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MATHEMATICAL CURIOSITIES

by Debra Lea Gutridge
Muskingum College

The study of mathematics often leads into some unusual territory. The mathematician is at times confronted with strange geometric figures, odd equations, misbehaved curves, and seemingly insoluble paradoxes. Here an attempt will be made to examine some of the curiosities that interest and challenge mathematicians.

1. Topological Curiosities

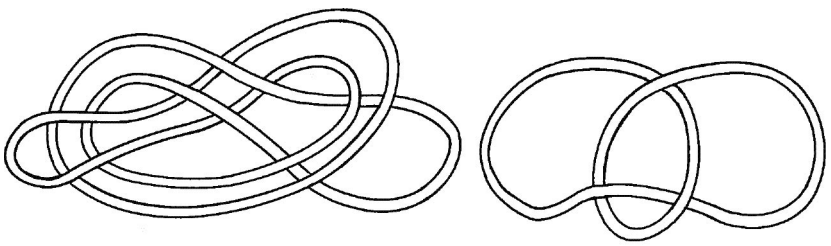
A. *Moebius Strips.* Several interesting mathematical curiosities have come from the area of topology. One very familiar example is the *Moebius strip,* named for August Moebius, a German mathematician. Most of the surfaces met in everyday life are bilateral, or two-sided. A fly placed on one side of a sheet of paper, for example, could not reach the other side unless he cut through the paper or crawled over an edge. However, the Moebius strip is a unilateral closed surface; it has only one side. A fly could crawl from any point on this surface to any other point on the surface without cutting through the strip or going over an edge.

Some properties of this surface can be explained by comparing it to a cylindrical surface made by taking a long rectangular piece of paper and pasting the ends together. This surface has two sides and two edges. A Moebius strip can be made in a similar manner, except that one end of the paper is given a half twist before the ends are joined. The resulting surface has *one side* and *one edge.*

To become convinced of these properties, one has only to start at a point and draw a line down the center of the strip without removing one's pencil from the paper until returning to the starting point. It will be found that a single line has been drawn on what had been *both aides* of the paper before the ends were glued together. Similarly, one can run a pencil around the edge of the strip and find that both edges of the original strip have been colored.

If the original cylindrical strip is cut along its center, it falls apart into two new cylindrical strips each half as wide as the first. However, if the Moebius strip is cut along its center, a single strip twice as long and half as wide as the original is obtained which has a whole twist in it. If this strip is again cut along its middle, two strips are formed which are interlocked. If a half-twisted Moebius strip is cut along a line one-third of the way from its edge, two interlocked strips are formed, one of which has a one-half twist and the other, a whole twist.

These properties of the Moebius strip can be explained in terms of the theory of knots. A closed curve is knotted if it cannot be transformed into a single simple closed curve without cutting and retying. Figure 1-a shows a string that is actually unknotted, while Figure 1-b shows a string that is knotted in the simplest way possible. The above discussion of the Moebius strip can be put in terms of knots as follows. The two edges of the cylindrical strip are neither knotted nor interlocked.



(a) (b)

FIGURE 1

This strip, cut down the middle, falls apart. The single edge of the one-half-twisted Moebius strip is not knotted. This strip, when cut down the middle, becomes one unknotted strip. The two edges of a strip with a whole twist are interlocked but not knotted. When this strip is cut down the center, it forms two interlocked strips. If one end of a strip is turned through one and one-half twists before pasting, the resulting strip has one side and one knotted edge. When this surface is cut down the center, a single strip of the half-twisted type is formed which is itself knotted,

B. *Klein's Bottle*. Another related curiosity is the *Klein* bottle, named for the German mathematician Felix Klein. We are all familiar with the sphere, which is a simple closed surface dividing space into two parts, one inside the sphere and the other outside it. The Klein bottle is also a closed surface like the sphere, but it has no *inside whatever*. If we start at a point outside a sphere and cut through its surface to the inside, we would have to cut through the surface again to get back outside. If we start anywhere and cut through the surface of a Klein bottle, we can follow a path which returns to the place we started without ever cutting the surface again.

A model of a Klein bottle is difficult to make with paper and is usually blown in glass. It is made by taking one end of a hollow glass tube, bending it around, inserting it through a hole in its own side, and joining the two open ends together. The resulting surface is closed, being unbroken in the usual sense at any point. A diagram of the bottle and its cross-section is shown in Figure 2. The Klein bottle, which was

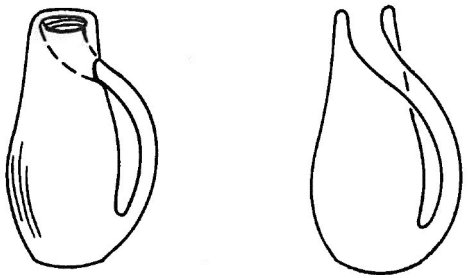


FIGURE 2

not invented originally only for fun, arises naturally from the consideration of a one-sided surface which is closed and has no boundary, but it does exert a certain undefined attraction for the mathematician that other ordinary bottles just do not have.

2. Fourth Dimensional Curiosities

A. *The Hypercube*. The Moebius strip and Klein's bottle are two-dimensional surfaces existing in three dimensions; some other interesting curiosities come from the world of four dimensions. Imagine a line segment moving in a direction perpendicular to itself; it generates a square. If this square moves perpendicular to all of its sides, it

generates a cube. Now imagine the cube moving in a new direction perpendicular to all of its faces; it generates a hypercube, or tesseract. Just as a cube can be represented by a two-dimensional perspective drawing, a hypercube can be represented by a three-dimensional perspective sculpture. The hypercube has 16 corners, 32 edges generated by the 8 corners of the original cube, 24 faces from the 12 edges of the cube, and 8 cubes from the 6 faces of the original cube. Each corner is common to 4 mutually perpendicular edges, to 6 faces, and to 4 cubes; each edge is common to 3 faces and 3 cubes, and each face is common to 2 cubes. Every cube has one face in common with 6 of the 7 others. A hypercube can be unfolded into its component cubes just as a cube can be unfolded into its component squares, as shown in Figure 3. Actually the hypercube is one

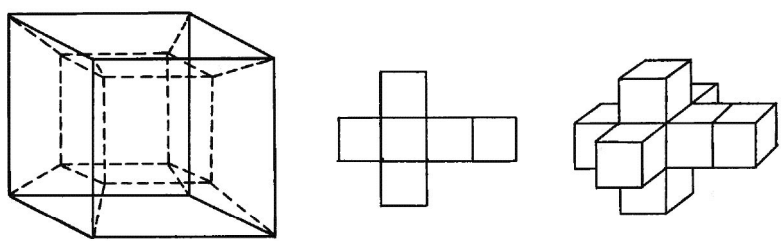


FIGURE 3

of a family of regular four-dimensional polyhedrons. While in three dimensions there are five regular polyhedrons bounded by regular polygons (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron), in hyperspace there are *six* regular hypersolids bounded by regular polyhedrons: C_5, bounded by 5 tetrahedrons; C_8, bounded by 8 cubes; C_{16}, by 16 tetrahedrons; C , by 24 octahedrons; C_{120}, by 120 dodecahedrons; and C_{600}, by 600 tetrahedrons. Models of their projections into three space have all been constructed. Our hypercube, C_8, has right angles throughout and is, therefore, the standard for measuring hyperspace.

8. Other Fourth Dimensional Surprises. The idea of hyperspace gives rise to several curiosities in addition to hypersolids. Just as a three-dimensional creature could look down on this square in its plane (Figure 4) and remove the circle from inside it without disturbing the sides of the square, so a four-dimensional creature could remove one's

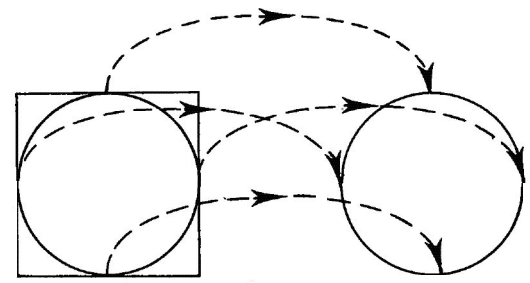


FIGURE 4

appendix without disturbing the skin. Such a creature could also enter any closed vault and become a perfect thief as well as a perfect surgeon. A four-dimensional creature could also untie knots in a string even though both of its ends were anchored (Figure 5). For example, in a

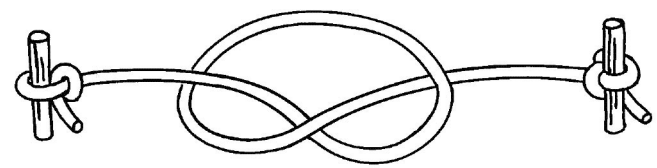


FIGURE 5

two-dimensional plane a knot could look like the diagram in Figure 6-a.

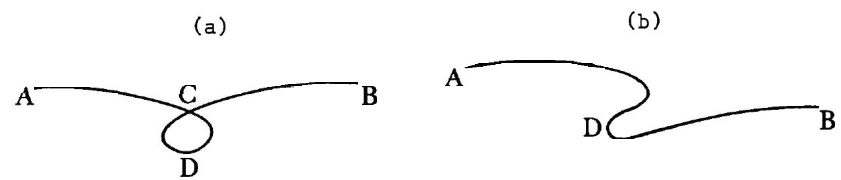


FIGURE 6

In order to untie it, end B must be rotated around *C*. However, a three-dimensional creature could untie the knot by moving the loop through a third dimension. Part *BD* would be turned one-half way over through this third dimension into the position shown in Figure 6-b.

A four-dimensional creature could untie three-dimensional knots similarly by moving some essential part of the knot through a fourth

dimension. Such a creature could also create mirror image reversals in three dimensions. If triangle A is in a plane, as in Figure 7, it cannot be rotated so as to coincide with triangle *B*. However, a three-dimensional

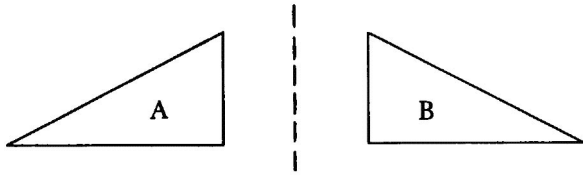


FIGURE 7

creature could pick up A out of its plane, turn it over through a third dimension, and put it back into its plane so that it would coincide with B.

Similarly, a four-dimensional creature could take prism A and make it coincide with prism *B* (Figure 8) by turning it through a fourth dimension. If it could actually be accomplished, this type of reversal

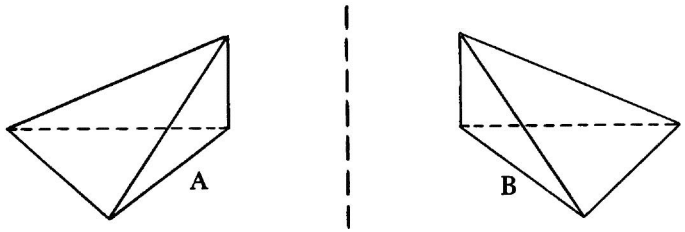


FIGURE 8

could have particularly interesting effects on a living animal since the activity of many biologically useful molecules depends on their three-dimensional orientation, and it would have interesting psychological effects on a human since his right-left orientation would be completely reversed. Of course one need not stop at such four-dimensional curiosities. Fifth, sixth, and n-dimensional curiosities are open to exploration as well.

3. *Curious Curves*

A. *Cycloids*. Some other mathematical curiosities arise from the

study of various curves. One interesting problem goes as follows. The large circle in Figure 9 has made one revolution in rolling without

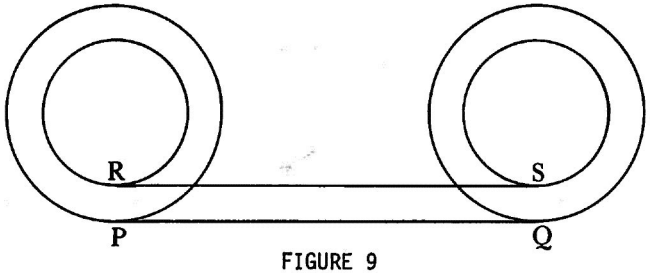


FIGURE 9

slipping along a straight line from P to Q, and so the distance PQ is equal to thecircumferenceof the large circle. However, the small circle, fixed to the large one, has also made one revolution, and so the distance RS is equal to the circumference of the small circle. Since *RS* is equal to PQ, the circumference of the two circles ***must*** be ***equal.***

This strange contradiction can be explained by the fact that while the large circle rolls without slipping, the small one does *slip* in a certain sense. Further explanation involves the concept of a rather curious curve known as the *cycloid.* This curve, shown in Figure 10, is

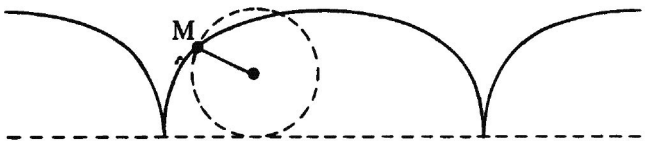


FIGURE 10

the path traced by a fixed point, *M*, on the circumference of a circle as the circle rolls without slipping along a straight line. A fixed point, N, *inside* the *circle* describes what is called a *curtate* cycloid, shown

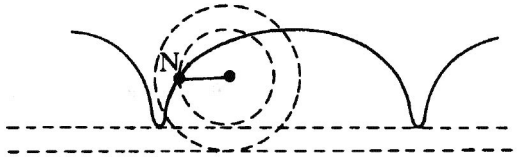


FIGURE 11

in Figure 11, and a point, O, *outside the circle* but attached to it describes a *prolate* cycloid, shown in Figure 12.

