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President

RVA:ar

(FIRST PRIZE PAPER)

THE PROBABILITY OF GENERATING A CYCLIC GROUP

by *Deborah L. Massari*
University of Akron

Often, interesting results in one area of mathematics may be viewed in the light of other mathematical interests. Such is the case between group theory and probability. In a recent paper in *The Mathematical Gazette*, MacHale [3] considered the question of determining the probability, $Pr(G)$, that a pair of elements of a finite non-commutative group G commute with each other. We will consider a related question; namely, determining the probability that a randomly chosen element of a finite cyclic group is a generator of the group.

Let $P(G)$ denote the probability that an arbitrary element chosen at random is a generator of a finite cyclic group G. We will let p and q denote distinct primes. Positive integers will be denoted by n, m, e, and f. Also $\pi(G) = \{p \mid p$ is a prime divisor of $|G|\}$. C_n denotes a cyclic group of order n.

We shall now consider the following situations. The first case will be $|G| = p$. Secondly, $|G| = p^n$ $(n > 1)$. The third case will be $|G| = p^n q^m$. Last, $|G| = \prod_{i=1}^{k} p_i^{e_i}$ where p_i is prime, e_i is a positive integer for each i. Our objective is to prove that $P(G)$ is independent of the order of G and that $P(G)$ depends only on $\pi(G)$.

A major result in the study of cyclic groups is if $G = \langle a\rangle$, $|G| = n$, then $G = \langle a^k\rangle$ if and only if $(n,k) = 1$. A corollary to this theorem is that the number of generators of a cyclic group G, $|G| = n$, is $\phi(n)$ where ϕ is the Euler-phi function. [1,p.44]. Thus, since $\phi(n)$ is the number of generators out of n elements, the probability that any one randomly chosen element of G is a generator is $\phi(n)/n$.

Let us now examine $|G| = p$. Then, $P(G) = \phi(p)/p$. Since, $\phi(p)/p = (p-1)/p = 1 - \frac{1}{p}$, $P(G) = 1 - \frac{1}{p}$. As an example, consider C_3 under addition with Cayley table:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Both 1 and 2 are generators of C_3. So, of three elements, two generate the group. Thus, $P(G) = 2/3$. Likewise according to the result just established, $P(C_3) = \phi(3)/3 = 1 - 1/3 = 2/3$.

If $|G| = p^n$, $P(G) = \phi(p^n)/p^n = p^n(1 - \frac{1}{p})/p^n = 1 - \frac{1}{p}$, [2, p. 29]. Hence, if $|G| = p^n$ and $|H| = p$, then $P(G) = P(H)$. Consider $C_8 = \{0,1,2,\cdots 7\}$ under addition. The generators are 1, 3, 5, 7. Thus, $P(C_8) = 4/8 = 1/2$. According to the formula established $|C_8| = 2^3$, $P(C_8) = 1 - 1/2 = 1/2$.

Thirdly, let $|G| = p^n q^m$.

$$P(G) = \frac{\phi(p^n q^m)}{p^n q^m} = \frac{\phi(p^n)\phi(q^m)}{p^n q^m} =$$

$$\frac{p^n(1 - \frac{1}{p})\, q^m(1 - \frac{1}{p})}{p^n q^m} = (1 - \frac{1}{p})(1 - \frac{1}{q}) ,$$

[2, p. 28]. As an example, consider C_6 under addition.

$$|C_6| = 2 \cdot 3,\ P(C_6) = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) = \tfrac{1}{3} .$$

Another example is C_{36} under addition.

$$C_{36} = 2^2 \cdot 3^2 \text{ so, } P(C_{36}) = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) = \tfrac{1}{3} .$$

Notice, $P(C_6) = P(C_{36})$.

The last case to be considered is $|G| = \prod_{i=1}^{k} p_i^{e_i}$. According to the general formula,

$$P(G) = \frac{\phi(\prod_{i=1}^{k} p_i^{e_i})}{\prod_{i=1}^{k} p_i^{e_i}} = \frac{\phi(p_1^{e_1})\phi(p_2^{e_2}) \cdots \phi(p_k^{e_k})}{p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}}$$

$$= \frac{p_1^{e_1}(1 - \frac{1}{p_1})p_2^{e_2}(1 - \frac{1}{p_2}) \cdots p_k^{e_k}(1 - \frac{1}{p_k})}{p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}}$$

$$= (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k}).$$

As an example, consider C_{60} under addition. $|C_{60}| = 2^2 \cdot 3 \cdot 5$. Thus,

$$P(C_{60}) = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(1 - \tfrac{1}{5}) = \tfrac{4}{15} .$$

Since $|C_{30}| = 2 \cdot 3 \cdot 5$, we observe that $P(C_{30})$ is also $\frac{4}{15}$. This is generalized by the following theorem.

__General Theorem__. Let G and H be two finite cyclic groups in which $\pi(G) = \pi(H)$. Then, $P(G) = P(H)$.

Proof? Let $|G| = \prod_{i=1}^{k} p_i^{f_i}$ and $|H| = \prod_{i=1}^{k} p_i^{e_i}$

$$P(G) = \frac{\phi(\prod_{i=1}^{k} p_i^{f_i})}{\prod_{i=1}^{k} p_i^{f_i}} = \frac{\phi(p_1^{f_1})\phi(p_2^{f_2}) \cdots \phi(p_k^{f_k})}{p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}}$$

$$= \left(\frac{\phi(p_1^{f_1})}{p_1^{f_1}}\right)\left(\frac{\phi(p_2^{f_2})}{p_2^{f_2}}\right) \cdots \left(\frac{\phi(p_k^{f_k})}{p_k^{f_k}}\right)$$

$$= (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k}).$$

In a similar manner, $P(H) =$

$$\frac{\phi\left(\prod_{i=1}^{k} p_i^{e_i}\right)}{\prod_{i=1}^{k} p_i^{e_i}} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

Hence, $P(G) = P(H)$

So, if for example $|G| = 1890$ and $|H| = 6300$, then $P(G) = P(H)$. This follows since $|G| = 1890 = 2 \cdot 3^3 \cdot 5 \cdot 7$ and $|H| = 6300 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7$. According to our result, $P(G)$ is independent of $|G|$ and $P(G)$ depends only on $\pi(G)$.

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This paper was awarded First Prize in the National Pi Mu Epsilon Paper Contest. The author was an undergraduate when the paper was submitted.

