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SYSTEMATIC SOLUTIONS OF THE FAMOUS 15-14 PUZZLES

by *Maw. G. Henney and Dagmar R. Henney*
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In the early 1870's Sam Loyd in his *Cyclopedia* of *Puzzles* popularized a type of puzzle that has continually fascinated the world for almost a century. Although numerous sliding-block-puzzles exist, one of the most popular is based upon the 15-14 puzzle. Most generally, fifteen numbered blocks are placed in a square box with the vacant square in the right-hand bottom corner as shown in Fig. 1. The problem is to slide the numbered blocks sequentially until one obtains a specified arrangement. The best solution is the one obtained in the minimum number of moves (*inputs*), which we shall abbreviate by *MNI*.

Sam Loyd's famous version is shown in Fig. 2. Another version (Fig. 3) was popularized by Kraitchik [1].

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

FIGURE 1

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

FIGURE 2

3	4	2	7
1	8	10	5
6	9	15	11
12	14	13	

FIGURE 3

The original version of the puzzle started a widespread puzzle-solving fad in Europe and America. Large sums of money were offered for the *MNI* solution to it. Although many people claimed to have found the *MNI* solution, no one was able to reproduce publically even one valid solution. Interest remained at fever-pitch until several mathematicians [2] published a proof that no solution to Sam Loyd's famous puzzle was possible. Examination of the proof reveals that a quick parity check can be used to determine whether a particular version of the puzzle is solvable. One switches pairs of numbers (by removing and replacing blocks) counting the number of switches until the desired pattern is

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[1]The authors would like to express their appreciation to the Computing Center of the George Washington University for the use of its IBM 360.

obtained. No efficiency is necessary in obtaining a value for the number of switches. If the resulting number is even, a solution exists, otherwise not.[2] Applying this parity check to Kraitchik's version one finds that it is solvable.

However, no analytical method is known through which to obtain the *MNI* solution. Until now, the only method available to find valid solutions has been by trial and error. Direct enumeration of all sequences having a specified length is not feasable for any non-trivial version such as Kraitchik's. One can obtain the exact number of sequences of given length by constructing a transition matrix. The transition matrix for any input after the first has the same form. Hence, by raising this matrix to the required power and multiplying by the initial transition matrix, one can obtain the number of sequences of any desired length. If one computes the number of 50 move-sequences for Kraitchik's version, it is seen that one must consider almost 10^{16} separate sequences. Although direct enumeration of all sequences is not feasable even with a high-speed computer, a method does exist to determine an *MNI* solution.

Versions of the 15-14 puzzle can be considered to be models of what are called sequential machines. Each movement of a block is an input, each arrangement or state of the blocks is an output. A number is assigned to an output configuration based primarily on the distance of each element from the position it will have in the final arrangement. The number, called the test-value, may be obtained by employing various rules, but it must possess three properties:

(1) The test-value is an integer less than or equal to the number of inputs required to obtain the final arrangement.

(2) After every input, the value of that integer remains the same or changes by one.

(3) When the required arrangement is obtained, the test-value is zero.

A test-value satisfying the above three requirements has several advantages. First, it may be used to find the length of *MNI* solutions without enumerating all possible sequences of that length. In order to visualize this, consider the following example. Suppose it is known that a certain matrix of elements can be transformed from the initial configuration into the final configuration with 12 inputs, but it is not known whether this sequence is an *MNI* solution. An initial solution is always required but it need not be very efficient. Now construct a test-value having the three required properties. Suppose a test-value of 10 was associated with the initial arrangement. Property (1) requires that it be less than or equal to the *MNI* length. Consider the initial configuration after 4 inputs. The best known solution will be the one which requires 8 more inputs. Compute the test-value associated with every possible subsequence of 4 inputs. By Property (2) this value may range from 6 to 14. Property (3) implies that any subsequence resulting in a test-value of 9 or greater could not lead to a solution in 8 moves. Therefore, all sequences starting with those subsequences can be eliminated from further consideration. By repeated application of this technique, it is possible to eliminate a large number of sequences before they have reached the length associated with the known solution.

The above discussion indicates that a test-value close to the value associated with an *MNI* solution would eliminate more sequences from consideration than would a smaller test-value. Therefore, one should attempt to devise rules that produce test-values close to the test-values associated with *MNI* solutions. Many different rules to obtain test-values could be devised. It is desirable to select rules which will eliminate a large number of sequences and which will require relatively few calculations. The following two rules have been selected for these reasons.

Rule 1. The minimum number of inputs required to translate each element to its final position in the diagram is computed. The translation is effected for each element as though it were the only one in the diagram, all other spaces being vacant. The sum of inputs obtained in this way is the test-value.

Rule 2. Each row and column is examined in turn. For each one, the number of elements that have reached the same row or column they will have in the final arrangement are considered. For each pair of elements that are in the reversed order two units are added to the sum computed by Rule 1. If more than two elements are reversed, care must be exercised so that Property (2) is not violated.

[2] See, for example, the article by Thomas Fournell, *The* 25-14 Puzzle, this *Journal*, 5, No. 8 (1973).--*Editor*.

As an example of how the two rules are applied consider Fig. 3. The element occupying matrix position (1,1) must be moved a minimum of two units to attain its final position. The element occupying position (1,2) must also be moved a minimum of two units, the element occupying position (1,3) must be moved one unit, etc. To illustrate Rule 2 consider the second row. The blocks numbered 5 and 8 are in the same row in the final arrangement but reversed in order; therefore, two units must be added to the sum computed by Rule 1.

It is easy to understand why Rule 1 will produce values less than or equal to the length of the MNI solutions. The vacant space is always assumed to be in the most favorable position. Rule 2 reflects the two inputs required in removing one of the two reversed elements out of the path of the other one.

Further rules could be developed. Most of them would be of relatively minor importance--especially for small matrices. For every rule, one must consider whether the number of sequences eliminated is worth the additional computation required to obtain a test-value. This feature is important in reducing the computer time required to obtain MNI solutions.

A FORTRAN program was written to effect the calculations. Through the use of the two stated rules, it was possible to solve Kraitchik's Problem in a few minutes of computer time (IBM 360). The MNI solution was unexpectedly short. Reference 1 lists Kraitchik's own solution to his problem, which required 114 inputs. Kraitchik suggests that his solution is very close to the MNI solution (mathematical insight?). The correct MNI solution actually has 58 inputs and is given below so that the reader may check it.

12, 8, 7, 6, 2, 3, 4, 7, 6, 2, 5, 1, 3, 4, 2, 11, 8, 6, 11, 5,
10, 8, 6, 11, 5, 10, 8, 9, 13, 14, 9, 13, 14, 9, 13, 6, 15, 13,
6, 14, 9, 6, 13, 12, 11, 15, 14, 13, 12, 14, 13, 9, 6, 12, 14,
13, 15, 11.

Thus it seems very difficult to arrive at an MNI solution without proceeding systematically. Other solutions to variations of this problem can be found in the literature and are often known to be in error. Consider for example the Problem Book by Dudney [3]. His Problem Number 254 can be reduced to a variation of the 15-14 Puzzle. Attacked systematically, it is found that Dudney's *MNI* solution of 30 inputs is incorrect. The correct MNI solution has 28 inputs.

In conclusion, in order to see the above problem in its proper perspective, we quote the editors of the *American Journal of Mathematics* [2]:

> The 15 puzzle has been prominently before the American public, and may safely be said to have engaged the attention of none out of ten persons of the community. The principle of the game has its roots in what all mathematicians of the present day are aware constitutes the most subtle and characteristic conception of modern algebra, viz: the law of dichotomy applicable to the separation of the terms of every complete system of permutations into two natural and indefeasible groups.

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REFEREES FOR THIS ISSUE

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ANNOUNCEMENT FOR AUTHORS OF RECENTLY SUBMITTED PAPERS

The *Journal* regretfully announces that some articles recently submitted for publication and certain *Journal* records were in a brief-case that was stolen in March, 1976. If you sent an article to us during the period January 1-March 31 and you have not heard from us regarding your paper, please send us a copy since it is probable that your article was among the stolen items.

GIFFEN'S PARADOX: AN EXERCISE IN MULTIVARIATE CALCULUS

by David W. Bash
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Most people have the belief that the so-called law of diminishing demand always holds for the rational consumer trying to maximize his purchases restricted by his income. That is, if the price for a product goes up, the quantity purchased at the new price is less than the quantity purchased at the old price. Mathematical economists were able to construct a mathematical model to display this common belief as a theorem. However, a celebrated report was made by Sir R. Giffen of an instance where consumption went up along with price (see reference [5], p. 132). Such goods for which the quantity purchased increases with price are now called *Giffen* goods. A Giffen good could be potatoes among lower income people, for example. As the price of potatoes goes up, the people have less real income and hence they cannot afford as much meat, a much more expensive good. Thus, to get enough to eat they must purchase more potatoes than they did before the price increase.

Some economists chose to ignore these goods and continued to use the dove-mentioned mathematical model without correcting for the existence of Giffen goods. However, E. E. Slutsky [7] about 1914 made a more careful derivation of the same basic model which can mathematically allow the existence of Giffen goods.

We shall let x stand for a commodity n-vector (row), the ith component x_i being the quantity of the ith good in a market with n goods. Also, let $U(\mathbf{x})$ be a continuous real-valued function defined on a set containing all possible commodity vectors $\mathbf{x}$; $U(\mathbf{x})$ is called the *utility* function. If p_i is the unit price of the ith good, CI the income of the consumer, then the consumer wishes to maximize $U(\mathbf{x})$ subject to the following constraints:

$$\sum_{i=1}^{n} p_i x_i = \mathbf{px}' = CI \,, \qquad x_i \geq 0 \qquad (1)$$

where p is the price vector (row) and $\mathbf{x}'$ is the transpose of x. A further constraint inequality of the form $\mathbf{px}' \leq CI$ is clearly needed to require that the consumer does not spend more than he makes. To simplify the analysis, we assume the consumer spends precisely his income (perhaps the nth good is a quantity of shares in a credit union). We wish to derive demand functions $x_i = d_i(p_1,\cdots,p_n,CI)$, $i = 1,2,\cdots,n$, from the solution $\mathbf{x}^*$ of the maximization problem. In particular, we want to determine the sign of $\partial x_i/\partial p_i$. The law of diminishing demand implies that $\partial x_i/\partial p_i$ is negative.

The utility function $U(\mathbf{x})$ is introduced primarily as a device to obtain the consumer demand functions d_i. We should note here that the economist adds several restrictions on the utility function or utility index, and the function's domain. These restrictions imply the consumer behaves in a rational manner in some sense. For an example of one axiom, if $\mathbf{x}_1$ bundle of goods is "preferred" over $\mathbf{x}_2$, then $U(\mathbf{x}_1) \geq U(\mathbf{x}_2)$. One really needs only an ordinal utility function instead of a cardinal function. Then one could consider all differentiable monotonically increasing cardinal transformations of the ordinal utility function. But since the analysis presented here is invariant under such transformations, we shall consider $U(\mathbf{x})$ to be a particular real-valued function of $\mathbf{x}$, a function for which we can calculate all the partial derivatives we need. For further discussion see [4, p. 6-8, 16-20].

