to (A \;\&\; \neg\neg B)$	$(m)(5,6)$
8.	$(A \;\&\; B) \to \neg(A \to \neg B)$.	$(l)(7,4)$

The converse is similar, reversing the implication in line 1. □

This completes our examination of the provability of L_3 from L_2. With similar effort we could complete the other proofs in our scheme above. It follows that from the rule system we examined at the start, we can develop any of the several axiomatizations.

As an interesting aside let us briefly consider the novel axiomatization due to J. Nicod [5]. He showed that it is possible to axiomatize the propositional calculus using one primitive connective, one axiom and one rule of inference. The connective, called the Sheffer stroke, and written "$|$" is defined as follows: $A \mid B$ is true unless both A and B are true; $|$ is thus alternate denial, with $A \mid B$ equivalent to $\neg A \vee \neg B$. From this stroke alone, it is possible to define the other connectives: $\neg A$ for instance, being equivalent to $A \mid A$. Nicod's axiom is the following:

$$(A \mid (B \mid C)) \mid ((D \mid (D \mid D)) \mid ((E \mid B) \mid ((A \mid E) \mid (A \mid E)))) \quad (1)$$

and his rule of inference:

If $A \mid (P \mid Q)$ and A, then Q.

Using this rule and (1), Nicod derived the axioms of Russell and Whitehead found in *Principia Mathematica* [7]. Later, Ackerman reduced the *Principia's* five axioms to four by showing one of them redundant. These four form the Hilbert, Ackerman system. Now taking (1) above and rewriting it as a sentence involving more familiar connectives we have:

$$(A \;\&\; (\neg B \vee \neg C)) \vee ((\neg D \vee D) \;\&\; ((E \;\&\; B) \vee (\neg A \vee \neg E)))$$

which is easily shown to be true regardless of the truth or falsity of A, B, C, D, or E and thus a tautology. By the tautology theorem, a standard proof of which may be found in Mendelson [4], we can establish (1) as a theorem. Now knowing that from (1) we can establish the Hilbert, Ackerman system and that from the same system we can prove (1) yields the equivalence of Nicod's system and the other axiomatizations. Therefore, we know that by some finite number of manipulations we could derive Nicod's system from our eight rules.

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REFERENCES

1. Gentzen, Gerhardt, *Untersuchungen über das logische Schliessen*, Mathematische Zeitschrift, 39 (1934-35), 176-210, 405-431.
2. Hilbert, D., and Ackerman, W., *Principles of Mathematical Logic*, Chelsea Publishing Company, New York, 1950.
3. Kleene, S. C., *Mathematical Logic*, John Wiley & Sons, Inc., New York, 1967.
4. Mendelson, Elliott, *Introduction to Mathematical Logic*, D. Van Nostrand Company, Inc., New York, 1964.
5. Nicod, J. G., *A Reduction in the Number of Primitive Propositions of Logic*, Proc. Camb. Phil. Soc., 19 (1917), 32-41.
6. Rosser, J. B., *Logic for Mathematicians*, New York, 1953.
7. Whitehead, A. N., and Russell, B., *Principia Mathematica*, Vol. I, University Press, Cambridge, 1925.

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A RELATIONSHIP BETWEEN APPROXIMATION THEORY AND STATISTICAL MEASUREMENTS

by *Herbert L. Dershem*
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1. *Introduction.*

There are many statistics which are used to measure central tendency and dispersion of a random variable. Each of several measures of central tendency is shown to be a best approximation to the variable over the set of all constant functions in a specified norm. The error of this best approximation serves as a corresponding measure of dispersion.

Hamming ([1], pp. 224-226) has noted this relationship between best approximation and measurements of central tendency. His work is extended here to include additional norms, measurements of dispersion, continuous distributions, and measurements which result from transformations of best approximations.

2. *Discrete Random Variables.*

Given a discrete random variable x which can take on any of a set of data values $\{x_i\}_{i=1}^{M}$, each value x_i occurring with respective probability f_i, and a given function norm $\|\ \|$, we define a ***measurement of central tendency*** to be the constant value c^* which, when substituted for c, minimizes

$$\|x - c\|.$$

This is the best approximation to x over the set of constant functions where x is the function defined on the discrete domain $\{1,2,\cdots,M\}$ by $x(i) = x_i$ and c is the constant function defined on the same domain by $c(i) = c$. We examine a number of choices of norm and show that these choices all correspond to commonly used measurements of central tendency. The error in the best approximation, $\|x - c^*\|$, is defined as the ***corresponding measurement of dispersion.***

The discrete L_2 norm with weight function f is defined by

$$\|u\|_{2,f} = \left\{\sum_{i=1}^{M} f_i u_i^2\right\}^{1/2}.$$

The measurement of central tendency associated with this norm is found by minimizing the function defined by

$$\phi(c) = \|x - c\|_{2,f} = \left\{\sum_{i=1}^{M} f_i(x_i - c)^2\right\}^{1/2}$$

The minimizing value of c, which we call $E(x)$ (the expected value of x), is found by setting the first derivative of 4 with respect to c equal to zero and solving the resulting equation for c^* to obtain

$$c^* = E(x) = \frac{\sum_{i=1}^{M} f_i x_i}{\sum_{i=1}^{M} f_i} = \sum_{i=1}^{M} f_i x_i$$

We have used the fact that $\Sigma f_i = 1$, that is, the sum of the probabilities is 1. This measurement is well known as the arithmetic mean or expected value of the random variable. The corresponding measurement of dispersion is the error of this expected value, given by

$$\|x - E(x)\|_{2,f} = \left\{\sum_{i=1}^{M} f_i(x_i - E(x))^2\right\}^{1/2}$$

which is the standard deviation of the random variable.

The discrete L_1 norm with weight function f is defined by

$$\|u\|_{1,f} = \sum_{i=1}^{M} f_i|u_i| \quad .$$

We show that the median is the measurement of central tendency associated with this norm. For ease of notation we define $S(a,b) = \{i \mid x_i \in (a,b)\}$ with similar definitions for semi-open intervals. If m is the median of the random variable defined by x and f, and ε is any positive number, then

$$\phi(m + \varepsilon) - \phi(m) = \|x - (m + \varepsilon)\|_{1,f} - \|x - m\|_{1,f}$$

$$= \sum_{S(-\infty,m]} f_i(m + \varepsilon - x_i) + \sum_{S(m,\, m + \varepsilon]} f_i(m + \varepsilon - x_i)$$

$$+ \sum_{S(m + \varepsilon,\, \infty)} f_i(x_i - m + \varepsilon) - \sum_{S(-\infty,m]} f_i(m - x_i)$$

$$- \sum_{S(m,\, m + \varepsilon]} f_i(x_i - m) - \sum_{S(m + \varepsilon,\, \infty)} f_i(x_i - m)$$

$$= \varepsilon \sum_{S(-\infty,m]} f_i + \sum_{S(m,\, m + \varepsilon]} f_i(2m - 2x_i + \varepsilon) - \varepsilon \sum_{S(m + \varepsilon,\, \infty)} f_i \,.$$

But for x_i in the interval $(m, m + \varepsilon]$,

$$2m - 2x_i + \varepsilon \geq -\varepsilon \,.$$

Therefore,

$$\phi(m + \varepsilon) - \phi(m) \geq \varepsilon \sum_{S(-\infty,m]} f_i - \varepsilon \sum_{S(m,\infty)} f_i \geq 0$$

where the last inequality follows from the definition of the median. A similar argument can be used to show that $\phi(m - \varepsilon) - \phi(m) \geq 0$. Hence, m is a minimum of ϕ and a measurement of central tendency with respect to the discrete weighted L_1 norm. In the case where the median is not one of the data values, this best approximation is not unique. The corresponding measurement of dispersion is

$$\|x - m\|_{1,f} = \sum_{i=1}^{M} f_i|x_i - m| \;,$$

the mean deviation.

Another common measurement of central tendency is derived from the discrete L_∞ norm which is defined by

$$\|u\|_\infty = \max_{1 \leq i \leq M} |u_i| \,.$$

In this case we wish to find the value of c which minimizes

$$\|x - c\|_\infty = \max_{1 \leq i \leq M} |x_i - c| \,.$$

It is easy to determine that the minimizing value must be located midway between the maximum and minimum values which x can take on, a value known as the *midrange*. If we denote the midrange by m_r, the corresponding measurement of dispersion is

$$\|x - m_r\|_\infty = \max_{1 \leq i \leq M} |x_i - m_r| \;,$$

which is half of the range.

We define one final discrete norm by

$$\|u\|_{m,f} = \sum_{i=1}^{M} f_i[1 - \delta(u_i)] \;,$$

where

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}.$$

The value of $\|x - c\|_{m,f}$ is the probability that the random variable is not equal to c, and any value of c which minimizes this is called a **mode.** The corresponding measurement of dispersion is the probability that the random variable is not equal to the mode.

3. *Statistics which are Transformations of Best Approximations.*

A number of measurements of central tendency do not arise directly as best approximations in some norm, but can be found as the transformation of a best approximation. Such measures are defined as follows.

Choose a norm $\|\ \|$ and a transformation function θ. Find c^* such that c^* is the best approximation to $\theta(x)$ in the given norm, that is,

$$\|\theta(x) - c^*\| \leq \|\theta(x) - c\| \text{ for all } c \in (-\infty,\infty).$$

Then a measurement of central tendency is given by $\theta^{-1}(c^*)$.

If the discrete L_2 norm is chosen, then the measurement of central tendency is

$$\theta^{-1}\left(\sum_{i=1}^{M} f_i\theta(x_i)\right) .$$

The geometric mean, harmonic mean, and root-mean-square are examples of this type of measurement when $\theta(x)$ equals $\log x$, $1/x$, and x^2, respectively, and the data values x_i are such that the appropriate function θ is defined.

4. *Continuous Random Variables.*

The above results can be extended to continuous random variables by replacing the discrete norms by the corresponding continuous norms. We now assume that we have a continuous random variable with probability density function $f(x)$. Therefore, f is such that $\int_{-\infty}^{\infty} f(x)dx = 1$ and $f(x) \geq 0$ for all x.

The best approximation in the continuous L_2 norm with weight function f is the constant c which minimizes

$$\phi(c) = \|x - c\|_{2,f} = \left\{\int_{-\infty}^{\infty} f(x)(x - c)^2dx\right\}^{1/2}.$$

By differentiation, the minimum of ϕ is found to be

$$c^* = \int_{-\infty}^{\infty} xf(x)dx .$$

This value of c^* is the expectation of a continuous random variable with probability density function f. The corresponding measurement of dispersion is

$$\|x - c^*\|_{2,f} = \left\{\int_{-\infty}^{\infty} f(x)(x - c^*)^2dx\right\}^{1/2} ,$$

the square root of the variance of the random variable.

If we consider the continuous L_1 norm with weight function f, then, since f is always nonnegative, we have

$$\phi(c) = \|x - c\|_{1,f} = \int_{-\infty}^{\infty} f(x)|x - c|dx$$

$$= \int_{-\infty}^{c} f(x)(c - x)dx + \int_{c}^{\infty} f(x)(x - c)dx ,$$

and

$$\phi'(c) = \int_{-\infty}^{c} f(x)dx - \int_{c}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{c} f(x)dx - \int_{-\infty}^{\infty} f(x)dx + \int_{-\infty}^{c} f(x)dx$$

$$= 2\int_{-\infty}^{c} f(x)dx - 1 .$$

Setting $\phi'(c)$ to zero, we find that ϕ is minimized when $\int_{-\infty}^{c} f(x)dx = 1/2$, that is, at that value of c for which the probability is exactly one-half that $x < c$. This is the natural extension of the median to a continuous distribution.

Unless $f(x)$ is zero everywhere outside of some bounded interval, there is no continuous extension of the midrange. If f is zero outside the interval (a,b) and positive somewhere in every neighborhood of a and b, then the continuous L norm defined by

$$\phi(c) = \|x - c\|_\infty = \sup_{a<x<b} |x - c|$$

has as its minimizing value and measurement of central tendency

$$c^* = (a + b)/2$$

which is the middle value of the interval in which f is non-zero. The corresponding measurement of dispersion is

$$\phi(c^*) = (b - a)/2 .$$

The continuous extension of the mode is found by minimizing

$$\phi(c) = \|x - c\|_{m,f} = 1 - \int_{-\infty}^{\infty} f(x)\delta(x)dx = 1 - f(c)$$

where δ is the Dirac delta function ([2], p. 6). The function ϕ is minimized at those values where f attains its maximum value. If m_0 is such a value, then the corresponding measurement of dispersion is $1 - f(m_0)$.

REFERENCES

1. Hamming, R. W., Numerical Methods for Scientists and Engineers, McGraw-Hill, New York, 1962.
2. Stakgold, I., Boundary Value Problems of Mathematical Physics, Volume I, Macmillan, New York, 1967.

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GENERALIZING BINARY OPERATIONS

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Most day to day calculations take place within the field of real numbers with the two binary operations of addition and multiplication. In this field, these two operations are definitionally independent of one another. However, if we approach binary operations from a different point of view, e.g. that of recursive formulae, we can develop multiplication from addition by use of the concept of repeated addition. Along similar lines, we can develop exponentiation from multiplication by repeated multiplication. The next logical step would be to try to develop another binary operation based on repeated exponentiation.

Professor D. F. Borrow of the University of Georgia in the American *Mathematical* Monthly, 43 (1936), p. 150, developed some theorems and a notation for repeated exponentiation. As Σ is used for summation and Π is used for products, he used E for repeated exponentiation. The development of a "fourth operation" would depend on all the indexed Terms of E being equal, similar to what is necessary in developing multiplication and exponentiation itself.