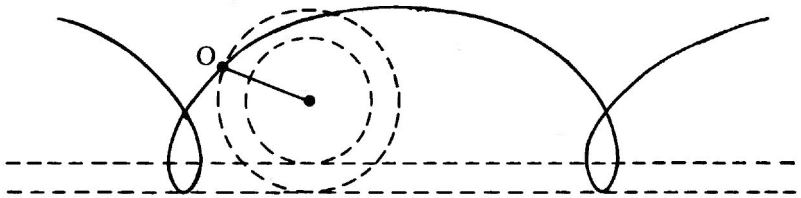


FIGURE 12

Returning to the two-circle problem, let us consider the motion of point M on the circumference of the large circle and that of point N on the circumference of the small circle. As the large circle rolls from P to Q, M describes a cycloid, and N describes a curtate cycloid. Although each wheel makes only one revolution, point M travels farther than point N (as can be seen from Figure 13), and only the common center of the circles travels a distance equal to the straight line PQ.

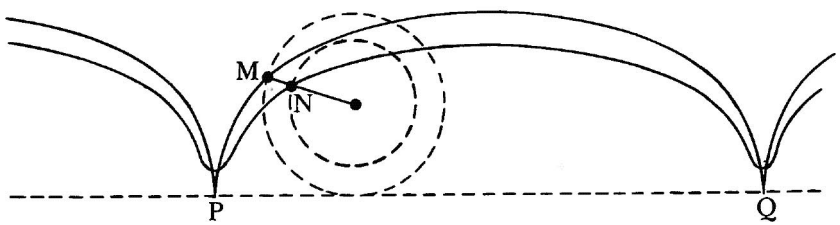


FIGURE 13

The cycloid has several other curious properties. The length of one arch of a cycloid is equal to the perimeter of a square circumscribed about the generating circle. (See Figure 14-a.)

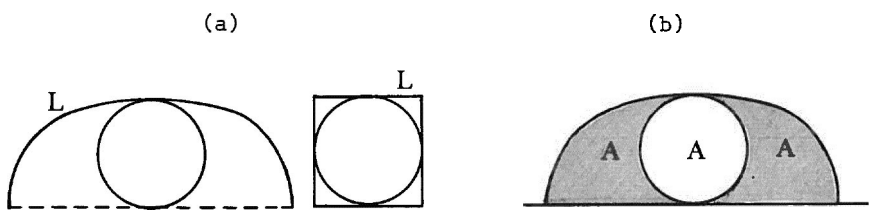


FIGURE 14

The area under one arch of a cycloid is equal to three times the area of the generating circle. Therefore, when the circle is in the position shown in Figure 14-b, the shaded areas on either side of it are each exactly equal to the area of the circle.

The arc of a cycloid is also the path of *quickest descent* between two points. For example, suppose that A and B are two points not in the same horizontal plane and that two balls are released simultaneously at A and are allowed to roll from A to B (Figure 15), and suppose the first

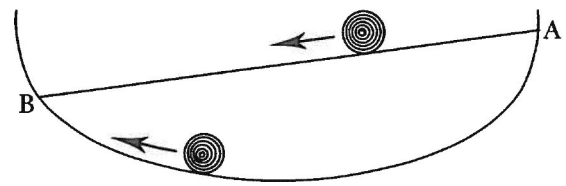


FIGURE 15

rolls along a plane, and the second rolls along a surface in the shape of an inverted cycloid. The second will reach B first in spite of the fact that this path is *longer* and that the second ball has to roll *uphill* before it gets to B. In fact if the plane from A to B is replaced by a surface of any other shape, the ball that rolls along that surface will always get to B later than the one that rolls along the cycloid. This problem, called the *brachistochrone problem*, was proposed to Jacob Bernoulli by his brother Johannes in 1696 and was solved by methods which developed into the field now called the *calculus of variations*.

4. Pathological Curves

Most curves dealt with by mathematicians are fairly innocent and, while they may exhibit a few idiosyncrasies like the cycloid, can be handled by using one technique or another. There are some curves, however, who simply will not behave themselves despite all efforts to bring them under control; these are the pathological curves of mathematics. Before we start to discuss them, the idea of a curve being the limit of a sequence of polygons must be introduced.

Let an equilateral triangle be inscribed in a circle (Figure 16-a); this triangle is curve C_1. As in Figures 16-b,c let C_2 be the regular hexagon obtained by bisecting the three resulting arcs of the circle and

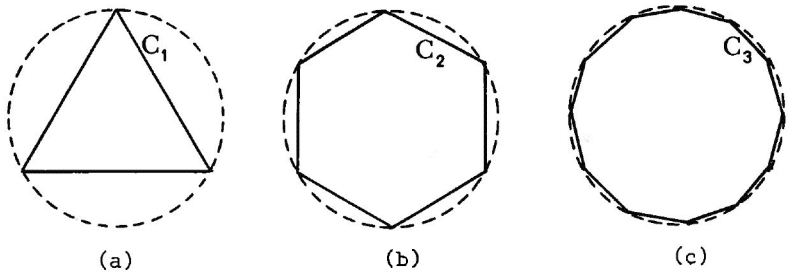


FIGURE 16

by joining, in order, the six vertices, Let C_3 be the regular dodecagon formed by bisecting the six arcs obtained and joining the twelve in order. If this process is repeated, the number of sides of the inscribed curve doubles each time. The curve approached as a limit in this process is the original circle, and so the circle is described as the limit curve of a sequence of curves or polygons. The pathological curves we shall discuss are limit curves like this one.

The first pathological example is the *snowflake* curve. Start with a triangle with sides each one unit in length (Figure 17). As in Figure 17-b, trisect each side of the triangle and erect on each of the

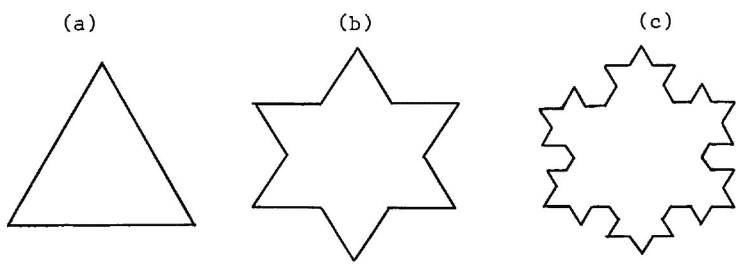


FIGURE 17

middle thirds an equilateral triangle pointing outward. Erase the parts common to the new and old triangles. This simple polygonal curve is obtained. Trisect each side of this curve and upon each middle third erect an equilateral triangle pointing outward. Erase the parts of the curve common to the new and old figures. The curve shown in Figure 17-c is obtained, Now continue this process. The limit curve of the process is the snowflake curve, (It obviously gets its name from the shape it

assumes at successive stages of its development.) This curve is considered pathological because although the curve has a perimeter of infinite length, its area is finite. At each stage of the construction, the perimeter increases, and the sequence of numbers representing the length of the perimeter at each stage does not converge.

This fact can be explained as follows. The perimeter of the triangle was 3. In constructing the second stage, we added six lines of length one-third unit each and subtracted three lines of length one-third unit each. The net result was that we added one unit to the perimeter. Therefore, the length of the second curve is $3 + 1$. Likewise, the perimeter of the third stage is

$$3 + 1 + 4/3;$$

of the fourth,

$$3 + 1 + 4/3 + (4/3)^2.$$

The perimeter at the nth stage is

$$3 + 1 + 4/3 + (4/3)^2 + \cdots + (4/3)^{n-2},$$

and so as n grows, the result grows, since this series does not converge, and the perimeter becomes infinite.

To show that the area is finite, think of a circle circumscribed around the original triangle. Then note that at no subsequent stage of the development will the curve ever extend beyond this circle. Therefore we are confronted by the strange fact that this curve of infinite length can be drawn *on* a small sheet of paper -- on a postage stamp, for instance. It is also not possible to tell at any point on the limit curve the direction in which it is going, and so the tangent line does not exist at any point.

There is also another pathological curve called the *anti-snowflake* curve, which is obtained by drawing triangles inward, not outward, in the above construction. It has the same properties as the snowflake. Its perimeter is infinite, but its area is finite, and no tangent line can be drawn to it. The first four stages are shown in Figure 18.

The in-and-out curve is another pathological example. Draw a circle with radius one and choose six points on it which divide the circumference into six equal parts. Take three alternate arcs and turn them inward. The original circle, C_1, is now the new figure C_2 (Figure 19). The

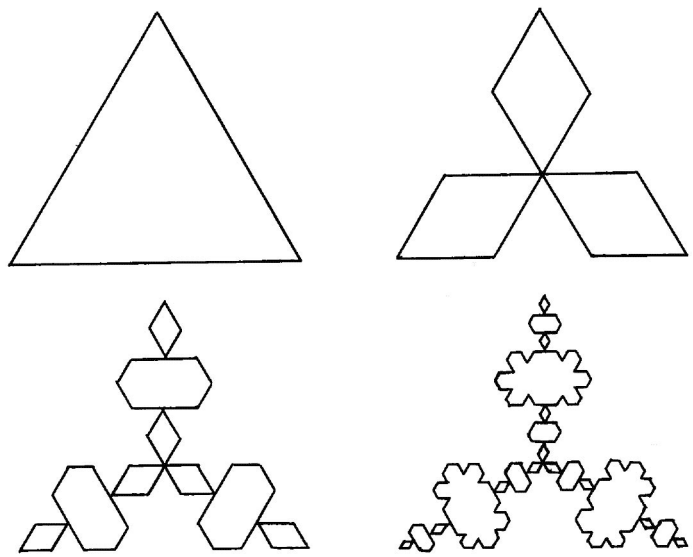


FIGURE 18

perimeter of C_2 is the same as the perimeter of C_1, since its length is not altered by turning three arcs inward, Next trisect each arc and

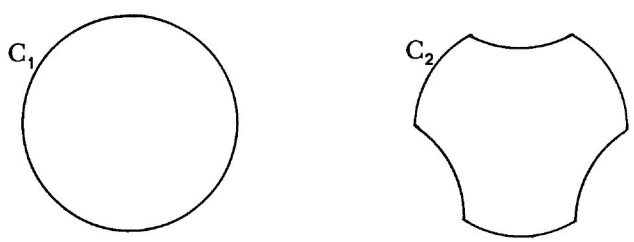


FIGURE 19

turn the middle third outward if it is now inward and turn it inward if it is now outward (Figure 20). This new curve is C_3, and its perimeter

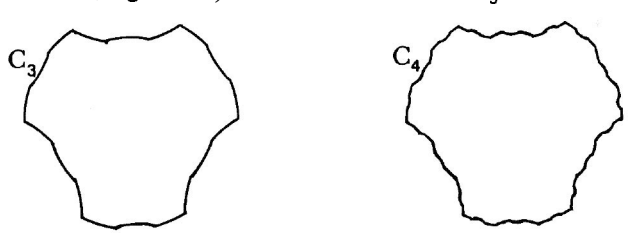
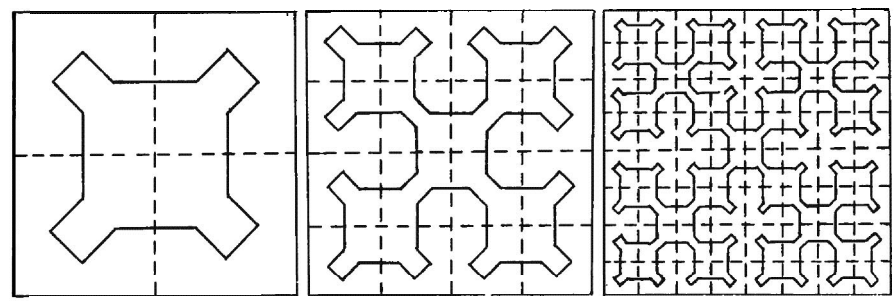


FIGURE 20

is also equal to that of the original circle. The area of C_3 is the same as that of C_2, since we alternately added and subtracted the same sized segments. Keep repeating this process. The limit curve has a perimeter equal to the perimeter of the circle, and its area is equal to that of C_2. While the curvature of the original circle can be computed without difficulty, the in-and-out curve presents a pathological problem in this respect. Consider an arbitrary point on it. Should we measure curvature at this point toward the center of the circle or away from it? There is no definite curvature, and the second derivative does not exist.

Our next pathological specimen is a *space-filling curve,* which seems to refute the idea that a point has zero dimensions and that a curve which is one-dimensional cannot fill a given space. The first member of the curve generating sequence is polygon P_1 inserted in a square as shown below in Figure 21-a. The square is then divided into four equal



(a) (b) (c)

FIGURE 21

squares, and four polygons similar to P_1 are formed and are joined together to form polygon P_2 (Figure 21-b). To get polygon P_3, each of the four squares is divided into four more, and sixteen polygons similar to P_1 are joined together (Figure 21-c). If this process is continued, the result is a sequence of polygon P_1, P_2, P_3, ···. This sequence approaches a limit curve. It can be rigorously shown that this curve passes through any specified point of the square in which it is inscribed, and so it must pass through every point of the square and must completely fill it. It can also be shown that such a one-dimensional curve could also fill an entire cube, hypercube, or figure corresponding to a cube in a space of any number of dimensions.