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{SECOND PRIZE PAPER}

NON-LINEAR ADDITIVE FUNCTIONS

by Julie D. Anderson
Hendrix College

Introduction

What functions $f:R \to R$ can be found satisfying $f(x+y) = f(x)+f(y)$ (*) for every $x, y \in R$? Clearly the function $f(x) = ax$ is one, but is every solution linear? The research generated by this question has led to some interesting results. [6]

A function which satisfies the condition (*) will be called additive. One for which $f(cx) = cf(x)$ for every $c \in R$ will be called homogeneous. If a function is both additive and homogeneous, it is linear. [6]

History

A survey of the research concerning additive functions is given by Green and Gustin [4, pp. 503-505] beginning with 1821 when Cauchy demonstrated that any additive function is rationally homogeneous, or $f(rx) = rf(x)$ for every rational number r [2]. Furthermore, he found that the only continuous additive functions are real homogeneous and thus linear, and any discontinuous additive function is continuous at no point. Therefore a non-linear additive function, if such exists, will be totally discontinuous. Further restrictions were placed on a non-linear additive function by Darboux who showed in 1875 that an additive function bounded above or below on some interval is continuous, hence linear [3].

The question of the existence of non-linear additive functions was resolved in 1905 by Hamel, who used the rational homogeneity of an additive function and a set which he constructed (now referred to as a Hamel basis) using Zermelo's well-ordering theorem. Hamel also noted that the graph of such a function is dense in the plane [6].

In summary, the following theorem demonstrates the slight conditions which cause an additive function to be linear [9,Th.5,p.6] :

Theorem. If $f:R \to R$ is additive then the following are equivalent:

1) f is linear.

2) f is continuous at some point.
3) There exists an interval on which f is bounded.
4) There exists an interval on which f is monotonic.
5) The graph of f is not dense in R^2.

Construction. One begins to see that the construction of a non-linear additive function is no trivial task. At the heart of it is the concept of a Hamel basis for the real numbers, the existence of which will be demonstrated using ideas concerning vector spaces.

Let X be a vector space over the field F and S be a subset of X. The subspace generated by S, denoted $\langle S\rangle$, is defined to be the smallest subspace of X containing S. It can be shown that

$$\langle S\rangle = \{\sum_{i=1}^{n} a_i x_i : x_i \in S,\ \ a_i \in F\}.$$

If $\langle S\rangle$ is generated by no proper subset of S, we say S is linearly independent. A Hamel basis in X is a linearly independent subset of X which generates X. If S is a Hamel basis in X it can be established that every x in X is a unique (finite) linear combination of members of S. These proofs are fairly standard fare in any linear algebra text.

The next step is to demonstrate that every vector space has a Hamel basis. One lemma is necessary.

Zorn's Lemma. If S is a nonempty partially ordered set such that every totally ordered subset has an upper bound in S, then S has a maximal element S_0.

The statement that S_0 is a maximal element of S means that if $S\varepsilon S$ and $S_0 \leq S$ then $S_0 = S$. Zorn's lemma is equivalent to the axiom of choice and Zermelo's well-ordering principle (see [5] or [8]).

Theorem. Let X be a vector space over the field F. Then X has a Hamel basis. [6]

Proof. Let S be the totality of linearly independent subsets of X (partially ordered by inclusion) and T be a totally ordered subset of S.

Claim: T has an upper bound in S.
Consider $U = \bigcup_{R\varepsilon T} R$. Clearly U is an upper bound of T, but is U in S ? Suppose U is not in S. Then U is not linearly independent, or $\langle U\rangle$ is generated by a proper subset of U. Therefore there exists $r\,\varepsilon U$ such that

$$r = \sum_{i=1}^{n} a_i r_i$$

where $a_i \varepsilon F$, $r_i \varepsilon U$, $r_i \neq r$. T is totally ordered by inclusion, so there exists $R_0 \varepsilon T$ such that $r\varepsilon R_0$ and $r_i \varepsilon R_0$ for $i=1,2,\ldots,n$. Then R_0 is not linearly independent, which is a contradiction. Therefore U is in S.

Thus by Zorn's lemma, S has a maximal element S_0. Suppose $x\varepsilon X$ such that $x \notin \langle S_0\rangle$. Then $S_0 \subset S_0 \cup \{x\}$ which in turn belongs to S. But since S_0 is a maximal element of S, this implies that $S_0 = S_0 \cup \{x\}$, which is clearly false. Thus if $x\varepsilon X$, then $x\varepsilon\langle S_0\rangle$, and we have that S_0 is a Hamel basis in X. □

Consider the vector space of the real numbers over the field of rational numbers, denoted R_Q, and let B be a Hamel basis for R_Q. If a function $f:R\to R$ is additive and $x = \sum r_i b_i$ where $r_i \varepsilon Q$ and $b_i \varepsilon B$, then

$$f(x) = f\left(\sum_{i=1}^{n} r_i b_i\right) = \sum_{i=1}^{n} f(r_i b_i) = \sum_{i=1}^{n} r_i f(b_i).$$

Thus the function is completely determined by the way it is defined on the Hamel basis.

To construct a non-linear additive function, we begin by defining $f:B\to R$ so that f is non-linear. Then the function is extended to $F:R\to R$ in such a way as to make F additive. To insure non-linearity, consider that for a linear function, $f(x)=ax$, the ratio $f(x)/x$ is the constant a for all $x\neq 0$. Therefore define $f:B\to R$ so there exist $b_1, b_2 \varepsilon B$ such that $f(b_1)/b_1 \neq f(b_2)/b_2$. Since every $x\varepsilon R$ can be expressed as $\sum_{i=1} r_i b_i$ where $r_i \varepsilon Q$ and $b_i \varepsilon B$, extend f to $F:R\to R$ by defining

$$F(x) = \sum_{i=1}^{n} r_i f(b_i).$$

The verification that F is additive is fairly straightforward.

Examples. In spite of the pathology of these functions, there are examples with some nice properties.

Example 1. Consider the function $f:B\to R$ defined by

$$f(x) = \begin{cases} 1 & x=b \\ 0 & x\neq b \end{cases},$$

where b is fixed in $\mathcal{B}$. Extend this to $F:\mathbb{R}\to\mathbb{R}$ as described above. Then $F(b)=1$, and the graph of F restricted to rational multiples of b is dense in the line $\{(x,y)\colon y=\frac{1}{b}x\}$. Where $c\varepsilon\mathcal{B}$ and $c\neq b$, $F(c)=0$ and the graph of F restricted to rational multiples of a is dense in the horizontal axis. For $r\varepsilon Q$, consider $F(b+rc)= f(b)+rf(c)=1$. The graph of F restricted to rational multiples of $b+rc$ is dense in the line $\{(x,y)\colon y=\frac{1}{b+rc}x\}$. Since $\{b+rc\colon r\varepsilon Q\}$ is dense in $\mathbb{R}$, $\bigcup_{r\varepsilon Q}\{(x,y)\colon y=\frac{1}{b+rc}x\}$ is dense in the plane. Consequently, the graph of F is dense in the plane. See Figure 1. Notice that this function takes on only rational values.