In order to clarify relations and notations, let us look at addition, multiplication, exponentiation, and a projected new fourth operation in terms of functions and recursive formulae. Let

$$f_1(n,m) = n + m$$

$$f_2(n,m) = n\cdot m$$

and

$$f_3(n,m) = n^m .$$

We know the following:

$$n\cdot m = n + [n\cdot(m - 1)] = \sum_1^m n_i \qquad \text{(where all } n_i = n)$$

and

$$n^m = n\cdot[n^{(m-1)}] = \prod_1^m n_i \qquad \text{(where all } n_i = n) .$$

Using our functional notation, we can write the above equations as recursive formulae:

$$f_2(n,m) = f_1[n, f_2(n, m - 1)]$$

$$f_3(n,m) = f_2[n, f_3(n, m - 1)] .$$

By comparing these two formulae, we can easily proceed to the definition of a fourth operation in terms of previous operations. Thus, let

$$f_4(n,m) = f_3[n, f_4(n, m - 1)] ,$$

and, in general, for a kth operation, let

$$f_k(n,m) = f_{k-1}[n, f_k(n, m - 1)] .$$

The question now arises, how does one define the first term in this recursive formula? In other words, what is $f_4(n,1)$? To answer this question, let us first look at $f_2(n,1)$, and $f_3(n,1)$, which are based on a similar process of recursive formulae and repeated operations. We know that $f_2(n,1) = \sum_1^1 n = n$ and we also know that $f_3(n,1) = \prod_1^1 n = n$. We can thus similarly define $f_4(n,1) = E_1^1 n$ to be equal to n by the same line of reasoning, that is, "one n" combined together by the process of [addition/multiplication/exponentiation] is still only "one n."

What about $f_4(n,2) = E_1^2 n$? This would be equal to

$$f_3[n, f_4(n, 2 - 1)] = f_3[n, f_4(n,1)] = f_3(n,n) = n^n .$$

Thus we see that our formulation of the recursive formula is consistent with what our initial intuitive feel was for what this new fourth function should be. Similarly, we obtain $f_4(n,3) = n^{(n^n)}$. At this point we might notice that, unlike our definitions of exponentiation and multiplication in terms of multiplication and addition respectively, our definition of f_4 does not allow associativity. In other words, $f_4(n,3) = n^{(n^n)} \neq (n^n)^n$, and, in general,

$$f_4(n,m) = n^{(n^{(n^{(n\cdots)})})} \neq ((((n^n)^n\cdots{}^n)^n)^n)^n = n^{n(m-1)} .$$

At this point, two questions may arise: What can one do with f and what about other operations? In particular, does there exist an f_0?

In answer to the first question, it is obvious that tables of f_4 are not readily available, and are not particularly useful, either. The numbers balloon quite rapidly. For example, $f_4(2,4) = 65{,}536$, and

$f_4(2,5) = f_3(2, 65{,}536) = 2^{65{,}536}$, while $f_4(3,3)$ exceeds ten digits. The only easily computable numbers are of the form $f_4(n,2) = n^n$. Even then, the numbers get fairly large, rather rapidly. For example, $f_4(8,2) = 16{,}777{,}216$.

There are other paths which can be taken with f_4 from here. As with an initial development of multiplication or exponentiation, we can develop definitions for $f_4(x,y)$ when y is zero, rational, real, or complex, and then develop definitions when x is zero, rational, real, or complex. For example, in developing exponentiation, one method of developing rational exponents is as follows:

Define $x = y^{(1/n)}$ to be equivalent to

$$y = x^n .$$

If one raises x to the power of m, then one has

$$z = x^m = y^{(m/n)} ,$$

and thus one has defined exponentiation for rational exponents.

Let us do something similar for f_4.

Define $x = f_4(y, 1/n)$ to be equivalent to

$$y = f_4(x,n) .$$

If we then operate on x by m, then we have

$$z = f_4(x,m) = f_4(y, m/n)$$

We can likewise work with negatives. In multiplication, $y = x\cdot(-n) = f_2(x,-n)$. But this is equivalent to saying $y + x\cdot n = 0 = I_1$ (the identity for f_1), or, using our functional notation, $f_1[y, f_2(x,n)] = I_1 = 0$. Likewise for exponentiation, $y = x^{-n} = \frac{1}{X}$ which is equivalent to saying $f_2[y, f_3(x,n)] = I_2 = 1$. Similarly, for our f_4, we can define $y = f_4(x,-n)$ as being equivalent to $f_3[y, f_4(x,n)] = I_3 = 1$.

Now, let us look at our other question--the possibility of f_0, that is, a binary operation more "basic" than addition. If it did exist, it would have to comply with our recursive formulae developed above and also to the general intuitive scheme of the functional notation. Now for any k, we saw that $f_k(n,m) = f_{k-1}[n, f_k(n, m - 1)]$. Let us take a closer look at what happens if k = 1. We would then have

$$f_1(n,m) = f_0[n, f_1(n, m - 1)] .$$

But f_1 is addition. Thus, we have

$$n + m = f_0[n, n + m - 1] .$$

If we now let $m = 1$, then we have

$$n + 1 = f_0[n,n] .$$

From the functional approach we know that $f_3(n,2) = n^2 = n \cdot n = f_2(n,n)$. Similarly, $f_2(n,2) = n \cdot 2 = n + n = f_1(n,n)$. If we are to be consistent, f_1 and f_0 should be similarly related (assuming f_0 exists). Thus, $f_1(n,2) = n + 2 = n \circ n = f_0(n,n)$ [where $f_0(n,n) = n \circ n$]. But above we showed that $f_0(n,n)$ was $n + 1$. From this contradiction resulting from the initial assumption that f_0 exists, we have shown that addition is the "most basic" operation we can have.

SPECIAL ANNOUNCEMENT FOR CHAPTER PRESIDENTS

The editorial staff of this *Journal* has prepared a special publication *Initiation Ritual* for use by local chapter's containing details for the recommended ceremony for initiation of new members. If you would like one, write to:

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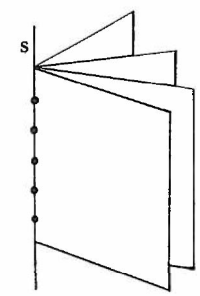
GRAPHS CRITICAL FOR MAXIMAL BOOKTHICKNESS

by *Charles V. Keys*
Louisiana State University

To "draw a graph G in a book" arrange the vertices of G in a fixed position along a line segment s (the spine) which is the intersection of a finite number of bounded half-planes (pages). Draw the edges of G on these pages such that each edge lies entirely on one page and no two edges cross (that is, intersect; except possibly at their endpoints). Then the minimum number of pages required to represent G in this way, considering all arrangements of $V(G)$ along she spine, is defined to be the ***bookthickness*** of G.



By coloring edges on separate pages different colors, one may see that we have the following equivalent characterization of bookthickness.

Arrange the vertices of a graph G along a circle and draw the edges of G as chords in the interior of the circle. Let m be the minimum number of colors required to color the edges of G such that no two edges of the same color cross. The bookthickness of G is then the minimum value of m over all arrangements of the vertices of G on the circle.

Note that since an edge joining vertices which are adjacent in a circular arrangement does not cross any other edge, it can always be given any color. We therefore do not have to worry about such edges, and they will not be shown in the colorings drawn in this paper.

Arrange the p vertices of G on a circle, numbered in order from 0 to $p - 1$. Define the *length* of an edge $\{u,v\}$ to be the length of the shortest path from u to v along the circle; that is,

$$\text{length } \{u,v\} = \min \{|u - v|, p - |u - v|\} .$$

Note that in any coloring on p vertices we can have at most $p - 3$ edges of the same color. This is clearly true for $p = 3, 4$. Assume true for $n < p$. Choose an edge e of one color, with say $p - n$ and $n - 2$ vertices above and below e, respectively. By induction, the maximum number of edges of this color above and below e, respectively, are $(p - n + 2) - 3$ and $n - 3$. This gives a maximum of $1 + (p - n + 2) - 3 + n - 3 = p - 3$ edges which may be colored the same.

The bookthickness of K_{2n} is n. We must color $\frac{2n(2n-1)}{2} - 2n = 2n^2 - 3n$ edges (the total number of edges minus the number of edges joining vertices adjacent in the ordering of $V(G)$). Since we can fit in at most $2n - 3$ edges of each color, we need at least $\frac{2n^2 - 3n}{2n - 3} = n$ colors.

Given a K_{2n}, we ask how many edges, and what configurations of edges, must be deleted so that the remaining edges may be colored with $n - 1$ colors. The above calculation shows that deleting less than $2n - 3$ edges from a K_{2n} does not reduce the bookthickness. If we delete exactly $2n - 3$ edges, the bookthickness may or may not drop to $n - 1$. The case of removing 5 edges from a K_8 will be discussed at the end of the paper.

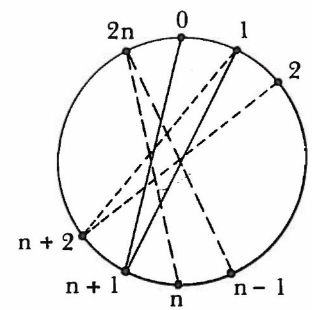
The bookthickness of K_{2n+1} is $n + 1$. We must color $\frac{(2n+1)2n}{2} - (2n + 1) = (n - 1)(2n + 1)$ edges. Since we can color at most $2n - 2$ edges a single color, we need at least $\frac{(n-1)(2n+1)}{2n-2} = \frac{2n+1}{2} = n + 1$ colors.

To reduce the bookthickness of K_{2n+1} to n, we must delete at least $(n - 1)(2n + 1) - n(2n - 2) = n - 1$ edges.

Theorem. If we delete $n - 1$ edges from a K_{2n+1}, one edge of each length $2,3,\cdots,n$, then there is a coloring for the remaining edges using n colors.

Proof. Arrange the vertices of K_{2n+1} on a circle, numbered in order from 0 to $2n$. We may assume that $\{0,n\}$ is the deleted edge of length n. Assign $\{j - 1, j + n\}$ and $\{j + n, j\}$ the color $j \pmod n$, $j = 1,2,3,\cdots,n$. Note: All arithmetic related to vertex numbering is modulo $2n + 1$.

This colors all the remaining edges of length n.



We shall now successively color all edges of length $n - 1, n - 2, \cdots, 2$ such that, after coloring the edges of length $n, n - 1, \cdots, \ell + 1$, the following properties hold:

1) The edges of length $\ell + 1$ are colored in order $1,2,\cdots,n,1,2,\cdots,n$ as we proceed around the circle: that is, if $\{i, i + \ell + 1\}$ is the deleted edge of length $\ell + 1$, and k is the color assigned to the edge $\{i - \ell, i + \ell\}$, then successively the edge $\{i + j, i + j + \ell + 1\}$ is colored $k + j \pmod n$, $j = 1,2,\cdots,2n$.

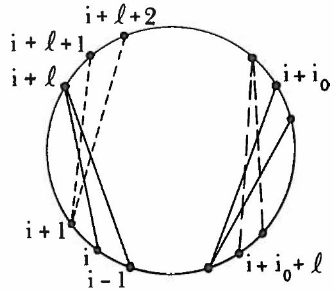
2) Every edge of length $\ell + 1 < n$ is adjacent to an edge of length $\ell + 2$ of the same color: that is, if $\{i, i + \ell + 1\}$ is colored k, then so is either $\{i, i + \ell + 2\}$ or $\{i - \ell, i + \ell + 1\}$ (but not both).

3) No edge of length $\ell + 1$ is colored the same as any crossing edge of length $\geq \ell + 1$; that is, no edge of length $\geq \ell + 1$ and incident to a vertex $i + p$ for $p = 1,2,\cdots,\ell$ has the same color as the edge $\{i, i + \ell + 1\}$.

Thus, inductively, no two crossing edges of length $\geq \ell + 1$ have the same color.

Suppose we have colored the edges of length $n, n - 1, \cdots, \ell + 1$ such that properties 1), 2), and 3) hold. Let $\{i, i + \ell + 1\}$ be the deleted edge of length $\ell + 1$, $\{i + i_0, i + i_0 + \ell\}$ the deleted edge of length ℓ, and the edge $\{i + j, i + j + \ell + 1\}$, of length $\ell + 1$ colored $k + j \pmod n$, $j = 1,2,\cdots,2n$, (k as above).

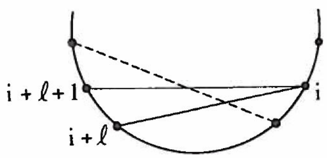
We now color the edges of length ℓ as follows: For $j = 1, 2, \cdots, i_0 - 1$, assign edge $\{i + j, i + j + \ell\}$ the color $k + j \pmod n$, and for $j = i_0 + 1, i_0 + 2, \cdots, 2n + 1$, assign $\{i + j, i + j + \ell\}$ the color $k + j - 1 \pmod n$.



Property 1) holds trivially in this coloring.

For $1 \le j \le i_0 - 1$, $\{i + j, i + j + A + 1\}$ and $\{i + j, i + j + \ell\}$ are both colored $k + j$ (mod n), while for $i_0 + 1 \le j$ 5 $2n + 1$ both $\{i + j - 1, i + j + \ell\}$ and $\{i + j, i + j + \ell\}$ are colored $k + j - 1$ (mod n). Thus property 2) holds.

Consider an edge e of length A colored a (mod n). By property 2) we have that any edge of length $\ge A + 1$ crossing e must also cross an edge of length $A + 1$ colored a (mod n), and must therefore be a different color.

