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/mathematical curiosities
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The study of mathematics often leads into some unusual territory. The mathematician is at times confronted with strange geometric figures, odd equations, misbehaved curves, and seemingly insoluble paradoxes. Here an attempt will be made to examine some of the curiosities that interest and challenge mathematicians.

1. Topological Curiosities

A. Moebius Strips. Several interesting mathematical curiosities have come from the area of topology. One very familiar example is the Moebius strip, named for August Moebius, a German mathematician. Most of the surfaces met in everyday life are bilateral, or two-sided. A fly placed on one side of a sheet of paper, for example, could not reach the other side unless he cut through the paper or crawled over an edge. However, the Moebius strip is a unilateral closed surface; it has only one side. A fly could crawl from any point on this surface to any other point on the surface without cutting through the strip or going over an edge.

Some properties of this surface can be explained by comparing it to a cylindrical surface made by taking a long rectangular piece of paper and pasting the ends together. This surface has two sides and two edges. A Moebius strip can be made in a similar manner, except that one end of the paper is given a half twist before the ends are joined. The resulting surface has one side and one edge.

To become convinced of these properties, one has only to start at a point and draw a line down the center of the strip without removing one's pencil from the paper until returning to the starting point. It will be found that a single line has been drawn on what had been both aides of the paper before the ends were glued together. Similarly, one can run a pencil around the edge of the strip and find that both edges of the original strip have been colored.
If the original cylindrical strip is cut along its center, it falls apart into two new cylindrical strips each half as wide as the first. However, if the Möbius strip is cut along its center, a single strip twice as long and half as wide as the original is obtained which has a whole twist in it. If this strip is again cut along its middle, two strips are formed which are interlocked. If a half-twisted Möbius strip is cut along a line one-third of the way from its edge, two interlocked strips are formed, one of which has a one-half twist and the other, a whole twist.

These properties of the Möbius strip can be explained in terms of the theory of knots. A closed curve is knotted if it cannot be transformed into a single simple closed curve without cutting and retying. Figure 1-a shows a string that is actually unknotted, while Figure 1-b shows a string that is knotted in the simplest way possible. The above discussion of the Möbius strip can be put in terms of knots as follows. The two edges of the cylindrical strip are neither knotted nor interlocked.

This strip, cut down the middle, falls apart. The single edge of the one-half-twisted Möbius strip is not knotted. This strip, when cut down the middle, becomes one unknotted strip. The two edges of a strip with a whole twist are interlocked but not knotted. When this strip is cut down the center, it forms two interlocked strips. If one end of a strip is turned through one and one-half twists before pasting, the resulting strip has one side and one knotted edge. When this surface is cut down the center, a single strip of the half-twisted type is formed which is itself knotted.

Another related curiosity is the Klein bottle, named for the German mathematician Felix Klein. We are all familiar with the sphere, which is a simple closed surface dividing space into two parts, one inside the sphere and the other outside it. The Klein bottle is also a closed surface like the sphere, but it has no inside whatever. If we start at a point outside a sphere and cut through its surface to the inside, we would have to cut through the surface again to get back outside. If we start anywhere and cut through the surface of a Klein bottle, we can follow a path which returns to the place we started without ever cutting the surface again.

A model of a Klein bottle is difficult to make with paper and is usually blown in glass. It is made by taking one end of a hollow glass tube, bending it around, inserting it through a hole in its own side, and joining the two open ends together. The resulting surface is closed, being unbroken in the usual sense at any point. A diagram of the bottle and its cross-section is shown in Figure 2. The Klein bottle, which was not invented originally only for fun, arises naturally from the consideration of a one-sided surface which is closed and has no boundary, but it does exert a certain undefined attraction for the mathematician that other ordinary bottles just do not have.

2. Fourth Dimensional Curiosities

A. The Hypercube. The Möbius strip and Klein's bottle are two-dimensional surfaces existing in three dimensions; some other interesting curiosities come from the world of four dimensions. Imagine a line segment moving in a direction perpendicular to itself; it generates a square. If this square moves perpendicular to all of its sides, it
generates a cube. Now imagine the cube moving in a new direction perpendicular to all of its faces; it generates a hypercube, or tesseract. Just as a cube can be represented by a two-dimensional perspective drawing, a hypercube can be represented by a three-dimensional perspective sculpture. The hypercube has 16 corners, 32 edges generated by the 8 corners of the original cube, 24 faces from the 12 edges of the cube, and 8 cubes from the 6 faces of the original cube. Each corner is common to 4 mutually perpendicular edges, to 6 faces, and to 4 cubes; each edge is common to 3 faces and 3 cubes, and each face is common to 2 cubes. Every cube has one face in common with 6 of the 7 others. A hypercube can be unfolded into its component cubes just as a cube can be unfolded into its component squares, as shown in Figure 3. Actually the hypercube is one of a family of regular four-dimensional polyhedrons. While in three dimensions there are five regular polyhedrons bounded by regular polygons (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron), in hyperspace there are six regular hypersolids bounded by regular polyhedrons: C₅, bounded by 5 tetrahedrons; C₈, bounded by 8 cubes; C₁₆, by 16 tetrahedrons; C₂₄, by 24 octahedrons; C₁₂₀, by 120 dodecahedrons; and C₆₀₀, by 600 tetrahedrons. Models of their projections into three space have all been constructed. Our hypercube, C₈, has right angles throughout and is, therefore, the standard for measuring hyperspace.

8. Other Fourth-dimensional Surprises. The idea of hyperspace gives rise to several curiosities in addition to hypersolids. Just as a three-dimensional creature could look down on this square in its plane (Figure 4) and remove the circle from inside it without disturbing the sides of the square, so a four-dimensional creature could remove one's appendix without disturbing the skin. Such a creature could also enter any closed vault and become a perfect thief as well as a perfect surgeon. A four-dimensional creature could also untie knots in a string even though both of its ends were anchored (Figure 5). For example, in a two-dimensional plane a knot could look like the diagram in Figure 6-a.

![Figure 4](image4.jpg)

![Figure 5](image5.jpg)

![Figure 6](image6.jpg)

In order to untie it, end B must be rotated around C. However, a three-dimensional creature could untie the knot by moving the loop through a third dimension. Part BD would be turned one-half way over through this third dimension into the position shown in Figure 6-b.

A four-dimensional creature could untie three-dimensional knots similarly by moving some essential part of the knot through a fourth...
Such a creature could also create mirror image reversals in three dimensions. If triangle $A$ is in a plane, as in Figure 7, it cannot be rotated so as to coincide with triangle $B$. However, a three-dimensional creature could pick up $A$ out of its plane, turn it over through a third dimension, and put it back into its plane so that it would coincide with $B$.

Similarly, a four-dimensional creature could take prism $A$ and make it coincide with prism $B$ (Figure 8) by turning it through a fourth dimension. If it could actually be accomplished, this type of reversal could have particularly interesting effects on a living animal since the activity of many biologically useful molecules depends on their three-dimensional orientation, and it would have interesting psychological effects on a human since his right-left orientation would be completely reversed. Of course one need not stop at such four-dimensional curiosities. Fifth, sixth, and $n$-dimensional curiosities are open to exploration as well.

3. Curious Curves

A. Cycloids. Some other mathematical curiosities arise from the study of various curves. One interesting problem goes as follows. The large circle in Figure 9 has made one revolution in rolling without slipping along a straight line from $P$ to $Q$, and so the distance $PQ$ is equal to the circumference of the large circle. However, the small circle, fixed to the large one, has also made one revolution, and so the distance $RS$ is equal to the circumference of the small circle. Since $RS$ is equal to $PQ$, the circumference of the two circles must be equal.

This strange contradiction can be explained by the fact that while the large circle rolls without slipping, the small one does slip in a certain sense. Further explanation involves the concept of a rather curious curve known as the cycloid. This curve, shown in Figure 10, is the path traced by a fixed point, $N$, on the circumference of a circle as the circle rolls without slipping along a straight line. A fixed point, $N$, inside the circle describes what is called a curtate cycloid, shown
in Figure 11, and a point, O, outside the circle but attached to it describes a prolate cycloid, shown in Figure 12.

Returning to the two-circle problem, let us consider the motion of point M on the circumference of the large circle and that of point N on the circumference of the small circle. As the large circle rolls from P to Q, M describes a cycloid, and N describes a curtate cycloid. Although each wheel makes only one revolution, point M travels farther than point N (as can be seen from Figure 13), and only the common center of the circles travels a distance equal to the straight line PQ.

The cycloid has several other curious properties. The length of one arch of a cycloid is equal to the perimeter of a square circumscribed about the generating circle. (See Figure 14-a.)

The area under one arch of a cycloid is equal to three times the area of the generating circle. Therefore, when the circle is in the position shown in Figure 14-b, the shaded areas on either side of it are each exactly equal to the area of the circle.

The arc of a cycloid is also the path of quickest descent between two points. For example, suppose that A and B are two points not in the same horizontal plane and that two balls are released simultaneously at A and are allowed to roll from A to B (Figure 15), and suppose the first rolls along a plane, and the second rolls along a surface in the shape of an inverted cycloid. The second will reach B first in spite of the fact that this path is longer and that the second ball has to roll uphill before it gets to B. In fact if the plane from A to B is replaced by a surface of any other shape, the ball that rolls along that surface will always get to B later than the one that rolls along the cycloid. This problem, called the brachistochrone problem, was proposed to Jacob Bernoulli by his brother Johannes in 1696 and was solved by methods which developed into the field now called the calculus of variations.

