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CAUSES OF MATH ANXIETY AT THE UNIVERSITY

by Kathleen Walker
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As students and teachers of mathematics, I'm sure all of you have been in the uncomfortable situation of being introduced to someone at your university; and after a short conversation, they ask you what your major or occupation is. When you respond by telling of your close relationship with mathematics, they stop, and then say, "I've always hated math" and your conversation practically ends.

Why do these negative attitudes against an area of such importance exist at a school of higher education? What causes these feeling of distress and uneasiness from the slightest contact with anything math oriented?

Jerrold Zacharias, a noted physicist and educator, calls the problem mathophobia: the fear of mathematics. Through its high level of social acceptability, mathophobia causes more mathophobia. Persons who are usually very proud of their education will tend to speak freely of their mathematical ignorance. They can say, "I'm terrible at math," almost with a sense of pride, as if being poor in mathematics shows good taste in failure.

This attitude is transferred in many ways. First, it is transferred to children in schools. Parents many times are not concerned when their child starts to do poorly in mathematics, as if to say, "I was never any good in math, so why should Bill or Sally be any better." The child has now lost all motivation from the home for success in mathematics. Most children in this situation will choose a game of baseball after school instead of staying home and finishing their math homework. The parents have paved the road to math failure. The lack of mathematical skills will certainly cause anxious and fearful moments for this child when he is faced with using the math he or she should have learned in school.

Secondly, teachers, especially in the elementary schools, are afflicted with mathophobia. Many have not taken a mathematics course since high school and carry into the classroom vague notions of what mathematics is or what it can do. They see math as merely a way of computing and are tense and ill at ease with it. It cannot be hard for children in these classes to be infected with the idea that math is hard and unpleasant.

Moreover, people who have some mathematical intelligence are many times viewed by society as being strange and difficult to communicate with. (This, of course, is not necessarily true!)

Our schools cause math anxiety, also. Sheila Tobias of Wesleyan University in Middletwon, Connecticut, explains part of the problem the following way.

> *"How confusing it is to learn arithmetic in elementary school when as a child in kindergarten one is told unequivocally zero is 'nothing'; in first grade that it is a 'placeholder' and in fifth grade that you can't divide by zero."*

Many students are bothered by what seem to be inconsistancies.

In a study done by Dr. Mitchell Lazarus of the Education Development Center in Newton, Massachusetts, many adults said that they enjoyed math 'until they did so-and-so in school" or, in other words, until they were exposed to some topic that seemed particularly difficult. Did the enjoyment of mathematics return after the hard topic passes? Almost never. The dislike is usually irreversible.

This is not surprising if one looks at the curricula in use at most schools. The mathematics taught at each level depends strongly on most of the work done in proceeding years. Therefore, trouble in any year, for any reason, is nearly certain to spell trouble in the future. This concept causes many problems for the classroom teacher who must make sure that all of the class has a sound understanding of all previously taught mathematics before moving on to new material. This is usually not the case in other subjects. A week out of school with the flu will not produce a case of history anxiety as easily as it will a case of math anxiety.

One of the most important problems in the school curriculum is the lack of connection between mathematics and everyday life. Many educators feel that "new math's" context is that of the professional mathematician; it is abstract, definitional, axiomatic, and supposedly rigorous. The result is to pull mathematics even farther from it's actual uses.

Also, symbols and abstractions for their own sakes, now very common in mathematics curricula, often strike students as pointless and confusing.

Dr. Lazarus has also done studies on the "memorize what to do approach" method.

This calls for much time and effort and eventually leads to the student having no understanding of what he is doing. When the student finally is approached with a demanding mathematics class in his late high school or early college years, he is totally lost as to what is happening because it has been some time since he last understood what he was doing. He lacks the necessary background and knowledge and will usually decide to totally give up in mathematics instead of returning to remedial classes for a thorough re-education.

The causes that have been discussed this far can affect anyone. But women have their own set of causes, in addition to those already mentioned. Ms. Tobias describes it as "a far more serious phenominon growing out of a culture that makes math ability a masculine attribute, that punishes women for doing well in math, and that soothes the slower learner by telling her that she does not have a 'mathematical mind'."

The problem of math anxiety in women of far above average intelligence can begin as early as the sixth grade. Professor Jerome Kagan of Harvard University found that children are inhibited in learning subjects they feel are inappropriate for their sex. Girls are indoctrinated by junior high school with messages such as math is masculine, women do not need math, and boys are better in math. The notion that mathematics is for boys and is not a feminine subject plays an important part in a young girl's conception of herself as not interested or competent in math.

When asked why they do poorly on a math exam, high school girls tend to attribute their failure to lack of ability, while high school boys usually say they did not work hard enough. Boys in junior and senior nigh do not like math any better than girls do. However, Dr. John Ernst of the University of California at Santa Barbara conjectures that boys are made aware that it will be necessary to the kinds of careers that envision for themselves.

There are other reasons for the "female strain" of math anxiety. Lenore Weitman in a 1975 study of school text books found that males were represented in more than two-thirds of the pictures showing science and/or math activities. Also, texts often used women and girls in math problems in ways that emphasixed their "stupidity."

The person who "hates" math could have many of these causes affecting their perception of what we do. Perhaps it would be interesting and helpful to stop and ask the next person who tells you that they "hate" math what causes these feelings and pick up your conversation from there.

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TWO APPLICATIONS OF CONTROLLABILITY AND OBSERVABILITY

by C. Gordon Huffman
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In many physical problems, we are interested in controlling some system, such as the motion of a spacecraft or missile, the neutron density in a nuclear reactor, or the current in a complicated circuit. That process which we try to control is called the state, and the means by which we try to control the state is called the control. Since the state may be very hard to directly measure, i.e. to observe, we may also want to find some indirect means of observing the state. In practice, the state, control, and observing quantities have several components, and the state is governed by a differential equation.

Let $x:\mathbb{R} \to \mathbb{R}^n$ be the state variable, let $u:\mathbb{R} \to \mathbb{R}^m$ be the control variable, and let $y:\mathbb{R} \to \mathbb{R}^p$ be the output variable (which is used to observe the state variable x). In this paper, we consider the case where the differential equation governing the state is the linear, time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

and the method of observing x is given by

$$y(t) = Cx(t), \tag{2}$$

where A, B, and C are real constant matrices of appropriate dimensions.

We shall give the definitions of controllability and observability, present some useful theory, and then give two applications of these concepts.[2]

Definition 1. The system (1) is *controllable* provided that for all $x_0, x_1 \in \mathbb{R}^n$, $t_0 \geq 0$ there is a $t_1 > t_0$ and a bounded measurable function $u: [t_0,t_1] \to \mathbb{R}^m$ such that the solution $x(t)$ satisfying (almost everywhere) the initial value problem (1) with the initial condition $x(t_0) = x_0$, also satisfies $x(t_1) = x_1$.

Definition 2. Let $t_1 > t_0 \geq 0$, let $u:[t_0,t_1] \to \mathbb{R}^m$ be bounded and measurable, and let $x_0, \tilde{x}_0 \in \mathbb{R}^n$. Let $x(t)$, $\tilde{x}(t)$ be the solutions of the initial value problems (1) together with $x(t_0) = x_0$, $\tilde{x}(t_0) = \tilde{x}_0$, respectively. The system (1) - (2) is *observable* provided that for all u, t_0, t_1, if

$y(t) = Cx(t) = C\tilde{x}(t) = \tilde{y}(t)$ on $[t_0, t_1]$, then $x_0 = \tilde{x}_0$.

Theorem 1. [1, pp. 81-84] The system (1) is controllable if and only if the $n \times nm$ *controllability* matrix $\Gamma = [B, AB, A^2B, \ldots, A^{n-1}B]$ has full rank n.

Example: Consider the n^{th} order linear differential equation with constant coefficients

(3) $$\xi^{(n)}(t) + a_1\xi^{(n-1)}(t) + \ldots + a_n\xi(t) = u(t)\ .$$

This can be represented as a linear system as in (1), with $x_i = \xi^{(i-1)}$, $B = [0, 0, \ldots, 0, 1]^T$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & \ddots & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix}$$

The controllability matrix Γ is a square matrix, and has the same rank as the square matrix $[A^{n-1}B, \ldots, AB, B]$, which is a lower triangular matrix with 1's on the diagonal, and hence has determinant 1. Γ thus has full rank n, and the system is controllable. Hence there is a function u that will force any given state $(\xi(t_0), \xi'(t_0), \ldots, \xi^{(n-1)}(t_0))$ to any given terminal state $(\xi(t_1), \xi'(t_1), \ldots, \xi^{(n-1)}(t_1))$, for some $t_1 > t_0$.

Definition 3. The *dual* of the system (1) - (2) is the system

(4) $$\dot{\tilde{x}}(t) = -A^T\tilde{x}(t) + C^T\tilde{u}(t)$$

(5) $$\tilde{y}(t) = B^T\tilde{x}(t),$$

where $\tilde{x} \in R^n$, $\tilde{u} \in R^p$, $\tilde{y} \in R^m$. (Note that the dual of (4) - (5) is (1) - (2).)

Theorem 2. [1, p.111] The system (1) - (2) is observable if and only if its dual is controllable.

Corollary. The system (1) - (2) is observable if and only if the $n \times np$ observability matrix $[C^T, A^TC^T, \ldots, (A^T)^{n-1}C^T]$ has full rank n.

Definition 4. The system (1) - (2) with matrices A, B, and C is *linearly* equivalent to a system (1) - (2) with matrices D, E, and F provided there is a real $n \times n$ invertible matrix T with $D = TAT^{-1}$, $E = TB$, and $F = CT^{-1}$. (Linear equivalence is an equivalence relation.)

If we set $\tilde{x} = Tx$, where x is as in (1) - (2), and T is a real, invertible $n \times n$ matrix, then $\dot{\tilde{x}} = (TAT^{-1})\tilde{x} + (TB)u$, and $y = (CT^{-1})\tilde{x}$. On the other hand, if we have two linearly equivalent systems as in Definition 4, it is clear that $\tilde{x} = Tx$ satisfies the D-E-F system (1) - (2). Thus two systems are linearly equivalent if and only if the state variables are related by $\tilde{x} = Tx$, for some nonsingular real matrix T.

Furthermore, the controllability matrix $\tilde{\Gamma} = [TB, (TAT^{-1})B, \ldots, (TAT^{-1})^{n-1}B]$ has the same rank as the controllability matrix $\Gamma = [B, AB, \ldots, A^{n-1}B]$, so that any system linearly equivalent to a controllable (observable) system is controllable (observable).

We now give two applications of controllability and observability.

Application 1: Identity Observers [2, 4]. If (1) - (2) is observable, then Definition 2 tells us that the output which results from any input uniquely determines the initial state. However, this does not tell us the initial state and subsequent behavior of the system. To get this information, we use another system called an *observer* which simulates the original system.

For example, let $z: \mathbb{R} \to \mathbb{R}^n$, and consider the system

(6) $$\dot{z} = Fz + Gy + Hu,$$

where F, G, and H are respectively $n \times n$, $n \times p$, and $n \times m$ real matrices. Notice that the state z of this system has as controls both the output and the control of the original system (1) - (2). We wish to choose F, G, and H so that z behaves like x. Setting $z = x$ in (6) and using (2), we get

(7) $$\dot{z} = \dot{x} = Ax + Bu = Fx + GCx + Hu\ ,$$

leading us to choose

(8) $$F = A - GC\ ,$$

(9) $$H = B\ .$$

Hence (6) becomes

$$\mathcal{L}\{y\} = C[sI - (A + BD)]^{-1}B \cdot \mathcal{L}\{u\} \quad . \tag{16}$$

The matrix function of s, $G(s) = C[sI - (A + BD)]^{-1} B$, is called the *transfer function matrix* of the system (13) - (2), and gives a relationship between the Laplace transforms of the control and of the output of the system. This representation is well known and widely used in engineering applications.

The matrix $H(s) = [sI - (A + BD)]^{-1}$ in the transfer function matrix is a matrix of rational functions with entries $H_{ij}(s) = R_{ij}(s)/\det[sI - (A + BD)]$, where $R_{ij}(s)$ are polynomials in s of degree less than n. One question that is asked of the transformed system (16) concerns the location of the poles of the system, i.e., the zeros of $\det[sI - (A + BD)]$. How much control do we have over the placement or assignment of these poles? Since the poles are just the eigenvalues of $A + BD$, the following theorem shows that we have the most control when (1) is controllable.

Theorem 4. [3] Given the system (1), and given an arbitrary set S of n or fewer complex numbers closed under complex conjugation, there exists a real $m \times n$ matrix D such that $A + BD$ has the spectrum S, if and only if (1) is controllable.

Note that Theorems 3 and 4 clearly illustrate the duality of the concepts of controllability and observability.

Example: Let $x = [\xi,\eta]^T$ and consider the real matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C = [c_1, c_2]$$

in (1) and (2). With $u(t) \equiv 0$ the state variables $\xi(t)$, $\eta(t)$ execute simple harmonic motion.

Here the controllability matrix is

$$\Gamma = [B, AB] = \begin{bmatrix} b_1 & b_2 \\ b_2 & -b_1 \end{bmatrix},$$

and the observability matrix is

$$[C^T, A^T C^T] = \begin{bmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{bmatrix}.$$

We easily see that the system (1) - (2) is controllable if and only if

$$\dot{z} = (A - GC)z + Gy + Bu \quad . \tag{10}$$

The system (10) is called an *identity observer*, because if $z(0) = x(0)$, then $z(t) = x(t)$, for all $t \geq 0$.

Now even if $z(0) \neq x(0)$, we would like the z-system (10) to behave like the x-system (1) as t increases. With this in mind, we use (6), (2), (8), and (9) to obtain:

$$\begin{aligned} \dot{z}(t) - \dot{x}(t) &= [Fz(t) + GCx(t) + Bu(t)] - [Ax(t) + Bu(t)] \\ &= Fz(t) + (GC - A)x(t) = F[z(t) - x(t)] \quad . \end{aligned} \tag{11}$$

Solving this system of differential equations for $z(t) - x(t)$ gives

$$z(t) - x(t) = e^{Ft}[z(0) - x(0)] \tag{12}$$

Hence to have $\lim_{t\to\infty} [z(t) - x(t)] = 0$ for $z(0) \neq x(0)$, we need all the eigenvalues of F to have negative real parts. Also, to have $z(t) \to x(t)$ as fast as possible as $t \to \infty$, we would like the real parts of the eigenvalues of F to be as large in modulus as possible. Thus we ask the following question: can we choose G so that $F = A - GC$ has any set of eigenvalues we wish? The following theorem answers the question favorably under the hypotheses of the observability of (1) -(2).