To solve problem (1) we use Lagrange multipliers and define the Lagrangian as $L(\mathbf{x},\lambda) = U(\mathbf{x}) + \lambda(CI - \mathbf{px}')$. The necessary conditions for a maximum are:

$$\begin{aligned} \frac{\partial U}{\partial x_i} - \lambda p_i &= 0 \,, \qquad i = 1,2,\cdots,n \\ CI - \mathbf{px}' &= 0 \,. \end{aligned} \qquad (2)$$

The second order, or sufficient, condition for a maximum to this problem to exist is that the following bordered Hessian matrix be negative definite:

$$J = \begin{bmatrix} H & -p' \\ -p & 0 \end{bmatrix}$$

where $H = [U_{ij}(\mathbf{x})]$ and $U_{ij}(\mathbf{x}) = \partial^2 U/\partial x_i \partial x_j$, $i, j = 1,\cdots,n$. A necessary condition for J to be negative definite is that all eigenvalues of the real symmetric matrix H be negative ([2], p. 126). Since the consumer tries to maximize his utility index, we assume J to be negative definite.

Then the necessary first order conditions (2) are also sufficient for a maximum. The system (2) with the non-singular negative definite Jacobian matrix J can be solved for the n + 1 unknowns x. and λ, and we denote the solutions by asterisks:

$$x^* = x^*(p, CI)$$

and

$$\lambda^* = \lambda^*(p, CI) \ .$$

These are also the demand functions for the n goods.

The effects of a change in a single price p_ℓ are obtained by partial differentiation of (2) with respect to p_ℓ at (x^*, λ^*). In matrix form the differentiated equations may be expressed in terms of a matrix $\partial x^*/\partial p$ with entries $\partial x_i/\partial p_j$ at x^* and a row vector $\partial x^*/\partial p$:

$$J^* \begin{bmatrix} \partial x^*/\partial p \\ \partial\lambda^*/\partial p \end{bmatrix} = \begin{bmatrix} \lambda^* I_n \\ x^* \end{bmatrix} \ . \tag{3}$$

Next, the effect of a change in price when the income is compensated so as to keep utility constant $(dU = \lambda^* p(dx)' = 0)$, is derived from (2):

$$J^* \begin{bmatrix} \partial x^*/\partial p|_c \\ \partial\lambda^*/\partial p|_c \end{bmatrix} = \begin{bmatrix} \lambda^* I_n \\ 0_{1\times n} \end{bmatrix} , \tag{4}$$

where the subscript c denotes the compensated income case.

Next, following the original contribution of Slutsky, consider the effects in (2) of a change of income CI at (x^*, λ^*). Differentiation with respect to CI yields

$$J^* \begin{bmatrix} (\partial x^*/\partial CI)' \\ \partial\lambda^*/\partial CI \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ -1 \end{bmatrix} \ . \tag{5}$$

The fundamental matrix equation in the theory of the household is obtained by combining (3), (4), and (5) as follows:

$$J^* \begin{bmatrix} \partial x^*/\partial p & \partial x^*/\partial p|_c & (\partial x^*/\partial CI)' \\ \partial\lambda^*/\partial p & \partial\lambda^*/\partial p|_c & \partial\lambda^*/\partial CI \end{bmatrix} = \begin{bmatrix} \lambda^* I_n & \lambda^* I_n & 0_{n\times 1} \\ x^* & 0_{1\times n} & -1 \end{bmatrix} \tag{5'}$$

Setting $D = -1/pH^{-1}p$, we may write the inverse of the Jacobian matrix J in the form (see page 212 of [1]):

$$J^{-1} = \begin{vmatrix} H^{-1}p'DpH^{-1} + H^{-1} & H^{-1}p'D \\ DpH^{-1} & D \end{vmatrix}$$

where we note that $D > 0$ since H is negative definite. Hence the solution of (5') is given by the three equations:

$$\partial x^*/\partial p = H^{-1}p'DpH^{-1}\lambda^* + H^{-1}\lambda^* + H^{-1}p'Dx^* \tag{6}$$

$$\partial x^*/\partial p|_c = H^{-1}p'DpH^{-1}\lambda^* + H^{-1}\lambda^* \tag{7}$$

and

$$(\partial x^*/\partial CI)' = -H^{-1}p'D \ . \tag{8}$$

Multiplying equation (8) on the right by x^* and then subtracting it from equation (7) we obtain the Slutsky equation

$$\partial x^*/\partial p = \partial x^*/\partial p|_c - (\partial x^*/\partial CI)'x^* ,$$

from which the diagonal terms may be extracted:

$$\partial x_i^*/\partial p_i = \partial x_i^*/\partial p_i|_c - (\partial x_i^*/\partial CI)x_i^* \ . \tag{9}$$

Here we have the result that the *total effect* of *a* change in p_i on x_i is the substitution effect of the *price* change on the quantity minus an *effect* due to *change* in income.

From the information $D > 0$, $\lambda^* = (\partial U/\partial x_i)/p_i > 0$ for $i = 1,\cdots,n$, and H symmetric and negative definite, one can show that $\partial x^*/\partial p|_c$ is negative definite. (Hint: Two useful lemmas are (a) a matrix A is positive definite if and only if A^{-1} is positive definite and (b) a symmetric matrix A is positive definite if and only if $A = P'P$ where P is a positive nonsingular matrix. Also see chapter 7 of [3].) Another easy exercise is $\partial x_i^*/\partial p_i|_c < 0$ for all i. Thus a compensated increase in the price of a good always results in a decrease in demand for the good. But the total effect according to the Slutsky equation (9) could be positive if the income effect is sufficiently negative. A good is called inferior if the consumption goes down as income rises, that is, if $\partial x_i^*/\partial CI < 0$. The Giffen goods, which do not obey the law of diminishing demand, are sufficiently inferior goods so that $\partial x_i^*/\partial p_i > 0$. Some goods may be inferior goods without being Giffen goods. Margarine is usually considered an example of one such inferior good.

Explicit utility functions for goods, one of which is a Giffen good, can be constructed as shown in reference [8] by Wold. His paper also contains the theorem that Giffen's phenomenon cannot occur in the complete feasible region for goods. For a different derivation of Slutsky's equation and a look at some other mathematically oriented economics, see reference [6].

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SUMMATION OF SPECIAL CLASSES OF SERIES

by Gerard P. Protomastro
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In a standard calculus text, like Leithold [1], the author illustrates the definitions of partial sums and sum of an infinite series by presenting an example like

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots .$$

One usually notes that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, and thus

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} .$$

Then the sum S of the infinite series is as follows:

$$S = \lim_{n\to\infty} s_n = 1$$

In this article we consider generalizing the above procedure. We shall develop formulas for the sum of two classes of infinite series. In all examples in this article we are asked to find the sum S of the given infinite series. It is not hard to show that S exists by the Limit Comparison Test.

Example 1.

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots .$$

Solution. First note that

$$\left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right) + \left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right) + \left(\frac{1}{3\cdot 4} - \frac{1}{4\cdot 5}\right) + \cdots = \frac{1}{2} .$$

Hence

$$\frac{2}{1\cdot 2\cdot 3} + \frac{2}{2\cdot 3\cdot 4} + \frac{2}{3\cdot 4\cdot 5} + \cdots = \frac{1}{2} ,$$

and thus $S = \frac{1}{4}$.

Example 2.

$$\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+\cdots t\ \frac{1}{(2n-1)(2n\ t\ 1)(2n\ t\ 3)}\ t\ \cdots .$$

Solution. First note that

$$\left(\frac{1}{1\cdot3}-\frac{1}{3\cdot5}\right)+\left(\frac{1}{3\cdot5}-\frac{1}{5\cdot7}\right)+\left(\frac{1}{5\cdot7}-\frac{1}{7\cdot9}\right)+\cdots=\frac{1}{3}.$$

Hence

$$\frac{4}{1\cdot3\cdot5}+\frac{4}{3\cdot5\cdot7}+\frac{4}{5\cdot7\cdot9}+\cdots=\frac{1}{3},$$

and thus $S=\frac{1}{12}$.

We now generalize the above examples and consider deriving a formula for the sum S of the following class of infinite series:

$$\frac{1}{a(a\ t\ b)(a\ t\ 2b)}+\frac{1}{(a+b)(a\ t\ 2b)(a+3b)}\ t\ \cdots .$$

First note that

$$\left(\frac{1}{a(a+b)}-\frac{1}{(a+b)(a+2b)}\right)+\left(\frac{1}{(a+b)(a+2b)}-\frac{1}{(a+2b)(a+3b)}\right)+\cdots$$

$$=\frac{1}{a(a+b)}.$$

Hence

$$\frac{2b}{a(a+b)(a+2b)}+\frac{2b}{(a+b)(a+2b)(a+3b)}+\cdots=\frac{1}{a(a+b)},$$

and thus

$$S=\frac{1}{2ab(a+b)}. \tag{1}$$

Example 3.

$$\frac{1}{2\cdot5\cdot8}+\frac{1}{5\cdot8\cdot11}+\frac{1}{8\cdot11\cdot14}+\cdots+\frac{1}{(3n-1)(3n+2)(3n+5)}+\cdots .$$

Solution. Note $a=2$ and $b=3$, and thus *by* substitution in (1) we have $S=\frac{1}{12.5}=\frac{1}{60}$.

We now consider a second class of infinite series which requires more ingenuity to find the sum S.

Example 4.

$$\frac{4}{1\cdot2\cdot3}+\frac{5}{2\cdot3\cdot4}+\frac{6}{3\cdot4\cdot5}+\cdots+\frac{n+3}{n(n+1)(n+2)}+\cdots .$$

Solution. First note that

$$\left(\frac{5}{1\cdot2}-\frac{7}{2\cdot3}\right)+\left(\frac{7}{2\cdot3}-\frac{9}{3\cdot4}\right)+\left(\frac{9}{3\cdot4}-\frac{11}{4\cdot5}\right)+\cdots=\frac{5}{2}.$$

Hence

$$\frac{8}{1\cdot2\cdot3}+\frac{10}{2\cdot3\cdot4}+\frac{12}{3\cdot4\cdot5}+\cdots=\frac{5}{2},$$

and thus $S=\frac{5}{4}$.

Example 5.

$$\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\frac{3}{5\cdot7\cdot9}+\cdots+\frac{n}{(2n-1)(2n+1)(2n+3)}\ t\ \cdots .$$

Solution. First note that

$$\left(\frac{3}{1\cdot3}-\frac{7}{3\cdot5}\right)+\left(\frac{7}{3\cdot5}-\frac{11}{5\cdot7}\right)+\left(\frac{11}{5\cdot7}-\frac{15}{7\cdot9}\right)+\cdots=1.$$

Hence

$$\frac{8}{1\cdot3\cdot5}+\frac{16}{3\cdot5\cdot7}+\frac{24}{5\cdot7\cdot9}+\cdots=1,$$

and thus $S=\frac{1}{8}$.

In deriving a formula for the above type of infinite series, the computations are more detailed. We first write their general form as follows:

$$\frac{c+d}{a(a+b)(a+2b)}\ \frac{c\ t\ 2d}{(a+b)(a+2b)(a+3b)}\ t\ \cdots .$$

Then note

$$\left(\frac{bc+ad+bd}{a(a\ t\ b)}-\frac{bc+ad+3bd}{(a+b)(a+2b)}\right)+\left(\frac{bc\ t\ a\ d\ t\ 3bd}{(a+b)(a\ t\ 2b)}-\frac{bc+ad+5bd}{(a+2b)(a\ t\ 3b)}\right)+\cdots$$

$$=\frac{bc+ad+bd}{a(a+b)}$$

Hence

$$\frac{2b^2(c\ t\ d)}{a(a\ t\ b)(a\ t\ 2b)}+\frac{2b^2(c\ t\ 2d)}{(a\ t\ b)(a\ t\ 2b)(a+3b)}+\cdots=\frac{bc\ t\ ad\ t\ bd}{a(a+b)},$$

and thus

$$S = \frac{bc + ad + bd}{2ab^2(a + b)} \qquad (2)$$

We conclude this article by considering an application of the above formula.