4. Pathological Curves

Most curves dealt with by mathematicians are fairly innocent and, while they may exhibit a few idiosyncrasies like the cycloid, can be handled by using one technique or another. There are some curves, however, who simply will not behave themselves despite all efforts to bring them under control; these are the pathological curves of mathematics. Before we start to discuss them, the idea of a curve being the limit of a sequence of polygons must be introduced.

Let an equilateral triangle be inscribed in a circle (Figure 16-a); this triangle is curve $C_1$. As in Figures 16-b,c let $C_2$ be the regular hexagon obtained by bisecting the three resulting arcs of the circle and...
by joining, in order, the six vertices. Let \( C_3 \) be the regular dodecagon formed by bisecting the six arcs obtained and joining the twelve in order. If this process is repeated, the number of sides of the inscribed curve doubles each time. The curve approached as a limit in this process is the original circle, and so the circle is described as the limit curve of a sequence of curves or polygons. The pathological curves we shall discuss are limit curves like this one.

The first pathological example is the snowflake curve. Start with a triangle with sides each one unit in length (Figure 17). As in Figure 17-b, trisect each side of the triangle and erect on each of the middle thirds an equilateral triangle pointing outward. Erase the parts common to the new and old triangles. This simple polygonal curve is obtained. Trisect each side of this curve and upon each middle third erect an equilateral triangle pointing outward. Erase the parts of the curve common to the new and old figures. The curve shown in Figure 17-c is obtained. Now continue this process. The limit curve of the process is the snowflake curve. (It obviously gets its name from the shape it assumes at successive stages of its development.) This curve is considered pathological because although the curve has a perimeter of infinite length, its area is finite. At each stage of the construction, the perimeter increases, and the sequence of numbers representing the length of the perimeter at each stage does not converge.

This fact can be explained as follows. The perimeter of the triangle was 3. In constructing the second stage, we added six lines of length one-third unit each and subtracted three lines of length one-third unit each. The net result was that we added one unit to the perimeter. Therefore, the length of the second curve is \( 3 + 1 \). Likewise, the perimeter of the third stage is

\[
3 + 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 3 + 1 + 4/3;
\]

of the fourth,

\[
3 + 1 + 4/3 + (4/3)^2 \]

The perimeter at the nth stage is

\[
3 + 1 + 4/3 + (4/3)^2 + \cdots + (4/3)^{n-2},
\]

and so as \( n \) grows, the result grows, since this series does not converge, and the perimeter becomes infinite.

To show that the area is finite, think of a circle circumscribed around the original triangle. Then note that at no subsequent stage of the development will the curve ever extend beyond this circle. Therefore we are confronted by the strange fact that this curve of infinite length can be drawn on a small sheet of paper — on a postage stamp, for instance. It is also not possible to tell at any point on the limit curve the direction in which it is going, and so the tangent line does not exist at any point.

There is also another pathological curve called the anti-snowflake curve, which is obtained by drawing triangles inward, not outward, in the above construction. It has the same properties as the snowflake. Its perimeter is infinite, but its area is finite, and no tangent line can be drawn to it. The first four stages are shown in Figure 18.

The in-and-out curve is another pathological example. Draw a circle with radius one and choose six points on it which divide the circumference into six equal parts. Take three alternate arcs and turn them inward. The original circle, \( C_1 \), is now the new figure \( C_2 \) (Figure 19). The
perimeter of $C_2$ is the same as the perimeter of $C_1$, since its length is not altered by turning three arcs inward. Next trisect each arc and turn the middle third outward if it is now inward and turn it inward if it is now outward (Figure 20). This new curve is $C_3$, and its perimeter is also equal to that of the original circle. The area of $C_3$ is the same as that of $C_2$, since we alternately added and subtracted the same sized segments. Keep repeating this process. The limit curve has a perimeter equal to the perimeter of the circle, and its area is equal to that of $C_3$.

While the curvature of the original circle can be computed without difficulty, the in-and-out curve presents a pathological problem in this respect. Consider an arbitrary point on it. Should we measure curvature at this point toward the center of the circle or away from it? There is no definite curvature, and the second derivative does not exist.

Our next pathological specimen is a space-filling curve, which seems to refute the idea that a point has zero dimensions and that a curve which is one-dimensional cannot fill a given space. The first member of the curve generating sequence is polygon $P_1$ inserted in a square as shown below in Figure 21-a. The square is then divided into four equal squares, and four polygons similar to $P_1$ are formed and are joined together to form polygon $P_2$ (Figure 21-b). To get polygon $P_3$, each of the four squares is divided into four more, and sixteen polygons similar to $P_1$ are joined together (Figure 21-c). If this process is continued, the result is a sequence of polygon $P_1, P_2, P_3, \ldots$. This sequence approaches a limit curve. It can be rigorously shown that this curve passes through any specified point of the square in which it is inscribed, and so it must pass through every point of the square and must completely fill it. It can also be shown that such a one-dimensional curve could also fill an entire cube, hypercube, or figure corresponding to a cube in a space of any number of dimensions.

We have discussed some of the oddities of mathematics -- the Moebius
strip, Klein's bottle, the weird world of the fourth dimension, and some very unusual curves. These are but a few of the strange and wonderful things with which mathematicians deal, and they represent just a few of the areas studied in mathematics. There are many more mathematical curiosities to be examined and explored or to be newly discovered. The mathematical curiosities are there just waiting for a sufficiently curious mathematician.

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THE PERFECT NUMBERS AND PASCAL'S TRIANGLE

by Robert Antol
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The fundamental theorem of arithmetic states that every positive integer can be represented uniquely as the product of prime factors. An integer $n > 1$ shall accordingly be written

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

(1)

where the $p_i$'s are the distinct prime factors and $a_i$ is the multiplicity of $p_i$ (the number of times $p_i$ occurs in the prime factorization).

A positive integer is called a perfect number if it is equal to the sum of all its positive divisors other than itself. The sum of divisors of a number $n$ with the prime factorization (1) is

$$\sigma(n) = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \frac{p_2^{a_2+1}-1}{p_2-1} \cdots \frac{p_r^{a_r+1}-1}{p_r-1} = \prod_{i=1}^{r} \frac{p_i^{a_i+1}-1}{p_i-1}$$

(2)

The condition for a perfect number may then be given by $n = \sigma(n) - n$ or equivalently, $\sigma(n) = 2n$.

Euclid argued that if $2^P - 1$ is prime for $P > 1$, then

$$P = 2^P - 1$$

is a perfect number. Euler showed later that all even perfect numbers must be of this type (see [4]). The number $2^P - 1$ is known as a Mersenne prime and is denoted by $M_P$, as in [3]. All perfect numbers known are even and the question of whether there is an odd perfect number is still unanswered. There is no evidence to prove or disprove the existence of an odd perfect number but if one does exist, it must be greater than $10^{100}$ [1].

For any positive integer $m$ and any integer $k$ satisfying $0 \leq k \leq m$,
Use will now be made of the configuration known as Pascal's triangle in which the binomial coefficient \( \binom{m}{k} \) appears as the \((k+1)\)st number in the \((m+1)\)st row, as in [5].

\[
\binom{m}{k} = \frac{m!}{k!(m-k)!} \tag{4}
\]

The borders of the triangle are composed of ones; a number not on the border is the sum of the two numbers nearest it in the row above.

All even perfect numbers can be shown to lie on the third diagonal of Pascal's triangle (see Figure 1). The restriction for \( m \) is that it must be equal to a Mersenne prime plus one; that is, \( m = M_p + 1 \).

Setting \( k \) equal to 2 (since the third diagonal of Pascal's triangle is \( k = 2 \)),

\[
\binom{m}{2} = \frac{(M_p+1)!}{21(M_p+1-2)!} = \frac{2^p}{2!} = \frac{2^p(2^p-1)(2^p-2)!}{2!} = \frac{2^p}{2} = 2^{p-1}(2^p-1) = P,
\]

which is an even perfect number by (3) above.

As in [5], we now note that each number in Pascal's triangle is the sum of the numbers in the preceding diagonal (see Figure 2):

\[
\sum_{i=k-1}^{m-1} \binom{m}{i}
\]

FIGURE 1

We have seen that all even perfect numbers are on the third diagonal of Pascal's triangle. Hence, the second diagonal would generate the perfect numbers. That is, every even perfect number is the sum of the first \( 2^p - 1 = M_p \) numbers:

\[
P = \sum_{i=1}^{M_p} i. \tag{5}
\]

We now observe that the elements of the third diagonal are the triangular numbers and every even perfect number is triangular in shape [2]. (See Figure 3.)

According to Burton [11] there are 24 even perfect numbers known to date (1976). The first 5 and their associated Mersenne primes are given in Table 1 on the next page.

We now have several different ways of computing perfect numbers. We must first compute Mersenne primes \( M_p \). Knowing the Mersenne primes,
we can:

(a) compute \( P = M_P(2^p - 1) \), using Euclid's formula.