We have discussed some of the oddities of mathematics -- the Moebius

strip, Klein's bottle, the weird world of the fourth dimension, and some very unusual curves. These are but a few of the strange and wonderful things with which mathematicians deal, and they represent just a few of the areas studied in mathematics. There are many more mathematical curiosities to be examined and explored or to be newly discovered. The mathematical curiosities are there just waiting for a sufficiently curious mathematician.

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LOCAL AWARDS

If your chapter has presented or will present awards this year to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal*.

REFEREES FOR THIS ISSUE

The *Journal* recognizes with appreciation the following persons who willingly devoted their time to evaluate papers submitted for publication prior to this issue, and to serve as judges in the Undergraduate Manuscript Contest: Patrick Lang, Old Dominion University, and members of the Mathematics Department at the University of Oklahoma, Marilyn Breen, Jeffrey Butz, Bradford Crain, John Green, Al Schwarzkopf and Dale Umbach.

The *Journal* also acknowledges with gratitude the expert typing performed by Theresa McKelvey.

THE PERFECT NUMBERS AND PASCAL'S TRIANGLE

by Robert Antol
Iowa State University

The fundamental theorem of arithmetic states that every positive integer can be represented uniquely as the product of prime factors. An integer $n > 1$ shall accordingly be written

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \tag{1}$$

where the p_i's are the distinct prime factors and α_i is the multiplicity of p_i (the number of times p_i occurs in the prime factorization).

A positive integer is called a perfect number if it is equal to the sum of all its positive divisors other than itself. The sum of divisors of a number n with the prime factorization (1) is

$$\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_r^{\alpha_r+1}-1}{p_r-1} = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1}-1}{p_i-1} \tag{2}$$

The condition for a perfect number may then be given by $n = \sigma(n) - n$ or equivalently, $\sigma(n) = 2n$.

Euclid argued that if $2^p - 1$ is prime for $p > 1$, then

$$P = 2^{p-1}(2^p - 1) \tag{3}$$

is a perfect number. Euler showed later that all even perfect numbers must be of this type (see [4]). The number $2^p - 1$ is known as a *Mersenne prime* and is denoted by M_p, as in [3]. All perfect numbers known are even and the question of whether there is an odd perfect number is still unanswered. There is no evidence to prove or disprove the existence of an odd perfect number but if one does exist, it must be greater than 10^{100} [1].

For any positive integer m and any integer k satisfying $0 \le k \le m$,

the binomial coefficient $\binom{m}{k}$ is defined by

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} \tag{4}$$

Use will now be made of the configuration known as *Pascal's triangle* in which the binomial coefficient $\binom{m}{k}$ appears as the $(k+1)$st number in the $(m+1)$st row, as in [5].

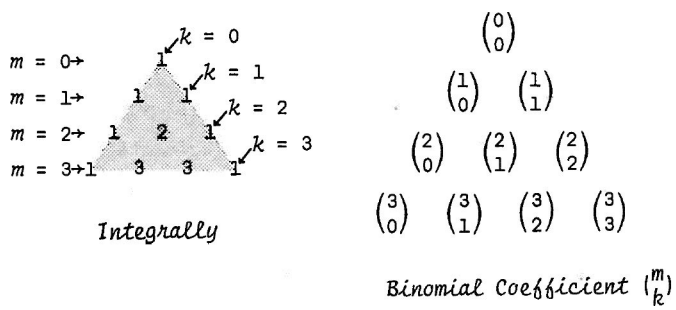


FIGURE 1

The borders of the triangle are composed of ones; a number not on the border is the sum of the two numbers nearest it in the row above.

All even perfect numbers can be shown to lie on the third diagonal of Pascal's triangle (see Figure 1). The restriction for m is that it must be equal to a Mersenne prime plus one; that is, $m = M_p + 1$. Setting k equal to 2 (since the third diagonal of Pascal's triangle is k = 2),

$$\binom{m}{k} = \frac{(M_p+1)!}{2!(M_p+1-2)!} = \frac{2^p!}{2!(2^p-2)!} = \frac{2^p(2^p-1)(2^p-2)!}{2!(2^p-2)!} = \frac{2^p(2^p-1)}{2} = 2^{p-1}(2^p-1) = P,$$

which is an even perfect number by (3) above.

As in [5], we now note that each number in Pascal's triangle is the sum of the numbers in the preceeding diagonal (see Figure 2):

$$\binom{m}{k} = \sum_{i=k-1}^{m-1} \binom{i}{k-1}$$

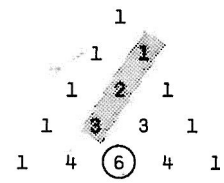
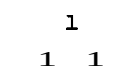


FIGURE 2

We have seen that all even perfect numbers are on the third diagonal of Pascal's triangle. Hence, the second diagonal would generate the perfect numbers. That is, every even perfect number is the sum of the first $2^p - 1 = M_p$ numbers:

$$P = \sum_{i=1}^{M_p} i. \tag{5}$$

We now observe that the elements of the third diagonal are the triangular numbers and every even perfect number is triangular in shape [2]. (See Figure 3.)



The perfect number 6 with base $M_p = 3$

FIGURE 3

According to Burton [1] there are 24 even perfect numbers known to date (1976). The first 5 and their associated Mersenne primes are given in Table 1 on the next page.

We now have several different ways of computing perfect numbers. We must first compute Mersenne primes M_p. Knowing the Mersenne primes,

TABLE 1

p	M_p	$P = M_p(2^{p-1})$
2	3	6
3	7	28
5	31	496
7	127	8,128
13	8,191	33,550,336

we can:

(a) compute $P = M_p(2^{p-1})$, using Euclid's formula,

(b) compute $P = \sum_{i=1}^{M_p} i$, summing up the first M_p positive integers, or

(c) with $m = M_p + 1$ and $k = 2$, compute $P = \binom{m}{k}$.

It is from (c) that we note all even numbers are on the third diagonal of Pascal's triangle.

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SOME INEQUALITIES FOR NON-NEGATIVE RANDOM VARIABLES INVOLVING THE MOMENT GENERATING FUNCTION

by J. Chang, C. Chen, B. Crain, W. Stockwell, M. Truskowski, and B. Wycherley
University of Oklahoma

Various inequalities on moments and probabilities can be derived using the moment generating function. In this article we give some results which may be new.

Let X be a non-negative random variable (r.v.) with cumulative distribution function (c.d.f.) $F(x)$. If $g(X)$ is a function of X, the expected value $E(g(X))$ of $g(X)$ (when it exists), is given by

$$E(g(X)) = \int_0^\infty g(x)dF(x).$$

The expected value of $g(X) = X^k$ is called the kth moment of X. The moment generating function (m.g.f.) $M(t)$ is the expected value of $exp(tX)$, or

$$M(t) = \int_0^\infty e^{tx}dF(x), \qquad t \in \text{domain}(M).$$

Throughout this article, we assume that $Pr(X \geq 0) = 1$. We also assume there exists a positive number h such that $M(t)$ exists (finite) on $-h < t < h$, or equivalently that $M(t)$ exists on a neighborhood of zero. (For a non-negative r.v. X, the m.g.f. $M(t)$ will automatically exist for all $t \geq 0$.)

The following results are not new, but are stated for reference purposes (they assume the existence of the moment generating function on a neighborhood of zero):

1) $E(X^k) < +\infty$ for $k = 1, 2, 3, \cdots$.

2) $M(t)$ is a $C^\infty(-h, h)$ function (continuous derivatives of all orders).

3) $M^{(k)}(t) = \int_0^\infty x^k e^{tx}dF(x)$ for $-h < t < h$, $k = 0, 1, 2, \cdots$.

4) $M^{(k)}(0) = E(X^k)$ for $k = 0, 1, 2, \cdots$.

5) $M(z) = \int_0^\infty e^{zx}dF(x)$ is analytic on $-h < Re(z) < h$.

6) $M(z) = \Sigma_{k=0}^\infty M^{(k)}(0)z^k/k!$ on $|z| < h$.

7) $M(t) = \Sigma_{k=0}^\infty M^{(k)}(0)t^k/k!$ on $-h < t < h$.

The proofs of these results do not appear often in statistical texts since they involve complex variables and real analysis, in particular, the Lebesgue Dominated Convergence Theorem. The reader may wish to consult the references given at the end (in particular, [1], pp. 52-53 and [2], Chapter 7).

Now for every k, and arbitrary positive A, we have

$$M^{(k)}(t) = \int_0^\infty x^k e^{tx}dF(x) = \int_0^A x^k e^{tx}dF(x) + \int_{A+0}^\infty x^k e^{tx}dF(x), \quad -h < t < h.$$

I. First consider the inequality $M^{(k)}(t) \geq \int_0^A x^k e^{tx}dF(x)$. If $0 \leq t < h$ we have

$$M^{(k)}(t) \geq \int_0^A x^k e^{tx}dF(x) \geq \int_0^A x^k dF(x). \tag{1.1}$$

Dividing through by $F(A) = Pr(0 \leq X \leq A)$, we have

$$\frac{M^{(k)}(t)}{F(A)} \geq \int_0^A x^k d\left[\frac{F(x)}{F(A)}\right], \qquad 0 \leq t < h, \tag{1.2}$$

where $F(x)/F(A)$ is the c.d.f. of the X-distribution, truncated to $[0, A]$. Taking the infimum over t, we have

$$\inf_{0\leq t<h} \frac{M^{(k)}(t)}{F(A)} \geq \int_0^A x^k d\left[\frac{F(x)}{F(A)}\right], \tag{1.3}$$

where the right-hand side is the kth moment of the truncated version of X.

Next consider $-h < t \leq 0$. We have

$$M^{(k)}(t) \geq \int_0^\infty x^k e^{tx}dF(x) \geq e^{At} \int_0^A x^k dF(x). \tag{1.4}$$

Dividing through by $e^{At}F(A)$, we have

$$\frac{e^{-At}M^{(k)}(t)}{F(A)} \geq \int_0^A x^k d\left[\frac{F(x)}{F(A)}\right], \qquad -h < t \leq 0. \tag{1.5}$$

Taking the infimum over t, this becomes

$$\inf_{-h<t\leq 0} \frac{e^{-At}M^{(k)}(t)}{F(A)} \geq \int_0^A x^k d\left[\frac{F(x)}{F(A)}\right]. \tag{1.6}$$

Taking the limit as $A \to \infty$, we have

$$\lim_{A\to\infty} \inf_{-h<t\leq 0} e^{-At}M^{(k)}(t) \geq E(X^k). \tag{1.7}$$

11 Second, consider. $M^{(k)}(t) \geq \int_{A+0}^\infty x^k e^{tx}dF(x)$. If $0 \leq t < h$, we have

$$M^{(k)}(t) \geq \int_{A+0}^\infty x^k e^{tx}dF(x) \geq A^k e^{At} \int_{A+0}^\infty dF(x) \tag{2.1}$$
$$= A^k e^{At} Pr(X > A).$$

After division, we obtain

$$\frac{e^{-At}M^{(k)}(t)}{A^k} \geq Pr(X > A). \tag{2.2}$$

By taking infimums, we then have

$$\inf_{0\leq t<h} \inf_{k\geq 0} \frac{e^{-At}M^{(k)}(t)}{A^k} \geq Pr(X > A) = 1 - F(A) \tag{2.3}$$

and

$$\inf_{k\geq 0}\ \inf_{0\leq t<h} \frac{e^{-At}M^{(k)}(t)}{A^k} \geq Pr(X > A) = 1 - F(A). \tag{2.4}$$

If $-h < t \leq 0$, the basic inequalities are

$$M^{(k)}(t) \geq \int_{A+0}^{\infty} x^k e^{tx} dF(x) \geq A^k \int_{A+0}^{\infty} e^{tx} dF(x). \tag{2.5}$$

After division by $A^k(1 - F(A))$, we then get

$$\frac{M^{(k)}(t)}{A^k(1 - F(A))} \geq \int_{A+0}^{\infty} e^{tx} d\left[\frac{F(x)}{1 - F(A)}\right], \tag{2.6}$$

where $F(x)/(1 - F(A))$ is the c.d.f. of X truncated to (A, ∞), and the right-hand side of (2.6) is the m.g.f. of such a distribution, which will exist at least on $(-h, h)$. Again taking the infimum on k, one sees that

$$\inf_{k\geq 0} \frac{M^{(k)}(t)}{A^k(1 - F(A))} \geq \int_{A+0}^{\infty} e^{tx} d\left[\frac{F(x)}{1 - F(A)}\right], \tag{2.7}$$

and

$$\lim_{A\to 0}\ \inf_{k\geq 0} \frac{M^{(k)}(t)}{A^k} \geq \int_0^{\infty} e^{tx} dF(x) = M(t), \tag{2.8}$$

provided X does not have an atom at zero, that is, provided $Pr(X = 0) = 0$.

