PI MU EPSILON JOURNAL

VOLUME 6 FALL 1977 NUMBER 7

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THE OFFICIAL PUBLICATION

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PI MU EPSILON JOURNAL is published at the University of Oklahoma twice a year — Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

A "GEOMETRIC" DEFINITION OF THE TANGENT TO A CURVE

by Louis I. Alpert[1]

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In most calculus courses, students are introduced to the notions of the tangent to a curve and the derivative of a function of one real variable, through definitions which are, essentially, based upon the concept of the limit of a sequence of real numbers. Furthermore, such definitions are generally expressed in terms of the particular function which, analytically, represents the curve, i.e., in terms of a particular parametrization of the given curve.

On the other hand, in the same calculus course, the related notions of the area enclosed by a curve, and the length of a curve, are often conveniently defined by means of upper and lower bounds. In this setting, these definitions are *not* dependent upon a particular parametrization. Surely, this suggests the question of whether there is a precise definition of the tangent to a curve (and the derivative of a function of one real variable) expressed in terms of upper and lower bounds.

In this note we answer the above in the affirmative by constructing such definitions. For this purpose we employ a very simple analogue of a concept of *deviation*, introduced in the study of surface area in (2), and generalized in (1).

1. *Plane Curves*

Let C be a curve and P a point of C. Let $N(P, \delta)$ be a neighborhood of P (a disc). By $D(C; N(P, \delta))$, the **deviation** of C on $N(P, \delta)$, we mean the L.U.B. of the acute angles between two chords $\overline{PQ}_1$ and $\overline{PQ}_2$ of C which are in $N(P, \delta)$. We define $D(C;$ P), the deviation of C at P to be the G.L.B. of the set {all $D(C; N(P, \delta))$ for all $\delta > 0$}.

Theorem 1. Let C be the graph of a function $f(x)$ and let p be in the domain of $f(x)$. Let P = $(p, f(p))$. If $D(C; P) = 0$, then f is

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[1]The author is grateful to the referee for his excellent suggestions, which have simplified some of the proofs of the theorems in this paper.

differentiable at p (in the general sense which permits $f'(p)$ to be infinite).

Proof, Let $(Q_1, Q_2, Q_3, \ldots)$ be an infinite sequence of points of C distinct from P, converging to P. Let $(V_{Q_1}, V_{Q_2}, V_{Q_3}, \ldots)$ be the corresponding sequence of unit vectors from P through Q_1, Q_2, Q , ... respectively.

If the set $V_{Q_1}, V_{Q_2}, V_{Q_3}, \ldots$ is finite, then there exists a convergent subsequence of $(V_{Q1}, V_{Q_2}, V_{Q_3}, \ldots)$. If the set is infinite, then there exists a vector limit point of the set. Then there exists a subsequence of $(V_{Q_1}, V_{Q_2}, V_{Q_3}, \ldots)$ which converges to this vector limit point. Thus, in either case, there exists a convergent subsequence.

Suppose now that there exist two subsequences $(V_1', V_2', V_3', \ldots)$ and $(V_1'', V_2'', V_3'', \ldots)$ which converge, respectively, to two distinct limit unit vectors V^* and V^{**}. Let 4α = the acute angle between V^* and V^{**} There exists a positive integer N such that if $n > N$, then the angle (V_n', V^*) between V_n and V^* is less than α and the angle (V_n'', V^{**}) between V_n'' and V^{**} is also less than α. It follows that $(V_n', V_n'') > \alpha$. This contradicts the hypothesis that $D(C; P) = 0$. Hence V, the limit unit vector, is unique. It follows that every subsequence of $(V_{Q_1}\ V_{Q_2}, \ldots)$ has a subsequence converging to V and hence, $(V_{Q_1}, V_{Q2}, \ldots)$ converges to V. Consequently, f is differentiable at P in the above indicated general sense of differentiability.

Theorem 2. Let C be the graph of a function f. If f is differentiable at p (whether $f'(p)$ is finite or infinite), then $D(C; P) = 0$, where $P = (p, f(p))$.

Proof. Let $X = (x, f(x))$ and let $f'(p) = \tan \alpha$, $0 \leq \alpha < \pi$. Let $\theta(X)$ denote the least positive angle from the x-axis and the chord through P and X.

Let $\varepsilon > 0$ be given.

Since f is differentiable at p, there exists $\delta > 0$ such that if $X \in C \cap N(P, \delta)$, then $|\theta(X) - \alpha| < \varepsilon/2$. If X_1 and X_2 are in $C \cap N(P, \delta)$ then $|\theta(X_1) - \alpha| < \varepsilon/2$ and $|\theta(X_2) - \alpha| < \varepsilon/2$. Hence $|\theta(X_1) - \theta(X_2)| < \varepsilon$. Since this is true for every $\varepsilon > 0$, $D(C; P) = 0$.

We now propose the following definition of the concept of tangent to a curve at a given point of the curve.

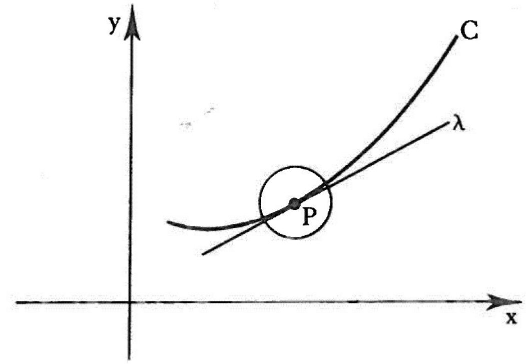

FIGURE 1

If through the point P of C there exists a straight line λ such that, for every $\varepsilon > 0$, there exists a neighborhood $N(P, \delta)$ such that for every chord through P and a point of C in $N(P, \delta)$, the acute angle between this chord and the line λ is less than ε, then this line λ is said to be tangent to C at P (See Figure 1).

As an example of the use of the foregoing definition, let us construct the tangent to C at P.

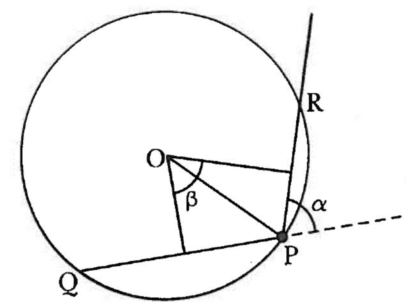

FIGURE 2

Referring to Figure 2, we wish the angle α to be small. It is seen that $\alpha = \beta$. Hence $\alpha = 0$ when $\beta = 0$. This implies that $\overline{PR}$ and $\overline{QP}$

be both perpendicular to $\overline{OP}$. It follows that the line through P and perpendicular to $\overline{OP}$ is tangent to C at P.

Theorem 3. A curve cannot have two distinct tangents at P.

Proof. Immediate.

Theorem 4. Let C be the graph of a function f. If f is differentiable at P (whether $f'(p)$ is finite or infinite), then there exists a tangent to C at $P:(p, f(p))$.

Proof. Suppose f is differentiable at P. Then $D(C; P) = 0$, so by the proof of Theorem 1, there is a unit vector V such that if $(Q_1, Q_2, Q_3, \ldots)$ is a sequence of points of C distinct from P converging to P and for each i V_{Q_i} is a unit vector from P through Q_i, then $(V_{Q_1}, V_{Q_2}, \ldots)$ converges to V.

We now propose the following definition of the derivative.

Let C be the graph of the given function f. If at $P \in C$, the curve C has a tangent line λ then $f'(p)$ is the trigonometric tangent of the acute angle between the x-axis and λ. Thus defined, $f'(p)$ may be finite or infinite.

One easily shows that $f'(p)$ as defined above is precisely equal to $f'(p)$ as defined by sequences in the standard manner. Thus, we have here a definition of the derivative which is, in a sense, expressed in terms of upper and lower bounds which occur in our newly introduced concept of the deviation of C at P, $D(C; P)$.

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INTERPOLATION USING THE CUBIC SPLINE FUNCTION

by Randolph J. Taylor
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Introduction

Consider the problem of finding an interpolating curve to a given set of $n + 1$ data points. One standard method of interpolation is to construct a polynomial of degree n which passes through the $n + 1$ points. To improve the accuracy of the interpolation, we increase the number of data points, which in turn, increases the degree of the polynomial. Generally, the higher the degree of a polynomial the more the function oscillates. Mathematically, we have interpolated the data points, but intuitively, an oscillating function is often unrealistic. The problem now becomes to find a smooth curve to interpolate the given data.

Draftsmen use splines to draw smooth curves through data points. A spline is a flexible strip of plastic or wood which can be bent to pass through the required points, yet constructed so that the spline straightens out as much as possible while still interpolating the data. The result is that the spline avoids sharp corners when shifting between intervals of data points. The mathematical cubic spline function is introduced as a model of the physical spline used by draftsmen.

Definition. The cubic spline is a function $y(x)$ defined on an interval $[a, b]$ which is a piecewise cubic polynomial on every subinterval $[x_i, x_{i+1}]$ of $[a, b]$, is twice continuously differentiable, and passes through the data points.

Theorem 1. The cubic interpolating spline function $y(x)$ given by the above definition exists and is unique.

Proof. Let the interval $[a, b]$ have nodes $a = x_0 < x_1 < \cdots < x_n = b$ with corresponding data points $f(x_0), f(x_1), \cdots, f(x_n)$ and let $y(x)$ be the spline function which interpolates these data points. For notational convenience, let

$$f_i = f(x_i) \qquad \text{for } i = 0, 1, \cdots, n,$$

$$h_i = x_{i+1} - x_i \qquad \text{for } i = 0, 1, \cdots, n - 1,$$

$$y_i(x) = y(x) \qquad \text{for } x \in [x_i, x_{i+1}],$$

$$s_i = y''(x_i) \qquad \text{for } i = 0, 1, \cdots, n.$$

Since $y(x)$ is a cubic polynomial on every subinterval $[x_i, x_{i+1}]$ of $[a, b]$, then $y''(x)$ is a linear polynomial on each $[x_i, x_{i+1}]$ and can be expressed as

$$y''(x) = s_i \frac{x_{i+1} - x}{h_i} + s_{i+1} \frac{x - x_i}{h_i} \qquad \text{for } i = 0, 1, \cdots, n - 1 \quad (1)$$

where $x \in [x_i, x_{i+1}]$. Since $y(x)$ is twice continuously differentiable, we obtain $y(x)$ by integrating (1) twice.

$$y'(x) = \int y''(x)\, dx = -\frac{s_i}{2h_i}(x_{i+1} - x)^2 + \frac{s_{i+1}}{2h_i}(x - x_i)^2 + C_1$$

$$y(x) = \int y'(x)\, dx = \frac{s_i}{6h_i}(x_{i+1} - x)^3 + \frac{s_{i+1}}{6h_i}(x - x_i)^3 + C_1 x + C_2 \quad (2)$$

From (2) we obtain

$$y(x_i) = \frac{s_i h_i^2}{6} + C_1 x_i + C_2 \quad (3)$$

$$y(x_{i+1}) = \frac{s_{i+1} h_i^2}{6} + C_1 x_{i+1} + C_2 \quad (4)$$

Since the interpolating spline function must pass through the data points, $y(x_i) = f_i$ and $y(x_{i+1}) = f_{i+1}$. Using this, we now determine the constants of integration, C_1 and C_2, from (3) and (4).

$$f_i = \frac{s_i h_i^2}{6} + C_1 x_i + C_2 \quad (5)$$

$$f_{i+1} = \frac{s_{i+1} h_i^2}{6} + C_1 x_{i+1} + C_2 \quad (6)$$

Subtracting (5) from (6) and solving for C_1 yields

$$f_{i+1} - f_i = \frac{s_{i+1} h_i^2}{6} - \frac{s_i h_i^2}{6} + C_1 (x_{i+1} - x_i)$$

$$C_1 = \frac{f_{i+1} - f_i}{h_i} - \frac{s_{i+1} h_i}{6} + \frac{s_i h_i}{6} \quad (7)$$

Substituting (7) into (5) and solving for C_2 yields

$$C_2 = f_i - \frac{s_i h_i^2}{6} - \left(\frac{f_{i+1}}{h_i} - \frac{f_i}{h_i} - \frac{s_{i+1} h_i}{6} + \frac{s_i h_i}{6}\right) x_i \quad (8)$$

Substituting (7) and (8) for C_1 and C_2 into (2), the spline becomes

$$y(x) = \frac{s_i}{6h_i}(x_{i+1} - x)^3 + \frac{s_{i+1}}{6h_i}(x - x_i)^3 + \left(\frac{f_{i+1}}{h_i} - \frac{s_{i+1} h_i}{6}\right)(x - x_i)$$

$$- \left(\frac{f_i}{h_i} - \frac{s_i h_i}{6}\right)(x - x_i) + f_i - \frac{s_i h_i^2}{6}$$

Substituting $x_i = x_{i+1} - h_i$ in the fourth term and simplifying we obtain

$$y(x) = \frac{s_i}{6h_i}(x_{i+1} - x)^3 + \frac{s_i h_i}{6h_i}(x - x_i)^3 + \left(\frac{f_{i+1}}{h_i} - \frac{s_{i+1} h_i}{6}\right)(x - x_i)$$

$$+ \left(\frac{f_i}{h_i} - \frac{s_i h_i}{6}\right)(x_{i+1} - x) \qquad \text{for } i = 0, \cdots, n - 1. \quad (9)$$

To obtain the interpolating spline $y(x)$ in (9) we need to determine s_i for $i = 0, 1, \cdots, n$. Differentiating (9) produces

$$y'(x) = -\frac{s_i}{2h_i}(x_{i+1} - x)^2 + \frac{s_{i+1}}{2h_i}(x - x_i)^2 + \frac{f_{i+1}}{h_i} - \frac{s_{i+1} h_i}{6}$$

$$- \frac{f_i}{h_i} + \frac{s_i h_i}{6} \quad (10)$$

On $[x_{i-1}, x_i]$ from (10) we obtain

$$y'_{i-1}(x_i) = \frac{s_i h_{i-1}}{2} + \frac{f_i}{h_{i-1}} - \frac{s_i h_{i-1}}{6} - \frac{f_{i-1}}{h_{i-1}} + \frac{s_{i-1} h_{i-1}}{6} \quad (11)$$

And, on $[x_i, x_{i+1}]$ from (10) we obtain

$$y'_i(x_i) = -\frac{s_i h_i}{2} + \frac{f_{i+1}}{h_i} - \frac{s_{i+1} h_i}{6} - \frac{f_i}{h_i} + \frac{s_i h_i}{6} \quad (12)$$

Since $y'(x)$ is continuous, the slope at the end of one subinterval must be the same as the slope at the beginning of the next subinterval. So, for subintervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ we have $y'_{i-1}(x_i) = y'_i(x_i)$ for $i = 1, 2, \cdots, n - 1$. Substituting (11) and (12) into this yields

$$\frac{s_ih_{i-1}}{2} + \frac{f_i}{h_{i-1}} - \frac{s_ih_{i-1}}{6} - \frac{f_{i-1}}{h_{i-1}} + \frac{s_{i-1}h_{i-1}}{6} = -\frac{s_ih_i}{2} + \frac{f_{i+1}}{h_i}$$
$$-\frac{s_{i+1}h_i}{6} - \frac{f_i}{h_i} + \frac{s_ih_i}{6}.$$

Grouping like terms and simplifying yields

$$s_{i+1} + 2\left(\frac{h_{i-1}+h_i}{h_i}\right)s_i + \frac{h_{i-1}}{h_i}s_{i-1} = \frac{6}{h_i}\left(\frac{f_{i+1}-f_i}{h_i} - \frac{f_i-f_{i-1}}{h_{i-1}}\right) \qquad (13)$$

for $i = 1, 2, \cdots, n-1$.