(b) compute \( P = \sum_{\substack{1 \leq i \leq M_P}} i \), summing up the first \( M_P \) positive integers, or

(c) with \( m = M_p + 1 \) and \( k = 2 \), compute \( P = \binom{m}{k} \).

It is from (c) that we note all even numbers are on the third diagonal of Pascal's triangle.

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### SOME INEQUALITIES FOR NON-NEGATIVE RANDOM VARIABLES INVOLVING THE MOMENT GENERATING FUNCTION

University of Oklahoma

Various inequalities on moments and probabilities can be derived using the moment generating function. In this article we give some results which may be new.

Let \( X \) be a non-negative random variable (r.v.) with cumulative distribution function (c.d.f.) \( F(x) \). If \( g(X) \) is a function of \( X \), the expected value \( E(g(X)) \) of \( g(X) \) (when it exists), is given by

\[
E(g(X)) = \int_0^\infty g(x) \, dF(x).
\]

The expected value of \( g(X) = x^k \) is called the \( k \)th moment of \( X \). The moment generating function (m.g.f.) \( M(t) \) is the expected value of \( \exp(tX) \), or

\[
M(t) = \int_0^\infty \exp(tx) \, dF(x), \quad t \in \text{domain}(M).
\]

Throughout this article, we assume that \( Pr(X \geq 0) = 1 \). We also assume there exists a positive number \( h \) such that \( M(t) \) exists (finite) on \(-h < t < h\), or equivalently that \( M(t) \) exists on a neighborhood of zero. (For a non-negative r.v. \( X \), the m.g.f. \( M(t) \) will automatically exist for all \( t \geq 0 \).)

The following results are not new, but are stated for reference purposes (they assume the existence of the moment generating function on a neighborhood of zero):

1) \( E(x^k) < +\infty \) for \( k = 1, 2, 3, \ldots \).

2) \( M(t) \) is a \( C^\infty(-h, h) \) function (continuous derivatives of all orders).

3) \[
M^{(k)}(t) = \int_0^\infty x^k \exp(tx) \, dF(x) \quad \text{for } -h < t < h, \ k = 0, 1, 2, \ldots.
\]
4) $M^{(k)}(0) = E(X^k)$ for $k = 0, 1, 2, \ldots$

5) $M(z) = \int_0^\infty e^{zt}dF(x)$ is analytic on $-h < Re(z) < h$.

6) $M(z) = \sum_{k=0}^{\infty} M^{(k)}(0) z^k/k!$ on $|z| < h$.

7) $M(t) = \sum_{k=0}^{\infty} M^{(k)}(0) t^k/k!$ on $-h < t < h$.

The proofs of these results do not appear often in statistical texts since they involve complex variables and real analysis, in particular, the Lebesgue Dominated Convergence Theorem. The reader may wish to consult the references given at the end (in particular, [1], pp. 52-53 and [2], Chapter 7).

Now for every $k$, and arbitrary positive $A$, we have

$I.$ First consider the inequality $M^{(k)}(t) \geq \int_0^A x^k e^{tx}dF(x)$. If $0 \leq t < h$ we have

$$M^{(k)}(t) \geq \int_0^A x^k e^{tx}dF(x) \geq \int_0^A x^k dP(A),$$

(1.1)

Dividing through by $P(A) = Pr(0 \leq X \leq A)$, we have

$$\frac{M^{(k)}(t)}{P(A)} \geq \int_0^A x^k d\frac{P(x)}{P(A)}, \quad 0 \leq t < h,$$

(1.2)

where $P(x)/P(A)$ is the c.d.f. of the $X$-distribution, truncated to $[0, A]$.

Taking the infimum over $t$, we have

$$\inf_{0 \leq t < h} \frac{M^{(k)}(t)}{P(A)} \geq \int_0^A x^k d\frac{P(x)}{P(A)},$$

(1.3)

where the right-hand side is the $k$th moment of the truncated version of $X$.

Next consider $-h < t \leq 0$. We have

$$M^{(k)}(t) \geq \int_0^A x^k e^{tx}dP(x) \geq e^{At} \int_0^A x^k dP(x).$$

(1.4)

Dividing through by $e^{At}P(A)$, we have

$$\frac{e^{At}M^{(k)}(t)}{P(A)} \geq \int_0^A x^k d\frac{P(x)}{P(A)}, \quad -h < t \leq 0,$$

(1.5)

Taking the infimum over $t$, this becomes

$$\inf_{-h < t \leq 0} \frac{e^{At}M^{(k)}(t)}{P(A)} \geq \int_0^A x^k d\frac{P(x)}{P(A)},$$

(1.6)

Taking the limit as $A \to \infty$, we have

$$\lim_{A \to \infty} \inf_{-h < t \leq 0} \frac{e^{At}M^{(k)}(t)}{P(A)} \geq E(X^k).$$

(1.7)

$\text{II}$ Second, consider $M^{(k)}(t) \geq \int_{A+0}^\infty x^k e^{tx}dP(x)$. If $0 \leq t < h$, we have

$$M^{(k)}(t) \geq \int_{A+0}^\infty x^k e^{tx}dP(x) \geq A^k e^{At} \int_{A+0}^\infty dP(x)$$

= $A^k e^{At} Pr(X > A)$.

After division, we obtain

$$\frac{e^{-At}M^{(k)}(t)}{A^k} \geq Pr(X > A).$$

(2.2)

By taking infimums, we then have

$$\inf_{0 \leq t < h} \inf_{k \geq 0} \frac{e^{-At}M^{(k)}(t)}{A^k} \geq Pr(X > A) = 1 - P(A)$$

(2.3)

and
\[ \inf \inf_{k \geq 0, 0 \leq t < h} e^{k A t} \mathbb{P}(X > A) \geq \Pr(X > A) = 1 - F(A). \tag{2.4} \]

If \(-h < t \leq 0\), the basic inequalities are

\[ M^{(k)}(t) \geq \int_{A+0}^{\infty} x^k e^{t x} dF(x) \geq A^k \int_{A+0}^{\infty} e^{t x} dF(x). \tag{2.5} \]

After division by \(A^k (1 - F(A))\), we then get

\[ \frac{M^{(k)}(t)}{A^k (1 - F(A))} \geq \int_{A+0}^{\infty} e^{t x} d\left[ \frac{F(x)}{1 - F(A)} \right], \tag{2.6} \]

where \(F(x)/(1 - F(A))\) is the c.d.f. of \(X\) truncated to \((A, \infty)\), and the right-hand side of (2.6) is the m.g.f. of such a distribution, which will exist at least on \((-h, h)\). Again taking the infimum on \(k\), one sees that

\[ \inf_{k \geq 0} \frac{M^{(k)}(t)}{A^k (1 - F(A))} \geq \int_{A+0}^{\infty} e^{t x} d\left[ \frac{F(x)}{1 - F(A)} \right], \tag{2.7} \]

and

\[ \lim \inf_{k \to 0} \frac{M^{(k)}(t)}{A^k} \geq \int_{0}^{\infty} e^{t x} dF(x) = M(t), \tag{2.8} \]

provided \(X\) does not have an atom at zero, that is, provided \(\Pr(X = 0) = 0\).

A multitude of inequalities become evident if one selects \(0 < A < B < \infty\) and then writes

\[ M^{(k)}(t) = \int_{0}^{A} x^k e^{t x} dF(x) + \int_{A+0}^{B} x^k e^{t x} dF(x) + \int_{B+0}^{\infty} x^k e^{t x} dF(x) \]

for \(k = 0, 1, 2, \ldots\) and \(-h < t < h\). Since there are so many cases, we leave the details to the reader. Various other generalities can be seen to exist, such as deleting the requirement that \(X\) be non-negative. In addition, for a non-negative \(r.v. X\), the moment generating function \(M(t)\) exists for all \(t \leq 0\), and hence some of the inequalities such as (1.6), (1.7) may be tightened.

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Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742
ANOTHER APPROACH TO A PAIR OF FAMILIAR SEQUENCES

by N. Schaumberger
Bronx Community College, CUNY

A simple proof that the sequence \( S_n = (1 + \frac{1}{n})^n \) approaches a limit can be based on:

1. \( S_n = (1 + \frac{1}{n})^n \) is an increasing sequence, and
2. \( T_n = (1 + \frac{1}{n+1})^{n+1} \) is a decreasing sequence.

The usual proofs of these statements employ either the binomial theorem or the inequality between the arithmetic and geometric means (see [1]). The first method is somewhat messy and the second uses an inequality that is rarely derived in elementary calculus. In this note we offer simple proofs of (1) and (2) that fit naturally into the sequence of calculus topics in a standard course.