Example. 6.

$$\frac{3}{2\cdot 5\cdot 8} + \frac{5}{5\cdot 8\cdot 11} + \frac{7}{8\cdot 11\cdot 14} + \cdots + \frac{2n+1}{(3n-1)(3n+2)(3n+5)} + \cdots .$$

Solution. Note a = 2, b = 3, $c = 1$, and $d = 2$. Then by substitution in (2) we have

$$S = \frac{3 + 4 + 6}{36\cdot 5} = \frac{13}{180}$$

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1976 NATIONAL MEETING IN TORONTO

There is still time for local chapters to be making plans for the national meeting in Toronto, Canada in conjunction with the Mathematical Association of America. Plan now to send your best undergraduate speaker or delegate (or both) to that meeting. Travel money for one approved speaker or delegate is available from Nationa. Send requests and proposed papers to:

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MENELAUS THEOREM IN A VECTOR SPACE[1]

by Ali R. Amir-Moez and *Patricia A. Stubbs*
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Techniques of vector algebra in geometry create some interests in both linear algebra and geometry [1]. As an example, we shall study Menelaus theorem. Since it involves incidence properties, it can be generalized to any n-dimensional vector space; that is, one does not need the concepts of length and angle.

1. Notations

We shall use standard notations of linear algebra. Capital letters are used for vectors and points. Small letters are elements of a field; in particular, we consider the field of real numbers.

2. Menelaus Theorem. Let the points L, M, and N be respectively on the sides BC, CA, and AB of the (nondegenerate) triangle ABC (Fig. 1). Then a necessary and sufficient condition for the points L, M, and N to be collinear is:

$$\frac{LB}{LC}\cdot\frac{MC}{MA}\cdot\frac{NA}{NB} = 1 , \qquad (1)$$

where directed line segments are considered.

Proof. We shall give a vector proof. Let A be chosen for the origin, that is, $A = \vec{0}$. Thus there exist h and k such that

$$M = hC \qquad \text{and} \qquad N = kB .$$

Since L is on the line BC, it follows that there are real numbers p and q such that

$$L = pB + qC , \qquad p + q = 1 . \qquad (2)$$

Now we compute the ratios of (1). Let $\frac{LB}{LC} = t$. Then we observe that

$$B - L = t(C - L) . \qquad (3)$$

[1]presented to the Mathematical Association of America, April 11, 1975, at San Anglelo, Texas. The theorem for the plane was proved by the second author named as a term paper in geometry.

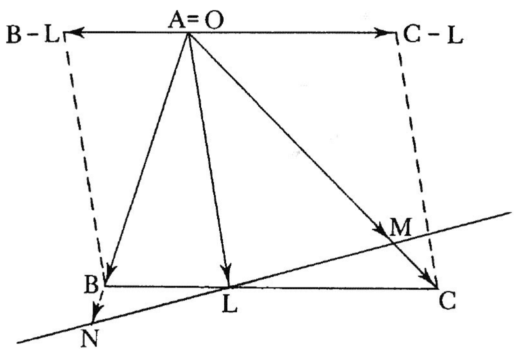


FIGURE 1

Thus

$$\frac{LB}{LC} = t = \frac{-q}{p} .$$

One observes that

$$C - M = (1 - h)C \qquad \text{and} \qquad A \quad M = -M = -hC$$

which imply

$$\frac{MC}{MA} = \frac{h - 1}{h} .$$

Similarly one obtains

$$\frac{NA}{NB} = \frac{k}{k - 1} .$$

Therefore (1) becomes

$$\frac{-q}{p} \cdot \frac{h - 1}{h} \cdot \frac{k}{k - 1} = 1$$

This equality is equivalent to

$$hp + kq = hk . \tag{4}$$

Now we shall present the proof. If N, M, and L are collinear, then there exist r and s such that

$$L = rM + sN , \qquad r + s = 1 .$$

Substituting for M and N their values in terms of B and C, we obtain

$$L = rhC + skB , \qquad r + s = 1 . \tag{5}$$

Comparing (2) and (5), we obtain

$$rhC + skB = pB + qC .$$

Since $\{B,C\}$ is taken to be linearly independent, we get

$$\begin{cases} hr - q = 0 \\ ks - p = 0 \\ r + s - 1 = 0 . \end{cases}$$

A necessary and sufficient condition for r and s to satisfy this set of equations is (as shown in [2]):

$$\begin{vmatrix} h & 0 & -q \\ 0 & k & -p \\ 1 & 1 & -1 \end{vmatrix} = 0 .$$

This equality is equivalent to

$$hp + kq = hk$$

which is the same as (4). Thus (4) is a necessary and sufficient condition for L to end at the point of intersection of lines BC and MN.

3. A *Generalization*

Let $\{A_1, \cdots, An\}$ be a set of linearly independent vectors in an n-dimensional vector space. Let L end on the n-dimensional linear element (coset) P which is generated by this set, i.e.,

$$L = \sum_{i=1}^{n} p_i A_i , \qquad \sum_{i=1}^{n} p_i = 1 . \tag{1}$$

Consider the vectors

$$M_i = h_i A_i , \qquad h_i \neq 0 , \qquad i = 1, \cdots, n .$$

Let the linear element (coset) generated by $\{M_1, \cdots, M_n\}$ be Q. Then a necessary and sufficient condition for L to end on $P \cap Q$ is:

$$\begin{vmatrix} h_1 & & 0 & -p_1 \\ & \ddots & & \vdots \\ 0 & & h_n & p_n \\ 1 & \cdots & 1 & -1 \end{vmatrix} = 0 .$$

(This equality is equivalent to

$$p_1 h_2 \cdots h_n + \cdots + p_n h_1 \cdots h_{n-1} = h_1 \cdots h_n .)$$

Proof. If L ends on Q, then

$$L = \sum_{i=1}^{n} r_i M_i \,, \qquad \sum_{i=1}^{n} r_i = 1 \qquad (2)$$

Comparing (1) and (2), we obtain

$$h_i r_i - p_i = 0 \,, \qquad i = 1,\cdots,n$$
$$\sum r_i = 1.$$

A necessary and sufficient condition for this system of equations to have a solution for $(r_1,\cdots,r_n)$ is:

$$\begin{vmatrix} h_1 & & 0 & -p_1 \\ & \ddots & & \vdots \\ 0 & & h_n & -p_n \\ 1 & \cdots & 1 & -1 \end{vmatrix} = 0 \,.$$

One may study the case $n = 3$ in order to understand the idea better. A variety of other generalizations is also possible.

REFERENCES

1. Amir-Moéz, A. R., Duran, B. S., Linear Algebra of the Plane, Western Printing Co., Lubbock, Texas, 1973.
2. Amir-Moéz, A. R., Fass, A. L., Elements of Linear Spaces, Pergamon Press. Oxford, England, 1962.

SPECIAL ANNOUNCEMENT

Conference on Recreational Mathematics

The Fourth Annual Conference in Mathematics and Statistics will be held at Miami University, Oxford, Ohio, on September 24-25, 1976. The speakers are well-known experts in recreational mathematics and prolific problem solvers, including Leon Bankoff, Problem Editor of this Journal. For more information write:

Professor Donald O. Koehler
Department of Mathematics and Statistics
Miami University
Oxford, Ohio 45056

A GENERALIZED GAME OF CRAPS

by Joe Van Austin
Emory University

The game of craps, played with a pair of dice, is one of the more interesting gambling games studied in elementary probability. This paper generalizes the usual rules to a game with two n-sided dice (or, more accurately, to a game using two n-sided spinners having all sides equally likely, since only five regular n-sided dice can be constructed). The probability of winning this generalized game is obtained as is the limit of this probability as n becomes infinite.

Conventional craps is played with two six-sided dice. Only the sum of the two dice is considered. The player rolls the dice and wins immediately if the sum is 7 or 11, loses immediately if the sum is 2, 3, or 12, and any other sum rolled is his point. If the first roll produces a point, the player continues to roll the two dice until the sum is his point or is 7. If his point is rolled first, he wins and if 7 is rolled first, he loses. The problem usually considered is to find the probability that the player wins the game. A solution is given in [1, p. 24-26].

Thus in the usual game of craps two regular 6-sided dice are used, $n = 6$. In the generalized game two n-sided dice are used. Again, only the sum of the two dice is considered. Rolling $n + 1$ or $2n - 1$ wins immediately, rolling 2, 3, or 2n loses immediately, and rolling any other sum is the point. If a point is rolled, the player continues to roll the two dice until he rolls his point or rolls $n + 1$. As above, if the point is rolled first, he wins, and if $n + 1$ is rolled first, he loses. We shall obtain the probability that the player wins this generalized game.

The solution is similar to that used in the regular game. The probability of winning is given by the following:

$$P_n[\text{win}] = P[S = n + 1] + P[S = 2n - 1] + \sum_{\substack{k=4 \\ k \neq n+1}}^{2n-2} P[S = k]\cdot P[\text{roll } k \text{ before } n+1 \mid S = k] \qquad (1)$$

where S is the sum of the two dice on the first roll. By considering which of the n^2 outcomes give various sums, it is easy to show that

$$P[S = n + 1 + j] = P[S = n + 1 - j] = \frac{n - j}{n^2} \tag{2}$$

for $j = 0, 1, 2, \cdots, n - 1$ and thus that

$$P[\text{roll } n + 1 \pm j \text{ before } n + 1 \mid S = n + 1 \pm j] = \frac{n - j}{n - j + n} \tag{3}$$

for $j = 1, 2, \cdots, n - 1$. Using the results of equations (2) and (3) in equation (1) yields, after a little algebra,

$$P_n[\text{win}] = \frac{2}{n^2} + \frac{2}{n^2}\sum_{k=3}^{n} \frac{k^2}{k + n} . \tag{4}$$

For $n = 6$ this gives the usual probability of a regular game of craps. Table 1 gives the $P_n[\text{win}]$ for the five regular polyhedral dice--i.e., the five possible regular dice.

n	$P_n[\text{win}]$
4	.5357
6	.4929
8	.4660
12	.4377
20	.4155

PROBABILITY OF WINNING GENERALIZED CRAP FOR THE FIVE REGULAR POLYHEDRAL VICE

TABLE 1

Next we investigate the limit of P [win] as n tends to infinity. Writing

$$\frac{k^2}{k + n} = k - n + \frac{n^2}{k + n}$$

in (4), evaluating the sums, and simplifying gives

$$P_n[\text{win}] = \frac{2}{n^2} + \frac{2}{n^2}\left[\frac{7n - n^2 - 6}{2}\right] + 2\sum_{k=3}^{n} \frac{1}{k + n} . \tag{5}$$

As $n \to \infty$, $2/n^2 \to 0$ and the second term $\to -1$. For the summation part of (5) $\frac{1}{k + n}$ is decreasing in k for n fixed. Using an integral to obtain upper and lower bounds on the summation for a given n yields

$$\int_3^{n+1} \frac{dk}{k + n} \le \sum_{k=3}^{n} \frac{1}{k + n} \le \int_2^{n} \frac{dk}{k + n} \tag{6}$$

Integrating and simplifying gives

$$\ln\left(\frac{2n + 1}{n + 3}\right) \le \sum_{k=3}^{n} \frac{1}{k + n} \le \ln\left(\frac{2n}{n + 2}\right) . \tag{7}$$

Letting $n \to \infty$ gives that

$$\lim_{n\to\infty}\left[\ln\left(\frac{2n + 1}{n + 3}\right)\right] = \ln\left[\lim_{n\to\infty}\left(\frac{2n + 1}{n + 3}\right)\right] = \ln 2 .$$

As both sides in (7) tend to ln 2 as n becomes infinite, we have,

$$\lim_{n\to\infty} \sum_{k=3}^{n} \frac{1}{k + n} = \ln 2.$$

Finally, using this limit in (5) we obtain

$$\lim_{n\to\infty} P_n[\text{win}] = 2 \ln 2 - 1 , \tag{8}$$

or about .3863 using a table of natural logarithms.

