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THE PAPPUS CONFIGURATION AND ITS GROUPS

by H.S.M. Coxeter[1]
University of Toronto

Consider, in a projective plane, a hexagon $AB'CA'BC'$ with its alternate vertices on two lines: ABC on one, $A'B'C'$ on the other. The theorem of Pappus states that the intersections of pairs of opposite sides, namely

$$L = BC'\cdot B'C, \quad M = CA'\cdot C'A, \quad N = AB'\cdot A'B,$$

lie on one line. Thus the complete figure consists of nine points and nine lines: a self-dual configuration 9_3. A more symmetrical notation is obtained by writing

$$8,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$$

instead of $A, B, C, A', B', C', L, M, N$. Then the conditions for points λ, μ, ν to be on one line are simply

$$\lambda + \mu + \nu \equiv 0 \pmod 9, \qquad \lambda \not\equiv \mu \pmod 3.$$

This Pappus configuration was rediscovered in 1839 by J.T. Graves, who regarded it as a cycle of three triangles, each inscribed in the next. From the three triangles (Figure 1)

$$012,\ 345,\ 678,$$

the complete set of six such Graves cycles is obtained by repeated doubling (and reduction modulo 9); for instance, the next cycle after the given one is

$$024,\ 681,\ 357.$$

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[1]The second lecturer in the J. Sutherland Frame Lecture Series of Pi Mu Epsilon. This article is a summary of Professor Coxeter's lecture, presented to the Fraternity at Toronto, Canada in August, 1976, reprinted by permission of Koninkl. Nederl. Akademie Van Wetenschappen, Amsterdam. The full text is to appear in the *Proceedings* of *the American Mathematical Society* -- *Editor*.

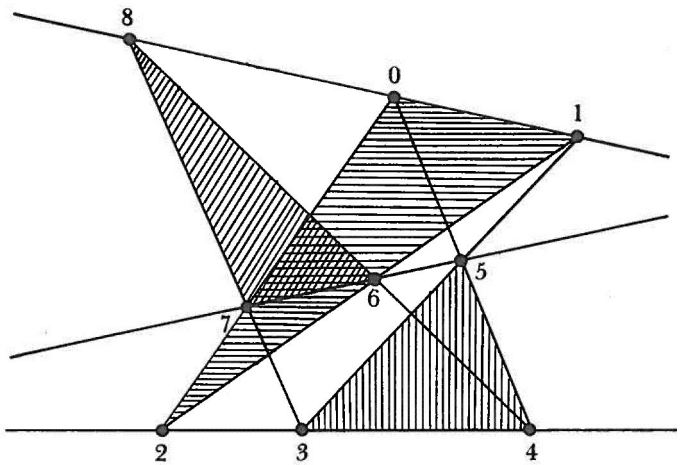

FIGURE 1

Such a configuration exists not only in the real plane but in any "Pappian" plane except $PG(2, 2)$ (which contains only seven points). For instance, the nine points of the finite affine plane $EG(2, 3)$ form a Pappus configuration when we omit three of its twelve lines: those in one direction. In this finite plane, each of the 18 Graves triangles forms (or is inscribed in) a parabola whose diameters are in that special direction. Products of pairs of polarities with respect to these parabolas generate a group of 108 affinities. These affinities, which are collineations of the projective plane $PG(2, 3)$, are precisely the 108 automorphisms of the configuration. The general Pappus configuration (in the real plane, for instance) has this same group of automorphisms. The above use of the finite field $GF[3]$ provides a convenient representation. Embedding the finite affine plane in a real Euclidean plane and then reducing the real coordinates modulo 3, we represent the 18 Graves triangles by the 18 faces of a regular map on a torus, and thus interpret the automorphism group as the symmetry group $G^{3,6,6}$ of that map. In terms of two generators

$$B = (056832)(47),\ C = (027)(165438),$$

this group has the presentation

$$B^6 = C^6 = (BC)^2 = (B^3C^2)^2 = (B^2C^3)^2 = 1.$$

Its commutator subgroup, generated by B^2 and C^2, is the Burnside group $B_{y,}$, of order 27.

Similarly, the 18 polarities of the configuration in the finite plane appear as dualities of the general Pappus configuration, and generate a group of order 216 in which $G^{3,6,6}$ occurs as a subgroup of index 2. Observing that B leaves one point invariant while C leaves one line invariant, we can identify one such duality with an involutory element D that transforms B into C (and C into B), so that the enlarged group, of order 216, has the presentation

$$B^6 = D^2 = (BD)^4 = (B^3DB^2D)^2 = 1.$$

In the group of 108 automorphisms of the general Pappus configuration, one subgroup of order 3, generated by the permutation (036)(147)(258), always consists of collineations. In general, however, at most six of the involutory dualities (such as D) can be polarities. If two Graves triangles belonging to the same cycle are a pair of perspective triangles, then each is likewise perspective with the remaining triangle in that cycle, and there are three conics such that each triangle is inscribed in one conic, circumscribed to another, and self-polar for the third. The corresponding polarities generate a dihedral group of order 6 whose cyclic subgroup is generated by the collineation (036)(147)(258). For a suitable triangle of reference and unit point, this collineation appears as a cyclic permutation of the three projective coordinates, and the conics have the equations

$$x^2 + 2yz = 0,\quad y^2 + 2zx = 0,\quad z^2 + 2xy = 0.$$

The vertices of the three triangles can then be written in the form

$$\begin{array}{lll} (2r, 2, -r^2) & (2s, 2, -a^2) & (2t, 2, -t^2) \\ (-r^2, 2r, 2) & (-s^2, 28, 2) & (-t^2, 2t, 2) \\ (2, -r^2, 2r) & (2, -s^2, 2s) & (2, -t^2, 2t) \end{array}$$

where r, s, t are the roots of the cubic equation

$$X^3 - 3kX + 2 = 0 \quad (k \neq 1).$$

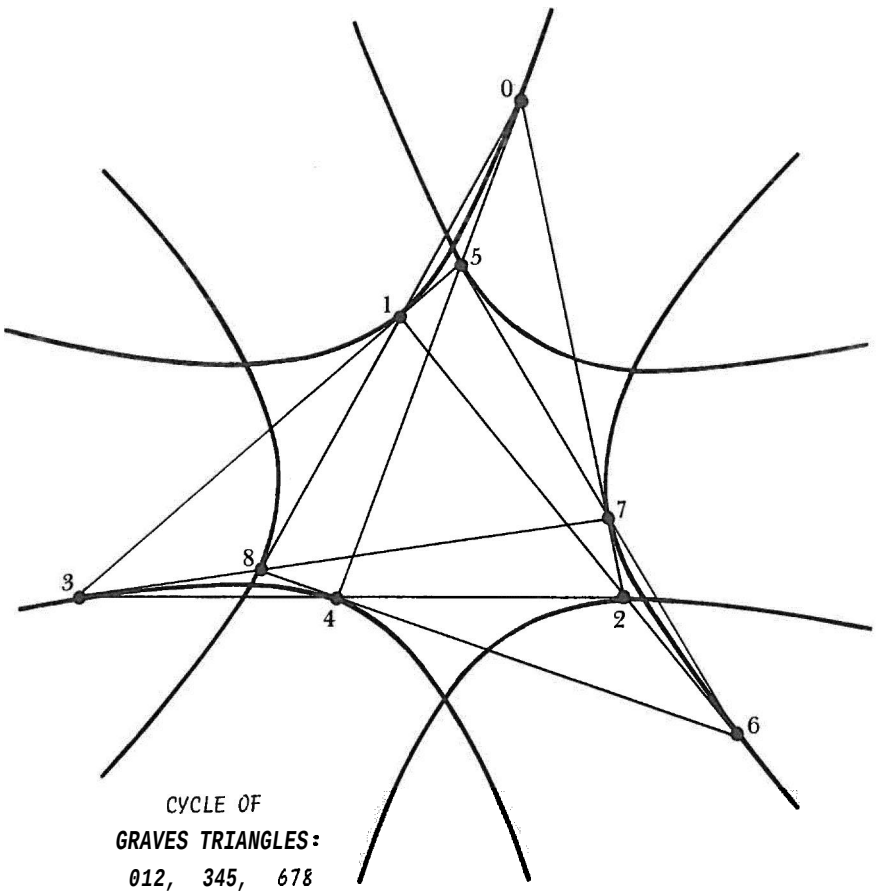

FIGURE 2

If the coordinates are real, k may take any value greater than 1.

In the Euclidean plane with areal or trilinear coordinates based on an equilateral triangle, the conics are rectangular hyperbolas and the whole figure has a very pleasing appearance (see Figure 2).

Another way to draw the same projective configuration is shown in Figure 3, where the conics consist of one rectangular hyperbola and two parabolas. A third way, shown in Figure 4, involves a rectangular hyperbola, a parabolaand a circle (due to P.W.L. Lemmens of Utrecht).