Let

\[ f(x) = (n+1)x - x^{n+1} \]

where \( n \) is a positive integer and \( a > 0 \), \( x > 0 \). Since

\[ f'(x) = (n+1)a - (n+1)x^n \]

and

\[ f''(x) = -n(n+1)x^{n-1} \]

it follows that \( f(x) \) attains an absolute maximum at \( x = \frac{1}{n} \). Thus if \( a \neq 1 \), we have

\[ f\left(\frac{1}{n}\right) > f(1) \]

or

\[ \frac{n+1}{n} > (n+1)a - 1. \]

Putting \( a = 1 + \frac{1}{n+1} \) gives

\[ n(1 + \frac{1}{n+1})^n > n + 1 \]

or

\[ (1 + \frac{1}{n+1})^{n+1} > (1 + \frac{1}{n})^n. \]

This proves (1). To prove (2), we put \( a = 1 - \frac{1}{n+1} \). Thus

\[ n(1 - \frac{1}{n+1})^n > n - 1 \]

or

\[ \left(\frac{n}{n+1}\right)^{n+1} > (\frac{n-1}{n})^n. \]

Inverting finally gives

\[ (1 + \frac{1}{n})^{n+1} < (1 + \frac{1}{n-1})^n. \]

REFERENCES

PI MU EPSILON STUDENT CONFERENCE

MIAMI UNIVERSITY
OXFORD, OHIO

CALL FOR STUDENT PAPERS

Saturday, September 30, 1978

Held in conjunction with the
Sixth Annual Conference in Mathematics and Statistics

This is a call for undergraduate student papers; we invite you to join us. The student conference will be Saturday afternoon. Student speakers will receive their picnic lunch free. Talks may be on any topic related to mathematics, statistics or computing. Do you have a favorite topic which you could share with us? We welcome items ranging from expository to research, interesting applications, problems, etc. Presentation time should be fifteen, thirty, or forty-five minutes. We need your title, presentation time, and a 50 (approx.) word abstract by September 18, 1978. Please send your material to Professor Cox (address below).

We also urge you to attend the Conference on Applications of Statistics and Mathematics, which begins Friday afternoon, September 29. Free overnight facilities for all students will be arranged with Miami students.

For more details write to:
Professor Milton D. Cox
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056

PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword puzzles and find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinguish material which might well be the domain of the Problem Department, but the Editor reserves the right to make an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate, or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to the Editor, Pi Mu Epsilon Journal, 601 Elm Avenue, Room 423, The University of Oklahoma, Norman, Oklahoma 73019. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March 31, and that for each Spring issue, the following September 30.

Mathacrostic No. 5

submitted by R. Robinson Rowe
Sacramento, California

Identify the 32 key words, matching their letters in order with the opposite sequence of numbers, and insert each letter of the key words in the square of the Mathacrostic with the same number. Words end at the blank squares, and some words extend on to the next line.
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**Definitions and Key**

A. Weigh anchor
B. Same in static energy
C. Towards Nome
D. Umpire
E. Stress-strain ratio
F. Calculus task
G. Absolute, in math
H. Relating integers
I. Geologic period
J. Mathematician 1642-1727
K. Crazy bone
L. Michigan campus
M. Greek letter
N. King of Judah ca 750 BC
O. Moon's age on Jan. 1
P. Melampyrum spp
Q. Mathematical snake
R. Most recent
S. Language in S. India
T. Kitchen tool
U. Dream
V. Famous cycloid
W. Maximum effort
X. The rabble
Y. Achillea spp
Z. Asimina triloba

---

- A. City of the Mohawk
- B. Achilles story
- C. Persian creed
- D. Scanned (a book)
- E. Mathematical inventor
- F. His invention
When completed, the Mathacrostic will be a 260-letter quotation, and the 32 initial letters of the key words will spell out the name of an author and title of his book from which the quotation was derived. It will tell you something human about yourself. There is a one-to-one correspondence between the 260 letters in the Mathacrostic and the 260 letters of the key words, so it is also an anagram.

Geometry From A Dozen Pennies

A mathematics teacher wants to illustrate some geometric patterns by arranging 12 pennies in various ways without having to use compass, straight-edge, measuring instruments or marking devices of any kind. To make it more of a challenge, the teacher has decided only to slide the pennies in arranging them and disallow the freedom of lifting the coins. Under these restrictions, show how each of the following figures may be created precisely.

An Equilateral Triangle:

Two Circles of Equal Radii:

Perpendicular Lines:

A Line and a Point Not on It:

A Regular Hexagon:

An Angle of 150° and Two Interior Points:

Mathematical Word Chains: One Letter Changes

Find a sequence of legitimate words (disallowing proper nouns and abbreviations) starting with the first word given and ending with the one below it if you are allowed to change only one letter at a time in proceeding from one word to the next.

(a) MATH
(b) LINE
(c) ZERO
(d) SEVEN

Example. A solution to MATH → NOTE is:

MATH
MATE
RATE
ROTE
NOTE

Mathematical Word Chains: Two Letter Changes

Follow the same instructions as in the preceding puzzle, except you are allowed (and required) to change exactly two letters at a time.

(a) LINES
(b) CURVE
(c) GROUP
(d) SLOPE
(e) LINEAR

Countdown

Find all possible solutions to the following long division problem:
Greek Crosses and Squares

Since no solutions were received for this puzzle which appeared in the Fall, 1976 issue of this Journal, we present it again. Solver must use one or more of the figures shown in the center column to piece together and form the figure on the left, then rearrange to form the figure on the right, in each case.

End with:  Start with:  End with:

(a)  
(b)  
(c)  
(d)  

Solutions

Missionaries and Cannibals  [Fall, 1976]
A late solution was received by ROGER E. KUEHL, Kansas City, Missouri, who observed the optimum number of 13 crossings and raised the question of whether all or some "river-crossing" problems have general solutions and whether it is indeed possible to prove that a (trial and error) solution to a particular one is minimum. (See Spring, 1977 issue for solution.)

Mathacrostic No. 3  [Spring, 1977]
Late solutions were received by MITCH ENTRICAN, University of Mississippi and SISTER STEPHANIE SLOYAN, Georgian Court College (Lakewood, New Jersey).

A Pair of Eights  [Spring, 1977]
Late solutions were received by MARK EVANS, LaMarque, Texas and B. FRANK WILLIAMS, Campbell, Texas.

Mathacrostic No. 4  [Fall, 1977]
Definitions and Key:

A.  Looking glass.  F.  Conjugated.  K.  Lethe.  P.  Halfway
B.  Esthete.  G.  Awash.  L.  Luffed.  Q.  Uther
D.  Isohyet.  I.  Rhythm.  N.  Habitude.  S.  Tetrahedron
U.  Nth.  Z.  Hashish.  c.  Rhomboid
V.  Gushed.  a.  Effete.  f.  Kowtows
W.  Oddity.  b.  Shorthand.  d.  Astronomy
X.  Fahrenheit.  c.  Nutty
Y.  Theory.  d.  Astronomy
First Letters: LEWIS CARROL THE HUNTING OF THE SNARK

Quotation: They found not a button or feather by which they could tell that they stood on the ground where the baker had met with the snark. In the midst of the word he was trying to say, in the midst of his laughter and glee, he had softly and suddenly vanished may, for the snark was a bogum you see.

Solved by JEANETTE BICKLEY, Webster Groves High School (Missouri); MITCHELL W. ENTRICAN, University of, Mississippi; BARBARA LEHMANN, St. Peters College; JODI L. LEVESQUE, University of, Florida; SIDNEY PENNER, Bronx Community College of CUNY; BOB PRIELIPP, University of, Wisconsin at Oshkosh; LEO SAUVE, Algonquin College (Ottawa, Canada); and ALLAN TUCHMAN and PATRICIA GROSS, University of Illinois.