REFERENCES

1. Mosteller, F., Fifty Challenging Problems in Probability, Addison-Wesley, Reading, 1965.

REGIONAL MEETINGS OF MAA

Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

COMMENTS ON CALCULUS **FROM** READERS

Occasionally, the Editor receives observations from readers about undergraduate mathematics which might be used in the classroom and could be helpful to a student in undergraduate mathematics. We have collected four such items to be presented here.

Ronald D. Rossi, Behrend College of Pennsylvania, points out a dynamic method for solving certain polynomial inequalities. Suppose it is desired to find the values of x for which

$$p(x) > 0$$

where $p(x)$ is a polynomial that is factorable into integral powers of linear factors of the form $ax + b$:

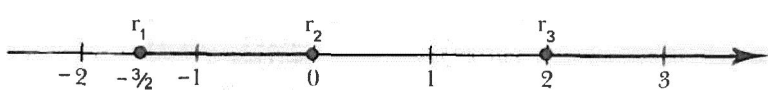
$$p(x) = c(a_1x + b_1)^{k_1}(a_2x + b_2)^{k_2}\cdots(a_nx + b_n)^{k_n} .$$

First eliminate all factors (except c) which occur to an even power, positive or negative. Then solve for the roots of the new polynomial $p(x)$ by setting each linear factor equal to zero. Next, line the roots up in numerical order on the real line: $r_1 < r_2 < \cdots < r_m$. If $c > 0$, the solution set is comprised of *alternating finite* open intervals determined by $r_1 < r_2 < r_3 < \cdots < r_m$ beginning at the left and proceeding to the right, ending with (r_m,∞) if m is odd. If $c < 0$, proceed as before, but take the complement of the set obtained above.

Example. $x^3(2x + 3)(x - 1)^2(x - 2) > 0$. Reduce lo the inequality

$$x^3(2x + 3)(x - 2) > 0 , \qquad c = 1 .$$

Since $c > 0$, the solution set is indicated by the shading in the diagram below:



$(-3/2,0) \cup (2,\infty)$

Brian Miller, Merrick, N.Y., proposes a shortcut to simplifying derivatives of algebraic expressions of the form U^mV^n. Assuming U and V are differentiable functions, upon differentiating the product U^mV^n, we find

$$(U^mV^n)' = U^{m-1}V^{n-1}(mU'V + nUV') .$$

This formula can be used consistently for a large class of derivatives normally encountered in elementary calculus courses.

Example. Find $f'(x)$ if

$$f(x) = \frac{\sqrt{x^2 + 1}}{(x^2 - 1)^{2/3}}$$

Here, $U(x) = x^2 + 1$, $V(x) = x^2 - 1$, $m = 1/2$, and $n = -2/3$, so

$$f'(x) = (x^2 + 1)^{\frac{1}{2} - 1}(x^2 - 1)^{-\frac{2}{3} - 1}\left[\frac{1}{2}(2x)(x^2 - 1) - \frac{2}{3}(x^2 + 1)(2x)\right]$$

$$= \frac{x}{3}(x^2 + 1)^{-1/2}(x^2 - 1)^{-5/3}[3(x^2 - 1) - 4(x^2 + 1)]$$

$$= -\frac{x}{3}(x^2 + 1)^{-1/2}(x^2 - 1)^{-5/3}(x^2 + 7)$$

Norman Schaumberger, Bronx Community College of CUNY, suggests first tackling the integral $\int \csc x\, dx$ before attempting $\int \sec x\, dx$. Adding the following two equations (which are well-known half-value formulas)

$$\tan\frac{x}{2} = \frac{1 - \cos x}{\sin x} , \qquad \cot\frac{x}{2} = \frac{1 + \cos x}{\sin x} ,$$

we find

$$2\csc x = \tan\frac{x}{2} + \cot\frac{x}{2} .$$

Therefore,

$$\int \csc x\, dx = \frac{1}{2}\int \tan\frac{x}{2}\, dx + \frac{1}{2}\int \cot\frac{x}{2}\, dx$$

$$= \log\left|\sin\frac{x}{2}\right| - \log\left|\cos\frac{x}{2}\right| + C$$

$$= \log\left|\tan\frac{x}{2}\right| + C$$

$$= \log\left|\frac{1 - \cos x}{\sin x}\right| + C$$

$$= \log|\csc x - \cot x| + C\,.$$

Since $\sec x = \csc(\pi/2 - x)$, the integral $\int \sec x\,dx$ can be readily worked out by substitution. The above identity can also be used to solve the integral $\int \csc^3 x\,dx$, hence $\int \sec^3 x\,dx$:

$$\int \csc^3 x\,dx = \frac{1}{8}\int\left(\tan\frac{x}{2} + \cot\frac{x}{2}\right)^3 dx$$

$$= \frac{1}{8}\int \frac{\left(\tan^2\frac{x}{2} + 1\right)^2 \sec^2\frac{x}{2}}{\tan^3\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\int \left[\tan\frac{x}{2} + 2\left(\tan\frac{x}{2}\right)^{-1} + \left(\tan\frac{x}{2}\right)^{-3}\right]\frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{8}\tan^2\frac{x}{2} + \frac{1}{2}\log\left|\tan\frac{x}{2}\right| - \frac{1}{8}\cot^2\frac{x}{2} + C$$

$$= -\frac{1}{2}\csc x \cot x + \frac{1}{2}\log|\csc x - \cot x| + C$$

Laura McCarten, University of Wisconsin at Janesville, suggests three methods of solving $\int \sec x\,dx$ which seem less artificial than conventional textbook presentations. The first is to use a change of variable: Set $\sec x = t$. Then

$$\int \sec x\,dx = \int \frac{t\,dt}{t\sqrt{t^2 - 1}} = \int \frac{dt}{\sqrt{t^2 - 1}}$$

which can be worked out with the help of the further substitution $t = \cosh\theta$.

The second is to use the identity $\cos x = \sin(\pi/2 - x)$:

$$\sec x = \frac{1}{\cos x} = \frac{1}{\sin\left(\frac{\pi}{2} - x\right)} = \frac{1}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \frac{\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right) + \cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)} + \frac{\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

(Since this last is equivalent to $\frac{1}{2}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + \frac{1}{2}\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)$, this method is only a slight variation of Schaumberger's method.) Thus the integral is:

$$\int \sec x\,dx = \log\left|\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| - \log\left|\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C$$

$$= \log|\sec x + \tan x| + C\,.$$

The third method is to use the identity

$$\sec x = \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}}$$

and then let $u = \tan\frac{x}{2}$:

$$\int \sec x\,dx = \int \frac{1 + u^2}{1 - u^2}\,\frac{2\,du}{1 + u^2} = \int \frac{2\,du}{1 - u^2}$$

$$= \log\left|\frac{1 + u}{1 - u}\right| + \quad = \log\left|\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right| + C$$

Now add the equations

$$\tan x = \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}}\,, \qquad \sec x = \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}}$$

to obtain

$$\sec x + \tan x = \frac{1 + 2\tan\frac{x}{2} + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = \frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}$$

and again the desired result follows.

WELCOME TO NEW CHAPTERS

The *Journal* welcomes the following new chapters of Pi Mu Epsilon recently installed:

CALIFORNIA KAPPA at Loyola Marymount University, installed April 29, 1975, by E. Allan Davis, Council President.

KENTUCKY GAMMA at Murray State University, installed May 1, 1975, by J. C. Eaves, Past Council President.

MISSOURI DELTA at Westminster College, installed May 7, 1975, by R. V. Andree, Council Vice-president.

PENNSYLVANIA XI at Saint Joseph's College, installed April 18, 1975, by Eileen Poiani, Councilor.

RHODE ISLAND GAMMA at Providence College, installed April 11, 1975, by Eileen Poiani, Councilor.

TEXAS IOTA at The University of Texas at Arlington, installed April 24, 1975, by R. V. Andree, Council Vice-president.

TEXAS KAPPA at West Texas State University, installed May 1, 1975, by J. C. Eaves, Past Council President.

MOVING??

BE SURE TO LET THE JOURNAL KNOW!

Send your name, old address with *zip code* and new address with *zip code* to:

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PUZZLE SECTION

With this issue the *Journal* is beginning a new feature on an experimental basis. Many readers who for a variety of reasons do not wish to compete in solving problems in our Problem Department may be more appropriately challenged by a puzzle. Puzzles take on many forms and occur prolifically in mathematics. We have chosen a peculiar kind of "double crossword" puzzle which was kindly submitted by R. Robinson Rowe, one of our active problem solvers. This type of puzzle, to our knowledge, is rare, having once been a feature of the *Technology Review* for several years, which was called a *double-crostic* in that publication. The present puzzle will be called a *mathematicrostic* for the obvious reasons.

Send your solutions directly to the Editor, 601 Elm, Room 423, The University of Oklahoma, Norman, Oklahoma 73019 (please note the new zip code). As in the Problem Department, the names of correct puzzle solvers and solution will be published in a forthcoming issue. Readers are urged to submit puzzles of their own which do not fall into the category of problems. The Editor reserves the right to use or not use such puzzles for future issues in any manner deemed appropriate.

Mathematicrostic No. 1

by R. Robinson Rowe
Sacramento, California

Instructions.

The 179 numbered squares of the acrostic on the following page are to be filled with letters corresponding to the same numbers in the defined words. The unnumbered squares are to be left blank; they simply indicate spaces between words. When completed, the acrostic will be a profound statement by a renowned mathematician. The initial letters of the 29 defined words will identify him and the source of the quotation. Including the author and two in the acrostic, the solution identifies 10 well-known mathematicians. If you find them, count yourself as the 11th.