Theorem 3. [4] Given the system (1) - (2), and given an arbitrary set S of n or fewer complex numbers closed under complex conjugation, there exists a real $n \times p$ matrix G such that $A - GC$ has the spectrum S, if and only if (1) - (2) is observable.

Application 77: Pole Assignment. [3], [4] Suppose we wish to modify the behavior of the state x in the system (1) by "linearly feeding back" the state into the system. We do this by replacing u with $u + Dx$, where D is some $m \times n$ real matrix. Then (1) becomes

$$\dot{x} = (A + BD)x + Bu \quad . \tag{13}$$

Taking the Laplace transform of both sides of (13), assuming that $x(0) = 0$, and rearranging terms yields

$$[sI - (A + BD)] \cdot \mathcal{L}\{x\} = B \cdot \mathcal{L}\{u\} \tag{14}$$

Taking the Laplace transform of (2) yields

$$\mathcal{L}\{y\} = C \cdot \mathcal{L}\{x\} \quad . \tag{15}$$

The "transformed" system derived from (14) and (15) is

$B \neq 0$ and it is observable if and only if $C \neq 0$.

If we let $G = [g_1, g_2]^T$, then

$$F = A - GC = \begin{bmatrix} -g_1c_1 & 1 - g_1c_2 \\ -1 - g_2c_1 & -g_2c_2 \end{bmatrix}.$$

The characteristic polynomial of F is

$$\lambda^2 + (g_1c_1 + g_2c_2)\lambda + 1 - g_1c_2 + g_2c_1 .$$

So, if s denotes the sum of the eigenvalues of F and p denotes the product, then we have

$$\begin{aligned} c_1g_1 + c_2g_2 &= -s \\ -c_2g_1 + c_1g_2 &= p - 1 . \end{aligned} \tag{17}$$

The eigenvalues of F (zeros of the characteristic polynomial) are uniquely detemined by their sum and product. We see that if $C \neq 0$ (the system (1) - (2) is observable) then for any s and p the system (17) has a solution for g_1, g_2. This illustrates Theorem 3. Similar computations for this example illustrate Theorem 4.

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[1] This is part of a paper written at the end of a research project sponsored by a grant from the National Science Foundation under its Undergraduate Research Program at Southern Illinois University, Carbondale. The research was directed by Professor Carl E. Langenhop.

[2] The theory and applications presented in this paper are a result of a literature survey of these topics.

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ANOTHER APPLICATION OF THE MEAN VALUE THEOREMS

by Norman Schaumberger
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The problem of determining whether e^{π} or π^e has the greater value never fails to stimulate the interests of students in a course in calculus. This problem, which was apparently first posed by the Swiss geometer, Jacob Steiner, in the last contury, can be solved in a variety of ways, most of which involve some clever trick. In this note we offer two related approaches that use the mean value theorems of calculus: Thus, they fit nicely into a first course.

Using $f(x) = \log x$, it follows from the mean value theorem of differential calculus that

$$\frac{\log \pi - \log e}{\pi - e} = \frac{1}{c}, \quad e < c < \pi.$$

Hence

$$\frac{\log \pi - \log e}{\pi - e} < \frac{1}{e}$$

or $\log \pi - 1 < \pi/e - 1$. Thus $e \log \pi < \pi < \pi \log e$ and, we have $\log \pi^e < \log e^{\pi}$. Therefore, $\pi^e < e^{\pi}$.

Now letting $f(x) = 1/x$ and using the mean value theorem of integral calculus, we get

$$\int_e^{\pi} \frac{dx}{x} = (\pi - e)\frac{1}{c}, \quad e \leq c \leq \pi.$$

Thus $\log \pi - \log e \leq (\pi - e)\frac{1}{e}$ and as above this produces the desired result.

It is interesting to note that these are not the only functions that will accomplish our purpose. Thus, using $f(x) = x \log x - x$ in the mean value theorem of differential calculus or fix) = $\log x$ in the mean value theorem of integral calculus will in each case give $e^{\pi} > \pi^e$. The details' are similar to those above.

ON DETERMINING FUNCTIONS OF MATRICES

by *Charles V. Allison*
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Since the time of Cayley and Sylvester, there has been great interest in the computation of matrix functions. For example, to compute the matrix exponential e^{At}, which satisfies the matrix differential equation with constant coefficients

$$\dot{X}(t) = AX(t),$$

methods have been developed which rely upon properties of differential equations, the Jordan canonical form, or results from linear algebra such as normality, diagonalizability, etc. ([Coddington 6 Levinson, 1955], [Fulmer, 1975],[Marcus 6 Ming, 1964]). Most techniques for calculating a function f of a matrix A express $f(A)$ as a polynomial in A.

Of all such methods, the simplest in concept are those based on an interpolation formula introduced by [Sylvester, 1883],

$$f(A) = \sum_{i=1}^{n} \Pi_{j\neq i}^{n} \frac{(A - \lambda_j I)}{(\lambda_i - \lambda_j)} f(\lambda_i) \tag{1}$$

which holds when A has distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, A\ )$, lying within the circle of convergence of $f(z)$.

The notion of a matrix function is usually seen for the first time in a matrix analysis course or in a course on the theory of ordinary differential equations, which are graduate courses at most schools. The purpose of this note is to give a development of Sylvester's formula accessible to the sophomore or junior in mathematics.

A proof of (1) follows from the following generalization of the division algorithm, which is a modification of a theorem of [Friedman, 19561.

<u>*Theorem 1.*</u> Let $p(z)$ be a polynomial with distinct roots, and let $f(z)$ be a function analytic in a domain D, which contains the roots of $p(z)$. Then there exists a unique polynomial $r(z)$, where $deg(r) = deg(p) - 1$, and a function $h(z)$, analytic in D, such that

$$f(z) = p(z)h(z) + r(z). \tag{2}$$

Proof. Denote the roots of $p(z)$ by λ_i, $i = 1, \ldots, n$, with $A_i = A_j \Leftrightarrow i = j$. Let $r(z)$ be the unique polynomial of degree n - 1 that agrees with $f(z)$ at each λ_i (this is the LaGrange interpolating polynomial, which is given by the formula

$$r(z) = \sum_{i=1}^{n} \Pi_{j\neq i}^{n} \frac{(z - \lambda_j)}{(\lambda_i - \lambda_j)} f(\lambda_j)\ \ )$$

and defines

$$h(z) = \frac{f(z) - r(z)}{p(z)}. \tag{3}$$

Since each zero of the denominator in (3) is also a zero of the numerator, the singularities of $h(z)$ are removable. i.e.

$$\lim_{z\to\lambda_i} (z - \lambda_i)\, h(z) = 0,$$

hence, $h(z)$ is analytic and the result follows.

In order to compute $f(A)$, we shall let $p(z)$ in the above theorem be the characteristic polynomial of A and consider equation (2). By the Cayley - Hamilton theorem, $f(A) = r(A)$ and

$$f(\lambda_i) = r(\lambda_i),\quad i = 1, \ldots, n. \tag{4}$$

The equations (4) represent a linear system which can be solved for the n-coefficients of $r(z)$, and the calculation of $f(A) = r(A)$ is straightforward.

Notice that the LaGrange interpolating formula for $r(z)$ satisfying (4) shows that $r(A)$ coincides with (1), by which $f(A)$ may be computed directly. We illustrate the two processes.

<u>Example 1</u>

i) Compute $f(A) = e^A$, where $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$. A has characteristic polynomial $p(\sim) = (1 - z)(2 - z)$, hence $\lambda_1 = 1$, $\lambda_2 = 2$. Since $r(z)$ is of the form $a_1 z + a$, we obtain the system

$$e^1 = a_1 + a_0$$

$$e^2 = 2a_1 + a_0$$

with solutions $a_1 = e^2 - e$, $a_0 = 2e - e^2$. Thus,

$$e^A = r(A) = a_1 A + a_0 I$$
$$= (e^2 - e)\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} + (2e - e^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e & 3(e^2 - e) \\ 0 & e^2 \end{pmatrix}.$$

ii) By (1)

$$f(A) = \sum_{i=1}^{2} \prod_{j\neq i}^{2} \frac{(A - \lambda_j I)}{(\lambda_i - \lambda_j)} f(\lambda_i)$$
$$= \frac{\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} e}{-1} + \frac{\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} e^2}{1}$$
$$= \begin{pmatrix} e & 3(e^2 - e) \\ 0 & e^2 \end{pmatrix}.$$

Note that ii) is more efficient for machine computation.

The procedure for the general case follows from Theorem 2, which is based on an extension of (1), first given by [Bucheim, 1886] (cf. [Marcus & Ming, 1964], [Rinehart, 1955]).

Theorem 2. Let $p(z)$ be a polynomial of degree n with k distinct roots, $k \leq n$, and let $f(z)$ be a function analytic in a domain D containing the roots of $p(z)$. Then $r(z)$ and $h(z)$ exist as in Theorem 1 and (2) holds.

Proof. Let m_i denote the multiplicity of each root λ_i of $p(z)$, so that $\sum_{i=1}^{k} m_i = n$. Let $r(z)$ be the polynomial of degree n - 1 that agrees with $f(z)$ at each λ_i, and whose derivatives of all orders up to $m_i - 1$ agree with those of $f(z)$ at each λ_i, i.e.,

$$f^{(j)}(\lambda_i) = r^{(j)}(\lambda_i), \quad j = 0, 1, \ldots, m_i - 1; \; i = 1, \ldots, k. \tag{5}$$

The polynomial $r(z)$ exists and is unique, being merely a form of the general Hermite osculating polynomial [Coddington & Levinson, 1955], [Rinehart, 1955].

We again form the quotient (3) and notice that if λ_i has multiplicity m_i, then λ_i is a zero of order at least $m_i - 1$ of the numerator of $h(z)$, and we apple L'Hospital's rule $m_i - 1$ times to obtain a finite limit,

$$\lim_{z\to\lambda_i} h(z) = \lim_{z\to\lambda_i} h^{(m_i - 1)}(z) < \infty, \quad i = 1, \ldots, k,$$

hence $h(z)$ is analytic in D and (2) follows.

Again we notice that (5) is a system of equations yielding the coefficients of $r(z)$, and we compute $f(A) = r(A)$ as before.

Example 2

Compute $\sin(A)$, where $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. A has characteristic polynomial $p(z) = (1 - z)^2$, hence $\lambda = 1$, $m = 2$. Since $r(z)$ and $r'(z)$ have respectively the forms $a_1 z + a_0$ and a_1, we obtain the system

$$\sin(1) = a_1 + a_0$$
$$\cos(1) = a_1$$

with solutions $a_1 = \cos(1)$, $a_0 = \sin(1) - \cos(1)$.
Then,

$$\sin(A) = \cos(1)\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + [\sin(1) - \cos(1)]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \sin(1) & 3\cos(1) \\ 0 & \sin(1) \end{pmatrix}.$$

For large matrices, it would be computationally more efficient to evaluate $f(A)$ directly from the Hermite formula.

Remarks

Notice that the foregoing development is valid if the minimum polynomial of A is used in place of the characteristic polynomial. The results likewise hold for any scalar function $f(z)$ provided that the right side of (2) is well defined for each characteristic root. The purpose for the requirement of analyticity here was to maintain an elementary exposition by avoiding the subtleties involved in shifting from a scalar to a matrix argument in $f(z)$ (see [Rinehart, 1955]). It is clear that any analytic function can support a matrix argument by virtue of its power series.

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Many regional meeting of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to *$50* for one local chapter to defray postage and other expenses. Send request to:

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HOW NOT TO SOLVE THE QUADRATIC FORMULA

by Professor V. C. Harris
San Diego State University

(Dedicated to William L. Robinson)

The quadratic equation can be written in the form $ax + bx + c = 0$ where a, b, c are assumed to be real and $a \neq 0$. We can divide by a, or instead, without loss of generality, assume $a = 1$, so that we have

$$x^2 + bx + c = 0$$

to solve for x. The solution consists of the values of x so that (x,b,c) lies on the surface given by

$$x^2 + xy + z = 0.$$

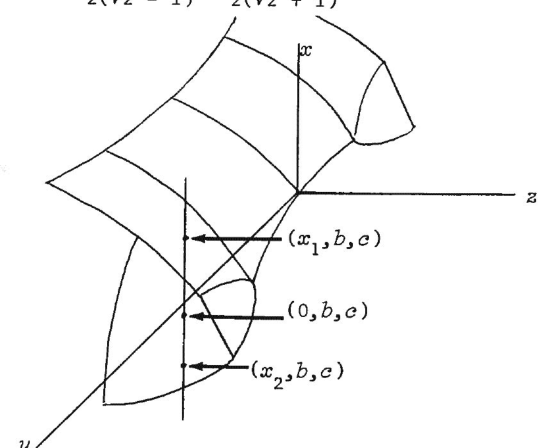
A rotation of $\pi/8$ in the xy-plane:

$$x = \frac{1}{2}\sqrt{2+\sqrt{2}}\,x' - \frac{1}{2}\sqrt{2-\sqrt{2}}\,y'$$

$$y = \frac{1}{2}\sqrt{2-\sqrt{2}}\,x' + \frac{1}{2}\sqrt{2+\sqrt{2}}\,y'$$

transforms $x^2 + xy + z = 0$ into

$$\frac{x'^2}{2(\sqrt{2}-1)} + \frac{y'^2}{2(\sqrt{2}+1)} = z$$

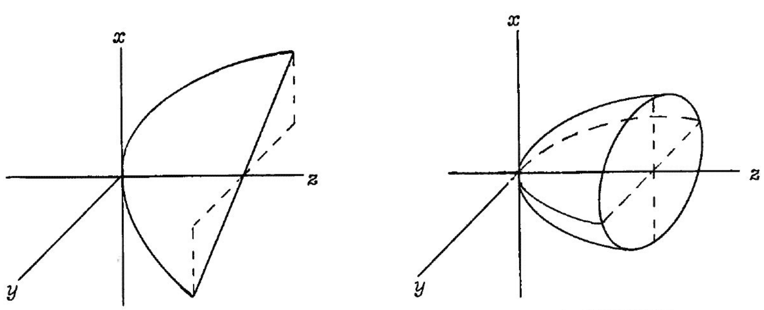


$$x^2 + xy + z = 0$$

FIGURE 1

The graph of this is a hyperbolic paraboloid, as shown in Figure 1; a vertical line through (b,c) in the horizontal yz-plane intersects the surface in the (two or one) real solutions. So the surface represents all solutions of all quadratics with real solutions, that is, for $y^2 - 4z \geq 0$. For a constant, as b varies, the roots move along a hyperbola; for & constant, as a varies, the roots move along a parabola.