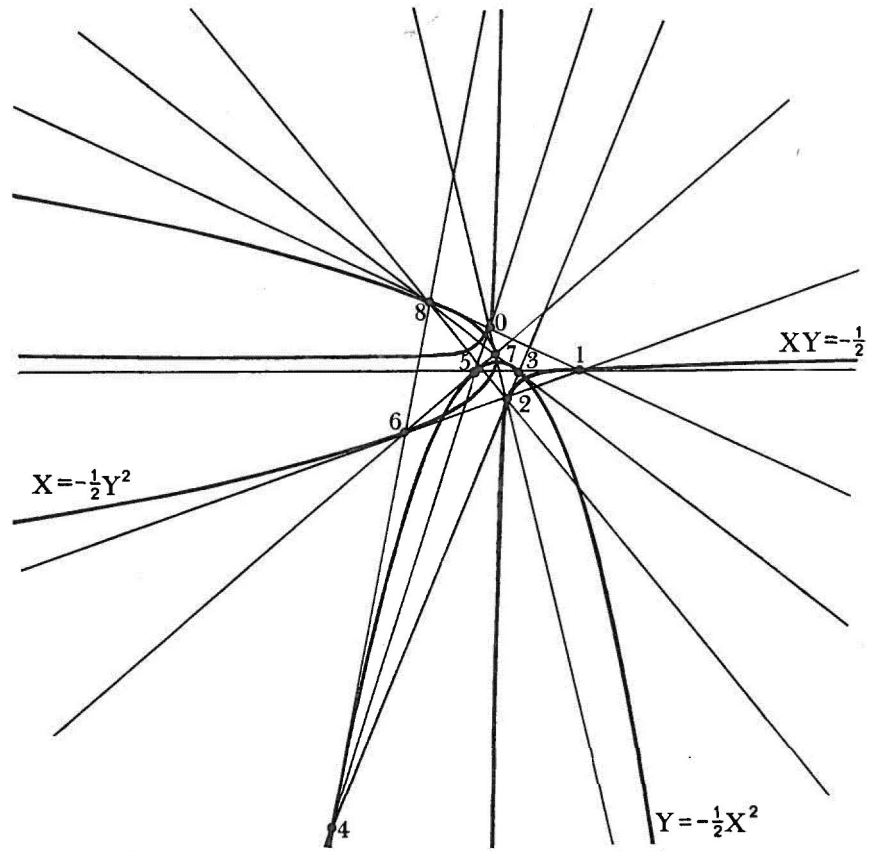

FIGURE 3

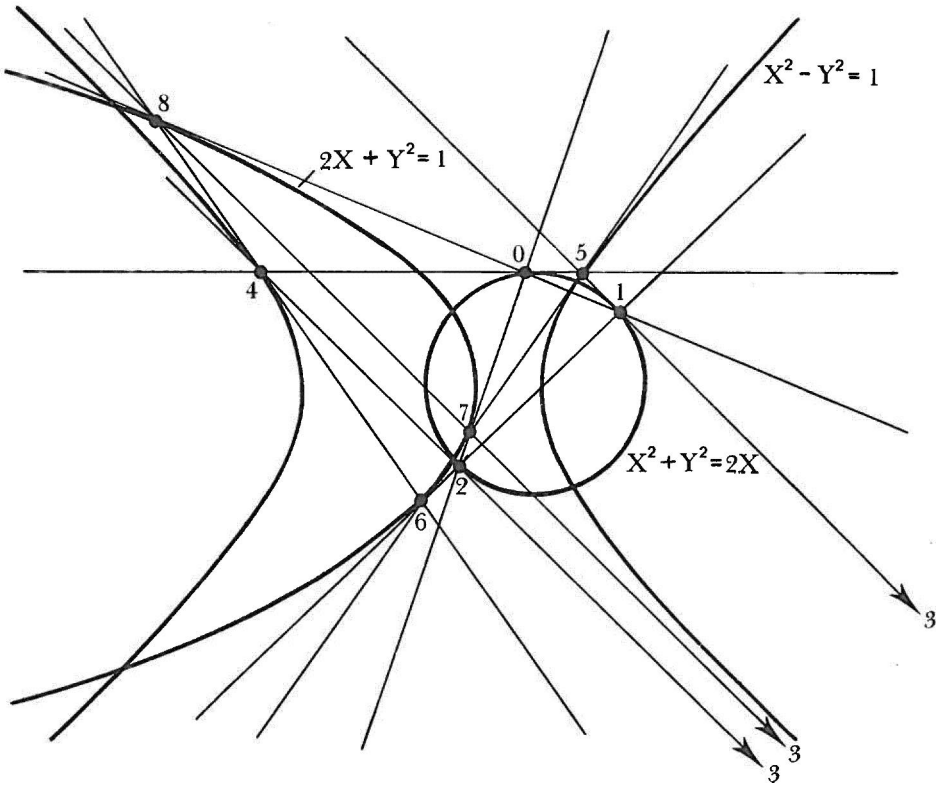

FIGURE 4

REFEREES FOR THIS ISSUE

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NUMERICAL INTEGRATION BY POLYNOMIAL INTERPOLATION

by Jackie L. Lawrence[1]
Western Kentucky University

1. Introduction

The interest in numerical integration by using the digital computer has prompted the development of many computational algorithms. Various techniques have gained popularity not as universal integration algorithms, but because of the varied, yet favorable characteristics that each possesses. To evaluate $\int_a^b f(x)dx$, the method presented here first produces an nth degree Chebyshev-based interpolatory polynomial $p_n(x)$ of the integrand over $[a,b]$. Using the Fundamental Theorem of Integral Calculus, the value of $\int_a^b p_n(x)dx$ is computed and used as an estimate of the original integral. By computerizing the entire algorithm and storing the coefficients of the antiderivative of $p_n(x)$, only two additional polynomial evaluations are required to evaluate $\int_c^d f(x)dx$ where $[c,d] \subset [a,b]$. Existing algorithms would require a complete reformulation of the problem. A theorem is derived which gives a maximum bound on the integration error. This error bound is valid for integration over the original interval or any subinterval of the original interval. The paper concludes with numerical comparisons between the new algorithm and several existing algorithms.

2. Development of the Method

Given $f(x)$ in the metric space of all continuous functions on $[a,b]$, denoted by $C[a,b]$, it is desired to evaluate $\int_a^b f(x)dx$. To this end, $f(x)$ is approximated over $[a,b]$ by an nth degree polynomial, $p_n(x)$, then the resulting approximation is integrated, viz.:

$$\int_a^b f(x)dx \approx \int_a^b p_n(x)dx = p_{n+1}(b) - p_{n+1}(a) \tag{1}$$

[1]The author wishes to express his appreciation to D. C. St. Clair for his help with this research project.

where $p'_{n+1}(x) = p_n(x)$. It should be emphasized that although $p_n(x)$ and $p_{n+1}(x)$ are defined over $[a,b]$, they are also valid approximations over any subinterval $[c,d] \subseteq [a,b]$.

The following theorem not only verifies the existence of p_n and p_{n+1}, but indicates a bound on the error produced by the approximation and integration steps. To the author's knowledge, this theorem represents a new contribution to the literature.

Theorem. Let $f(x) \in C[a,b]$. Given $\varepsilon > 0$, there exists a polynomial $p_{n+1}(x) \in C[a,b]$ such that $|[p_{n+1}(b) - p_{n+1}(a)] - \int_a^b f(x)dx| \le \varepsilon (b - a)$.

Proof. Let $f(x) \in C[a,b]$ and let $\varepsilon > 0$ be given. Then by the Weierstrass Approximation Theorem [1], there exists $p_n(x)$ such that $|p_n(x) - f(x)| \le \varepsilon$ where $a \le x \le b$, and b # a. Since $f(x) \in C[a,b]$ and since $p_n(x)$ is a polynomial, both are in $R[a,b]$, the set of Riemann integrable functions on $[a,b]$. Thus

$$\left|\int_a^b p_n(x)dx - \int_a^b f(x)dx\right| = \left|\int_a^b [p_n(x) - f(x)]dx\right|$$

$$\le \int_a^b |p_n(x) - f(x)|dx \le \int_a^b \varepsilon\, dx = \varepsilon\,(b - a).$$

Clearly, for $p'_{n+1}(x) = p_n(x)$, we have by the Fundamental Theorem of Integral Calculus

$$|[p_{n+1}(b) - p_{n+1}(a)] - \int_a^b f(x)dx| \le \varepsilon\,(b - a)$$

Several useful conclusions follow from the theorem: (1) one may approximate $f(x)$ by a polynomial and integrate that approximation, (2) the degree of the antiderivative $p_{n+1}(x)$ is one higher than the degree of the interpolation polynomial $p_n(x)$, (3) the error bound on integration is $\varepsilon\,(b - a)$ where ε is an error bound on the polynomial that approximates $f(x)$, and (4) if $(b - a) = 1$, the error is not increased by integration while if $(b - a) < 1$, the error is decreased by integration. Hence, the

error is a function of interval width and the error of the interpolating polynomial.

In his proof of the Weierstrass Approximation Theorem, Goldberg [1] uses Bernstein Polynomials to construct $p_n(x)$. However, to make the method computationally feasible, it was decided to obtain

$$p_n(x) = \sum_{i=0}^{n} \alpha_i x^{n-i}$$

by the interpolatory method of undetermined coefficients. This technique is equivalent to solving the system of $n + 1$ linear equations

$$\Phi\alpha = \beta \tag{2}$$

where

$$\Phi = \begin{pmatrix} x_0^n & \cdots & x_0 & 1 \\ x_1^n & \cdots & x_1 & 1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ x_n^n & \cdots & x_n & 1 \end{pmatrix}$$

is the matrix of powers of the interpolation points x_i, and α and β are the column vectors

$$\alpha = [\alpha_i], \qquad \beta = [f(x_i)].$$

Having determined the coefficients α_i of $p_n(x)$, the desired solution is obtained by polynomial integration, viz.:

$$\int_a^b f(x)dx \approx \int_a^b p_n(x)dx = p_{n+1}(b) - p_{n+1}(a) \tag{1}$$

where

$$p_{n+1}(x) = \sum_{i=0}^{n} \frac{\alpha_i}{n + 1 - i} x^{n+1-i}. \tag{3}$$

Remember that the approximation $p_{n+1}(x)$ is good for the interval $[a,b]$ and for any subinterval of $[a,b]$. Thus, if $[c,d] \subseteq [a,b]$

$$\int_c^d f(x)dx \approx P_{n+1}(d) - P_{n+1}(c) \tag{4}$$

which is easily calculated using equation (3).

As indicated by the Theorem, the error in equations (1) and (4) is bounded by $\varepsilon\,(b - a)$ where $\varepsilon \le |f(x) - p_n(x)|$ is the error for the general interpolating polynomial. Pizer [2] states this error as

$$\varepsilon = \frac{H(x)}{(n+1)!}\,|f^{n+1}(\zeta)| \tag{5}$$

for $\zeta \in [a,b]$ and $H(x) = \prod_{i=0}^{n} (x - x_i)$.

For a given value of x, a given function $f(x)$, and a given polynomial p (~) the error in equation (5) is a function of the x_i, the interpolation points. Thus, to minimize the maximum value of ε, it is desirable to choose x_i in such a way as to minimize the maximum value of $H(x)$. Pizer shows that the best choice of x_i for this purpose are the zeros of the $(n + 1)st$ Chebyshev polynomial. Since Chebyshev polynomials are defined on [-1, 1] and we are interested in the interval $[a,b]$, the x_i are chosen by the translation

$$x_i = \frac{b - a}{2} \cos\left(\frac{2i + 1}{2(n + 1)}\pi\right) + \frac{b + a}{2}$$

where $i = 0, 1, \cdots, n$ and n is the degree of $p_n(x)$ used in the interpolation.

Pizer verifies that choosing the x. as zeros of the $(n + 1)st$ degree Chebyshev polynomial yields the following bound on the interpolation error

$$\varepsilon \le \frac{2}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1} \max_{[a,b]} |f^{(n+1)}(\zeta)|.$$

Hence, the error produced by the integration in equations (1) and (4) is bounded by

$$\varepsilon\,(b - a) \le \frac{8}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+2} \max_{[a,b]} |f^{(n+1)}(\zeta)|.$$

This implies that the new method is of order n.

3. Numerical Examples

In this section, performance of the new method is compared with performances of the well-established Simpson, Romberg, and Gauss-Chebyshev[2] methods of numerical quadrature. In order to allow for error comparisons between algorithms, three problems having known analytic solutions were selected for testing. The results presented were obtained to seven significant digits using algorithms written in BASIC-PLUS and executed on a PDP 11/45.

The first problem considered was that of evaluating the integral

$$\int_1^{\pi} [4e^x + \frac{1}{x} - \sin(\sin x) \cos x]dx$$

whose solution is

$$[4e^x + \ln(x) + \cos(\sin x)]\,\Big|_1^{\pi} = 83.1680.$$

Table 1 contains computational results for this problem.

In addition to an estimate of the integral value, the new method applied over n subintervals also produces the $(n + 1)st$ degree polynomial P_{n+1} of equation (3). This makes the new method particularly attractive when it is desired to integrate over a subinterval such as $[1, \pi/2]$. Using $P_{17}(x)$ produced by the original integration over sixteen subintervals, the new integral can be computed by calculating $P_{17}(\pi/2) - P_{17}(1)$. The error is 0.30×10^{-4}. Evaluation of the integral over [2.5, 3.11 with the same polynomial gave an error of 0.38×10^{-5}. In both cases, no additional evaluations of the original function were required.

For each of the conventional algorithms, evaluation of the integral over $[1, \pi/2]$ and [2.5, 3.1] would require two complete reformulations of the problem. In tests, the Gauss-Chebyshev method required 100 additional function evaluations for each additional integration to produce

[2] This method is of interest since it also uses the Chebyshev zeros as interpolation points. An excellent development of this algorithm can be found in Hildebrand [3].