Cross-number Puzzle. [Fall, 1977]

```
9 2 1 6 7 6 2 4 1
1 6 0 3 9 8 6 2
1 1 0 4 2 4 9 4 9
9 6 8 1 4 3 6 6
2 7 6 1 2 5 6
4 8 5 6 6 6 6 6
6 7 6 5 5 6 6
2 5 6 1 1 7 3 6 1
4 6 7 7 7 7 1 5 9
0 2 4 6 5 2 8 9
1 3 3 1 1 1 0 5 8
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Solved by JEANETTE BICKLEY, Webster Groves High School (Missouri); MITCHELL W. ENTRICAN, University of, Mississippi; MICHAEL HANEY, ALLAN TUCHMAN, and PATRICIA GROSS. University of, Illinois; CLARK HEISER, Yonkers, New York; BECKY HENNING, University of California at Los Angeles; SIDNEY PENNER, Bronx Community College of, CUNY; BOB PRIELIPP, University of, Wisconsin at Oshkosh; and R. ROBINSON ROWE, Sacramento, California.

A number of elementary facts may be easily observed in the sequence of diagrams below. For example, if it is assumed that the original tri-
angle has sides of length one, the total perimeter of each figure and area enclosed may be computed.

1. Carry this out, and determine what happens when these computations are performed for each of the figures as the sequence of constructions is carried on ad infinitum.

2. What would happen if the new triangles pointed inward instead of out? What would the new figures look like? What would be the answer to the above question in this case?

3. What would happen if we started with some regular polygon other than the triangle (say a square, pentagon or hexagon) and erected similar (but smaller) n-gons on the middle one-third of each side in each step? What would the six figures look like for \( n = 4, 5, \) or \( 6 \)? What would the answer be for each of the above questions?

4. What if the triangles (or n-gons) were pointed inward on one step, outward on the next, alternating at each step? Generalize.

Comment by Editor
See in this connection the article in this issue, "Mathematical Curiosities" by Debra Gutridge (14).

MATCHING PRIZE FUND

If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of mathematical films. To apply, or for more information, write to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on a separate sheet properly identified with name and address and mailed before November 1, 1978.

Address all communications concerning this department to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90042.

Problems for Solution

412. Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, California.

Are there examples of angles which are trisectible but not constructible? That is, can you find an angle \( \alpha \) which is not constructible with straight edge and compass, but such that when \( \alpha \) is given, \( \alpha/3 \) can be constructed from it with straight edge and compass?

413. Proposed by R. Robinson, 1025 Naukaway, Michigan and Sacramento, California.

In a variation of the crossed-ladders-in-an-alley classic, the new tall building on one side of the alley was vertical, but on the other side the old low building, having settled, leaned toward the alley. Projected, its face would have met the top of the tall building and would have been one foot longer than the height of the tall building.
The ladders, unequal in length, rested against the buildings 21 feet above the ground and crossed 12 feet above the ground. How high was the tall building and how wide was the alley?


In discussing the discriminant of a quadratic equation, a certain textbook says, "...if $a$, $b$ and $c$ are integers with $a \neq 0$ and if $b^2 - 4ac = 79$, the roots of $ax^2 + bx + c = 0$ will be real, irrational and unequal." Explain why this is incorrect.

415. Proposed by Charles W. Trigg, San Diego, California.

A hexagonal number has the form $2n^2 - n$. In base 9, show that the hexagonal number corresponding to an $n$ that ends in 7 terminates in 11.

416. Proposed by Scott Kim, Rolling Hills Estates, California.

Each of the three figures shown above is composed of two isosceles right triangles, $\triangle ABC$ and $\triangle EBD$, where $\angle ABC$ and $\angle DBE$ are right angles, and $B$ is between points A and D. Points C and E coincide in Figure 1a, so that $CB/EB = 1$. In Figure 1b, we are given that $CB/EB = 2$, and in Figure 1c, we are given that $CB/EB = 3$. Consider each pair of triangles as a single shape and suppose that the areas of the three shapes are equal. (The figures are not drawn to scale.) Problem: For each pair of figures, find the minimum number of pieces into which the first figure must be cut so that the pieces may be reassembled to form the second figure. Pieces may not overlap, and all pieces must be used in each assembly.

417. Proposed by Clayton W. Dodge, University of Maine, Orono, Maine.

1) Prove that the line joining the midpoints of the diagonals of a quadrilateral circumscribed about a circle passes through the center of the circle.

2) Let the incircle of triangle $ABC$ touch side $BC$ at $X$. Prove that the line joining the midpoints of $AX$ and $BC$ passes through the incenter $I$ of the triangle.


Find all angles $\theta$ other than zero such that $\tan 116 = \tan 1119 = \tan 11116 = \cdots$.

419. Proposed by Michael W. Ecker, City University of New York.

Seventy-five balls are numbered 1 to 75 and are partitioned into sets of 15 elements each, as follows: $B = \{1, \cdots, 15\}$, $I = \{16, \cdots, 30\}$, $N = \{31, \cdots, 45\}$, $G = \{46, \cdots, 60\}$, and $O = \{61, \cdots, 75\}$, as in Bingo.

Balls are chosen at random, one at a time, until one of the following occurs: At least one from each of the sets $B$, $I$, $G$, $O$ has been chosen, or four of the chosen numbers are from the set $N$, or five of the numbers are from one of the sets $B$, $I$, $G$, $O$.

Problem: Find the probability that, of these possible results, four $N$'s are chosen first. (Comment: The result will be approximated by the situation of a very crowded bingo hall and will give the likelihood of what bingo players call "an $N$ game", that is, bingo won with the winning line being the middle column $N$.)

420. Proposed by Herbert Taylor, South Pasadena, California.

Given four lines through a point in 3-space, no three of the lines in a plane, find four points, one on each line, forming the vertices of a parallelogram. (This is a variation of problem B-2 on the December 1977 William Lowell Putnam Mathematical Competition.)

421. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

If $F(x, y, z)$ is a symmetric increasing function of $x$, $y$, $z$, prove
that for any triangle, in which \( w_a, w_b, w_c \) are the internal angle
bisectors and \( m_a, m_b, m_c \), the medians, we have
\[
P(w_a, w_b, w_c) \leq P(m_a, m_b, m_c)
\]
with equality if and only if the triangle is equilateral.

422. **Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.**

If perpendiculars are erected outwardly at \( A, B \) of a right triangle
\( ABC (C = 90^\circ) \), and at \( A' \) the midpoint of \( AB \), and extended to points \( P, Q, R \) such that \( AP = AQ = AB/2 \), show that triangle \( PQR \) is perspective with triangle \( ABC \).

423. **Proposed by Richard S. Field, Santa Monica, California.**

Find all solutions in positive integers of the equation \( \alpha^d - \beta^d = \gamma^d \), where \( d \) is an odd integer.

424. **Proposed by R. S. Luther, University of Wisconsin, Janesville.**

Prove that
\[
(x^{1/n} + y^{1/n})^n > \left( \frac{x - y}{\ln x - \ln y} \right)(2n + 2)
\]
where \( n \) is an odd integer \( \geq 3 \) and \( 0 < y < x \).

**Solutions**

386. [Spring 1977] **Proposed by Charles W. Trigg, San Diego, California.**

Show that the volume of Kepler's Stella Octangula (a compound of two interpenetrating tetrahedrons) is three times that of the octahedron that was stellated.

**FIGURE 1**

Solution by Kenneth W. Wilke, Topeka, Kansas, with practically identical solutions by Clayton W. Dodge, University of Maine, Orono; R. Robinson Pasco and the proposer, Charles W. Trigg.

Since a plane through the midpoints of the three edges of a tetrahedron issuing from one vertex cuts off a smaller tetrahedron whose volume is one-eighth that of the larger (similar) tetrahedron, repeating this process three more times decomposes the given tetrahedron into four smaller identical tetrahedrons and a regular octahedron whose volume is one-half the volume of the initial tetrahedron. In the formation of the Stella Octangula by interpenetrating two identical tetrahedrons, the octahedrons contained therein occupy the same space, leaving a solid composed of eight smaller tetrahedrons and one octahedron. Hence the volume of the Stella Octangula is one and one-half times the volume of the initial tetrahedron and three times the volume of the stellated octahedron contained therein.