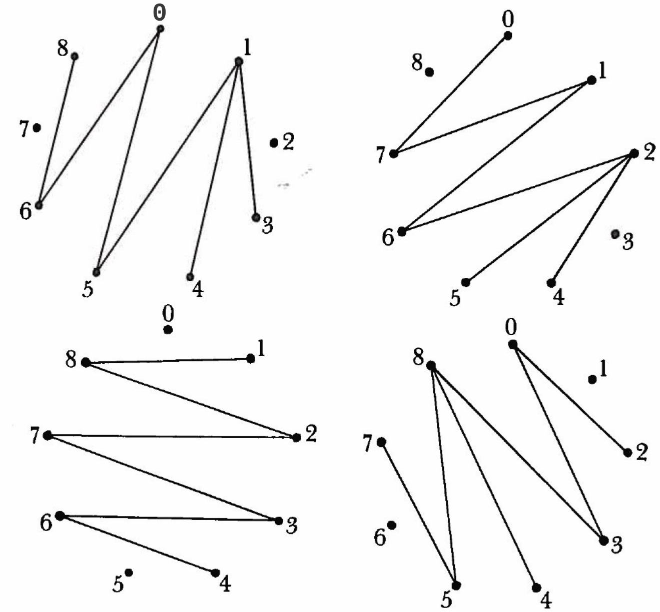


Additionally, an edge $\{i, i + \ell\}$ of length I colored a crosses precisely the edges $\{i - A + 1, i + 1\}$, $\{i - A + 2, i + 2\}$, $\cdots$, $\{i - 1, i + \ell - 1\}$, $\{i + 1, i + \ell + 1\}$, $\cdots$, $\{i + \ell - 1, i + 2\ell - 1\}$ among all edges of length I. These edges are colored a $\pm$ p (mod n) for $p = 1, 2, \cdots, A - 1$. But $p \le \ell - 1 < n$ implies that a $\neq$ a $\pm$ p (mod n), so we have that property 3) holds in this coloring.

Continue this process to color the edges of lengths A - 1, I - 2, $\cdots$, 2. Then by property 3) no two crossing edges will have the same color, which proves the theorem. □

As an illustration of the theorem, suppose we delete the edges $\{0,4\}$, $\{4,7\}$, and $\{3,5\}$ from a K_9. The theorem gives the following coloring

for the remaining edges:



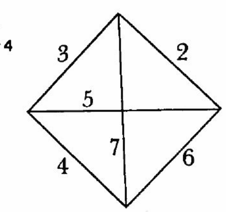
Thus asking whether deleting a graph with $n - 1$ edges from K_{2n+1} reduces bookthickness is reduced to the problem of fitting the graph on $2n + 1$ vertices arranged on a circle such that it contains one edge of length $2,3,\cdots,n$.

This can be done for all graphs on $n - 1$ edges for $n = 2,3,4,5,6$. There are no known cases in which this cannot be done.

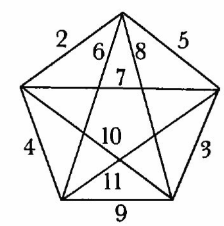
In addition we raise the question whether deleting the complete graph K_m from $K_{2\binom{m}{2}+3}$ reduces bookthickness. This is the case for $m = 2,3,4,5$.

Example. Deleting a K_4 and K_5 from a K_{15} and K_{23}, respectively, does reduce bookthickness, as shown below. (The numbers assigned to edges are their respective lengths.)

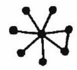
K_4

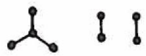


K_5



In the case of a K_8, we have shown that deleting the following graphs (and their subgraphs) does not reduce bookthickness to 3:

I  (8,8) graph with a vertex of degree 7

II  $K_{1,3}$, K_2, K_2

III  K_4

IV  $K_{1,4}$, P_3

V  K_3, $K_{1,3}$

VI  K_3, K_3

VII  $K_{1,3}$, $K_{1,3}$

Deleting all other configurations of 5 or more edges reduces bookthickness.

As an example, we give the argument for I: Deleting an (8,8) graph with a vertex of degree 7 from a K_8 does not reduce bookthickness.

Suppose there is a coloring of K_8 with G deleted using only 3 colors. Let the vertex of G with degree 7 be placed at a and $\{b,c\}$ the edge of G not incident to a. Then by deleting the vertex a and edges incident to a in the coloring, we obtain a coloring for K_7 with only one edge ($\{b,c\}$) deleted which uses only 3 colors. But this is impossible.

In general, since deleting less than $n - 2$ edges from a K_{2n-1} does not reduce the bookthickness to $n - 1$, deleting a $(2n, (2n - 1) + (n - 3))$ graph with a vertex of degree $2n - 1$ from a K_{2n} does not reduce the book-thickness to $n - 1$.

LOCAL AWARDS

If your chapter has presented awards to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal* so they may receive recognition in this publication. Also, the national office can supply you with attractive award certificates for use in presentations.

THE BASIC MATHEMATICS OF THE FARO SHUFFLE

by S. Brent Morris
Duke University

Among magicians the operation of cutting a deck by shuffling into two equal portions, and these portions perfectly interlaced by shuffling is called a *faro* shuffle, referring to the card game faro. For further classification, a shuffle where the top card is left out or on top is called an Out shuffle, while shuffling the top card in or to the second position is called an In shuffle.

In studying these shuffles, certain mathematical properties present themselves. This paper will concern itself with the following questions:

1) In a deck of size $2k$, where will a card in position p be taken following one In or one Out shuffle?

2) In a deck of size $2k$, where will a card in position p be taken following an In and then an Out shuffle or an Out following an In shuffle?

3) Can the Out and In faro shuffles be generalized to decks of size $2k - 1$, and how will this affect the answers to 1) and 2)?

All three questions will be answered. Surprisingly, decks of size $2k - 1$ yield much more elegant answers to 1) and 2), although it is perhaps unnatural to perform a faro shuffle with an odd sized deck.

Throughout this paper, certain notational conventions will be used. The letters p and q will always be used to indicate the position of a card. Deck and *card* will be used rather than linear **away** and *element*. Odd deck and even deck will refer to decks with $2k - 1$ and $2k$ cards respectively. A card in position p will be in position $O(p)$ or $I(p)$ following an Out or an In shuffle respectively. Finally, a series of Out and In shuffles will be indicated by concatenation and read from left to right (rather than the usual right to left). Thus $OI(p) = I(O(p))$, and indicates an Out shuffle followed by an In shuffle.

<u>*Proposition 1.*</u> In a deck of size $2k$, the effect of one Out faro shuffle on a card in position p is to move it to position

$$O(p) \equiv 2p - 1 \pmod{2k - 1}\text{[1]} . \qquad (1)$$

Proof. We first consider the effect of an Out shuffle on a card in the upper half of a deck, and thus temporarily avoid the problem of a card being taken to a position with a lower value than it began with; that is, $O(p) < p$.

A card in position p has $p - 1$ cards above it, and following an Out shuffle its new position $O(p)$ will be $p - 1$ greater since an Out shuffle places a card between 1 and 2, 2 and 3, $\cdots$, $p - 1$ and p. We thus have for $1 \le p \le k$,

$$O(p) = p + p - 1 = 2p - 1 ,$$

and clearly (1) holds.

For a card in the lower half of the deck, we observe that $O(k + 1) = 2$, $O(k + 2) = 4$, $\cdots$, $O(2k - 1) = 2k - 2$, and $O(2k) = 2k$; or more clearly:

$$O(k + 1) = 2 \equiv 2k - 1 + 2 \equiv 2(k + 1) - 1 \pmod{2k - 1}$$
$$O(k + 2) = 4 \equiv 2k - 1 + 4 \equiv 2(k + 2) - 1 \pmod{2k - 1}$$

$$O(2k - 1) = 2k - 2 \equiv 2k - 1 + 2k - 2 \equiv 2(2k - 1) - 1 \pmod{2k - 1}$$
$$O(2k) = 2k \equiv 2k - 1 + 2k \equiv 2(2k) - 1 \pmod{2k - 1} .$$

We have just demonstrated that for $k + 1 \le p \le 2k$, and hence all p, formula 1 holds. □

Proposition 2. In a deck of size $2k$, the effect of one In faro shuffle on a card in position p is to move it to position

$$I(p) \equiv 2p \pmod{2k + 1} . \qquad (2)$$

Proof. We again first consider a card in the upper half of the deck. For a card at position p, an In shuffle will increase its position by p, since an In shuffle places a card between 1 and 2, 2 and 3, $\cdots$, p - 1 and p, plus a card on top of 1. Thus we have for $1 \le p \le k$,

$$I(p) = p + p = 2p ,$$

and clearly (2) holds.

---

[1] As zero is not assigned as a position number, the card in position k will be taken to position $2k - 1$. Further, the card in position $2k$ may be effectively ignored as an Out shuffle does not affect it, though formula (1) does apply if we agree that $O(2k)$ is $2k$ and not 1, for,
$$O(2k) \equiv 2(2k) - 1 = 4k - 1 \equiv 2k \pmod{2k - 1} .$$

For a card in the lower half of the deck, we observe that $I(k + 1) = 1$, $I(k + 2) = 3$, $\cdots$, $I(2k - 1) = 2k - 3$, $I(2k) = 2k - 1$. As before, we can see that formula (2) holds for $k + 1 \le p \le 2k$, and hence all p. □

In considering decks for mathematical purposes, the terms *top* and *bottom* are purely arbitrary. It makes as much sense for our purposes to number a deck from bottom to top as to number it from top to bottom.

We now wish to define a transformation that reverses the ordering of an even deck. Simple calculation shows that the desired transformation is

$$Tp = 2k + 1 - p .$$

Further, T is idempotent, since

$$T^2(p) = 2k + 1 - (2k + 1 - p) = p .$$

Proposition 3. In a deck of size $2k$, the following relations hold:

$$O(Tp) = T[O(p)] \qquad (3)$$
$$I(Tp) = T[I(p)] \qquad (4)$$

Proof.

$$\begin{aligned} O(Tp) &\equiv 2Tp - 1 \pmod{2k - 1} \\ &= 2(2k + 1 - p) - 1 \pmod{2k - 1} \\ &= 4k + 2 - 2p - 1 \pmod{2k - 1} . \end{aligned}$$

Hence

$$\begin{aligned} O(Tp) - 2k - 1 &\equiv 2k + 1 - 2p - 1 \pmod{2k - 1} \\ &\equiv -2p + 1 \pmod{2k - 1} \\ &= -O(p) . \end{aligned}$$

Thus

$$O(Tp) = 2k + 1 - O(p) = T[O(p)] .$$

The proof of (4) follows with similar ease. □

The significance of Proposition 3 is that we need only determine the action of a shuffle on the first k cards on an even deck to be able to determine the action on the final k cards.

Proposition 4. In a deck of size $2k$, the effect of one Out followed by one In faro shuffle on a card in position p is to move it to position

$$OI(p \begin{cases} 4p - 2 \pmod{2k + 1} , & 1 \le p \le k \\ 4p + 2 \pmod{2k + 1 , & k + 1 \le p \le 2k \end{cases} .$$

Proof. We shall first show that we need only consider the action upon the first k cards, and then apply the transformation T to determine the action on the last k:

$$OI(Tp) = T[OI(p)] .$$

Using (3) and (4) we have

$$\begin{aligned} OI(Tp) &= I(O(Tp)) \\ &= I(T[O(p)]) \\ &= T(I[O(p)]) \\ &= T[OI(p)] . \end{aligned}$$

Thus we need only consider the action upon the first k cards. By Proposition 1 we know that

$$O(p) \equiv 2p - 1 \pmod{2k - 1} .$$

However, since we are dealing with the first k cards, we may write

$$O(p) = 2p - 1 , \quad 1 \le p \le k .$$

Applying an In shuffle, we have

$$OI(p) = I(O(p)) \equiv 4p - 2 \pmod{2k + 1} , \quad 1 \le p \le k .$$

Now, if $k + 1 \le p \le 2k$, we can write $p = 2k + 1 - q$, where $1 \le q \le k$. By our first conclusions, we have

$$OI(p) = OI(2k + 1 - q) = 2k + 1 - OI(q) ,$$

or equivalently

$$\begin{aligned} OI(p) - 2k - 1 &= OI(q) \\ &\quad -4q + 2 \pmod{2k + 1} . \end{aligned}$$

Thus

$$\begin{aligned} OI(p) &\equiv 2k + 1 - 4q + 2 \pmod{2k + 1} \\ &\quad 8k + 4 - 4q + 2 \pmod{2k + 1} \\ &\equiv 4p + 2 \pmod{2k + 1} , \quad k + 1 \le p \le 2k . \end{aligned}$$

Thus our proposition is proven for all p. □

Proposition 5. In a deck of size $2k$, the effect of one In followed by one Out faro shuffle on a card in position p is to move it to position

$$IO(p \begin{cases} 4p - 1 \pmod{2k - 1} , & 1 \le p \le k \\ 4p - 5 \pmod{2k - 1} , & k + 1 \le p \le 2k . \end{cases}$$

Proof. As in Proposition 4, we can show that $IO(Tp) = T[IO(p)]$, and then proceed with parallel arguments. □

Thus far we have answered questions 1) and 2). There remains the problem of defining a faro shuffle for an odd deck, which we shall now proceed to do.

The distinguishing property of a faro shuffle is that after the deck is separated into two portions, each portion is perfectly interlaced into the other, and any two cards adjacent before the shuffle are now separated by one card (cards k and $k + 1$ excepted). Hence any definitions for faro shuffling an odd deck must incorporate these features.