	1V	2S		3a	4C	5B	6H	7a		8C	9b	10c
11J		12A	13a		14A	15I	16T	17Y	18D	19Q	20R	21K
	22E	23K	24Q	25c	26a	27C	28Z		29S	30U	31M	32L
33A	34Q	35U		36T	37J	38C		39N	40c	41T	42G	43D
44V		45a	46a	47C	48T	49P	50K	51S		52F	53C	54Y
55W	56U		57I	58b	59b	60V		61C	62R	63B	64A	65c
66Q		67H	68L		69P	70F	71E	72Z	73Y	74K	75M	76X
77D		78P	79L		80R	81c		82Y	83H	84A	85M	
86Q	87P	88C		89D		90L	91I	92Z	93R	94X		95K
96X	97F	98H	99E		100G	101G	102T		103S	104Y	105C	
106D	107M	108J	109Q	110a		111P	112K	113N	114B		115X	116W
117S	118J	119D	120b	121U	122E	123H	124V		125W	126B	127L	
128U	129H	130M	131Z		132P	133a	134K		135J	136U	137I	
138P	139W	140F	141N		142O	143Q	144Q	145A	146K	147T	148S	149L
	150O	151M	152P		153J	154H		155V	156L	157R		158X
159c	160A	161I	162b	163S		164O	165C	166F		167G	168N	169E
170K	171A	172D	173H	174L		175B	176Y		177J	178O	179G	

Definitions and *Key*

A. Direct path on a curved surface	171 64 12 160 33 145 84 14
B. Mathematician of 3rd century	114 5 63 175 126
C. Percentage points	4 47 165 8 88 27 38 105 53 61
D. Secondary mathematics	119 18 77 172 106 43 89
E. Linear mercator course	122 99 169 71 22
F. Means to duplicate what precedes	166 97 52 140 70
G. Produce	167 101 42 179 100
H. Cartesian coordinate	123 67 173 98 129 154 6 83
I. Problem-solving procedure	57 161 91 118 15 137
J. Mathematicienne 1718-1799	177 108 37 135 11 153
K. This century	146 95 50 170 21 112 23 134 74
L. Pleasure lover	156 68 127 79 149 32 90 174
M. Mathematician, 3rd century B. C.	107 151 31 130 75 85
N. Any tineid	39 168 113 141
O. Norse mathematician, 1802-1829	164 150 142 178
P. Congruent pair	152 111 87 78 138 49 132 69
Q. Beyond the googol	19 144 86 109 34 143 24 66
R. Third-powered	20 62 80 157 93
S. Brief time	148 103 51 163 29 2 117
T. Part of a sum	36 102 41 147 16 48
U. English mathematician, 1642-1727	136 56 128 35 121 30
V. Stunt-like	124 60 1 44 155
W. Astronomer-mathematician, 1801-1892	125 139 55 116
X. Fake	115 96 158 76 94
Y. Modern printing method	104 17 176 82 54 73
Z. Spherical angle	131 92 28 72
a. Host frequent	133 13 3 26 110 7 45 46
b. German mathematician, 1777-1855	120 9 58 59 162
c. 5800 Ångströms	10 81 159 65 40 25

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Lob Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally, we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions if available and by any information that will assist the editor.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of November 1976.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

362. *Ptopobed by Zelda Katz, Beverly Hills, California.*

As shown in Fig. 1, a diameter AB of a circle (O) passes through C, the midpoint of a chord DE. M is the midpoint of arc AB and the chord MP passes through C. The radius OP cuts the chord DE at Q. The tangent circles (O_1), (O_2), (W_1) and (W_2) are drawn as shown. Show that $DQ = W_1W_2$.

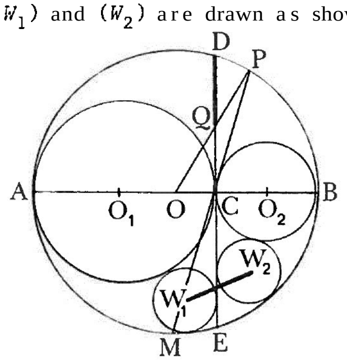


FIGURE 1

363. *Ptopobed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Does $\frac{\sin 1}{1} + \frac{\sin 2}{2} + \frac{\sin 3}{3} + \cdots$ converge, and if so, to what?

364. *Proposed by Charles W. Trigg, San Diego, California.*

Show that there is only one third-order magic square with positive prime elements and a magic constant of 267.

365. *Proposed by Clayton W. Dodge, University of Maine, Orono, Maine.*

Find all fractions $\frac{abc}{cde}$ such that cancelling the digit c yields an equivalent fraction, such as $\frac{166}{664} = \frac{16\cancel{5}}{\cancel{6}64} = \frac{16}{64}$. As in the illustration, not all the digits a, b, c, d, e need be distinct, but they should not be all equal.

366. *Proposed by Richard Field, Santa Monica, California.*

Let $Q = \left[\frac{10^n}{p}\right]$, where p is a prime > 5, and n is the cycle length of the repeating decimal $1/p$; $[x]$ denotes the greatest integer in x. Can Q be a prime?

367. *Proposed by R. Robinson Rowe, Sacramento, California.*

A box of unit volume consists of a square prism topped by a pyramid, Find the side of the square base and heights of prism and pyramid to minimize the surface area.

368. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

Given a triangle ABC with its inscribed circle (I). Lines AI, BI, CI cut the circle in points D, E, F respectively. Prove that $AD + BE + CF \geq$ (Perimeter of triangle $DEF)/\sqrt{3}$.

369. *Proposed by Paul Erdös, Spaceship Earth.*

Determine all solutions of $\binom{n}{k} = \prod_{p \leq n} p$.

370. *Proposed by David L. Silverman, West Lot, Angeles, California.*

Able, Baker and Charlie take turns cyclically, in that order, tossing a coin until three successive heads or three successive tails appear. With what probabilities will the game terminate on Able's turn? On Baker's?

371. *Proposed by 1. P. Scalisi, State College at Bridgewater, Massachusetts.*

A unit fraction is any rational number of the form $1/n$, where n is a positive integer. Write $2/n$ as the sum of 4 (or 6 or 10 or 14) *distinct* unit fractions.

372. *Proposed by Sidney Penner, Bronx Community College of CUNY.*

Prove the following:

Theorem: Let (X_1,τ_1) and (X_2,τ_2) be topological spaces and let f be a function from a subset of X_1 into X_2. The function f is continuous in the relative topology on its domain if and only if for every a $\in \tau_2$ there exists b $\in \tau_1$ such that

(i) Dom $f \cap$ b $\subset f^{-1}(a)$

(ii) if $c \subset a \cap$ Ran f then $f^{-1}(c) \subset$ Dom f $\cap$ b.

373. *Proposed by Joe Van Austin, Emory University, Atlanta, Georgia.*

Assume that the number of shots at the goal in a hockey game is a random variable Y that has a Poisson distribution with parameter λ. Each shot is either blocked or is a goal. Assume each shot is independent of the other shots and $p = P$[a shot is blocked] for each shot. Find the probability there are exactly k goals in a game for k = 0, 1, 2, $\cdots$.

Solutions

338. [Spring 1975] *Proposed by Hung C. Li, Southern Colorado State College.*

Let $(O)a$ be a circle centered at 0 with radius a. Let P, any point on the circumference of (O), be the center of circle (P). What is the radius of (P) such that it divides the area of (O) into two regions whose areas are in the ratio $s:t$?

Solution by *R. Robinson Rome, Sacramento, California.*

We note first that OPQ is isosceles (Fig. 2) so that its base angles ϕ are one half the exterior angle θ. Let A = Lime PQR and

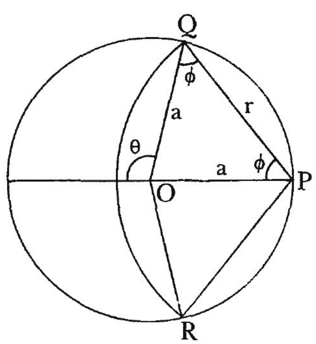


FIGURE 2

A = Circle(O) - A_s; then

$$\text{Lune } PQR = \text{Sector } OQR + \text{Sector } PQR - OQPR$$

$$A = a^2(\pi - \theta) + r^2\phi - a^2 \sin\theta \,. \tag{1}$$

Now

$$r = 2a\cos\phi \quad \text{and} \quad \phi = \frac{1}{2}9 \,, \tag{2}$$

$$r^2\phi = \frac{1}{2}6 \cdot 4a^2\cos^2\frac{1}{2}\theta = a^2\theta(1 + \cos\theta) \,, \tag{3}$$

making

$$A = a^2(\pi + \theta\cos\theta - \sin\theta) \,. \tag{4}$$

Then

$$A_t = \pi a^2 - A = a^2(\sin 9 - \theta\cos\theta) \tag{5}$$

and the given ratio

$$\frac{s}{t} = \frac{A_s}{A_r} = \frac{\pi + \theta\cos\theta - \sin\theta}{\sin\theta - \theta\cos\theta} = \frac{\pi}{\sin 9 - \theta\cos\theta} - 1 \tag{6}$$

whence

$$\sin\theta - \theta\cos\theta = \frac{\pi}{1 + s/t} = k \,. \tag{7}$$

For a particular problem, k can be computed from the given ratio s/t, (7) used by "cut-try" to find 9, then

$$r = 2a\cos\frac{1}{2}\theta \tag{8}$$

For example, suppose $s/t = 1$, making k = $\pi/2$ = 1.57079 6327.

Try 9 =	sin 9	$-\theta$ cos 9	Sum	Sum - k
1.7	.9917	.2190	1.2107	-.3601
1.8	.9738	.4090	1.3828	-.1880
1.9	.946300088	.614250176	1.560550264	-.010246063
1.91	.943019932	.635526229	1.578546161	+.007749834
1.9056	.944474838	.626149195	1.570624093	-.000172294
1.9057	.9444419745	.626362038	1.570804013	+.000007686
1.9056957	.9444433880	.626352886	1.570796274	-.000000053
1.9056958	-9444433549	.626353098	1.570796453	+.000000126
1.9056957296	.9444433779	.626352948	1.570796327	.000000000

Then, using (8), r = 1.15872 8473 a.

It is, of course, possible to deduce an implicit relation between r and the given data, such as

$$\frac{r}{2}\sqrt{4a^2 - r^2} - (r^2 - 2a^2)\cos^{-1}\frac{r}{2a} = \frac{\pi a^2}{1 + s/t} \qquad (9)$$

but it would be awkward for a cut-try solution for r directly.

Also solved by R. ROBINSON ROWE, Sacramento, California.

339. *[Spring 1975] Proposed by Paul Erdös, Budapest, Hungary.*

Let $a_1 < a_2 < \cdots$ be a sequence of integers $(a_i, a_j) = 1$; $a_{i+2} - a_{i+1} \geq a_{i+1} - a_i$. Prove that $\sum \frac{1}{a_i} < \infty$.

Solution by Ernst G. Straus, University of California at Los Angeles.

Set $d_n = a_{n+1} - a_n$. Then we have $1 \leq d_1 \leq d_2 \leq \cdots$. The number of times the same difference d can occur is $< 2p$ where p is the smallest prime which does not divide d. To see this we only need to observe that among p consecutive numbers in an arithmetic progression, a, a + d, a + $2d$, $\cdots$, there is one term divisible by p. Thus in $2p$ consecutive terms there would be two numbers divisible-by p, contrary to the condition $(a_i, a_j) = 1$ for $i \neq j$.

To estimate the magnitude of the smallest prime p which does not divide d, we could use some number theoretic considerations to prove $p < c \log d$ for a suitable constant a. However, we restrict ourselves to using the following:

Bertrand's Postulate. For every $x > 1$ there exists a prime number between x and $2x$.

Now d can have at most one prime divisor $p > \sqrt{d}$ since the product of two such primes would be too big. But by Bertrand's Postulate, there are at least two primes between $\sqrt{d}$ and $4\sqrt{d}$ and thus $p < 4\sqrt{d}$. (Of course, by an exactly analogous argument we can prove $p < 8\sqrt[3]{d}$, $p < 16\sqrt[4]{d}$, etc. but we don't need this).

So, if we let $m_1, m_2, \cdots, m_d, \cdots$ be the multiplicities of the differences 1, 2, $\cdots$, d, $\cdots$ then $m_d < 2p < 8\sqrt{d}$ and for $n = m_1 + m_2 + \cdots + m_{d-1} + m_{d'}$, with $0 \leq m_{d'} \leq m_d$, we have $d_n \geq d$. Then

$$n < 8(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{d}) \leq 8d^{3/2}$$

and $d_n \geq d = \frac{1}{4}(8d^{3/2})^{2/3} > \frac{1}{4}n^{2/3}$.