When $y^2 - 4z < 0$ the roots are imaginary, the real part of the root is $R(x) = -\frac{1}{2} b$ and the imaginary part is $I(x) = \pm \frac{1}{2}\sqrt{4c - b^2}$. Now two surfaces are given, the part of the plane $x = -\frac{1}{2} y$ for which $z > y^2/4$ and the elliptic paraboloid $x = \pm \frac{1}{2}\sqrt{4z - y^2}$ for the real and the imaginary parts separately;



$x = -\frac{1}{2}y,\ z > \frac{1}{4}y^2$ $\qquad$ $x = \pm\frac{1}{2}\sqrt{4z - y^2}$

FIGURE 2

Lines parallel to the x axis through points (b,c) in the yz-plane give the real and imaginary parts of the solution, respectively. The manner in which the roots vary as b or c varies is evident from the figures.

Thus in these surfaces you see all solutions of all quadratics with real coefficients; any such quadratic can be solved graphically using the figures in Figure 2. But since as every schoolboy knows, in the general case (not necessarily a = 1)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

it is clear that the method of this paper shows how not to solve the quadratic equation.

A GENERALIZATION OF APPOLONIUS' THEOREM

by Hamid Ghannadian
Texas Tech University

In a paper of P. Jordan and J. Van Neumann (3), a theorem of Appollonius was studied very carefully. In fact this theorem is a necessary and sufficient condition for a normed vector space to be an inner product space. In this paper we study a generalization of Appolonius Theorem to Euclidean space of dimension n, for $n \geq 3$.

1. *Notation and Definition*

A Euclidean space of dimension n will be denoted by E_n. Vectors are indicated by Capital Letters. The inner product of two vector A,B will be denoted by (A,B). We shall also use the standard notation of the elementary linear algebra.

2. *Theorem (Apollonius)*

Let $\{A,B\}$ be linearly independent; then $\|A+B\|^2 + \|A-B\|^2 = 2\|A\|^2 + 2\|B\|^2$.

Proof. The Theorem means that: For the parallelogram OA BC, where $C = A + B$, the sum of squares of the diagonals is equal to the sum of squares of the four sides (this is a generalization of the Pythagorem Theorem due to Apollonicus). We observe that

$$\|A+B\|^2 = (A+B,A+B)$$
$$= (A,A) + 2(A,B) + (B,B).$$
$$\|A-B\|^2 = (A-B,A-B)$$
$$= (A,A) + -2(A,B) + (B,B).$$

Therefore: $\|A+B\|^2 + \|A-B\|^2 = 2\|A\|^2 + 2\|B\|^2$.

3. *A Generalization to E_3*

Let $\{A, A_2, A_3\}$ be linearly independent. The parallelepiped generated by this set of vectors has four diagonals which can be denoted by vectors

$$D_1 = A_1 + A_2 + A_3, \qquad D_2 = A_1 + A_2 - A_3,$$
$$D_3 = A_1 + A_3 - A_2, \qquad D_4 = A_2 + A_3 - A_1,$$

and

$$\sum_{j=1}^{4} \| D_j \|^2 = 4 \sum_{k=1}^{3} \| A_k \|^2.$$

Proof. We note that

$$\| D_1 \|^2 = \| A_1 + A_2 + A_3 \|^2.$$
$$= \| A_1 \|^2 + \| A_2 \|^2 + \| A_3 \|^2 + 2(A_1,A_2) + 2(A_1,A_3) + 2(A_2,A_3),$$
$$\| D_2 \|^2 = \| A_1 \|^2 + \| A_3 \|^2 + \| A_2 \|^2 - 2(A_1,A_3) - 2(A_2,A_3) + 2(A_3,A_2),$$
$$\| D_3 \|^2 = \| A_1 \|^2 + \| A_3 \|^2 + \| A_2 \|^2 + 2(A_1,A_3) - 2(A_1,A_2) - 2(A_3,A_2),$$
$$\| D_4 \|^2 = \| A_2 \|^2 + \| A_3 \|^2 + \| A_1 \|^2 + 2(A_2,A_3) - 2(A_2,A_1) - 2(A_1,A_3).$$

Thus:

$$\| D_1 \|^2 + \| D_2 \|^2 + \| D_3 \|^2 + \| D_4 \|^2 = 4\| A_1 \|^2 + 4\| A_2 \|^2 + 4\| A_3 \|^2$$

or

$$\sum_{j=1}^{4} \| D_j \|^2 = 4 \sum_{k=1}^{3} \| A_k \|^2.$$

4. *The Number of Diagonals of a Hyperparallelepiped*

Let $\{ A_1,\ldots,A_n \}$ be a set of linearly independent vectors in E_n. Then the number of diagonals of the hyperparallelepiped generated by this set is 2^{n-1}.

Proof. One observes that the diagonals can be expressed by vectors.

$$D_o = \sum_{j=1}^{n} A_j$$

$$D_k = \sum_{\substack{j \neq k \\ j=1}}^{n} A_j - A_k, \quad k = 1, \ldots, n$$

$$\ldots \qquad\qquad \ldots$$

$$D_{(n-1)k_1,\, k,\, \ldots,\, k_{n-1}} = A_j - \sum_{\substack{k \neq j \\ k=1}}^{n} A_k, \quad j = 1, \ldots, n.$$

$$D_3 = - \sum_{j=1}^{n} A_j$$

But we have considered each vector and its negative. In reality the number of D_3's expressed above is twice as many as the number of diagonals. Now it is clear that the number of elements of the form $Dpk, \ldots kp$ s $\binom{n}{p}$. Therefore the number of all D's is $\sum_{p=0}^{n} \binom{n}{p} = 2^n$. Consequently the number of the diagonals is

$$\tfrac{1}{2} (2^n) = 2^{n-1}.$$

5. Lemma

Let $\{A_1, \ldots, A_n\}$ be a set of linearly independent vectors in E_n. Then the number of terms of the form (A_i, A_3), $i \# j$ of $\| \sum_{j=1}^{n} A_j \|^2$ is $n^2 - n$.

Proof. It is clear that

$$\| \sum_{j=1}^{n} A_j \|^2 = \sum_{j=1}^{n} \|A_j\|^2 + \sum_{\substack{j \neq k \\ j,k=1}}^{n} (A_j, A_k).$$

Since the number of all the terms is n^2 and the number of elements of the form $\|A_j\|^2$ is n, it follows that the number of terms of the form (A_i, A_3), $i \# j$, is $n^2 - n$.

6. Lemma

Let $\{A_1, \ldots, A_n\}$ be a set of linearly independent vectors in E_n. Then the number of negative terms, in

$$\| \sum_{j=1}^{k} A_j - \sum_{j=k+1}^{n} A_j \|^2 \text{ is } 2k(n-k).$$

The proof is quite clear and will be omitted.

7. Lemma

Let us consider the vector D_j's of 54, which are the diagonals of the hyperparallelpiped generated by $\{A_1, \ldots, A_n\}$. In considering $\sum_{i=1}^{2^{n-1}} \|D_j\|^2$, the number of all the terms of the form (A_i, A_3), $i \# j$ is $2^{n-1}\,(n^2-n)$. We shall prove that half of them are negative. In order to prove this we observe that for D_1 we have $(1)\,(n-1)\binom{n}{1}$ negative terms and in general we have $h(n-h)\binom{n}{h}$ in D_h which have h negative vectors. Thus the number of negative terms is

$$N = \sum_{k=1}^{h-1} k(n-k) \binom{n}{k}.$$

One can obtain

$$n \sum_{k=1}^{n-1} k\binom{n}{k} = n(2^n)(\tfrac{n}{2}) - n^2,$$

and

$$\sum_{k=1}^{n-1} k^2\binom{n}{k} = \frac{2^n}{2^2} (n+n^2) - n^2.$$

Therefore

$$N = 2^{n-2} (n^2 - n).$$

8. *Theorem (Generalization of 2)*

Let $D_1, \ldots, D_{(2^n-1)}$ be the diagonals of the parallelepiped generated by $\{A, \ldots, A\}$, a set of linearly independent vectors in E^n. Then

$$\sum_{j=1}^{2^n-1} \|D_j\|^2 = 2^{n-1} \sum_{j=1}^{n} \|A_j\|^2 .$$

Proof. In order to obtain $\sum_{j=1}^{2^n-1} \|D\|^2$ by Lemma 7, we observe the terms of the form (A_i, A_j), $i \neq j$ cancel each other which proves the theorem.

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REFEREES FOR THIS ISSUE

The Editor and the Journal recognize with appreciation the following persons who graciously devoted their time to evaluate papers considered for this issue, and those who served as judges in the Undergraduate Manuscript Contest. Without their help, our work could not be done.

DAVID KAY, University of Oklahoma; **JEFFREY BUTZ**, University of Oklahoma; **JOHN GREEN**, University of Oklahoma; **AL SCHWARZKOPF**, University of Oklahoma; **DUANE PORTER**, University of Wyoming; **LEROY PICKEY**, University of Waterloo; **MARILYN BREEN**, University of Oklahoma; **PATRICK LANG**, Old Dominion University; **DAVID ANDERSON**, Central Washington State College; **CHARLES PARRY**, Virginia Polytechnic Institute and State College; **AL GRIMM**, South Dakota School of Mines and Technology; **ROGER OPP**, South Dakota School of Mines and Technology; **RON WEGER**, South Dakota School of Mines and Technology; **DAVID ROSELLE**, Virginia Polytechnic Institute and State College; **BRUCE PETERSON**, Middlebury College; **JAMES PATTERSON**, South Dakota School of Mines and Technology; **L.CARLITZ**, Duke University; **JOHN HODGES**, University of Colorado; **ROBERT PRIELIPP**, University of Wisconsin-Oshkosh; **E. M. BEESLEY**, University of Neveda-Reno; **IRVING REINER**, University of Illinois; **WARD BOUWSMA**, University of Southern Illinois.

A RESEARCH PROBLEM FOR COMPUTER ASSISTED INVESTIGATION

by Dr. Richard Andree
University of Oklahoma

The following problem is suggested for your investigation. A computer may be helpful in the investigation, but the use of (un)common sense (i.e., mathematical skill) is also desirable. Of course the problem can be investigated without computer assistance if desired.

4 by 4 Prime Squares

There are many arrangements of digits into the 16 cells of a 4 by 4 array such that each row and each column contains a four-digit prime. Two such arrangements might be:

3	3	5	9
5	0	2	1
2	2	7	3
9	3	3	7

3	4	4	9
3	0	2	3
5	2	7	3
9	1	3	7

Your problem is to investigate the possibility of finding one or more such squares which not only contain four-digit primes in each row and each column, but the diagonals (upper left to lower right and lower left to upper right) are also four-digit primes. In the cases here, the diagonals are divisible by 17 and 11 respectively.

You may also wish to investigate similar possibilities for 5-digit and 6-digit primes in appropriate size arrays.

WHAT HE REALLY MEANT WAS .

Any student who has ever sat or slept through a mathematics course knows that certain words and phrases occur very frequently. This glossary might eliminate some confusion.

When the instructor says:	He really means:
(1) trivial	The student might be able to do it in three hours or so.
(2) simple	An "A" student can do it in a week or so.
(3) easy	This topic would make a good Master's Thesis.
(4) clear	The instructor can do it (he thinks).
(5) obvious	The instructor is sure it is in his notes somewhere.
(6) certainly	The instructor saw one of *his* instructors do it but has completely forgotten how it was done.
(7) left as an exercise for the student	The instructor lost his notes.
(8) is well known	The instructor heard that someone once did it.
(9) can be shown	The instructor thinks it might be true, but has no idea how to prove it.
(10) the diligent student can show	It's an unsolved problem--probably harder than Fermat's Last Theorem.

the Editor

PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinguish material which might well be the domain of the Problem Department, but the Editor reserves the right to make an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate, or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March 1, and that for each Spring issue, the following September 15.

THE JOURNAL WISHES TO NOTE THE PASSING OF R. ROBINSON ROWE THIS PAST SPRING. HE WAS A FREQUENT CONTRIBUTOR TO THE JOURNAL AND A GREAT MATHACROSTIC PUZZLE MAKER.