Also solved by LOUIS H. CAIROLI, Graduate Student, Kansas State University, Manhattan, Kansas; MK JAEGER, Chicago, Illinois; and SISTER STEPHANIE SLOYAN, Georgian Court College, Lakewood, New Jersey.
**Comments by the Problem Editor:**

1) R. Robinson Rowe noted that if each of the equilateral triangular faces has an area $A$, the Stella has a surface area of $24A$ and the octahedron an area of $8A$, so again the ratio of Stella to octahedron, by area, is three.

2) The two interpenetrating tetrahedrons are, of course, regular and the eight vertices are the vertices of a cube. Furthermore, the twelve edges of the two tetrahedrons are the diagonals of the six faces of the cube. The interested reader can find additional material on Kepler's *Stella Octangula* in Regular Figures, by L. Fejes Toth, Mathematical Essays and Recreations, by W. W. Rouse Ball and H. S. M. Coxeter, Mathematical Snapshots, by H. Steinhaus, Introduction to Geometry, by H. S. M. Coxeter, and in Cundy and Rollett's Mathematical Models.

---

**Solution by Charles U. Trigg, San Diego, California.**

\[ AC = BC, \quad DC = FC \quad \text{and} \quad \angle ACD = 60^\circ = \angle DCB = \angle BCP. \] Hence, triangles \( ACD \) and \( BCP \) are congruent with \( BP = AD \equiv AE \equiv AQ \) and \( \angle BPC = \angle DCP = 60^\circ \). Therefore \( \angle RBP = 180^\circ \), which makes \( RP \) a straight line segment = \( RQ \). Thus triangle \( PQR \) is an isosceles triangle with a vertex angle of \( 60^\circ \), so it is equilateral and \( QE \) falls along \( QP \).

Let the midpoints of \( PE, AQ, RD, RA \) and \( QE \) be \( X, Y, Z, M \) and \( N \) respectively. \( MZ \) is parallel to \( AD \), and \( YN \) is parallel to \( AE \), so
\[ MZ = AD/2 = AE/2 = YN, \quad \text{and} \quad \angle ZMA = 120^\circ - \angle YNE. \] Also, \( MY = QG/2 = FQ/2 = ZN \), so triangles \( ZMY \) and \( YNX \) are congruent with \( ZY \equiv YX \) and \( \angle ZMY = \angle YXN \). Now \( \angle AYW = 120^\circ = \angle YNX \), so \( \angle ZXY = \angle AYN = \angle MYZ = \angle XYN = \angle AYN - \angle YXW = \angle XYN = 120^\circ - 60^\circ = 60^\circ \). Therefore triangle \( ZXY \) is equilateral.

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**Solution by Charles U. Trigg, San Diego, California.**

In the game of "Larger, But Not That Large" two players each write down a positive integer. The numbers are then disclosed and the winner (who is paid a dollar by the loser) is the player who wrote the larger number, unless the ratio of larger to smaller is three or more, in which...
case the player with the smaller number wins. If the same number is
picked by both players, no payment is made.

a) What is the optimal strategy?

b) Suppose instead that the players are not restricted to integers
but to the set \([1, \omega]\) and that the larger number wins provided the larger-
to-smaller ratio is less than \(r\) (for some \(r > 1\)); otherwise the larger
number loses. Find an optimal strategy.

**Solution by the proposers.**

a) The unique optimal strategy is to choose 1, 2 and 5 randomly
and with equal frequency, shunning all other numbers. Optimality follows
by observing that this strategy ties each of the pure strategies 1, 2 and
5 and beats every other pure strategy. Since all pure strategies other
than 1, 2 and 5 are thus ruled out, the above strategy, since it is the
solution of the reduced 3 by 3 matrix, is the unique optimal strategy.
Generalizations to the case in which violation of the critical ratio \(r\)
involves a penalty of \(p\) is not difficult.

b) Consider the intervals \([1, r]\) and \([r, r^2]\) and the following
strategy: choose either of the two intervals with probability 1/2 and
then from the interval chosen select a number \(x\) from the uniform random
distribution. That this is an optimal strategy is easily confirmed by
checking that it ties with any pure strategy \(y\) in \([1, r^2]\) and beats any
\(y > r^2\).

**Solutions with differing conclusions were offered by Mark Evans,
LaMarque, Texas; Kenneth N. Wilke, Topeka, Kansas; R. Robinson Rowe,
Sacramento, California; and Donald Canard, Anaheim, California.**


Find a sequence of positive integers \(1 \leq a_1 < a_2 < \ldots\) which omits
infinitely many integers from every arithmetic progression (in fact it
has density 0) but which contains all but a finite number of terms of
every geometric progression. Prove also that there is a set \(S\) of real
numbers which omits infinitely many terms of any arithmetic progression
but contains every geometric progression (disregarding a finite number of
terms).

No solutions, with the exception of the proposer's, have been submitted.
Proposer's solution will be published in the next issue of the Journal
unless other qualifying solutions are received.

390. [Spring 1977] Proposed by Robb Koether and David C. Kay,
University of Oklahoma, Norman, Oklahoma.

Let the diagonals of a regular \(n\)-sided polygon of unit side be
drawn. Prove that the \(n - 2\) consecutive triangles thus formed which
have their bases along one diagonal, their legs along two others or a
side, and one vertex in common with a vertex of the polygon each have
the property that the product of two sides equals the third.

**Solution by the Proposers.**

Let \(A\) be the common vertex of the triangles and \(BC\) the diagonal
containing the bases (see Figure 4). From elementary properties of
regular polygons and their circumscribed circles, the angles at \(A\) are
equal and the first and last triangles have equal base angles. In
particular, triangle \(ABD\) is isosceles, with \(AD = BD\). Let triangles
\(AXY\), \(AYZ\) be any two consecutive triangles, and let the side-lengths be
as indicated. Since \(AY\) bisects angle \(XAZ\), \(ax/u = az/v\) or \(ay/u = yz/v\).
Hence, by induction, the ratio of the product of two sides of a triangle
to the third is constant. Therefore, \(ab/a = ay/u\), and since \(a = b, s = ay/u\)
or \(ay = su\). Thus, the product of two sides of each of the
triangles equals \(s\) times the third, where \(s\) is the side of the regular
polygon. (If \(s = 1\) the stated result then obviously follows.)

Solve this alphametic where, of course, NINE is divisible by 9:

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<td>E</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>N</td>
<td>I</td>
<td>N</td>
<td>E</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T</td>
<td>H</td>
<td>I</td>
<td>R</td>
</tr>
</tbody>
</table>

Solve this alphametic where, of course, NINE is divisible by 9:

T W E L V E
N I N E
N I N E
T H I R T Y

Solution by Charles W. Trigg, San Diego, California.

Clearly, T ≠ 0, N ≠ 0, and E ≠ 0 or 5. Proceeding from the right, the columns establish the following equations:

\[ E = Y + 10k \]
\[ V + 2W + k = T + 10m \]
\[ L + 2T + m = R + 10n \]
\[ E + 2N + n = I + 10p \]
\[ W + p = H \]

where k, m, n, p are non-negative integers < 3 and p ≠ 0.

Since 9 divides N I N E, then

\[ E + 2N + I \equiv 0 \pmod{9} \]

whereupon, from (4):

\[ 21 + 10p + n \equiv 0 \pmod{9} \]

Thus, taking into consideration (1), (4), and (3), the following possibilities with distinct integers exist:

<table>
<thead>
<tr>
<th>I</th>
<th>P</th>
<th>m</th>
<th>N</th>
<th>E</th>
<th>Y</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>8</td>
<td>7</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>7</td>
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</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

Further consideration of (2), (5) and (4) reduces the possibilities to

I = 5, N = 6, E = 1, Y = 3, V = 2, T = 4, W = 7, H = 8, L = 9 and R = 0.

Consequently, the unique solution is 471921 + 2(6561) = 485043.

Also solved by LOUIS H. CAIROLI, (graduate student) Kansas State University, Manhattan, Kansas; VICTOR G. FESER, Mary College, Bismarck, North Dakota; HOWARD FORMAN, Bucknell University, Lewisburg, Pennsylvania; JOHN M. HOWELL, Littlerock, California; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer, CLAYTON W. DODGE.


Solve in distinct positive integers.

\[ \frac{1}{a+1} - \frac{3}{b+1} = \frac{1}{2} \]

Solution by Kenneth M. Wilke, Topeka, Kansas.

Let \( x = \frac{b+1}{a+1} \).

Then by considering the convergents of this continued fraction in the usual manner, we find \( x \) defined by the equation \( ax^2 - abx - b = 0 \) so that

\[ x = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2a} \],

rejecting the negative root.

Hence the first term on the left side of the given equation can be replaced with

\[ -ab + \frac{\sqrt{a^2b^2 + 4ab}}{2a} \]

By symmetry the given equation becomes