This is a linear system of $n-1$ equations in $n+1$ unknowns, $s_0, s_1, \cdots, s_n$. Allowing the spline to stick out past the end points $x_0 = a$ and $x_n = b$, causes the spline to straighten out and have a zero curvature. But, the second derivative is zero at all points on a curve having zero curvature. Thus, $s_0 = 0$ and $s_n = 0$. We now have $n-1$ equations and $n-1$ unknowns. Rewriting (13) we obtain

$$c_is_{i-1} + 2(1+c_i)s_i + s_{i+1} = d_i \qquad \text{for } i = 1, 2, \cdots, n-1 \qquad (14)$$

$$\text{where } c_i = \frac{h_{i-1}}{h_i} \text{ and } d_i = \frac{6}{h_i}\left(\frac{f_{i+1}-f_i}{h_i} - \frac{f_i-f_{i-1}}{h_i}\right).$$

We need only show that s_i exists and is unique for $i = 1, 2, \cdots, n-1$; then by (9), the spline exists and is unique. Writing (14) in matrix form we obtain a tridiagonal system $As = d$ given by

$$\begin{pmatrix} 2(1+c_1) & 1 & 0 & 0 \dots 0 & 0 & 0 \\ c_2 & 2(1+c_2) & 1 & 0 & 0 & 0 \\ 0 & c_3 & 2(1+c_3) & 1 & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 \quad c_{n-2} & 2(1+c_{n-2}) & 1 \\ 0 & 0 & 0 & 0 \quad 0 & c_{n-1} & 2(1+c_{n-1}) \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{pmatrix} \qquad (15)$$

where $A = (a_{ij})$, $s = (s_i)^T$, and $d = (d_i)^T$ for $i, j = 1, 2, \cdots, n-1$. Clearly,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n-1} |a_{ij}| \quad \text{for } i = 1, 2, \cdots, n-1.$$

Thus, A is diagonally dominant and hence A is ncnsingular. Thus, $As = d$

has a unique solution. Therefore, s_i exists, is unique, and the spline can be calculated from (9) once each s_i is determined. ∎

Computation of s_i.

To determine the solution of the tridiagonal system $As = d$ given by (15) we use a recursion algorithm. Let

$$s_{i-1} = p_is_i + q_i \qquad \text{for } i = 1, 2, \cdots, n. \qquad (16)$$

Substituting this into (14) yields

$$c_i(p_is_i + q_i) + 2(1+c_i)s_i + s_{i+1} = d_i$$

Thus,

$$s_i = \frac{-1}{c_ip_i + 2(1+c_i)}s_{i+1} + \frac{d_i - c_iq_i}{c_ip_i + 2(1+c_i)}.$$

This has the same form as (16) with

$$p_{i+1} = \frac{-1}{c_ip_i + 2(1+c_i)} \text{ and } q_{i+1} = \frac{d_i - c_iq_i}{c_ip_i + 2(1+c_i)} \qquad (17)$$

Letting $p_1 = 0$ and $q_1 = 0$ in (16) produces $s_0 = 0$ which is desired. Now using p_1, q_1, and (17) recursively we obtain p_i and q_i for $i = 1, \cdots, n-1$ providing that no demoninator is zero. Then, using $s_n = 0$ and (16) recursively we compute s_i in the reverse order. We now prove that

$$c_ip_i + 2(1+c_i) \neq 0 \qquad \text{for } i = 1, 2, \cdots, n. \qquad (18)$$

Proof. Since $p_0 = 0$, $|p_1| < 1$, which in turn results in $|p_2| < 1$. Continuing this we get $|p_i| < 1$. Then $|c_ip_i + 2(1+c_i)| = |c_i(p_i+2)+2| \geq |c_i + 2| > 1$ so (18) holds for case i. Thus,

$$|p_{i+1}| = \frac{1}{|c_ip_i + 2(1+c_i)|} < 1$$

and $|c_{i+1}p_{i+1} + 2(1+c_{i+1})| > 1$. Hence, by induction, (18) holds for the case $i+1$. ∎

Theorem 2. The cubic spline function is the smoothest interpolating curve in terms of the mean square curvature of all twice continuously differentiable interpolating functions.

Proof. Let $f(x)$ and $g(x)$ be twice continuously differentiable on

$[a, b]$. Let $y(x)$ be the interpolating cubic spline to $f(x)$ at $a = x_0 < x_1 < \cdots < x_n = b$ with $y''(x_0) = 0$ and $y''(x_n) = 0$. Let $g(x)$ also interpolate $f(x)$ at $a = x_0 < x_1 < \cdots < x_n = b$.

$$\int_a^b [g''(x)]^2 dx = \int_a^b [g''(x) - y''(x)]^2 dx + 2\int_a^b [g''(x) - y''(x)]y''(x)dx + \int_a^b [y''(x)]^2 dx. \tag{19}$$

Consider,

$$\int_a^b [g''(x) - y''(x)]y''(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} [g''(x) - y''(x)]y''(x)dx.$$

Integrating by parts with $u = y''(x)$ and $dv = [g''(x) - y''(x)]dx$ yields

$$\int_a^b [g''(x) - y''(x)]y''(x)dx = \sum_{i=0}^{n-1} [g'(x) - y'(x)]y''(x)\Big|_{x_i}^{x_{i+1}} - \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} [g'(x) - y'(x)]y'''(x)dx. \tag{20}$$

First,

$$\sum_{i=0}^{n-1} [g'(x) - y'(x)]y''(x)\Big|_{x_i}^{x_{i+1}} = [g'(x_n) - y'(x_n)]y''(x_n) - [g'(x_0) - y'(x_0)]y''(x_0) = 0$$

since $y''(x_n) = y''(x_0) = 0$. Next, since $y'''(x)$ is a constant

$$\int_{x_i}^{x_{i+1}} [g'(x) - y'(x)]y'''(x)dx = y'''(x)\int_{x_i}^{x_{i+1}} [g'(x) - y'(x)]dx$$

$$= y'''(x)[(g(x) - y(x))\Big|_{x_i}^{x_{i+1}}]$$

$$= y'''(x)[(g(x_{i+1}) - y(x_{i+1})) - (g(x_i) - y(x_i))]$$

$$= 0$$

since $y(x)$ and $g(x)$ both interpolate $f(x)$ at the points x_i and x_{i+1}, so that $y(x_i) = g(x_i)$ and $y(x_{i+1}) = g(x_{i+1})$. Thus (20) becomes

$$\int_a^b [g''(x) - y''(x)]y''(x)dx = 0.$$

Substituting this into (19) we obtain

$$\int_a^b [g''(x)]^2 dx = \int_a^b [g''(x) - y''(x)]^2 dx + \int_a^b [y''(x)]^2 dx \ge \int_a^b [y''(x)]^2 dx$$

since $\int_a^b [g''(x) - y''(x)]^2 dx \ge 0$. Thus, the mean square curvature of $y(x)$ is less than or equal to the mean square curvature of $g(x)$. □

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LOCAL AWARDS

If your chapter has presented or will present awards this year to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal*.

REFEREES FOR THIS ISSUE

The *Journal* recognizes with appreciation the following persons who graciously devoted their time to evaluate papers submitted for publication prior to this issue, and to serve as judges in the Undergraduate Manuscript Contest: J. Harold Ahlberg, Brown University; David W. Ballew, South Dakota School of Mines and Technology; John Green, University of Oklahoma; and Patrick Lang, Old Dominion University.

The Journal also acknowledges with gratitude the expert typing performed by Theresa McKelvey.

ON DIFUNCTIONAL AND CIRCULAR RELATIONS

by Alma E. Posey[1]
Hendrix College

The purpose of this paper is to investigate relationships between the notions of difunctional relations and circular relations. Among such relationships we will prove that a reflexive relation is difunctional (circular) if and only if it is an equivalence relation. The graph G of a function is difunctional but is circular if and only if G^3 is the diagonal relation. Transitive closure, difunctional closure, and circular closure for relations are also examined. Finally, we show that subgroups of the product of two groups are difunctional relations, but subsemigroups of the product of two semigroups need not be difunctional.

We begin with some introductory remarks and definitions concerning relations. Let X and Y be sets. A relation R *from* X to Y means R is a subset of $X \times Y$. The inverse of such a relation is defined by $R^{-1} = \{(y, x) | (x, y) \in R\}$ and is a relation from Y to X. If S is a relation from Y to a set Z, then the composition of R with S, denoted $R \circ S$, is defined by $R \circ S = \{(x, z) | \text{there is a } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$. It should be noted that $R \circ S$ is a relation from X to Z. It is routine to show the composition of relations is *associtive*, (if T is a relation from Z to W, then $(R \circ S) \circ T = R \circ (S \circ T)$). The inverse operation and composition respect each other via: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. The verification is routine and is omitted.

If R is a relation from X to X, we say R is a relation on X. We denote the diagonal relation on X by $\Delta_X = \{(x, x) | x \in X\}$. A relation R on X is called reflexive if $\Delta_X \subset R$, symmetric if $R^{-1} = R$, and transitive if $R \circ R \subset R$. R is called an equivalence *relation* if R is reflexive, symmetric, and transitive.

We next consider the two types of relations we wish to examine in this paper. Let S be a relation on X. S is called *circular* if and only if $S \circ S \subset S^{-1}$. If R is a relation from X to Y, then R is called

[1] The author acknowledges the help and inspiration of Professor Temple H. Fay in writing this paper.

difunctional if and only if $R \circ R^{-1} \circ R \subset R$.

Theorem 1. For any relation R from X to Y, $R \subset R \circ R^{-1} \circ R$. Thus R is difunctional if and only if $R = R \circ R^{-1} \circ R$.

Proof. Let $(x, y) \in R$; then $(y, x) \in R^{-1}$ and hence $(x, y) \in R \circ R^{-1} \circ R$. Thus for any relation R, $R \subset R \circ R^{-1} \circ R$. Thus, R is difunctional if and only if $R = R \circ R^{-1} \circ R$.

Theorem 2. If R is a relation from X to Y, then R is difunctional if and only if R^{-1} is difunctional.

Proof. If R is difunctional, then $R \circ R^{-1} \circ R = R$, so $(R \circ R^{-1} \circ R)^{-1} = (R)^{-1} = R^{-1}$. Thus $R^{-1} \circ R \circ R^{-1} = R^{-1}$ and R^{-1} is difunctional. It follows that if R^{-1} is difunctional, then $(R^{-1})^{-1} = R$ is also.

We have an analogous theorem for circular relations.

Theorem 3. If S is a relation on X, then S is circular if and only if S^{-1} is circular.

Proof. If S is circular, then $S \circ S \subset S^{-1}$, so $S^{-1} \circ S^{-1} = (S \circ S)^{-1} \subset (S^{-1})^{-1} = S$. Thus $S^{-1} \circ S^{-1} \subset S$ and S^{-1} is circular. It follows that if S^{-1} is circular, then $(S^{-1})^{-1} = S$ is also.

Theorem 4. If R is a reflexive relation on X, then the following are equivalent.

1. R is an equivalence relation.
2. R is a difunctional relation.
3. R is a circular relation.

Proof. Assume R is an equivalence relation. Then R is symmetric; hence $R = R^{-1}$. Consequently, since R is transitive, $R \circ R^{-1} \circ R = R \circ R \circ R \subset R \circ R \subset R$. Hence R is difunctional.

To that see R being difunctional implies R is circular, recall that R^{-1} must also be difunctional and reflexive. Therefore $R \circ R = \Delta_X \circ R \circ \Delta_X \circ R \circ \Delta_X \subset R^{-1} \circ R \circ R^{-1} \circ R \circ R^{-1} \subset R^{-1} \circ R \circ R^{-1} \subset R^{-1}$; thus R is circular.

Next assume R is circular and reflexive. Then $R^{-1} = \Delta_X \circ R^{-1} \subset R^{-1} \circ R^{-1} \subset R$, since R^{-1} is also circular. Hence R is symmetric. To see R is transitive, note $R \circ R \subset R^{-1} \subset R$.

Theorem 4 illustrates a connection between difunctional and circular relations on a set X. Using this theorem for motivation, we investigate further properties in order to clarify connections between the notions.

Theorem 5. If R is a symmetric circular relation on X then R is difunctional.

Proof. Since R is symmetric, $R = R^{-1}$, so $R \circ R^{-1} \circ R = R \circ R \circ R$. Applying circularity, $R \circ R \circ R \subset R^{-1} \circ R = R \circ R \subset R^{-1}$. Thus R is difunctional.

The converse of Theorem 5 can be shown not to hold through the following example. Let $X = \{x, y, z, w\}$ and $R = \{(x, y)(y, x)(z, y)(y, z)(w, z)(z, w)(x, w)(w, x)\}$. R is symmetric and difunctional, but if R is circular, (x, y) and $(y, x) \in R$ would imply $(x, x) \in R^{-1}$ which is false.

Theorem 6. If R is a difunctional relation from X to Y with the property that for every $x \in X$ there exist a $y \in Y$ such that $(x, y) \in R$, then $R \circ R^{-1}$ is an equivalence relation on X.

Proof. Let $x \in X$. Then there exist $y \in Y$ such that $(x, y) \in R$. Then $(y, x) \in R^{-1}$, so $(x, x) \in R \circ R^{-1}$. Thus $R \circ R^{-1}$ is reflexive. Since $(R \circ R^{-1})^{-1} = (R^{-1})^{-1} \circ R^{-1} = R \circ R^{-1}$, then $R \circ R^{-1}$ is symmetric. Now, $(R \circ R^{-1}) \circ (R \circ R^{-1}) = (R \circ R^{-1} \circ R) \circ R^{-1} = R \circ R^{-1}$ using Theorem 1, hence $R \circ R^{-1}$ is transitive. Thus $R \circ R^{-1}$ is an equivalence relation.

For the next theorem it is useful to define the set $RY = \{x \mid$ there exists $y \in Y$ such that $(x, y) \in R\}$ and $XR = \{y \mid$ there exists $x \in X$ such that $(x, y) \in R\}$. It should be noted that if R is a relation from X to Y, then R is also a relation from RY to XR, so there is no loss in generality in assuming $X = RY$ and $Y = XR$. Also note that if $f: X \to Y$ is a function, then its graph is denoted by $G_f = \{(x, y) \mid y = f(x)\}$.

Theorem 7 (Riguet). If R is a difunctional relation from X to Y, then there exist a unique bijection $h : RY/R \circ R^{-1} \to XR/R^{-1} \circ R$ making the following diagram cummutative where f and g are canonical surjections (See also [3, 4]).

$$\begin{tikzcd}
R \arrow[r, "\mathrm{Proj}_1"] \arrow[dd, "\mathrm{Proj}_2"'] & RY \arrow[d, "f"] \\
 & RY/R \circ R^{-1} \arrow[d, "h"] \\
XR \arrow[r, "g"'] & XR/R^{-1} \circ R
\end{tikzcd}$$

Moreover, it follows $R = G_{hf} \circ G_g^{-1}$, hence the term difunctional.