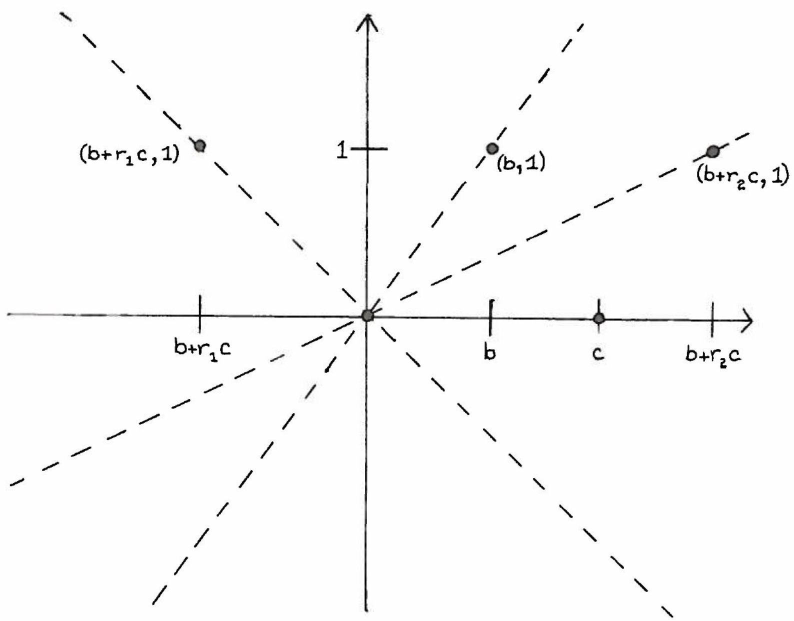

FIGURE 1

Example 2. Another example is described by Wilansky In [8,Ex.6, p.116]. Again, b is fixed in $\mathcal{B}$ and $f:\mathcal{B}\to\mathbb{R}$ and $g:\mathcal{B}\to\mathbb{R}$ are defined by

$$f(x)=\begin{cases} b & x=b \\ \frac{1}{2}x & x\neq b\end{cases}, \quad g(x)=\begin{cases} b & x=b \\ 2x & x\neq b\end{cases}$$

Extend f and g to F and G respectively. Let $x=\sum_{i=1}^{n} r_ib_i + rb$ where $b\notin\{b_1,b_2,\ldots,b_n\}$.

$$\text{Then}\quad F(G(x)) = F\left[\sum_{i=1}^{n} r_ig(b_i)+rg(b)\right]$$

$$= F\left[\sum_{i=1}^{n} r_i2b_i+rb\right]$$

$$= \sum_{i=1}^{n} r_i2f(b_i) + rf(b)$$

$$= \sum_{i=1}^{n} r_i2(\tfrac{1}{2})\, b_i+rb$$

$$= x.$$

Similarly, $G(F(x))=x$. The existence of an inverse of F implies that F is one-to-one and onto $\mathbb{R}$ -- a totally discontinuous function, dense in the plane, yet mapping to every real number exactly once.

Remark. It is of interest to note that the graphs of the functions in Examples 1 and 2 are disconnected. In Example 1, the function takes on only rational values, so a horizontal line through an irrational point on the vertical axis separates the graph. As for Example 2, according to a result of F. B. Jones, the connected graph of a discontinuous additive function must intersect every continuum in the plane not lying wholly in a vertical line [7,Th.2,pp.116,117]. Since $F(x)$ in Example 2 isone-to-oneand onto $\mathbb{R}$, every horizontal line contains only one point of the graph. Choose a segment of a horizontal line such that the segment does not contain a point of the graph. Then this continuum, the segment, is not intersected by the graph. Thus the graph is not connected. Jones demonstrates that there does exist a discontinuous additive function whose graph is connected.

Example 3. Consider the usefulness of these functions in studying another functional equation, $h(x+y)= h(x)h(y)$. The obvious solution is exponential, $h(x)=e^{ax}$, but using non-linear additive functions instead of ax yields discontinuous solutions. For example, consider $h(x)=e^{F(x)}$ where $F(x)$ is as defined in Example 1. The graph of h exhibits much of the pathology of the graph of $F(x)$. Consider the fixed b: $h(b)=e^{F(b)}=e^1$, and the graph of h restricted to rational multiples of b is dense in the exponential curve $\{(x,y): y=exp(\frac{x}{b})\}$. Where $c\epsilon B$ and of b, $h(c)=e^0=1$, so the graph of h restricted to rational multiples of c is dense in the line $\{(x,y): y=1\}$. $h(b+rc)=e$, and the graph of h restricted to rational multiples of $b+rc$ is dense in the curve $\{(x,y): y=exp(\frac{x}{b+rc})\}$. Evidently, the graph of h is the homeomorphic image of the graph of F under the mapping $(x,y)\to(x,e^y)$.

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This paper was written under the direction of *Professor Robert Eslinger*, Hendrix College, Conway, Arkansas.

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THE DOUBLE FERRIS WHEEL PROBLEM

by Al Parish
College of Charleston

In certain real-world situations, mathematical models may prove to be quite accurate and at the same time have reasonable assumptions. Granted, this case is most unusual, but we will be interested in just such a situation. The problem may be stated quite simply in the following manner: describe the equation of motion of the double Ferris wheel with arbitrary physical dimensions.

For those of you who have never had the dubious pleasure of riding the device, or for those of you who have and have attempted to forget it, a basic description is in order. A double Ferris wheel consists of a large supporting beam with a rotating beam attached at some convenient point on the supporting one. At each end of the rotating beam is a wheel with chairs attached to it for riders to sit in. Since a picture is worth a thousand words, the following one may help:

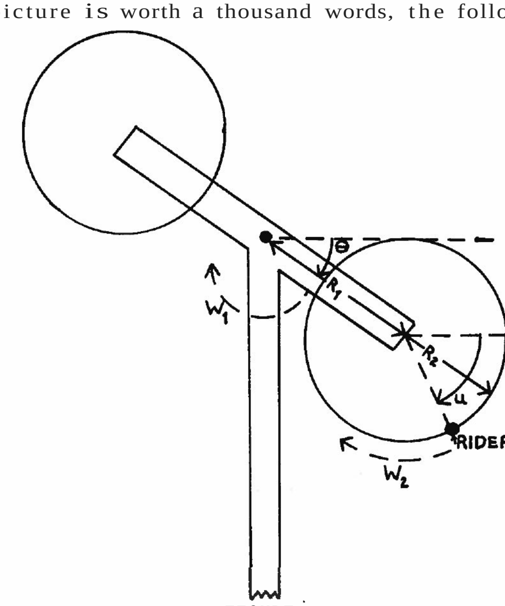

FIGURE 1

We shall refer to the rotating and supporting beams together as the central structure. As pictured in Figure 1, we shall define the following variables:

R_1: radius of the central structure

R_2: radius of a single wheel

W_1: angular velocity of the rotating beam

W_2: angular velocity of the rider about the center of the wheel

θ : angle of the rotating beam at time t with the horizontal

U : angle of the rider to the horizontal

The following assumptions must be made for this model:

i) There is no force of friction between the chair and the steel axis on which the chair sits.

ii) Angular velocity of both the wheel and rotating beam is uniform. This assumption eliminates initial and final "jerk" and acceleration which cause the rocking motion of the chair at start and stop of the ride.

iii) No external forces such as wind interfere with the motion.