Method	Number of Intervals	No. of Function Values	Problem #1 CPU Time	Problem #1 Error	Problem #2 CPU Time	Problem #2 Error	Problem #3 CPU Time	Problem #3 Error
New	10	11	21	0.43×10^{-4}	21	0.35×10^{-3}	21	0.0
	16	17	68	0.29×10^{-4}	63	0.12×10^{-2}	62	0.30×10^{-6}
	20	21	110	0.65×10^{-4}	110	0.20×10^{-2}	110	0.36×10^{-6}
Simpson	10	15	2	50.34	1	0.10	1	0.80×10^{-6}
	16	24	3	49.87	1	0.10	1	0.24×10^{-6}
	20	30	3	49.65	2	0.10	2	0.12×10^{-6}
	500	750	106	46.63	52	0.10×10^{-1}	40	0.10×10^{-6}
	1000	1500	208	45.73	113	0.10×10^{-1}	81	0.36×10^{-6}
Romberg	6	126	19	49.11	10	0.10	8	0.0
	10	2046	295	46.34	147	0.10	110	0.18×10^{-6}
	16	131070	18071	42.18	9278	0.10	6944	0.42×10^{-6}
Gauss-Chebyshev	10	10	4	0.46	2	0.50×10^{-2}	2	0.28×10^{-2}
	16	16	5	0.18	4	0.20×10^{-2}	4	0.11×10^{-2}
	20	20	6	0.12	6	0.13×10^{-2}	5	0.70×10^{-3}
	500	500	166	0.16×10^{-3}	129	0.64×10^{-4}	118	0.48×10^{-6}
	1000	1000	336	0.76×10^{-5}	264	0.75×10^{-4}	235	0.30×10^{-6}

Note: One unit of CPU time is equivalent to about 0.1 second.

TABLE 1

errors of 0.37×10^{-3} and 0.17×10^{-2} respectively. This computation required an additional 64 units of computer time.

The second example tested was that of integrating

$$\int_{0.267}^{6.859} \frac{\Gamma(3)}{\sqrt{5\pi}\,\Gamma(2.5)} \left(1 + \frac{t^2}{5}\right)^{-3} dt$$

where $\Gamma(t)$ is the Gamma function. The integrand is Student's t probability density function with five degrees of freedom. The value of the integral, 0.3995, represents the probability that an observation falls in the interval [0.267, 6.859]. As Table 1 indicates, numerical results were competitive with the Gauss-Chebyshev method.

Since the nature of this type of problem often requires evaluation of the integral over a subinterval, the value of the integral over [1.476, 2.5711 was determined by using $P_{11}(x)$ from the method. The resulting error was 0.16×10^{-3}. Evaluation of this integral using the Gauss-Chebyshev method over ten subintervals required ten additional function evaluations

and two additional units of computer time to produce an error of 0.36×10^{-3}.

The third problem

$$\int_0^1 e^{-x^2} dx$$

whose solution is 0.746824, was suggested by Conte [4]. Table 1 indicates the new method produced results similar to those obtained for Simpson's method.

4. *Conclusions*

The numerical quadrature method presented here produces a polynomial interpolation of the integrand in question. This polynomial is computer integrated to obtain an antiderivative which not only allows one to integrate over the original interval, but to perform additional integrations over subintervals by simply evaluating polynomials. Hence in problems where a number of integrals are to be computed within a specified interval, the new method represents a considerable savings in computer time. This savings is even more pronounced when function values are expensive to evaluate. The author observed that applying the new method repeatedly to subintervals of the original interval required only a modest increase in the amount of programming but produced better answers.

The numerical results cited indicate the new method is competitive with, and in some cases better than the other methods tested. Due to the ill-conditioning inherent in the linear system of equation (2), several numerical methods were tried for solving this system. Gaussian elimination [5] was found to be the most consistently successful. In addition, the use of more significant digits in computation could be used to further reduce round-off errors.

Further study in this area seems worthwhile. In addition, other function approximations such as Fourier series might be used to replace the original integrand.

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A GENERALIZATION TO ALMOST DIVISIBLE GROUPS

by Karen M. Lesko
Central Missouri State University

Divisible groups and discussions of their properties arise in abelian group theory. By altering the definition of a divisible group to form a group whose structure is less restrictive, one may see how this might change the theorems regarding divisible groups. The purpose of this paper is to offer a definition of what will be called an "almost divisible" (which will be abbreviated "AD") group and to show how it might affect two theorems which hold for divisible groups.

Definition. An abelian group G is said to be *almost divisible* if, given a decomposition of G of the form $D \oplus K$, where D is a maximal divisible subgroup of G and K is reduced, a maximal independent set of K contains at most a finite number of elements.

One of the equivalent definitions of a divisible group is that D is divisible if for any non-zero integer n, $nD = D$. This result will not hold for any AD group G. However, if one places certain restrictions on G, then it will be possible to find an infinite number of integers such that $nG = G$ will hold, as the following theorem shows.

Theorem 1. Suppose G is an AD group. If every decomposition of G of the form $D \oplus K$, where D is divisible and K is reduced, is such that K has bounded order, then there exists an infinite number of primes p such that $pG = G$.

Proof. Suppose K has bounded order n. Then by definition of boundedness, every element k_i in K has order n_i, where n_i divides n. Let p be any prime such that n is less than p. Then pK is a subgroup of K.

Since K has bounded order, K is also expressible as a direct sum of cyclic groups (see [3]). Hence pK is expressible as a direct sum of p times each of the cyclic groups in the decompositon of K.

Let $gp(a)$ be any cyclic group in the direct sum of K, with the order of a being m. Then m is necessarily less than p. Also,

$$p(gp(a)) = \{pa, 2pa, \cdots, (m-1)pa, 0\}$$

are the elements in $p(gp(a))$. Suppose $ipa = jpa$, where i and j are integers between 0 and m, with $i > j$. Then $(i - j)pa = 0$, or $p(i - j) = km$ for some integer k. Since p and m are relatively prime, this implies that m divides $(i - j)$. However, this is a contradiction, since $(i - j)$ is an integer between 0 and m. Therefore, all of the elements above are distinct. Since $p(gp(a))$ contains m distinct elements and a has finite order which is relatively prime to p, $p(gp(a)) = gp(a)$. Then pK will equal K since $gp(a)$ was chosen at random. Thus there exists an infinite number of primes p such that $pK = K$, and also, $pG = p(D \oplus K) = pD \oplus pK = D \oplus K = G$.

By this theorem, one can see that divisible groups and AD groups need not behave in a similar manner. This is not to say, however, that they never act alike. It is already known [2] that the homomorphic image of a divisible group is divisible, and this result will also hold for AD groups.

Theorem 2. Any homomorphic image of an AD group is AD.

Proof, By definition, any decomposition of an AD group G as $D \oplus K$, where D is divisible and K is reduced, is such that K contains at most a finite number of independent elements. Therefore, under any homomorphism, the image of the divisible portion of G will still be divisible and the image of the reduced portion of G will contain at most a finite number of independent elements, as the homomorphic image of any group must contain fewer or the same number of independent elements as its pre-image. Therefore, the homomorphic image of G will still be AD.

Thus divisible groups and AD groups may act alike under certain circumstances, but AD groups are a more general class in which some properties for divisible groups are lost.

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d'ALEMBERT ENUMERATION AND PROBABILITY

by Louis G. Vargo
University of Missouri

1. *Introduction*

Jean d'Alembert (1717-1783) is known in the history of probability as one of a group of "prominent mathematicians [who] sometimes committed errors in solving elementary probabilistic problems" (see [1, p. 123-129]). His "errors" were not casual; they arose in principle. Simply put, d'Alembert believed that the enumeration of possible outcomes of an experiment depends on the probabilistic event under consideration. Classical probability theory rests on the contrary. The author asserts that d'Alembert was not wrong, and that his position is directly analogous to the views of those who first questioned the inviolability of the fifth postulate in Euclid's geometry. Parallel to what happened in geometry, this note shows that a non-classical probability theory can be generated when d'Alembert's ideas are applied to a class of finite outcome experiments.

2. *Are Example.*

Consider two successive tosses of a coin. Call the possible outcomes of a toss H and T. Classical enumeration of the two-toss outcomes gives HH, HT, TH and TT before any event is specified. Let $2H$, Iff and OH represent the events that exactly two, one and no H's respectively occur in the two tosses. These events are mutually exclusive and exhaustive in terms of the number of H's which appear. Thus, classical probability has

$$P(2H) + P(1H) + P(0H) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1. \qquad (1)$$

Again consider two tosses of a coin. For the event $2H$, d'Alembert would enumerate all the two-toss outcomes as HH, HT and Tb, where b indicates that the outcome of the second toss is ignored since 2H cannot occur if the first toss resulted in T. Only one of the outcomes is 2H, and there are three *possible* outcomes. Letting Q symbolize the d'Alembert probability, $Q(2H) = 1/3$. For the event IS, the d'Alembert outcomes are the same as the classical. $Q(1H) = 2/4 = 1/2$. For OH, a d'Alembert enumeration gives Hb, TH and TT, and hence $Q(0H) = 1/3$.

The historical record does not show that d'Alembert inquired into the probability of compound events. We shall see that this forces a decision regarding the rejection of one or the other of the following two axioms of classical probability:

I. $Q(S) = 1$, where S is the sample space, the set consisting of all elementary outcomes; S = {OH, Iff, $2H$} in this two-toss example.

II. Q(A or B) = $Q(A) + Q(B)$, where A and B are any two mutually exclusive events, that is, A and B cannot occur in a single experiment. Again for the two-toss case, an example of the application of this axiom is

$$(\text{"less than two heads"}) = Q(0H \text{ or Iff}) = Q(0H) + Q(1H) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Both axioms cannot be retained since II implies

$$\begin{aligned} Q(S) &= Q(0H \text{ or } 1H \text{ or } 2H) = Q(0H \text{ or Iff}) + Q(2H) \\ &= Q(0H) + Q(1H) + Q(2H) = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6} \end{aligned} \qquad (2)$$

contradicting I. We choose to reject I because the values of $Q(S)$ serve as convenient measures of the disparity between classical and d'Alembert systems. In this approach, II is used to define the probability of any non-elementary event. Enumeration is required only in the determination of elementary outcome probabilities. Further, it might be argued that II has more structural significance than I. Specification of a fixed value for $Q(S)$ indicates a restrictive arbitrariness comparable to that included in the fifth postulate.

3. *The General* Case.

Now consider n tosses of a coin. In the event that exactly i H's occur,

$$P(iH) = \binom{n}{i} 2^{-n}, \qquad i = 0, 1, \ldots, n.$$

$Q(iH)$ is not so easily obtained. A d'Alembert enumeration considers successive toss outcomes until iH cannot occur. If on the jth toss, the $(i+1)$st H appears, then the remaining toss outcomes would be labeled

b. This is the case of too *many* H's. For each of some j's, there are $\binom{j-1}{i}$ of these sequences; j must be at least $i + 1$ and can be as large as n. Thus

$$\sum_{j=i+1}^{n} \binom{j-1}{i} = \frac{n!}{(i+1)!(n-i-1)!} \qquad (3)$$

gives the number of experiment outcomes in this case.

Consider now the case in which there are not enough H's. If on the jth toss, the $(n - i + 1)$st T appears, then again the remaining outcomes are b's. There are $\binom{j-1}{n-i}$ of these sequences for a given j, j has a range from $n - i + 1$ to n. This sum is

$$\sum_{j=n-i+1}^{n} \binom{j-1}{n-i} = \frac{n!}{(i-1)!(n-i+1)!} \qquad (4)$$

We can now write the d'Alembert probability of the event iH as

$$Q(iH) = \frac{\binom{n}{i}}{\binom{n}{i} + \frac{n!}{(i+1)!(n-i-1)!} + \frac{n!}{(i-1)!(n-i+1)!}}$$

$$= \frac{(n+1) + i(n-i)}{(n+1)^2 - i(n-i)}, \qquad i = 0, 1, \cdots, n. \qquad (5)$$

From (5), note that $Q(iH) = Q[(n - i)H]$ holds as in classical probability.