Definition. For a deck of size $2k - 1$, an Out *faro shuffle* is performed by adding a conventional 2kth card, performing a standard Out faro shuffle on the pack of $2k$, and then removing the conventional card. For a deck of size $2k - 1$, an In *faro shuffle* is performed by removing the $(2k - 1)$st card, performing a standard In faro shuffle on the remaining $2k - 2$, and then returning the $(2k - 1)$st card.

Brief consideration will show that the odd decks are separated into portions and each interlaced into the other. Further, any two cards adjacent prior to the shuffle are now separated by one card (cards k and $k + 1$ excepted for Out shuffles, and cards $k - 1$ and k excepted for In shuffles).

Proposition 6. In a deck of size $2k - 1$, the effect of one Out faro shuffle on a card in position p is to move it to position

$$O(p) \equiv 2p - 1 \pmod{2k - 1} .$$

Proof. By the definition of an odd Out faro shuffle, we are Out shuffling a deck of size $2k$, with a conventional 2kth card. As this card is never affected by an Out shuffle, we need not consider it in our calculations. The movement of the other $2k - 1$ cards with which we are concerned is according to the formula in Proposition 1. Thus for our deck of size $2k - 1$:

$$O(p) \equiv 2p - 1 \pmod{2k - 1} . \ \square$$

Proposition 7. In a deck of size $2k - 1$, the effect of one In faro shuffle on a card in position p is to move it to position

$$I(p) \equiv 2p \pmod{2k - 1} .$$

Proof. By the definition of an odd In faro shuffle, the $(2k - 1)$st card remains in position $2k - 1$ and the position of the remaining cards is determined by the formula in Proposition 2 using a deck of size $2k - 2$ Hence for our deck of size $2k - 1$:

$$I(p) \equiv 2p \pmod{2k - 1}\ . \ \square$$

Proposition 8. In a deck of size $2k - 1$, the effect of one Out followed by one In faro shuffle on a card in position p is to move it to position

$$OI(p) \equiv 2p \pmod{2k - 1}\ .$$

Proof. First, if $1 \le p \le k$, we have by Proposition 6:

$$O(p) = 2p - 1\ .$$

Applying an In shuffle, we have by Proposition 7

$$OI(p) \equiv 4p - 2 \pmod{2k - 1}\ , \quad 1 \le p \le k\ .$$

Now if $k + 1 \le p \le 2k - 1$, we can write $p = k + q$, where $1 \le q \le k - 1$. Thus by Proposition 6:

$$\begin{aligned} O(p) &\equiv 2(k + q) - 1 \pmod{2k - 1} \\ &= 2k + 2k - 1 \pmod{2k - 1} \\ &\equiv 2q \pmod{2k - 1}\ . \end{aligned}$$

Since $1 \le q \le k - 1$, we thus have:

$$O(k + q) = I(2q)\ , \quad 1 \le q \le k - 1\ .$$

Now applying an In shuffle yields by Proposition 7:

$$\begin{aligned} OI(k + q) &= I(2q) \\ & 4q \pmod{2k - 1} \\ & 4k + 4q - 2 \pmod{2k - 1} \\ & 4(k + q) - 2 \pmod{2k - 1}\ . \end{aligned}$$

Thus

$$OI(p) \equiv 4p - 2 \pmod{2k - 1}\ , \quad k + 1 \le p \le 2k - 1\ ,$$

proving the proposition for all p. $\square$

With almost identical arguments, we can also prove the following proposition.

Proposition 9. In a deck of size $2k - 1$, the effect of one In followed by one Out faro shuffle on a card in position p is to move it to position

$$IO(p) \equiv 4p - 1 \pmod{2k - 1}\ .$$

It should be noted that proofs of Proposition 8 and Proposition 9

could have been developed parallel to the proofs of Proposition 4 and" Proposition 5 by the introduction of a transformation $T^*(p) = 2k - p$. The significant identities for decks of size $2k - 1$ are

$$O(T^*p) = T^*[I(p)] \quad \text{and} \quad I(T^*p) = T^*[O(p)]\ .$$

Similarly Proposition 4 and Proposition 5 could have been proven by the elementary method. However by so doing, the important symmetry of the faro shuffle with respect to the top and bottom of a deck would have been overlooked.

As one final problem of interest, we consider the action of a series of Out and In faro shuffles upon the top card. We shall place one restriction upon the series of shuffles, namely that prior to any shuffle the top card will not be in the bottom portion of the deck. This restriction allows us to ignore the various moduli and use equalities rather than congruences.

For a deck sufficiently large, we can easily see from our various equations that

$$O^n(p) = 2^n p - (2^n - 1) \quad \text{and} \quad I^n(p) = 2^n p\ .$$

Further, as the top card is in position $p = 1$, our equations are even simpler.

Since any number of Out shuffles does not affect the top card, we shall assume that our series of shuffles begins with a series of In shuffles. A few direct calculations show that

$$\begin{aligned} I^a(1) &= 2^a\ , \\ I^aO^b(1) &= 2^{a+b} - 2^b + 1\ , \\ I^aO^bI^c(1) &= 2^{a+b+c} - 2^{b+c} + 2^c\ , \end{aligned}$$

and so forth.

With this information in hand, we consider the problem from a slightly different point of view. Rather than being concerned with the position of the top card, we wish to know how many cards are on top of it. Thus, following a series of a In shuffles, there are $2^a - 1$ cards above the top card. Using a prime to indicate the number of cards above the top one, we get the following table:

$$\begin{aligned} [I^a(1)]' &= 2^a - 1 \\ &= 2^{a-1} + 2^{a-2} + \cdots + 2^2 + 2 + 1\ , \\ \ulcorner I^aO^b(1)]' &= 2^{a+b} - 2^b \end{aligned}$$

$$= 2^{a+b} - 1 - (2^b - 1)$$
$$= 2^{b+(a-1)} + \cdots + 2^{b+1} + 2^b + 2^{b-1} + \cdots + 2^2 + 2 + 1$$
$$- 2^{b-1} - \cdots - 2^2 - 2 - 1$$
$$= 2^{b+(a-1)} + \cdots + 2^{b+1} + 2^b = \cdots .$$

Expressing the number of cards above the top card in terms of sums of powers of 2 brings to mind the thought of using base 2 notation. Thus we have:

$$[f(1)]_2' = [\underbrace{II\cdots I}_{a \text{ terms}}(1)]_2' = \underbrace{11\cdots 1}_{a \text{ terms}}$$

$$[I^aO^b(1)]_2' = [\underbrace{II\cdots I}_{\substack{a\\ \text{terms}}}\underbrace{OO\cdots O}_{\substack{b\\ \text{terms}}}(1)]_2' = \underbrace{11\cdots 1}_{\substack{a\\ \text{terms}}}\underbrace{00\cdots 0}_{\substack{b\\ \text{terms}}} = \cdots$$

These equations can be formulated in a general rule which we state without proof: To determine the position of the top card following a series of In and Out faro shuffles, first transform the I's to 1's and the O's to 0's. This number (given in base 2) tells how many cards are above the top card. Similarly, to place m cards above the top card, convert m into base 2 notation and then transform the 1's to I's and 0's to O's. By applying the resultant series of shuffles to the deck, the top card will be in the required position, m + 1.

REFERENCES[2]

1. Elmsley, Alex, "Mathematics of the Weave Shuffle," *The* Pentagram, 11, No. 9 (1957), pp. 70-71; 11, No. 10 (1957), pp. 77-79; 11, No. 11 (1957), p. 85.
2. Gardner, Martin, "Mathematical Games," Scientific American, 215, No. 4 (1961), pp. 267-280.
3. Golomb, Solomon W., "Permutations by Cutting and Shuffling," SIAM Review, 3, No. 4 (1961), pp. 114-118.
4. Hugard, Jean and F. Braue, Expert Card Technique, Carl Waring Jones, Minneapolis, Minnesota, 1940.
5. Marlo, Edward, The Faro Shuffle, Privately Published, 1958.
6. Marlo, Edward, Faro Notes, Privately Published, 1959.
7. Swinford, Paul, Faro Fantasy, The Haley Press, Connersville, Indiana, 1968.

[2]References 4, 6 are magic books that contain no mathematics, but do have card tricks utilizing the properties of the faro shuffle.

APPLICATIONS OF FINITE DIFFERENCES TO THE SUMMATION OF SERIES

by H. Joseph Straight
Western Michigan University

Basically, Finite Differences deals with the changes which take place in the value of a function when the variable is increased by a certain amount. This classical branch of mathematics has many interesting applications; in this case to the problem of finding a nice, compact formula for the sum of n terms of a particular series.

Definition 1. Let $f(x)$ denote a function of x. Define the *first* difference of $f(x)$, written $\Delta f(x)$, to be:

$$\Delta f(x) = f(x+1) - f(x).$$

Similarly, we can define the second difference of $f(x)$:

$$\Delta^2 f(x) = \Delta(\Delta f(x)).$$

And we can define inductively the kth difference, $k \geq 2$:

$$\Delta^k f(x) = \Delta(\Delta^{k-1} f(x)).$$

Theorem 1. Let $f(x)$ and $g(x)$ be functions of x. Then:

1. $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$.
2. $\Delta[f(x)\cdot g(x)] = f(x)\Delta g(x) + g(x+1)\Delta f(x)$.
3. If c is a constant,
   (a) $\Delta c = 0$,
   (b) $\Delta cf(x) = c\Delta f(x)$.

Proof. The proofs of 1 and 3 are straightforward and follow directly from the definition. The proof of 2 involves the trick of adding a convenient zero, and is given here.

$$\Delta[f(x)g(x)] = f(x+1)g(x+1) - f(x)g(x) + [f(x)g(x+1) - f(x)g(x+1)]$$
$$= f(x)g(x+1) - f(x)g(x) + g(x+1)f(x+1) - g(x+1)f(x)$$
$$= f(x)\Delta g(x) + g(x+1)\Delta f(x).$$

Example 1. Find Δa^x where a is some constant.

$$\Delta a^x = a^{x+1} - a^x$$
$$= a^x(a - 1).$$

Definition 2. Let a and b be constants and n a non-negative integer.

$$(a + bx)^{(n)} = [a + bx][a + b(x - 1)]\cdots[a + b(x - n + 1)]$$
$$(a + bx)^{(0)} = 1.$$

(Note when a = 0, b = 1 we have $x^{(n)} = x(x - 1)\cdots(x - n + 1)$.)

Theorem 2. $\Delta(a + bx)^{(n)} = bn(a + bx)^{(n-1)}$.

The proof of this theorem again follows from the definitions.

Corollary. $\Delta x^{(n)} = nx^{(n-1)}$.

Next we come to a very important theorem, due to Newton, which gives a formula for expanding any polynomial in terms of its difference functions evaluated at 0.

Theorem 3 (Newton). Let $p(x)$ be a polynomial of degree n. Then

$$p(x) = p(0) + \frac{\Delta p(0)x^{(1)}}{1!} + \frac{\Delta^2 p(0)x^{(2)}}{2!} + \cdots + \frac{\Delta^n p(0)x^{(n)}}{n!}$$

The right side of the above is called the factorial form of $p(x)$. (Note the similarity between Newton's theorem and Maclaurin's expansion for $p(x)$.)

Proof. Assume $p(x) = a_0 + a_1x^{(1)} + a_2x^{(2)} + \cdots + a_nx^{(n)}$. Differencing $p(x)$ n times, we obtain the following identities:

$$\Delta p(x) = a_1 + 2a_2x^{(1)} + 3a_3x^{(2)} + \cdots + na_nx^{(n-1)}$$
$$\Delta p^2(x) = 2a_2 + 6a_3x^{(1)} + \cdots + n(n - 1)a_nx^{(n-2)}$$
$$\Delta p^3(x) = 6a_3 + \cdots + n(n - 1)(n - 2)a_nx^{(n-3)}$$
$$\vdots$$
$$\Delta p^n(x) = n!a_n .$$

Setting x = 0 in the above equations we have

$$a_0 = p(0), \quad a_1 = \Delta p(0), \quad a_2 = \frac{\Delta^2 p(0)}{2!}, \cdots \quad a_n = \frac{\Delta^n p(0)}{n!}$$

Substituting these values back into the expression for $p(x)$, we obtain the result of the theorem as stated.

Example 2. Use Newton's theorem to express $p(x) = x^3 - 2x^2 + 3x - 1$ in factorial form. Also find $\Delta p(x)$.

Method 1. Construct a Difference Table.

x	$p(x)$	$\Delta p(x)$	$\Delta^2 p(x)$	$\Delta^3 p(x)$
0	-1	2	2	6
1	1	4	8	6
2	5	12	14	
3	17	26		
4	43			

$$\therefore\ p(x) = -1 + 2x^{(1)} + \frac{2x^{(2)}}{2!} + \frac{6x^{(3)}}{3!}$$
$$= x^{(3)} + x^{(2)} + 2x^{(1)} - 1.$$

Now that we have $p(x)$ expressed in factorial form, we can use Theorem 2 to find $\Delta p(x)$.

$$\Delta p(x) = 3x^{(2)} + 2x^{(1)} + 2.$$

Method 2. Synthetic Division.

$$\begin{array}{r|rrrl}
1 & 1 & -2 & 3 & -1 = f(0) \\
 & 0 & 1 & -1 & \\ \hline
2 & 1 & -1 & 2 = \Delta f(0) & \\
 & 0 & 2 & & \\ \hline
 & 1 & 1 = \frac{\Delta^2 f(0)}{2!} & & \\
 & 0 & & & \\ \hline
 & 1 = \frac{\Delta^3 f(0)}{3!} & & &
\end{array}$$

$$\therefore\ p(x) = x^{(3)} + x^{(2)} + 2x^{(1)} - 1 .$$

At this point, given $F(x)$, we can compute $\Delta F(x) = f(x)$. Consider the reverse problem; that is, given $f(x)$, can we find $F(x)$ such that $\Delta F(x) = f(x)$?