A multitude of inequalities become evident if one selects $0 < A < B < \infty$ and then writes

$$M^{(k)}(t) = \int_0^A x^k e^{tx} dF(x) + \int_{A+0}^{B} x^k e^{tx} dF(x) + \int_{B+0}^{\infty} x^k e^{tx} dF(x)$$

for $k = 0, 1, 2, \cdots$ and $-h < t < h$. Since there are so many cases, we leave the details to the reader. Various other generalities can be seen to exist, such as deleting the requirement that X be non-negative. In addition, for a non-negative r.v. X, the moment generating function $M(t)$ exists for all $t \leq 0$, and hence some of the inequalities such as (1.6), (1.7) may be tightened.

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REGIONAL MEETINGS OF MAA

Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

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Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

ANOTHER APPROACH TO A PAIR OF FAMILIAR SEQUENCES

by N. *Schaumberger*
Bronx Community College, CUNY

A simple proof that the sequence $S_n = (1 + \frac{1}{n})^n$ approaches a limit can be based on:

(1) $S_n = (1 + \frac{1}{n})^n$ is an increasing sequence, and

(2) $T_n = (1 + \frac{1}{n})^{n+1}$ is a decreasing sequence.

The usual proofs of these statements employ either the binomial theorem or the inequality between the arithmetic and geometric means (see [1]). The first method is somewhat messy and the second uses an inequality that is rarely derived in elementary calculus. In this note we offer simple proofs of (1) and (2) that fit naturally into the sequence of calculus topics in a standard course.

Let

$$f(x) = (n + 1)ax - x^{n+1}$$

where n is a positive integer and $a > 0$, $x > 0$. Since

$$f'(x) = (n + 1)a - (n + 1)x^n$$

and

$$f''(x) = -n(n + 1)x^{n-1}$$

it follows that $f(x)$ attains an absolute maximum at $x = a^{\frac{1}{n}}$. Thus if $a \neq 1$, we have

$$f(a^{\frac{1}{n}}) > f(1)$$

or

$$na^{\frac{n+1}{n}} > (n + 1)a - 1.$$

Putting $a = 1 + \frac{1}{n+1}$ gives

$$n(1 + \frac{1}{n+1})^{\frac{n+1}{n}} > n + 1$$

or

$$(1 + \frac{1}{n+1})^{n+1} > (1 + \frac{1}{n})^n.$$

This proves (1). To prove (2), we put $a = 1 - \frac{1}{n+1}$. Thus

$$n(1 - \frac{1}{n+1})^{\frac{n+1}{n}} > n - 1$$

or

$$(\frac{n}{n+1})^{n+1} > (\frac{n-1}{n})^n.$$

Inverting finally gives

$$(1 + \frac{1}{n})^{n+1} < (1 + \frac{1}{n-1})^n.$$

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Saturday, September 30, 1978

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This is a call for undergraduate student papers; we invite you to join us. The student conference will be Saturday afternoon. Student speakers will receive their picnic lunch free. Talks may be on any topic related to mathematics, statistics or computing. Do you have a favorite topic which you could share with us? We welcome items ranging from expository to research, interesting applications, problems, etc. Presentation time should be fifteen, thirty, or forty-five minutes. We need your title, presentation time, and a 50 (approx.) word abstract by September 18, 1978. Please send your material to Professor Cox (address below).

We also urge you to attend the Conference on Applications of Statistics and Mathematics, which begins Friday afternoon, September 29. Free overnight facilities for all students will be arranged with Miami students.

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PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword puzzles and find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinquish material which might well be the domain of the Problem Department, but the Editor reserves the right to make an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate, or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to the Editor, Pi Mu Epsilon Journal, 601 Elm Avenue, Room 423, The University of Oklahoma, Norman, Oklahoma 73019. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March 31, and that for each Spring issue, the following September 30.

Mathacrostic No. 5

submitted by R. Robinson Rowe
Sacramento, California

Identify the 32 key words, matching their letters in order with the opposite sequence of numbers, and insert each letter of the key words in the square of the Mathacrostic with the same number. Words end at the blank squares, and some words extend on to the next line.

V 1	B 2		c 3	c 4		V 5	b 6	c 7	V 8	c 9	V 10	c 11	E 12	F 13
c 14	V 15	V 16	E 17		P 18	V 19	c 20	V 21		F 22	B 23	V 24	V 25	c 26
F 27	F 28	B 29	B 30	c 31	B 32		G 33		E 34	V 35	c 36	F 37		B 38
B 39	c 40	N 41	E 42	B 43		V 44	b 45	c 46		F 47	B 48	c 49		b 50
	T 51	F 52	B 53	e 54	F 55		W 56	H 57	E 58	L 59		H 60	F 61	E 62
b 63	B 64	F 65		P 66	F 67		N 68	G 69	E 70	Z 71		F 72	e 73	b 74
	G 75	H 76	B 77	G 78	T 79		e 80	G 81		V 82	e 83		G 84	X 85
	e 86	b 87	F 88	U 89	G 90	L 91	G 92		G 93	c 94	L 95	b 96	e 97	H 98
I 99	G100	V101	e102		Y103	e104	L105		C106	H107		T108	T109	T110
	H111	E112	Z113	T114	L115	d116	T117	G118		T119	b120		S121	N122
E123	C124	N125	C126		H127	C128		P129	b130	b131		Y132	C133	C134
e135		J136	P137		N138	X139	A140	D141		E142	E143	G144	P145	D146
U147	H148	D149	d150		P151	N152	D153	X154	U155		L156	b157		Y158
	Z159	K160	U161	Z162	e163	E164	X165	N166		Z167	H168	Q169		d170
	e171	J172	H173	L174	R175		e176	d177		C178	Q179	J180	L181	R182
X183	A184	O185	R186	N187	H188	L189		W190	P191	M192	W193		O194	D195
R196		S197		B198	E199	X200	M201	d202		W203	G204	X205	D206	U207
a208	S209	d210	V211	M212		D213	A214	R215	a216	U217	Q218		X219	C220
U221		P222	W223	Y224		O225	I226	J227	a228	T229	S230	A231		K232
f233	M234	C235		Z236	A237	O238	J239		S240	M241	Q242	Q243	f244	
a245	f246		J247	W248		O249	Y250	f251	b252	f253		a254	K255	
K256	I257		K258	Y259	R260									

Definitions and Key

A. Weigh anchor
B. Same in static energy
C. Towards **Nome**
D. Umpire
E. Stress-strain ratio
F. Calculus task
G. Absolute, in math
H. Relating integers
I. Geologic period
J. Mathematician 1642-1727
K. Crazy bone
L. Michigan campus
M. Greek letter
N. King of Judah ca 750 BC
O. Moon's age on Jan. 1
P. **Melampyrum** spp
Q. Mathematical snake
R. Most recent
S. Language in S. India
T. Kitchen tool
U. Dream
V. Famous cycloid
W. Maximum effort
X. The rabble
Y. **Achillea** spp
Z. **Asimina triloba**
a. City of the Mohawk
b. Achilles story
c. Persian creed
d. Scanned (a book)
e. Mathematical inventor
f. His invention

237 214 184 140 231
43 198 39 30 38 48 2 53 32 29 77 23 64
235 220 126 124 133 106 134 178 128
195 141 206 146 213 149 153
17 62 143 58 34 123 + 112 70 12 164 42 199 142
37 13 22 47 61 52 65 27 55 88 28 72 67
204 78 144 81 69 118 90 75 93 84 100 33 92
148 111 188 60 76 57 168 98 127 173 107
257 226 99
239 172 136 180 227 247
232 160 256 255 258
59 95 174 156 115 91 189 105 181
201 241 234 212 192
138 166 41 125 68 187 122 152
238 249 194 185 225

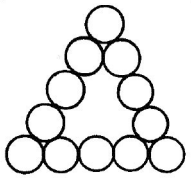
145 222 18 66 191 137 129 151
179 169 218 242 243
260 175 215 196 182 186
240 197 121 209 230
110 79 51 229 114 108 119 117 109
217 147 89 155 221 207 161
82 8 15 25 19 1 24 21 10 101 44 16 35 211 5
223 190 56 248 203 193
154 200 85 183 165 139 205 219
103 158 224 250 259 132
113 162 71 159 167 236
228 254 245 208 216
252 74 130 96 157 + 63 45 87 50 131 120 6
40 31 11 36 9 4 7 49 3 20 14 26 46 94
116 202 170 177 210 150
83 97 54 80 176 163 86 73 102 135 104 171
246 251 244 233 253

When completed, the Mathacrostic will be a 260-letter quotation, and the 32 initial letters of the key words will spell out the name of an author and title of his book from which the quotation was derived. It will tell you something human about yourself. There is a one-to-one correspondence between the 260 letters in the Mathacrostic and the 260 letters of the key words, so it is also an anagram.

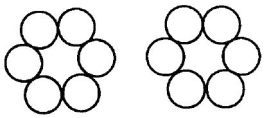
Geometry From A Dozen Pennies

A mathematics teacher wants to illustrate some geometric patterns by arranging 12 pennies in various ways without having to use compass, straight-edge, measuring instruments or marking devices of any kind. To make it more of a challenge, the teacher has decided only to slide the pennies in arranging them and disallow the freedom of lifting the coins. Under these restrictions, show how each of the following figures may be created precisely.

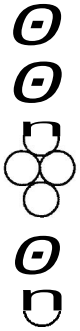
An Equilateral Triangle:



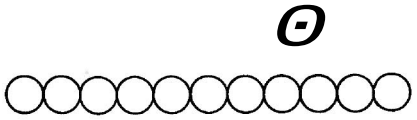
Two Circles of Equal Radii:



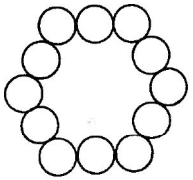
Perpendicular Lines:



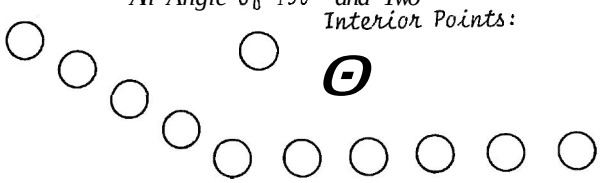
A Line and a Point Not on It:



A Regular Hexagon:



An Angle of 150° and Two Interior Points:



Mathematical Word Chains: One Letter Changes

Find a sequence of legitimate words (disallowing proper nouns and abbreviations) starting with the first word given and ending with the one below it if you are allowed to change only one letter at a time in proceeding from one word to the next.

(a) **MATH** / **NOTE** (b) **LINE** / **SURD** (c) **ZERO** / **ROOT** (d) **SEVEN** / **PRIME**

Example. A solution to **MATH → NOTE** is:

MATH
MATE
RATE
ROTE
NOTE

Mathematical Word Chains: Two Letter Changes

Follow the same instructions as in the preceding puzzle, except you are allowed (and required) to change exactly two letters at a time.

(a) **LINES** / **CURVE** (b) **CIRCLE** / **BEAUTY** (c) **GROUP** / **FIELD** (by way of **RINGS**)

(d) **SLOPE** / **EQUAL** (e) **LINEAR** / **ORDERS**

Countdown

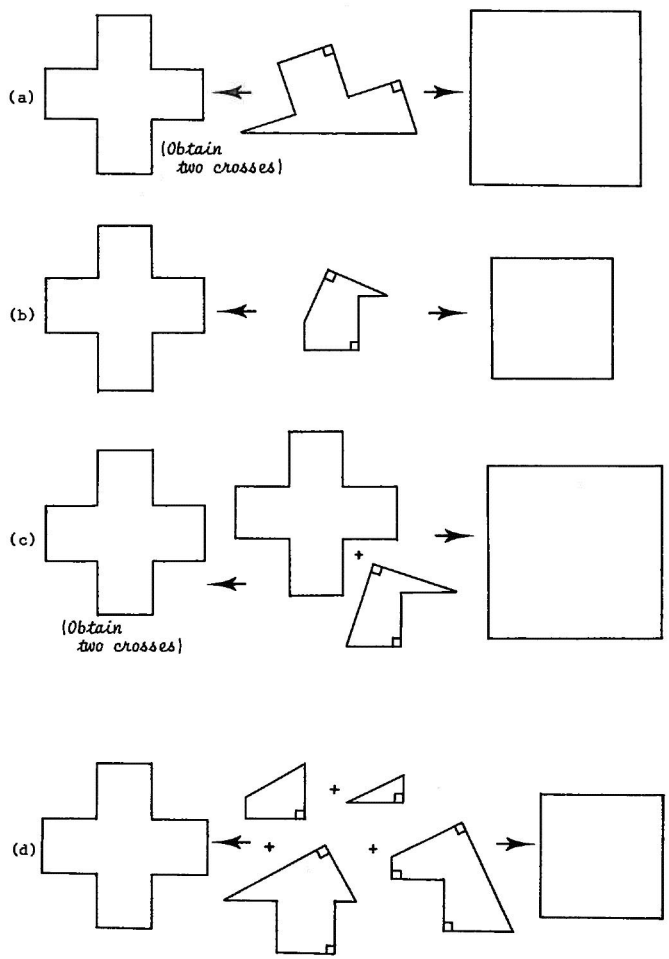
Find all possible solutions to the following long division problem:

```
                 9 * * * * * * * *
* * * * ) * * 8 * * * * * * * * *
          * * * *
            * 7 * *
            * * * *
              * 6 * *
              * * * *
                * 5 * *
                * * * *
                  * 4 * *
                  * * * *
                    * 3 * *
                    * * * *
                      * 2 * *
                      * * * *
                        * 1 * *
                        * * * *
                          * 0 * *
                          * * * *
```

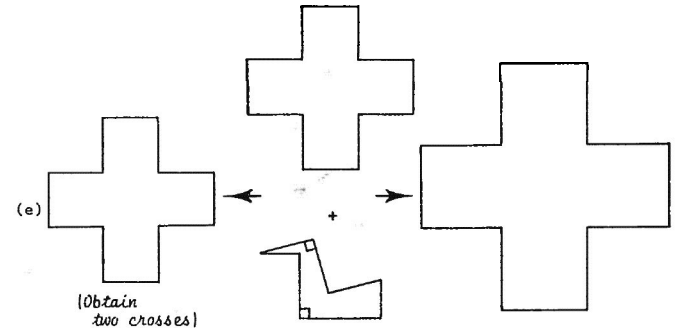
Greek Crosses and Squares

Since no solutions were received for this puzzle which appeared in the Fall, 1976 issue of this *Journal*, we present it again. Solver must use one or more of the figures shown in the center column to piece together and form the figure on the left, then rearrange to form the figure on the right, in each case.