Proof. Choose a $z \in RY/R \circ R^{-1}$; then $z = f(x)$ for some $x \in RY$, and there exists y such that $(x, y) \in R$. Define $h(z) = g(y)$. Let (x, y) and $(x', y') \in R$, so $(x, x') \in R \circ R^{-1}$, and $(y, x) \in R^{-1}$. Using Theorem 1, $R^{-1} \circ R = R^{-1} \circ (R \circ R^{-1} \circ R)$. Now take $(y, x) \in R^{-1}$, $(x, x') \in R \circ R^{-1}$ and $(x', y') \in R$; then $(y, y') \in R^{-1} \circ R$. Thus h is well defined.

Now choose $w \in XR/R^{-1} \circ R$; then $w = g(y)$ for some $y \in XR$, and there exists x such that $(x, y) \in R$. Consider $f(x) \in RY$; then $h(f(x)) = g(y) = w$. Hence h is surjective. Now suppose $h(z) = h(z')$. Then there exist (x, y) and (x', y') in R such that $z = f(x)$ and $z' = f(x')$. Thus $g(y) = h(z) = h(z') = g(y')$. From $(x, y) \in R$, $(y, y') \in R^{-1} \circ R$, and $(y', x') \in R^{-1}$, taken with $R \circ R^{-1} = (R \circ R^{-1} \circ R) \circ R^{-1}$, we see $(x, x') \in R \circ R^{-1}$. Hence h is injective.

Now let $(x, y) \in G_{hf} \circ G_g^{-1}$; then there exists w such that $(x, w) \in G_{hf}$ and $(w, y) \in G_g^{-1}$. Therefore, $h(f(x)) = w$ and $g(y) = w$. There exists a y' such that $(x, y') \in R$ and $h(f(x)) = g(y') = w$ since $g(y') = g(y)$, which also implies $(y, y') \in R^{-1} \circ R$ and $(y', y) \in R^{-1} \circ R$. Since $(x, y') \in R$ and $(y', y) \in R^{-1} \circ R$, then $(x, y) \in R \circ R^{-1} \circ R = R$. Thus $G_{hf} \circ G_g^{-1} \subset R$. Now let $(x, y) \in R$; then since $h(f(x)) = g(y)$, $(x, h(f(x))) \in G_{hf}$ and $(y, g(y)) \in G_g$ with $(g(y), y) \in G_g^{-1}$. Therefore $(x, y) \in G_{hf} \circ G_g^{-1}$. Thus $R = G_{hf} \circ G_g^{-1}$.

The graphs of functions are interesting relations to consider.

Theorem 8. The graph G of a function g is difunctional.

Proof. Let $(x, w) \in G \circ G^{-1} \circ G$; then there exist y and z such that $(x, y) \in G$, $(y, z) \in G^{-1}$, and $(z, w) \in G$. Since (x, y) (z, y) and $(z, w) \in G$, by definition of a function, $y = w$ so $w = f(x)$ and $(x, w) \in G$. Thus G is difunctional.

Due to the similarities between difunctional and circular relations noted throughout this paper, it is a natural question to ask if the graphs of functions from X to X are circular. This question is answered by the next theorem.

Theorem 9. If G is the graph of a function, $f : X \to Y$, G is circular if and only if $f^3 = 1_X$; that is, f^3 is the identity function on X.

Proof. Assume G is circular and let $x \in X$; then $(x, f(x)) \in G$ so

$f(x) \in X$ and $(f(x), f^2(x)) \in G$. Since G is circular, $(x, f^2(x)) \in G^{-1}$ which implies $(f^2(x), x) \in G$ thus stating $f^3(x) = x$. Therefore $f^3 = 1_X$.

Let $(x, y) \in G \circ G$; then since G is the graph of a function, $(x, f(x)) \in G$ and $(f(x), y) \in G$ which implies $f^2(x) = y$. Now $f(y) = f^3(x) = x$, therefore $(y, f(y)) \in G$ which is the same as $(y, x) \in G$. Hence G is circular.

Difunctional relations arise frequently in algebra, in particular, as subgroups of products of groups. More percisely:

Theorem 70. Let G and H be groups and R a subgroup of $G \times H$ with coordinate-wise operations. Then R is difunctional.

Proof. Let e denote the identity of H. Let $(x, y) \in R \circ R^{-1} \circ R$; then there exist z and w such that $(x, z) \in R$, $(z, w) \in R^{-1}$, and $(w, y) \in R$. Since R is a subgroup, for every a $\in$ R, then $a^{-1} \in R$, so consequently (x^{-1}, z^{-1}), (w^{-1}, z^{-1}), and $(w^{-1}, y^{-1}) \in R$. Also since R is a subgroup, for every c and $d \in R$, $cd^{-1} \in R$. Take $(x, z) \in R$ and $(w^{-1}, z^{-1}) \in R$. It follows that $(xw^{-1}, zz^{-1}) = (xw^{-1}, e) \in R$. Since $(w, y) \in R$, it follows that $(xww^{-1}, ey) = (x, y) \in R$, hence R is difunctional.

Unfortunatly, relations which are subsemigroups of the product of two semigroups need not be difunctional. If S and T are semigroups having left trivial multiplication, then $S \times T$ with coordinate-wise operations has left trivial multiplication and any subset is a subsemigroup. Simply choose a subset R of $S \times T$ which is not a difunctional relation to find an example.

It will be interesting to look at the intersection of families of certain classes of relations. It will be shown that these classes are closed under the formation of intersections.

Theorem 11. If $\{R_i : i \in I\}$ is a set of difunctional relations from X to Y, then $\cap R_i$ is a difunctional relation.

Proof. Let (x, w), (z, w) and $(a, y) \in \cap R_i$; then by definition of intersection, (x, w), (z, w) and $(a, y) \in R_i$ for every $i \in I$. Since R_i is difunctional, then $(x, y) \in R_i$, so $(x, y) \in \cap R_i$ and hence $\cap R_i$ is difunctional.

Similar proofs will establish the following two results.

Theorem 72. If $\{R_i : i \in I\}$ is a set of circular relations on X, then $\cap R_i$ is a circular relation.

Theorem 13. If $\{R_i : i \in I\}$ is a set of transitive relations on X, then $\cap R_i$ is a transitive relation,

Since the indicated classes of. relations are closed under the formation of intersections, we may obtain closure operations in the following way. If $R \subset X \times Y$, define difunctional closure, dR, to be the intersection of all difunctional relations from X to Y containing R. Analogously define circular closure, cR, and transitive closure, tR, of a relation R.

Corollary 1. Let R and S be relations from X to Y. Then

1. $d(R \cup S) = d(R) \cup d(S)$.
2. $R \subset S$ implies $dR \subset dS$.
3. $dR = R$ if and only if R is difunctional.
4. $dR^{-1} = (dR)^{-1}$.

Similar statements hold for circular and transitive closures.

It is known that there exist relationships between difunctional closure and transitive closure. More precisely, a result due to Riguet [6] states that $dR \circ dR^{-1} = t(R \circ R^{-1})$. It would be interesting to know if such relationships exist for circular closure.

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HOW TO DESIGN AND IMPLEMENT A PI MU EPSILON STUDENT CONFERENCE

by Milton Cox[1]
Miami University

The present policy of the Council of Pi Mu Epsilon is to subsidize the organizing or host chapter of at least a two-college student conference to the extent of $50. In the few years since this offer was made, few chapters have participated. Yet, a student conference can be a unique and fulfilling experience for participants, and an excellent way to carry out the fraternity motto: *To promote scholarship* ***and*** *mathematics,*

It is a good idea to schedule a student conference in conjunction with a mathematics meeting or conference of interest to faculty, and certainly there are a prolific number of these. Examples are: National annual conferences on special topics (**e.g.,** the topology conferences), regularly- or irregularly-scheduled regional conferences (see the *Notices of the American Mathematical Society* for announcements of such meetings), the sectional meetings of the American Mathematical Society, the regional meetings of the Mathematical Association of America (see the *American Mathematical Monthly* for dates and locations), or when a speaker of national prominence visits your school. Thus, it is possible to sponsor a student conference either as a one-time event or as an ongoing annual event. The regional conferences hold the advantage of involving less travel expense.

After your chapter decides to organize a conference the first task is to prepare an attractive one-page announcement of the event and a tentative schedule, and possibly, what accomodations and travel arrangements are being made available, and then mail these to all schools in the appropriate region. Naturally, the up to $50 available to defray expenses is not enough to support travel or lodging, but occasionally either a local chapter or the corresponding mathematics department can find such funds if it is decided to sponsor speakers, or, as a more common arrangement, faculty members can bring their top students along with them. In addition (or in lieu of), a chapter could sponsor a special Pi Mu Epsilon Travel Award for promising student(s) to present papers; under this arrangement matching funds are available from the national coffers. Such awards, however, should be made formal and presented at a special meeting.

It has been the author's experience, however, that if announcements" are mailed out calling for student papers and this is all that is done, then there will be little response. The chapter advisor must make telephone contact with faculty friends at nearby schools in order to create interest. Or, at your next regional MAA meeting, why not make contacts and plan for such a conference at a future regional meeting? Follow-up calls are sometimes necessary regardless of the mutual agreements or the excellent plans which may have been formulated.

It is recommended that the student session not conflict with the faculty session so that the student can experience the faculty-level sessions *and* present a paper, and faculty members can be present at the student sessions to guide and support their students.

Student participation by one's own chapter members has been no problem at Miami University (where the fourth annual student conference was just recently held). Usually one or two are willing to present papers and all are eager to help at the conference. The student conference is held on Saturday afternoon following an annual faculty level conference sponsored by the department on Friday and Saturday morning. The chapter members of Ohio Delta share their dorm rooms or apartments with overnight student visitors so that no lodging expense is incurred. This arrangement is advertised and some students are encouraged to come because they feel that they will not only be learning mathematics at the sessions but also making new friends and exploring a new school.

It has been found in running the conferences at Miami University that the $50 from the national treasury adequately covers the advertising and printing costs. When possible, a free lunch has been provided for student speakers.

Our first conference attracted five speakers, the second eight, the third nine, and this year, eleven. The experience and confidence gained in presenting a talk has encouraged many students to present talks at the National Pi Mu Epsilon meetings; others in the audience are encouraged to try the next year. It is hoped that you will soon try a student Pi Mu

[1]Professor Cox is presently a national Councillor, and is also the advisor of the *Ohio Delta Chapter.*

Epsilon Conference in your area.

REGIONAL MEETINGS OF MAA

Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

MATCHING PRIZE FUND

If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of mathematical films. To apply, or for more information, write to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

1975-1976 MANUSCRIPT CONTEST WINNERS

The judging for the best expository papers submitted for the 1975-76 school year has now been completed. The winners are:

FIRST PRIZE ($200): Brent Hailpern, University of Denver, for his paper "Continuous Non-differentiable Functions" (this *Journal*, Vol. 6, No. 5, pp. 249-260).

SECOND PRIZE ($100): Philip D. Olivier, Texas Tech University, for his paper "Two Applications of Pseudoinverses" (this *Journal*, Vol. 6, No. 5, pp. 261-265).

THIRD PRIZE ($50): Karen M. Lesko, Central Missouri State University, for her paper "A Generalization to Almost Divisible Groups" (this *Journal*, Vol. 6, No. 6, pp. 345-347).

1977-78 CONTEST

Papers for the 1976-77 contest are now being judged, and we are receiving papers for this year's contest, so be sure to send us your paper, or your chapter's papers (at least 5 entries must be received from the same chapter in order to qualify, with a $20 prize for the best paper in each chapter). For all manuscript contests, in order for authors to be eligible, *they must not have received a Master's degree at the time they submit their paper.*

INITIATION CEREMONY

The editorial staff of the *Journal* has prepared a special publication entitled *Initiation Ritual* for use by local chapters containing details for the recommended ceremony for initiation of new members. If you would like one, write to the National Office.

WELCOME TO NEW CHAPTERS

The *Journal* welcomes the following new chapters of Pi Mu Epsilon, recently installed:

NORTH CAROLINA ETA at Appalachian State University, installed in November of 1976 by Robert M. Woodside, Councillor.

SOUTH CAROLINA GAMMA at The College of Charleston, installed March 31, 1977 by Robert M. Woodside, Councillor.

TEXAS MU at East Texas State University, installed April 12, 1977 by R. V. Andree, Council Vice-president.

CONNECTICUT BETA at The University of Hartford, installed April 26, 1977 by R. A. Good, Council Secretary-Treasurer.

TEACHING OPPORTUNITIES OVERSEAS

More than 1000 English-language oriented schools and colleges in over 150 foreign countries offer teaching and administrative opportunities to American and Canadian educators. Positions exist in most all fields, on all levels, from kindergarten to the university. Salaries vary from school to school, but in most cases they are comparable to those in the U.S. Vacancies occur and are filled throughout the year. Some schools overseas do not require previous teaching experience or certification. If you are interested in a position with an overseas school or college, contact:

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Friends of World Teaching is an independent teachers' information agency, dedicated entirely to assisting American educators in securing teaching or administrative positions overseas and serving American educators since 1969. FOWT is an active member of the San Diego Better Business Bureau.

PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword pussies or who find an occasional mathematical puzzle attractive, We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinquish material which might well be the domain of the Problem Department, but the Editor reserves the right to make an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate, or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to the Editor, Pi Mu Epsilon Journal, 601 Elm Avenue, Room 423, The University of Oklahoma, Norman, Oklahoma, 73019. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March, and that for each Spring issue, the following September.