Definition 3. Let $f(x)$ be given. If there exists a function $F(x)$ such that $F(x)$ = fix), then $F(x)$ is called the finite *integral* of fix) and we write

$$F(x) = \Delta^{-1}f(x) .$$

Theorem 4. Let $f(x)$ and $g(x)$ be functions and c a constant. Then:

1. $\Delta^{-1}[f(x) + g(x)] = \Delta^{-1}f(x) + \Delta^{-1}g(x)$
2. $\Delta^{-1}cf(x) = c\Delta^{-1}f(x)$
3. $\Delta^{-1}[f(x)\Delta g(x)] = f(x)g(x) - \Delta^{-1}[g(x+1)\Delta f(x)]$.

From our previous theorems and examples on differencing, we know the following:

1. $\Delta^{-1}a^x = \dfrac{a^x}{a-1}, \quad a \neq 1.$
2. $\Delta^{-1}[(a+bx)^{(n)}] = \dfrac{(a+bx)^{(n+1)}}{b(n+1)}$.
3. $\Delta^{-1}[x^{(n)}] = \dfrac{x^{(n+1)}}{n+1}$.

We are now ready to tackle the problem of finding a compact formula for the sum of n terms of a given series. Let $f(x)$ be a function and suppose we wish to find $\Sigma_{i=0}^{n} f(i)$. Suppose we can find a function $g(x)$ such that $\Delta g(x) = f(x)$. Then $g(x+1) - g(x) = f(x)$. Hence:

$$\begin{aligned} g(1) - g(0) &= f(0) \\ g(2) - g(1) &= f(1) \\ &\vdots \\ g(n+1) - g(n) &= f(n) . \end{aligned}$$

Adding these n equations we obtain:

$$\sum_{i=0}^{n} f(i) = g(n+1) - g(0) = g(x)\Big|_0^{n+1} ,$$

or in general

$$\sum_{i=a}^{n} f(i) = g(x)\Big|_a^{n+1} .$$

The summation problem thus becomes a problem in finite integration!

Example. Find $\Sigma_{i=1}^{n} (i^2 + 4i + 3)$.

$$\begin{aligned} \sum_{i=1}^{n} (i^2 + 4i + 3) &= \Delta^{-1}(x^2 + 4x + 3)\Big|_1^{n+1} \\ &= \Delta^{-1}[x^{(2)} + 5x^{(1)} + 3]\Big|_1^{n+1} \\ &= \left[\frac{x^{(3)}}{3} + \frac{5x^{(2)}}{2} + 3x^{(1)}\right]_{1}^{n+1} \\ &= \frac{2n^3 + 15n^2 + 31n}{6} \end{aligned}$$

Thus, given any sum where the general term is a polynomial, use Newton's theorem to express the polynomial in factorial form, and then use the formulas to find the finite integral.

Example. Find $\Sigma_{i=1}^{n} i2^i$.

$$\sum_{i=1}^{n} i2^i = \Delta^{-1}[x2^x]\Big|_1^{n+1}$$

In order to evaluate this finite integral we need to use integration by parts:

$$\Delta^{-1}[f(x)\Delta g(x)] = f(x)g(x) - \Delta^{-1}[g(x+1)\Delta f(x)].$$

Let $f(x) = x$ and $\Delta g(x) = 2^x$; then $\Delta f(x) = 1$ and $g(x+1) = 2^{x+1}$, $g(x) = 2^x$. Hence

$$\begin{aligned} \sum_{i=1}^{n} i2^i &= [x2^x - \Delta^{-1}(2^{x+1})]\Big|_1^{n+1} \\ &= (x2^x - 2^{x+1})\Big|_1^{n+1} \\ &= 2^{n+1}(n-1) + 2. \end{aligned}$$

Example. $\Sigma_{i=1} \dfrac{2i-1}{2^{i-1}}$.

From our knowledge of differencing we assume that

$$\Delta^{-1}\frac{2x-1}{2^{x-1}} = \frac{f(x)}{2^{x-1}} .$$

This implies that $\frac{f(x+1)}{2^x} - \frac{f(x)}{2^{x-1}} = \frac{2x-1}{2^{x-1}}$ or

$$f(x+1) - 2f(x) = 4x - 2.$$

$f(x)$ must be linear, so assume $f(x) = ax + b$. Substituting in above we obtain $-ax + a - b = 4x - 2$, and so $a = -4$ and $b = -2$. Thus

$$\sum_{i=1}^{\infty} \frac{2i-1}{2^{i-1}} = \lim_{n\to\infty}\left[\Delta^{-1}\frac{2x-1}{2^{x-1}}\right]\Bigg|_1^{n+1}$$

$$= \lim_{n\to\infty}\left[-\frac{(2x+1)}{2^{x-2}}\right]\Bigg|_1^{n+1}$$

$$= \lim_{n\to\infty}\left[-\frac{2n+3}{2^{n-1}} + 6\right]$$

$$= 6.$$

In this paper we have attempted to give the reader a brief introduction to the calculus of finite differences and to one of the applications in the area. For further study, the interested reader is referred to the references given. The books by Miller and Richardson make excellent texts for independent study, while the book by Boole is of a more advanced nature.

REFERENCES

1. Boole, George, *Calculus of Finite Differences*, 4th Ed., Chelsea Publishing Co., New York, 1958.
2. Hiller, Kenneth S., *An Introduction to the Calculus of Finite Differences and Difference Equations*, Henry Holt & Co., New York, 1960.
3. Richardson, C. H., *An Introduction to the Calculus of Finite Differences*, D. Van Nostrand Co., Princeton, New Jersey, 1954.

BOOK REVIEWS

Edited by Roy B. Deal
Oklahoma University Health Sciences Center

Niels Henrik Abel, Mathematician Extraordinary. By *Oystein Ore.* Chelsea Publishing Company, New York. 1974. (First edition 1957). 274 pages. $8.50.

One of the bonuses in studying mathematics is the deeper appreciation for the literature on the lives of some of its creative geniuses. Few, if any, have had more impact, or have been more creative, than Abel. The excellent writing and scholarship of Oystein Ore makes this book one that every *Pi Mu Epsilon Journal* reader should enjoy.

Stochastic Analysis. *By O. G. Kendall and E. F. Harding (Editors).* John Wiley and Sons, Inc., New York. 1973. xiii + 465 pages. $29.95.

Stochastic Geometry. *By E. F. Harding and V. G. Kendall (Editors).* John Wiley and Sons, Inc., New York. 1974. xiii + 400 pages. $29.95.

These two volumes were assembled as a tribute to the memory of the brilliant young mathematical statistician Rollo Davidson who died prematurely in a mountain climbing accident. The editors have gone to great lengths to provide all the material necessary for an up-to-date account of these subjects. They provide an excellent introduction in reference work for one prepared to do serious research in these areas.

Stochastic Differential Equations, Theory and Applications. *By Ludwig Arnold.* John Wiley and Sons, Inc., New York. 1974. xvi + 228 pages. $17.95.

"This book is geared for mathematicians, physicists, engineers, economists, and all those interested in noise problems in dynamical systems." For those with the graduate level mathematics background in these applied areas, including some probability theory, this book provides a textbook, introduction and reference work to this currently popular subject, including the use and exposition of It6 integrals.

Uniform Distribution of Sequences. *By L. Kuipers and H. Niederreiter.* John Wiley and Sons, Inc., New York. 1974. xiv + 390 pages. $24.50.

The basic theory on this subject first appeared in a famous paper by Hermann Weyl in 1916 and dealt mainly with the fractional parts of real numbers in the unit interval (0,1). This scholarly work surveys the development from then to now, listing over 900 references, and shows some of the fascinating ramifications into such topics as probability theory, ergodic theory and topological algebra. The extensive list of exercises serves to complement the text, along with the historical notes, in surveying the literature. It should provide a graduate student or mathematician from a related field with an excellent preparation for doing research in this area.

Characterization Problems in Mathematical Statistics. *By A. M. Kagan, Yu. V. Linnik and C. Radhakrishna Rao.* John Wiley and Sons, Inc., New York. 1973. xii t 499 pages. $22.50.

'This extensive work on mathematical statistics deals with problems in estimation, testing hypotheses, linear models, factor analysis, sequential estimation, among others, by systematically studying the important properties of the families of parent distributions and their relations to the distributions of the statistics used in the statistical inferences." Many characterizations of classes of distributions are given along with some analysis techniques of interest in their own right. The approaches should lead to some more interesting research.

Fundamentals of Queueing Theory. *By Donald Gross and Carl M. Harris.* John Wiley and Sons, Inc., New York. 1974. xvi + 556 pages. $22.50.

'This book provides a comprehensive and current treatment of queueing theory--from the development of standard queueing models and general queueing methodology, to applications and implementation in industry and government. The undergraduate background of most engineering, physical science, and mathematics majors, as well as some economics, business administration, and social science majors, would be adequate." Besides the large variety of models and bibliography, there are discussions of statistical inference and design and control of queues and discussion of various aspects of simulation models including Monte-Carlo generation, bookkeeping aspects, simulation programming languages and statistical consideration.

Traffic Science. *Edited by Denos C. Gazis.* John Wiley and Sons, Inc., New York. 1974. viii + 293 pages. $19.95.

For those with similar backgrounds and interests required for the previous book, this is "everything you wanted to know about traffic and were afraid to ask." With interesting details and extensive references, five experts have covered the subject very well in the following four chapters: Flow Theories, Delay Problems for Isolated Intersections, Traffic Control--Theory and Application, and Traffic Generation, Distribution, and Assignment.

Handbook of Applied Mathematics, Selected Results and Methods. *Edited by Carl E. Pearson.* Van Nostrand Reinhold Company, New York. 1974. xiii + 1265 pages. $37.50.

Twenty authors have combined to provide basic mathematical tools for the engineering and science type applications of mathematics from high school algebra to spectral theory. Although any reader will have some of the material at his command, he may find it handy to have it available in a reference work that is almost certain to have details on some subjects that he has never seen and may well need. In this way he is likely to find any background for the more advanced subjects in this same book. Some of the advanced subjects are quite thoroughly covered but there are always good references if additional material is needed. Much of the material is classical from a mathematical point of view but still useful, and thus current, in many applications. This may cause a modern student to have some difficulty with the tensor analysis since the references are also mainly classical. He may find it a useful reference, however. Coverage is indicated by the following list of chapters and their authors: Formulas from Algebra, Trigonometry and Analytic Geometry by H. Lennart Pearson, Elements of Analysis by H. Lennart Pearson, Vector Analysis by Gordon C. Dates, Tensors by Bernard Budiansky, Functions of a Complex

Variable by A. Richard Seebass, Ordinary Differential and Difference equations by Edward R. Benton, Special Functions by Victor Barcilon, First Order Partial Differential Equations by Jirair Kevorkian, Partial Differential Equations of Second and Higher Order by Carl E. Pearson, Integration, Linear Operators, Spectral Analysis by Frank H. Brownell, Transform Methods by Gordon E. Latta, Asymptotic Methods by Frank W. J. Olver, Oscillations by Richard E. Kronauer, Perturbation Methods by G. F. Carrier, Wave Propagation by Wilbert Lick, Matrices and Linear Algebra by Tse-Sun Chow, Functional Approximation by Robin Esch, Numerical Analysis by A. C. R. Newbery, Numerical Solution of Partial Differential Equations by Burton Wendroff, Optimization Techniques by Juris Vagners, Probability and Statistics by L. Fisher.

Linear and Nonlinear Waves. *By G. B. Whitham.* John Wiley and Sons, Inc., New York. 1974. xvi + 636 pages. $22.50.

For students in applied mathematics, engineering, physics, or geophysics with a mathematical background including such subjects as transform techniques, asymptotic expansion of integrals, solutions of standard boundary value problems, and related topics, this is a beautiful, modern, thorough and well written discussion of the subject. It covers a wide variety of applications and emphasizes throughout the relationships and results of the non-linear theories. It should be fascinating to a wide variety of readers with applied mathematical interests.

Computational Methods for Matrix Eigenproblems. *By A. R. Gourlay and G. A. Watson.* John Wiley and Sons, Inc., New York. 1973. xi + 132 pages. $9.95.

For mathematics, physics, engineering or other students having a need for the computational aspects of matrix eigenproblems, and who have had an elementary course in matrix theory, this is a excellent introduction to the subject. For those who do not have a separate course available in such problems, this book should make a good complement to a course in numerical analysis or applied mathematics or a basis for a reading course.

WELCOME TO NEW CHAPTERS

The *Journal* welcomes the following new chapters of Pi Mu Epsilon which were recently installed:

FLORIDA ETA at the University of North Florida at Jacksonville, installed May 22, 1974 by Houston Karnes, Council President.

GEORGIA GAMMA at Armstrong State College, installed April 2, 1974, by Houston Karnes, Council President (also reported in the last issue).

MISSISSIPPI BETA at Mississippi College, installed May 7, 1974, by Houston Karnes, Council President.

NORTH CAROLINA ZETA at the University of North Carolina at Wilmington, installed May 16, 1974 by Houston Karnes, Council President.

PENNSYLVANIA NU at Edinboro State College, installed May 4, 1974, by Eileen Poiani, councilor (also reported in the last issue).

TEXAS THETA at the University of Houston, installed October 3, 1974, by Houston Karnes, Council President.