Thus

$$a_n = a_1 + d_1 + \cdots + d_{n-1} \geq 1 + \frac{1}{4}\left(1 + 2^{2/3} + \cdots + (n-1)^{2/3}\right)$$

$$> \frac{1}{4}\int_0^{n-1} x^{2/3}dx = \frac{3}{20}(n-1)^{5/3}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{a_n} < 1 + \frac{20}{3}\sum_{n=1}^{\infty}\frac{1}{n^{5/3}} < \infty .$$

Also solved by the PI MU EPSILON CALIFORNIA ETA PROBLEM SOLVING GROUP, University of Santa Clara, Santa Clara, California; ZAZOU KATZ, Beverly Hills, California; and the Proposer. Two incorrect solutions were received.

340. *[Spring 1975] Proposed by Charles W. Trigg, San Diego, California.*

The arithmetic mean of the twin primes 17 and 19 is the heptagonal number 18. Heptagonal numbers have the form $n(5n - 3)/2$. Are there any other twin primes with a heptagonal mean?

Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

If $H_n = n(5n - 3)/2$, then $H_n - 1 = (5n + 2)(n - 1)/2$ which is composite for $n = 2$ and $n > 3$. Therefore (17,19) is the only twin prime pair having a heptagonal mean.

Also solved by DAVID B. ANDERSON, student, University of California, Davis; JEFFREY BERGEN, Undergraduate, Brooklyn College; LOUIS H. CAIROLI, Syracuse University, S. Euclid, Ohio; CLAYTON W. DODGE, University of Maine at Orono; ALIZA DUBIN, Far Rockaway, New Yotk; STEVEN J. EBERHARD, Cambridge, Massachusetts; VICTOR G. FESER, Mary College, Bismarck, North Dakota.; PETER A. LINDSTROM, Genesee Community College., Batavia, New York; HENRY OSNER, Modesto Junior College, California; PI MU EPSILON CALIFORNIA ETA PROBLEM SOLVING GROUP, University of Santa Clara, California; BOB PRIELIPP, University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; I. PHILIP SCALISI, Bridgewater State College, Bridgewater, Massachusetts; RICHARD SIBNER, Los Angeles, California; BRUCE A. YANOSHEK, Cincinnati, Ohio; and the Proposer.

341. *[Spring 1975] Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

Prove that the following construction trisects an angle of 60°.

Triangle ABC is a $30°$-$60°$-$90°$ right triangle inscribed in a circle. Median CM is drawn to side AB and extended to M' on the circle. Using a marked straightedge, point N on AB is located such that CN extended to N' on the circle makes $NN' = MM'$. Then CN trisects the $60°$ angle ACM.

Solution by Charles W. Trigg, San Diego, California.

As shown in Figure 3, $r = MC = MA$, so triangle CMA is isosceles with $\angle MCA = \angle MAC = 60°$, whereupon $\angle CMA = 60°$.

Also, $r = MM' = NN' = MN'$, so triangles $N'MC$ and $MN'N$ are isosceles with $\angle MCN = \angle MN'N = x$, and $\angle N'MN = \angle N'NM = (180° - x)/2$.

The vertical angle of $\angle N'NM$ is an exterior angle of triangle CMN and therefore is equal to $60° + x$.

Consequently $(180° - x)/2 = 60° + x$, and $x = 20°$, which is $\angle ACM/3$.

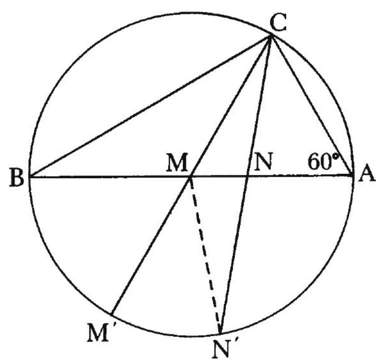


FIGURE 3

Also solved in a similar manna by JEFFRY BERGEN, *Brooklyn College;* JEANETTE BICKLEY, *Webster Groves High School, Missouri;* VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh. Trigonometric solutions were offered by* CLAYTON W. DODGE, *University of Maine at Orono, and by the Proposer.*

Bob Prielipp comments that readers interested in this problem would probably enjoy the following article: A. H. Lightstone, "A Construction for Trisecting the Angle," Mathematics Magazine, March-April (1962), 99-102.

342. [Spring 1975] *Proposed by David L. Silverman, West Lob Angeles, California.*

In The Game of the Century two players alternately select dates of the Twentieth Century (1 January 1901 - 31 December 2000) subject to the following restrictions:

1. The first date chosen must be in 1901.

2. Following the first play, each player, on his turn, must advance his opponent's last date by changing exactly one of the three "components" (day, month, year).

3. Impossible dates such as 31 April or 29 February of a non-leap year are prohibited.

The player able to announce 31 december 2000 is the winner.

a. What are the optimal responses by the second player to first player openings of 4 July 1901? 25 December 1901?

b. Who has the advantage and what is the optimal strategy?

c. What is the maximum number of moves that can occur if both players play optimally?

Solution by Zelda Katz, Beverly Hills, California.

a. 4 July 1970. 25 December 1994.

b. If every month had 30 days, this would be a simple realization of Nim with three piles of 12, 30, and 100 counters. The fact that the months are not of uniform length complicates things somewhat, but the Nim property of "safe leaves" remains intact. Since it constitutes the winning stroke, 31 December 2000 is the most obvious "safe" or "winning" date. Consider the dale 31 October 1999. It is also winning, since the only legal replies to it are 31 December 1999 or 31 October 2000, both of which can be converted to 31 December 2000. This suggests a systematic method for determining all winning dates. Using a two-dimensional day-month coordinate axis and working South and West from December 31st (see table on next page), the winning year for each day-month pair (obviously unique) is generated as follows: first, the six illegal day-month pairs are blocked out. The year 2000 (abbreviated 0 in the table with all other years abbreviated by their last two digits) is entered in the 31 December position. Henceforth, the year entered in each position is the most advanced year that does not already appear either directly North or Last, with the special proviso that in the case of 29 February, the year entered must be a leap year. That this in fact generates winning dates is clear by definition of the algorithm. If a player reaches a day-month pair with a year smaller than that indicated in the diagram, the winning play is to advance the year to the latter. If instead he reaches a day-month pair with a year larger than that indicated in the diagram, the algorithm ensures that his opponent will find the year in question, either North or

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	36	27	28	29	30	31
December	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	35	96	97	38	39	0
November	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	36	97	98	99	0	X
October	69	73	71	72	76	74	75	79	77	78	82	80	81	85	83	84	88	86	87	91	89	90	94	92	93	37	95	96	0	98	99
September	72	70	74	75	73	77	78	76	80	81	79	83	84	82	86	87	85	89	90	88	93	94	91	95	92	38	99	0	96	97	X
August	73	74	70	71	72	78	79	75	76	77	83	84	80	81	82	88	89	85	86	87	92	93	95	90	91	34	0	99	97	96	98
July	68	69	75	70	77	73	74	80	81	76	78	79	85	86	87	82	83	84	91	92	88	89	90	96	99	0	98	93	94	95	97
June	74	75	76	77	71	72	73	81	82	83	84	78	79	80	88	89	90	91	85	86	87	96	97	98	0	39	93	92	95	94	X
May	67	68	69	78	79	80	81	73	74	75	76	77	87	88	89	83	82	90	84	85	86	97	99	0	98	31	94	95	92	93	96
April	66	67	77	76	78	79	72	74	75	85	86	87	88	89	80	81	84	82	83	95	96	98	0	99	97	30	91	94	93	92	X
March	76	77	78	68	69	70	71	72	73	84	85	88	86	87	79	80	81	83	82	94	99	0	98	97	96	33	92	89	90	91	95
February	75	76	68	69	80	71	83	84	85	82	77	86	78	79	81	93	94	95	97	98	0	99	96	87	89	32	90	91	88	X	X
January	65	66	67	79	70	81	80	82	83	74	75	76	77	78	93	92	96	97	99	0	98	95	84	85	86	37	89	88	91	90	94
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	36	27	28	29	30	31

WINNING DATES IN THE GAME OF THE CENTURY

East of the day-month position (or both) and can, therefore, retain his advantage by advancing either the day or the month.

The "contour map" represented by the diagram of winning dates shows a lack of apparent structure but indicates clearly that there is exactly one winning year for each day-month pair and that the second player has the advantage. In fact, he has the advantage even if the first player Is allowed to select any date in the first 64 years of the century.

c. An optimally played game can be prolonged to 72 moves (36 moves each) by forcing the second player to advance the year 36 times, from 1965 to 2000. One such sequence involves the successive day-month pairs: 1 January, 2 January, 3 January, 3 February, 3 May, 3 August, 3 October, and 3 through 31 December.

Solutions were also offered by R. ROBINSON ROWE, *Sacramento, California, and the Proposer.*

343. [Spring 1975] *Proposed by R. Robinson Rome., Sacramento, California.*

Current serious promotion of a tunnel under the English Channel, combined with the energy crunch, has renewed interest in a fall-through tunnel under Bering Strait. From Cape Prince of Wales on Alaska's Seward Peninsula to Mys Dezhneva (East Cape) on Siberia's Chukuski Peninsula is 51 miles. A straight tunnel 58 miles long could be driven in earth below the bed of the Strait, which is 20 fathoms deep near each shore and 24 fathoms near mid-Strait. A frictionless vehicle could "fall" through such a tunnel without motive power. How long would it take? (At latitude 66° North, the earth's radius is 3954 miles and the acceleration of gravity $g = 32.23$ ft/sec^2.)

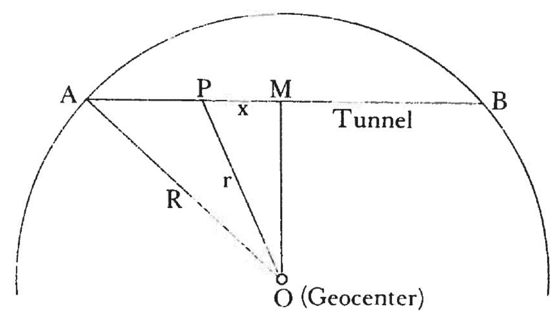


FIGURE 4

Solution by the Proposer.

Subterranean acceleration due to gravity is proportional to distance from the geocenter. In Figure 4, O is the geocenter, AB the tunnel, M its midpoint, and P a point distant x from M. Let the earth's radius AO = R and $OP = r$. Then gravitational attraction of a mass m at P is F = mgr/R and the component along AB is $F' = F\cdot x/r = mgx/R$. This will produce an acceleration at P toward M of $a = gx/R$. Now g and R are constants, so acceleration is proportional to distance from the midpoint and the motion is simple harmonic. Like a pendulum with small amplitude, the half-period is $\pi\sqrt{R/g}$. This will also be the elapsed time for traverse of the tunnel, so

$$T = \pi\sqrt{\frac{3954\cdot 5280}{32.23}} = 2528.45375 \text{ sec}$$

$$= 42 \text{ min, } 8.45375 \text{ sec.}$$

Comment

Paradoxically, T is independent of the length of the tunnel, whether that length be an inch or 7908 miles through to the antipodes. Of course, if the tunnel be too long, the earth's infernal heat and/or molten rock would make it impracticable, but here the maximum depth at M is only 561.5227 feet. If this depth at any point is h, the velocity there is

$$v = \sqrt{gh(2 - h/R)}$$

which becomes the familiar $v^2 = 2gh$ at the surface. At point M, this velocity is 129.716277 mph, which is not impractical.