N 1	U 2		Z 3	R 4	D 5	A 6	M 7	U 8	V 9	H 10	L 11	B 12	C 13
	T 14	I 15	P 16		R 17	L 18	C 19	T 20	U 21	X 22	J 23	B 24	E 25
F 26		T 27	X 28		O 29	N 30	D 31	E 32	W 33	G 34		Y 35	I 36
E 37		W 38	J 39		Z 40	C 41	N 42	S 43	M 44	K 45	O 46	Y 47	F 48
I 49		H 50	V 51		M 52	F 53	J 54	L 55	Q 56	H 57	A 58	W 59	
R 60	U 61	Z 62	K 63	W 64	V 65		I 66	Z 67	H 68		U 69	O 70	
G 71	S 72		R 73	W 74	O 75	F 76	J 77	H 78	C 79	S 80		V 81	U 82
D 83	S 84	W 85	L 86	B 87	Q 88	N 89	Z 90	J 91		K 92	W 93	T 94	S 95
E 96	M 97	Q 98	Y 99		T 100	E 101	I 102	P 103	C 104	O 105	K 106	L 107	Q 108
G 109		I 110		S 111	O 112	Z 113	C 114	L 115		F 116	V 117		D 118
B 119	Q 120	V 121	C 122	P 123	K 124	A 125	R 126		L 127	Z 128	H 129	B 130	N 131
Q 132		Q 133	Y 134	I 135		E 136	U 137	J 138	W 139	B 140	M 141	P 142	
K 143	M 144		Y 145	X 146		Z 147	N 148	A 149	C 150	U 151	H 152	X 153	
G 154	S 155		F 156	W 157	Y 158		D 159	H 160	K 161	C 162	V 163		P 164
R 165		N 166	Q 167		L 168	I 169	J 170	H 171	D 172	N 173	C 174	B 175	F 176
O 177		O 178	S 179	L 180	B 181	A 182	Y 183						

A. socket in head of golf club for shaft — 6 149 58 125 182

B. bent at the tip like a hook — 181 119 12 175 24 87 130 140

C. perplexes; puzzles — 150 41 162 19 104 122 174 114 79 13

D. join forces (2 wds.) — 5 172 31 159 118 83

E. garden in which Aristotle taught — 96 37 25 101 32 136

F. envelope of the normals to a curve — 26 176 116 76 53 156 48

G. double-disked toy with string to connecting shaft (comp.) — 109 71 34 154

H. act to suit the time or occasion — 10 68 152 129 50 171 57 78 160

I. in 1896 he and de la Vallée Poussin independently published the first proofs of the Prime Number Theorem — 15 36 49 66 169 110 102 135

J. exposed to capture — 23 91 170 54 77 138 39

K. to whom the intermediate value theorem for derivatives is due — 63 161 92 124 143 45 106

L. unproved assertion; dogmatic statement (2 wds.) — 107 55 180 115 127 11 18 168 86

M. in quaternion algebra, an operator which alters the direction of a vector but not its length — 97 7 144 52 44 141

N. local and temporary anemia due to obstruction of circulation — 89 173 148 131 30 42 1 166

O. de-alcoholized Prohibition drink (comp.) — 46 105 75 70 29 177 112 178

P. characteristic of the null set — 16 123 103 164 142

Q. clues to conjectures — 120 133 88 108 98 56 167 132

R. Bohemian dance in two forms — 73 17 126 165 60 4

S. 'must' pitch for effective hurler — 95 72 155 111 43 179 84 80

T. system with elements related by a reflexive, transitive and anti-symmetric relation — 100 27 94 20 14

U. conjoining of contradictory terms — 61 82 137 8 151 21 69 2

V. E. W. Hornung's roguish cracksman — 65 9 117 51 121 81 163

W. upside down or sideways relative to one another (comp.) — 33 64 139 74 38 59 85 157 93

X. abounding in contingencies — 22 28 146 153

Y. article from a broken set — 145 47 99 35 158 134 183

Z. developer of atmospheric steam engine — 62 147 67 40 90 3 128 113

Mathacrostic No. 6

submitted by Joseph P. E. Konhauser
Macalester College, St. Paul, Minnesota

Like the preceeding five, this acrostic is a keyed anagram. The 183 letters to be entered in the diagram in the numbered spaces will be identical with those in the 26 key words at matching numbers and the key letters have been entered in the diagram to assist in correlation during your solution. When completed, the initial letters will give a famous author and the title of his equally famous book. The diagram will be a quotation from that book. (See preceeding two pages.)

A Single Cut

submitted by Pier Square
University of Inter Polation

Is it possible to make a single cut in a 9 × 16 rectangle, rearrange the two parts and get a 12 × 12 rectangle?

The Bridge Game

submitted by Pier Square
University of Inter Polation

Four men named Banker, Waiter, Baker and Farmer are playing bridge. Each man's name is another man's job. If the baker is Mr. Baker's partner, if Mr. Banker's partner is the farmer and if at Mr. Farmer's right is the waiter, who is sitting on the banker's left?

Solutions

Mathacrostic No. 4 [Fall, 1977]

Also solved by JOSEPH KONHAUSER, *Macalester College;* VICTOR FESER, *Mary College;* RICHARD STRATTON, *Colorado Springs;* CHARLES W. TRIGG, *San Diego.* (See Spring issue for solution.)

Mathacrostic No. 5 [Spring, 1978]

Definitions and key:

A.	Heave	E.	Youngs + modulus	I.	Era	M.	Theta
B.	Equipotential	F.	Differentiate	J.	Newton	N.	Hezekiah
C.	Northwest	G.	Unconditional	K.	Elbow	O.	Epact
D.	Referee	H.	Diophantine	L.	Ypsilanti	P.	Cowwheat
Q.	Adder	V.	Brachistochrone	Z.	Pawpaw	c.	Zoroastrianism
R.	Newest	W.	Utmost	a.	Utica	d.	Leafed
S.	Tamil	X.	Riffraff	b.	Zenos + paradox	e.	Eratosthenes
T.	Eggbeater	Y.	Yarrow			f.	Sieve

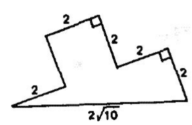
First letters: HENRY DUDENEY THE CANTERBURY PUZZLES

Quotation: ***It is extraordinary what fascination a good puzzle has for a great many people. We know the thing to be of trivial importance yet we are impelled to master it and when we have succeeded there is a pleasure and a sense of satisfaction that are a quite sufficient reward for our trouble even when there is no prize to be won.***

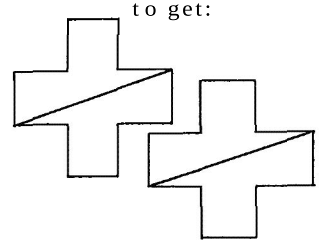
Solved by RICHARD D. STRATTON, *Colorado Springs, Colorado;* VICTOR FESER, *Mary College;* ROBERT PRIELIPP, *University of Wisconsin-Oshkosh;* JOSEPH KONHAUSER, *Macalester College;* SIDNEY PENNER, *B-tonx Community College;* ELENA SABA, *Loqoia University, New Orleans;* LOUIS CAIROLI, *Kansas State University;* SISTER STEPHANIE SLOYAN, *Georgian Court College;* JEANETTE BICKLEY, *Webster Groves High School, Missouri;* DEBRA MULLER, *ADelphi;* ROLAN CHRISTOFFERSON; *the Proposer,* R. ROBINSON ROWE; *and the Editor.*

Greek Crosses and Squares [Fall, 1977]

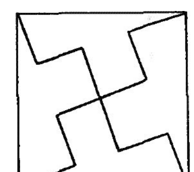
(a) Use 4 of these:



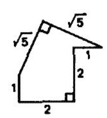
to get:



or to get:



(b) Use 4 of these:



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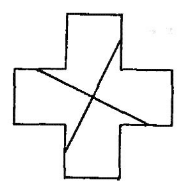

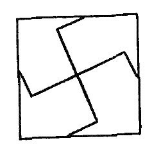

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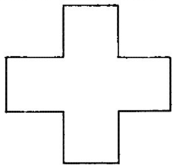

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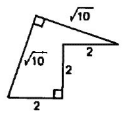

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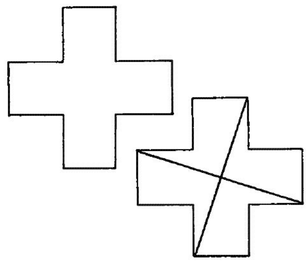
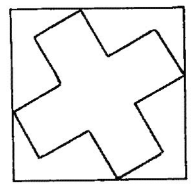
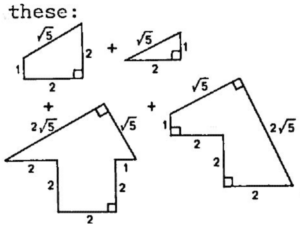


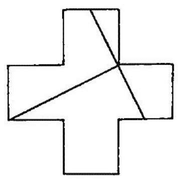
or to get:



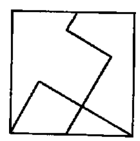
(d) Use 1 each of these:



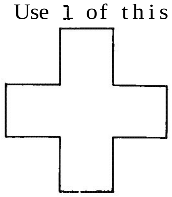
to get:



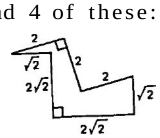
or to get:



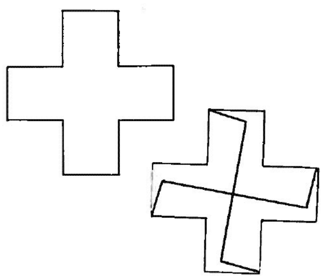
(e) Use 1 of this:



and 4 of these:



to get:



No one has yet found the solution to the giant cross. We leave it open.

Partial solutions by: VICTOR FESER, *Mary College;* RANDALL SCHEER, *SUNY at Potsdam;* RANDY ISTRANEK, *University of Wisconsin-Parkside;* KATHLEEN HENRY, *Iona College;* LOUIS CAIROLI, *Kansas State University; and* ROLAN CHRISTOFFERSON.

Cross-number Puzzle [Fall 1977]

Late solutions were received from PHILIP A. LIPPERT, *Carleton College and* VICTOR FESER, *Mary College.* (See Spring, 1978 issue for solution.)

Countdown [Spring, 1978]

All possible solutions to the following long division problem

```
                9 * * * * * * * *
          -----------------------
* * * * ) * * 8 * * * * * * * * *
          * * * *
          -------
            * 7 * *
            * * * *
            -------
              * 6 * *
              * * * *
              -------
                * 5 * *
                * * * *
                -------
                  * 4 * *
                  * * * *
                  -------
                    * 3 * *
                    * * * *
                    -------
                      * 2 * *
                      * * * *
                      -------
                        * 1 * *
                        * * * *
                        -------
                          * 0 * *
                          * * * *
                          -------
```

are:

1001
917654321
927654321
937654321
947654321
957654321
967654321
977654321
987654321
997654321

1012
917544321
927544321
937544321
947544321
956544321
966544321
976544321
986544321

1013
946544321
956544321

1023
917443321
936543321
946543321
956543321

1034
936443221
965443221

1035
916443221
926443211
945443221
955443211

1045
926342211
936342211
955433211

1046
916342211
926342211
935433211
945433211

1056
926332211
945332211
955433211

1057
916332211
935332211

1067
935332211

1068
925332211
935332211

1079
916141211
925232211

1090
916151211

The top number is the divisor and the list represents the dividends.

Solved by ROLAN CHRISTOFFERSON.

Mathematical Word Chains [Spring, 1978]

Solutions -- One Letter Changes

(a) MATH, MATE, MOTE, NOTE (Bob Prielipp)

(b) LINE, SINE, SIRE, SURE, SURD (Louis Cairoli)

(c) ZERO, HERO, HERD, HELD, HOLD, HOOD, HOOT, ROOT (Kathleen Henry)

(d) SEVEN, SEVER, SAVER, SAYER, PAYER, PRYER, PRIER, PRIES, PRIMS, PRIME (Louis Cairoli)

Solutions -- Two Letter Changes

(a)	(b)	(c)	(d)	(e)
LINES	CIRCLE	GROUP	SLOPE	LINEAR
LURES	CHICLE	GROWS	STORE	LINERS
CURDS	CHICKS	FROGS	STOAT	CIDERS
CURVE	SHACKS	FANGS	STEAD	ORDERS
	SHARDS	RINGS	SQUAD	
	BEARDS	FINES	EQUAL	
	BEASTS	FILLS		
	BEAUTY	FIELD		

(all by Victor Feser)

Solved by VICTOR FESER, *Mary College;* LOUIS CAIROLI, *Kansas State University;* KATHLEEN HENRY, *Iona College;* ROBERT PRIELIPP, *University of Wisconsin-Oshkosh;* RANDY ISTVANEK, *University of Wisconsin-Parkside;* VIRGINIA DWYER, *Clemson University.*

WILL YOUR CHAPTER BE REPRESENTED IN DULUTH?

It is time to be making plans to send an undergraduate delegate or speaker from your chapter to attend the annual meeting of Pi Mu Epsilon in Duluth, Minnesota during August, 1979. Each speaker who presents a paper will receive travel funds of up to $400, and each delegate, up to $200.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

At the suggestion of the Pi Mu Epsilon Council we have had a supply of 10 x 14-inch Fraternity crests printed. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock - - - - - $1.50/dozen,
(2) Purple and lavendar on goldenrod - $2.00/dozen.

LOCAL AWARDS

If your chapter has presented or will present awards this year to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the *Journal.*

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA at Hendrix College heard a variety of papers during the year. *Beth Posey* reported on her trip to the national Pi Mu Epsilon meeting. *Mark Burton, Beth Posey,* Jon *VanDen Heuvel, Janet Dillahunty, Louis Beck, Julie Anderson,* John *Merrill,* and Sandy *Scrimshire* presented papers to the chapter, at the Oklahoma-Arkansas MAA meeting, and in colloquia at the University of Arkansas, the University of the South, and Southwestern at Memphis. Students from the University of Arkansas at Pine Bluff and the University of Mississippi presented colloquia at Hendrix. As a result of this activity, a joint colloquium has been scheduled in the spring of 1979 hosted by Southwestern at Memphis.

Guest speakers and their titles were: *Elizabeth Taylo* , "Least Squares Orthogonal Gram Polynomial Approximations of Commodity Market Prices"; *Dr.* Stephen *Puckette*(University of the South), "Continuous Square Roots of Functions"; *Dr. Gordon Johnson*(University of Houston), "Convex Sets"; *Dr. Satya Deo* (University of Arkansas), "Some Aspects of Cantor Sets"; *Dr. Steve Smith* (Harding College), "The Lighter Side of Mathematics".

FLORIDA *GAMMA* at Eckerd College heard twelve papers: *Robert Meacham,* "Some Problems in Plasticity"; *Bryan Wallace,* "Search for a Viable First Principle of Physics"; *Robert Meacham,* "What Does an Applied Mathematician Do?"; *Irving Foster,* "Variations on a Theme--Living with a 17th Century Mind"; *Douglas Boynton* and *Lars Olson,* "Limits to Growth"; *George Lofquist,* "How Long Should an Amber Light Stay Yellow?"; *Joel Browley,* "Gambler's Ruin"; *Sid Smith,* "Time and Thought Patterns"; *Prof. J. Sutherland Frame,* "Matrix Functions and Applications"; *Rick Parrish,* "Harmonic Series"; *Billy* Maddox, "Transition Matrices"; and *Janet Coursey,* "Linear Programming Approach to the School Bussing Problem".

FLORIDA EPSILON at the University of South Florida heard a variety of papers and talks including: *Dr. Oscar Garcia*, "Mini Computers, Microprocessors and Their Simulation"; *Joni Hersch*, "Infinitesimal Calculus"; *Dr. Don Hill*, "Teaching Mathematics in Africa: A Peace Corps Experience"; *Dr. Robert Shannon*, "Some Observations on the Relationship of Mathematics to Economics"; *Dr. Charles Zaiontz*, "Computability or Uncomputability"; *Dr. Sylvan Block*, "The USF Comstar Satellite Experiment"; *Dr. Kent Nagle*, "Problems of Pursuit: The Submarine-Destroyer Experiment"; *Dr. James Higgins*, "Subjective Probability: An Introduction to Bayesian Statistics"; *Mario Pita*, "Analytic Functions and Elementary Particles"; *Professor A. W. Goodman*, "Chromatic Graphs"; *Professor Seymour Schuster*, "Topics in Graph Theory"; *Dr. John Turner*, "An Introduction to Robust Estimators of Location"; *Dr. Sonia Forseth*, "An Artist Looks at Mathematics"; *Professor Hans Zassenhaus*, "The Work of Gauss in Number Theory and Applied Mathematics"; *Dr. Heinrick Eichorn*, "Least Squares Adjustments of Probabilistic Constraints and Other Oddities"; *Dr. Joseph Liang*, "The Four Color Problem and Its Recent Computer Solution"; *David Williams*, "Fractional Differentiation". In addition. three students presented twenty-five minute talks at the Florida Section of the MAA meeting.