End with: *Start with:* *End with:*



End with: *Start with;* *End with:*



Solutions

Missionaries and Cannibals [Fall, *1976*]

A late solution was received by ROGER E. KUEHL, *Kansas City, Missouri, who observed the optimum number of* 13 *crossings and raised the question of whether all or some "river-crossing" problems have general solutions and whether it is indeed possible to prove that a (trial and error) solution to a particular one is minimum. (See Spring,* 1977 *issue for solution.)*

Mathacrostic No. 3 [Spring, 19771

Late. solutions were received by MITCH ENTRICAN, *University of Mississippi and* SISTER STEPHANIE SLOYAN, *Georgian Court College (Lakewood, New Jersey).*

A Pair of Eights [Spring, 19771

Late solutions were received by MARK EVANS, *LaMarque, Texas and* B. FRANK WILLIAMS, *Campbell, Texas.*

Mathacrostic No. 4 [Fall , 1977]

Definitions and Key:

A.	Looking glass	F.	Conjugated	K.	Lethe	P.	Halfway
B.	Esthete	G.	Awash	L.	Luffed	Q.	Uther
C.	Wonderland	H.	Rebaked	M.	Twenty thousandths	R.	Norman
D.	Isohyet	I.	Rhythm	N.	Habitude	S.	Tetrahedron
E.	Symbolic	J.	Ottawa	O.	Eureka	T.	Ivy
U.	Nth	Z.	Hashish	c.	Rhomboid		
V.	Gushed	a.	Effete	f.	Kowtows		
W.	Oddity	b.	Shorthanded				
X.	Fahrenheit	c.	Nutty				
Y.	Theory	d.	Astronomy				

First Letters: LEWIS CARROL THE HUNTING OF THE SNARK

Quotation: *They found not a button or feather by which they could tell that they stood on the ground where the baker had met with the snark. In the midst of the word he was trying to say, in the midst of his laughter and glee, he had softly and suddenly vanished may, for the snark was a bojum you see.*

Solved by JEANETTE BICKLEY, *Webster Groves High School (Missouri);* MITCHELL W. ENTRICAN, *University of, Mississippi;* BARBARA LEHMANN. *St. Peters College;* JODI L. LEVESQUE, *University of, Florida;* SIDNEY PENNER, *Bronx Community College of CUNY;* BOB PRIELIPP, *University of, Wisconsin at Oshkosh;* LÉO SAUVÉ, *Algonquin College (Ottawa, Canada);* and ALLAN TUCHMAN and PATRICIA GROSS, *University of Illinois.*

Cross-number Puzzle. [Fall, 1977]

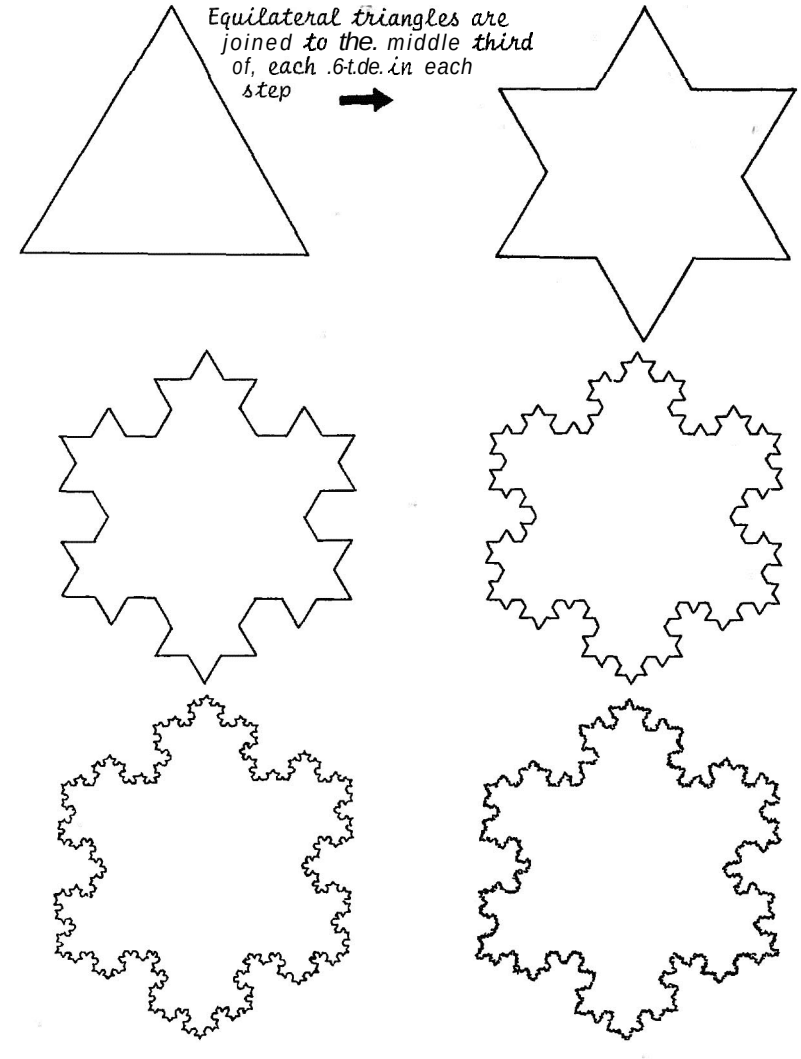
9	2	1	6		7		6	2	4	1
1		6	0	3	9	3	9	6		2
1	1		9	4	2	4	9		4	9
9	6	8		3	4	3		6	6	6
	2	7	6		1		2	5	6	
4	9	8	5	6		6	6	5	6	6
	6	7	6		5		6	5	6	
2	5	6		1	1	2		5	6	1
4	6		4	7	7	7	1		6	9
0		2	4	0	3	6	2	8		9
1	3	3	1		6		1	0	5	8

Solved by JEANETTE BICKLEY, *Webster Groves High School (Missouri);* MITCHELL W. ENTRICAN, *University of, Mississippi;* MICHAEL HANEY, ALLAN TUCHMAN, *and* PATRICIA GROSS, *University of, Illinois;* CLARK HEISER, *Yonkers, New York;* BECKY HENNING, *University of California at Lob Angeles;* SIDNEY PENNER, *Bronx Community College of, CUNY;* BOB PRIELIPP, *University of, Wisconsin at Oshkosh; and* R. ROBINSON ROWE, *Sacramento, California.*

UNDERGRADUATE RESEARCH PROPOSAL

by James Jones and James Trott
University of, Oklahoma

A number of elementary facts may be easily observed in the sequence of diagrams below. For example, if it is assumed that the original tri-



angle has sides of length one, the total perimeter of each figure and area enclosed may be computed.

1. Carry this out, and determine what happens when these computations are performed for each of the figures as the sequence of constructions is carried on ad infinitum.
2. What would happen if the new triangles pointed inward instead of out? What would the new figures look like? What would be the answer to the above question in this case?
3. What would happen if we started with some regular polygon other than the triangle (say a square, pentagon or hexagon) and erected similar (but smaller) n-gons on the middle one-third of each side in each step? What would the six figures look like for n = 4, 5, or 6? What would the answer be for each of the above questions?
4. What if the triangles (or n-gons) were pointed inward on one step, outward on the next, alternating at each step? Generalize.

Comment by Editor

See in this connection the article in this issue, "Mathematical Curiosities" by Debra Gutridge (§4).

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Lob Angeles, California

This department welcomes problems believed -to be. n w and at a level appropriate for the. readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The. choice. of proposals for publication will be. bated on the. editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be. accompanied by solutions if available and by any information that will assist the. editor. Challenging conjectures and problem proposals not accompanied by solutions will be. designated by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on a separate sheet properly identified with name. and address and mailed before November 1, 1978.

Address all communications concerning this department to Dr. Lion Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90042.

Problems for Solution

412. *Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, California.*

Are there examples of angles which are trisectible but not constructible? That is, can you find an angle a which is not constructable with straight edge and compass, but such that when a is given, $\alpha/3$ can be constructed from it with straight edge and compass?

413. *Proposed by R. Robinson Rome., Naubinway, Michigan and Sacramento, California.*

In a variation of the crossed-ladders-in-an-alley classic, the new tall building on one side of the alley was vertical, but on the other side the old low building, having settled, leaned toward the alley. Projected, its face would have met the top of the tall building and would have been one foot longer than the height of the tall building.

The ladders, unequal in length, rested against the buildings 21 feet above the ground and crossed 12 feet above the ground. How high was the tall building and how wide was the alley?

414. *Proposed by Steven S. Conrad, Benjamin N. Cardozo High School, Bayside, New York.*

In discussing the discriminant of a quadratic equation, a certain textbook says, "...if a, b and c are integers with $a \neq 0$ and if $b^2 - 4ac = 79$, the roots of $ax^2 + bx + c = 0$ will be real, irrational and unequal." Explain why this is incorrect.

415. *Proposed by Charles W. Trigg, San Diego, California.*

A hexagonal number has the form $2n^2 - n$. In base 9, show that the hexagonal number corresponding to an n that ends in 7 terminates in 11.

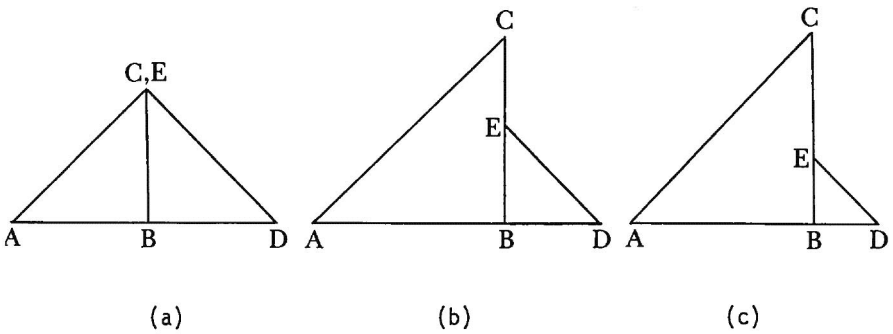


FIGURE 1

416. *Proposed by Scott Kim, Rolling Hills Estates, California.*

Each of the three figures shown above is composed of two isoceles right triangles, ΔABC and W E, where $\angle ABC$ and $\angle DBE$ are right angles, and B is between points A and D. Points C and E coincide in Figure la, so that $CB/EB = 1$. In Figure lb, we are given that $CB/EB = 2$, and in Figure lc, we are given that $CB/EB = 3$. Consider each pair of triangles as a single shape and suppose that the areas of the three shapes are equal. (The figures are not drawn to scale,) Problem: For each pair of figures, find the minimum number of pieces into which the first figure

must be cut so that the pieces may be reassembled to form the second figure. Pieces may not overlap, and all pieces must be used in each assembly.

417. *Proposed by Clayton W. Dodge., University of Maine., Orono, Maine.*

1) Prove that the line joining the midpoints of the diagonals of a quadrilateral circumscribed about a circle passes through the center of the circle.

2) Let the incircle of triangle ABC touch side BC at X. Prove that the line joining the midpoints of AX and BC passes through the incenter I of the triangle.

418. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find all angles 9 other than zero such that tan 116 = tan 1119 = tan 11119 = tan 111116 = ···.

419. *Proposed by Michael W. Ecker, City University of New York.*

Seventy-five balls are numbered 1 to 75 and are partitioned into sets of 15 elements each, as follows: $B = \{1, \cdots, 15\}$, $I = \{16, \cdots, 30\}$, $N = \{31, \cdots, 45\}$, $G = \{46, \cdots, 60\}$, and $O = \{61, \cdots, 75\}$, as in Bingo.

Balls are chosen at random, one at a time, until one of the following occurs: At least one from each of the sets B, I, G, O has been chosen, or four of the chosen numbers are from the set N, or five of the numbers are from one of the sets B, I, G, O.