Comment by the Problem Editor

According to information supplied by the proposer, the "fall-thru" tunnel problem first appeared in Civil Engineering, 18, No. 9 (1948), 70. Other references pertaining to this and related problems were located by your editor, namely:

1. The Mathematical Gazette, (1968), 376.
2. The *Mathematical* Gazette, (1970), 352.
3. The American Mathematical Monthly, (1968), 708.

By a weird coincidence, Rowe's proposal arrived at this editor's desk on January 19, 1975, exactly one day before the appearance in the *Wall* Street *Journal* of an article announcing that Paris was "stunned" at Britain's rejection of channel-tunnel plans.

Also solved by MARK JAEGER, *Madison, Wisconsin, and the Proposer.*

344. [Spring 1975] *Proposed by J. A. H. Hunter, Toronto, Canada.*

Three circles whose radii are a, b and c are tangent externally in pairs and are enclosed by a triangle each side of which is an extended tangent of two of the circles. Find the sides of the triangle. (See Figure 5.)

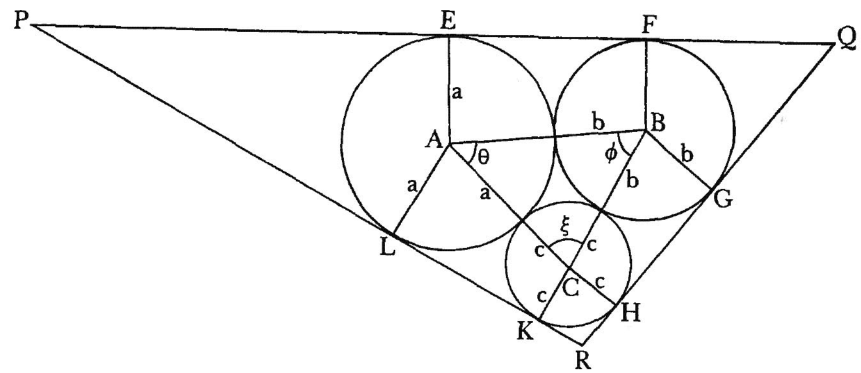


FIGURE 5

Solution by the Proposer.

We have in Figure 6a

$$EF = \sqrt{(a+b)^2 - (a-b)^2} = 2\sqrt{ab}\ .$$

Similarly, GH = $2\sqrt{bc}$, and $KL = 2\sqrt{ac}$.

$$\sin\alpha = \frac{a-b}{a+b} \qquad \sin\beta = \frac{b-c}{b+c} \qquad \sin\gamma = \frac{a-c}{a+c}\ ,$$

$$\cos\alpha = \frac{2ab}{a+b} \qquad \cos\beta = \frac{2bc}{b+c} \qquad \cos\gamma = \frac{2ac}{a+c}\ .$$

Also,

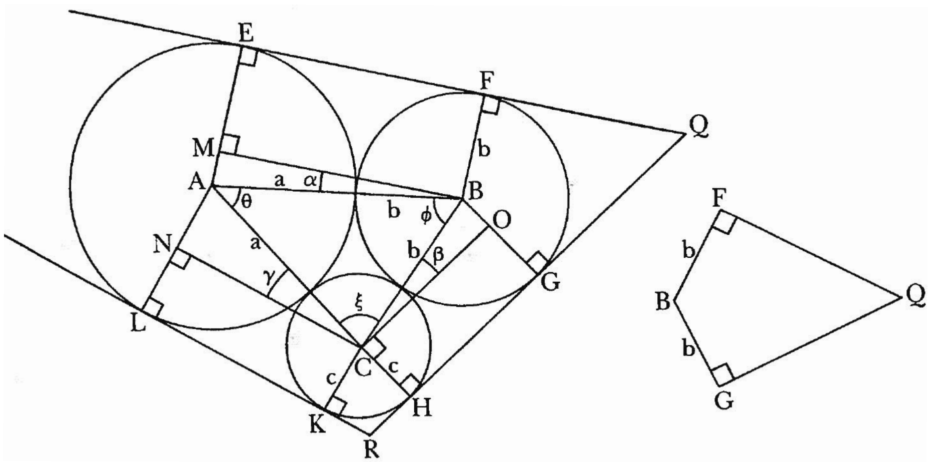
$$\sin\theta = \frac{2\sqrt{s(s-[a+b])(s-[b+c])(s-[a+c])}}{(a+b)(a+c)}$$

where

$$s = a + b + c\ .$$

so

$$\sin\theta = \frac{2\sqrt{abc(a+b+c)}}{(a+b)(a+c)}$$



(a) (b)

FIGURE 6

Similarly,

$$\sin\phi = \frac{2\sqrt{abc(a+b+c)}}{(a+b)(b+c)} \quad \text{and} \quad \sin\xi = \frac{2\sqrt{abc(a+b+c)}}{(a+c)(b+c)}.$$

By inspection: $\angle FQG = (\phi + a - \beta)$, $\angle EPL = (\theta - a - \gamma)$ so $\angle KRH = 180^\circ - ([\theta + \phi] - [\beta + \gamma])$.

Obviously, in Figure 6b, FQ = GQ; similarly, KR = HR, and EP = LP. Then,

$$FQ = b\cot\frac{\angle FQG}{2} = b\cot\frac{\phi + \alpha - \beta}{2}$$

Similarly,

$$KB = c\cot\frac{\angle KRH}{2} - c\cot\left(90^\circ - \frac{[\theta+\phi] - [\beta+\gamma]}{2}\right)$$
$$= a\tan\frac{[\theta+\phi] - [\beta+\gamma]}{2}$$

and

$$EP = a\cot(\angle LPE) = a\cot\frac{\theta - \alpha - \gamma}{2}.$$

Thence

$$PQ = a\cot\frac{6 - a - y}{2} + b\cot\frac{\phi + \alpha - \beta}{2} + 2\sqrt{ab}$$

$$QR = b\cot\frac{\phi + \alpha - \beta}{2} + c\tan\frac{[\theta + \phi] - [\beta + \gamma]}{2} + 2\sqrt{bc}$$

$$RP = a\cot\frac{\theta - \alpha - \gamma}{2} + c\tan\frac{[\theta + \phi] - [\beta + \gamma]}{2} + 2\sqrt{ac}$$

where

$$\sin\theta = \frac{2\sqrt{abc(a+b+c)}}{(a+b)(a+c)}, \qquad \sin\phi = \frac{2\sqrt{abc(a+b+c)}}{(a+b)(b+c)},$$

$$\sin a = \frac{a - b}{a + b}, \qquad \sin\beta = \frac{b - a}{b + c}, \qquad \sin\gamma = \frac{a - a}{a + c},$$

with $a \ge b \ge c$, whichisthe general solution.

Example. Say $a = 5$, $b = 4$, $c = 3$.

Then $6 \approx 48^\circ\ 12'$, $\phi \approx 58^\circ\ 24'$, $a \approx 6^\circ\ 23'$, $\beta \approx 8^\circ\ 13'$, $\gamma \approx 14^\circ\ 29'$.

Thence, the triangle sides are approximately 38.94, 17.06, and 31.01.

Also solved by **R. ROBINSON ROWE**, *Sacramento, California.*

345. [Spring 1975] *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

Resolve the paradox:

$$i(\sqrt{i} + \sqrt{-i}) = i\sqrt{i} + i\sqrt{-i} = \sqrt{-i} + \sqrt{i} = \sqrt{i} + \sqrt{-i}.$$

Solution by the Pi Mu Epsilon California Eta Problem Solving Group.

The roots of $\sqrt{i}$ are $\pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ and of $\sqrt{-i}$ are $\pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$ so that the possible number of expressions for $\sqrt{i} + \sqrt{-i}$ is four:

$$\sqrt{i} + \sqrt{-i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \qquad (1)$$

$$= \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \qquad (2)$$

$$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \qquad (3)$$

$$= \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \qquad (4)$$

On multiplying each expression by i we get the corresponding sums:

for (1) $\qquad i(\sqrt{i} + \sqrt{-i}) = -\sqrt{-i} + \sqrt{i}$

for (2) $\qquad i(\sqrt{i} + \sqrt{-i}) = -\sqrt{-i} + \sqrt{i}$

for (3) $\qquad i(\sqrt{i} + \sqrt{-i}) = \sqrt{-i} - \sqrt{i}$

for (4) $\qquad i(\sqrt{i} + \sqrt{-i}) = \sqrt{-i} - \sqrt{i}$

In no case is $i(\sqrt{i} + \sqrt{-i}) = \sqrt{-i} + \sqrt{i}$; therefore, $i(\sqrt{i}+\sqrt{-i}) \neq \sqrt{i} + \sqrt{-i}$. The mistake stems from the fact that the complex numbers are not ordered.

Also solved by JEFFREY BERGEN, *Brooklyn College, New York;* LOUIS H. CAIROLI, *Syracuse University, S. Euclid, Ohio;* CLAYTON W. DODGE, *University of, Maine at Orono;* VICTOR G. FESER, *Mary College, Bismarck, North Vakota;* RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* MARK JAEGER, *Madison, Wisconsin;* JEAN LANE, *West New York, New Jersey;* PAUL W. MERRIAM, *Lob Angeles, California;* C. B. A. PECK, *State College, Pennsylvania;* BOB PRIELIPP, *The University of, Wisconsin-Oshkosh;* and *the Proposer.*

346. [Spring 1975] *Proposed by* R. S. Luthar, *University of Wisconsin, Janesville.*

The internal angle bisectors of a convex quadrilateral $ABCD$ enclose another quadrilateral $EFGH$. Let FE and GH meet in M and let GF and HE meet in N. If the internal bisectors of angles EMH and ENF meet in L, show that angle NLM is a right angle.

Solution by Charles W. Trigg, *San Diego, California.*

The sum of the interior angles of a quadrilateral is $360°$. From triangles AHD and BFC in Fig. 7,

$$\angle AHD = 180° - \frac{1}{2}(\angle BAD + \angle ADC),$$

$$\angle BFD = 180° - \frac{1}{2}(\angle DCB + \angle CBA).$$

Then $$\angle AHD + \angle BFD = 360° - \frac{1}{2}(360°) = 180°,$$

so quadrilateral $EFGH$ is inscriptible.