IOWA ALPHA at Iowa State University heard *Professor Richard Sprague's* lecture on "Mathematical Words" at the Annual Initiation banquet.

KENTUCKY GAMMA at Murray State College heard two talks during the year: "Bwornian Movement" by *Dr. Hagood* and "The Relationship Between Mathematics and Computer Science" by *Ross Snyder*. The chapter helped with the Annual Western Kentucky Regional High School Mathematics and Science Fair.

MASSACHUSETTS GAMMA at Bridgewater State College held monthly problem solving sessions and worked as high school and college tutors for the PROGRESS/OUTREACH Program. The chapter heard a lecture by *Dr. Murray Abramson* and *Susan Marshall* on "Mathematics and Education in England."

MISSOURI GAMMA at St. Louis University heard talks by *Barnard Smith*, "Magic Squares, Cubes and Hypercubes"; *Thomas Sweeney*, "0,1 is Not Compact: A Discussion of Hyperreals"; and *Becky Kirkpatrick*, "Variations on Closure of a Set". The chapter participated in the Fourth Annual Pi Mu Epsilon Bi-State Student Conference sponsored by Illinois Delta at SIU-Carbondale.

Missouri Gamma speakers were: *Kathleen Cain*, "Atonal Music - Organized Chaos"; *Alfredo Garcia*, "The Digital Curve - Techniques in Printer Plotting"; *William Kottmeyer*, "A Method for the Solution of Stiff Ordinary Differential Equations"; *Barbara Reynold*, "The Rise and Fall of Roman Numerals"; and *Marie Moranarco*, "Map Projections".

MISSOURI DELTA at Westminster College heard *Professor M. Z. Williams* speak on "Mathematical Games and Puzzles"; *Dr. Kent Palmer* speak on "The Clouds of Venus"; and at the initiation, *Professor Charles Stuth* spoke on "Perfect Numbers - Two Thousand Years of Mathematics".

NEW JERSEY DELTA at Seton Hall held four meetings during the year. The chapter heard *Ronald Infante* speak on "Continued Harmonies and Continued Fractions" and saw two films on mathematics.

NORTH CAROLINA GAMMA at North Carolina State University heard papers by *Professor Armstrong Maltbie*, "Mathematical Potpourri"; *James Bergin*, "The Actuarial Profession - Performance, Training and Opportunities"; *Dr. J. M. Ortega*, "Mathematics and the Real World". The chapter also viewed two mathematical films.

OHIO NU at the University of Akron presented an orientation for newly enrolled mathematics majors and heard *Dr. Jeffrey McLean* speak on "The Artist as a Geometer."

PENNSYLVANIA NU at Edinboro State College heard programs given by: *Dr. Richard Reese*, "Solution to the Four Color Problem"; *Dr. John Lane*, "Mathematical Nonsense"; *Dr. Aiyappan Nair*, "Functions of Regular Variation"; *William Efling*, "Generalized Inverses"; *Dannal Platt*, "Maxwell Distribution". Six chapter members attended the Alleghany Mountain meeting of the MAA.

PENNSYLVANIA THETA at Drexel heard *John Staib* speak on "Solved: The Four Color Problem" and heard *Bruce Wetzel* speak on "Queuing Theory."

SOUTH CAROLINA GAMMA at the College of Charleston heard *Nancy Rallis* speak on topology, *William Caldwell* speak on computers, and *Karen Bell* speak on stochastic processes. In addition, the chapter saw the film, "Let Us Teach Guessing" by Polya.

SOUTH DAKOTA BETA at the South Dakota School of Mines and Technology heard a talk by *Professor David Ballew* on "Finding the Volume of a Hypercube." The chapter helped sponsor the Western South Dakota Mathematics Contest.

TEXAS EPSILON at Sam Houston State (1976-77) heard *Dr. Hunsucker* speak on "Recreational Mathematics" and *Dr. Loeffler* on the topic, "Future Jobs." The chapter also viewed demonstrations by the Chemistry and Physics Departments.

TEXAS EPSILON at Sam Houston State (1977-78) heard *Dr. Rich* speak on four-dimensional geometry and *Dr. Luning* speak on Galileo's and Newton's studies of gravity. The chapter also viewed films and had a job seminar where six people from different fields discussed how education related to their work.

VIRGINIA GAMMA at James Madison University heard *William Sanders* speak at the annual banquet. In addition, the chapter sponsored a book sale as a fund raising activity.

ANECDOTES WANTED

We wish to publish a collection of anecdotes about *well-known mathematicians.* If you are interested in contributing, please write to:

Peter Borwein
Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada
V6T 1W5

or

Maria Klawe
Department of Computer Science
University of Toronto
Toronto, Ontario, Canada
M5S 1A7

For each anecdote please include your source and your assessment of its truth (as a probability between 0 and 1).

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on a separate sheet properly identified with name and address and mailed before the end of June 1979.

Address all communications concerning this department to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

425. *Proposed by Charles W. Trigg, San Diego, California.*

Without using its altitude, compute the volume of a regular tetrahedron by the prismoidal formula.

426. *Proposed by the late R. Robinson Rowe, Sacramento, California.*

With some oversimplification of an actual event, after a cold dry snow had been falling steadily for 72 hours, a niphometer showed a depth of 340 cm., compared to a reading of 175 cm. after the first 24 hours. Assuming that underlying snow had been compacted only by the weight of its its snow overburden, so that the depth varied as a power of time, what would have been the depths after 12 and 48 hours?

427. *Proposed by Jackie E. Fritts, Texas A&M University, College Station, Texas.*

If a, b, c, d are integers and $u = \sqrt{a^2 + b^2}$, $v = \sqrt{(a - c)^2 + (b - d)^2}$ and $w = \sqrt{c^2 + d^2}$, then $\sqrt{(u + v + w)(u + v - w)(u - v + w)(-u + v + w)}$ is an even integer.

428. *Proposed by Solomon W. Golomb, University of Southern California.*

One circle of radius a may be "exactly surrounded" by 6 circles of radius a. It may also be exactly surrounded by n circles of radius t, for any $n \geq 3$, where

$$t = a(\csc \frac{\pi}{n} - 1)^{-1} .$$

Suppose instead we surround it with $n + 1$ cirlces, one of radius a and n of radius b (again $n \geq 3$). Find an expression for b/a as a function of n. (Note: For $n = 3$, $b/a = (3 + \sqrt{17})/2$, and of course for $n = 5$, $b/a = 1$. What about $n = 4$ and $n = 6$ as individual special cases?)

429. *Proposed by Richard S. Field, Santa Monica, California.*

Let P denote the product of n random numbers selected from the interval 0 to 1. Question: Is the expected value of P greater or less than the expected value of the n-th power of a single number randomly selected from the interval 0 to 1?

430. *Proposed by John M. Howell, Littlerock, California.*

Given any rectangle, form a new rectangle by adding a square to the long side. Repeat. What is the limit of the long side to the short side?

431. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, Nw Yohk.*

In a right triangle ABC, with sides a, b, and hypotenuse c, show that $4(ac + b^2) \leq 5c^2$.

432. *Proposed by Erwin Just, Bronx Community College of CUNY, Bronx, New York.*

Does there exist an integer m for which the equation

$$\sum_{i=0}^{m} 3^{ix} = 7^y$$

has solutions in positive integers?

433. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Pay this bill for four. That is, solve for *BILL*, which is divisible by 4.

```
 PAY
  MY
BILL
```


434. *Proposed by Sidney Penner, Bronx Community College of Xke City University of Nw York.*

Consider $(2n + 1)^2$ hexagons arranged in a "diamond" pattern, the kth column from the left and also from the right consisting of k hexagons, $1 \leq k \leq 2n + 1$. Show that if exactly one of the six hexagons adjacent to the center hexagon is deleted then it is impossible to tile the remaining hexagons by trominoes as in Figure 2. (Figure 1 illustrated the 5^2 case in which each of the hexagons adjacent to the center one is labeled A.)

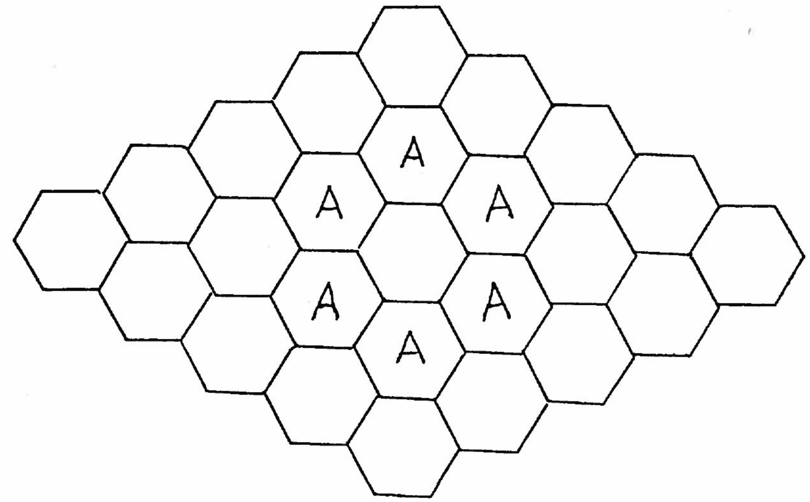


Figure 1

Figure 2

435. *Proposed by David R. Simonds, Rensselaer Polytechnic Institute.*

Two non-congruent triangles are "almost congruent" if two sides and three angles of one triangle are congruent to two sides and three angles of the other triangle. Clearly two such triangles are similar. Show that the ratio of similarity k is such that $\phi^{-1} < k < \phi$, where $\phi = (1 + \sqrt{5})/2$, the familiar golden ratio.

Editor's Note: This old problem is being reopened with the hope of eliciting fresh insights.

436. *Proposed by Carl Spangler and Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

P_1 and P_2 are distinct points on lines L_1 and L_2 respectively. Let L_1 and L_2 rotate about P and P_2 respectively with equal angular velocities Describe the locus of their intersection.

437. *Proposed by Zelda Katz, Beverly Hills, California.*

In times gone by, it was fairly well-known that N, the Nagel point of a triangle, is the intersection of the lines from the vertices to the points of contact of the opposite escribed circles. In the triangle whose sides are AB = 5, BC = 3, and CA = 4, show that the areas of triangles ABN, CAN and BCN are 1, 2 and 3 respectively.

405. [Fall 1976, Corrected] *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

Locate a point P in the interior of a triangle such that the product of the three distances from P to the sides of the triangle is a maximum.

Solutions

399. [Fall 1977] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

Show that arc sin $(\frac{x-3}{3})$ + 2 arc cos $\sqrt{x/6} = \pi/2$, (3 2 x ≤ 6).

I. *Solution by Léo Sauvé, Algonquin College, Ottawa, Canada.*

Let a = arc sin $(\frac{x-3}{3})$ and β = arc cos $\sqrt{x/6}$; then

$$0 \le a\ 5\ \pi/2,\ \sin a = (\frac{x-3}{3}),$$

and $$0\ 5\ \beta\ 2\ \pi/4,\ \cos\beta = \sqrt{x/6},\ \cos 23 = 2\cos^2\beta - 1 = (\frac{x-3}{3}).$$

Since $0 \le a\ 2\ \pi/2$, $0 \le 2\beta \le \pi/2$, and sin a = cos 2B, we must have $a + 2\beta = \pi/2$, which is the required identity.

II. *Solution by Peter A. Lindstrom, Genesee Community College, Batavia, New Yolk.*

Consider the function f defined on [3,6], where

$$f(x) = \text{arc sin } (\frac{x-3}{3}) + 2 \text{ arc cos } \sqrt{x/6}.$$

Differentiating and simplifying,

$$f'(x) = 0,$$

so that $f(x) = c$, a constant. If x = 3, then $c = \pi/2$. Hence arc sin $(\frac{x-3}{3})$ + 2 arc cos $\sqrt{x/6} = \pi/2$, when 3 2 a 5 6.

III. *Solution by Solomon W. Golomb, University of Southern California, Lo'. Angeles, California.*

We prove that for all $\alpha \in [0,1]$,

$$\text{arc sin a t 2 arc cos}\sqrt{\frac{\alpha+1}{2}} = \frac{\pi}{2}\ . \qquad (1)$$

As a corollary, if a = $\frac{x}{3} - 1 = \frac{x-3}{3}$, then $\sqrt{\frac{\alpha+1}{2}} = \sqrt{\frac{x}{6}}$, where now a 6 [0,1] is equivalent to x 6 [3,6]. Hence

$$\text{arc sin } \frac{x-3}{3} \text{ t 2 arc cos } \sqrt{\frac{x}{6}} = \sqrt{\frac{\pi}{2}}, \qquad (2)$$

for all 6 [3,6].

The proof is quite simple. From the definition of the cosine function, if 0 5 9 ≤ $\pi/2$ then sin 9 = a = cos($\pi/2 - \theta$) with 0 2 a 2 1. Thus: arc sin a = 9, arc cos a = $\pi/2 - \theta$, and:

$$\text{arc sin a + arc cos a} = \frac{\pi}{2}\ , \qquad (3)$$

for all a 6 [0,1].

To obtain (1), we simply solve for β in the relation

$$\text{arc cos a} = 2 \text{ arc cos } \beta. \qquad (4)$$

We have: arc cos β = 1/2 arc cos a, and taking the cosine of both sides, $\beta = \cos(\frac{\text{arc cos }\alpha}{2})$. For $\alpha \in [0,1]$, we note that arc cos $\alpha \in [0,\pi/2]$, and in this quadrant the "half angle formula" $\cos\frac{\varphi}{2} = \sqrt{\frac{\cos\varphi + 1}{2}}$ pertains. Thus

$$\beta = \cos(\frac{\text{arc cos a}}{2}) = \sqrt{\frac{\cos(\text{arc cos }\alpha) + 1}{2}} = \sqrt{\frac{\alpha+1}{2}}, \qquad (5)$$

as asserted. Using this in (3) and (4), we obtain (1).