Problem: Find the probability that, of these possible results, four N's are chosen first. (Comment: The result will be approximated by the situation of a very crowded bingo hall and will give the likelihood of what bingo players call "an N game", that is, bingo won with the winning line being the middle column N.)

420. *Proposed by Herbert Taylor, South Pasadena, California.*

Given four lines through a point in 3-space, no three of the lines in a plane, find four points, one on each line, forming the vertices of aparallelogram. (This is a variation of problem B-2 on the December 1977 William Lowell Putnam Mathematical Competition.)

421. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

If $F(x, y, z)$ is a symmetric increasing function of x, y, a, prove

that for any triangle, in which w_a, w_b, w_c are the internal angle bisectors and m_a, m_b, m_c, the medians, we have

$$F(w_a, w_b, w_c) \le F(m_a, m_b, m_c)$$

with equality if and only if the triangle is equilateral.

422. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

If perpendiculars are erected outwardly at A, B of a right triangle ABC ($C = 90°$), and at M, the midpoint of AB, and extended to points P, Q, R such that $AP = BQ = MR = AB/2$, show that triangle PQR is perspective with triangle ABC.

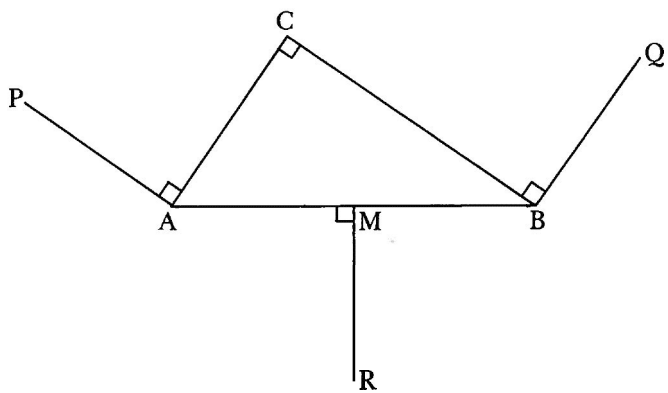


FIGURE 2

423. *Proposed by Richard S. Field, Santa Monica, California.*

Find all solutions in positive integers of the equation $a^d - b^d = c^c$, where d is an odd integer.

424. *Proposed by R. S. Luther, University of Wisconsin, Janesville.*

Prove that

$$\left(x^{1/n} + y^{1/n}\right)^n > \left(\frac{x - y}{\ln x - \ln y}\right)(2n + 2)$$

where n is an odd integer ≥ 3 and $0 < y < x$.

Solutions

386. *[Spring 1977] Proposed by Charles W. Trigg, San Diego, California.*

Show that the volume of Kepler's Stella Octangula (a compound of two interpenetrating tetrahedrons) is three times that of the octahedron that was stellated.

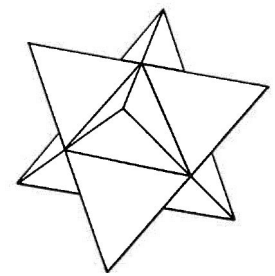


FIGURE 1

Solution by Kenneth M. Wilke, Topeka, Kansas, with practically identical solutions by Clayton W. Dodge, University of Maine, Orono; R. Robinson Rouse, and the proposer, Charles W. Trigg.

Since a plane through the midpoints of the three edges of a tetrahedron issuing from one vertex cuts off a smaller tetrahedron whose volume is one-eighth that of the larger (similar) tetrahedron, repeating this process three more times decomposes the given tetrahedron into four smaller identical tetrahedrons and a regular octahedron whose volume is one-half the volume of the initial tetrahedron. In the formation of the Stella Octangula by interpenetrating two identical tetrahedrons, the octahedrons contained therein occupy the same space, leaving a solid composed of eight smaller tetrahedrons and one octahedron. Hence the volume of the Stella Octangula is one and one-half times the volume of the initial tetrahedron and three times the volume of the stellated octahedron contained therein.

Also solved by LOUIS H. CAIROLI, Graduate Student, Kansas State University, Manhattan, Kansas; MARK JAEGER, Chicago, Illinois; and SISTER STEPHANIE SLOYAN, Georgian Court College, Lakewood, New Jmey.

Comments by the Problem Editor:

1) R. Robinson Rowe noted that if each of the equilateral triangular faces has an area A, the Stella has a surface area of 24A and the octahedrom an area of 8A, so again the ratio of Stella to octahedron, by area, is three.

2) The two interpenetrating tetrahedrons are, of course, regular and the eight vertices are the vertices of a cube. Furthermore, the twelve edges of the two tetrahedrons are the diagonals of the six faces of the cube. The interested reader can find additional material on Kepler's *Stella Octangula* in Regular Figures, by L. Fejes Toth, Mathematical Essays and Recreations, by W. W. Rouse Ball and H. S. M. Coxeter, Mathematical Snapshots, by H. Steinhaus, Introduction *to* Geometry, by H. S. M. Coxeter, and in Cundy and Rollett's Mathematical Models.

387. [Spring 19771 *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

On the sides AB and AC of an equilateral triangle ABC mark the points D and E respectively such that AD = AE. Erect equilateral triangles on CD, AE and AB, as in the figure, with P, Q, R as the respective third vertices. Show that triangle PQR is equilateral. Also show that the midpoints of PE, AQ and RD are vertices of an equilateral triangle.

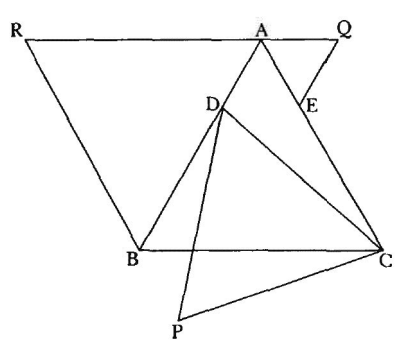


FIGURE 2

Solution by Charles U. Trigg, San Diego, California.

AC = BC, DC = PC and ∠ ACD = 60° - ∠ DCB = ∠ BCP. Hence, triangles ACD and BCP are congruent with BP = AD = AE = AQ and ∠ PBC = ∠ DAC = 60°. Therefore ∠ RBP = 180°, which makes RP a straight line segment = RQ. Thus triangle PQR is an isoceles triangle with a vertex angle of 60°, so it is equilateral and QE falls along QP.

Let the midpoints of PE, AQ, RD, RA and QE be X, Y, Z, M and N respectively. MZ is parallel to AD, and YN is parallel to AE, so MZ = AD/2 = AE/2 = YN, and ∠ ZMA = 120° - ∠ YNE. Also, MY = RQ/2 = PQ/2 = ZN, so triangles ZMY and YNX are congruent with ZY = YX and ∠ MYZ = ∠ YXN. Now ∠ AYW = 120° = ∠ YNX, so ∠ ZYX = ∠ AYN - ∠ MYZ - ∠ XYN = ∠ AYN - ∠ YXW - ∠ XYN = 120° - 60° = 60°. Therefore triangle ZYX is equilateral.

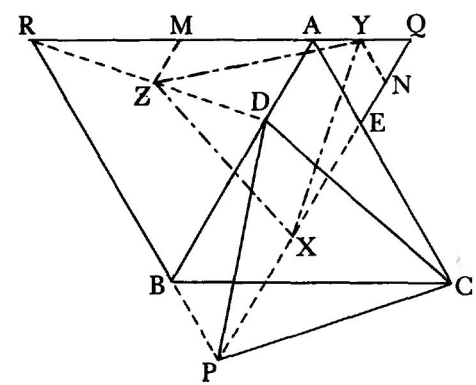


FIGURE 3

Also solved by KENNETH M. WILKE, Topeka, Kansas; CLAYTON W. DODGE, University of Maine, Orono; FLORA N. FONA, Kew Gardens, Long Island; ZAZOU and ZELDA KATZ (jointly), Beverly Hills, California; and the Proposer.

388. [Spring 19771 *Proposed by David L. Silverman, West Lob Angeles, California.*

In the game of "Larger, But Not That Large" two players each write down a positive integer. The numbers are then disclosed and the winner (who is paid a dollar by the loser) is the player who wrote the larger number, unless the ratio of larger to smaller is three or more, in which

case the player with the smaller number wins. If the same number is picked by both players, no payment is made.

a) What is the optimal strategy?

b) Suppose instead that the players are not restricted to integers but to the set $[1, \infty]$ and that the larger number wins provided the larger-to-smaller ratio is less than r (for some $r > 1$); otherwise the larger number loses. Find an optimal strategy.

Solution by the proposer.

a) The unique optimal strategy is to choose 1, 2 and 5 randomly and with equal frequency, shunning all other numbers. Optimality follows by observing that this strategy ties each of the pure strategies 1, 2 and 5 and beats every other pure strategy. Since all pure strategies other than 1, 2 and 5 are thus ruled out, the above strategy, since it is the solution of the reduced 3 by 3 matrix, is the unique optimal strategy. Generalizations to the case in which violation of the critical ratio r involves a penalty of p is not difficult.

b) Consider the intervals $[1, r]$ and $[r, r^2]$ and the following strategy: choose either of the two intervals with probability 1/2 and then from the interval chosen select a number x from the uniform random distribution. That this is an optimal strategy is easily confirmed by checking that it ties with any pure strategy y in $[1, r^2]$ and beats any $y > r^2$.

Solutions with differing conclusions were offered by **MARK EVANS**, *LaMarque, Texas;* **KENNETH M. WILKE**, *Topeka, Kansas;* **R. ROBINSON ROWE**, *Sacramento, California; and* **DONALD CANARD**, *Anaheim, California.*

388. [Spring 19771 *Proposed by Paul Erdös, Spaceship Earth.*

Find a sequence of positive integers $1 \le a_1 < a_2 < \cdots$ which omits infinitely many integers from every arithmetic progression (in fact it has density 0) but which contains all but a finite number of terms of *every* geometric progression. Prove also that there is a set S of real numbers which omits infinitely many terms of any arithmetic progression but contains every geometric progression (disregarding a finite number of terms).

No solutions, with the exception of the proposer's, have been submitted. Proposer's solution will be published in the next issue of the Journal unless other qualifying solutions are received.

390. [Spring 1977] *Proposed by Robb Koether and David C. Kay, University of Oklahoma, Norman, Oklahoma.*

Let the diagonals of a regular n-sided polygon of unit side be drawn. Prove that the $n - 2$ consecutive triangles thus formed which have their bases along one diagonal, their legs along two others or a side, and one vertex in common with a vertex of the polygon each have the property that the product of two sides equals the third.

Solution by the Proposers.

Let A be the common vertex of the triangles and BC the diagonal containing the bases (see Figure 4). From elementary properties of regular polygons and their circumscribed circles, the angles at A are equal and the first and last triangles have equal base angles. In particular, triangle ABD is isosceles, with $AD = BD$. Let triangles AXY, AYZ be any two consecutive triangles, and let the side-lengths be as indicated. Since AY bisects angle XAZ, $x/u = z/v$ or $xy/u = yz/v$. Hence, by induction, the ratio of the product of two sides of a triangle to the third is constant. Therefore, $sb/a = xy/u$, and since $a = b$, $s = xy/u$ or $xy = su$. Thus, the product of two sides of each of the triangles equals s times the third, where s is the side of the regular polygon. (If $s = 1$ the stated result then obviously follows.)

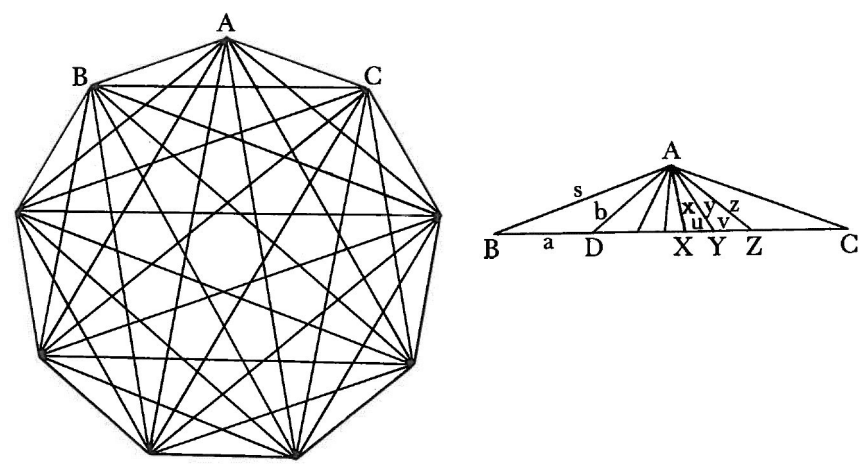


FIGURE 4

391. *[Spring 19771 Proposed by Clayton W. Dodge, University of Maine, at Orono.*

Solve this alphametic where, of course, N I N E is divisible by 9:

$$\begin{array}{r} \text{TWELVE} \\ \text{NINE} \\ \underline{\text{NINE}} \\ \text{THIRTY} \end{array}$$

Solution by Charles W. Trigg, San Diego, California.

Clearly, T # 0, N # 0, and E # 0 or 5. Proceeding from the right, the columns establish the following equations:

$$3E = Y + 10k \qquad (1)$$
$$V + 2N + k = T + 10m \qquad (2)$$
$$L + 2I + m = R + 10n \qquad (3)$$
$$E + 2N + n = I + 10p \qquad (4)$$
$$W + p = H \qquad (5)$$

where k, m, n, p are non-negative integers < 3 and $p \neq 0$.