First we consider the motion of the rotating beam about the center of the central structure. Clearly the endpoint of the beam (also the center of a wheel) rotates in a circle, which we chose to be centered at the origin, of radius R_1 and therefore satisfies:

(1) $x^2 + y^2 = R_1^{\,2}$, or in polar coordinates -

(2) $\left.\begin{array}{l} x = R_1 \cos\theta \\ y = R_1 \sin\theta \end{array}\right\}$, where the angle θ is defined above.

Now consider the motion of the rider on the wheel. The equation of motion here is similar to that in the previous derivation. The rider rotates in a circle, centered about the endpoint of the rotating beam of radius R_2 and so satisfies:

(3) $\underline{x}^2 + \underline{y}^2 = R_2^{\,2}$, or in polar coordinates,

(4) $\left.\begin{array}{l} \underline{x} = R_2 \cos U \\ \underline{y} = R_2 \sin U \end{array}\right\}$, where U is previously defined.

A standard translation of axes yields the following two equations:

(5) $\left.\begin{array}{l} x = R_1 \cos\theta + R_2 \cos U \\ y = R_1 \sin\theta + R_2 \sin U \end{array}\right\}$.

These two parametric equations describe the equation of motion of the rider given the two angles θ and U. But these two angles may be difficult to find at some moment in time, so a stopwatch may be used to measure the two variables W_1 and W_2. For pure convenience, we shall assume that the rider is at point A in Figure 2 at time $t = 0$. Now the angles θ and U may be expressed in terms of the following simple differential equations:

(6) $\dfrac{d\,\theta(t)}{dt} = W_1$ and $\dfrac{d\,U(t)}{dt} = W_2$.

Integrating with respect to time, we obtain the following solutions:

(7) $\theta(t) = W_1 t + C$ and $U(t) = W_2 t + E$.

But we assumed that $\theta = U = 0$ at time $t = 0$, so that $C = E = 0$. Recall that W_1 and W_2 are angular velocities and are hence measured in revolutions per time period. Radian measure is far more desirable so we may write:

(8) $W_1 = 2\pi F_1$ and $W_2 = 2\pi F_2$.

The two previous parametric equations of (5) become the following two equations:

(9) $x = R_1 \cos(2\pi F_1 t) + R_2 \cos(2\pi F_2 t)$

$y = R_1 \sin(2\pi F_1 t) + R_2 \sin(2\pi F_2 t)$.

These last two equations describe the motion of the rider in terms of the easily measured parameter time. We must now concern ourselves with the physical interpretation of these equations. The graph for this situation is shown in Figure 3. It could be described as a three-leaved rose with a hyperbolic triangular center.

Return to Figure 1 for a moment. The angles θ and U are measured in the same direction, implying the beam and rider rotate in the same direction. Obviously this need not be the case. So let us replace U by $-U$, or equivalently, applying (7) and (8), replace W_2 by $-W_2$ and F_2 by $-F_2$. Since the parametric equation of x in (9) involved only the even cosine function, it remains the same. However, the equation involving y contains the odd sine function so that this equation becomes:

$$(10)\quad y = R_1 \sin(2\pi F_1 t) - R_2 \sin(2\pi F_2 t).$$

This apparently minor change creates an entirely new motion as depicted in Figure 4.

Let us now consider the displacement of the rider from the origin at time t. Squaring the parametric equations and adding yields the following rather complicated expression:

$$(11)\quad x^2 + y^2 = R_1^2 \cos^2(2\pi F_1 t) + R_2^2 \cos^2(2\pi F_2 t) + 2R_1R_2 \cos(2\pi F_1 t) \cos(2\pi F_2 t) + R_1^2 \sin^2(2\pi F_1 t) + R_2^2 \sin^2(2\pi F_2 t) + 2R_1R_2 \sin(2\pi F_1 t)\sin(2\pi F_2 t).$$

The square root of this expression is the equation for displacement we desire. Fortunately, we may simplify this expression by using the two familiar trigonometric identities:

$$(12)\quad \cos^2 A + \sin^2 A = 1 \quad \text{and} \quad \cos(A-B) = \cos A\cos B + \sin A \sin B.$$

Applying these to (11) yields the following:

$$(13)\quad D = x^2 + y^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos 2\pi(F_1 - F_2)\, t.$$

The square root of this expression is the displacement we desire. When the angles θ and U are in opposite directions, a similar analysis yields the same displacement as above except that the final term involving cosine has $F_1 + F_2$.

In conclusion, if any reader still wishes to ride one of these machines, all that the writer can say is "HAVE FUN!"; as for the author, he has never been on one and now that he knows what the motion is, he will never ride one in the future!

Editor's Note: It might be of interest to consider accelerations at various points on the curves. The Journal would be interested in hearing from its readers on this subject.

The author was an undergraduate when the paper was submitted.