To prove that

$$T(n) \equiv \sum_{i=0}^{n} Q(iH) > 1,$$

observe that $Q(iH) \geq 1/(n + 1)$ for all i. This last inequality is equivalent to $(n + 2)(n - i)i \geq 0$, which is true. The equality holds if and only if $i=0$ or n. Hence for $n > 1$, the proof is complete.

An exact closed-form expression for $T(n)$ has proved intractable. By retaining only the first three terms in the division expansion of (5) and then summing.

$$T(n) \approx 1 + \frac{(n-1)n(n+2)}{6(n+1)^2}\left[1 - \frac{(n^2+1)}{5(n+1)^2}\right]. \qquad (6)$$

Exact and approximate (using (6)), values of $T(n)$ for $n = 2$ through 7 are listed below.

n	$T(n)$-exact	$T(n)$-approximate
2	7/6 $\approx$ 1.167	1.165
3	19/14 $\approx$ 1.357	1.352
4	599/385 $\approx$ 1.556	1.545
5	211/120 $\approx$ 1.758	1.742
6	30984/15785 $\approx$ 1.963	1.940
7	26417/13572 $\approx$ 2.169	2.138

Note: $T(n)$ increases almost linearly with n.

It is not the purpose of this note to discuss the philosophical aspects of alternative probability systems. It should be mentioned, however, that the above d'Alembert system is a Carnap c^*: a uniform prior distribution on a structure description. Readers are referred to Hacking's excellent book [2], and his chapters 14 and 15 in particular, for this discussion.

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2. Hacking, I., The Emergence *of Probability*, Cambridge University Press, 1975.

ANOTHER PROOF OF THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY

by Norman Schaumberger and Bert Kabak
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The classical inequality connecting the arithmetic and geometric means of n quantities states that if $a_1 > 0$, $a_2 > 0$, $\cdots$, $a > 0$ then $(a_1 + a_2 + \cdots + a_n)/n \geqq (a_1 a_2 \cdots a_n)^{1/n}$ where equality holds if and only if $a_1 = a_2 = \cdots = a_n$. There are many proofs of this fundamental theorem none of which is particularly simple. In this note we offer a new proof which uses only basic algebra and the principle of mathematical induction. Instead of proving the above inequality we shall prove that if $x_1 > 0$, $x_2 > 0$, $\cdots$, $x_n > 0$ then

$$x_1^n + x_2^n + \cdots + x_n^n \geqq n x_1 x_2 \cdots x_n$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. The equivalence between the two statements becomes apparent if we put $x_1 = a_1^{1/n}$, $x_2 = a_2^{1/n}$, $\cdots$, $x_n = a_n^{1/n}$.

Let $x_1 > 0$, $x_2 > 0$ and $x_3 > 0$. Since $(x_1 - x_2)(x_1 - x_2)$ is nonnegative it follows that $x_1^2 + x_2^2 \geqq 2x_1x_2$ and the theorem is true for $n = 2$. Furthermore each term in

$$(x_1^2 - x_2^2)(x_1 - x_2) + (x_1^2 - x_3^2)(x_1 - x_3) + (x_2^2 - x_3^2)(x_2 - x_3)$$

is nonnegative and so the entire expression is nonnegative. Expanding, we get

$$\begin{aligned} 2(x_1^3 + x_2^3 + x_3^3) &\geqq x_1x_2^2 + x_2x_1^2 + x_1x_3^2 + x_3x_1^2 + x_2x_3^2 + x_3x_2^2 \\ &= x_1(x_2^2 + x_3^2) + x_2(x_1^2 + x_3^2) + x_3(x_1^2 + x_2^2) \\ &\geqq x_1(2x_2x_3) + x_2(2x_1x_3) + x_3(2x_1x_2) \\ &= 6x_1x_2x_3 \end{aligned}$$

Consequently

$$x_1^3 + x_2^3 + x_3^3 \geqq 3x_1x_2x_3,$$

which is the desired result for $n = 3$. Clearly equality holds if $x_1 = x_2 = x_3$, and it is easy to show that if $x_1^3 + x_2^3 + x_3^3 = 3x_1x_2x_3$ then

$$(x_1^2 - x_2^2)(x_1 - x_2) + (x_1^2 - x_3^2)(x_1 - x_3) + (x_2^2 - x_3^2)(x_2 - x_3) = 0.$$

Hence $x_1 = x_2 = x$ since each term must vanish.

Essentially the same procedure can be used to prove the general case: Let $x_i \geqq 0$, $(i = 1, 2, \ldots, n, n+1)$ and assume the arithmetic-geometric mean inequality holds for any n of the x_i. First observe that since each term of the summation is nonnegative,

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (x_i^n - x_j^n)(x_i - x_j) \geqq 0.$$

Hence

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (x_i^{n+1} + x_j^{n+1}) \geqq \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (x_i x_j^n + x_j x_i^n)$$

or

$$2(n+1) \sum_{j=1}^{n+1} x_j^{n+1} \geqq 2 \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} x_i x_j^n = 2 \sum_{\substack{i=1 \\ (i \neq j)}}^{n+1} \sum_{j=1}^{n+1} x_i x_j^n + 2 \sum_{\substack{i=1 \\ (i = j)}}^{n+1} \sum_{j=1}^{n+1} x_i x_j^n.$$

It follows that

$$n \sum_{j=1}^{n+1} x_j^{n+1} \geqq \sum_{\substack{i=1 \\ (i \neq j)}}^{n+1} \sum_{j=1}^{n+1} x_i x_j^n = \sum_{i=1}^{n+1} x_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_j^n \right).$$

Using the inductive hypothesis, we have

$$\sum_{i=1}^{n+1} x_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_j^n \right) \geqq \sum_{i=1}^{n+1} x_i \left(n \prod_{\substack{j=1 \\ j \neq i}}^{n+1} x_j \right) = n(n+1) \prod_{j=1}^{n+1} x_j.$$

Consequently,

$$\sum_{j=1}^{n+1} x_j^{n+1} \geqq (n+1) \prod_{j=1}^{n+1} x_j.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_{n+1}$ since each term in $\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (x_i^n - x_j^n)(x_i - x_j)$ is nonnegative.

THE RELATIONSHIP BETWEEN SOME DISCRETE AND CONTINUOUS PROBABILITY MODELS

by Genovevo Lopez and Joseph M. Moser
San Diego State University

1. Summary

This paper presents simple, direct proofs of theorems which show the respective relationships between various probability models:

(1) discrete uniform and continuous uniform;

(2) geometric and exponential; and

(3) negative binomial and gamma.

2. Introduction

It is known that in the discrete case the discrete uniform, the geometric, and the negative binomial probability models have a role very similar to that in the continuous case for the continuous uniform, the exponential and the gamma probability models. (See Freeman [1] and Parzen [2], for instance). This fact, however, is not usually exploited in introductory courses.

Feller [3] has shown the basic relationships among the geometric, the exponential, the negative binomial and the gamma probability models by considering appropriate discrete densities and then performing a limiting process to obtain the corresponding continuous densities. In [4], Prochaska shows the relationship between the geometric and the exponential probability models by considering their respective distribution functions cumulative distribution functions, However, a serious preliminary effort will convince one that this method becomes very difficult to use in consideration of the discrete uniform, the continuous uniform, the negative binomial and the gamma probability models. Therefore, three theorems which relate the respective densities already mentioned are proved here. The method used in the present paper consists of showing that the discrete moment generating function becomes, by a suitable limiting process, that of the corresponding probability model. Because the moment generating function of the geometric probability model cannot be obtained directly from the negative binomial probability

model, a separate theorem must be stated and proved. This is done in Theorem 2.

3. Principal Theorems

Theorem 1. Let $Y_n = \frac{X_n}{n}$, where X_n is a random variable following the negative binomial probability distribution with parameters $p_n = \frac{u}{n}$ and r (u is a fixed positive real number), where p_n is the probability of success on independent trials and X_n is the number of failures which occur before the rth success [that is,

$$f(X_n) = \binom{r + X_n - 1}{X_n} p_n^r q_n^{x_n}, \quad X_n = 0, 1, 2, \cdots].$$

Then

$$\lim M_{y_n}(t) = \left(\frac{u}{u-t}\right)^t$$

where $M_{y_n}(t) = E[e^{t y_n}]$. Equivalently, y_n converges in distribution to a gamma random variable with parameters u and r.

Proof. The moment generating function of a random variable following a negative binomial probability distribution with parameters p_n and r is given by the formula

$$E[e^{tX_n}] = \sum_{X_n=0}^{\infty} \binom{r+X_n-1}{X_n} p_n^r q_n^{X_n} e^{tX_n}$$

Then,

$$E[e^{tX_n}] = p^r \sum_{X_n=0}^{\infty} \binom{r+X_n-1}{X_n} (q_n e^t)^{X_n} = p_n^r \left(\frac{1}{1-q_n e^t}\right)^r = \left(\frac{p_n}{1-q_n e^t}\right)^r,$$

where of course $p_n + q_n = 1$. Next,

$$1 - q_n e^t = p_n + q_n - qe^t = p_n - q_n(e^t - 1)$$

$$= p_n - (1 - p_n)(e^t - 1).$$

Now if $u = np_n$, then

$$\frac{p_n}{1 - q_n e^t} = \frac{\frac{u}{n}}{\frac{u}{n} - (1 - \frac{u}{n})(e^t - 1)} = \frac{u}{u - (1 - \frac{u}{n})\, n(e^t - 1)}.$$

Therefore, the moment generating function of the random variable X_n/n is given by the expression

$$M_{Y_n}(t) = \left(\frac{u}{u - (1 - \frac{u}{n})\, n(e^{t/n} - 1)}\right)^r,$$

where $Y_n = X_n/n$.

Moreover,

$$\lim_{n\to\infty} M_{Y_n}(t) = \left(\frac{u}{u - t}\right)^r.$$

Theorem 2. Let $W_n = X_n/n$, where X_n is a random variable following a geometric probability distribution with parameter $A = np_n$; [that is, $f(X_n) = p_n q_n^{X_n}$, $X_n = 0, 1, 2,$]. Then

$$\lim_{n\to\infty} M_{W_n}(t) = \frac{A}{A - t}$$

where $M_{W_n}(t)$ denotes the moment generating function of W_n and the right side of the equality is the moment generating function of a random variable following an exponential probability model with parameter A.

Proof. Theorem 2 is a special case of Theorem 1 with $r = 1$.

Theorem 3. Let $Z_n = X_n/n$, where the random variable X_n follows a discrete uniform probability distribution; [that is, $f(X_n) = 1/n$, $X_n = 0, 1, \ldots, n$]. Then

$$\lim_{n\to\infty} M_{Z_n}(t) = \frac{e^t - 1}{t}$$

where $M_{Z_n}(t)$ denotes the moment generating function of Z_n and the right side of the equality is the moment generating function of a random variable following a continuous uniform probability model defined on the unit interval.