Next, we observe that we need not restrict a to [0,1] in (1), nor x to [3,6] in (2). The same identities hold for -IS $\alpha \le 1$. and for 0 2 x ≤ 6. What happens is that for $-1 \le a \le 0$, arc sin a is in $(-\pi/2,0)$ and arc cos a is in $[\pi/2,\pi]$, but still in such a way that the sum is $\pi/2$. To take the extreme case,

$$\arcsin(-1) + \arccos(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}.$$

Thus with $a = -1$ in (1), we have

$$\arcsin(-1) + 2\arccos(0) = -\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right) = \frac{\pi}{2},$$

and the same occurs with $x = 0$ in (2).

Also solved by JEFFREY BERGEN, *Chicago, Illinois;* LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas;* CLAYTON W. DODGE, *University of Maine at Orono;* MICHAEL W. ECKER, *City University of New York;* MARK EVANS, *La Marque, Texas;* VICTOR G. FESER, *Mary Cottage, Bismarck, North Dakota;* SAMUEL GUT, *Brooklyn, New York;* BRUCE KING, *Schenectady County Community College, Schenectady, New York;* CHARLES H. LINCOLN, *Goldsboro, North Carolina;* LANNIE LIPKE, *Milton, Wisconsin;* C.B.O. PECK, *State College, Pennsylvania;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* KENNETH M. WILKE, *Topeka, Kansas;* CHARLES ZIEGENFUS, *James Madison University, Harrisburg, Virginia; and the Proposer,* JACK GARFUCKEL.

Garfunkel explained how this problem came about. In a recent examination he asked his students to integrate $\int \frac{dx}{\sqrt{6x - x^2}}$. Most of the students completed the square and obtained the correct result:

$$\int \frac{dx}{\sqrt{6x - x^2}} = \arcsin\left(\frac{x-3}{3}\right) + C.$$

One student, however, let $x = 6\cos^2\theta$ and, proceeding correctly, obtained $-2\arccos\sqrt{(x/6)} + C$ as his answer. Hence, within the permissible values for x, these two expressions could differ only by a constant, and the result follows as in Solution II above.

400. [Fall 1977] *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Evaluate $\sum_{k=1}^{m} ([kn/m] + \{kn/m\})$, where m and n are positive integers, $[x]$ is the greatest integer not exceeding x and $\{x\} = -[-x]$ is the smallest integer not less than x.

I. Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.

$\sum_{k=1}^{m} ([kn/m] + \{kn/m\}) = \sum_{k=1}^{m} ([kn/m] - [-kn/m]) = n - (-n) +$

$\sum_{k=1}^{m-1} [kn/m] - \sum_{k=1}^{m-1} [-kn/m] = 2n + \sum_{k=1}^{m-1} [kn/m] - \sum_{k=1}^{m-1} [-(m-k)n/m] = 2n +$

$\sum_{k=1}^{m-1} [kn/m] - \sum_{k=1}^{m-1} [-n+kn/m] = 2n + \sum_{k=1}^{m-1} [kn/m] - \sum_{k=1}^{m-1} (-n+[kn/m])$

[since $[j+x] = j + [x]$ where j is an integer and x is a real number] $=$

$2n + \sum_{k=1}^{m-1} [kn/m] + (m-1)n - \sum_{k=1}^{m-1} [kn/m] = mn + n.$

II. Solution by Richard A. Gibbs, the Proposer, with a practically identical solution by Kenneth M. Wilke, Topeka, Kansas.

Consider the $m \times n$ rectangle in the first quadrant with diagonal joining $(0,0)$ to (m,n). This rectangle contains $(m+1)$ by $(n+1)$ lattice points. For $j = 1, \ldots, m$, $[jn/m]$ enumerates the lattice points in column j above the X-axis which are on or below the diagonal. Also, $\{jn/m\}$ enumerates the lattice points in column j on or above the X-axis which are below the diagonal. By symmetry then, $\{jn/m\}$ enumerates the lattice points in column $m-j$ on or below the line $y = n$ which are above the diagonal. Hence the desired sum enumerates all of the lattice points contained in the rectangle with the exception of the $m+1$ on the X-axis. Therefore

$$\sum_{k=1}^{m} [kn/m] + \{kn/m\} = n(m+1).$$

Also solved by JEFFREY BERGEN, *Chicago, Illinois;* LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas;* MARK EVANS, *LaMarque, Texas;* C.B.A. PECK, *State College, Pennsylvania; and* TERRY J. WOODFIELD, *Beaumont, Texas.*

401. [Fall 1977] *Proposed by Zelda Katz, Beverly Hills, California.*

From a point 250 yards due north of Tom, a pig runs due east. Starting at the same time, Tom pursues the pig at a speed 4/3 that of the pig and changes his direction so as to run toward the pig at each instant. With each running at uniform speed, how far does the pig run before being caught?

This is Problem 28 of ***The Mathematical Puzzles*** of *Sam Loyd,* Volume Two, Dover Publications, 1960. (Selected and edited by Martin Gardner). Loyd's solution is based on the average of the distance traveled by the pig if both ran forward on a straight line and the distance traveled if both ran directly toward each other. How did Loyd arrive at what he calls this "simple rule for problems of this kind" and how can we justify it?

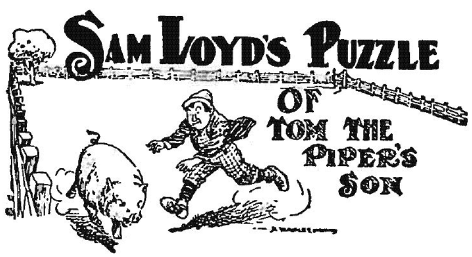


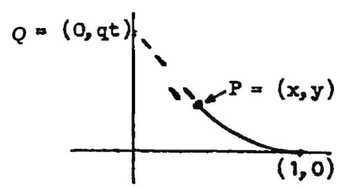
FIGURE 1

Solution by Tom Apostol, California Institute of Technology, Pasadena, California.

A point Q starts at the origin and moves along the positive y-axis with constant speed q. Another point P starts at $(1,0)$ and pursues Q with constant speed p. The problem is to find the curve of pursuit traversed by P.

Denote the coordinates of P at time t by (x,y), where $x = X(t)$ and $y = Y(t)$ are unknown functions of t to be determined. These can be considered as parametric equations of the curve of pursuit. We will obtain a cartesian equation expressing y as a function of x.

At time t, Q is located at the point $(0,qt)$. The tangent line of the pursuit curve at P is always directed towards Q so its slope, dy/dx, is the same as the slope of the line segment PQ. (See the accompanying figure.) Hence



$$\frac{dy}{dx} = \frac{y - qt}{x}$$

from which we get $xy' = y - qt$, where $y' = dy/dx$. Differentiating with respect to x we find

$$xy'' + y' = y' - q\frac{dt}{dx} \quad \text{and hence} \quad \frac{dx}{dt} = -\frac{q}{xy''} .$$

Now q and x are positive and dx/dt is negative so y'' is positive and we have

$$(1) \qquad \left|\frac{dx}{dt}\right| = \frac{q}{xy''} .$$

On the other hand, P moves at constant speed p, which means that

$$(2) \qquad p = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left|\frac{dx}{dt}\right| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} .$$

Using (1) in (2) we obtain

$$(3) \qquad p = \frac{q}{xy''}\sqrt{1 + (y')^2}$$

This is a second-order differential equation satisfied by y. We can solve this by putting

$$v = y', \quad \frac{dv}{dx} = y'' .$$

Then (3) can by written as

$$\frac{q}{p}\sqrt{1 + v^2} = x\frac{dv}{dx}, \quad \text{or } r\frac{dx}{x} = \frac{dv}{\sqrt{1 + v^2}}$$

where $r = q/p$ is the ratio of the two speeds. The equation is now seperable and we can integrate it to obtain

$$r \log x = \log(v + \sqrt{1 + v^2}) + C.$$

But $v = 0$ when $x = 1$ so $C = 0$. Hence $\log x^r = \log(v + \sqrt{1 + v^2})$ so

$$x^r = v + \sqrt{1 + v^2}.$$

Now

$$x^{-r} = \frac{1}{v + \sqrt{1 + v^2}} = -v + \sqrt{1 + v^2}.$$

Subtracting the last two equations we find $2v = x^r - x^{-r}$, so

$$v = y' = \tfrac{1}{2}(x^r - x^{-r}).$$

Integrating again we get

$$y = \frac{1}{2}\left[\frac{x^{r+1}}{r + 1} - \frac{x^{1-r}}{1 - r}\right] + C \text{ if } r \neq 1$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \log x\right] + C \quad \text{if } r = 1.$$

Now $y = 0$ when $x = 1$ and this determines C. We find $C = r/(1 - r^2)$ if $r \neq 1$ and $C = -\tfrac{1}{4}$ if $r = 1$. Hence the curve of pursuit is given by the formula:

$$(4) \qquad y = \frac{1}{2}\left[\frac{x^{r+1}}{r + 1} - \frac{x^{1-r}}{1 - r}\right] + \frac{r}{1 - r^2} \text{ if } r \neq 1 \text{ (that is, if } p \neq q)$$

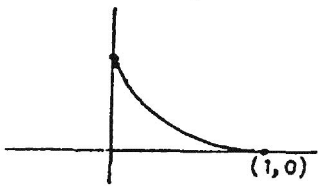
and

(5) $$y = \frac{1}{2}\frac{x^2}{2} - \log x - \tfrac{1}{4} \quad \text{if } r = 1 \text{ (if } p = q\text{)}.$$

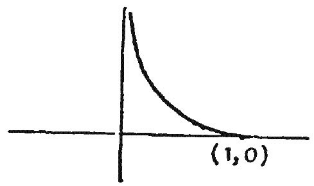
Note. If $r < 1$, both x^{r+1} and x^{1-r} tend to 0 as $x \to 0$ and $y \to r/(1 - r^2)$. In this case, P overtakes Q.

If $r \geq 1$, then $y \to +\infty$ as $x \to 0$ and P never overtakes Q. Therefore the pursuit curves look like this:

$r < 1$ $(q < p)$



$r \geq 1$ $(q \geq p)$



In the foregoing discussion, the point P was initially at (1,0). If it was initially at $(a,0)$, the problem can be reduced to the previous case by changing the scale on both the x and y axes by a factor $1/a$. In other words, if P starts at $(a,0)$, simply replace x by x/a and y by y/a in formulas (4) and (5) to obtain the curve of pursuit.

Now we obtain the arc-length L of the pursuit curve. If $r \geq 1$ this length is infinite, so we consider the case $r < 1$. Again, it suffices to consider the case where P is initially at (1,0). The arc-length L is given by the integral

$$L = \int_0^1 \sqrt{1 + (y')^2}\, dx .$$

Instead of using Equation (4) to calculate the integrand we use the differential equation (3) which gives us

$$\sqrt{1 + (y')^2} = xy''/r .$$

Hence

$$L = \frac{1}{r}\int_0^1 xy''\, dx .$$

This integral, in turn, is easily calculated by using integration by parts. We find

$$L = \frac{1}{r}(xy')\Big|_0^1 - \frac{1}{r}\int_0^1 y'\, dx .$$

Now both y and y' are 0 when $x = 1$ so

$$L = -\frac{1}{r}\int_0^1 y'\, dx = -\frac{1}{r}\{y(1) - y(0)\} = \frac{y(0)}{r} = \frac{1}{1 - r^2} .$$

This can also be written as

$$L = \frac{1}{2}\Big(\frac{1}{1 - r} + \frac{1}{1 + r}\Big) = \frac{1}{2}\Big(\frac{1}{1 - q/p} + \frac{1}{1 + q/p}\Big)$$

$$= \frac{1}{2}\Big(\frac{p}{p - q} + \frac{p}{p + q}\Big) .$$

Now $p/(p + q)$ is the distance P would travel if Q moved along the x-axis directly toward the intial point (1,0) and $p/(p - q)$ is the distance P would travel if Q moved directly away from the initial point (1,0). Thus, we see that the distance traveled by P is the average of these two distances, as asserted by Sam Loyd.

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; MURRAY S. KLAMKIN, University of, Alberta, Edmonton; CONRAD MEMBRINO, Waterbury, Connecticut; LEON MACDUFF, Edinburgh, Scotland; and the. Proposer.

Comment by the. Problem Editor

The charm of this problem lies in the insidious way the stated result defies our intuitive notions. At first we suspect that the result, if true, might accidentally apply only to the special case involving the given distances and the relative speeds but then the denouement affirms the generality of Loyd's method of solution.

Klamkin called attention to his article, co-authored with D.J. Newman, entitled Flying in a Wind Field, published in two parts in the American Mathematical Monthly, January 1969, 16-23 and November 1969, 1013-1019. This paper treats related problems and contains a short list of useful references.

Membrino used the triple-barreled weapons of differential equations, a Texas Instuments SR-56 calculator program and a graphical solution to verify the validity of Loyd's method of solution.

The interested pusuit-problem buff may enjoy looking up the following problems published in the American Mathematical Monthly: 3573 [1932, 549; 1933,4361; E 387 [1939,513; 1940,3201; 3942 [1940,114; 1941,4841. See also a short paper entitled A Pursuit *Problem* by Gerald Crough, Mathematics Magazine, March-April 1971, pp. 94-97.

402. [Fall 19771 Proposed by Charles W. Trigg, San Diego, California.

The first eight non-zero digits are distributed on the vertices of a cube. Addition of the digits at the extremities of each edge forms twelve edge-sums. Find distributions such that every edge-sum is the same as the sum on the opposite (non-cofacial) edge. [The solution to the related problem 304 appears on pages 36-37 of the Fall 1974 *PI* MU EPSILON **JOURNAL.**]

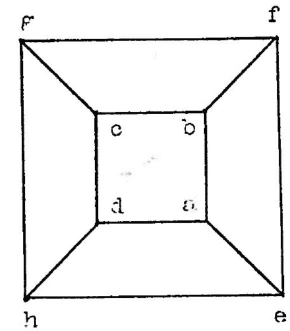
Solution by Clayton W. Dodge, University of Maine at Orono.