\[ -ab + \frac{\sqrt{a^2b^2 + 4ab}}{2a} - 3 \left( -\frac{cd + \sqrt{c^2d^2 + 4cd}}{2c} \right) = \frac{1}{2} \]

which is equivalent to the system of equations

\[ 3d = b + 1 \quad \text{and} \quad a^2(a^2b^2 + 4ab) = 9a^2 \left( a^2b^2 + 4ab \right) \]

Since \( a^2b^2 + 4ab = (ab + 2)^2 - 4 \), it is easily shown that this expression...
cannot be a perfect square unless \( ab = 0 \). Hence the radicals must be equal.) The second equation of the system reduces to \( 2ba(2 - a) = a0 + 12(b + 1) \).

Since all variables are positive integers we must have \( a = 1 \) and \( c = 12(b + 1)/(2b - 1) = 6 + 18/(2b - 1) \). Since \( 2b - 1 \) divides 18 we obtain \( (a, b, c, d) = (1, 2, 12, 1) \) and \( (1, 5, 8, 2) \), because \( d = (b + 1)/3 \).

Hence the unique solution is \( (a, b, c, d) = (1, 5, 8, 2) \), corresponding to

\[
\begin{align*}
-5 + \sqrt{5} & \quad \text{and} \\
-3 - 16 \sqrt{210} & = -5 + 3 \sqrt{5} \\
-2 + \sqrt{5} & = 2
\end{align*}
\]

Also solved by JEFFREY BERGEN, Chicago, Illinois; JOHN N. HOWELL, Littleton, California; FLORA N. FONA, Kew Gardens, Long Island; DONALD CANARD, Anaheim, California; and the proposer, R. ROBINSON ROME. Some of the solvers neglected to notice that the solution \((1, 2, 12, 1)\) was invalidated by the restriction requiring distinct integers.


Consider the sequence \( f(n) = n^2 - n + 41 \). Find the GCD of \( f(n) \) and \( f(n+1) \).

**I. Solution by Kenneth M. Wilke, Topeka, Kansas, with practically verbatim solutions offered by RONY ABOUDI, Florida Atlantic University; JEFFREY BERGEN, Chicago, Illinois; CLAYTON W. DODGE, University of Maine; at Oxford; RICHARD A. GIBBS, Fort Lewis College; VICTOR G. FESER, Mary College, Bismarck, North Dakota; MWK JAEGER, University of Wisconsin, Madison, Wisconsin; EBO PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROME, Sacramento, California; LEO SAUVE, Algonquin College, Ottawa, Canada; CHARLES W. TRIGG, San Diego, California and the proposer, PETER A. LINDSTROM.

Note that \( f(n) \) is always an odd integer. \( f(n+1) = n^2 + n + 41 \). Let \( d = GCD(f(n), f(n+1)) \). Then \( d | (f(n) + f(n+1)) = 2(n^2 + 41) \) and \( d | (f(n+1) - f(n)) = 2n \). Then since \( (n, n^2+41) = 1 \) unless \( n = 4k \) for some integer \( k \), we have \( d = 41 \) whenever \( n = 4k \) and \( d = 1 \) otherwise.


Using an argument similar to that of Solution I, Ecker shows that it is just as easy to solve the more general situation with \( f(n) = n^2 - n + p \), where \( p \) is any prime \( \neq 2 \). He arrives at the conclusion that if \( n \) is a multiple of \( p \), it is immediate that \( p \) is a divisor of both \( f(n) \) and \( f(n+1) \) and is equal to their GCD. If \( n \) is not a multiple of \( p \), the assumption that \( GCD = p \) leads to the contradiction that \( p \) divides both \( n - 1 \) and \( n + 1 \), leading to the conclusion that the \( GCD = 1 \).

Comment by the Problem Editor

Louis H. Cairoli, Kansas State University, calls attention to the article on "The Generation of Prime Numbers" in Mathematical Gems II by Ross Honsberger, published by the Mathematical Association of America. Other material relating to this problem may be found in the October 1976 issue of Eureka, published by Algonquin College, Ottawa, Canada, Problem 142, page 175 et. seq., and in the accompanying references.

Clayton W. Dodge also cited the references listed by Cairoli and added his own article, "A Prime-Generating Trinity", published in the October 1977 issue of Eureka.


394. [Spring 1977] Proposed by Erwin Just and Bertram Kabak, Bronx Community College.

Prove that if \( A_1, A_2 \) and \( A_3 \) are the angles of a triangle, then

\[
3 \sum_{i=1}^{3} \sin^2 A_i - 2 \sum_{i=1}^{3} \cos^3 A_i \leq 6
\]

I. Solution by Louis H. Cairoli, Graduate Student, Kansas State University, Manhattan, Kansas.

The result follows immediately from the known relations

\[
\sum_{i=1}^{3} \sin^2 A_i \leq 2 + 2 \cos A \cos B \cos C
\]

and

\[
\sum_{i=1}^{3} \cos^3 A_i \leq 3 \cos A \cos B \cos C
\]

II. Solution by Murray S. Klambkin, University of Alberta, Edmonton, Alberta.

We will establish the more general and stronger inequality

\[
\cos^3 A_1 + \cos^3 A_2 + \cos^3 A_3 \geq 3(1/2)^n
\]
where \( \pi \) is an integer \( \geq 2 \) and \( \{ \pi \} \) denotes with equality iff the triangle is equilateral.

The given inequality can be rewritten as
\[
2 \sum \cos^3 A_x + 3 \sum \cos^2 A_x \geq 3
\]
and thus can be gotten immediately from linear combinations of (1) for \( \pi = 2 \) and 3.

To prove (1), we use the known special case of it for \( \pi = 2 \) [1].
This latter case is also a special case \( \pi = 2, x = y = \pi = 1 \) of the inequality [2]
\[
x^2 + y^2 \geq (-1)^{n+1} [2xy \cos nA_1 + 2ax \cos nA_2 + 2ay \cos nA_3] \tag{2}
\]
and which is easily established from a sum of squares (here \( x, y, a \) are arbitrary real numbers). One can also obtain other \( n \)th order trigonometric triangle inequalities from (2), [2].

Proof. Case (1). The triangle is non-obtuse: By the power mean inequality [3],
\[
\left( \frac{\cos^n A_1 + \cos^n A_2 + \cos^n A_3}{3} \right)^{1/n} \geq \left( \frac{\cos^2 A_1 + \cos^2 A_2 + \cos^2 A_3}{3} \right)^{1/2} \geq \frac{1}{2}, \{ \pi \},
\]
for all real \( \pi \geq 2 \). Also (2) is then valid immediately for all triangles if \( \pi \) is an even integer \( \geq 2 \).

Case (2). The triangle is obtuse (let \( A_3 > \pi/2 \)): We now have to show that
\[
\cos^n A_1 + \cos^n A_2 \geq 3(1/2)^n + \cos^n (A_1 + A_2) \tag{3}
\]
where \( 0 < A_1 + A_2 < \pi/2 \) and \( \pi \) is an odd integer \( \geq 3 \). If either \( \cos^n A_1 \) or \( \cos^n A_2 \geq 3(1/2)^n \), the inequality is then obviously valid (since \( \cos (A_1 + A_2) > \cos A_1 \) and \( \cos A_2 \)). Since at least one of \( A_1, A_2 \) is \( \pi/4 \) and \( \cos^{\pi/4} \geq 3(1/2)^n \) for \( \pi \geq (log 9)/(log 2) \approx 3.1699 \), (3) is valid for all real \( \pi \geq 3.17 \). For \( \pi = 3 \), we have \( \cos^{-1} \sqrt[3]{3/5} > 43.85^\circ \). Thus, for (3) to be valid for all real \( \pi \geq 3 \) (and all non-acute triangles), it suffices to show that
\[
\cos^{45^\circ} + \cos^{45^\circ} > 3/8 + \cos^{60^\circ}
\]
or

Using calculus, one can show that
\[
\cos^2 A_1 + \cos^2 A_2 - \cos^2 (A_1 + A_2) \geq 1
\]
and that (3) is valid for all real \( \pi \geq 2 \). An open problem here is to determine the minimum \( \pi \) such that (3) is valid. Clearly, (3) is invalid for \( \pi = 1 \). Also, \( \pi \) must be \( 2[(\log 3)/(\log 2) - 1] \approx 1.1699 \) (just let \( A_1 = A_2 = \pi/4 \)).

REFERENCES


Also solved by BOB PRIELIPP, University of Wisconsin at Oshkosh; and the Proposers.

395. [Spring 1977] Proposed by Joe Van Austin, Emory University, Atlanta, Georgia.

Assume that \( \pi \) independent Bernoulli experiments are made with \( p = P \{ \text{success} \} \). \( 1 - p = P \{ \text{failure} \} \). \( 0 < p < 1 \). Intuitively it seems that \( P \{ \text{success on the first trial} \} \) exactly one success is always less than \( P \{ \text{success on the first trial} \} \) at least one success. Verify directly that this is indeed the case.

Solution by Louis H. Cairoli, Graduate Student, Kansas State University, Manhattan, Kansas.

Let \( A = \text{success on first trial} \); \( B = \text{exactly one success} \); \( C = \text{at least one success} = 1 - \text{probability all failures} \). By Bayes' Formula, we see that
\[
P[A|B] = \frac{P[B|A]P[A]}{P[B]} = \frac{(1-p)^{\pi-1} \frac{p}{np(1-p)^{\pi-1}}} = \frac{1}{\pi}
\]
and
\[
P[A|C] = \frac{P[C|A]P[A]}{P[C]} = \frac{1p}{1 - (1-p)^n}
\]
Hence we must show that for \( n > 1 \), \( \frac{1}{\pi} < p/[1 - (1 - p)] \). But this is equivalent to \((1 - p)^n > 1 - n p \), a result easily shown by induction.
If $a_j = 2 \cos(j\pi/n)$, prove that

$$
\prod_{j=1}^{n} (1 + 3a_j^2) = (3^n - 3^{n/2} \cdot 2 \cos(5\pi/6) + 1)^2
$$

and more generally that

$$
\prod_{j=1}^{n} (s^4 + a_j^2) = (x^n + x^{n-1} - x^n - x^{-n})^2 = P_n(t)
$$

where $P_n(t)/F_1(t)$ is a polynomial in $t^2$ with integral coefficients and

$$
x = u + u^{-1},
$$

and $u + u^{-1} = t e^{\pi i/n}$.

Solution by the Proposer.

Setting $a = e^{\pi i/n}$ and factoring (2) we have

$$
\prod_{j=1}^{n} (t^2 - 2^{1/2} t a_j + a_j^2) = \prod_{j=1}^{n} (t^2 + 2^{1/2} t a_j^2) = F_n(t)/F_1(t)
$$

Now the determinant of order $n-1$

$$
d_{n-1}(s) = \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1
\end{vmatrix}
$$

satisfies the second order recurrence relation

$$
d_n = ad_{n-1} - d_{n-2},
$$

so if $s = u + u^{-1}$, we see by induction that

$$
d_{n-1}(s) = (u^n - u^{-n})/(u - u^{-1}).
$$

Since $d_{n-1}$ vanishes for $u^{2n-1} = 1$, its zeros are $s = a_j$, $j = 1, 2, \cdots, n - 1$. Thus