Proof. The moment generating function of a random variable following a discrete uniform probability model is given by the formula

$$E[e^{tX_n}] = \sum_{X_n=1}^{n} \frac{1}{n} e^{tX_n}.$$

The moment generating function of Z_n is then given by the formula

$$M_{Z_n}(t) = \sum_{X_n=1}^{n} \frac{1}{n} e^{\frac{t}{n}X_n}.$$

Now,

$$\sum_{X_n=1}^{n} \frac{1}{n} e^{\frac{t}{n}X_n} = 1 + \frac{t}{n^2}\sum_{i=1}^{n} i + \frac{t^2}{2!n^3}\sum_{i=1}^{n} i^2$$

$$+ \cdots + \frac{t^k}{k!n^{k+1}}\sum_{i=1}^{n} i^k + \cdots .$$

Moreover, recall the fact that

$$\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1} + \sum_{j=1}^{k} a_{k_j} n^j, \qquad (k = 1, 2, \ldots).$$

Substituting this into the preceding relation, one obtains the following expression.

$$M_{Z_n}(t) = 1 + \frac{t}{2}\left(1 + \frac{1}{n}\right) + \frac{t^2}{3!}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right) + \cdots .$$

It now follows readily that

$$\lim_{n\to\infty} M_{Z_n}(t) = 1 + \frac{t}{2} + \frac{t^2}{3!} + \cdots$$

$$= \frac{e^t - 1}{t}.$$

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PUZZLE SECTION

Mathacrostic No. 3

submitted by R. Robinson Rowe
Sacramento, California

Like the preceding two, this acrostic is a keyed anagram. The 204 letters to be entered in the diagram in the numbered spaces will be identical with those in the 32 key words at matching numbers and the key letters have been entered in the diagram to assist in correlation during your solution (see next two pages).

When completed, the initial letters of the 32 key words will spell a famous mathematician and the title of a classic he wrote. The diagram will then quote two sentences from the English translation of the book. This year (1977) is his bicentennial.

Dissecting the Letter E

A block letter E has a uniform thickness of 3 in., stands 15 in. tall, is 9 in. wide, and the horizontal middle stroke is half as long as the top and bottom strokes. Show how this letter can be cut into 5 pieces so they will fit together to form a perfect square without turning over any of the pieces. Can the number of pieces be reduced to 4 if the freedom of turning pieces over is granted?

Puzzle: A Pair of Eights

The following long division problem, in which the known digits appear nowhere else, has only one solution. Find it.

```
              * 8 * * *
* * * ) * * * * * * * *
        * * * *
        -------
          * * * *
            * * *
          -------
              * 8 *
              * * *
              -----
                * * * *
                * * * *
                -------
```

			1 T	2 D		3 D		4 T	5 T	6 T	7 D								
8 f	9 T		10 H	11 T		12 T	13 T	14 V	15 V	16 H		17 D							
18 V	19 D	20 H	21 K	22 a		23 H	24 T		25 A	26 H	27 D	28 O							
	29 K	30 T	31 V	32 V		33 a	34 H	35 f	36 A	37 a	38 V	39 O							
	40 V	41 D	42 a	43 a	44 T	45 D	46 I	47 V	48 K	49 W		50 K							
51 M	52 K	53 c	54 V	55 W	56 a	57 Y	58 M	59 O		60 f	61 T	62 H							
	63 Y	64 A		65 V	66 D	67 M	68 X	69 O	70 A	71 f	72 K	73 I							
74 U	75 H		76 B	77 f	78 D	79 U		80 O	81 L		82 Y	83 M							
84 I	85 W	86 M	87 d		88 f	89 I	90 d	91 L	92 X	93 c	94 f	95 N							
96 O	97 d		98 R	99 N	100 d	101 Q	102 e		103 E	104 W	105 C	106 U							
107 G	108 c	109 O	110 X		111 U	112 H	113 P	114 U	115 A	116 W	117 E	118 c							
	119 S	120 H	121 Q	122 I		123 A	124 d	125 K	126 W	127 N	128 B	129 J							
	130 Q	131 I	132 K	133 L	134 Y	135 L	136 b	137 I	138 N		139 X	140 K							
141 d	142 e	143 c	144 R	145 R	146 N	147 M	148 C		149 Q	150 K	151 A	152 G							
153 Y	154 L	155 C	156 F		157 F	158 B		159 g	160 M	161 S	162 J	163 X							
164 Q	165 R	166 B		167 R	168 a	169 Z	170 U		171 G	172 S	173 C	174 F							
175 e	176 E		177 A	178 e	179 P	180 g	181 Z	182 B	183 b	184 B	185 J	186 b							
	187 Q	188 M		189 P	190 g	191 g		192 E	193 b	194 Z	195 G	196 e							
		197 a	198 G		199 M	200 F	201 Z	202 E	203 J	204 A									

Definitions and Key

A. In math, reciprocally related
B. Mathematician 1601-1665
C. Watered rum
D. Geometer 260-200 BC
E. An integration limit
F. Exhibit
G. Continuous part of a surface
H. Mathematician, c. 200-270 AD
I. Number like 39, 79 or 116
J. Slender
K. Number expressed with i, j and k
L. Speak
M. A reverse transformation
N. Trunk cover
O. Unlawful
P. Involuntary twitch
Q. Like the eye's rainbow
R. Proprietor
S. Game, to draw last
T. Mathematician, c. 276-194 BC
U. With connectivity number 1
V. Mathematician 287-212 BC
W. What plenty did to twenty
X. Pruritic
Y. Dense and compact
Z. What Beatrice was shown
a. Mathematician 1588-1648
b. First name of a math author
c. Members of math expressions
d. Pacific
e. Shout of joy
f. Math from Arabia
g. Draw with acid

25 64 70 36 115 123 177 151 204
76 158 166 182 128 184
173 155 105 148
3 17 78 27 41 19 45 66 7 2
202 103 192 117 176
156 174 200 157
195 171 107 152 198
62 10 120 20 75 26 112 23 34 16
73 131 46 122 84 137 89
162 129 185 203
132 72 50 29 21 48 125 52 140 150
133 81 91 154 135
58 188 199 86 51 67 83 160 147
138 99 146 127 95
69 28 109 80 59 96 39
189 179 113
130 121 187 164 149 101
167 98 145 165 144
161 172 119
5 4 1 11 24 6 9 30 13 61 44 12
74 111 79 106 114 170
47 18 38 54 31 15 40 65 14 32
104 85 49 55 126 116
163 68 139 92 110
57 134 153 82 63
181 194 169 201
43 37 22 33 42 197 168 56
183 193 136 186
53 143 108 93 118
90 87 124 141 100 97
142 102 175 196 178
60 8 71 94 35 77 88
191 180 159 190

Solutions

Mathacrostic No. 1 [Spring, 1976]

A late solution was received by Greg Foley, Austin, Texas.

Missionaries and Cannibals [Fall, 1976]

The optimal number of crossings is 13, consisting of the following moves (*M* for missionaries, C for cannibals): 3*C*, 1*C*, 2*C*, 1*C*, 3*M*, 1*M* and 1*C*, 1*M* and 1*C*(rower), 1*M* and 1*C*(non-rower), 3*M*, 1*C*, 3*C*, 1*C*, and 2*C*. (If more than one cannibal could row the boat then 11 crossings would suffice.)

Solved by Sidney Penner, Bronx Community College of CUNY.

One incorrect solution was received.

Mathacrostic No. 2 [Fall, 1976]

Definitions and Key:

A.	Eighty-two	F.	Ethyl	K.	Boole
B.	René Descartes	G.	Modulus	L.	Ellipse
C.	Inversion	H.	Photon	M.	Lavoisier
D.	calculus	I.	Leader	N.	Loathed
E.	Todhunter	J.	Eddied	O.	Mighty
P.	Effete	U.	Asphodel	Z.	Arithmetic
Q.	Napier	V.	Twelve	a,	Thistle
R.	Oshkosh	W.	Haversine	b.	Integer
S.	Fifty-two	X.	Eggheads	c.	Cauchy
T.	Moffat	Y.	Mayo	d.	shrubby

First letters: ERIC TEMPLE BELL MEN OF MATHEMATICS

Quotation: *Though the idea behind it all is childishly simple, yet the method of analytic geometry is so powerful that very ordinary boys of seventeen can use it to prove results which would have baffled the greatest of the Greek geometers: Euclid, Archimedes, and Apollonius.*

Five Mathematicians: Descartes, Todhunter, Boole, Napier, and Cauchy.

Solved by Jeanette Bickley, Webster Groves High School, Webster Groves, Missouri; Ezra Brown, Virginia Polytechnic Institute and State University, Blacksburg, Virginia; Bradford E. Carter, Middle Tennessee State University, Murfreesboro, Tennessee; Edwin Comfort, Ripon College, Ripon, Wisconsin; Eleanor S. Elder, New Orleans, Louisiana; Joseph D. E. Konhauser, Macalester College, St. Paul, Minnesota; Barbara Lehmann, St. Peters College, Jersey City, New Jersey; Sidney Penner, Bronx Community College of CUNY, Bronx, New York; Bob Prielipp, University of Wisconsin at Oshkosh, Oshkosh, Wisconsin; and Richard D. Stratton, Colorado Springs, Colorado.

One solver did not give the source nor the 5 mathematicians called for.

WELCOME TO NEW CHAPTERS

The *Journal* welcomes the following new chapters of Pi Mu Epsilon which were recently installed:

TEXAS LAMBDA at the University of Texas, installed October 30, 1975 by E. Allan Davis, Council President.

VIRGINIA EPSILON at Longwood College, Farmville, Virginia, installed January 28, 1976 by R. A. Good, Council Secretary-treasurer.

ALABAMA ZETA at Alabama State University, installed March 25, 1976 by Milton D. Cox, Councilor.

ARKANSAS BETA at Hendrix College, Conway, Arkansas, installed April 12, 1976 by Robert M. Woodside, Councilor.

MATCHING PRIZE FUND

If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of mathematical films. To apply, or for more information, write to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

INITIATION CEREMONY

The editorial staff of the *Journal* has prepared a special publication entitled *Initiation Ritual* for use by local chapters containing details for the recommended ceremony for initiation of new members. If you would like one, write to the National Office.

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree, Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions if mailable and by any information that will assist the editor.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of November, 1977.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

386. Proposed by Charles W. Trigg, San Diego, California.

Show that the volume of Kepler's Stella Octangula (a compound of two interpenetrating tetrahedrons) is three times that of the octahedron that was stellated.

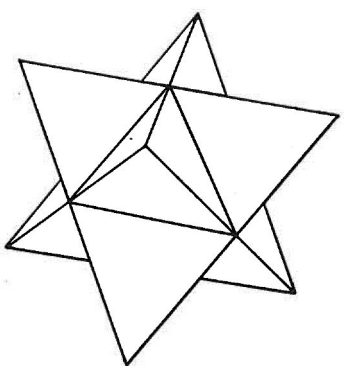


FIGURE 1

387. Phopobed by Jack Garfunkel, Forest Hills High School, Flushing, New Yohk.

On the sides AB and AC of an equilateral triangle ABC mark the points D and E respectively such-that AD = AE. Erect equilateral triangles on CD, AE and AB, as in the figure, with P, Q, R as the respective third vertices. Show that triangle PQR is equilateral. Also show that the midpoints of PE, AQ and RD are vertices of an equilateral triangle.