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PROBLEM DEPARTMENT

Edited by Lwn Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall, publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if available, and by any information that will assist the editor.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of November 1975.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

338. *Proposed by Hung C. Li, Southern Colorado State College.*

Let $(O)a$ be a circle centered at O with radius a. Let P, any point on the circumference of (O), be the center of circle (P). What is the radius of (P) such that it divides the area of (0) into two regions whose areas are in the ratio $s:t$?

339. *Proposed by Paul Erdös, Budapest, Hungary.*

Let $a_1 < a_2 < \cdots$ be a sequence of integers $(a_i, a_j) = 1$; $a_{i+2} - a_{i+1} \geq a_{i+1} - a_i$. Prove that $\sum \frac{1}{a_i} < \infty$.

340. *Proposed by Charles W. Trigg, San Diego, California.*

The arithmetic mean of the twin primes 17 and 19 is the heptagonal number 18. Heptagonal numbers have the form $n(5n - 3)/2$. Are there any other twin primes with a heptagonal mean?

341. *Proposed by Jack Garfunkel, Forest Hills High School, New York.*

Prove that the following construction trisects an angle of 60°. Triangle **ABC** is a 30°-60°-90° right triangle inscribed in a circle. Median CM is drawn to side AB and extended to M' on the circle. Using a marked straightedge, point N on AB is located such that CN extended to N' on the circle makes $NN' = MM'$. Then CN trisects the 60° angle ACM.

342. *Proposed by David L. Silverman, West Los Angeles, California.*

In *The Game of the Century* two players alternately select dates of the Twentieth Century (1 January 1901 - 31 December 2000) subject to the following restrictions;

1. The first date chosen must be in 1901.

2. Following the first play, each player, on his turn, must *advance* his opponent's last date by changing exactly one of the three "components" (day, month, year).

3. Impossible dates such as 31 April or 29 February of a non-leap year are prohibited.

The player able to announce 31 December 2000 is the winner.

a. What are the optimal responses by the second player to first player openings of 4 July 1901? 25 December 1901?

b. Who has the advantage and what is the optimal strategy?

c. What is the maximum number of moves that can occur if both players play optimally?

343. *Proposed by R. Robinson Rowe, Sacramento, California.*

Current serious promotion of a tunnel under the English Channel, combined with the energy crunch, has renewed interest in a fall-through tunnel under Bering Strait. From Cape Prince of Wales on Alaska's Seward Peninsula to Mys Dezhneva (East Cape) on Siberia's Chukuski Peninsula is 51 miles. A straight tunnel 58 miles long could be driven in earth below the bed of the Strait, which is 20 fathoms deep near each shore and 24 fathoms near mid-Strait. A frictionless vehicle could 'fall' through such a tunnel without motive power. How long would it take? (At latitude 66° North, the earth's radius is 3954 miles and the acceleration of gravity, g = 32.23 ft/sec^{-2}.)

344. *Proposed by J. A. H. Hunter, Toronto, Canada.*

Three circles whose radii are a, b and c are tangent externally in pairs and are enclosed by a triangle each side of which is an extended tangent of two of the circles. Find the sides of the triangle.

345. *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

Resolve the paradox:

$$i(\sqrt{i} + \sqrt{-i}) = i\sqrt{i} + i\sqrt{-i} = \sqrt{-i} + \sqrt{i} = \sqrt{i} + \sqrt{-i}\ .$$

346. *Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

The internal angle bisectors of a convex quadrilateral ABCD enclose another quandrilateral EFGH. Let FE and GH meet in M and let GF and HE meet in N. If the internal bisectors of angles EMH and ENF meet in L, show that angle NLM is a right angle.

347. *Proposed by Joe. Van Austin, Emory University, Atlanta, Georgia.*

It is easy to show that $f(x) = \dfrac{\sin x}{x} - \dfrac{99x}{4} + 1$ for $x > 0$,

(i) has a linear asymptote $y = -\dfrac{99x}{4} + 1$, and

(ii) $f(x)$ crosses this asymptote for all $x = n\pi$ for $n = 1,2,\cdots$.

Show that the derivative $f'(x)$ is never zero for $x > 1$.

348. *Proposed by Bob Prielipp and N. J. Kuenzi, The University of Wisconsin-Oshkosh.*

When the digits of the positive integer N are written in reverse order, the positive N' is obtained. Let $N + N' = S$. Then S is called the sum after one reversal addition. A palindromic number is a positive integer that reads the same from right to left as it does from left to right. The nth triangular number $T_n = n(n+1)/2$, $n = 1,2,3,\cdots$.

Prove that there are infinitely many triangular numbers which have a palindromic sum after one reversal addition in the base b, where b is an arbitrary positive integer ≥ 2.

349. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

If 2 $(n > 1)$ is the highest power of 2 dividing an even perfect number m, prove that $\sigma(m^2) + 1 \equiv 0 \pmod{2^{n+1}}$, where $\sigma(m)$ denotes the sum of the divisors of m.

SOLUTIONS

292. [Spring 1973; Spring 1974] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

If perpendiculars are constructed at the points of tangency of the incircle of a triangle and extended outward to equal lengths, then the joins of their endpoints form a triangle perspective with the given triangle.

I. Revised solution and *comments* by Clayton W. Dodge, *Orono,* Maine.

The solution published in the Spring 1974 issue is invalid since in general, perspective is not transitive. The following solution avoids that error.

Let AP cut BC at S and let T and U be the corresponding points for BQ and CR (Figure 1). Let D be the foot of the altitude from A to BC, F the area of triangle ABC, a, b, c the lengths of its sides, and s its semiperimeter. Let m be the common length of PX, QY and RZ. Then $AD = 2F/a$. From similar triangles ASD and PSX, $DS/SX = AD/PX = 2F/am$, whence $SX/DX = SX/(DS + SX) = am/(2F + am)$. From right triangle ABD, $BD = (c^2 - AD^2)^{1/2} = (1/a)(a^2c^2 - 4F^2)^{1/2}$. Since $BX = s - b$, then $DX = BX - BD$ (in this figure), so

$$SX = \frac{am}{2F + am}(s - b) - \frac{1}{a}(a^2c^2 - 4F^2)^{1/2}\ ,$$

$$BS = BX - SX = \frac{2F(s - b) + m(a^2c^2 - 4F^2)^{1/2}}{2F + am}$$

with similar expressions for CS, CT, AT, AU and BU.

Since S, T, U all divide sides BC, CA and AB internally, it is easy to see that $BS\cdot CT\cdot AU/CS\cdot AT\cdot BU = +1$, whence the three Cevians AP, BQ, CR concur by Ceva's theorem. Their point of concurrence is the center of perspective for triangles ABC and PQR.

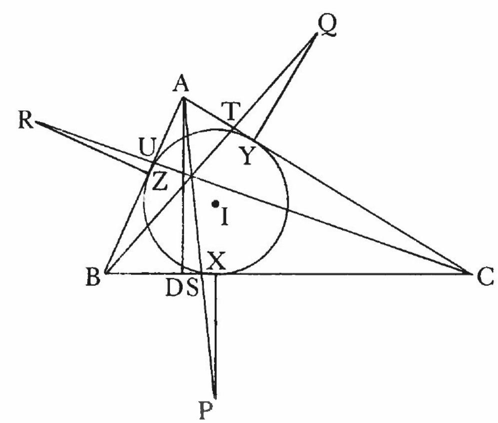

FIGURE 1

If the perpendiculars are erected inward instead of outward, the proof holds with m replaced by $-m$. In this case one must also consider whether any of the points S, T, U are external and adjust the signs of the affected segments accordingly.

II. Comment by Howard Eves, University of Maine at Orono.

Although homothety is transitive, perspective in general is not. In the figure, lines BP and BR are isogonal conjugates with respect to vertex B of triangle ABC, since triangles BPX and BRZ are congruent by SAS. Similarly for the lines CP and CQ and for lines AQ and AR. Now it is known that if at each vertex of a given triangle ABC a pair of isogonal lines be drawn, then the triangle $A'B'C'$, whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle. (See Howard Eves, Concerning some perspective triangles, Amer. Math. Monthly, 51 (1944), 324-331.) This theorem establishes the desired result.

III. Solution by the Proposer.

Let D, E, F, denote the intersections of AP, BQ, and CR with the sides BC, CA and AB respectively. Through P, Q and R draw lines parallel to BC, CA and AB terminated by JK, LM and NO as in Figure 2.

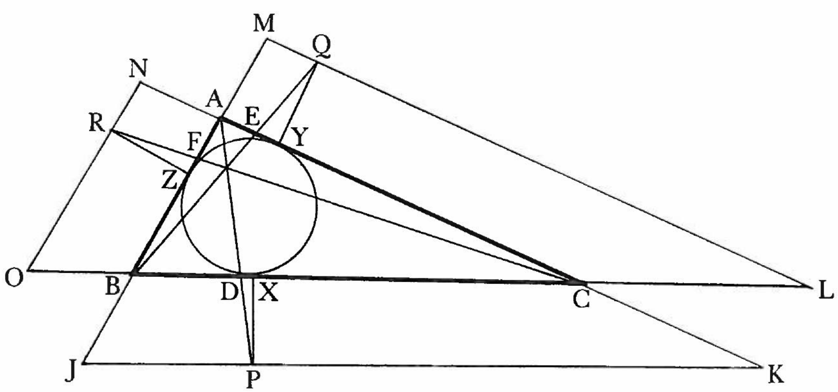

FIGURE 2

By the congruence of quadrilaterals BJPX and ZROB, we establish the equality of RO and JP. Similarly PK = LQ and QM = NR. Hence $JP \cdot LQ \cdot NR / RO \cdot PK \cdot QM = 1$. Then $(NR/RO)(JP/PK)(LQ/QM) = 1$. It follows by similar triangles that $(AF/FB)(BD/DC)(CE/EA) = 1$. Hence triangles ABC and PQR are in perspective.

313. [Fall 1973; Fall 1974 (corrected)] *Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.*

Give an elementary proof that

$$(1 + 8\cos^2 A)(1 + 8\cos^2 B)(1 + 8\cos^2 C) \geq 64\sin^2 A \sin^2 B \sin^2 C,$$

where A, B, C are the angles of an acute triangle ABC.

Remark. J. Gillis gave a proof using calculus techniques in Problem E 2119, American Mathematical Monthly, (1969), p. 831.

Solution by the Proposer.

Expand the given inequality into the form

$$64\cos^2 A\cos^2 B\cos^2 C + 8(\cos^2 A + \cos^2 B + \cos^2 C) \geq 7$$

Now using the known identity

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2(\cos A\cos B\cos C)$$

for the angles of an arbitrary triangle, we obtain

$$(8\cos A\cos B\cos C - 1)^2 \geq 0$$

with equality only if A = B = C.

Comment by the Problem Editor.

As mentioned in the Fall 1974 issue, Victor G. Feser of St. Louis, Missouri and R. Robinson Rowe of Sacramento, California found the erroneous version unsolvable. Rowe subsequently offered an algebraic proof.

314. [Spring 1974] *Proposed by J. A. H. Hunter, Toronto, Canada.*

Show that

$$\frac{\sin^2 45^\circ - \sin^2 15^\circ}{\sin^2 30^\circ - \sin^2 10^\circ} = \frac{\sin 80^\circ}{\sin 30^\circ}$$

Solution by Zazou Katz, Beverly Hills, California.

$$\frac{\sin^2 45^\circ - \sin^2 15^\circ}{\sin^2 30^\circ - \sin^2 10^\circ} = \frac{(\sin 45^\circ + \sin 15^\circ)(\sin 45^\circ - \sin 15^\circ)}{(\sin 30^\circ + \sin 10^\circ)(\sin 30^\circ - \sin 10^\circ)}$$

$$= \frac{\sin 30^\circ \cos 15^\circ \sin 15^\circ \cos 30^\circ}{\sin 20^\circ \cos 10^\circ \sin 10^\circ \cos 20^\circ}$$

$$= \frac{\sin 60^\circ \sin 30^\circ}{\sin 20^\circ \sin 40^\circ}$$

Letting $\theta = 20^\circ$ in the formula

$$\sin 3\theta = 4 \sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) ,$$

we find that $\sin 60^\circ = 4 \sin 20^\circ \sin 40^\circ \sin 80^\circ$. Hence,

$$\frac{\sin^2 45^\circ - \sin^2 15^\circ}{\sin^2 30^\circ - \sin^2 10^\circ} = \frac{4 \sin 20^\circ \sin 40^\circ \sin 80^\circ \sin 30^\circ}{\sin 20^\circ \sin 40^\circ} = \frac{\sin 80^\circ}{\sin 30^\circ} .$$

Also solved by JEFFREY BERGEN, *Undergraduate, Brooklyn College, N. Y.*; ROBERT CALCATERRA, *Brooklyn, N. Y.*; TOMMY R. CHRISTIAN, *Louisiana Tech University, Ruston, Louisiana*; CLAYTON W. DODGE, *University of Maine at Orono*; R. C. GEBHARDT, *Hopatcogn, N. J.*; N. J. KUENZI, *The University of Wisconsin-Oshkosh*; CHARLES H. LINCOLN, *Raleigh, N. C.*; BOB PRIELIPP, *The University of Wisconsin-Oshkosh*; PAOLO RANALOI, *Akron, Ohio*; R. ROBINSON ROWE, *Sacramento, California*; GREGORY WULCZYN, *Bucknell University, Lewisburg, Pa.*; *and the Proposer.*

315. [Spring 1974] *Proposed by Charles W. Trigg, San Diego, California.*

One type of perpetual calendar consists of two white plastic cubes resting on a tilt-back base. On each face of each cube is a single digit. The digits are so distributed that the cubes can exhibit any date from 01 to 31 on their front faces.