Mathacrostic No. 4

submitted by R. Robinson Rome
Sacramento, California

Like the preceding three, this acrostic is a keyed anagram. The 235 letters to be entered in the diagram in the numbered spaces will be identical with those in the 32 key words at matching numbers and the key letters have been entered in the diagram to assist in correlation during your solution (see next two pages). When completed, the initial letters

		1 M	2 M	3 M	4 M		5 X	6 M	7 M	8 A	9 C		10 F	11 S
12 M		13 A		14 N	15 c	16 M	17 M	18 b	19 M		20 d	21 S		22 L
23 S	24 C	25 N	26 S	27 b	28 X		29 e	30 b		31 e	32 M	33 S	34 f	
35 e	36 d		37 C	38 N	39 X	40 F	41 X		42 S	43 X	44 S	45 c		46 E
47 f	48 N	49 A	50 M		51 S	52 X	53 A	54 C		55 X	56 b	57 N	58 b	
59 f	60 e	61 X	62 Y		63 A	64 d	65 A	66 C	67 S		68 F	69 C		70 c
71 Z	72 a		73 A	74 e	75 A	76 V	77 C	78 F		79 M	80 M	81 N	82 d	83 a
	84 a	85 Z	86 Y		87 H	88 R	89 A	90 a	91 Y		92 Z	93 S	94 C	
95 d	96 F	97 c		98 f	99 A	100Y	101b		102W	103Y	104B		105A	106M
107F	108H	109O		110N	111S		112U	113V	114C		115E	116e	117N	118M
119K		120e	121a		122J	123U	124H		125f	126d	127C	128b		129D
130V		131P	132X	133B		134F	135R	136W	137Z	138X	139F		140Q	141E
	142f	143P	144T		145W	146b		147J	148G	149Q		150I	151E	152e
153d	154B		155D	156a		157Q	158T	159G		160P	161b	162Q	163A	164B
165D	166L	167Q		168J	169c	170b		171V	172E	173K	174O		175I	176O
	177P	178d	179W		180D	181Y	182L	183I	184L	185P		186P	187d	188W
	189b	190F	191H	192V	193B	194R	195K	196I		197T	198G	199U	200D	201Z
202K	203H	204L		205Z	206G	207O	208E		209P	210f	211O		212B	213I
214D		215V	216R	217J	218I	219H		220J	221G	222Z		223H		224E
225J	226W	227F	228O	229R		230D	231R	232L		233E	234K	235B		

Definitions and Key

A.	Imaginary domain boundary	49 65 75 89 99 8 73/163 53 13 63 105
B.	Nature and beauty lover	104 133 154 164 193 212 235
C.	Author's imaginary domain	37 66 77 94 114 127 54 24 69 9
D.	Locus of equal rainfall	200 180 155 129 230 214 365
E.	Author's logic	233 208 115 224 141 172 E1 46
F.	Related interchangeably in mathematics	40 68 10 227 190 139' 107 134 96 78
G.	Without freeboard	221 206 198 159 148
H.	Real meaning of "biscuit"	106 324 87 223 215 203 391
I.	Metrical form	218 213 396 183 175 150
J.	Headquarters of new mathematics journal	225 147 122 217 220 168
K.	River Hades	195 234 119 202 173
L.	Sailed to Windward	184 232 182 22 166 204
M.	Two hundredths	1 79 3 19 12 4/16 2 6 7 118 32 106 50 17 80 142
N.	Customary way of life	38 57 14 110 25 48 117 81
O.	New mathematics journal	174 228 211 176 309 207
P.	Midway	177 186 160 209 131 143 185
Q.	King Arthur's dad	162 140 157 149 167
R.	Headquarters of this *Journal*	216 231 135 229 88 394
S.	Platonic solid	42 23 51 21 93 26 44 67 33 11 111
T.	Sports league	158 197 144
U.	General power or term	199 112 123
V.	Flowed copiously	171 76 215 113 130 192
W.	Singularity	226 179 188 145 102 136
X.	Thermal scale	5 132 41 28 52 138 43 61 39 55
Y.	General explanation	100 103 86 181 91 62
Z.	Hemp derivative	92 205 222 85 137 201 71
a.	Worn out	83 121 156 90 84 72
b.	Understaffed	189 101 18 30 58 56 161 146 128 27 170
c.	Crazy	169 15 97 70 45
d.	Heavenly mathematics	178 153 64 82 20 187 126 95 36
e.	Parallelogram	74 60 120 31 35 29 116 152
f.	Act obsequiously	34 47 98 59 210 125 142

Note. The 2 words of A and M are separated by /. A, C and E are key words in titles of 3 other books by the same author.

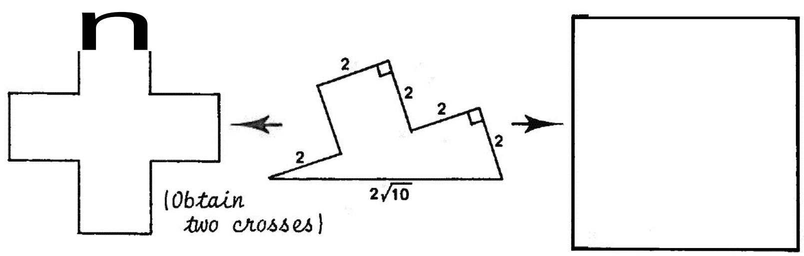
of the 32 key words will spell a famous mathematician and the title of his book that was featured in its centennial in 1976. The diagram will then quote the surprising conclusion of the book.

Greek Crosses and Squares

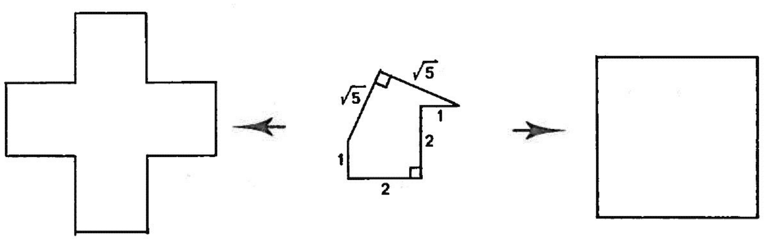
In the diagrams below are illustrated a number of figures of various shapes and dimensions. Discover in each case the arrangement which will produce either the Greek crosses or squares indicated. Solver must determine the correct number of figures which must be used.

End with: *Start* with: End *with:*

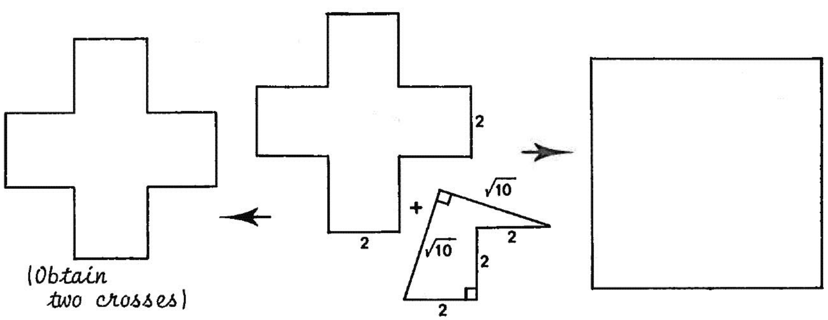
(a)



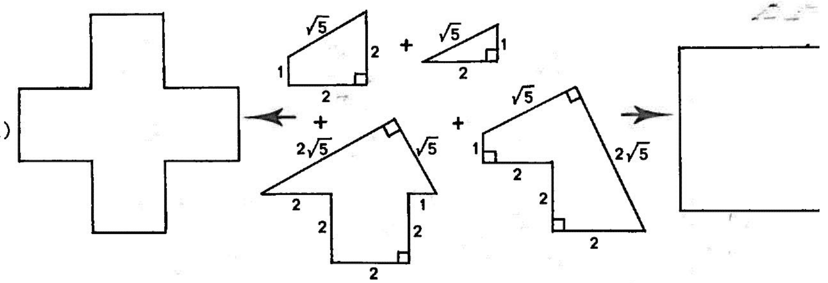
(b)



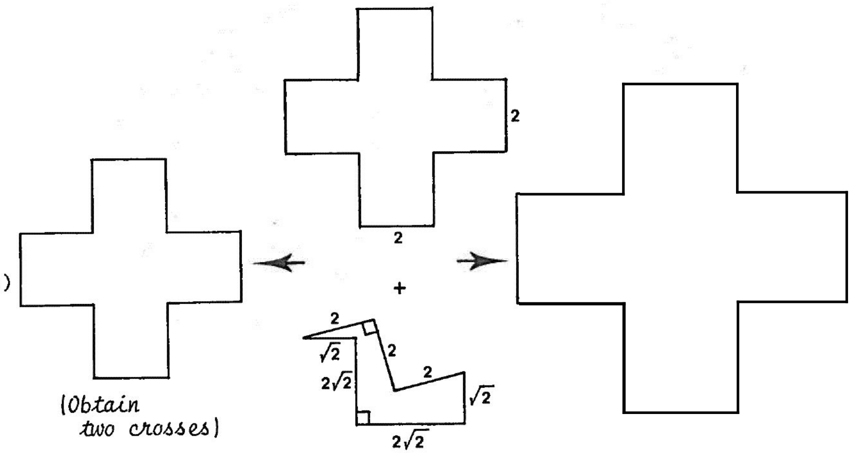
(c)



(d)

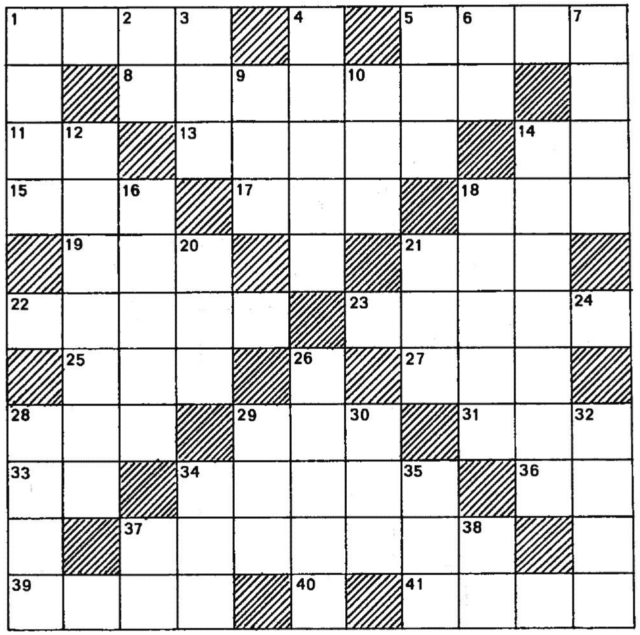


(e)



A Cross-number Puzzle

The blanks in the 11 × 11 square on the following page are to be filled in with single digits as in a crossword puzzle so that the resulting numbers satisfy the conditions given.



Across

1. A square number
4. Square root of 14 across
5. A square number
8. Sum of digits is 36
11. Cube root of 39 across
13. A square number which reads same both ways
14. A square number
15. A multiple of 35 down
17. The three-halves power of 14 across
18. All digits the same
19. Product of 24 down and 33 across
21. A square number
22. A multiple of both 19 and 41
23. All digits alike except central one
25. Perfect square ending in 6
27. See 20 down
28. A fourth power
29. Product of 2 down and 4 across
31. A triangular number
33. Two-thirds of 36 across
34. Digits sum to 26, and middle three numbers are 7
36. An odd number
37. All digits even, except one, and their sum is 25
39. An odd cube
40. A perfect number
41. Twice the square of 37 down

Down

1. Reads same both ways
2. Square root of 28 across
3. Sum of 17 across and 21 down
4. Sum of digits is 37 down
5. Digits sum to 24
6. Difference of 14 across and 37 down
7. A fourth power
9. A cube number
10. Product of 4 across and 14 across
12. Digits sum to 35
14. All digits the same except the first
16. Last 3 digits in arithmetic progression
18. All digits the same except the first
20. A multiple of 41 which evenly divides 22 across
21. A multiple of 19
22. Square number
24. An even number
26. Sum of digits is 37 down
28. Fourth power of 4 across
29. Sum of 14 across and 35 down
30. A triangular number
32. Digits sum to 27 and end in 8
34. A square number
35. A square number
37. The ninth prime
38. A cube times 10

Solutions

Mathacrostic No. 1 [Spring, 1976]

Also solved by VICTOR G. FESER, *Mary College, Bismark, North Dakota*. (*inadvertently omitted from original list of solvers; see Fall,* 1976 *issue for solution*).

Mathacrostic No. 2 [Fall, 1977]

Late *solutions were submitted* by EZRA BROWN, *Virginia Polytechnic Institute* and *State University;* LOUIS H. CAIROLI, *Kansas State University;* DONALD G. CASCI, *Providence,* Rhode *Island;* VICTOR G. FESER, *Mary College, Bismark, North* Dakota; ROBERT C. GEBHARDT, *Hopatcong, New Jersey;* SHARON E. GORDON, *University of North Carolina* at *Chapel Hill;* PATRICIA GROSS and ALLAN TUCHMAN; *University of Illinois;* and DAVID DEL SESTO, *North Providence,* Rhode *Island.* (See *Spring, 1977 issue for solution.*)

Missionaries and *Cannibals* [Fall , 1976]

Late solutions were submitted by LOUIS H. CAIROLI, *Kansas State University;* ROGER E. KUEHL, *Kansas City, Missouri;* and STEVE LEELAND, *University of* South *Florida,* Tampa, *Florida.* (See *Spring,* 1977 *issue for solution.*)

Mathacrostic No. 3 [Spring, 1977]

Definitions and key:

A.	Conjugate	E.	Upper	I.	Integer	M.	Inversion
B.	Fermat	F.	Show	J.	Slim	N.	Shirt
C.	Grog	G.	Sheet	K.	Quaternion	O.	Illicit
D.	Apollonius	H.	Diophantus	L.	Utter	P.	Tic
Q.	Iridic	V.	Archimedes	Z.	Hell	d.	Irenic
R.	Owner	W.	Rhymed	a.	Mersenne	e.	Cheer
S.	Nim	X.	Itchy	b.	Eric	f.	Algebra
T.	Eratosthenes	Y.	Tight	c.	Terms	g.	Etch
U.	Simply						

First letters: C F GAUSS DISQUISITIONES ARITHMETICAE

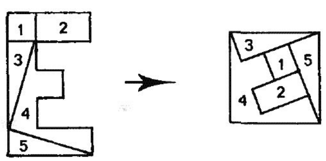
Quotation: *As a result, it seems proper to call this subject Elementary Arithmetic and to distinguish from it Higher Arithmetic which properly includes more general inquiries concerning integers. We consider only Higher Arithmetic in the present volume.*

Solved by GORDON R. BAKER, *Houston, Texas;* JEANETTE BICKLEY, *Webster Groves* High *School, St. Louis, Missouri;* CHRISTOPHER CACAS, *East Texas*

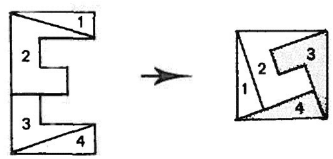
State University, Commerce, Texas; LOUIS CAIROLI, *Kansas State University;* DAREN CLINE, *Harvey* Mudd Collage, *Claremont, California;* EDWIN COMFORT, *Ripon College, Ripon, Wisconsin;* ELEANOR S. ELDER, *University of New Orleans;* VICTOR G. FESER, *Mary* College, *Bismark, North Dakota;* PATRICIA GROSS and ALLAN TUCHMAN, *University of Illinois;* MARK JAEGER, *University of Wisconsin;* JOSEPH D. E. KONHAUSER, *Macalester College,* St. Paul, *Minnesota;* LISA J. LASHER, St. *Lawrence University,* Canton, *New* Yohk; BARBARA LEHMANN, *Brigantine, New Jersey;* MARIANNE MANCUSI, *Adelphi University, Garden City, New York;* BETTY MCELROY, *Southern Illinois University at Edwardsville;* SIDNEY PENNER, *Bronx Community College of* CUNY; BOB PRIELIPP, *University of Wisconsin* at *Oshkosh;* RITA PRINCI, *Holly Hill, Florida;* EDITH E. RISEN, Oregon *City, Oregon;* FORREST K. RUSSELL, *University of North Carolina;* RICHARD D. STRATTON, *Colorado Springs, Colorado;* and CHARLES W. TRIGG, Son *Diego, California.*

Dissecting the Letter E [Spring, 19771

The desired dissections are as shown below:



Five Piece *Dissection*



Four Piece *Dissection* (*2 parts inverted*)

Solved by VICTOR G. FESER, *Mary College, Bismarck, North Dakota* and CHARLES W. TRIGG, San *Diego, California.*

Editor's Note.

Charles W. Trigg comments that this puzzle is precisely Problem 184, p. 55 of H. E. Dudeney's Puzzles and Curious Problems, Thomas Nelson and Sons, 1931, appears as Problem 330, p. 114 in H. E. Dudeney's 536 Pussies and Curious *Problems*, edited by Martin Gardner, Scribner, 1967, and is found in Harry Lindgren's Geometric Dissections, Dover, 1972, p. 78.

A *Pair of Eights*

There are a pair of solutions in spite of the erroneous claim of uniqueness made in the original statement:


```
          98409                  98309
119 ) 11710671         124 ) 12190316
      1071                   1116
      1000                   1030
       952                    992
        486                    383
        476                    372
         1071                   1116
         1071                   1116
```

Solutions were accepted as correct if either one (or both) of these was submitted.