Label the vertices as shown in the accompanying figure. Then we must have

$$a + b = g + h, \qquad a + d = f + g, \qquad a + e = c + g,$$
$$c + d = e + f, \qquad b + c = e + h, \qquad b + f = d + h.$$

If we solve the first three equations for h, f, and c respectively and substitute these values into the last three equations, we obtain identities, showing that the last three equations are redundant. Thus we solve each of the first three equations for $a - g$, obtaining $\delta = a - g = h - b = f - d = c -$ To have four equal differences using the digits 1 to 8, we can take only $|\delta| = 1$, 2, or 4. To show, for example, that we cannot have $\delta = 3$, note that 8 must be paired with 5 and 7 with 4. But also 1 must be paired with 4, so $\delta = 3$ cannot be permitted. Now, once δ has been chosen and a permissible value assigned to a, then there are just three values from which to choose b, d, and e. Then the other four values are determined. These solutions are all equivalent since Euclidean transformations of the cube map the solutions one to the other. It is seen, then, that there are just three distinct solutions:

a	b	c	d	e	f	g	h
8	5	2	3	1	4	7	6
8	5	3	2	1	4	6	7
8	1	7	2	3	6	4	5



Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; MARK EVANS, LaMarque, Texas; VICTOR G. FESER, Mary College, Bismarck, North Dakota; CLAUDIA WILCOX, LaMarque, Texas; CHARLES H. LINCOLN, Goldsboro, North Carolina; KENNETH M. WILKE, Topeka, Kansas; and the Proposer, CHARLES W. TRIGG.

404. [Fall 1977] Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.

Let x be a positive integer of the form $24n - 1$. Prove that if a and b are positive integers such that $x = ab$, then $a + b$ is a multiple of 24.

Solution by Clayton W. Dodge, University of Maine at Ohono.

If $ab = 24n - 1$, then $ab \equiv -1 \pmod{24}$. Permissible solutions $\{a,b\}$, modulo 24, are $\{1,-1\}$, $\{5,19\}$, $\{7,17\}$, and $\{11,13\}$. In each case we see that $a + b \equiv 0 \pmod{24}$. The theorem follows.

Also solved by RONNY ABOUDI, student at Florida Atlantic University, Coral Springs, Florida; JEFFREY BERGEN, Chicago, Illinois; M.J. OELEON, Florida Atlantic University, Boca Raton, Florida; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; VICTOR G. FESER, Mary College, Bismarck, North Dakota.; F. DAVID HAMMER, University of California, Davis; CHARLES H. LINCOLN, Goldsboro, Notvth Carolina; BLACKWELL SAWYER, student a Florida Atlantic University; DALE WATTS, Denver University, Colorado Springs, Colorado; CHARLES ZUEGENFUS, James Madison University, Harrisburg, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the Proposer, BOB PRIELIPP.

405. [Fall 1977] Corrected version appears in the Proposal Section of this issue. Solution will appear in the Spring 1979 issue.

406. [Fa11 19771 *Phopobed by Paul Erdös, Spaceship Earth.*

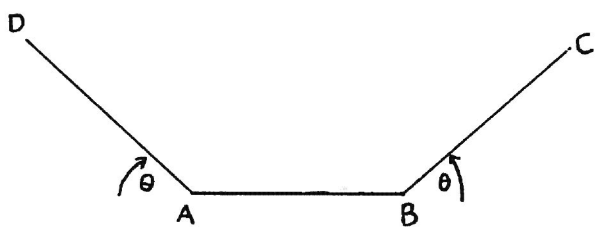
Let there be given 5 distinct points in the plane. Suppose they determine only two distances. Is it true that they are the vertices of a regular pentagon?

Solution by Kenneth M. Wilke, Topeka, Kansas.

The answer is "yes".

Proof: Let the smaller distance be taken as a unit distance and let l denote the longer distance. Then if one point is removed from the configuration, the remaining four points also determine only two distances. For otherwise if the four points determine only one distance, then three of these points likewise determine only one distance; hence these three points are the vertices of an equilateral triangle. Then if the fourth point lies inside or outside this triangle, a second distance (at least) is determined. If the fourth point lies on the perimeter of the triangle, either a second distance is determined or the fourth point coincides with one of the other three, which is impossible since the points are distinct. Hence the four points determine exactly two distances.

Next let A and B be two of the four points which are separated by the unit distance on the coordinate axis. Let C and D be the other two points which lie on the unit radii AD and BC. Let $DABC$ begin as a straight line with radius DA rotating clockwise at the same rate CB rotates counter-clockwise as shown in the figure. From the figure, $BD = AC$. It remains to determine the positions in which $BD = AC = CD$ or $CD = AB$.



This analysis produces three cases: I) Points C and D coincide, in which case point D can be taken as the reflection of C in line AB, which produces a figure formed by two equilateral triangles having common edge AB; II) $CD = AB$, in which case we have $ABCD$ as a square with diagonals AC and BD; and III) an isosceles trapezoid $ABCD$, in which $AC = BD = BC$. In these three cases, we have $l = \sqrt{3}$, $\sqrt{2}$ and $(1+\sqrt{5})/2$ respectively.

Case I. ($l = \sqrt{3}$). Since ABC and ABD are equilateral triangles with common edge AB and $BD = l = \sqrt{3}$, symmetry requires consideration of possible placement of the fifth point in only three cases. First consider possible placements equidistant from B and D. Then the fifth point lies on the perpendicular bisector of BD. Regardless of whether the distance from the fifth point to B or D is 1 or l, a third distance is introduced. Similar analysis reveals similar results when A and B or A and D replace B and D.

Case 11. ($l = \sqrt{2}$). Here $ABCD$ forms a square. If E is a fifth point on the perimeter or inside the square, only if E is the intersection of the diagonals BD and AC does it deserve consideration. But then $AE = BE = CE = DE = \sqrt{2}/2$, a third distance. If E lies outside the square, analysis similar to that used in Case I establishes that a third distance is introduced regardless of the location of E.

Case 111. ($l = (1+\sqrt{5})/2$). Here $ABCD$ forms an isosceles trapezoid in which $AB = BC = AD$ and the longer parallel side (parallel to AB) $CD = AC = BD$, the diagonals. Third distances arise for all possible locations of the fifth point E, equidistant from A and D or from B and C. Points equidistant from A and B or from C and D lie on the perpendicular bisctors of AB and CD respectively. (These perpendicular bisectors coincide.) In either case the unique location for E which does not introduce a third distance places E one unit from C and D and l units from A and B. This is procisely the location of the fifth vertex of the regular pentagon.

Case IV. The only other configuration of four points involving exactly two distances occurs when A, B and C are the vertices of an equilateral triangle and D is its centroid. Here $l = \sqrt{3}$ and the smaller distance is a unit distance. Analysis similar to that used in the other cases shows that each possible location of the fifth point E requires the introduction of a third distance.

Hence if five distinct points in the plane ditermine exactly two distances, these points lie at the vertices of a regular pentagon.

Comment by Louis H. Cairoli, Kansas State University.

Chapter 12 of *Mathematical Gems II* by Ross Honsberger is entitled "The Set of Distances Determined by n Points in the Plane". This chapter contains an excellent summary of the early work (1946) of Erdös on this interesting subject.

The results include: For n points in the plane, $n = 3, 4, 5, \ldots$

(I) There are at least $\sqrt{n - 3/4} - 1/2$ different distances.

(II) The mimimum distance can occur not more than $3n - 6$ times.

(III) The maximum distance can occur only n times.

(IV) No distance can occur as often as $n^{3/2}/\sqrt{2} + n/4$ times.

Comment by Joe Konhauser, Macalester College, Saint Paul, Minnesota.

The answer is "Yes". The result is Theorem 4 in the paper "On Euclidean Sets Having Only Two Distances Between Points" by S.J. Einhorn and I.J. Schoenberg, which appeared in the *Proceedings*, Series A, 69, No. 4 and *Indagationes Math.*, 28, No. 4, 1966.

There are exactly 26 2-valued 5-point sets in E_3, none of which are in E_2.

407. [Fall 1977] *Proposed by Ben Gold, John Howell and Vance Stine, Los Angeles City College.*

Two sets of dice are rolled. $(n = 1, 2, 3, 4, 5, 6)$. What is the probability of k matches? $(k = 0, 1, \ldots, n)$

Solution by Mitchell Entrican, University of Mississippi.

The probability of two thrown dice matching is 1/6; then the probability of two thrown dice matching is 5/6. Therefore, for two sets of n dice thrown, the probability of k matches is $(1/6)^k\,(5/6)^{n-k}$ times the number of possible ways to arrange the matches, which is the combination of n dice taken k at a time. This combination of n dice taken k at a time can be denoted $\binom{n}{k}$. Since $(1/6)^k\,(5/6)^{n-k}$ is equal to

$$\frac{1}{6^k}\cdot\frac{5^{n-k}}{6^{n-k}}\cdot\frac{5^{n-k}}{6^n},$$

the solution will then be

$$\binom{n}{k}\frac{5^{n-k}}{6^n}.$$

408. [Fall 1977] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Squares are erected on the sides of a triangle, wither all externally or all internally. A circle is centered at the center of each square with each radius a fixed multiple $k > 0$ of the side of that square. Find k so that the radical center of the three circles falls on the Euler line of the triangle and find where on the Euler line it falls. (See Fig.)

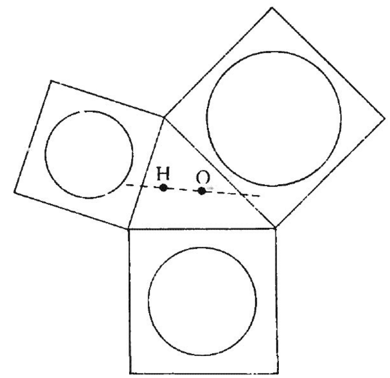


FIGURE 2

Solution by the Proposer.

Place triangle ABC, oriented counterclockwise, in the Gauss plane so that $|a| = |b| = |c| = 1$. Let $\alpha = (1/\sqrt{2})\exp(i\pi/4) = 1/2 + i/2$. Let D, E, F be the centers of the externally-erected squares on sides BC, CA, and AB respectively. Then

$$d = ab + \bar{\alpha}c, \qquad e = \alpha c + \bar{\alpha}a, \quad \text{and} \quad f = \alpha a + \bar{\alpha}b.$$

We must find a point P on the Euler line so that, for some k, the power of P with respect to each circle is the same; that is,

$$PD^2 - (k\cdot BC)^2 = PE^2 - (k\cdot CA)^2 = PF^2 - (k\cdot AB)^2.$$

It is sufficient to find k and P so that a cyclic permutation of a, b, and c in the expression for, say, $PD^2 - (k\cdot BC)^2$ leaves the expression unaltered.

Since P is to be taken on the Euler line and the circumcenter of triangle ABC is at the origin in the complex plane, then

$$p = m(a + b + c)$$

for some real m. (If $m = 0$, 1/3, 1/2, or 1, then P is the circumcenter, centroid, ninepoint center, or orthocenter.) Now, recalling that $a\bar{a} = b\bar{b} = c\bar{c} = 1$ and that $|z|^2 = z\bar{z}$, we have

$$\begin{aligned}
&PD^2 - (k\cdot BC)^2\\
&= |m(a + b + c) - (\alpha b + \bar{\alpha}c)|^2 - |k(b - c)|^2\\
&= |ma + (m - \alpha)b + (m - \bar{\alpha})c|^2 - k^2|b - c|^2\\
&= (ma + (m - \alpha)b + (m - \bar{\alpha})c)(m\bar{a} + (m - \bar{\alpha})\bar{b} + (m - \alpha)\bar{c}) -\\
&\quad k^2(b - c)(\bar{b} - \bar{c})\\
&= m^2 + 2(m - \alpha)(m - \bar{\alpha}) + m(m - \bar{\alpha})(a\bar{b} + \bar{a}c) + m(m - \alpha)(a\bar{c} + \bar{a}b) +\\
&\quad (m - \alpha)^2\, b\bar{c} + (m - \bar{\alpha})^2\, \bar{b}c - k^2(2 - \bar{b}c - b\bar{c}).
\end{aligned}$$

The cyclic permutations of $a\bar{b}$ are $b\bar{c}$ and $c\bar{a}$. Equating the coefficients of these terms, we get

$$m(m - \bar{\alpha}) = (m - \alpha)^2 + k^2 = m(m - \bar{\alpha}).$$

Similarly, for the permutations $\bar{a}b$, $\bar{b}c$, $\bar{c}a$, we must have

$$m(m - \alpha) = (m - \bar{\alpha})^2 + k^2 = m(m - a),$$

just the conjugate of Equations (1). Subtracting these two left hand equations, we find that

$$m(m - \bar{\alpha}) - m(m - \alpha) = (m - \alpha)^2 - (m - \bar{\alpha})^2,$$
$$m(\alpha - \bar{\alpha}) = (2m - \alpha - \bar{\alpha})(\bar{\alpha} - \alpha),$$
$$m = \alpha + \bar{\alpha} - 2m,$$
$$3m = \alpha + \bar{\alpha} = 1,$$
$$m = 1/3.$$

So $P = G$, the controid of triangle ABC. Substituting $m = 1/3$ into Equations (1), we get

$$(1/3)(1/3 - \bar{\alpha}) = (1/3 - \alpha)^2 + k^2,$$
$$k^2 = (1/3)(-1/6 + i/2) - (-2/9 + i/6)$$
$$= -1/18 + i/6 + 2/9 - i/6 = 1/6,$$

so $k = 1/\sqrt{6}$.

We find then that the power of G with respect to each of the three circles is given by

$$1/18 + (-1/18 + i/6)(a\bar{b} + b\bar{c} + c\bar{a}) + (-1/18 - i/6)(\bar{a}b + \bar{b}c + \bar{c}a).$$

If the squares are to be erected internally, then orient triangle ABC clockwise, and the above proof holds unaltered.

409. *[Fall 1977] Proposed by Zelda Katz, Beverly Hills, California.*

A point E is chosen on side CD of a trapezoid $ABCD$, (AD $\parallel$ BC), and is joined to A and B. A line through D parallel to BE intersects AB in F. Show that FC is parallel to AE. (See Fig. 3)

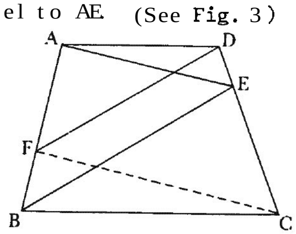


Figure 3

Solution by Kenneth M. Wilke, Topeka, Kansas.

Let AE intersect BF at G and let BE cut FC at H. Let EF and GH intersect at I.