Could this type of calendar be constructed if a base of numeration smaller than ten were employed?

I. Solution by R. Robinson Rowe, Sacramento, California.

The answer is "Yes". Bases 2, 3, 4 and 5 can be eliminated as requiring 3 or more digits, but for Base 6, each cube can be faced with the digits 0, 1, 2, 3, 4, 5, to exhibit the 31 days with the sequence 01, 02, 03, 04, 05, 10, 11, ···, 41, 42, 43, 44, 45, 50, 51.

II. Solution by Irwin Jungreis, Age 11, Brooklyn, N. Y.

No arrangement is possible for a radix less than 6 since the number 25 requires at least 3 digits for those, and only 2 6-face cubes are present.

Radix 6: 0, 1, 2, 3, 4, 5, and 0, 1, 2, 3, 4, 5.

Radix 7: 0, 1, 2, 3, 4, 5, and 1, 2, 3, 4, 5, 6.

Radix 8: 0, 1, 2, 3, 4, 5, and 0, 1, 2, 3, 6, 7.

Radix 9 and Radix 10 are ***impossible***, even though the problem presumes a solution for Radix 10, ***unless*** numbers less than 10 do not have a leading zero.

Proof for Radix 9: The numbers 11, 22 and 33 are required, using 6 of the available 12 faces. The 6 digits 0, 4, 5, 6, 7, 8 remaining preclude repeating any other digit. The 0 can be paired with only 6 of the 8 digits 1 through 8 so not all of the 01_9 to 08_9 can occur.

Proof for Radix 10: Ten separate digits and the need for 11 and 22 uses twelve faces so 01 through 09 cannot all occur.

Also solved by LOUIS H. CAIROLI, *John Carroll University, Cleveland, Ohio*; CLAYTON W. DODGE, *University of Maine at Orono*; VICTOR G. FESER, *St. Louis, Missouri*; R. C. GEBHARDT, *Hopatcong, N. J.*; ARTHUR M. KELLER, *New York Gamma, Brooklyn, N. Y.*; CHARLES H. LINCOLN, *Raleigh, N. C.*; BOB PRIELIPP, *The University of Wisconsin-Oshkosh*; THERESA PRATT, *N. Easton, Maine*; *and the Proposer.*

Feser, Keller, Beghardt and Dodge suggested solutions for Base 10 in which numbers less than 10 ***have*** a leading zero but with 9 being obtained by turning 6 upside down.

316. [Spring 1974] *Proposed by Zazou Katz, Beverly Hills, California.*

If you were marooned on a desert island without a calculator or tables of trigonometric functions, how would you go about determining which is greater:

$$2 \tan^{-1}(\sqrt{2} - 1) \text{ or } 3 \tan^{-1}(1/4) + \tan^{-1}(5/99) \text{ ?}$$

Comment by the Problem Editor.

Because of the remarkable similarity displayed by the nine correct solutions that were received, it would be appropriate to consider the published version an amalgam of those offered by the listed solvers.

Solution.

Let $A = \tan^{-1}(\sqrt{2} - 1)$, $B = \tan^{-1}(1/4)$, $C = \tan^{-1}(5/99)$.

Then
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2(\sqrt{2} - 1)}{1 - (3 - 2\sqrt{2})} = 1$$

$$\tan 2B = \frac{2 \tan B}{1 - \tan^2 B} = \frac{1/2}{1 - 1/16} = \frac{8}{15}$$

$$\tan 3B = \frac{\tan 2B + \tan B}{1 - \tan 2B \tan B} = \frac{47}{52}$$

Hence
$$\tan(3B + C) = \frac{\tan 3B + \tan C}{1 - \tan 3B \tan C} = 1.$$

Therefore, $3B + C = \tan^{-1}(1) = 2A$ and the expressions are equal.

Solved by JEFFREY BERGEN, *Brooklyn College, Brooklyn, N. Y.;* LOUIS H. CAIROLI, *Syracuse University, S. Euclid, Ohio;* ROBERT CALCATERRA, *Brooklyn, N. Y.;* CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *St. Louis, Missouri;* ROSALIE JUNGREIS, *Brooklyn, N. Y.;* CHARLES H. LINCOLN, *Raleigh, N. C.;* R. ROBINSON ROWE, *Sacramento, California;* GREGORY WULCZYN, *Bucknell University, Lewisburg, Pa.;* and *the Proposer.*

317. *[Spring 1974] Proposed by the Editor of the Problem Department.*

A rectangle ADEB is constructed externally on the hypotenuse AB of a right triangle ABC (Fig. 3). The lines CD and CE intersect the line AB in the points F and G respectively. a) If $DE = AD\sqrt{2}$, show that $AG^2 + FB^2 = AB^2$. b) If AD = DE, show that $FG^2 = AF \cdot GB$.

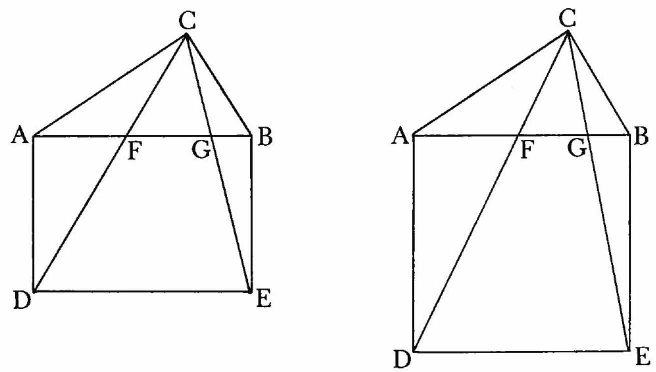

FIGURE 3

Solution by Leonard Barr, Beverly Hills, California.

At F and G erect perpendiculars to AB, cutting AC and CB in J and K. By triangle similarity

$$\frac{AD}{JF} = \frac{CD}{CF} = \frac{DE}{FG} \qquad \text{and} \qquad \frac{DE^2}{FG^2} = \frac{AD^2}{JF^2}$$

Also, $JF/AF = GB/KG = GB/JF$. Hence $JF^2 = AF \cdot GB$, and $FG^2/DE^2 = AF \cdot GB/AD^2$.

a) If $DE^2 = 2AD^2$, $FG^2 = 2AF \cdot GB = 2AF(FB - FG)$ or $FG^2 + 2AF \cdot FG = 2AF \cdot FB$. Then $FG^2 + 2AF \cdot FG + (AF^2 + FB^2) = 2AF \cdot FB + (AF^2 + FB^2)$ or $(FG + AF)^2 + FB^2 = (AF + FB)^2$, and $AF^2 + FB^2 = AB^2$.

b) If DE = AD, $FG^2 = AF \cdot GB$.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* ZAZOU KATZ, *Beverly Hills, California;* ALIZA DUBIN, *Far Rockaway, New York;* CHARLES H. LINCOLN, *Raleigh, N. C.;* R. ROBINSON ROWE, *Sacramento, Calif.;* and GREGORY WULCZYN, *Bucknell University, Lewisburg, Pa.*

318. *[Spring 1974] Proposed by R. Robinson Rowe, Sacramento, California.*

Two equal cylindrical tanks, Tank A above Tank B, have equal orifices in their floors, capable of discharging water at the rate of $13\sqrt{h}$ gallons per minute, where h is the depth of water in feet. At 10:20 a.m. Tank B is empty and water is 10 feet deep in Tank A, as discharge begins. At noon Tank A is just emptied. What was the maximum depth in Tank B, and when? How deep is the water in Tank B at noon, and when will it be empty?

Solution by the Proposer.

Let the varying depth of water be a in Tank A and b in Tank B. Let m be the unit volume of each tank in gallons per foot of depth. Let q be the discharge of either tank in feet of depth per minute. Let t be the elapse of time in minutes.

Then for Tank A:

$$q = -\frac{da}{dt} = \frac{13}{m}\sqrt{a} \tag{1}$$

And for Tank B, from inflow minus outflow:

$$q = \frac{db}{dt} = \frac{13}{m}(\sqrt{a} - \sqrt{b}) \tag{2}$$

Then, dividing (2) by (1):

$$-\frac{db}{da} = 1 - \sqrt{b/a} \tag{3}$$

Now let $b = au^2$ so that (3) becomes $db = -(1 - u)da$ (4)

Differentiating, $db = u^2 da + 2audu = -(1 - u)da$

$$2audu = -(1 - u + u^2)da\ .$$

Whence, to integrate:

$$\frac{2udu}{1 - u + u^2} = -\frac{da}{a} \tag{5}$$

$$\log(1 - u + u^2) + \frac{2}{\sqrt{3}}\tan\frac{2u - 1}{\sqrt{3}} = -\log a + C$$

When $a = 10$, $b = u = 0$, $C = \log 10 - \pi/3\sqrt{3}$. Then:

$$\log \frac{10}{a(1 - u + u^2)} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}}{2 - u} \tag{6}$$

$$a = \frac{10}{1 - u + u^2} \exp\left(- \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}}{2 - u}\right) . \tag{7}$$

Thus, lacking an explicit relation between a and b, we can use (7) and (4) with any value of u and obtain a contemporaneous set (a,b). For what follows it will now be convenient to let the constant

$$\exp(-\pi/3\sqrt{3}) = k = 0.546293016 . \tag{8}$$

To find the maximum value of b, it should be obvious that b will increase with time as shown in (2) until b = a, when outflow equals inflow, then decrease. Hence at this maximum, $u = 1$ and from (7)

$$b = a = 10k^2 = 2.9843\ 6059 \text{ feet} . \tag{9}$$

To find the depth in Tank B at noon, we have $a = 0$ and $u = \infty$, which cannot be substituted in (7), but in (6) $a(1 - u + u^2) = b$ and $u\sqrt{3}/(2 - u) = -\sqrt{3}$ whence

$$b = 10k^4 = 0.8906\ 4081 \text{ feet} . \tag{10}$$

For determination of time of two events, it will be convenient to set $t = 0$ at noon. Then from (1)

$$dt = - \frac{m\,da}{13\sqrt{a}} \tag{11}$$

$$t = - \frac{2m\sqrt{a}}{13} + C .$$

When $a = 0$, $t = 0$, so $C = 0$. When $a = 10$, $t = -100$, $m = 65\sqrt{10}$. Thus,

$$t = -10\sqrt{10a} . \tag{12}$$

So when b is a maximum and $a = b = 10k^2$,

$$t_1 = -100k = -54.6293016 \text{ minutes} \tag{13}$$

and the clock time is

$$T_1 = 11{:}05{:}22.2419 \text{ am} . \tag{14}$$

For the run-out of Tank B after noon, we note that (12) is equivalent to saying that for any depth h the run-out time is $10\sqrt{10h}$. At noon, there is no longer any inflow into Tank B and the depth is $10k$. Hence the run-out time is

$$t_2 = 100k^2 = 29.8436\ 0592 \text{ minutes} \tag{15}$$

and the clock time is

$$T_2 = 12{:}29{:}50.6164 \text{ pm}. \tag{16}$$

Comment.

Note that the 4 required quantities, given in (9), (10), (13) and (15) are respectively $10k^2$, $10k^4$, $-100k$ and $100k^2$ and that (8) defines k in a closed form. I believe these are the only items of the tandem tank problem which can be so simply expressed.

I purposely put 13 in the text of the problem as a red herring. It cannot be used without some other data like m, but this can be computed from the runout time as in deriving (12).

Also solved by R. C. GEBHARDT, Hopatcong, N. J.

319. [Spring 1974] *Proposed by Professor M. S. Longuet-Higgins, Cambridge, England.*

Let A', B', C' be the images of an arbitrary point in the sides BC, CA, AB of a triangle ABC. Prove that the four circles $AB'C'$, $BC'A'$, $CA'B'$, ABC are all concurrent.

I. *Solution by R. Robinson Rowe, Sacramento, California.*

The general triangle can be oriented with any one of its vertices, A, at a Cartesian origin, another, C, on the positive x-axis, and the third, B, with a positive ordinate. Then if it can be shown that the circumcircle ABC and two image circles $AB'C'$ and $CA'B'$ are concurrent at some point Q, it would follow that the triangle could be reoriented with B at the origin and A on the x-axis to find circumcircle ABC and image circles $BC'A'$ and $AB'C'$ also concurrent and necessarily at the same point Q.

Hence with full generality, let the coordinates be $A = (0,0)$, $C = (r,0)$, $B = (s,t)$ and the arbitrary point $P = (p,q)$. Then the equation of the circumcircle ABC is:

$$x^2 + y^2 - rx + \left(\frac{rs - s^2 - t^2}{t}\right)y = 0 . \tag{1}$$

The image points of P reflected in AC and BC are: $B' = (p,-q)$ (2) and using the equations for

$$AB:\quad sy - tx = 0 \tag{3}$$
$$PC':\quad sx + ty - sp - tq = 0 , \tag{4}$$

$$C' = \left(\frac{ps^2 - pt^2 + 2stq}{s^2 + t^2}, \frac{qt^2 - qs^2 + 2stp}{s^2 + t^2}\right) . \qquad (5)$$

The general form for a circle through the origin is

$$x^2 + y^2 + ax + by = 0 . \qquad (6)$$

Substituting the coordinates of B' and C' for x and y and solving for the coefficients a and b derives the equation of circle $AB'C'$,

$$x^2 + y^2 - \left(\frac{sp + tq}{s}\right)x + \left(\frac{sq - tp}{s}\right)y = 0 . \qquad (7)$$

Solving (1) and (7) simultaneously for the two intersections of circles ABC and $AB'C'$ checks that one is at (0,0). The other at point Q has the coordinates:

$$x_Q = (s^3 + st^2 + qst - pt^2 - rs^2)F$$
$$y_Q = t(qt + ps - rs)F \qquad (8)$$

in which F is the fraction

$$F = \frac{qt + ps - pr}{(rs - s^2 - qt)^2 + (st - pt)^2}$$

Next, by steps analogous to (2) to (7), the equation of image circle $CA'B'$ is derived as

$$x^2 + y^2 - \frac{(r + p)(r - s) - qt}{r - s}x + \frac{q(r - s) - t(r - p)}{r - s}y$$
$$+ \frac{pr(r - s) - qrt}{r - s} = 0 \qquad (10)$$

Finally, substitution of the coordinates of Q in (8) confirms that Q is indeed a point on image circle $CA'B'$ with its equation (1). Using the argument of the opening paragraph, this completes the proof.