Solved by GORDON R. BAKER, Houston, *Texas;* LOUIS H. CAIROLI,, *Kansas State University;* JOHN M. FERRO, St. *John's University, Jamaica, New* Yolk; VICTOR G. FESER, *Mary College, Bismark, North Dakota;* HOWARD FORMAN, *Bucknell University, Lewisburg, Pennsylvania;* KEVIN KARPLUS and GARY LIEBEMAN, *Stanford University;* BOB PRIELIPP, *University* of, *Wisconsin at Oskosh;* EDITH E. RISEN, *Oregon City, Oregon;* NGUYEN THI THANH NHA, *Louisiana* State *University; and* CHARLES W. TRIGG, San *Diego, California.*

WILL YOUR CHAPTER BE REPRESENTED IN PROVIDENCE?

It is time to be making plans to send an undergraduate delegate or speaker from your chapter to attend the annual meeting of Pi Mu Epsilon in Providence, Rhode Island during August 8-12, 1978. Each speaker who presents a paper will receive travel funds of up to $400, and each delegate, up to $200.

MOVING??

BE SURE TO LET THE JOURNAL KNOW!

Send your name, old address with *zip code* and new address with *zip code* to:

Pi Mu Epsilon Journal
601 Elm Avenue, Room 423
The University of Oklahoma
Norman, Oklahoma 73019

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA at Hendrix College heard several student members speak during the year, *Tom Connor*, *Janet Dillahunty*, *Lisa Orton*, *Don Haqman*, *Bill Orton*, and *Mike Tiefenback*, and *Professor William O. Murray* also presented *a* lecture. In addition, *Professor David Peterson* of the University of Central Arkansas discussed an interesting problem in probability, and Professor Kiyoshi Iseki, from Kobe University in Japan visiting at the University of Arkansas, presented some of his research findings on fixed point theorems. Several members participated in the regional MAA meeting at Oral Roberts University in April.

IOWA ALPHA at Iowa State University heard *Professor A. M. Fink* talk on "How to Probate a Peanut Butter and Jelly Coffeecake" at the 54th Annual Initiation Banquet.

KENTUCKY GAMMA at Murray State University heard talks by *Ross L. Snider* on "Divergence and Curl of a Vector Field" and *Jeffrey Cates* on "The Mento Wheel".

LOUISIANA EPSILOS at McNeese State University heard a paper on the correlation of college dropouts with background information by *Albert Palachek* and *Leigh Erin Joneb.*

MICHIGAN ALPHA at Michigan State University heard several student papers: *Michael Arnold* on "Number Wheels", *Carl Page* on "Representation of Informal Knowledge in Computers", and *Steven Fuller* on "Dyadic Vector Products" and "Surreal Numbers". The chapter also heard *Professor Davis* speak on "The First Theorem" at the annual initiation banquet.

MISSOURI DELTA at Westminster College witnessed a talk and slide presentation given by *Professor S. R. Filippone* from the University of Wisconsin at Parkside on "A Linear Algebra Approach to Fibonacci Numbers", and as part of the spring initiation activities, heard student papers on "A Random Survey of Missouri Voters in the 1976 Election" by *Mark R. Rudoff* and "A Statistical Evaluation of the Peer Counselling Program at Westminster College for 1976-77" by *Theodore S. Wilson.*

NEW JERSEY DELTA at Seton Hall University heard lectures by *Professor J. W. Andrushkiw* on "Some Problems Related to the Triangles Inscribed in a Given Triangle and Their Generalizations", and *Professor Ronald Infante*, on "Generalized Sine and Elliptic Functions".

NEW JERSEY EPSILON at Saint Peter's College sponsored lectures by *Larry Bernstein* of Bell Laboratories who presented the Second Annual Collins Lecture, and *Professor William G. Lister* of SUNY at Stony Brook. The chapter also heard *Professor Thomas, Marlow* speak on the topic "Infinite Sums and Finite Differences" and *Professor Larry E. Thomas*, on "Atonal Music and Bizarre Set Theory".

NORTH CAROLINA GAMMA at North Carolina State University heard talks by *Scott Ross*, a student, on "Looking at Mosaics" and *Professor R. E. Chandler* on "A Rolling Circle Gathers No Moss".

OKLAHOMA BETA at Oklahoma State University participated in a problem-solving event in November where 20 members tackled Putnam Examination problems, and 6 members competed in the 1976 Putnam Examination in December with the following 4 scoring on the test: *Rolan Christofferson*, *Kathy Sullivan*, *Emily Wonderly*, and *Kathy Stewart.* The February meeting consisted of a quiz game in which the undergraduate members quizzed their professors on mathematicians and the history of mathematics.

PENNSYLVANIA NU at Edinboro State College heard *Professor Freitag* lecture on "Tiling" and *Professor Lane.*, on "Pascal's Triangle--Another Look". Student members who presented talks were *Michael Lynn* on "Impossible Scores", *Daniel Platt* on "Fibonacci Sequences and the Golden Ratio", and *Mr. Watson* presented a lecture on "Optimal Controls".

SOUTH CAROLINA GAMMA at the College of Charleston heard 3 student speakers at the initiation banquet: *Lonita Spivey* on "How to Model for Politicians", *Jo Ella Rentz* on "A History of Numerals", and *Durward Rogers* on "The Game of Life".

TEXAS BETA at Lamar University heard *Professor Richard Alo* discuss tayloring of degree plans for mathematics related job opportunities and *Lieutenant Colonel Thomas J. Hardy* on "Job Opportunities in the Air Force for Mathematics Majors".

TEXAS DELTA at Stephen F. Austin State University heard lectures given by *Professor Pat Boston* on "What I Know About Recursions in

Combinatorial Analysis", *Professor Wayne Proctor* on "Order by Choice", *Melanie Damron* on "Women in Mathematics", *Mr. Bunch* on "Latin Squares". The guest speaker at the spring banquet in February was *Professor Kenneth Reid* of Louisiana State University who spoke on "After-Dinner Combinatorics".

TEXAS IOTA at the University of Texas at Arlington heard lectures by *Shakuntala Devi* on "Amazing Feats with Numbers", *Carl Franklin* from Southwestern Bell on "What Do Employers Look For?", and *Tom Shelby* and *Ken Haynes* on "Opportunities and Problems in Actuarial Science".

TEXAS MU at East Texas State University held its formal installation ceremony in April, 1977. Guest speaker was *Professor Richard V. Andree* of the University of Oklahoma who spoke on "Secret Writing: Codes, Ciphers and Cryptarithms". The chapter also heard *Bill Copeland*, honors student, present his paper on "Numerical Quadrature: The Trapezoidal Rule and Modifications",

WEST VIRGINIA ALPHA at West Virginia University co-hosted the student sessions of the Allegheny Mountain Section of the Mathematical Association of America meeting at St. Francis College, Loretto, Pennsylvania. The following students presented papers at this meeting: *James Bretti*, *Michael Monnett*, *Edward Weismann*, and *Robert Goldrick* (Allegheny College), *Michael Kuchinski*, *Suzanne Rex*, and *Clayton Tenney* (West Virginia university), *James Snyder* and *Douglas Bartley* (Butler County community College), *Brother Joe Robinson* (St. Francis Seminary), and *David Loth* (St. Francis College).

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a 'Level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be distinguished by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on a separate sheet properly identified with name and address and mailed before the end of June, 1978.

Address all communications concerning this department to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

399. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

Show that $\arcsin\left(\frac{x-3}{3}\right) + 2\arccos\sqrt{x/6} = \pi/2$, $(3 \le x \le 6)$.

400. *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Evaluate $\sum_{k=1}^{m} [kn/m] + \{kn/m\}$,

where m and n are positive integers, $[x]$ is the greatest integer not exceeding x and $\{x\} = -[-x]$ is the smallest integer not less than x.

401. *Proposed by Zelda Katz, Beverly Hills, California.*

From a point 250 yards due north of Tom, a pig runs due east. Starting at the same time, Tom pursues the pig at a speed 4/3 that of the pig and changes his direction so as to run toward the pig at each instant. With each running at uniform speed, how far does the pig run before being caught?

This is problem 28 of The *Mathematical* Puzzles *of Sam* Loyd, Volume Two, Dover Publications, 1960. (Selected and Edited by Martin Gardner). Loyd's solution is based on the average of the distance traveled by the pig if both ran forward on a straight line and the distance traveled if both ran directly toward each other. How did Loyd arrive at what he calls this "simple rule for problems of this kind" and how can we justify it?

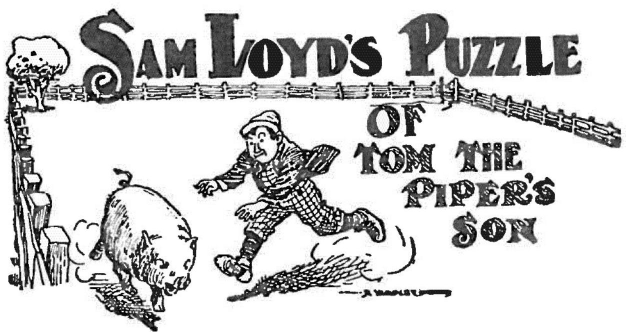


FIGURE 1

402. *Proposed by Charles W. Trigg, San Diego, California.*

The first eight non-zero digits are distributed on the vertices of a cube, Addition of the digits at the extremities of each edge forms twelve edge-sums. Find distributions such that every edge-sum is the same as the sum on the opposite (non-cofacial) edge. [The solution to the related problem 304 appears on pages 36-37 of the Fall 1974 *Pi Mu* Epsilon *Journal*.]

403. *Phopobed by David L. Silverman, West Los Angeles, California.*

Two players play a game of "Take It or Leave It" on the unit interval (0, 1). Each player privately generates a random number from the uniform distribution and either keeps it as his score or rejects it and generates a second number which becomes his score. Neither player knows, prior to his own play, what his opponent's score is or whether it is the result of an acceptance or a rejection. (However variants based on modifying this condition, either unilaterally or bilaterally are interesting).

The scores are compared and the player with the higher score wins $1 from the other,

a. What strategy will give a player the highest expected score?

b. What strategy will give a player the best chance of winning?

c. If one player knows that his opponent is playing so as to maximize his score, how much of an advantage will he have if he employs the best counter-strategy?

404. *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

Let x be a positive integer of the form $24n - 1$. Prove that if a and b are positive integers such that $x = ab$, then a + b is a multiple of 24.

405. *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

Locate a point P in the interior of a triangle such that the product of the three distances from P to the sides of the triangle is a minimum.

406. *Proposed by Paul Erdös, Spaceship Earth.*

Let there be given 5 distinct points in the plane. Suppose they determine only two distances. Is it true that they are the vertices of a regular pentagon?

407. *Proposed by Ben Gold, John H. Howell and Vance Stine, Los Angeles City College.*

Two sets of n dice are rolled. $(n = 1, 2, 3, 4, 5, 6)$. What is the probability of k matches? $(k = 0, 1, \cdots, n)$.

408. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Squares are erected on the sides of a triangle, either all externally or all internally. A circle is centered at the center of each square with each radius a fixed multiple k > 0 of the side of that square. Find k so that the radical center of the three circles falls on the Euler line of the triangle and find where on the Euler line it falls. (See Fig. 2.)

409. *Phopobed by Zazou Katz, Beverly Hills, California.*

A point E is chosen on side CD of a trapezoid ABCD, $(AD \| BC)$, and is joined to A and B. A line through D parallel to BE intersects AB in F. Show that FC is parallel to AE. (See Fig. 3.)

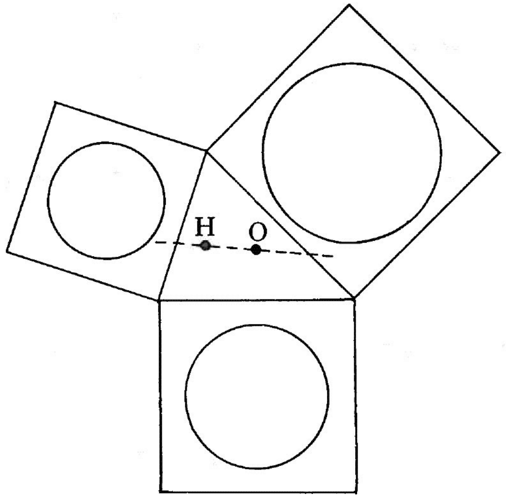


FIGURE 2

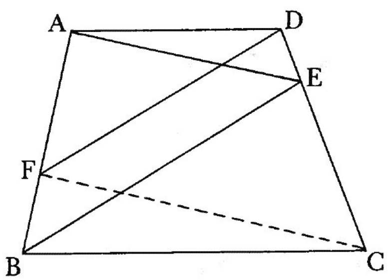


FIGURE 3

410. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

If x, y, z are the distances of an interior point of a triangle ABC to the sides BC, CA, AB, show that

$$1/x + 1/y + 1/z \geq 2/r$$

where r is the inradius of the triangle.

411. *Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

Find all polynomials $P(x)$ such that

$$P(x^2 + 1) - [P(x)]^2 - 2x[P(x)] = 0 \text{ and } P(0) = 1.$$

Solutions

374. [Fall 1976] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

In a triangle ABC inscribed in a circle (O), angle bisectors AT_1, BT_2, CT_3 are drawn and extended to the circle (see Fig. 1). Perpendiculars T_1H_1, T_2H_2, T_3H_3 are drawn to sides AC, BA, CB respectively. Prove that $T_1H_1 + T_2H_2 + T_3H_3$ does not exceed $3R$, where R is the radius of the circumcircle.

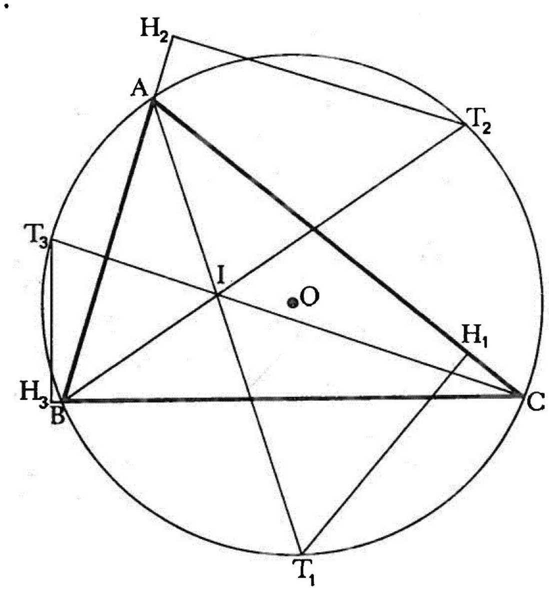


FIGURE 1

Amalgam of solutions by Clayton W. Dodge, University of Maine at Orono and Léo Sauvé, Algonquin College, Ottawa, Canada.

From $T_1H_1 = AT_1 \sin(A/2) \leq 2R \sin(A/2)$ and two similar results, we get

$$T_1H_1 + T_2H_2 + T_3H_3 \leq 2R[\sin(A/2) + \sin(B/2) + \sin C/2)],$$

and the required result follows from $\sin(A/2) + \sin(B/2) + \sin(C/2) \leq 3/2$, which can be found on page 20 (formula 2.9) of O. Bottema et al, Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.

Also solved by STEVE FROM, Creighton University, Omaha, Nebraska; DAVID C. KAY, University of Oklahoma; GERRIANNE VOGT, St. Louis University, St. Louis, Missouri; and the proposer.

375. *[Fall 1976] Proposed by Richard S. Field, Santa Monica, California.*

Approximate the value of $2^{10,000}$ without using pencil and paper (or chalk and blackboard or similar equipment).