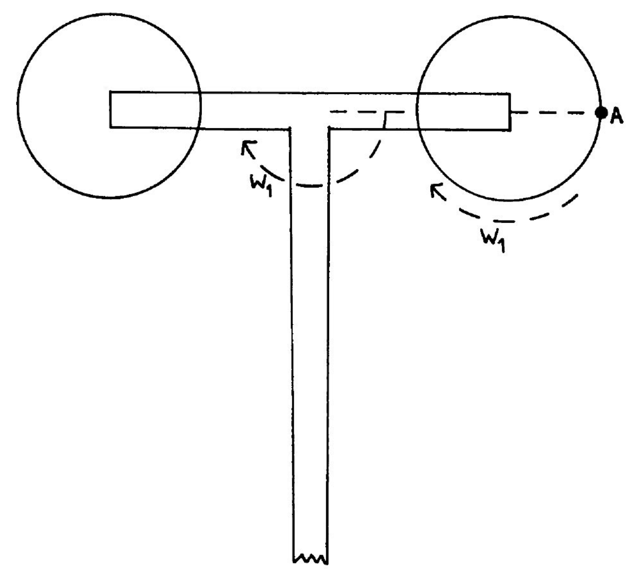

FIGURE 2

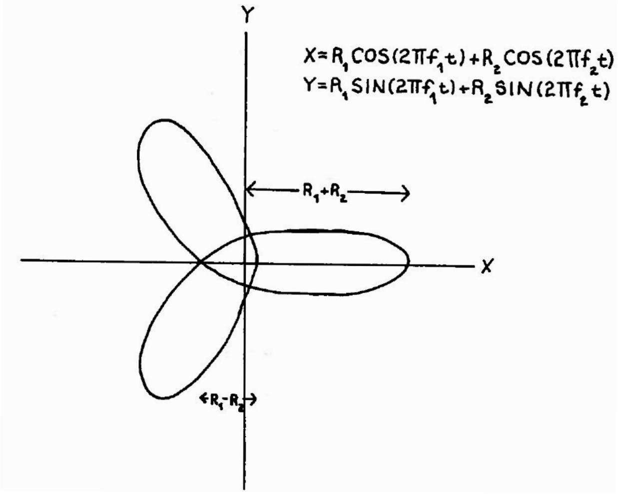

FIGURE 3

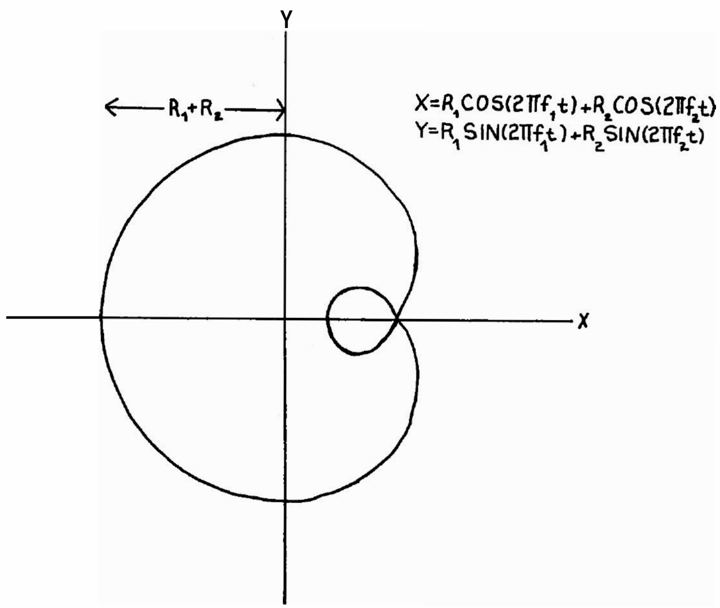

FIGURE 4

LOOPS IN SOME SEQUENCES OF INTEGERS

by Victor G. Feser
Mary College

A common topic of recreational mathematics deals with manipulating the digits of an integer by means of some function, and then studying the results. One of the oldest, best-known, and most striking examples of this: start with the integer 153; cube its digits and add them; the result is 153 again. In this way, 153 can be called a function of its digits; such an integer is known in the literature as a narcissistic number [2], [3].

Of course, a wide variety of "functions" can be used here. Also, since the use of base 10 is a mathematical accident, one really ought to consider other bases too--this is not always done in the literature. The most fertile extension of the idea consists of generating sequences of integers; beginning with an integer N in base B, and a function f, one sets $N = N_0$ and defines recursively: $N_{i+1} = f(N_i)$, $i = 0,1,2, \cdots$.

Thus, using the function mentioned above, and working in base 10, one may begin at random, say with N = 76. The resulting sequence is 76,559,979,1801,514,190,730,370,370,$\cdots$. Obviously the sequence has become repeating; and this would occur at any time a duplication occurs anywhere in the sequence.

The question now is: under what conditions for N, B, and f does the sequence repeat? This paper will show that the answer is -- under most conditions!

Some preparatory remarks: a repeating block will be called a loop; if the loop is of length one, it will be called a self-loop. (A self-loop is a narcissistic number.) Examples will be given in various bases, but the treatment is general for base B. In base B, the integer N is written:

$a_0B^m + a_1B^{m-1} + \cdots + a_mB + a_{m+1}$, where the digits are the a_i, with $a_0 \neq 0$, and N has $m+1$ digits. N will always be non-negative.

The first section of the paper presents the general method in detail with a number of remarks. The remaining sections can then be made quite brief, since the same method is used in each with obvious modifications.

POWER-SUMS

Given N in base B, let k be a positive integer. Then the k-power-sum of N, written $P(N,B,k)$ or simply $P(N)$, is defined:

$$P(N) = a_0^k + a_1^k + \cdots + a_m^k + a_{m+1}^k \qquad [1].$$

The example at the beginning of this paper is a 3-power-sum. Other examples, with $B = 7$ and $k = 3$, are: loop 466, 1306, 466, $\cdots$; self-loop 505. Self-loops in base 10 are also called digital invariants [5], [6], [7].

Theorem 1. For any $B \geq 2$, for any $k \geq 1$, for any $N \geq 0$: the sequence defined by $P(N)$ has a loop.

Proof. If N has $m+1$ digits, then $N \geq B^m$. From the definition, $P(N) \leq (B-1)^k(m+1)$, since the digit $a_i \leq B-1$ for each i. Then $P(N)/N \leq (B-1)^k(m+1)/B^m$: let the latter be $f(m)$. We now consider $f(x)$, where x is any real number in $(0,\infty)$; after analyzing $f(x)$, we will return to integer values of x. (cf. [4], ch. 4).

Clearly $f(x) = (B-1)^k(x+1)/B^x \to 0$ as $x \to \infty$; i.e., for any $\varepsilon > 0$, there exists x_0 such that $x > x_0$ implies $f(x) < \varepsilon$. Choosing $\varepsilon = 1$ and returning to integer values, we have for some integer Z, N > Z implies $f(m) < 1$, or $P(N) < N$. So, if in the sequence $N_i > Z$, then $N_{i+1} < N_i$, and succeeding terms of the sequence decrease until for some j (possibly equal to $i+1$) $N_j < Z$. After this point we have no direct information on the next terms; but whenever a term appears that is greater than Z, the same decreasing behavior occurs.

Therefore in the infinite sequence $\{N_i\}$ we have infinitely many terms less than Z; but all these terms are positive integers, and so a repetition must occur. Therefore the sequence has a loop. Q.E.D.

Corollary. Every power-sum loop contains an integer less than Z.

Here is an illustration: For $B = 10$ and $k = 3$, we have $f(x) = 729(x+1)/10^x$, which is less than 1 for $x \geq 4$. This yields $Z = 10000$. In fact, this is far larger than necessary; the optimum value of Z, it can be shown, is 2189. Note that $P(1999) = 2188$. Note also that the theorem does not say $N > Z$ implies $P(N) < Z$ (since, e.g., $P(9999) = 2916 > Z$) but rather $N > Z$ implies $P(N) < N$.

Remarks:

1) If $k = 1$, then the only loops are the trivial self-loops consisting of the one-digit integers in that base. In any base B, there are the trivial self-loops 0 and 1.

2) This theorem, and the following ones, unfortunately do not guarantee the existence of loops other than trivial ones.