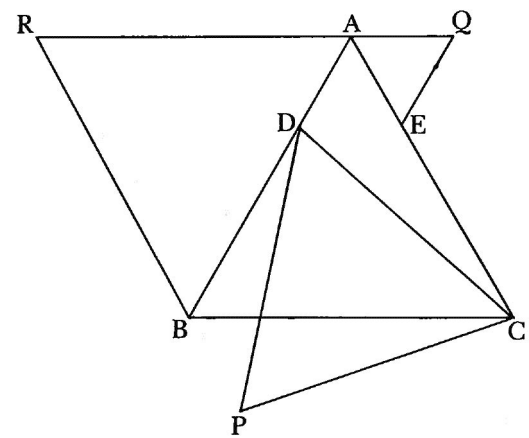


FIGURE 2

388. Proposed by David L. Silverman, West Lob Angeles, California.

In the game of "Larger, But Not That Large" two players each write down a positive integer. The numbers are then disclosed and the winner (who is paid a dollar by the loser) is the player who wrote the larger number, unless the ratio of larger number to smaller is three or more, in which case the player with the smaller number wins. If the same number is picked by both players, no payment is made.

a) What is the optimal strategy?

b) Suppose instead that the players are not restricted to integers but to the set $[1, \infty)$ and that larger number wins provided the larger-to-smaller ratio is less than r (for some $r > 1$); otherwise larger number loses. Find an optimal strategy.

389. *Phopobed by Paul Erdös, Spaceship Earth.*

Find a sequence of positive integers $1 \le a_1 < a_2 < \cdots$ which omits infinitely many integers from every arithmetic progression (in fact it has density 0) but which contains all but a finite number of terms of every geometric progression. Prove also that there is a set S of real numbers which omits infinitely many terms of any arithmetic progression but contains every geometric progression (disregarding a finite number of terms).

390. *Phopobed by Robb Koether and David C. Kay, University of Oklahoma.*

Let the diagonals of a regular n-sided polygon of unit side be drawn. Prove that the $n - 2$ consecutive triangles thus formed which have their bases along one diagonal, their legs along two others or a side, and one vertex in common with a vertex of the polygon each have the property that the product of two sides equals the third.

391. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve this alphametic where, of course, NINE is divisible by 9:

$$\begin{array}{r} \text{TWELVE} \\ \text{NINE} \\ \text{NINE} \\ \hline \text{THIRTY} \end{array}$$

392. *Proposed by R. Robinson Rowe, Sacramento, California.*

Solve in distinct positive integers,

$$\frac{1}{a+\frac{1}{b+\frac{1}{a+\frac{1}{b+\cdots}}}} - \frac{3}{c+\frac{1}{d+\frac{1}{c+\frac{1}{d+\cdots}}}} = 1/2$$

393. *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, New York.*

Consider the sequence $f(n) = n^2 - n + 41$. Find the GCD of $f(n)$ and $f(n+1)$.

394. *Proposed by Erwin Just and Bertram Kabak, Bronx Community College.*

Prove that if A_1, A_2 and A_3 are the angles of a triangle, then

$$3\sum_{i=1}^{3} \sin^2 A_i - 2\sum_{i=1}^{3} \cos^3 A_i \le 6.$$

395. *Proposed by Joe Van Austin, Emory University, Atlanta, Georgia.*

Assume that n independent Bernoulli experiments are made with $p = P$ [success], $1 - p = P$ [failure], and $0 < p < 1$. Intuitively it seems that P [success on the first trial | exactly one success] is always less than P [success on the first Trial | at least one success]. Verify directly that this is indeed the case.

396. *Proposed by David R. Simonds, Rensselaer Polytechnic Institute, Troy, New York.*

Let $[m]_n$ denote the integral part of the quotient when m is divided by n. Prove that

$$[m]_{nk} \equiv [m]_n^k, \ \forall\ m, n, k \in N,$$

where $[m]_n^k$ means $[[\cdots[m]_n\cdots]_n]_n$ (k sets of brackets).

397. *Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.*

If $c_j = 2\cos(j\pi/n)$, prove that

$$\prod_{j=1}^{n} (1 + 3c_j^4) = (3^n - 3^{n/2}\cdot 2\cos(5\pi n/6) + 1)^2 \qquad (1)$$

and more generally that

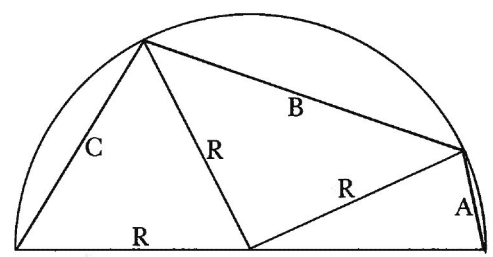
$$\prod_{j=1}^{n} (t^4 + c_j^4) = (x^n + x^{-n} - z^n - z^{-n})^2 = F_n^2(t)$$

where $F_n(t)/F_1(t)$ is a polynomial in t^2 with integral coefficients, and

$$x = u\bar{u} \ge 1,\ z = u/\bar{u},\ \text{and}\ u + u^{-1} = t\, e^{\pi i/4}. \qquad (3)$$

398. *Proposed by Richard S. Field, Santa Monica, California.*

Find solutions in integers $A = B \ne C \ne R$ and $A \ne B \ne C = R$ for the quadrilateral inscribed in a semicircle of radius R, as shown in the



$A \ne B \ne C = R$

FIGURE 3

figure. Also, find solutions in integers A # B ≠ C # R or prove that none exist.

Solutions

362. [Spring 19761 *Proposed by Zelda Katz, Beverly Hills, California.*

As shown in Figure 1, a diameter AB of a circle (O) passes through C, the midpoint of a chord DE. M is the midpoint of arc AB and the chord MP passes through C, The radius OP cuts the chord DE at Q. The tangent circles (O_1), (O_2), (W_1) and (W_2) are as shown. Show that $DQ = W_1W_2$.

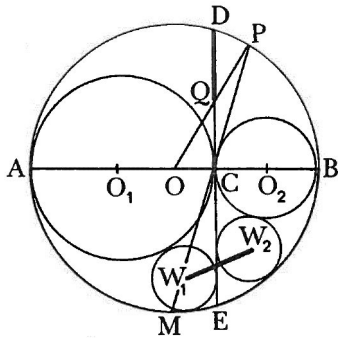


FIGURE 1

I. *Solution by R. Robinson Rowe, Sacramento, California.*

Let the radii be a, b and a + b and the unknown radii be r. In triangles O_1W_1F and OW_1F_1,

$$(a + r)^2 - (a - r)^2 = (a + b - r)^2 - (a - b - r)^2.$$

Whence $r = ab/(a+b)$ and $W_1F = 2a\sqrt{b/(a + b)}$.
Similarly in triangles O_2W_2G and OW_2G,

$$r = ab/(a+b) \text{ and } W_2G = 2b\sqrt{a/(a + b)}.$$

Then

$$W_1W_2 = \sqrt{(2r)^2 + \left[\frac{2}{\sqrt{a + b}}(a\sqrt{b} + b\sqrt{a})\right]^2}$$

$$= 2\sqrt{ab} - 2ab/(a+b).$$

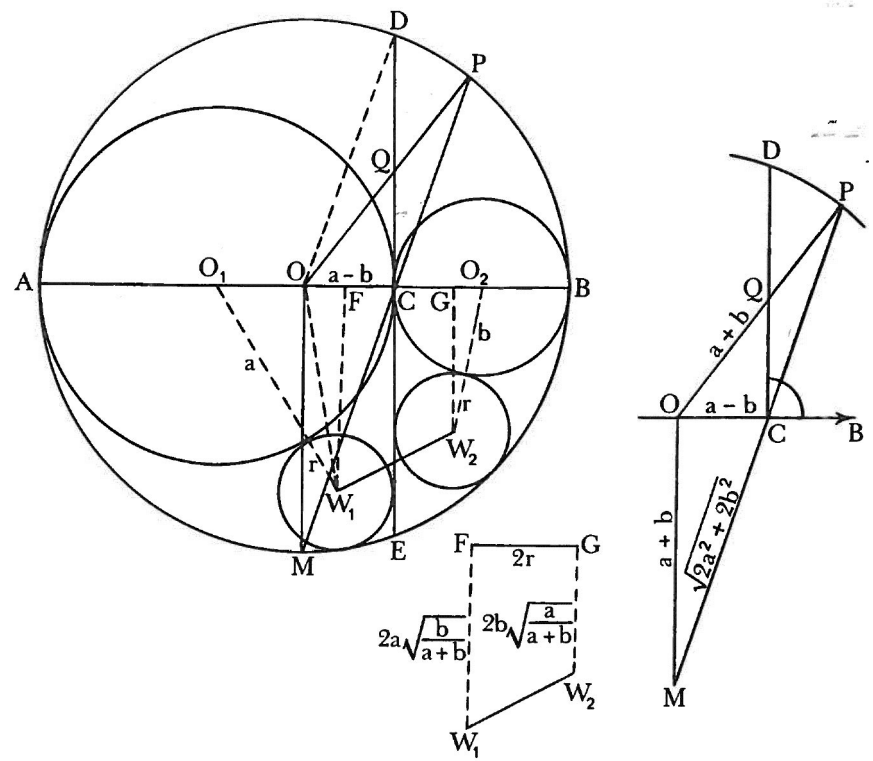


FIGURE 2

In triangle OCD, OD = a + b, OC = a - b, CD = $2\sqrt{ab}$. In triangle OMC, sin C = $(a + b)/\sqrt{2a^2 + 2b^2}$. In triangle OPC, sin P = $(a - b)/(a + b)$· sin C = $(a - b)/\sqrt{2a^2 + 2b^2}$, and ∠0 = ∠C - ∠$P$. Hence sin 0 = $2ab/(a^2 + b^2)$, tan 0 = $2ab/(a^2 - b^2)$. In triangle OCQ, CQ = $(a - b)$tan 0 = $2ab/(a + b)$; DQ - DC - CQ = $2\sqrt{ab} - 2ab/(a + b) = W_1W_2$.

11. *Solution by Leon Bankoff, Los Angeles, California.*

The configuration is a Shoemaker's Knife (or arbelos) and its reflection in the diameter AB. We make use of properties described on pages 116 and 117 of Roger A. Johnson's *Advanced Euclidean* Geometry (*Modern* Geometry), Dover Reprint, 1960. 1) The circles (W_1) and (W_2) inscribed in the curvilinear triangles ACE and BCE are equal, the diameter of each being equal to $AC \cdot BC/AB$, or half the harmonic mean of AC and CB. 2) The smallest circle that is tangent to and circumscribes

the two circles (W_1) and (W_2) is equal to the circle on CD. By these two properties it is seen that W_1W_2 is equal to CD minus half the harmonic mean of AC and CB. It now remains to show that QC is also equal to half the harmonic mean of AC and CB.

We first show that $QC = QP$. In triangle QOC, $\angle OQC + \angle QOC = 90° = \angle QPC + \angle PCQ + \angle OAP + \angle APO$. Now $\angle APO + \angle QPC = 45°$. Hence $\angle PCQ + \angle OAP = 45° = \angle OAP + \angle QPC$. Therefore $\angle PCQ = \angle QPC$ and $QC = QP$.

It follows that Q is the center of a circle tangent to AB at C and to the circumference of the outer circle (O) at P. Then $OC^2 = OP(OP - 2PQ)$, from which we obtain PQ (or PC) $= (OP^2 - OC^2)/2(OP) = (OP + OC)(OP - OC)/2(OP) = AC \cdot CB/AB$, thus proving that QC is also equal to half the harmonic mean of AC and CB.

Excellent solutions were also offered by Clayton W. Dodge, University of, Maine at Orono; Dr. John T. Hurt, Bryan, Texas; Barbara Seville, Rossini Conservatory, Bologna, Italy; and the. proposer, Zelda Katz.