Solutions resulting in the best approximations were submitted by Clayton W. Dodge, University of Maine at Orono; Mark Jaeger, Madison, Wisconsin; Theodore Jungreis, Brooklyn, New York; R. Robinson Rowe, Sacramento, California; Kenneth M. Wilke, Topeka, Kansas, and Richard Field, the proposer.

The most succinct version, submitted by Kenneth M. Wilke, goes as follows:

Since $\log_{10}2 \approx .30103$, $\log_{10}2^{10,000} \approx 3010.3$, so that $2^{10,000} \approx 2 \cdot 10^{3010}$.

Also solved by VICTOR G. FESER, Mary College, Bismarck, North Dakota; STEVEN FROM, Creighton University, Omaha, Nebraska; JOHN HOWELL, Littlerock, California; DAVID C. KAY, University of Oklahoma; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Canada; STEVEN B. LEELAND, Lauderdale Lakes, Florida; EDITH E. RISEN, Oregon City, Oregon; and GERRIANNE VOGT, St. Louis University, St. Louis, Missouri.

The proposer remarks that almost everybody tries to solve this problem by using the approximation $2^{10} \cong 1000$. This lead to $2^{10,000} \cong 10^{3000}$ which is off by more than 10 orders of magnitude!

Clayton Dodge marvels at the accuracy available to us by using the result $2 \cdot 10^{3010}$ considering that a 12-digit calculator yields the result

$$2^{10,000} = (2^{392})^{25} \cdot 2^{196} \cdot 2^{4} \cong 1.99506311505 \cdot 10^{3010}.$$

376. *[Fall 1976] Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, California.*

Let the sequence $\{a_n\}$ be defined inductively by $a_1 = 1$ and $a_{n+1} = \sin(\arc\tan a_n)$ for $n \geq 1$. Let the sequence $\{b_n\}$ be defined inductively by $b_1 = 1$ and $b_{n+1} = \cos(\arc\tan b_n)$ for $n \geq 1$. Give explicit expressions

for a_n and b_n, and find $\lim a_n$ and $\lim b_n$ as n approaches ∞.

I. Solution by David C. Kay, University of Oklahoma.

The values of the two sequences may be deduced from the right triangles in the two familiar spiral-like figures below:

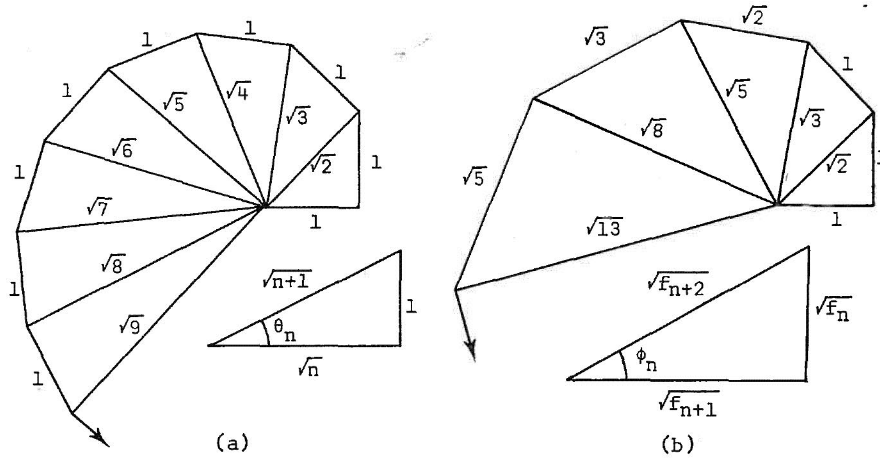


FIGURE 2

The inductive relations $a_1 = 1$ and $a_{n+1} = \sin(\arc\tan a_n)$ are implicit in a right triangle in (a) with acute angle θ_n and sides 1, $\sqrt{n}$, and $\sqrt{n+1}$: For, suppose it has been proved that $\tan\theta_n = a_n = 1/\sqrt{n}$; then $a_{n+1} = \sin\theta_n = 1/\sqrt{n+1}$ and $\tan\theta_{n+1} = a_{n+1} = 1/\sqrt{n+1}$, so induction carries. Similarly, $b_1 = 1$ and $b_{n+1} = \cos(\arc\tan b_n)$ are implicit in a right triangle with acute angle ϕ_n and whose sides are the square roots of 3 consecutive members of the Fibonacci sequence: For, suppose $\tan\phi_n = b_n = \sqrt{f_n}/\sqrt{f_{n+1}}$; then $b_{n+1} = \cos\phi_n = \sqrt{f_{n+1}}/\sqrt{f_{n+2}}$ and $\tan\phi_{n+1} = b_{n+1} = \sqrt{f_{n+1}}/\sqrt{f_{n+2}}$. Hence we have

$$\lim a_n = \lim \sqrt{n} = 0$$

and

$$\lim b_n = \lim \frac{\sqrt{f_n}}{\sqrt{f_{n+1}}} = \frac{1}{\sqrt{\mu}} = \sqrt{\mu - 1}$$

where $\mu = \dfrac{1+\sqrt{5}}{2}$ is the Golden section. Geometrically this means that in the sequences of right triangles in figures (a) and (b), $\theta_n \to 0$ while $\phi_n \to \arc\tan\sqrt{\mu - 1} = 38.17271\cdots^\circ$.

II. Solution by R. Robinson Rowe, Sacramento, California.

If ϕ is the angle in $a_{n+1} = \sin(\arctan a_n)$, then $a_n = \tan\phi$ and $a_{n+1} = \sin\phi$, whence

$$a_{n+1} = a_n/\sqrt{1 + a_n^2} \tag{1}$$

Using (1) recursively from $a_1 = 1$, we have generally,

$$a_n = n^{-\frac{1}{2}} \tag{2}$$

and the limit at infinity is

$$\lim a_n = 0 \tag{3}$$

Similarly, with $b_n = \tan\phi$ and $b_{n+1} = \cos\phi$,

$$b_{n+1} = 1/\sqrt{1 + b_n^2} \tag{4}$$

and using (4) recursively from $b_1 = 1$

$$b_n = \sqrt{F_n/F_{n+1}} \tag{5}$$

where F's are the Fibonacci numbers,

$$1, 1, 2, 3, 5, 8.....$$

For the limit, let $b_{n+1} = b_n$ in (4),

$$\text{and } b^4 + b^2 - 1 = 0 \tag{6}$$

from which

$$b = \sqrt{\frac{\sqrt{5} - 1}{2}} = \sqrt{G} = 0{,}786\ 151\ 378 \tag{7}$$

where G is the Golden Ratio and at this limit

$$\cos\phi = \tan\phi \text{ and } \phi = 38.172\ 70762^\circ$$

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* STEVE FROM, *Creighton University, Omaha, Nebraska;* JOHN HOWELL, *Littlerock, California;* MARK JAEGER, *Madison, Wisconsin;* THEODORE JUNGREIS, *Brooklyn, NY Yohk;* MURRAY S. KLAMKIN, *University of Alberta, Edmonton, Canada;* LEO SAUVÉ, *Algonquin College, Ottawa, Canada;* GERRIANNE VOGT, *St. Louis University, St. Louis, Missouri;* and *the proposer,* SOLOMON W. GOLOMB.

377. *[Fall 1976] Proposed by Charles W. Trigg, San Diego, California.*

From the following square array of the first 25 positive integers, choose five, no two from the same row or column, so that the maximum of

the five elements is as small as possible. Justify your choice.

2	13	16	11	23
15	1	9	7	10
14	12	21	24	8
3	25	22	18	4
20	19	6	5	17

I. Solution by Sidney Penner, Bronx Community College of CUNY, Bronx, New Yohk.

The five we choose are 1, 3, 6, 8, 11. Crossing out the numbers 12, 13, ..., 25, we see that all the numbers in the third row are crossed out except 8. Choosing 8 forces us to cross out the other entries in the 5th column, Whence the only entry left in the 4th row is 3. Choosing 3 forces us to cross out the other entry in the 1st column. Whence the only entry left in the 1st row is 11.

II. Computer solution by Jeanette Bickley, Webster Groves Senior High School, Webster Groves, Missouri. (Submitted at the request of the problem editor).

I used a Digital Equipment Corporation PDP 11/70 computer, with the program written in BASIC. This program searches the given matrix for five elements (no two from the same row or column) and prints both the maximum and the five chosen elements. It examines each possible remaining choice of five elements and prints only those choices that give a lower maximum element. The last row of the output shows the best choice---11, 1, 8, 3, 6. (See following page for display of program.)

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR FESER, *Mary College, Bismarch, North Dakota;* ROBERT C. GEBHARDT, *Hopatcong, NY Jersey;* JOHN HOWELL, *Littlerock, California;* THEODORE JUNGREIS, *Brooklyn, NY York;* DAVID C. KAY, *University of Oklahoma, Norman, Oklahoma;* HARRY NELSON, *Livermore, California;* BOB PRIELIPP, *University of Wisconsin-Oshkosh;* EDITH E. RISEN, *Oregon City, Okegon;* GERRIANNE VOGT, *St. Louis University, St. Louis, Missouri;* DALE WATTS, *Denver, Colorado;* and *the proposer,* CHARLES W. TRIGG.

378. *[Fall 1976] Proposed by M. L. Glasser and M. S. Klamkin, University of Waterloo, Waterloo, Ontario, Canada.*

Show that

$$\left\{\frac{x^x}{(1 + x)^{1+x}}\right\}^x > (1 - x) + \left\{\frac{x}{1 + x}\right\}^{1+x} > \frac{1}{(1 + x)^{1+x}}$$

for $1 > x > 0$.

COMPUTER PROGRAM FOR PROBLEM 377

```
5   T=0
10  MAT READ M(5,5)
20  DATA 2, 13, 16, 11, 23
30  DATA 15, 1, 9, 7, 10
40  DATA 14, 12, 21, 24, 8
50  DATA 3, 25, 22, 18, 4
60  DATA 20, 19, 6, 5, 17
70  FOR H=1 TO 5
80  A(1)=M(1,H)
90  FOR I=1 TO 5
100  IF I=H THEN 260
110  A(2) = M(2,I)
120  FOR J=1 TO 5
130  IF J=H OR J=I THEN 250
140  A(3)=M(3,J)
150  FOR K=1 TO 5
160  IF K=H OR K=I OR K=J THEN 240
170  A(4) = M(4,K)
180  FOR L = 1 TO 5
190  IF L=H OR L=I OR L=J OR L=K
       THEN 230
200  A(5) = M(5,L)
210  IF T=0 GO TO 500
220  GO SUB 600
230  NEXT L
240  NEXT K
250  NEXT J
260  NEXT I
270  NEXT H
280  PRINT "THE REQUIRED ELEMENTS
       ARE IN THE LINE ABOVE."
290  END
500  B=A(1)
510  FOR C=2 to 5
520  IF B>A(C) THEN 540
530  B=A(C)
540  NEXT C
560  T=1
570  GO TO 230
600  D=A(1)
610  FOR E=2 TO 5
620  IF B>A(E) THEN 640
630  D=A(E)
640  NEXT E
650  IF D>=B GO TO 680
660  PRINT D,M(1,H);M(2,I);M(3,J);
       M(4,K);M(5,L)
670  B=D
680  RETURN

READY

RUN
 18     2   1   8   1   8   6
 12     2   9   1   2   4   5
 11     1   1   1   8   3   6
THE REQUIRED ELEMENTS ARE IN THE
LINE ABOVE

READY
```

Solution by Gerrianne Vogt, St. Louis University, St. Louis, Missouri.

When $x = .5$ is substituted in the stated inequalities we obtain: Left term = .6204032394; Middle term = .6924500897; Right term = .544331054. In fact, within the entire range of x given, the second term is the largest of the three. Therefore the first inequality does not hold.

Solutions were offered also by ZELDA KATZ, *Beverly Hills, California;* C. B. A. PECK, *State College, Pennsylvania; and the proposers,* M. L. GLASSER *and* MURRAY S. KLAMKIN. *Zelda Katz substituted .25 and .75 to obtain the sequences .855237768, .883748061, .756593287 and .386641537, .477007720, .375563532 respectively. The proposers rewrote the second inequality as*

$$G(x) = (1 - x)F(1 + x) + xF(x) > 1$$

where $F(x) = x^x$. Since $F(x)$ is convex for $x \geq 0$,

$$G(x) > F(1 - x^2 + x^2) = 1.$$

379. [Fall 1976] *Proposed by David L. Silverman, West Los Angeles, California.*

You play in a non-symmetric two-man subtractive game in which the players alternately remove counters from a single pile, the winner being the player who removes the last counter(s). At a stage when the pile contains k counters, if it is your-opponent's move, he may remove 1, 2, $\cdots$, up to $[\sqrt{k}]$ counters, where $[x]$ is the largest integer $\leq x$. If it is your move, you may remove 1, 2, $\cdots$, up to $\phi(k)$ counters, where ϕ is the Euler totient function. If you play first on a pile of 1776 counters, can you assure yourself of a win against best play by your opponent?

Solution by the proposer.

Yes, and the strategy is simple. Remove, at each stage, enough counters so as to leave your opponent any number of counters other than 1, 3 or 4. It is readily verified that these are unsafe leaves by you and that 2 is a safe leave by your opponent. By induction it can then be shown that every leave by you > 4 is safe for you, while every leave > 2 by your opponent is unsafe for him. The first follows directly from the fact that if n is a safe leave for you, then $n + 1$ is an unsafe leave for him.

The end-play alone is crucial and the game is heavily stacked in favor of the ϕ-player, namely "you".

Comment, by the proposer.

Perhaps some reader can design a non-symmetric two-man subtractive game that is not so trivial.

Also solved by HARRY NELSON, *Livermore, California and by* R. ROBINSON ROWE, *Sacramento, California.*

380. [Fall 1976] *Proposed by V. F. Ivanoff, San Carlos, California.*

Form a square from a quadrangle $(ABCD)$ by bisecting segments and the angles.

Combination of solutions by Clayton W. Dodge, University of Maine at Orono, and the proposer, V. F. Ivanoff, San Carlos, California.

Let the midpoints of the sides of quadrangle $ABCD$ be E, F, G, H as shown in Figure 3. Then $EFGH$ is a parallelogram since, for example, EH and FG are each parallel to and half the length of BD (when E is the midpoint of AB, F is the midpoint of BC, etc.). Let the internal bisectors of angles E, F, G, H meet one another at J, K, L, M. (If we draw the external angle bisectors, we get another rectangle, not shown on the figure.) If the internal bisectors of angles E and H meet at J, then

$$\angle JHE + \angle JEH = (\angle GHE + \angle FEH)/2 = 180^\circ/2 = 90^\circ,$$

from which it follows that $\angle J = 90^\circ$. Thus $JKLM$ is a rectangle. Finally, let the internal bisectors of angles J, K, L, M meet one another at P, Q, R, S. By symmetry, $PQRS$ is a square.

Another square is formed by the intersections of the external angle bisectors of angles J, K, L, M.