II. *Solution by Clayton W. Dodge, University of Maine at Orono.*

Inscribe the triangle in the unit circle centered at the origin of the Gauss plane. From the opposite similarity of triangles $A'BC$ and PBC, we obtain

$$a' = b + c - \bar{p}bc$$

with similar expressions for points B' and C'. Letting Z denote the second point of intersection of circles ABC and $AB'C'$, we have

$$|a| = |b| = |c| = |z| = 1$$

and

$$\begin{vmatrix} z\bar{z} & z & \bar{z} & 1 \\ a\bar{a} & a & \bar{a} & 1 \\ b'\bar{b}' & b' & \bar{b}' & 1 \\ c'\bar{c}' & c' & \bar{c}' & 1 \end{vmatrix} = 0 .$$

Under the assumption that A, B, C, Z, *and* P are distinct points and with the aid of considerable algebra, we solve this determinant for a, obtaining

$$z = abc\,\frac{\bar{a} + \bar{b} + \bar{c} - \bar{p}}{a + b + c - p} .$$

Since this expression is symmetric in A, B, and C, it follows that Z lies on the other two stated circles also.

Observe that there is no need for the point P to lie inside the triangle; it is necessary only that P not coincide with a vertex. Of course, for some positions of P, one or more of the given circles become straight lines, but the proof holds for these cases too.

III. *Comment by Howard Eves and Clayton Dodge, University of Maine at Orono.*

Call the point of concurrence the *Longuet-Higgins* point L for the arbitrary point P with respect to triangle ABC, and inscribe triangle ABC in the unit circle centered at the origin rotated so that the orthocenter H lies on the real axis. Then $h = a + b + c$ is real and

$$\ell = abc\,\frac{\bar{a} + \bar{b} + \bar{c} - \bar{p}}{a + b + c - p} .$$

In H. Eves, *A Survey of Geometry*, Vol. 2 (Allyn and Bacon, 1965), Theorems 12.4.15 to 12.4.17 show that if $f = abc$, then F is the in-Feuerbach point for the tangential triangle; that is, F is the point of tangency of circle ABC and the ninepoint circle of the triangle whose sides are tangent to circle ABC at A, B, and C respectively. Since we have taken $h = a + b + c$ real, then the real axis is the Euler line for triangle ABC. If point P lies on the real axis, then its Longuet-Higgins point L is given by $\ell = abc$ since $a + b + c - p$ is real. That is, the Longuet-Higgins point for any-point on the Euler line is the in-Feuerbach point for the tangential triangle. Furthermore, the complex representation for L shows that the locus of all points P whose Longuet-Higgins point is a given point L on circle

ABC is a line through H (and not just that portion of the line inside circle *ABC*); if HP makes an angle θ with the Euler line, then L is rotated through angle -2θ from F. Hence, any point on circle ABC is the Longuet-Higgins point for some line of points through H. Observe that, if P = H, then all four stated circles coincide, so every point on circle *ABC* is a Longuet-Higgins point for H.

Comment by the Problem Editor.

The interested reader would do well to refer to the following articles pertaining to this fascinating problem:

1. M. S. Longuet-Higgins, Reflections on a Triangle, Mathematical Gazette, Vol. 57, No. 402 (1973), 293-296.

2. M. S. Longuet-Higgins, Reflections on Reflections, Mathematical Gazette, Vol. 58, No. 406 (1974), 257-263.

3. S. N. Collings and H. Martyn Cundy, Reflections on Reflections, Mathematical Gazette, Vol. 58, No. 406 (1974), 264-272.

320. [Spring 1974] Proposed by H. S. M. Coxeter, Toronto, Canada.

Prove that the projectivity $ABC \barwedge BCD$ (for four collinear points) is of the period 4 if and only if $H(AC, BD)$.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

Since $ABC \barwedge BCD$, the projectivity maps A to B, B to C, and C to D.

If the period is 4, then we must have that D maps to A, so $ABCD \barwedge BCDA$. Since a projectivity preserves cross ratio, then

$$(AB, CD) = (BC, DA) .$$

$$\frac{AC}{CB}\cdot\frac{DB}{AD} = \frac{BD}{DC}\cdot\frac{AC}{BA} ,$$

$$DC\cdot BA = -CB\cdot AD ,$$

$$\frac{CD}{DA}\cdot\frac{BA}{CB} = -1 ,$$

which is equivalent to $H(AC, BD)$. This argument reverses to establish the converse.

II. *Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pa.*

$$H(ABCD) \barwedge H(BCDX) \barwedge H(CDXY) \barwedge H(DXYZ) \barwedge H(XYZU)$$

The following analysis then applies:

From $H(ABCD) \barwedge H(BCDX)$, it follows that X ≡ A

From $H(BCDA) \barwedge H(CDAY)$, it follows that Y ≡ B

From $H(CDAB) \barwedge H(DABZ)$, it follows that Z ≡ C.

Hence $ABC \barwedge BCD$ is a projectivity of period 4.

321. [Spring 1974] Proposed by Nosmo King, Raleigh, North Carolina. (Dedicated to the memory of Leo Moser.)

According to Merten's Theorem

$$\prod_{p\le n}\left(1 - \frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\log n}$$

where γ denotes Euler's constant (0.57721...) and where the product on the left is taken over all primes not exceeding n. (See Hardy and Wright, The Theory of Numbers, p. 351, or Trygve Nagell's Introduction to Number Theory, p. 298). Can you estimate

$$\prod_{p\le n}\left(1 - \frac{2}{p}\right) ?$$

Solution by R. Robinson Rome., Sacramento, California.

This is a nicely concealed hoax: $\prod\left(1 - \frac{2}{p}\right) = 0\cdot\frac{1}{3}\cdot\frac{2}{5}\cdots = 0.$

Also solved by VICTOR G. FESER, St. Louis, Mo.; CHARLES H. LINCOLN, Raleigh, N. C.; and the Proposer.

322. [spring 1974] Proposed by Jack Garfunkel, Forest Hills High Schoot, New Yolk.

It is known that the ratio of the perimeter of a triangle to the sum of its altitudes is greater than or equal to $2/\sqrt{3}$. (See American *Mathematical Monthly*, Problem E 1427, 1961, pp. 296-297.) Prove the stronger inequality for the internal angle bisectors t_a, t_b and t_a:

$$2(t_a + t_b + t_c) \le \sqrt{3}(a + b + c)$$

equality holding if and only if the triangle is equilateral.

Solution by Louis H. Cairoli, Syracuse University, South Euclid, Ohio.

The value of the internal angle bisector t in terms of the sides of a triangle ABC is given by

$$t_a = \frac{2}{b + c}\sqrt{sbc(s - a)}$$

where s is the semiperimeter of the triangle. [See D. Kay, College *Geometry*, Holt, Rinehart and Winston, 1969, p. 199.] Combining this expression with the inequality $2\sqrt{bc} \le b + a$, we obtain $t_a \le \sqrt{s(s - a)}$, with the result that $t_a + t_b + t_c \leq \sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c})$. It is known that the sum of the radicals in parenthesis does not exceed $\sqrt{3s}$. [See O. Bottema et al., Geometries Inequalities, p. 16.] The stated result follows, with equality if and only if a = b = a.

Also solved by ZAZOU KATZ, *Beverly Hills, California and the Proposer. Katz pointed out that this problem was considered by Luis A. Santaló in his paper* Some Inequalities Between the Elements of a Triangle, *published in Math. Notae* 3 (1943), 65-73.

323. *[Spring 1974] Proposed by David L. Silverman, Los Angeles, California.*

Call plane curves such as the circle of radius 2, the square of side 4, or the 6 × 3 rectangle in Fig. 4 *isometric* if their perimeter is numerically equal to the area they enclose. What is the maximum area that can be enclosed by an isometric curve?

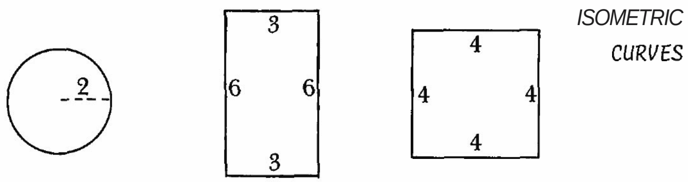

FIGURE 4

I. *Solution by N. J. Kuenzi and Bob Prielipp, The University of Wisconsin-Oshkosh.*

Given a positive real number N, there is an isometric curve which encloses an area greater than N. For example, consider the isometric rectangle whose adjacent sides have measures a and b where $2 \leq a < b$. Since $ab = 2a + 2b$, $a = 2b/(b-2)$. Hence $ab = 2b^2/(b-2)$, so the area enclosed increases without bound as b increases without bound. (A similar example can be obtained using an isometric right triangle.)

II. *Solution by Clayton W. Dodge, University of Maine at Orono.*

There is no upper limit to the enclosed area as seen in the rectangle of length $2 + x$ and width $(4 + 2x)/x$ with $x > 0$. The common value of its area and perimeter is $(8 + 8x + 2x^2)/x$, which increases without limit as x approaches zero.

Also solved by LOUIS H. CAIROLI, *Syracuse University, S. Euclid, Ohio;* VICTOR G. FESER, *St. Lam., Missouri;* R. C. GEBHARDT, *Hopatcong, N. J.;* JOHN TOM HURT, *Texas A & M University;* THEODORE JUNGREIS, *Brooklyn, N. Y.;* ARTHUR M. KELLER, *New Yohk Gamma;* CHARLES H. LINCOLN, *Raleigh, N. C;* SIDNEY PENNER, *Bronx Community College of CUNY;* PAOLO RANALDI, *Akron, Ohio; and* R. ROBINSON ROWE, *Sacramento, California.*

324. *[Spring 1974] Proposed by R. S. Luthar, University of Wisconsin, Janesville, Wisconsin.*

Evaluate

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \frac{j}{j+1}}{n}$$

I. *Solution by Henry J. Ricardo, Manhattan College, Bronx, N. Y.*

It is a well-known theorem due to Cauchy (see, for example, Knopp's Infinite Sequences and Series, p. 33) that if a sequence $\{a_k\}$ converges to L, then so does the sequence $\{\sum_{k=1}^{n} a_k/n\}$ of arithmetic means. Since $j/(j+1) \to 1$ as $j \to \infty$, the desired limit is also equal to 1.

II. *Solution by Clayton W. Dodge., University of Maine at Orono.*

We have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{j}{j+1} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(1 - \frac{1}{j+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(n - \sum_{j=1}^{n} \frac{1}{j+1}\right)$$

$$= 1 - \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j+1}$$

$$= 1$$

since $\sum_{j=1}^{n} 1/(j+1) < \ln n$ and $\lim_{n\to\infty} (\ln n)/n = 0$.

Also solved by KEN BLACKSTEIN, *Mamaroneck, N. Y.;* R. C. GEBHARDT, *Hopatcong, N. J.;* ARTHUR M. KELLER, *Now Yoik Gamma;* CHARLES H. LINCOLN, *Raleigh, N. C.;* PETER A. LINDSTROM, *Genesee Community College, Batavia, New York;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh; and the Proposer.*

325. *[Spring 1974] Proposed by Charles W. Trigg, San Diego, California.*

Show that there is only one third-order magic square with positive prime elements and a magic constant of 267.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pa.

Since 267 is the magic constant, the middle prime must be 267/3 = 89. There are six prime triples with 89 as one element and with sum equal to 267. They are

5, 89, 173
11, 89, 167
29, 89, 149
41, 89, 137
47, 89, 131
71, 89, 107

From these triples can be formed the magic square

29	167	71
131	89	47
107	11	149

It is not possible to use the two other triples (5, 89, 173) and (41, 89, 137) either with elements at the end or in the middle to form a magic square with sum 267. Hence this magic square is unique.

Also solved by LOUIS H. CAIROLI, *John Carroll University, Cleveland, Ohio;* CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *St. Louis, Missouri;* CHARLES H. LINCOLN, *Raleigh, N. C.;* BOB PRIELIPP, *The University of Wisconsin, Oshkosh;* R. ROBINSON ROWE, *Sacramento, California; and the Proposer.*

Comments by the Problem Editor.

Apologies are due to Jean J. Pedersen of The University of Santa Clara, California for the inadvertent omission of credit for her solution to Problem 304 [Fall 1973]. Accompanying a thorough analysis of the problem were colored, construction paper models illustrating extensions of the problem theme to truncated forms of the five Platonic Solids.

ERRATA FOR LAST ISSUE

In the diagram on page 7 of Volume 6, No. 1, $vt'/2$ should be changed to $ct'/2$. The *Journal* regrets the typographical error in one of the names in the list of manuscript award winners, page 23. The name Dennis C. Swolarski should have been Dennis C. Smolarski.