Since 9 divides N I N E, then

$$E + 2N + I \equiv 0 \pmod 9$$

whereupon, from (4):

$$21 + 10p - n \equiv 0 \pmod 9$$

Thus, taking into consideration (1), (4), and (3), the following possibilities with distinct integers exist:

I	P	n	N	E	Y	k
0	1	1	4	1	3	0
4	2	1	8	7	1	2
5	1	2	2	9	7	2
5	1	2	3	7	1	2
5	1	2	6	1	3	0
9	1	1	5	8	4	2
9	1	1	7	4	2	1
9	1	1	8	2	6	0

Further consideration of (2), (5) and (4) reduces the possibilities to I = 5, N = 6, E = 1, Y = 3, V = 2, T = 4, W = 7, H = 8, L = 9 and R = 0. Consequently, the unique solution is 471921 + 2(6561) = 485043.

Also solved by LOUIS H. CAIROLI, (graduate student) Kansas State University, Manhattan, Kansas; VICTOR G. FESER, Mary College, Bismarck, North Dakota; HOWARD FORMAN, Bucknell University, Lewisburg, Pennsylvania; JOHN M. HOWELL, Littlerock, California; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer, CLAYTON W. DODGE.

392. [Spring 1977] Proposed by R. Robinson Rowe, Sacramento, California.

Solve in distinct positive integers,

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cdots}}}} - \cfrac{3}{c+\cfrac{1}{d+\cfrac{1}{c+\cfrac{1}{d+\cdots}}}} = 1/2$$

Solution by Kenneth M. Wilke, Topeka., Kansas.

Let $x = \cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cdots}}}}$.

Then by considering the convergents of this continued fraction in the usual manner, we find x defined by the equation $ax^2 - abx - b = 0$ so that

$$x = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2a},$$

rejecting the negative root.

Hence the first term on the left side of the given equation can be replaced with

$$\frac{-ab + \sqrt{a^2b^2 + 4ab}}{2a}$$

By symmetry the given equation becomes

$$\frac{-ab + \sqrt{a^2b^2 + 4ab}}{2a} - 3\left(\frac{-cd + \sqrt{c^2d^2 + 4cd}}{2c}\right) = \frac{1}{2}$$

which is equivalent to the system of equations

$$3d = b + 1 \quad \text{and} \quad c^2(a^2b^2 + 4ab) = 9a^2(c^2d^2 + 4cd)$$
$$= a^2[c^2(b+1)^2 + 12c(b+1)].$$

(Since $a^2b^2 + 4ab = (ab + 2)^2 - 4$, it is easily shown that this expression

cannot be a perfect square unless $ab = 0$. Hence the radicals must be equal.) The second equation of the system reduces to $2bc(2 - a) = a[c + 12(b + 1)]$.

Since all variables are positive integers we must have $a = 1$ and $c = 12(b + 1)/(2b - 1) = 6 + 18/(2b - 1)$. Since $2b - 1$ divides 18 we obtain $(a, b, a, d) = (1, 2, 12, 1)$ and $(1, 5, 8, 2)$, because $d = (b + 1)/3$. Hence the unique solution is $(a, b, c, d) = (1, 5, 8, 2)$, corresponding to

$$\frac{-5 + \sqrt{45}}{2} - 3\left(\frac{-16 + \sqrt{320}}{16}\right) = \frac{-5 + 3\sqrt{5}}{2} - 3\left(\frac{-2 + \sqrt{5}}{2}\right) = \frac{1}{2}.$$

Also solved by JEFFREY BERGEN, Chicago, Illinois; JOHN N. HOWELL, Littlerock, California; FLORA N. FONA, Kew Gardens, Long Island; DONALD CANARD, Anaheim, California; and the proposer, R. ROBINSON ROWE. Some of the solvers neglected to notice, that the solution (1, 2, 12, 1) was invalidated by the restriction requiring distinct integers.

393. *[Spring 1977] Proposed by Pita A. Lindstrom, Genesee Community College, Batavia, New Yohk.*

Consider the sequence $f(n) = n^2 - n + 41$. Find the GCD of $f(n)$ and $f(n+1)$.

I. *Solution by Kenneth M. Wilke, Topeka, Kansas, with practically verbatim solutions offered by RONNY ABOUDI, Florida Atlantic University; JEFFREY BERGEN, Chicago, Illinois; CLAYTON W. DODGE, University of Maine at Orono; RICHARD A. GIBBS, Fort Lewis College, VICTOR G. FESER, Mary College, Bismarck, North Dakota; MARK JAEGER, University of, Wisconsin, Madison, Wisconsin; BOB PRIELIPP, The University of, Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; LEO SAUVÉ, Algonquin College, Ottawa (Canada); CHARLES W. TRIGG, San Diego, California and the proposer, PETER A. LINDSTROM.*

Note that $f(n)$ is always an odd integer. $f(n+1) = n^2 + n + 41$. Let $d = (f(n), f(n+1))$. Then $d|(f(n) + f(n+1)) = 2(n^2 + 41)$ and $d|(f(n+1) - f(n)) = 2n$. Then since $(n, n^2+41) = 1$ unless $n = 41k$ for some integer k, we have $d = 41$ whenever $n = 41k$ and $d = 1$ otherwise.

11. *Solution by Michael W. Ecker, Mew York, New York.*

Using an argument similar to that of Solution I, Ecker shows that it is just as easy to solve the more general situation with $f(n) = n^2 - n + p$, where p is any prime $\neq 2$. He arrives at the conclusion that if n is a multiple of p, it is immediate that p is a divisor of both $f(n)$ and $f(n+1)$ and is equal to their GCD. If n is not a multiple of p, the assumption that GCD $= p$ leads to the contradiction that p divides both $n - 1$ and $n + 1$, leading to the conclusion that the GCD $= 1$.

Comment by the Problem Editor

Louis H. Cairoli, Kansas State University, calls attention to the article on "The Generation of Prime Numbers" in Mathematical Gems *II* by Ross Honsberger, published by the Mathematical Association of America. Other material relating to this problem may be found in the October 1976 issue of EUREKA, published by Algonquin College, Ottawa, Canada, Problem 142, page 175 et. seq., and in the accompanying references.

Clayton W. Dodge also cited the references listed by Cairoli and added his own article, "A Prime-Generating Trinity", published in the October 1977 issue of EUREKA.

Albert H. Beiler discusses formulas for primes in his Recreations in the Theory of Numbers, Dover Publications, pp. 219-221.

394. *[Spring 19771 Proposed by Erwin Just and Bertram Kabak, Bronx Community College.*

Prove that if A_1, A_2 and A_3 are the angles of a triangle, then

$$3\sum_{i=1}^{3}\sin^2 A_i - 2\sum_{i=1}^{3}\cos^3 A_i \le 6.$$

I. *Solution by Louis H. Cairoli, Graduate Student, Kansas State University, Manhattan., Kansas.*

The result follows immediately from the known relations

$$\sum_{i=1}^{3}\sin^2 A_i = 2 + 2\cos A \cos B \cos C$$

and

$$\sum_{i=1}^{3}\cos^3 A_i \ge 3\cos A \cos B \cos C$$

11. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

We will establish the more general and stronger inequality

$$\cos^n A_1 + \cos^n A_2 + \cos^n A_3 \ge 3(1/2)^n, \{E\}, \qquad (1)$$

where n is an integer ≥ 2 and $\{E\}$ denotes with equality iff the triangle is equilateral.

The given inequality can be rewritten as

$$2 \sum \cos^3 A_i + 3 \sum \cos^2 A_i \geq 3$$

and thus can be gotten immediately from linear combinations of (1) for $n = 2$ and 3.

To prove (1), we use the known special case of it for $n = 2$ [1]. This latter case is also a special case ($n = 2$, $x = y = z = 1$) of the inequality [2]

$$x^2 + y^2 + z^2 \geq (-1)^{n+1}\{2yz \cos nA_1 + 2zx \cos nA_2 + 2xy \cos nA_3\} \quad (2)$$

and which is easily established from a sum of squares (here x, y, a are arbitrary real numbers). One can also obtain other nth order trigonometric triangle inequalities from (2), [2].

Proof. Case (1). The triangle is non-obtuse: By the power mean inequality [3],

$$\left\{\frac{\cos^n A_1 + \cos^n A + \cos A_3}{3}\right\}^{1/n} \geq \left\{\frac{\cos^2 A_1 + \cos^2 A_2 + \cos^2 A_3}{3}\right\}^{1/2} \geq \frac{1}{2}, \{E\},$$

for all real $n \geq 2$. Also, (2) is then valid immediately for all triangles if n is an even integer ≥ 2.

Case (2). The triangle is obtuse (let $A_3 > \pi/2$): We now have to show that

$$\cos^n A_1 + \cos^n A_2 \geq 3(1/2)^n + \cos^n(A_1 + A) \quad (3)$$

where $0 < A, + A_2 < \pi/2$ and n is an odd integer ≥ 3. If either $\cos^n A_1$ or $\cos^n A_2 \geq 3(1/2)^n$, the inequality is then obviously valid (since $\cos(A_1 + A_2) < \cos A_1$ and $\cos A_2$). Since at least one of A_1, A_2 is $\pi/4$ and $\cos^n \pi/4 \geq 3(1/2)^n$ for $n \geq (\log 9)/(\log 2) \approx 3.1699$, (3) is valid for all real $n \geq 3.17$. For $n = 3$, we have $\cos^{-1} \sqrt[3]{3/8} > 43.85°$. Thus, for (3) to be valid for all real $n \geq 3$ (and all non-acute triangles), it suffices to show that

$$\cos^3 45° + \cos^3 45° > 3/8 + \cos^3 60°$$

or

Using calculus, one can show that

$$\cos^2 A_1 + \cos^2 A_2 - \cos^2(A_1 + A_2) \geq 1$$

and that (3) is valid for all real $n \geq 2$. An open problem here is to determine the minimum n such that (3) is valid. Clearly, (3) is invalid for $n = 1$. Also, n must be $\geq 2[(\log 3)/(\log 2) - 1] \approx 1.1699$ (just let $A_1 = A, = \pi/4$).

REFERENCES

1. Kooistra, R., Nieuw Tijdschr. Wisk., 45 (1957/58), 108-115.
2. Klamkin, M. S., Publ. Electrotech. Fac. Ser. Mat. Fiz. Univ. Beograd, No. 357-380 (1971), 33-44.
3. Beckenbach, E. F., Bellman, R., *Inequalities*, Springer-Verlag, Heidelberg, 1965, 15-16.

Also solved by BOB PRIELIPP, *University of Wisconsin at Oshkosh; and the Proposers.*

395. [Spring 1971] *Proposed by Joe. Van Austin, Emory University, Atlanta, Georgia.*

Assume that n independent Bernoulli experiments are made with $p = P$ [success], $1 - p = P$ [failure], and $0 < p < 1$. Intuitively it seems that P [success on the first trial | exactly one success] is always less than P [success on the first trial | at least one success]. Verify directly that this is indeed the case.

Solution by Louis H. Cairoli, Graduate Student, Kansas State University, Manhattan, Kansas.

Let A = success on first trial; B = exactly one success; C = at least one success = 1 - probability all failures. By Bayes' Formula, we see that

$$P[A|B] = \frac{P[B|A]\,P[A]}{P[B]} = \frac{(1-p)^{n-1}\,p}{np(1-p)^{n-1}} = \frac{1}{n}$$

and

$$P[A|C] = \frac{P[C|A]\,P[A]}{P[C]} = \frac{1p}{1-(1-p)^n}$$

Hence we must show that for $n > 1$, $1/n < p/[1 - (1 - p)]$.

But this is equivalent to $(1 - p)^n \geq 1 - np$, a result easily shown by induction.

Also solved by MAX J. CLEVELAND, *National Mine Health and Safety Academy, Beckley, West Virginia;* JOHN HOWELL, *Littlerock, California;* HOWARD FORMAN, *Bucknell University, Lewisburg, Pennsylvania;* MARK JAEGER, *Department of Statistics, University of Wisconsin at Madison;* N. J. KUENZI, *University of Wisconsin at Oshkosh;* BOB PRIELIPP, *University of Wisconsin at Oshkosh;* MICHAEL STEPHEN SCHWARTZ, *Bronx, NW Yoik;* and *the proposer,* JOE DAN AUSTIN.

396. [Spring 19771 Proposed by David R. Simonds, Rensselaer Polytechnic Institute, Troy, N w York.

Let $[m]_n$ denote the integral part of the quotient when m is divided by n. Prove that

$$[m]_{n^k} \equiv [m]_n^k, \ \forall\, m, n, k \in N,$$

where $[m]_n^k$ means $[[\cdots[m]_n\cdots]_n]_n$ (k sets of brackets).

Solution by Bob Prielipp, The University of Wisconsin at Oshkosh.