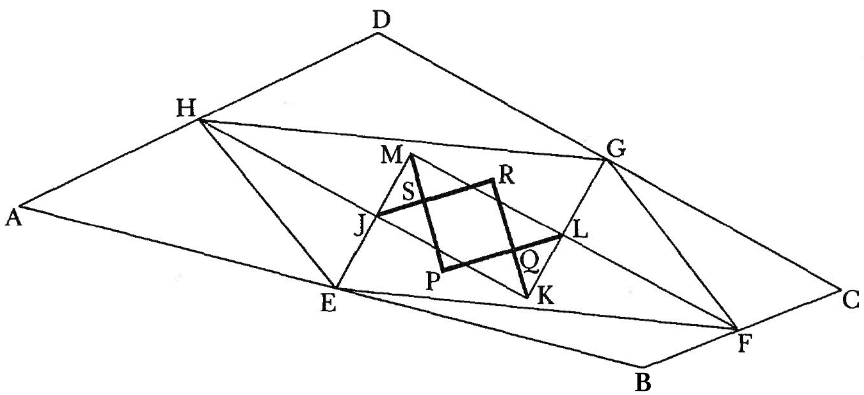


FIGURE 3

381 [Fall 1976] *Proposed by Clayton W. Dodge, University of Maine, Orono, Maine.*

Solve the following wintery, slippery alphametrics (also known as cryparithms and alphametics):

$$(ICE)^3 = ICYWHEEE.$$
$$(ICE)^3 = ICYOHOH.$$

Solution by Charles W. Trigg, San Diego, California.

Since the first cube has eight digits, $215 < IC < 465$. The common beginning, IC, of the number and its cube reduces the span for consideration to 315-320. The unique reconstruction is $(320)^3 = 32768000$.

Since the second cube has seven digits, $99 < ICE < 216$. The common beginning, IC, of the number and its cube restricts the span to 100-103. The unique reconstruction is $(103)^3 = 1092727$.

Solutions were offered also by JEANETTE BICKLEY, *St. Louis, Missouri;* JOHN FERRO, *South Ozone Park, New York and the* NEW YORK PHO PROBLEM SOLVING TEAM, *St. John's University, New York;* VICTOR C. FESER, *Bismarck, North Dakota;* ROBERT C. GEBHARDT, *Hopatcong, New Jersey;* JOHN HOWELL, *Littlerock, California;* THEODORE JUNGREIS, *Brooklyn, New York;* DAVID C. KAY, *University of Oklahoma, Norman, Oklahoma;* JAMES METZ, *Springfield, Illinois;* HARRY NELSON, *Livermore, California;* R. ROBINSON ROWE, *Sacramento, California;* KENNETH M. WILKE, *Topeka, Kansas;* GERRIANNE VOGT, *St. Louis, Missouri; and the proposer,* CLAYTON W. DODGE.

Some of these solutions were derived by the limitation of the number of trials while others were obtained by direct table-searching. In questions of this sort, answers are of hardly any interest; our quest is for the most satisfying logical analysis. Unquestionably, solutions supersede answers.

382. [Fall 1976] *Proposed by R. Robinson Rowe, Sacramento, California.*

Two cows, Lulu and Mumu, are tethered at opposite ends of a 120-foot rope threaded thru a knothole in a post of a straight fence separating 2 uniform pastures. How much area can they graze, presuming they eat, nap and ruminate on identical schedules and the rope length is also the extreme reach from muzzle to muzzle of Lulu and Mumu? As a sequel, if Mumu is replaced by the heifer Nunu with half the appetite, what is the area accessible to Lulu and Nunu?

Solution by the proposer.

In the first diagram, K is the knothole in fence FG. When Lulu grazes an arc AB of infinitesimal width and radius u, Mumu grazes arc CD of the same length and radius $120 - u$ (Fig. 4-a), and hence of central angle in reciprocal. For u less than 60, Mumu's grazing limit is defined by $2r\phi = (120 - r)\pi$ and for u greater than 60 Lulu is limited by a similar curve. These curves are the arcs of 4 spirals EF, EG, FH and GH. In the first quadrant we have $r = 120\pi/(2\phi + \pi)$ in polar coordinates and since in general the area is $\frac{1}{2}\int r^2 d\phi$, the area $EFGH$ grazed is $A = 4 \cdot \frac{1}{2} \cdot 14400\pi^2\int_0^{\pi/2} d\phi/(2\phi + \pi)^2 = 7200\pi$.

In the second diagram (Fig. 4-b), Nunu's arc CD is only half as long as Lulu's AB, and the relation continues until Lulu has 80 feet of tether and Nunu only 40 feet. Their spiral bounds differ, equations are shown on the diagram, and areas derived as before are 4800π for Lulu and

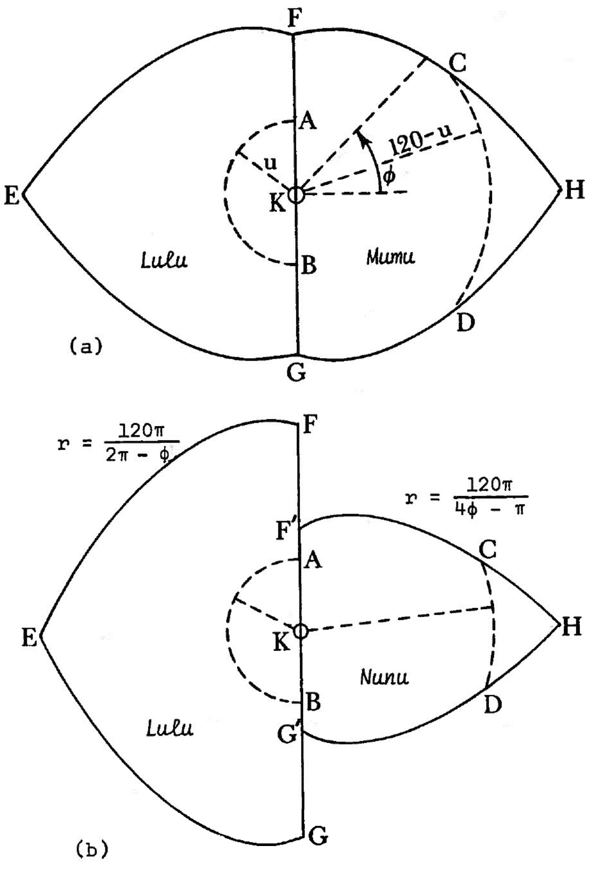


FIGURE 4

2400π for **Nunu,** adding again to 7200π.

Comment: With less elegance, one can note that when **Lulu** has less than half the rope, she can graze a semicircle with an area of 1800π and **Mumu** has a wider range to graze an equal area. Vice versa, **Mumu** with less than half the rope grazes 1800π while Lulu has a wide range to match it. All this adds to 7200π. **Similarly** for Lulu and **Nunu,** dividing the rope 2/3 to 1/3. However, these short cuts do not develop the curves I call 'bovine spirals'.

383. [Fall **1976]** ***Proposed by Norman Schaumberger, Bronx Community College, New York.***

Find a pentagon such that the sum of the squares of its sides is equal to four times its area.

I. Solution by Clayton W. Dodge,, University of Maine, Orono, Maine.

There are many families of solutions to this problem, and we shall consider just three of them. We let Σs^2 and K denote the sum of the squares of the sides and the area of the pentagon respectively. We first consider a square of side $a + b$ with an isosceles right triangle of leg b cut off from one corner, as shown in Figure 5-a. We have

$$\Sigma s^2 = 2(a + b)^2 + 2a^2 + (\sqrt{2}b)^2 = 4a^2 + 4ab + 4b^2$$

and

$$4K = 4((a + b)^2 - b^2/2) = 4a^2 + 8ab + 2b^2.$$

Now $\Sigma s^2 = 4K$ reduces to $2b^2 = 4ab$, so $b = 0$ or $b = 2a$. That is, a square satisfies the equation $\Sigma s^2 = 4K$; the figure for $b = 2a$ is shown in Figure 5-a.

Figure 5-b shows an equilateral triangle of side $a + b + c$ with an equilateral triangle of side a cut from one corner and one of side c cut from another. Since the area of an equilateral triangle of side a is $s^2\sqrt{3}/4$, we have

$$\Sigma s^2 = a^2 + b^2 + c^2 + (a + b)^2 + (b + c)^2$$
$$= 2a^2 + 3b^2 + 2c^2 + 2ab + 2bc$$

and

$$4K = \sqrt{3}((a + b + c)^2 - a^2 - c^2) = \sqrt{3}(b^2 + 2ab + 2bc + 2ca)$$

The equation $\Sigma s^2 = 4K$ can now be written in the form

$$(3 - \sqrt{3})b^2 + (2 - 2\sqrt{3})(a + c)b + 2(a^2 + c^2 - \sqrt{3}ac),$$

which we solve for b to obtain

$$b = \frac{2\sqrt{3}(a + c) \pm \sqrt{2}(3 + \sqrt{3})[(1 + \sqrt{3})ac - a^2 - c^2]^{1/2}}{6}$$

This solution does not seem especially exciting, but we set $a = c$ to obtain the solution for that symmetric figure, obtaining

$$\frac{b}{a} = \frac{4\sqrt{3} \pm \sqrt{2}(3 + \sqrt{3})(\sqrt{3} - 1)^{1/2}}{6} = 1.1547 \pm .9543$$
$$= 2.1090 \text{ or } .2004.$$

The third case we consider is a rectangle of base a and altitude b surmounted by an isosceles triangle of altitude c, as shown in Figure 5-c.

By the Pythagorean theorem we have

$$d^2 = (a/2)^2 + c^2 = a^2/4 + c^2,$$

so that

$$\Sigma s^2 = a^2 + 2b^2 + 2(a^2/4 + c^2) = \tfrac{3}{2}a^2 + 2b^2 + 2c^2$$

and

$$4K = 4(ab + ac/2) = 4ab + 2ac.$$

Then $\Sigma s^2 = 4K$ can be written in the form

$$4c^2 - 4ac + (3a^2 - 8ab + 4b^2) = 0.$$

Assuming a and b are given, we solve this quadratic equation for c:

$$c = \frac{a \pm \sqrt{2a^2 - 4(a-b)^2}}{2}$$

It seems of interest to note that, when $a = 2b$, we get $c = 0$ or $a = 2b$. The first case is degenerate while the second one gives the isosceles triangle sides of length $b\sqrt{5}$.

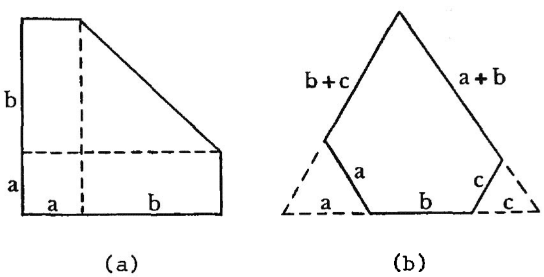


FIGURE 5

II. Solution by R. Robinson Rome, Sacramento, California.

We are to find a pentagon such that S, the sum of the squares of its 5 sides, equals $4A$, where A is its area. It cannot be a regular pentagon, for which $S \approx 2.9A$. Lacking other specifications, it need not be convex, nor inscript, nor integral, so there are infinitely many solutions. Here are a few of each kind, illustrated in Figure 6, a–f.

(a) A right triangle a, b, c atop a $c \times d$ rectangle satisfies with $d = c \pm \sqrt{ab}$. However ab is never a square. With $a = 3$, $b = 4$, $c = 5$, $d = 1.5359$ or 8.4641 and A = 13.68 or 48.32.

(b) Add to (a) a right triangle d, e, f on one side, and $d^2 - d(2c + e) - ab + c^2 + ce + e^2 = 0$. This is satisfied with $a = 3$, $b = 4$, $c = 5$, $e = 2$,

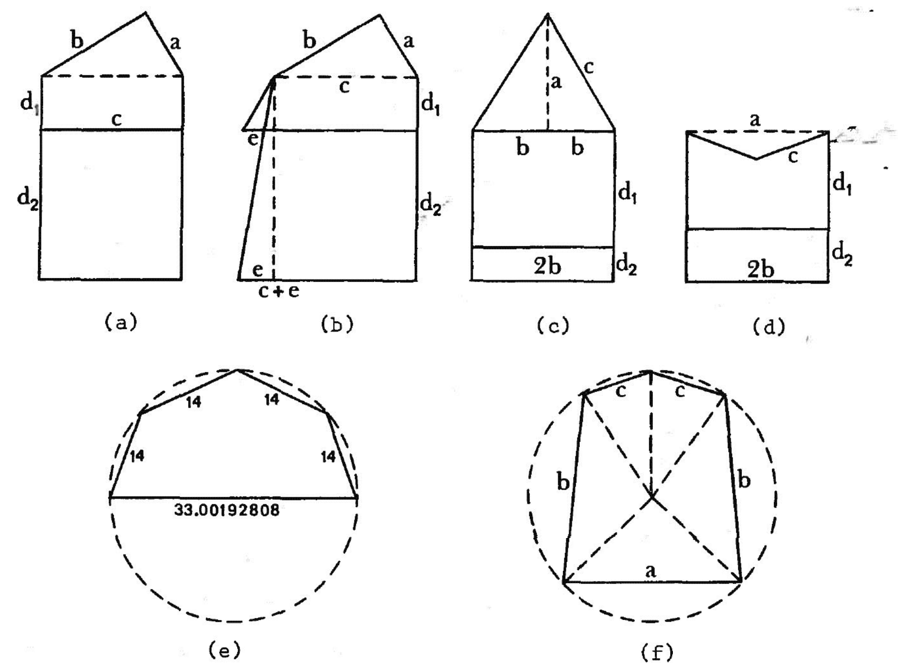


FIGURE 6

and $d = 3$ or 9. Then $A = 24$ or 60. All are integers except f.

(c) Two right triangles back-to-back atop a rectangle finds $d = 2b \pm \sqrt{b^2 + 2ab - a^2}$. This is satisfied with $a = 12$, $b = 5$, $c = 13$ and $d = 9$ or 11. A = 150 or 170. Hence these two solution are entirely in integers, which may have been the author's intent.

(d) Like (c) with the triangles re-entrant, finding $d = 2b \pm \sqrt{b^2 - 2ab - a^2}$. The radicand is negative unless $b > a(1 + \sqrt{2})$. Using $a = 7$, $b = 24$, $c = 25$, $d = 48 \pm \sqrt{191}$, $A = 2136 \pm 48\sqrt{191}$. Using $a = 2$, $b = 5$, $c = \sqrt{29}$, $d = 9$ or 11 and $A = 80$ or 100, which is nearly integral -- all except c.

(e) Investigating an inscript solution, I started with the symmetrical figure with chords a, b, b, c, c, with corresponding central angles $2A$, $2B$ and $2C$, leading to $\sin^2 A + 2\sin^2 B + 2\sin^2 C = \tfrac{1}{2}\sin 2A + \sin 2B + \sin 2C$. If A is chosen arbitrarily between limits, this can be solved for B and C. Choosing $A = 45^\circ$, $2\sin^2 B + 2\sin^2 C = \sin 2B + \sin 2C = \sqrt{2}\sin B$, $C = \tfrac{1}{2}\sqrt{\sqrt{2} \pm \sqrt{\sqrt{2}}}$; that is, B and C are interchangeable, using one sign of $\pm$ for B and the other for C. For $a = 1$, $b = 1.14092521$, $c = 0.335415004$, $S = 1 + 2\sqrt{2}$. The lower limit for A is 0° when the pentagon becomes a square with a null side $a = 0$. At the upper limit, $B = C$, when $v = \sin 2\delta = 0.760776413$, being the only real root of $8v^5 + 8v^4 - 8v^3 + 5v - 5 = 0$ and $A = 80.9346148^\circ$, $B = C = 24.7663463$.

(f) With 4 equal sides, which can be made integers, this is an interesting solution. It happens that if we make $b = c = 14$, then $a = 33.00192808$ and nearly an integer. $A = 468.2818143$. The circumradius is 16.70968173.