SUM-POWERS

Given N in base B and let k be a positive integer. Then the sum-k-power of N, written $S(N,B,k)$ or simply $S(N)$, is defined:

$$S(N) = (a_0 + a_1 + \cdots + a_m + a_{m+1})^k \qquad [2].$$

Examples in base 6, with k = 3, are: loop 1331, 2212, 1331, $\cdots$; self-loop 3213. The search for sum-power self-loops is fairly easy to do empirically. Since a sum-k-power is precisely a kth power, all one needs to do is look through a table of powers for the base in question. For any base B and any k, there are, of course, the trivial self-loops 0 and 1.

Theorem 2. For any $B \geq 2$, for any $k \geq 1$, for any $N \geq 0$: the sequence defined by $S(N)$ has a loop.

Proof. We follow exactly the pattern of the proof of Theorem 1. We have $S(N)/N \leq ((B-1)(m+1))^k/B^m$, and this is $f(m)$. Then clearly $f(x) \to 0$ as $x \to \infty$ (again we have a polynomial in x over an exponential in x), and the rest follows as before.

SELF-POWERS

Given N in base B, the self-power of N, written $L(N,B)$ or simply $L(N)$, is defined:

$$L(N) = a_0^{a_0} + a_1^{a_1} + \cdots\ a_m^{a_m} + a_{m+1}^{a_{m+1}} \quad [2], [3].$$

This definition requires us to first define a value for 0^0. The obvious choices are 0 and 1. Either one may be used, and in fact both choices are usually considered in discussing this function [3].

Examples in base 5 are as follows: defining $0^0 = 0$, loop: 11, 2, 2011, 11, $\cdots$; self-loops: 103 and 2024. Defining $0^0 = 1$: loop 104, 2013, 113, 104, $\cdots$; but no nontrivial self-loops exist.

Under either definition of 0^0 we have

Theorem 3. For any $B \geq 2$, for any N 20: the sequence defined by $L(N)$ has a loop.

Proof: We have $L(N)/N$ 2 $(m+1)(B-1)^{B-1}/B^m \to 0$, etc.

FACTORIAL-SUMS

We now introduce a new type of function. Given N in base B, the factorial-sum of N, writtin $F(N,B)$ or simply $F(N)$, is defined:

$$F(N) = a! + a! + \cdots a_m! + a_{m+1}! \quad [2], [3].$$

In base 6, two examples of self-loops are 41 and 42. In any base, there is the trivial self-loop 1. If $B > 2$, there is also the trivial self-loop 2. (For this function, 0 is not a self-loop.)

Theorem 4. For any $B \geq 2$, for any $N \geq 0$: the sequence defined by $F(N)$ has a loop.

Proof! We have $F(N)/N \leq (m+1)((B-1)!)/B^m \to 0$, etc.

SUM-FACTORIALS

Finally we present a function for which the general method does not work. Given N in base B, the sum-factorial of N, written $G(N,B)$ or simply $G(N)$, is defined:

$$G(N) = (a_0 + a_1 + \cdots + a_m + a_{m+1})!$$

Trivial self-loops exist: 1 is such in any base B; 2 is such in any base $B > 2$.

If we follow the pattern used so far, we set up the ratio $G(N)/N \leq ((B-1)(m+1))!/B^m$; but the latter ratio is divergent! This can be seen quickly as each term is greater than $m!/B^m$; now this is the reciprocal of $B^m/m!$, and the series resulting from such a sequence is convergent (it defines the number e). But then the terms $B^m/m! \to 0$ as $m \to \infty$, and therefore the terms $m!/B^m \to \infty$ as $m \to \infty$.

Now, of course, this divergence does not settle the question of the ratio $G(N)/N$: the terms here are term-wise less than those of a divergent sequence, and so it might still be convergent, after all. I have not been able to decide the question, but empirical evidence supports my closing conjectures:

1. The sequence $G(N)/N$ is divergent.
2. Given $N \geq 0$ in base $B \geq 2$, the sequence defined by $G(N)$ has no loops except the trivial ones.

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The **Journal** sadly announces that *Professor Carl G. Townsend* recently died in an automobile accident. *Professor Townsend* was the advisor of the *Illinois* Delta chapter.

LINEAR APPROXIMATIONS TO SQUARE ROOTS

by *Stewart M. Venit*
California State University, Los Angeles

Assume that we are given a positive number, a, with $n^2 < a \le (n+1)^2$, where n is a nonnegative integer. One simple means of approximating the positive square root of a is to evaluate at $x = a$ the linear polynomial interpolating $f(x) = \sqrt{x}$ using the points (n^2, n) and $((n+1)^2, n+1)$. A somewhat more sophisticated way is to form the linear Taylor polynomial for $f(x) = \sqrt{x}$, expanded about either $x = n^2$ or $x = (n+1)^2$, and to evaluate it at $x = a$. Although the latter approximation is normally used only "near" the center of expansion, we will show here that it is "usually" a better one than the former throughout the interval $(n^2, (n+1)^2)$.

First, we will more precisely define our approximations. The interpolating polynomial described above is simply the line through the points (n^2, n) and $((n+1)^2, n+1)$; and we can obtain its equation $y = n + (x - n^2)/(2n+1)$, by using the two-point form. Evaluating this function at $x = a$, we obtain an approximation we shall call $s(a)$ ("s" for "secant"):

(1) $s(a) = n + (a - n^2)/(2n+1)$ if $n^2 < a \le (n+1)^2$.

(This equation also represents one iteration of the method of "false position," $z_{k+2} = z_k - (z_k - z_{k+1})g(z_k)/(g(z_k) - g(z_{k+1}))$, for $g(x) = x^2 - a$ with $z_k = n$ and $z_{k+1} = n+1$.)

The two linear Taylor polynomials, the one expanded about $x = n^2$ and the other about $x = (n+1)^2$, are given by, respectively, $y = n + (x - n^2)/(2n)$ and $y = (n+1) + (x - (n+1)^2)/(2n+2)$. These lines intersect at the point $(n(n+1), n + 1/2)$ (see Figure 1).

Thus, in the interval $\left(n^2, (n+1)^2\right]$, the error in this type of approximation will be minimized by choosing the first of these lines if a is such that $n^2 < a \le n(n+1)$ and choosing the second if $n(n+1) \le a \le (n+1)^2$. We thus arrive at the following procedure to find an approximation $t(a)$ ("t" for "tangent") to $\sqrt{a}$ based upon the

Taylor lines. When $a > 1$, take

(2) $t(a) = n + (a - n^2)/(2n)$ if $n^2 < a \le n(n + 1)$,

(3) $t(a) = (n + 1) + (a - (n + 1)^2)/(2n + 2)$ if $(n + 1) \le a \le (n + 1)^2$.