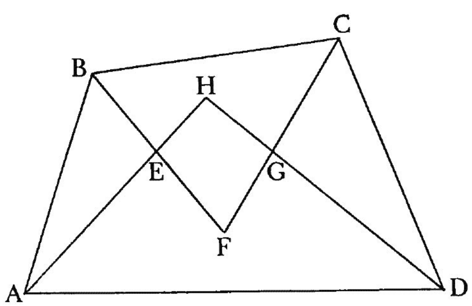


FIGURE 7

In Fig. 8, $\angle LMN = \frac{1}{2}(\angle FMN + \angle HMN)$ and $\angle LNM = \frac{1}{2}(\angle FNM + \angle HNM)$. Then from triangles MLN, MFN and MHN,

$$\begin{aligned}\angle MLN &= 180° - \angle LMN - \angle LNM\\ &= 180° - \frac{1}{2}(\angle FMN + \angle FNM + \angle HMN + \angle HNM)\\ &= 180° - \frac{1}{2}(180° - \angle MFN + 180° - \angle MHN)\\ &= \frac{1}{2}(\angle MFN + \angle MHN) = \frac{1}{2}(\angle EFG + \angle EHG)\\ &= 90°.\end{aligned}$$

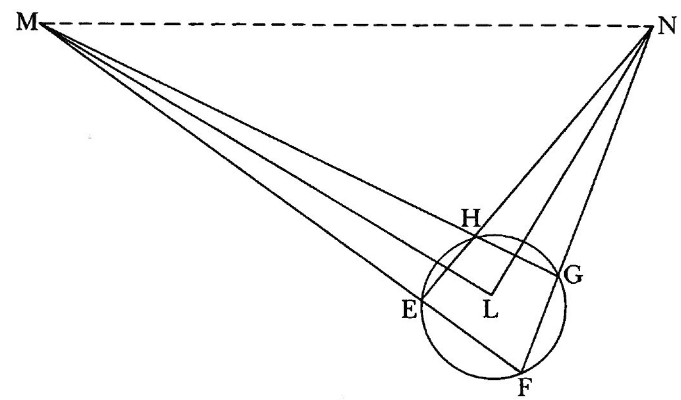


FIGURE 8

Also solved by JEFFREY BERGEN, *Brooklyn College, New Yohk;* CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* and *the Proposer.*

347. [Spring 1975] *Proposed by* Joe Van Austin, *Emoty University, Atlanta, Georgia.*

It is easy to show that $f(x) = \frac{\sin x}{x} - \frac{99x}{4} + 1$ for $x > 0$,

(i) has a linear asymptote $y = -\frac{99x}{4} + 1$, and

(ii) $f(x)$ crosses this asymptote for all $x = n\pi$ for $n = 1,2,\cdots$.

Show that the derivative $f'(x)$ is never zero for $x > 1$.

Solution by Paul W. Merriam, Los Angeles, California.

For $x > 0$, we have

$$f'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} - \frac{99}{4}.$$

Then since $\cos x \le 1$ and $\sin x \ge -1$, we have, for $x > 1$,

$$f'(x) \le \frac{1}{x} + \frac{1}{x^2} - \frac{99}{4} \le 1 + 1 - \frac{99}{4} = -\frac{91}{4} < 0.$$

Also solved by VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* MARK JAEGER, *Madison, Wisconsin;* THE PI MU EPSILON CALIFORNIA ETA PROBLEM SOLVING GROUP, *University of Santa Clara, California; and the Proposer, who remarks: "When studying linear asymptotes, it is often felt that if $f'(x)$ is never zero then $f(x)$ does not cross the asymptote. While this is true for horizontal asymptotes, this example shows that this is not true in general."*

348. [Spring 1975] *Proposed by Bob Prielipp and N. J. Kuenzi, The University of Wisconsin-Oshkosh.*

When the digits of the positive integer N are written in reverse order, the positive N' is obtained. Let $N + N' = S$. Then S is called the sum after one reversal addition. A palindromic number is a positive integer that reads the same from right to left as it does from left to right. The n'th triangular number $T_n = n(n+1)/2$, $n = 1,2,3,\cdots$.

Prove that there are infinitely many triangular numbers which have a palindromic sum after one reversal addition in the base b, where b is an arbitrary positive integer ≥ 2.

Solution by Pi Mu Epsilon California Eta Problem Solving Group, University of Santa Clara.

It is sufficient to show that there exists an infinite sequence $\{a_i\}$ for which $\frac{a_i(a_i + 1)}{2}$ has a palindromic sum after one reversal addition in the arbitrary base b for all i. Take

$$\{a_i\} = \{b^2, b^3, b^4, \cdots\}.$$

To show this satisfies the conditions, consider two cases.

(i) When b is even.

(ii) When b is odd.

(i) If b is even, then $\frac{b^k(b^k+1)}{2} = \frac{b}{2} \cdot b^{2k-1} + \frac{b}{2} \cdot b^{k-1}$, which when written in base b is $\frac{b}{2}\,00\cdots0\,\frac{b}{2}\,00\cdots0$, a number with $2k$ digits.

The first digit is $\frac{b}{2}$, followed by $k - 1$ zeros, then another $\frac{b}{2}$, followed by $k - 1$ zeros. After one reversal addition, the sum is

$$\frac{b}{2}\,0\cdots0\,\frac{b}{2}\,\frac{b}{2}\,0\cdots0\,\frac{b}{2},$$

a number with $2k$ digits which is obviously palindromic.

(ii) If b is odd, then the representation for $\frac{b^k(b^k+1)}{2}$ is not so simple.

$$\frac{b^k(b^k+1)}{2} = \left(\frac{b-1}{2}\right)b^{2k-1} + \left(\frac{b-1}{2}\right)b^{2k-2} + \cdots + \left(\frac{b-1}{2}\right)b^{k+1} + \left(\frac{b+1}{2}\right)b^k,$$

which when written in base b is

$$\frac{b-1}{2}\,\frac{b-1}{2}\cdots\frac{b+1}{2}\,00\cdots0,$$

again a number with $2k$ digits, the last k of which are zeros. After one reversal addition, the sum is

$$\frac{b-1}{2}\cdots\frac{b-1}{2}\,\frac{b+1}{2}\,\frac{b+1}{2}\,\frac{b-1}{2}\cdots\frac{b-1}{2},$$

a number with $2k$ digits which is obviously palindromic.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* CHARLES W. TRIGG, *San Diego, California; and the Proposers.*

349. [Spring 1975] *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

If 2^n $(n \ge 1)$ is the highest power of 2 dividing an even perfect number m, prove that $\sigma(m^2) + 1 \equiv 0 \pmod{2^{n+1}}$, where $\sigma(m)$ denotes the sum of the divisors of m.

Solution by Clayton W. Dodge, University of Maine at Orono.

We know that $m = 2^n(2^{n+1} - 1)$ where $2^{n+1} - 1 = p$ is a prime number. Now $m^2 = 2^{2n}p^2$, so

$$\sigma(m^2) = (2^{2n+1} - 1)(1 + p + p^2),$$

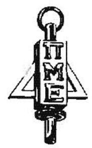
which reduces to

$$\sigma(m^2) = 2^{4n+3} - 2^{3n+2} - 2^{2n+1} + 2^{n+1} - 1.$$

It is easily seen that 2^{n+1} divides $\sigma(m^2) + 1$.

Also solved by JEFFREY BERGEN, *Brooklyn College, New York;* LOUIS H. CAIROLI, *Syracuse University, S. Euclid, Ohio;* VICTOR G. FESER, *Mary*

College, Bismarck, North Dakota; RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* I. PHILIP SCALISI, *Bridgewater State College, Bridgewater, Massachusetts;* C. B. A. PECK, *State College, Pennsylvania;* PI MU EPSILON CALIFORNIA ETA PROBLEM SOLVING GROUP, *University of Santa Clara, California;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* CHARLES W. TRIGG, *San Diego, California;* and the *Proposer.*



FRATERNITY KEY-PINS

Gold key-pins are available at the National Office (the University of Maryland) at the special price of $5.00 each, post paid to anywhere in the United States.

Be sure to indicate the chapter into which you were initiated and the approximate date of initiation.

ERRATA FOR LAST ISSUE

The name of Edith Risen, Portland State University, erroneously appeared as Edith Kisen in the list of solvers of Problem 326, p. 181.

LOCAL AWARDS

If your chapter has presented or will present awards to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal.*

INITIATION CEREMONY

The editorial staff of the *Journal* has prepared a special publication entitled ***Initiation Ritual*** for use by local chapters containing details for the recommended ceremony for initiation of new members. If you would like one, write to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

At the suggestion of the Pi Mu Epsilon Council we have had a supply of 10 × 14-inch Fraternity crests printed. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock - - - - - - $1.50/dozen,
(2) Purple and lavendar on goldenrod- - - $2.00/dozen.

LOCAL CHAPTER AWARDS WINNERS

CALIFORNIA ETA (University of Santa Clara). An award for outstanding achievement in upper division mathematics courses was presented to

Melissa Burns
James Dechene
James Hafner

Recognition for distinguished service to the Fraternity went to

Anne Mulligan

COLORADO DELTA (University of Northern Colorado). The *Outstanding Freshman Award* was presented to

James Fluke

The *Outstanding Senior Award* was presented to

Sam Sandh

FLORIDA EPSILON (University of South Florida). The two winners of the *Outstanding Scholar Award* were

Witold Kosmala
Mary Glynn Porter

GEORGIA GAMMA (Armstrong State College). College sponsored memberships in the American Mathematical Society were presented to

Donald Braffitt | *John Finders*
Philip Strenshi | *Anne Hudson*

The *Outstanding Freshman* and *Outstanding Senior* mathematics majors were, respectively,

Benjamin Zipper
Donald Braffitt

Special recognition was made to

Philip Strenski

who was among the top five participants nationally in the 1974 William Lower Putnam competition and was a member of the college's Putnam team which ranked *44th* nationally.

MISSOURI BETA (Washington University). The *Pi Mu Epsilon Prize* for the graduating member of Pi Mu Epsilon with the best mathematics record was awarded to

William E. Moerner

NEW YORK PHI (State University College at Potsdam). The *Outstanding Graduating Senior in Mathematics* for 1973-74 was

Barney L. Watson

NEW YORK PSI (Iona College). Nominated for the *Sullivan Award* and the *Julia Friedman Award* was

Elizabeth Reischer,

and the *Joseph E. Power Award* nomination went to

Michael Sasso

OHIO DELTA (Miami University). A test consisting of 14 calculus and linear algebra problems was administered, and awards based on this test were presented to

Lawrence Rogers (First Prize)
Cheryl Blausey (Second Prize)
Frederick Davenport, and
John Nelson (Third Prize, Tie)

OHIO EPSILON (Kent State University). The recipient of the 1975 *Pi Mu Epsilon Mathematics Award* was

David Wilson

OHIO NU (University of Akron). Recognition for outstanding academic records was given to

Gregory Davis | *Luke Maki*
Judith Harrison | *Valentina Ranaldi*
Norma Hoffmaster | *Elizabeth Schleier*
Denise Jones | *Gary Sponseller*
Gayleen Kolaczewski | *Roger Whiddon*
David Kuntz

For outstanding achievement in mathematics a membership in the Mathematical Association of America was awarded to

Robert Burnell | *W. Keith Shiflett*
Nancy Channell | *Gary Sponseller*
Elizabeth Schleier | *Roger Whiddon*

A $25 savings bond was presented to

Corinne Coons,

the first place winner in the Akron Public School's Science Fair, Mathematics Division.

A plaque was presented to

Professor Louis Ross,

permanent faculty correspondent, in appreciation for his guidance and efforts in establishing the local chapter.

OHIO ZETA (University of Dayton). The *Outstanding Sophomore Award* was presented to

Randy Smith

OREGON ALPHA (University of Oregon). The *De Cou Prize* awarded to outstanding mathematics students in honor of Edgar E. De Cou, professor of mathematics 1902–1944 and the first chairman of the department, was won by

David Jarrett
Patrick Keef
Ronald Siegel

RHODE ISLAND BETA (Rhode Island College). A prize consisting of a $50 savings bond was initiated as the *Mitchell Award.* The recipient for 1974–75 was

Lisa Taglianetti

TEXAS DELTA (Stephen F. Austin State University). The Outstanding Senior Mathematics student for 1974–75 was

Mike McPhail

VIRGINIA GAMMA (Madison College). A $25 award each for outstanding senior mathematics students was presented to

Marilyn Lawson
Victoria Brown

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