Comment: I was surprised by the variety of shapes and enjoyed the extended recreation. The all-integer solution (c) is probably what was wanted, but I like (f) the most.

In addition to the foregoing solution, an amazing variety of configurations were contributed by JEANETTE BICKLEY, Webster Grove High School, Missouri; STEVE FROM, Council Bluffs, Iowa; JOHN M. HOWELL, Littlerock, California; MARK JAEGER, Madison, Wisconsin; THEODORE JUNGREIS, Brooklyn, New York; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta, Canada; C. B. A. PECK, State College, Pennsylvania; CHARLES W. TRIGG, San Diego, California; GERRIANNE VOGT, St. Louis University, St. Louis, Missouri; and the proposer, NORMAN SCHAUMBERGER.

384. *[Fall 1976] Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

Discuss the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{p_n^2}$$

where p_n means the nth prime.

Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.

It is known that $p_n > \frac{n \ln n}{4}$ for $n > 1$. [For a derivation of this result, see pp. 148-150 of Sierpinski, Elementary Theory of Numbers, Hafner Publishing Company, New York, 1964.] Thus

$$\frac{n}{p_n^2} < \frac{16}{n(\ln n)^2}$$

for $n > 1$. But it is well-known that

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^\alpha}$$

is convergent if and only if $\alpha > 1$. Hence

$$\sum_{n=1}^{\infty} \frac{n}{p_n^2}$$

converges by the Comparison Test.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; MARK JAEGER, Madison, Wisconsin; C. B. A. PECK, State College, Pennsylvania; EDITH E. RISEN, Oregon City, Oregon; R. ROBINSON ROWE, Sacramento, California; GERRIANNE VOGT, St. Louis University, St. Louis, Missouri; and the proposer, R. S. LUTHAR, University of Wisconsin-Janesville.

385. *[Fall 1976] Proposed by John T. Hurt, Bryan, Texas.*

Solve: $\sin\alpha = \tan(\alpha - \beta) + \cos\alpha \tan\beta$.

Solution by the proposer.

Rewrite as:

$$\sin\alpha - \cos\alpha \frac{\sin\beta}{\cos\beta} = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

from which

$$\frac{\sin(\alpha-\beta)}{\cos\beta} = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}.$$

From the numerator, $(\alpha - \beta) = n\pi$; from the denominator, $\pm\beta = \alpha - \beta + 2n\pi$, and from these we have all the solutions:

$$\alpha \equiv 0 \bmod 2\pi$$
$$\alpha \equiv \beta \bmod \pi$$
$$\alpha \equiv 2\beta \bmod 2\pi.$$

Also solved by JACKIE E. FRITTS, Texas A & M University; STEVE FROM, Council Bluffs, Iowa; CLAYTON DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest Hills High School, Flushing, New York; ROBERT C. GEBHARDT, Hopatcong, New Jersey; C. B. A. PECK, State College, Pennsylvania; R. ROBINSON ROWE, Sacramento, California; GERRIANNE VOGT, St. Louis University, St. Louis, Missouri; and ZELDA and ZAZOU KATZ (jointly), Beverly Hills, California.

386. *[Spring 1976] Proposed by Richard Field, Santa Monica, California.*

Let $Q = [10^n/p]$, where p is a prime > 5, and n is the cycle length of the repeating decimal $1/p$; $[x]$ denotes the greatest integer in x. Can Q be a prime?

I. Solution by Kenneth S. Williams, Carleton University, Ottawa, Canada.

Since $p \neq 2, 5$, p^{-1} has a periodic decimal expansion of the form

$$p^{-1} = 0.\dot{a}_1 a_2 \cdots \dot{a}_n.$$

Hence, with $Q = a_1 a_2 \cdots a_n$ (in decimal notation), we have

$$p^{-1} = Q\left(\frac{1}{10^n} + \frac{1}{10^{2n}} + \cdots\right) = \frac{Q}{10^n - 1},$$

that is,

$$10^n - 1 = Qp.$$

Now $3^2 = 9 = 10 - 1 \mid 10^n - 1$, so $3^2 \mid Qp$. But $p \neq 3$; hence $3^2 \mid Q$. That is, $9 \mid Q$, and so Q is never a prime.

II. Comment by Léo Sauvé, Algonquin College, Ottawa, Canada.

If the cycle length n of $1/p$ is even--that is, if in decimal notation

$$Q = a_1 a_2 \cdots a_r a_{r+1} \cdots a_{2r}$$

(this occurs, in particular, if 10 is a primitive root of p, when $n = p - 1$)--then we have the stronger result

$$a_1 + a_{r+1} = a_2 + a_{r+2} = \cdots = a_r + a_{2r} = 9.$$

For a discussion and proof of this statement, see, for example, Higher Algebra, by S. Barnard and J. M. Child, Macmillan, London, 1969, pp. 439-443.

363. [Spring, 1976] *Proposed by Robert C. Gebhart, Hopatcong, New Jersey.*

Does $\frac{\sin 1}{1} + \frac{\sin 2}{2} + \frac{\sin 3}{3} + \cdots$ converge, and if so, to what?

Solution by Léo Sauvé, Algonquin College, Ottawa, Canada.

A routine expansion of the function $\frac{1}{2}(\pi - x)$ in a Fourier series gives

$$\tfrac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad 0 < x < 2\pi, \tag{1}$$

and setting $x = 1$ gives the required sum

$$\frac{\sin 1}{1} + \frac{\sin 2}{2} + \frac{\sin 3}{3} + \cdots = \tfrac{1}{2}(\pi - 1).$$

It may be of interest to find at the same time the sum of the corresponding cosine series by setting $x = 1$ in the following Fourier expansion:

$$-\log(2 \sin \tfrac{1}{2}x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}; \tag{2}$$

this gives

$$\frac{\cos 1}{1} + \frac{\cos 2}{2} + \frac{\cos 3}{3} + \cdots = -\log(2 \sin \tfrac{1}{2}).$$

All that this shows is that if one knows the answer it is very easy to justify it. It is not so easy to find the left side of (1) and (2) when only the right side is given. This can be done in several ways, as can be seen in An Introduction to the Operations with Series, by I. J. Schwatt, Second Edition, Chelsea, New York, 1961, pp. 211-214.

A NEW PUBLICATION DEVOTED TO UNDERGRADUATE MATHEMATICS

An informal bimonthly publication printed in the form of a newsletter has recently come to the attention of this Journal, and we recommend it highly to our readers. It is the Eureka, sponsored by the Carleton-Ottowa Mathematics Association (a Chapter of the Ontario Association for Mathematics Education). The editor is Professor Léo Sauvé, Agonquin College, Ottawa. Send inquiries regarding subscriptions to:

F. G. B. Maskell
Algonquin College
200 Lees Avenue
Ottowa, Ontario K1S 0C5

LOCAL CHAPTER AWARDS WINNERS

ARKANSAS BETA (Hendrix College). The *McHenry-Lane Freshman Mathematics Award* was presented to

Mary Sudman,

the *Hogan Senior Mathematics Award* went to

Donald Hayman,
Wit Orton,

and special recognition was given to

Michael Tiefenback

for ranking highest in the Oklahoma-Arkansas Section on the 1976 Putnam Examination.

GEORGIA BETA (Georgia Institute of Technology). Book awards were presented to the following outstanding graduates in mathematics (grade point of at least 3.8, with 4.0 perfect, in all mathematics courses taken):

Roger L. Gray,
Margaret Anne Reimer.

INDIANA DELTA (Indiana State University). An award of $100 was presented to

Ross McKenna,

a junior, who had the highest index (GPA of 4.0) in mathematics.

IOWA ALPHA (Iowa State University). *Pi Mu Epsilon Scholarship Awards* of $50 each were presented to

Robert Gebhardt,
John Spohnheimer,

who were the top two scorers of a competitive examination conducted by members of the mathematics department. For outstanding achievement in mathematics, an additional $50 award was given to

Kent Probst.

The Dia Lewis Hall Award of *$50* for an outstanding graduating senior mathematics major went to

Bill Rockenbach.

The *Gertrude Herr Adamson Awards* of $50 each for demonstrated ingenuity in mathematics were presented to

Galen Aswegan,	*F. George Janvrin,*
John Briggs,	*Patrick Ryan,*
David Challener,	*Van Scott,*
Marjorie faddy,	*Susan Scott.*

MARYLAND ALPHA (University of Maryland). The *Milton Abramowitz Memorial Prize* for a senior mathematics major who has demonstrated superior competence and promise for the future in the field of mathematics, was won by

Chewier Collins.

The *Higginbotham Award* for the outstanding junior student majoring in mathematics was presented to

Glenn Joseph Galfond.

NEW JESSES BETA (Douglass College). Two juniors qualified for gift certificates for mathematics books because of excellence and achievement in mathematics. They were

Kathy Chapman,
Lesley Chapman.

NEW JERSEY EPSILON (Saint Peter's College). The *Janes B. Collin Award* was won by

Thomas Murphy.

NEW YORK EPSILON (St. Lawrence University). The *Dr. O. Kenneth Bates Mathematics Award* has been instituted in honor of Professor Bates who served as mathematics chairman at St. Lawrence 1933-1967 and was a charter member of the Epsilon chapter. The award is presented annually to a senior for superior performance in mathematics courses, departmental activities and general interest in and enthusiasm for the discipline. The first annual award was presented to

Constance R. Nelson.

OHIO EPSILON (Kent State University). Cash certificates of $25 each for books went to

Patricia Weinmann,
Howard Fraser

who were tie winners of the 1977 *Pi Mu Epsilon Mathematics Award.*

OHIO THETA (Xavier University). The *Kramer-Miller Mathematics Award* for outstanding seniors was presented to

Thomas Fagedes,
Richard W. Hack.

The *Richard J. Wehrmeyer Memorial Pi Mu Epsilon Award* for excellence in problem solving went to

Barry T. Neyer,

and the *Comer Memorial Fund Award* for the outstanding student in statistics was won by

James R. Schott.

The Robert F. Cissell Memorial Award for exceptional undergraduates went to

Robert F. Niemoeller,
Danial J. Roessner,
Lisa M. Schoettinger,
Mary Jo Stentz.

OKLAHOMA BETA (Oklahoma State University). The *O. H. Hamilton Award* of $100, in honor of the late Professor Olan H. Hamilton and presented to the outstanding graduate student in mathematics, went to

Max Hibbs.

The *Mathematical Sciences Alumni Award* for outstanding sophmores was presented to

Suzanne. McCoy,
Celeste White,

and an *Alumni Certificate of Merit* was given to

Queta Ann Barnes.

The winners of the *W. R. Pogue Award* for outstanding juniors were

Janette Moyer,
Karen Sonder,

and a *Pogue Certificate of Merit* was awarded to

Lisa Love..

The *Mathematical Sciences Faculty Achievement Award* for outstanding seniors went to

Marcia Currie,
Christy Gelmers, Deborah Huffman Willis,
Kathy Sullivan, Emily Wonderly,

with the two outstanding seniorswho received this award recognized as

Robert Hayes,
Mary Stone..

RHODE ISLAND BETA (Rhode Island College). The *Mitchel Award in Mathematics* was presented to

Jeanne Duguay.

SOUTH CAROLINA GAMMA (College of Charleston). The *Harrison Randolph Calculus Award* was won by

Eric Seth Webb,

and the *Outstanding Mathematics Major Awards* were presented to

Clarence Michael Phillips,
Lonita Spivey.

TEXAS BETA (Lamar University). High school students who exhibited outstanding ability in mathematics competition and who were each awarded R. S. Burlington's *Handbook of Mathematical Tables and Formulas* were

Mark Rippetoe,
Chris Erickson,
Pout Skinner.

In the annual *Homer A. Dennis Freshman Contest,* a competition consisting of 6 problems ranging from algebra to calculus, the winners were

Joseph Bouchard, *First Place,*
John Matson, *Second Place.*

TEXAS DELTA (Stephen F. Austin State University). The *Outstanding Senior Mathematics Student* for 1976–77 was

Sherry Sweat.

PI MU EPSILON AWARD CERTIFICATES

Is your chapter making use of the excellent award certificates to help you recognize mathematical achievements? For further information write:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

SUMMER MEETING IN SEATTLE

Pi Mu Epsilon held its annual summer meeting in conjunction with the American Mathematical Association in Seattle, Washington August 12-16, 1977 on the University of Washington campus. On Monday, the Governing Council held its annual luncheon and business meeting at Husky Den. After approving the minutes the Council was advised of the continuing adequate financial status of the Fraternity by the Secretary-Treasurer, Richard A. Good, and of a substantial increase in *Journal* subscriptions by the Journal Editor, David C. Kay. Allan Davis, President, reported on new chapters of the Fraternity. Eileen Poianni suggested holding a pre-meeting with student speakers and delegates, and Milton Cox discussed student sessions at regional meetings and conferences. It was moved by Maurice Beeseley, seconded and passed that the Fraternity (1) have an actual banquet each year rather than cafeteria type meals and (2) subsidize any banquet costs beyond $4.00 per member. It was decided to give travel support to past presidents and staunch supporters of the Fraternity equal to present delegate support, that the $400 limit for travel support of delegates and speakers to the Seattle meeting be continued indefinitely, and that travel expenses include ground transportation to and from the airport, depot or terminal. It was moved by Richard Good, seconded and passed that as of July 1, 1979 the price of the gold Fraternity pins be raised to $8.00 and that this be announced well in advance of that date.

Monday evening the annual banquet was held at the Sherwood Inn, and at 8:00 the third J. Sutherland Frame Lecture was delivered by Professor Ivan Niven, University of Oregon, on the topic "Techniques of Solving Extremal Problems".

The Dutch Treat Breakfast was held 8:00 a.m. Tuesday morning at Husky Inn.

The following student papers were presented in Guggenheim Hall on Monday and Tuesday afternoons:

1. *Paradox in the Development of Mathematics*, Wayne Heym, Ohio Delta.

2. *The Value of Mathematics in Pre-legal Education*, Bruce Fox, Illinois Alpha.
3. *Commutative Rings with Fields*, Kathyrn Dowdell, Pennsylvania Xi.
4. *Continuous Convergence of Functions*, Robert Childs, North Carolina Delta.
5. *Continuous Convergence in* $C(X)$, James Lewis, North Carolina Delta.
6. *A Characterization of* G_δ *Sets in Metric Spaces*, Jeff Thompson, New York Theta.
7. *How to Model for Politicians*, Lonita B. Spivey, South Carolina Gamma.
8. *Mathematical Modeling of a Sewage Plant*, John Gimbel, Michigan Gamma.
9. *The Structure of the Solution Space of 5th Order Linear Differential Equations*, Ernest Lowery, Texas Gamma.
10. *Sink Like Structures in Compartmental Analysis*, James Bellinger, Illinois Delta.
11. *A Computer Application of Linguistics*, Alfredo Garcia, Missouri Gamma.
12. *Set Simplification Simplified*, Dale Watts, Colorado Beta.
13. *The Effect of Finite Infinities on Rational Numbers*, Victor Meyer, Michigan Beta.
14. *Mathematics Field Day*, Kenneth Pitz, Nebraska Beta.
15. *An Introduction to, and an Application of Elliptical Integrals*, Mark Goldsmith, Ohio Delta.

ERRATA FOR VOLUME 6, NUMBER 5

Page 268, line -7: Replace "Q' is $1/\bar{n}$" by "R is $1/\bar{n}$". Page 269, line 1: Replace "Q' is $1/\bar{n}$" by "R is $1/\bar{n}$".

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