Since $f'(0)$ does not exist, we use formula (3) for any a lying in the interval $(0, 1$. (Equations (2) and (3) also represent one iteration of Newton's method, $z_{k+1} = a, - g(z_k)/g'(z_k)$ for the function $g(x) = x^2 - a$ with $z_k = n$ and $a, = n + 1$, respectively.)

The errors in approximating $\sqrt{a}$ by $t(a)$ and $s(a)$, $|t(a) - \sqrt{a}|$ and $|s(a) - \sqrt{a}|$, respectively, are obtained by subtracting $\sqrt{a}$ from both sides of equations (1), (2), (3). If $a > 1$ and $n^2 < a \le n(n + 1)$,

$$\begin{aligned} t(a) - \wedge a &= n - \sqrt{a} + (a - n^2)/(2n) \\ &= n - \wedge a + (\sqrt{a} - n)(\sqrt{a} + n)/(2n) \\ &= (n - \sqrt{a})\left[1 - (\sqrt{a} + n)/(2n)\right] \\ &= (n - \sqrt{a})\left[(n - \sqrt{a})/(2n)\right]. \end{aligned}$$

Thus,

(4) $t(a) - \sqrt{a} = (n - \sqrt{a})^2/(2n)$ if $a > 1$ and $n^2 < a \le n(n + 1)$.

Similarly,

(5) $t(a) - \sqrt{a} = (n + 1 - \sqrt{a})^2/(2n + 2)$ if $0 < a \le 1$ or $n(n + 1) \le a \le (n + 1)^2$,

(6) $s(a) - \sqrt{a} = (n - \sqrt{a})(n + 1 - \sqrt{a})/(2n + 1)$ if $n^2 < a \; 5 \; (n + 1)^2$.

Now, $t(a)$ is a better approximation than $s(a)$ for those a for which $|t(a) - \sqrt{a}| < |s(a) - \sqrt{a}|$. From equations (4), (5) and (6), or from Figure 1, we see that $t(a) - \sqrt{a} \ge 0$, and $s(a) - \sqrt{a} \le 0$. Hence, $t(a)$ is more accurate if $t(a) - \sqrt{a} < \sqrt{a} - s(a)$; that is, if $(t(a) - \sqrt{a}) + (s(a) - \sqrt{a}) < 0$. Now, for $a > 1$ and $n^2 < a \; 5 \; n(n + 1)$, equations (4) and (6) yield

$$(t(a) - \sqrt{a}) + (s(a) - \sqrt{a}) = (n - \sqrt{a})\left[\frac{n - \sqrt{a}}{2n} + \frac{n + 1 - \sqrt{a}}{2n + 1}\right]$$

$$= \frac{(\sqrt{a} - n)\left[\sqrt{a}(4n + 1) - (4n^2 \; t \; 3n)\right]}{2n(2n + 1)}$$

This expression will be negative, as desired, if $\sqrt{a} < (4n^2 + 3n)/(4n + 1)$; that is, if $a < (4n^2 + 3n)^2/(4n + 1)^2$. Performing the division, we see that this will be true if $a < n(n + 1) - n/(4n + 1)^2$. So for $a \gneq 1$ and $n^2 < a \le n(n + 1)$, $t(a)$ yields a better approximation than $s(a)$ for those a which satisfy

(7) $a < n(n + 1) - n/(4n + 1)^2$.

For $0 < a \; 5 \; 1$ or $n(n + 1) \le a \le (n + 1)^2$, a similar computation using (5) and (6) gives a negative value for $(t(a) - \sqrt{a}) + (s(a) - \sqrt{a})$ when

(8) $a > n(n + 1) + (n + 1)/(4n + 3)^2$.

Combining the results of (7) and (8), we see that the Taylor lines will produce a more accurate approximation to $\sqrt{a}$ than the interpolating line except when a is such that

$$\begin{cases} \dfrac{n}{(4n + 1)^2} \le (a - n(n + 1)) \le \dfrac{n + 1}{(4n + 3)^2} & \text{if } a > 1, \\ a \le 1/9 \text{ (from (8) with } n = 0) & \text{if } 0 < a \; 5 \; 1. \end{cases}$$

For example, the approximation $t(a)$ is more accurate when:

$n = 1$ $(1 < a \; 5 \; 4)$, unless a lies in the interval $[2 - 1/25, 2 + 2/49]$;

$n = 2$ $(4 < a \le 9)$, unless a lies in the interval $[6 - 2/81, 6 + 3/121]$;

$n = 9$, $(81 < a \; 5 \; 100)$, unless a lies in the interval [89.993, 90.997]

The table below further illustrates these results.

			Interpolating Line		Taylor Lines	
a	n	$\sqrt{a}$	$s(a)$	$\sqrt{a} - s(a)$	$t(a)$	$t(a) - \sqrt{a}$
.01	0	.10000	.01000	.09000	.50500	.40500
1/9	0	.33333	.11111	.22222	.55555	.22222
.5	0	.70710	.50000	.20710	.75000	.04290
1.5	1	1.22474	1.16666	.05807	1.25000	.02526
1.96	1	1.40000	1.32000	.08000	1.48000	.08000
2	1	1.41421	1.33333	.08088	1.50000	.08579
90	9	9.48683	9.47368	.01315	9.50000	.01317
95	9	9.74679	9.73684	.01006	9.75000	.00321

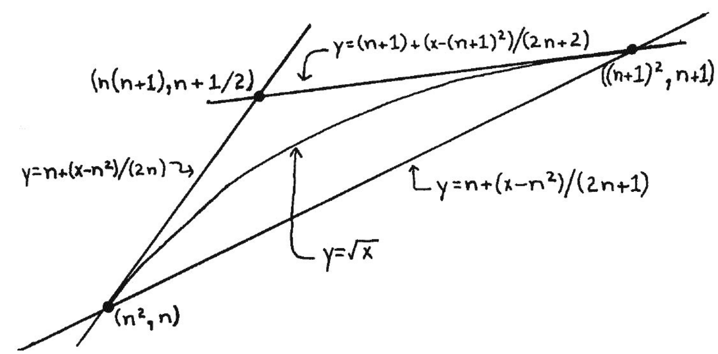

FIGURE 1

LARGEST PRIME FOUND TO DATE

Scientists. *Harry Nelson* and *David Slowinski*, at Lawrence Livermore have shown that $2^{44497} - 1$ is prime. The number is 13,395 digits in length.