Our solution will be by induction on k and will employ the fact that if x is a real number and n is a positive integer then $[x/n] = [[x]/n]$. (For a proof of this result see Theorem 4.1(f) on pp. 78-79 of Niven and Zuckerman, *An Introduction to the Theory of Numbers,* Third Edition, John Wiley & Sons, Inc., 1972.) Clearly the desired result holds when k = 1. Assume that $[m]_{n^j} = [m]_n^j$. Then

$$[m]_n^{j+1} = \left[\frac{[m]_n^j}{n}\right] = \left[\frac{[m]_{n^j}}{n}\right] = \left[\frac{[\frac{m}{n^j}]}{n}\right] = \left[\frac{\frac{m}{n^j}}{n}\right]$$

and by the result cited above,

$$\left[\frac{m}{n^{j+1}}\right] = [m]_{n^{j+1}},$$

and our solution is complete.

Also solved by JEFFREY BERGEN, *Chicago, Illinois;* LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas;* CLAYTON W. DODGE, *University of Maine out Orono;* RICHARD A. GIBBS, *Fort Lewis College, Durango, Colorado;* RAY HAERTEL, *Central Oregon Community College, Bind, Oregon;* KENNETH M. WILKE, *Topeka, Kansas; and the proposer,* DAVID R. SIMONDS, *Troy, New York.*

397. [Spring 19771 Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

If $c_j = 2\cos(j\pi/n)$, prove that

$$\prod_{j=1}^{3} (1 + 3c_j^4) = (3^n - 3^{n/2} \cdot 2\cos(5\pi n/6) + 1)^2 \tag{1}$$

and more generally that

$$\prod_{j=1}^{n} (t^4 + c_j^4) = (x^n + x^{-n} - z^n - z^{-n})^2 = F_n^2(t) \tag{2}$$

where $F_n(t)/F_1(t)$ is a polynomial in t^2 with integral coefficients and

$$x = u\bar{u} \geq 1, \ z = u/\bar{u}, \text{ and } u + u^{-1} = t\, e^{\pi i/4}. \tag{3}$$

Solution by the Proposer.

Setting $a = e^{\pi i/4}$ and factoring (2) we have

$$\prod_{j=1}^{n-1} (t^2 - 2^{1/2} t\, c_j + c_j^2) = \prod_{j=1}^{n-1} (t^2 + 2^{1/2} t\, c_j^2) = F_n(t)/F_1(t) \tag{4}$$

$$= \prod_{j=1}^{n-1} (ta - c_j)(t\bar{a} - c_j).$$

Now the determinant of order $n - 1$

$$d_{n-1}(s) = \begin{vmatrix} s & 1 & 0 & \cdots & 0 \\ 1 & s & 1 & \cdots & 0 \\ 0 & 1 & s & 1\cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0\ 1 & s \end{vmatrix} \tag{5}$$

satisfies the second order recurrence relation

$$d_n = s d_{n-1} - d_{n-2}, \qquad d_o = 1, \ d_1 = s \tag{6}$$

so if $s = u + u^{-1}$, we see by induction that

$$d_{n-1}(s) = (u^n - u^{-n})/(u - u^{-1}). \tag{7}$$

Since d_{n-1} vanishes for $u^{2n} = 1$, its zeros are $s = c_j$, $j = 1, 2, \cdots, n - 1$. Thus

$$\prod_{j=1}^{n-1} (ta - c_j) = (u^n - u^{-n})/(u - u^{-1}),$$

if $u + u^{-1} = ta$. Now (3) implies

$$(u - u^{-1})^2(\bar{u} - \bar{u}^{-1})^2 = (a^2t^2 - 4)(\bar{a}^2t^2 - 4) = t^4 + 16 = F_1^2(t) \quad (9)$$

Hence, by (4), (8), and (9), with x, z, u given by (3),

$$F_n(t) = (u^n - u^{-n})(\bar{u}^n - \text{if''}) = x^n + x^{-n} - z^n - z^{-n}. \quad (10)$$

We check directly from (4) that

$$F_1/F_1 = 1,\ F_2/F_1 = t^2,\ F_3/F_1 = t^4 + 1,\ F_4/F_1 = t^6 + 4t^2. \quad (11)$$

Using (3) and (10) we verify that x and z satisfy the relations

$$a; + x^{-1} = (t^2 + F_1)/2,\ z + z^{-1} = (t^2 - F_1)/2 \quad (12)$$

$$x^2 - 2 + x^{-2} = t^2(x + x^{-1}),\ z^2 - 2 + z^{-2} = t^2(z + z^{-1}) \quad (13)$$

Hence, by (10) and (13) the functions $F_n(t)$ satisfy the fourth order recurrence relation

$$F_{n+2} - 2F_n + F_{n-2} = t^2(F_{n+1} + F_{n-1}) \quad (14)$$

We conclude inductively from (11) and (14) that $F_n(t)/F_1(t)$ is a polynomial in t^2 with integral coefficients.

To prove (1), we now set $t^4 = 1/3$ in (2) and multiply by 3''. By (9) we obtain $F_1 = 7/3^{1/2}$, and from (12) and (13) we obtain $x + x^{-1} = 4/3^{1/2}$, $x - x^{-1} = 2/3^{1/2}$, $z + z^{-1} = -3^{1/2}$, $z - z^{-1} = i$ or $-i$. Hence,

$$x^n = 3^{n/2},\ z^n + z^{-n} = 2\cos(5n\pi/6), \quad \text{when } t^4 = 1/3, \quad (15)$$

and (1) is proved.

Another interesting case is $t = 1$, when $F_1(1) = 17^{1/2}$ and

$$x = 2.081019\cdots,\ z^n + z^{-n} = 2\cos n(141.3317\cdots)^\circ. \quad (16)$$

Then, if [ ] denotes the greatest integer function, we have

$$\prod_{j=1}^{[n/2]} (1 + c_j^4) = [(17)^{-1/2}x^n + 1/2], \quad (17)$$

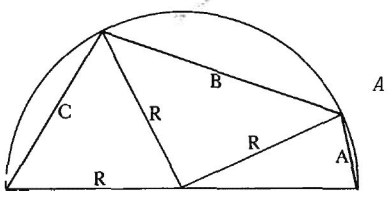
where $x = 2.08101899662\cdots$. These integers are the sums of the coefficients in the polynomials $F_n(t)/F_1(t)$.

Editor's Note.

The proposer remarked that the problem arose in trying to evaluate and factor some of the symmetric functions of roots of unity that he discussed on pp. 132-135 of the Fall 1975 issue of this *Journal* in his article on Matrix Functions.

398. [Spring 19771 *Proposed by Richard S. Field, Santa Monica, California.*

Find solutions in integers A = B = C ≠ R and A # B ≠ C = R for the quadrilateral inscribed in a semicircle of radius R, as shown in the figure. Also find solutions in integers A ≠ B ≠ C ≠ R or prove that none exist.



$A \neq B \neq C = R$

FIGURE 5

I. Solution by Kenneth M. Wilke, Topeka, Kansas.

Since one side of the quadrilateral is a diameter of the circle, it is well known that A, B, C and 2R are related by the equation:

$$(2R)^3 - 2R(A^2 + B^2 + C^2) - 2ABC = 0$$

or

$$4R^3 - R(A^2 + B^2 + C^2) - ABC = 0 \quad (1)$$

a) $A = B \neq C \neq R$.

Equation (1) becomes $4 - (2a^2 + c^2) - a^2c = 0$ or $c = 2 - a^2$, where $c = C/R$ and $a = A/R$. By letting $a = p/q$ for some arbitrary integers p and q, we obtain the parametric solution of (1): $A = B = pq$, $C = 2q^2 - p^2$ and $R = q^2$, where p and q are positive and $p < q\sqrt{2}$ and $p \# q$.

b) $A \neq B \neq C = R$.

Equation (1) becomes $4C^3 - C(A^2 + B^2 + C^2) - ABC = 0$ or $3C^2 = A^2 + AB + B^2$. Then since $3 = 1^2 + 1\cdot 1 + 1^2$ and since $k^2 = m^2 + mn + n^2$ has the parametric solution $k = p^2 + pq + q^2$, $m = p^2 - q^2$ and $n = 2pq + q^2$, equation (1) has the parametric solution $C + R = p^2 + pq + q^2$, $A = p^2 - 2pq - 2q^2$ and $B = p^2 + 4pq + q^2$, where p, q are arbitrary integers such that A, B, C and R are positive.

c) $A \neq B \neq C \neq R$.

Let $a = A/R$, $b = B/R$ and $c = C/R$. Then equation (1) becomes

$$4 - (a^2 + b^2 + c^2) - abc = 0$$

or equivalently,

$$(4 - a^2)(4 - b^2) = (2c + ab)^2 \qquad (2)$$

Letting $a = 4p/(p^2 + 1)$ and $b = 4q/(p^2 + 1)$ implies $c = [2(p^2 - 1)\cdot(q^2 - 1) - 8pq]/(p^2 + 1)(q^2 + 1)$, where p and q are integers chosen so that c is positive. Then we have the parametric solution of (1): $A = 4p(q^2 + 1)$, $B = 4q(p^2 + 1)$, $C = 2(p^2 - 1)(q^2 - 1) - 8pq$ and $R = (p^2 + 1)(q^2 + 1)$.

Equation (1) can be found in Dickson's History of the *Theory* of *Numbers*, Vol. II, p. 220, in the section devoted to rational quadrilaterals, and is attributed to Isaac Newton.

II. Solution by R. Robinson Rowc, Sacramento, California.

Apparently this solver anticipated the problem with his paper "Primitive Semi-Inscribed Quadrilateral", published in the *Journal* of Recreational Mathematics, 3, No. 3, July 1970, pp. 151-157. The solutions listed are derived from Table 1, with A, B, $C = a$, b, c and $R = d/2$.

$A = B \neq C \neq R$			
2	2	7	4
3	3	17	9
6	6	14	9
12	12	2	9
4	4	31	16
12	12	2	9
20	20	7	16
5	5	49	25
10	10	46	25
15	15	41	25
20	20	34	25
30	30	14	25
35	35	1	25
6	6	71	36
30	30	14	25
42	42	23	36
7	7	97	49
14	14	94	49
21	21	89	49
28	28	82	49
35	35	73	49
42	42	62	49
56	56	34	49
63	63	17	49

$A \neq B \neq C = R$			
211	7	7	
1	22	13	13
11	26	19	19
13	46	31	31
26	47	37	37
22	61	43	43
23	71	49	49

$A \neq B \neq C \neq R$			
2	912		8
3	14	25	15
8	17	22	16
12	22	28	21
12	19	33	22
10	21	45	27
11	39	46	33
22	34	42	33
17	28	53	34
6	25	63	35
14	38	50	35
3	26	66	36
19	44	49	38
13	43	57	39
28	41	50	40
10	55	62	44
11	24	81	44
11	38	74	44
9	35	79	45
35	42	57	45
23	43	68	46
29	36	69	46
14	23	91	49

These lists have been limited to primitive sets with R up to 50. Since all multiples are eligible, there is an infinitude for each of the three categories. Probably there are an infinitude of primitives in each category, and this is obvious for the first, with the sequence A, B, C, R = n, n, $2n^2 - 1$, n^2. So far, at least, R is a square in the first and congruent to 1 (mod 6) in the second, but rather random in the third category.

A very thorough analysis of the problem was also offered by CLAYTON W. DODGE, University of Maine at Orono.

Comments by the Phoblem Editor

Charles W. Trigg, the world's champion indefatigable proofreader, never fails to supply a list of errors and omissions that somehow manage to creep into the Problem Department. The first two paragraphs of Solution II on page 371 of the Spring 1977 issue should have read:

The magic constant of a third order magic square is three times the central element, which therefore is 89.

The nine elements of a third order magic square can be rearranged into a square array in which the elements of the rows are in arithmetic progression with the same common difference, and likewise for the elements of the columns, and conversely.

Trigg also supplied additions to the list of periodicals that contain problem departments, given on page 381 of the Spring 1977 issue:

1) *EUREKA*, published by Algonquin College, Mathematics Department, 200 Lees Avenue, Ottawa, Ontario, K1S OC5. Managing Editor: F. G. B. Maskell. Ten issues per year for $8.00.

2) The Pentagon. Subscription rate: $3.00 for two years. Business Manager: Wilbur J. Waggoner, Central Michigan University, Mount Pleasant, Michigan.

3) The Fibonacci Quarterly. Published by the Fibonacci Association. Subscription rate: $15.00 per year. Address Professor Leonard Klosinski, Mathematics Department, University of Santa Clara, Santa Clara, California 95053.

4) *DELTA*. Published by the Waukesha Mathematical Society, 1550 University Drive, Waukesha, Wisconsin 53186.

5) *NABLA*. The Bulletin of the Malayan Mathematical Society, Dept. of Mathematics, University of Singapore, Singapore, 10.

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INITIATION CEREMONY

The editorial staff of the *Journal* has prepared a special publication entitled *Initiation Ritual* for use by local chapters containing details for the recommended ceremony for initiation of new members. If you would like one, write to the National Office.

OMISSION IN LAST ISSUE

We regret to report that in our account of the annual meeting in Seattle we omitted the following student presentations:

16. *Acceleration of Root Fading Algorithms Through Chebyshev Interpolation*, Richard Daugherty, Kentucky Beta.
17. *Simple Continued Fractions*, David Miyashiro, Ohio Alpha.
18. *On Distance Attaining Sets*, Robert Goggins, Mississippi Alpha.
19. *An Introduction to Coding Theory*, Bill Heidler, Ohio Delta.

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