363. [Spring 19761 Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.

Does $\frac{\sin 1}{1} + \frac{\sin 2}{2} + \frac{\sin 3}{3} + \cdots$ converge, and if so, to what?

Solution by Clayton W. Dodge, University of Maine at Orono.

A Fourier series expansion for the interval $0 < x < \pi$ in sine term only yields the equation

$$\frac{\pi - x}{2} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots$$

for $0 < x < \pi$. When $x = 1$ we get that

$$\frac{\pi - 1}{2} = \sin 1 + \frac{1}{2}\sin 2 + \frac{1}{3}\sin 3 + \frac{1}{4}\sin 4 + \cdots$$

whose decimal value is 1.070796327...

Similar solutions were submitted by Fred Ahrens, Pomona, California; P. Bloemendaal, Technological University, Eindhoven, The. Netherlands; Michael W. Ecker, City University of, New Yohk; Jackie E. Fritts, Rocky Mount, N. C.; John T. Hurt, Bryan, Texas; Bob Prielipp, The. University of Wisconsin-Oshkosh; Henry J. Ricardo, Manhattan College, Bronx, N. Y.; R. Robinson Rowe, Sacramento, California; 1. Philip Scalisi, Bridgewater Stain. College, Bridgewater, Massachusetts; and the. proposer, Robert C. Gebhardt.

Bob Prielipp suggested that readers who enjoyed this problem would probably find "A Monotonic Trigonometric Sum" by Richard Askey and John Steinig found on pp. 357-365 of the Summer 1976 issue of *American Journal of Mathematics* of interest. Ricardo and Scalisi cited Bromwich's *An Introduction to the Theory of Infinite Series*, pp. 356, 383, 386, while Rowe called attention to Smithsonian Publication 2672, Formula 6.813. Dodge proposed the question: What is "Yea + yea + yea + yea"? the answer to which is "A Fourier Series".

364. [Spring 1976] Proposed by Charles W. Trigg, Son Diego, California.

Show that there is only one third-order magic square with positive prime elements and a magic constant of 267.

I. *Solution by Clayton W. Dodge, University of, Maine at Orono.*

Since the constant is 267, the center element is 267/3 = 89. Now we need prime pairs adding to 267 - 89 = 178. We find that

$$178 = 5 + 173 = 11 + 167 = 29 + 149$$
$$= 41 + 137 = 47 + 131 = 71 + 107.$$

For any element k there must be at least one sum of two other primes adding to 267 - k. This eliminates 5, 41, 137 and 173. Corner elements must have two such sums. Fortunately we find that

$$267 - 29 + 238 = 71 + 167 = 107 + 131,$$
$$267 - 71 = 29 + 167 = 47 + 149,$$
$$267 - 107 = 11 + 149 = 29 + 131,$$
$$267 - 149 = 11 + 107 = 47 + 71.$$

The corner elements then are 29, 71, 107 and 149. We easily obtain the unique magic square:

29	131	107
167	89	11
71	47	149

11. Solution by- the proposer.

The magic constant of a third-order magic square can be rearranged into a square array in which the elements of the rows are in arithmetic progression with the same common difference, and likewise for the elements of the columns, and conversely.

In any arithmetic progression of primes with a first term > 3, the common difference d is a multiple of 6.

Now 9 - 4 = 5 and 9 + 6 = 15, so if 89 is the arithmetic mean, d cannot terminate in 4 or 6. Furthermore, 89 - 12 = 77, 89 + 30 = 119, and 89 + 72 = 161. Consequently there are only five arithmetic pro-

gressions of three positive primes with 89 as a middle term, namely: $A(11, 89, 167)$, $B(29, 89, 149)$, $C(41, 89, 137)$, $D(47, 89, 131)$, and $E(71, 89, 107)$. These can be paired in ten ways. However, the pairings AB, AC, AD, AE, BC, BD and CD produce square arrays with negative elements, and CE and DE produce arrays with an element of the form $5k$.

Therefore the only magic square with prime elements and a magic constant of 267 derives from BE. That is

11	29	47
71	89	107
131	149	167

which leads to

107	11	149
131	89	47
29	167	71

Also solved by Victor G. Feser, Mary College, Bismarck, North Dakota; John T. Hurt, Swan, Texas; R. Robinson Rowe, Sacramento, California; and Kenneth M. Wilke, Topeka, Kansas.

Charles W. Trigg called attention to the duplication of his proposal which first appeared as problem 325 in the Spring 1974 issue with a solution published a year later. Your problem editor has often wondered how many times he must make the same mistake twice.

365. *[Spring 1976] Proposed by Clayton W. Dodge, University of Maine, Orono, Maine.*

Find all fractions abc/cde such that cancelling the digit c yields an equivalent fraction, such as $166/664 = 16\cancel{6} / \cancel{6}4 = 16/64$. As in the illustration, not all the digits a, b, c, d, e need be distinct, but they should not be all equal.

Solution by Charles W. Trigg, San Diego, California.

$(10x + c)/(100c + y) = x/y$, where x and y are two-digit integers, can be manipulated into the form

$$x = cy/(100c - 9y).$$

For $c = 1$, 2, 3, 4, 5, and 8, the only solutions in two-digit integers are those which give fractions with six like digits. Otherwise:

$c = 6$, $(x,y) = (16,64)$ and $(26,65)$;

$c = 7$, $(x,y) = (21,75)$;

$c = 9$, $(x,y) = (19,95)$, $(24,96)$, and $(49,98)$.

Thus there are six fractions with three-digit numerators and denominators that can be reduced to equivalent fractions by illegal "cancellation" in the manner specified. They are: 166/664, 266/665, 199/995, 249/996, 499/998, and 217/775. In none of these does the specified "cancellation" reduce it to lowest terms, although a second "cancellation" will do so in the first three cases.

There are seven more proper fractions with like digits in the c positions that can be reduced to lowest terms by two illegal "cancellations", namely: 163/326, 316/632, 244/427, 455/546, 127/762, 139/973, and 187/748.

Also, there are eight fractions with like digits in the c position that can be reduced to lowest terms by resisting the impulse and "cancelling" b and d instead. They are: 143/341, 253/352, 154/451, 374/473, 275/572, 176/671, 385/583 and 187/781. In each of these fractions the denominator is the reverse of the numerator.

The 15 fractions given in the two paragraphs above are among the 116 proper fractions with denominators less than 1000 that can be reduced to lamest terms by illegal "cancellation" as given in my solution to problem 434, Mathematics Magazine, 34 (September 1961), 367-368.

Also solved by Jackie E. Fritts, Rocky Mount, North Carolina; Robert C. Gebhardt, Hopatcong, New Jersey; John T. Hurt, Bryan, Texas; Edith E. Risen, Oregon City, Oregon; R. Robinson Rowe, Sacramento, California; Kenneth M. Wilke, Topeka, Kansas; and the proposer, Clayton W. Dodge.

Robert C. Gebhardt found the required solution through a quick search by a programmed desktop electronic calculator and offered the following comment:

I rather wonder about problems like No. 365. I'm not sure that they "demand no greater ability in problem-solving than that of the average member of the Fraternity". What they do seem to require is the ability to locate a programmable electronic calculator and to return to it later to discover what answers it has found in its search. That is, I wonder about the usefulness of problems in which it is sufficient to let a'high-speed machine do a search for answers instead of solving by traditional means.

A reply to this contention can be found in the article "Reflections of a Problem Editor" published in the Fall 1975 issue of this Journal. To quote:

> It may be hard to believe, but your problem editor occasionally receives an answer to a problem instead of a solution. Participants in this arena are not really concerned with answers; their primary interest is in the

way the solution was found--the train of thought that led to the solution, the transparency of the solver's heuristic approach to the problem, essentially, the solver's ability to take the reader by the hand and literally lead him over the various steps of the proof.

Charles W. Trigg remarks as follows:

The value of a problem should not be judged solely on the basis of whether or not it can be solved by a machine. Frequently problems which may appear to be suitable only for solution by machine can, with the proper insight, be solved by hand in the time that it would take to write a program. Admittedly, one is more likely to make errors than would a properly programmed machine. In that case the machine can act as a check on the results. There is a modern danger that dependence on computer solutions is likely to reduce the possibility of hitting upon an elegant approach.

366. [Spring 19761 *Phopobed by Richard Field, Santa Monica, California.*

Let $Q = [10^n/p]$, where p is a prime > 5, and n is the cycle length of the repeating decimal $1/p$; $[x]$ denotes the greatest integer in x. Can Q be a prime?

All the solutions received were practically identical so credit should be divided among the following solvers: Jeffrey Bergen, Brooklyn, New York; Clayton W. Dodge, University of Maine at Orono; Michael W. Ecker, City University of New York; Victor G. Feser, Mary College, Bismarck, North Dakota.; Richard A. Gibbs, Fort Lewis College, Durango, Colorado; Kenneth M. Wilke, Topeka, Kansas and the proposer, Richard Field, whose solution is as follows:

Q is the repeating part of the decimal representation of $1/p$, expressed as an integer. (Example: for $p = 13$, $1/p = .0769230769...$; $n = 6$, $Q = [10^6/13] = 76923$.) Thus $1/p$ may be written as the geometric series $Q/10^n + Q/10^{2n} + Q/10^{3n} + \cdots$. Summing, we obtain $pQ = 10^n - 1$. Since $10^n - 1$ is divisible by 9 and since p is prime, Q must be divisible by 9, so cannot be prime.

367. [Spring 1976] *Proposed by R. Robinson Rowe, Sacramento, California.*

A box of unit volume consists of a square prism topped by a pyramid. Find the side of the square base and heights of prism and pyramid to minimize the surface area.

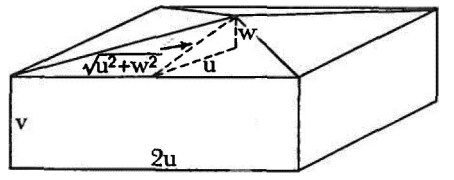


FIGURE 3

Solution by the. Proposer.

Let the dimensions of the prism be $2u \times 2u \times v$ and the height of the pyramid be W.

Then the surface area is

$$A = 4u^2 + 8uv + 4u\sqrt{u^2 + w^2} \tag{1}$$

and the volume

$$V = 1 = 4u^2v + 4u^2w/3. \tag{2}$$

From (2)

$$v = 1/4u^2 - w/3. \tag{3}$$

Then with (3) in (1)

$$A = 4u^2 + 2/u - 8uw/3 + 4u\sqrt{u^2 + w^2}. \tag{4}$$

Differentiating,

$$\frac{dA}{dw} = -\frac{8u}{3} + \frac{4uv}{\sqrt{u^2 + w^2}} = 0. \tag{5}$$

Times $3\sqrt{u^2 + w^2}/4u$:

$$3w = 2\sqrt{u^2 + w^2}. \tag{6}$$

Whence

$$5w^2 = 4u^2 \text{ and } W = 2u/\sqrt{5} \tag{7}$$

$$u^2 + w^2 = 9u^2/5 \text{ and } \sqrt{u^2 + w^2} = 3u/\sqrt{5}. \tag{8}$$

With (7, 8) in (4)

$$A = (4 + 4\sqrt{5}/3)u^2 + 2/u \tag{9}$$

$$dA/du = 8u(1 + \sqrt{5}/3) - 2/u^2 = 0 \tag{10}$$

$$u^3 = 3(3 - \sqrt{5})/16 \tag{11}$$

$$2u = \frac{1}{2}\sqrt[3]{12(3 - \sqrt{5})} = 1.046\ 442\ 3918\ldots \tag{12}$$

From (7)

$$w = \frac{1}{10}\sqrt[3]{60(3\sqrt{5} - 5)} = 0.467\ 983\ 2643\ldots \tag{13}$$

From (3)

$$v = \frac{1}{10}(5 + 2\sqrt{5})\sqrt[3]{12(47 - 21\sqrt{5})} = 0.757\ 212\ 8273 \tag{14}$$

Not asked for,

A = 5.733 712 667, which is less than A = 6 for a unit cube and more than $A = \sqrt[3]{36\pi} = 4.835\ 975\ 86\ldots$ for a unit sphere.