It is easily seen that triangles FGI and EHI are similar so that $GI/HI = IF/IE$ with the result that triangles FIH and GIE are also similar. It follows that FGEH is a parallelogram and that FC and AE are parallel.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* ROBERT C. ERTLE, *Racine, Wisconsin;* DONALD CANARD, *Anaheim, California; and* ZELDA KATZ, *the Proposer.*

410. [Fall 1977] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

If x, y, z are the distances of an interior point of a triangle ABC to the sides BC, CA, AB, show that

$$1/x + 1/y + 1/z \geq 2/r$$

where r is the inradius of the triangle.

Solution by the Proposer.

Since $ax + by + cz = 2\Delta$, where a, b, c are the sides opposite the vertices A, B, C, it follows from Cauchy's inequality that

$$1/x + 1/y + 1/z \geq (\sqrt{a} + \sqrt{b} + \sqrt{c})^2/2\Delta$$

with equality iff

$$x\sqrt{a} = y\sqrt{b} = z\sqrt{c} = 2\Delta/(a + b + c).$$

Consequently

$$\min\{1/x + 1/y + 1/z\}\, r = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2/(a + b + c).$$

We now show that the minimum of the r.h.s. over all triangles is two. Let $d = \sqrt{a}$; $e = \sqrt{b}$; $f = \sqrt{c}$; then d, e, f are the sides of a non-obtuse triangle and

$$d^2 \geq (e - f)^2, \quad e^2 \geq (f - d)^2, \quad f^2 \geq (d - e)^2.$$

Adding and rearranging gives

$$\frac{(d + e + f)^2}{d^2 + e^2 + f^2} \geq 2$$

with equality iff $def = 0$.

Remark: In a similar fashion, it is easy to obtain the known inequality

$$h_1/x + h_2/y + h_3/z \geq 9$$

where h_i is the altitude from A_i

411. [Fall 1977] *Proposed by* R.S. *Luthar, University of Winsconsin, Janesville.*

Find all polynomials $P(x)$ such that

$$P(x^2 + 1) - [P(x)]^2 - 2x[P(x)] = 0 \text{ and } P(0) = 1.$$

I. Solution by M.S. Klamkin, University of Alberta.

Letting $P(x) + x = G(x)$, we obtain

(1) $$G(x^2 + 1) = G(x)^2 + 1, \quad G(0) = 1.$$

One obvious solution is $G(x) = x^2 + 1$.

Now let $G(x) = x^2 + 1 + H(x)$, giving

$$H(x^2 + 1) = 2(x^2 + 1)H(x) + H(x)^2, \quad H(0) = 0.$$

It now follows that $H(x_n) = 0$ where

$$x_{n+1} = x_n^2 + 1, \quad x = 0. \quad \text{Whence, } H(x) \equiv 0 \text{ and thus}$$

$P(x) = x^2 - x + 1$ uniquely.

As an extension of (1), consider finding all polynomials $P(x)$ satisfying

$$P[Q(x)] = Q[P(x)]$$

where $Q(x)$ is a given polynomial and $P(0) = Q(0) = 1$. Then if the sequence defined by $a_{n+1} = Q(a_n)$, $a_0 = 0$, consists of an infinite number of different values, $P(x) = Q(x)$.

Proof: Let $P(x) = Q(x) + F(x)$; thus $F(0) = 0$. Then

$$F[Q(x)] = Q[Q(x) + F(x)] - Q[Q(x)],$$

and $F(a_n) = 0$ for $n = 0, 1, 2, \ldots$. Whence, $F(x)$ is identically zero.

Comment: The problem (1) where $G(0) = 0$, leading to $G(x) = x$, was set as a problem in the 1971 Putnam Intercollegiate Mathematics Competition by D.J. Newman. I had given this problem as practice to the 1975 USA International Mathematical Olympiad team. One of the members of the team, Bernard B. Beard, had extended the problem and it appeared as Problem 965 in the Mathematics Magazine 50(1977), 166: "Find all polynomials $P(x)$ such that $P[F(x)] = F[P(x)]$, $P(0) = 0$, where $F(x)$ is a given function satisfying $F(x) > x$ for all $x \geq 0$.

Also, in the extension of (1) given above we can relax the condition that $Q(x)$ be a polynomial if we let $P(a_0) = a_0$ instead of $a_0 = 0$ and $P(0) = Q(0) = 1$.

II. Solution by the Proposer.

$$P(0) = 1 = (0 - 1)^2 + 0$$

$$P(1) = 1 = (1 - 1)^2 + 1$$

$$P(2) = 3 = (2 - 1)^2 + 2$$

$$P(5) = 21 = (5 - 1)^2 + 5$$

$$P(26) = 651 = (26 - 1)^2 + 26.$$

Thus the polynomial in question agrees with $(x - 1)^2 + x$ for more values of x than the degree of $P(x)$. Therefore $P(x) \equiv (x - 1)^2 + x = x^2 - x + 1$.

Also solved by JEFFREY BERGEN, *Chicago, Illinois;* L. CARLITZ, *Duke University, Durham, North Carolina;* LOUIS CAIROLI, *Kansas State University, Manhattan, Kansas;* MARK EVANS, *La Marque, Texas;* CHARLES H. LINCOLN, *Goldsboro, North Carolina;* KENNETH M. WILKE, *Topeka, Kansas.*

ERRATA (Discovered by Charles W. Trigg)

Vol. 6:7 - Fall 1977 - p. 428, 6th line from bottom, "215 IC 465" should read "215 ICE 465".

Vol. 6:8 - Spring 1978 - p. 487 -line 9- should start with PQ/2 = XN instead of ZN. On page 488, problem numbered 388 should read 389.

In the Spring 1978 issue, MICHAEL W. ECKER *should have received credit for solutions to problems* 395 *and* 396. *In the same issue,* CLAYTON W. DODGE *was inadvertently omitted from the list of solvers of problems* 392, 394, *and* 395.

FRATERNITY KEY-PINS

Gold clad key-pins are available at the National Office (the University of Maryland) at the special price of $5.00 each, post paid to anywhere in the United States. The price will raise to $8.00 effective July 1, 1979. Order soon!

Be sure to indicate the chapter into which you were initiated and the approximate date of initiation.

LOCAL CHAPTER AWARDS WINNERS

ARKANSAS BETA (Hendrix College). The *McHenry-Lane Freshman Math Awards* were given to

Michael Pinter,
and *Sandra Cousins.*

The *Hogan Senior Math Awards* were presented to

Mark Burton,
and *Beth Posey.*

GEORGIA BETA (Georgia Institute of Technology). Book awards were given to cutstanding graduates in mathematics:

James Herndon III,
James Novotnak,
and *Maria Santana.*

FLORIDA EPSILON (University of South Florida). The *Outstanding Scholar Award* was given to

David Ian Williams.

ILLINOIS DELTA (SIU/Carbondale) (1977-78). The *Outstanding Senior Award* was given to

Steven Winker.

The *SIU Putnam Competition* winner was

Ying-Chen Kwong.

The outstanding talk at the Regional Pi Mu Epsilon meeting was given by

Susan Long.

ILLINOIS DELTA (SIU/Carbondale) (1976-77). The *Outstanding Senior Award* winners were

Joe Gibson
and *Gordon Huffman.*

The *SIU Putnam Competition* winner was

James Bellinger.

IOWA ALPHA (Iowa State University). *Pi Mu Epsilon Scholarship*

Awards of $50 each were presented to

Timothy Tjarks
and *Robert Cmelik,*

who scored highest on a competitive examination.

MISSOURI GAMMA (St. Louis University). The *James Garneau Award* (based on grade point average) was given to

John Sladen.

The *Francis Regan Scholarships* were presented to

Gerrianne Vogt
and *Gary Szatkowski.*

The Senior Pi Mu Epsilon Contest ($25 in cash, $25 in mathematics books) was won by

Michele Pomash (Maryville College).

The Junior Pi Mu Epsilon Contest ($25 in cash, $25 in mathmatics books) was collected by

John Roth (Maryville College).

The *Missouri Gamma Awards* were presented to

Karin Angeli (Frontbonne College),
Michele Pomash (Maryville College),
and *Delores Flores* (Maryville College).

The *Missouri Gamma Graduate Award for Scholarship and Service to the Fraternity* was given to

Fr. Joseph Raj.

The *John H. Andrews Graduate Service Award* for active participation was given to

Tom Sweeney.

The *Al and Sheely Beradino Fraternityship Award* for helpfulness, friendliness and concern was won by

Jeanette Medewitz.

NEW YORK PHI (State College of New York, Potsdam). The *Outstanding Senior Award* (Vols. 2 and 3 of Knuth: *The Art* of Computer *Programming*) was presented to

Ron Olsson.

OHIO EPSILON (Kent State University). The *Pi Mu Epsilon Awards* ($25 in books plus a plaque) were presented to

Robert Ulrich

and Kenneth Weber.

OHIO NU (University of Akron) (1976-77). Awards were given to Akron Regional Science Fair winners and mathematics majors who excelled in their course of study.

OHIO NU (University of Akron) (1977-78). The Akron Regional Science Fair winners were

Ben Chang

and Brenda Smith.

The *Outstanding Undergraduate Students* who were awarded student memberships to MAA were

Donald Asher,

Gary Giorgio,

Chris Kolaczewski,

Ruth Nielsen,

Mary Ruckeh,

and Jerry Young.

The *Samuel Selby Mathematics Scholarship Award* was given to

Donald Asher.

SOUTH DAKOTA ALPHA (University of South Dakota). The *William Ekman Awards* were presented to

Thomas Bylander

and William Even.

The *Mascott Awards* for deserving students majoring in mathematics went to

Melamie Morgan

and Kurt Lovrien.

The *Merton Hasse Award* for a minority student majoring in mathematics went to

Ip wang Chan.

The *Thomas Emery McKinney Awards,* to senior mathematics majors who have shown the most power and originality in mathematics went to

Denis Guenthner

and Dennis Freidel.

The *Pell Scholarships* given to promising students were awarded to

David Barnes

and Rick Wiese.

The *Pi Mu Epsilon Award* which is presented to a senior mathematics major for outstanding scholarship and service to the department was given to

Colleen Locken

and Thomas Severson.

The *Robert L. Walter Award* presented to students interested in mathematics education went to

Jean Groeber Johnson

and Cindy Rohde.

TEXAS EPSILON (Sam Houston State University). The outstanding freshman mathematics major was

Julie Montgomery.

The outstanding junior mathematics major was

Sandy Mlcak.

The outstanding senior mathematics major was

Ronnie Webb.

VIRGINIA GAMMA (James Madison University). The outstanding senior mathematics student was

Willaim Grubbs.

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PAPERS PRESENTED AT THE 1978 PROVIDENCE MEETING OF THE PU MU EPSILON FRATERNITY

1. *Why I Like Graph Theory*	NANCY L. BURGER NY Phi State University of Colorado at Postdam
Calulation of the Period of the Lotka-Volterra Predator Prey Model	KATHY STUEWE TN Delta University of Tennessee
3. *The Unique Number 15*	J. B. ZIPPERER, JR. GA Gamma Armstrong St. College
4. *A Paradox in Quantum Theory*	LINDA WHILEYMAN TX Epsilon Sam Houston State University
5. *Is It Possible to Lose the Ol' Magic?*	DOUGLAS W. BOONE OH Delta Miami University
6. *The Double Ferris Wheel Problem*	ALBERT E. PARISH SC Gamma College of Charleston
7. *The Use of Fractional Calculus in Solving Certain Difference Equations*	DAVID CHALLENER IA Alpha Iowa State University
8. *Causes of Math Anxiety at the University*	KATHLEEN V. WALKER IL Zeta Southern Illinois University at Edwardsville
9. *Non-linear Additive Functions*	JULIE D. ANDERSON AR Beta Hendrix College
10. *The Converse of Lagrange's Theorem and Finite Nilpotent Groups*	MARCIA JAMES NC Delta East Carolina University
11. *Matrix Models in Biology*	MICHAEL YOUNG OR Gamma Portland State University

1978 J. SUTHERLAND FRAME LECTURE

The Statistics of Incidents and Accidents	PROFESSOR HERBERT E. ROBBINS

12. *Multivariate Discriminant Analysis and the Prediction of Loan Defaults*	NICK BELLOIT FL Eta University of Northern Florida
13. *Scaling in Mammals*	TIMOTHY O'SHEA NJ Epsilon St. Peter's College
14. *Concerning Irreducible Compact Continua*	W. DWAYNE COLLINS TX Theta University of Houston
15. *Exam Scheduling: An Example of Math Modeling*	CAROLE H. COOK OH Delta Miami University
16. *Generalized Lipschitz Criteria for First Order Differential Eauations*	MARK L. BURTON AR Beta Hendrix College
17. *A Method of Finding the Complement of a Sequence*	SUSAN MCCLINTOCK NC Delta East Carolina University
18. *A Pulse-time Model for Mathematics Class Enrollments*	JULIE MONTGOMERY TX Epsilon Sam Houston State University
19. *Division Algorithms for Prime Factorization*	JOHN ANDERSON IN Gamma Rose-Hulman Institute of Technology
20. *Problems: Stimulation to Research and Application*	STEVEN FROM NE Beta Creighton University
21. *Numerical Treatment of Meteorological Data*	GREGORY BATTLE MO Beta Washington University
22. *Magic Card Squares, Cubes, and Hypercubes*	BERNARD P. SMITH MO Gamma St. Louis University

1978-79 CONTEST

Papers for the 1977-78 contest have now been ***Judged,*** and we are receiving papers for this year's contest, so be sure to send us your paper, or your chapter's papers (at least 5 entries must be received from the same chapter in order to qualify, with a $20 prize for the best paper in each chapter).

For all manuscript contests, in order for authors to be eligible, *they must not have received a Master's degree at the time they submit their paper.*

1977-1978 MANUSCRIPT CONTEST

The judging for the best expository papers submitted for the 1977-78 school year has been completed. The winners are:

First prize ($200), ROBERT ANTOL, Iowa State University, *The Perfect Numbers and Pascal's Triangle.* (this *Journal,* Vol. 6, No. 8, pp. 459-462).

Second prize ($100), DEBRA GUTRIDGE, Muskingum College, *Mathematical Curiosities.* (this *Journal,* Vol. 6, No. 8, pp. 445-458).

Third prize ($50), JACKIE LAWRENCE, Western Kentucky University, *Numerical Integration by Polynomial Interpolation.* (this *Journal,* Vol. 6, No. 6, pp. 336-344).

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