$$
\prod_{j=1}^{n} (t^2 - 2^{1/2} t a_j + a_j^2) = (u^n - u^{-n})/(u - u^{-1}),
$$

if $u + u^{-1} = t$. Now (3) implies
\[(u - u^{-1})^2(a - a^{-1})^2 = (a^2 + u^2 - 4)(a^2 + u^{-2} - 4) = t^4 + 16 = F(t) (9)\]

Hence, by (4), (8), and (9), with \(u, a, u\) given by (3),

\[F_n(t) = (u^n - u^{-n})(a^n - a^{-n}) = x^n + x^{-n} - z^n - z^{-n}. \tag{10}\]

We check directly from (4) that

\[F_1/F_1 = 1, F_2/F_1 = t^2, F_3/F_1 = t^4 + 1, F_4/F_1 = t^6 + 4t^2. \tag{11}\]

Using (3) and (10) we verify that \(x\) and \(z\) satisfy the relations

\[x + z^{-1} = (t^2 + F_1)/2, z + z^{-1} = (t^2 - F_1)/2 \tag{12}\]

\[x^2 - 2 + z^{-2} = t^2(x + z^{-1})^2, z^2 - 2 + z^{-2} = t^2(z + z^{-1}) \tag{13}\]

Hence, by (10) and (13) the functions \(F_n(t)\) satisfy the fourth order recurrence relation

\[F_{n+2} = 2F_n + F_{n-2} = t^2(F_{n+1} + F_{n-1}) \tag{14}\]

We conclude inductively from (11) and (14) that \(F_n(t)/F_1(t)\) is a polynomial in \(t^2\) with integral coefficients.

To prove (1), we now set \(t^4 = 1/3\) in (2) and multiply by \(3^n\). By (9) we obtain \(F_n = 7/3^{n/2}\). and from (12) and (13) we obtain \(x + z^{-1} = u^3/3^{n/2}, x = -z^{-1} = 3/3^{n/2}, z = -z^{-1} = 6^{n/2}\). Hence,

\[x^n = 3^{n/2}, x^n + z^{-n} = 2\cos(5n\pi/6), \quad \text{when} \quad x^h = 1/3, \tag{15}\]

and (1) is proved.

Another interesting case is \(t = 1\), when \(F_1(1) = 17^{1/2}\) and

\[x = 2.081019\cdots, x^n + z^{-n} = 2\cos(n(141.3317\cdots))\tag{16}\]

Then, if \([ \cdot ]\) denotes the greatest integer function, we have

\[\lfloor x^n \rfloor = \lfloor (1 + \sigma^n)^{1/2} \rfloor^{1/2}, \tag{17}\]

where \(x = 2.08101899662\cdots\). These integers are the sums of the coefficients in the polynomials \(F_n(t)/F_1(t)\).

Editor's Note.

The proposer remarked that the problem arose in trying to evaluate and factor some of the symmetric functions of roots of unity that he discussed on pp. 132–135 of the Fall 1975 issue of this Journal in his article on Matrix Functions.

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398. [Spring 1977] Proposed by Richard S. Field, Santa Monica, California.

Find solutions in integers \(A = B = C \neq R\) and \(A \neq B \neq C \neq R\) for the quadrilateral inscribed in a semicircle of radius \(R\), as shown in the figure. Also find solutions in integers \(A \neq B \neq C \neq R\) or prove that none exist.

![Figure 5](image_url)

**FIGURE 5**

1. Solution by Kenneth M. Wilke, Topeka, Kansas.

Since one side of the quadrilateral is a diameter of the circle, it is well known that \(A, B, C\) and \(R\) are related by the equation:

\[(2R)^3 - 2R(A^2 + B^2 + C^2) - 2ABC = 0 \tag{1}\]

or

\[uR^3 - R(A^2 + B^2 + C^2) - ABC = 0 \tag{1}\]

a) \(A = B \neq C \neq R\).

Equation (1) becomes \(4 - (2a^2 + 2^2) - 2^2 \sigma = 0\) or \(\sigma = 2 - a^2\), where \(\sigma \geq 0\) and \(a = A/R\). By letting \(a = p/q\) for some arbitrary integers \(p\) and \(q\), we obtain the parametric solution of (1): \(A = B = pq, C = 2q^2 - p^2\) and \(R = q^2\), where \(p\) and \(q\) are positive and \(p < q\) and \(p \neq q\).

b) \(A \neq B \neq C \neq R\).

Equation (1) becomes \(uC^3 - C(A^2 + B^2 + C^2) - ABC = 0\) or \(3C^2 = A^2 + AB + B^2\). Then since \(3 = 1^2 + 1^2 + 1^2\) and since \(k^2 = m^2 + mn + n^2\) has the parametric solution \(k = p^2 + pq + q^2, m = p^2 - q^2\) and \(n = 2pq + q^2\), equation (1) has the parametric solution \(C + R = p^2 + pq + q^2, A = p^2 - 2pq - 2q^2\) and \(B = 2p^2 + 4pq + q^2\), where \(p, q\) are arbitrary integers such that \(A, B, C\) and \(R\) are positive.

c) \(A \neq B \neq C \neq R\).

Let \(a = A/R\), \(b = BR\) and \(\sigma = C/R\). Then equation (1) becomes
or equivalently,
\[(4 - a^2)(4 - b^2) = (2a + ab)^2\] (2)

Letting \(a = 4p/(p^2 + 1)\) and \(b = 4q/(p^2 + 1)\) implies \(c = 2(p^2 - 1)\).
\[(q^2 - 1) - 8pq]/(p^2 + 1)(q^2 + 1),\] where \(p\) and \(q\) are integers chosen so that \(c\) is positive. Then we have the parametric solution of (1):
\[A = 4p(q^2 + 1), B = 4q(p^2 + 1), C = 2(p^2 - 1)(q^2 + 1) - 8pq \text{ and } R = (p^2 + 1)(q^2 + 1)\.

Equation (1) can be found in Dickson’s History of the Theory of Numbers, Vol. II, p. 220, in the section devoted to rational quadrilaterals, and is attributed to Isaac Newton.

II. Solution by R. Robinson Row. Sacramento, California.

Apparently this solver anticipated the problem with his paper “Primitive Semi-Inscribed Quadrilateral”, published in the Journal of Recreational Mathematics, 3, No. 3, July 1970, pp. 151–157. The solutions listed are derived from Table I, with \(A, B, C = a, b, c\) and \(R = d/2\).

<table>
<thead>
<tr>
<th>(A \neq B \neq C \neq R)</th>
<th>(A \neq B \neq C = R)</th>
<th>(A \neq B \neq C \neq R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 2 7 4</td>
<td>2 1 1 7 7</td>
<td>2 4 1 2 8</td>
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</tr>
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</tr>
<tr>
<td>12 12 2 9</td>
<td>11 3 46 31 31</td>
<td>12 2 22 28 21</td>
</tr>
<tr>
<td>4 4 3 4 16</td>
<td>4 2 2 26 37 37</td>
<td>12 19 33 22</td>
</tr>
<tr>
<td>12 12 2 9</td>
<td>22 6 1 43 43</td>
<td>10 11 25 45 27</td>
</tr>
<tr>
<td>20 20 7 16</td>
<td>23 7 1 49 49</td>
<td>11 39 46 33</td>
</tr>
<tr>
<td>5 5 4 1 25</td>
<td></td>
<td>22 3 42 33</td>
</tr>
<tr>
<td>10 10 4 6 25</td>
<td></td>
<td>17 28 5 35</td>
</tr>
<tr>
<td>15 15 4 1 25</td>
<td></td>
<td>6 25 6 35</td>
</tr>
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<td></td>
<td>14 38 5 35</td>
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<td>6 6 7 1 36</td>
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<td>13 43 5 39</td>
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<td></td>
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<tr>
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<td>14 2 3 9 49</td>
</tr>
</tbody>
</table>

These lists have been limited to primitive sets with \(R\) up to 50. Since all multiples are eligible, there is an infinitude for each of the three categories. Probably there are an infinitude of primitives in each category.

and this is obvious for the first, with the sequence \(A, B, C, R = n, n, 2n^2 - 1, n^2\). So far, at least, \(R\) is a square in the first and congruent to 1 (mod 6) in the second, but rather random in the third category.

A very thorough analysis of the problem was also offered by CLAYTON W. DODGE, University of Maine at Orono.

Comments by the Problem Editor

Charles W. Trigg, the world’s champion indefatigable proofreader, never fails to supply a list of errors and omissions that somehow manage to creep into the Problem Department. The first two paragraphs of Solution II on page 371 of the Spring 1977 issue should have read:

The magic constant of a third order magic square is three times the central element, which therefore is 89.

The nine elements of a third order magic square can be rearranged into a square array in which the elements of the rows and in arithmetic progression with the same common difference, and likewise for the elements of the columns, and conversely.

Trigg also supplied additions to the list of periodicals that contain problem departments, given on page 381 of the Spring 1977 issue:

1) EUREKA, published by Algonquin College, Mathematics Department, Ten issues per year for $8.00.
2) DELTA. Published by the Waukesha Mathematical Society, 1550 University Drive, Waukesha, Wisconsin 53186.
3) NABLA. The Bulletin of the Malayan Mathematical Society, Dept.
5) Published by the Fibonacci Association. Subscription rate: $15.00 per year. Address Professor Leonard Klosinski, Mathematics Department, University of Santa Clara, Santa Clara, California 95053.
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Gold key-pins are available at the National Office (the University of Maryland) at the special price of $5.00 each, post paid to anywhere in the United States. Be sure to indicate the chapter into which you were initiated and the approximate date of initiation.

INITIATION CEREMONY

The editorial staff of the Journal has prepared a special publication entitled Initiation Ritual for use by local chapters containing details for the recommended ceremony for initiation of new members. If you would like one, write to the National Office.

OMISSION IN LAST ISSUE

We regret to report that in our account of the annual meeting in Seattle we omitted the following student presentations:

17. Simple Continued Fractions, David Miyashiro, Ohio Alpha.
18. On Distance Attaining Sets, Robert Goggins, Mississippi Alpha.
19. An Introduction to Coding Theory, Bill Heidler, Ohio Delta.
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