Also solved by John T. Hurt, *Bryan, Texoi.*

368. [Spring 1976] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

Given a triangle ABC with its inscribed circle (I). Lines AI, BI, CI cut the circle in points D, E, F respectively. Prove that AD + BE + CF ≥ (Perimeter of triangle DEF)/$\sqrt{3}$.

Solution by Clayton W. Dodge, University of Maine at Orono.

In O. Bottema et al, *Geometric Inequalities*, we find that

$$AI + BI + CI \geq 6r \qquad \text{Item 12.3}$$

and

$$a + b + c \leq 3R\sqrt{3} \qquad \text{Item 5.3}$$

where r is the inradius, R the circumradius, and a, b, a the side lengths of triangle ABC. From item 12.3 it immediately follows that

$$AD + BE + CF \geq 3r$$

since DI = EI = FI = r. Now apply item 5.3 to triangle DEF and its circumcircle, the incircle of triangle ABC, whose radius is r. We get

$$DE + EF + FD \leq 3r\sqrt{3}.$$

Dividing this last inequality by $\sqrt{3}$ and combining it with the preceding inequality, we get the desired result:

$$AD + BE + CR \geq 3r \geq (DE + EF + FD)/\sqrt{3}.$$

Also solved by John T. Hurt *and the Proposer,* Jack Garfunkel.

369. [Spring 1976] *Proposed by Paul Erdös, Spaceship Earth.*

Determine all solutions of $\binom{n}{k} = \prod_{p \leq n} p$.

Solution by John T. Hurt, Bryan, Texas.

Clearly there can be no solutions unless

$$\max_{0 \leq k \leq n} \binom{n}{k} \geq \prod_{p \leq n} p.$$

Since $\binom{n}{k}$ are the binomial coefficients the maximum value is $n!/(n/2)!\,(n/2)!$ if n is even and $n!/(\frac{n-1}{2})!(\frac{n+1}{2})!$ for n odd. From the table

n	$\max\binom{n}{k}$	Πp
1	1	1
2	2	2
3	3	6
4	6	6
5	10	30
6	20	210
7	35	210
8	70	210
9	126	210
10	252	210
11	462	2310
12	924	2310
•	⋮	⋮
17	24310	510510

we see that n = 1, 2, 4, 10 will give the solutions

$n = 1$	$n = 1$	$n = 2$	$n = 4$	$n = 10$	$n = 10$
$k = 0$	$k = 1$	$k = 1$	$k = 2$	$k = 4$	$k = 6$

From the above, the only non-trivial solutions are

$$\binom{4}{2} = 2 \times 3 = 6$$

$$\binom{10}{4} = 2 \times 3 \times 5 \times 7 = 210$$

No solutions exist for $n > 10$.

Also solved by the proposer, Paul *Erdos.*

370. *[Spring* 1976] *Proposed by David L. Silverman, West Los Angeles, California.*

Able, Baker and Charlie take turns cyclically, in that order, tossing a coin until three successive heads or three successive tails appear. With what probabilities will the game terminate on Able's turn? On Baker's?

Solution by the. Proposer.

At Charlie's first turn he is confronted with one of two equally likely situations: first two throws the same or different. If the same, he has half a chance of ending the game on the first toss and half a chance of starting a new run, putting him in Able's position. If different, he is effectively the second player in a newrun, that is, in Baker's position.

Thus

$$c = \frac{1}{2}\left(\frac{1}{2} + \frac{a}{2}\right) + \frac{b}{2}.$$

Of the 3 out of 4 cases (OOX, *OXO*, *OXX*) in which Able gets a second toss, the first two cases place him in Baker's position and the third affords him equal chances of ending the game or of resuming his initial state. Thus

$$a = \frac{b}{2} + \frac{1}{4}\left(\frac{1}{2} + \frac{a}{2}\right).$$

From the equation $a + b + c = 1$, the solution $a = 9/31$, $b = 8/31$, $c = 14/31$ is obtained.

Also solved by R. Robinson Rowe. Two incorrect solutions were received.

371. *[Spring* 19761 *Proposed by I. P. Scalisi, State. College at Bridgewater, Massachusetts.*

A unit fraction is any rational number of the form $1/n$, where n is a positive integer. Write $2/n$ as the sum of 4 (or 6 or 10 or 14) distinct unit fractions.

I. *Solution by Clayton W. Dodge, University of Maine ad. Orono.*

There are many ways to obtain a solution for this problem. One is to utilize several times the well-known equation

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

Thus, for example, we might write

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n+2} + \frac{1}{(n+1)(n+2)}$$

or

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n(n+1)+1} + \frac{1}{[n(n+1)][n(n+1)+1]}.$$

Extending the idea of the second example, for k terms we have

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+k-3)(n+k-2)} + \frac{1}{n+k-2}.$$

II. *Solution by R. Robinson Rowe, Sacramento, California.*

$2/3 = 1/2 + 1/12 + 1/20 + 1/30$

$2/5 = 1/6 + 1/8 + 1/15 + 1/24$

$2/7 = 1/5 + 1/15 + 1/60 + 1/420$

$2/3 = 1/3 + 1/4 + 1/30 + 1/40 + 1/60 + 1/120$

$2/3 = 1/4 + 1/6 + 1/10 + 1/15 + 1/30 + 1/40 + 1/60 + 1/120$

$2/3 = 1/6 + 1/8 + 1/10 + 1/12 + 1/15 + 1/24 + 1/30 + 1/40 + 1/60 + 1/120$

which are 3 solutions with 4 terms adding differently and 3 more to the same sum but 6, 8 and 10 terms'respectively. Enuf?

Also solved by Fred Ahrens, Pomona, California; Gordon R. Baker, Houston, Texas; Alize Dubin, Far Rockaway, New York; Michael W. Ecker, City University of New York; Victor G. Feser, Bismarck, North Dakota; Mike Khalil, Cherry Hill, New Jersey; Robert C. Gebhardt, Hopatcong, New Jersey; John T. Hurt, Bryan, Texas; Edith E. Risen, Oregon City, Oregon; Kenneth M. Wilke, Topeka, Kansas; and the. proposer, I. P. Scalisi.

372. *[Spring* 1976] *Proposed by Sidney Penner, Bronx Community College of CUNY.*

Prove the following theorem:

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let f be a function from a *subset* of X_1 into X . The function f is continuous in the relative topology on its domain if and only if for every $a \in \tau_2$ there exists $b \in \tau_1$ such that

(*i*) $\mathrm{Dm}\, f \cap b \subset f^{-1}(a)$

(*ii*) if $c \subset a \cap \mathrm{Ran}\, f$ then $f^{-1}(c) \subset \mathrm{Dm}\, f \cap 6$.

Solution by David Del Sesto, North Providence, Rhode Island.

Necessity: Let $a \in \tau_2$. Then as f is continuous on Dm f, there

is a $b \in \tau_1$ such that $f^{-1}(a) = \text{Dom } f \cap b$; and so, at least $\text{Dom } f \cap b \subset f^{-1}(a)$.

Also, if $c \subset a \cap \text{Ran } f$, then $f^{-1}(c) \subset f^{-1}(a \cap \text{Ran } f)$. But $a \cap \text{Ran } f \subset a$; hence $f^{-1}(a \cap \text{Ran } f) \subset f^{-1}(a) = \text{Dom } f \cap b$.

Sufficiency: Let $a \in \tau_2$; by (i), it suffices to show $f^{-1}(a) \subset \text{Dom } f \cap b$, for $b \in \tau$.

Let $x \in f^{-1}(a) \subset \text{Dom } f$. Therefore, $f(x) \in a$ and $f(x) \in f(\text{Dom } f) = \text{Ran } f$; i.e., $f(x) \in a \cap \text{Ran } f$ or, $f[\{x\}] \subset a \cap \text{Ran } f$. By (ii) we get $f^{-1}[f[\{x\}]] \subset \text{Dom } f \cap b$. But $\{x\} \subset f^{-1}[f[\{x\}]]$, and thus $x \in \text{Dom } f \cap b$.

Also solved by **Fred Ahrens**, *Pomona, California;* **Jackie E. Fritts**, *Rocky Mount, North Carolina; and the proposer,* **Sidney Penner.**

373. **[Spring 1976]** *Proposed by Joe. Van Austin, Emory University, Atlanta, Georgia.*

Assume that the number of shots at the goal in a hockey game is a random variable Y that has a Poisson distribution with parameter λ. Each shot is either blocked or is a goal. Assume each shot is independent of the other shots and $p = P[\text{a shot is blocked}]$ for each shot. Find the probability there are exactly k goals in a game for $k = 0, 1, 2, \cdots$.

Solution by the Proposer.

The possible ways to have exactly k goals are as follows: exactly k shots and none blocked, exactly $k + 1$ shots and 1 blocked, $\cdots$, exactly $k + j$ shots and j blocked, $\cdots$. These are all disjoint and the probability is

$$P[\text{exactly } k + j \text{ shots} \cap j \text{ blocked}]$$
$$= P[\text{exactly } k + j \text{ shots}], P[j \text{ blocked} | k+j]$$
$$= e^{-\lambda} \frac{\lambda^{k+j}}{(k+j)!} \cdot \binom{k+j}{j} p^j (1-p)^k = e^{-\lambda}(1-p)^k \lambda^k \frac{(\lambda p)^j}{(k+j)!} \cdot \frac{(k+j)!}{j!\ k!}$$

Summing the probabilities for $j = 0, 1, 2, \ldots$ gives

$$P[\text{exactly } k \text{ goals}] = e^{-\lambda} \cancel{[\lambda(1-k)!]} \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{}$$

$$= e^{-\lambda} \frac{[\lambda(1-p)]^k}{k!} \cdot e^{\lambda p} = e^{-\lambda(1-p)} \cdot \frac{[\lambda(1-p)]^k}{k!}$$

for $k = 0, 1, 2, \cdots$. Thus the number of goals scored is also Poisson with parameter $\lambda(1 - p)$.

Comments by the Problem Editor:

a) Only one misprint was detected in the problem Department of the Fall 1976 issue: Gusses on page 315 should read guesses.

b) The solution to part (c) of problem 360, tentatively scheduled for this issue, is not yet ready for presentation.

c) Readers with an insatiable addiction and an uncontrollable proclivity to the highly civilized activity of problem solving are encouraged to liquidate their frustrating compulsions and obsessions by participating in the problem departments of the following mathematical journals:

The American Mathematical Monthly
The Mathematics Magazine
The Two-Year College Mathematics Journal
Eureka (Algonquin College, Ottawa, Ontario, Canada)
The Mathematics Association of Two-Year Colleges Journal
School Science and Mathematics
The Ontario Secondary School Mathematics Bulletin
The Mathematics Student Journal
The Journal of Recreational Mathematics
The Problem Department of the Technology Review.

Readers are urged to communicate with me regarding suggestions for extending this list. Subscription information is available on request.

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