REFEREES FOR THIS ISSUE

The following mathematicians have served as referees for papers considered since the last issue. The Journal appreciates their help and contributions. Almost every article published is revised and improved by the referee's comments and suggestions. Many authors have requested that these contributions be specifically acknowledged. *Bruce Peterson*, *Middlebury College*; *Leroy J. Dickey*, *University of Waterloo*; *L. Carlitz*, *Duke University*; *E. A. &own*, *Virginia Polytechnic Institute and State University*; *Wallace Growney*, *Susqehanna University*; *David Roselle*, *Virginia Polytechnic Institute and State University*; *J. Sutherland Frame*, *Michigan State University*; *Patrick Lung*, *Old Dominion University*; *James Bardsdale*, *Western Kentucky University*; *Donald Bushnell*, *Ft. Lewis College*; *David Anderson*, *Central Washington State College*; *Dennis Burke*, *Miami University*; *Gary Chartrand*, *Western Michigan University*; *Hudson V. Kronk*, *SUNY at Binghamton*; *Don R. Lick*, *Western Michigan University*; *Frieda Holley*, *Metropolitan State College*; *Dean C. Benson*, *C. A. Grimm*, *Dale Rognlie*, *Ronald Weger*, *Sailes Sengupta*, *Sumedha Sengupta*, *Rogm Opp* and *the Editor*, all of *The South Dakota School of Mines and Technology*; *Mary Ellen Rudin*, *University of Wisconsin*.

AN EMBEDDING THEOREM FOR SEPARABLE METRIC SPACES

by Roger C. McCann
Mississippi State University

The concept of homeomorphism is fundamental in topology. One of the most important applications of the use of homeomorphism is in proving Urysohn's metrization theorem: a second countable, regular space X is metrizable. This theorem can be proved by constructing a homeomorphism from X onto a subset of infinite dimensional Euclidean space R^∞, ([1, page 138]), i.e. onto a subset of the set of all sequences $a = (a_1, a_2, \cdots)$ of real numbers such that $\Sigma_{n=1}^{\infty} a_n^2 < \infty$, endowed with the metric $\rho(a,b) = \Sigma_{n=1}^{\infty}(a_n - b_n)^2$ where $a = (a_1, a_2, \cdots)$ and $b = (b_1, b_2, \cdots)$. The proof of this theorem is not elementary. An equivalent result (The proof of the equivalence is essentially the proof of Urysohn's metrization theorem.), whose proof requires only basic properties of separability and continuity of a mapping on a metric space is "A separable metric space is homeomorphic to a subset of R^∞." The elementary nature of the proof of this result permits the result to be presented early in a beginning course on topology as either a theorem or modified to be an exercise.

Henceforth, (X,d_1) will denote a separable metric space and $\{x_n\}$ will denote a countable dense subset of X. It is well known that the function $d:X\times X\to[0,1]$ defined by

$$d(x,y) = \frac{d_1(x,y)}{d_1(x,y) + 1}$$

is a metric on X which is equivalent to d_1. Hence, without loss of generality, we may consider the metric space (X,d).

We begin by defining a countable number of functions $f_n:X\to[0,1)$ by

$$f_n(x) = d(x,x_n).$$

Evidently each f_n is continuous and $0 \le f_n(x) < 1$ for every $x \in X$ and $n = 1,2,\cdots$. A mapping $h:X\to R^\infty$ may now be defined by

$$h(x) = (f_1(x), \tfrac{1}{2}f_2(x), \cdots, \tfrac{1}{n}f_n(x), \cdots).$$

Lemma 1. Let $x \varepsilon X$ and $\varepsilon > 0$. Then there is a positive integer k such that $f_k(y) - f_k(x) \geq \frac{1}{2}\varepsilon$ whenever $d(x,y) \geq \varepsilon$. Hence, h is one-to-one.

Proof. Let $y \varepsilon X$ be such that $d(x,y) \geq \varepsilon$. Set $B = \{z \mid d(z,x) < \frac{1}{4}\varepsilon\}$ Since $\{x_n\}$ is dense in X and B is a neighborhood of x, there is a positive integer k such that $x_k \varepsilon B$. Then

$f_k(x) = d(x,x_k) \leq \frac{\varepsilon}{4} < \frac{3}{4}\varepsilon \leq d(y,x_k) = f_k(y)$. Hence, $f_k(y) - f_k(x) \geq \frac{1}{2}\varepsilon$.

Lemma 2. h is continuous.

Proof. Let $x \varepsilon X$ and $\varepsilon > 0$. Since $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, there is an $N > 1$ such that $\Sigma_{n=N}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{2}$. For each n there is a $\delta_n > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon^{1/2} \left\{2\Sigma_{n=1}^{N-1} \frac{1}{n^2}\right\}^{-1/2}$ whenever $d(x,y) < \delta_n$. Set $\delta = \min\{\delta_1, \delta_2, \cdots, \delta_{N-1}\}$. Then

$$\rho(h(x),h(y)) = \sum_{n=1}^{\infty} \left[\frac{f_n(x) - f_n(y)}{n}\right]^2$$

$$= \sum_{n=1}^{N-1} \left[\frac{f_m(x) - f_m(y)}{n}\right]^2 + \sum_{n=N}^{\infty} \left[\frac{f_m(x) - f_m(y)}{n}\right]^2$$

$$\leq \sum_{n=1}^{N-1} \left[\frac{\varepsilon^{1/2}\left\{2\sum_{i=1}^{N-1} \frac{1}{i^2}\right\}^{-\frac{1}{2}}}{n}\right]^2 + \sum_{n=N}^{\infty} \frac{1}{n^2}$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

whenever $d(x,y) < \delta$. Since x is an arbitrary element of X and ε is an arbitrary positive number, h must be continuous.

lemma 3. h^{-1} is continuous.

Proof. Let $y \varepsilon X$ and $\{y_i\}$ by any sequence in X such that $\{h(y_i)\}$ converges to $h(y)$. We will show that $\{y_i\}$ converges to y. Suppose the contrary. Then there is a subsequence $\{z_i\}$ of $\{y_i\}$ and an $\varepsilon > 0$ such that $d(z_i,y) > \varepsilon$ for every i. By Lemma 1 there is a positive integer k such that $f_k(z_i) - f_k(y) \geq \frac{1}{2}\varepsilon$ for every i. Hence,

$$\rho(h(y),h(z_i)) = \sum_{n=1}^{\infty} \left\{\frac{f_n(y) - f_n(z_i)}{n}\right\}^2$$

$$\geq \left\{\frac{f_n(y) - f_n(z_i)}{k}\right\}^2$$

$$\geq \left[\frac{\varepsilon}{2k}\right]^2$$

for every . This is impossible because $\{h(z_i)\}$ is a subsequence of $\{h(y_i)\}$ and $\{h(y_i)\}$ converges to $h(y)$. It follows that h^{-1} is continuous at $h(y)$. Since y is an arbitrary point of X, h^{-1} is continuous.

Combining these three lemmas we have

Theorem. A separable metric space is homeomorphic to a subset of R^{∞}.

REFERENCES

1. Simmons, G.F., *Introduction to Topology and Modem Analysis*, McGraw Hill Book Co., New York, 1963.

A graduate student at Trinity
Computed the square of infinity
*But **it** gave him the fidgets*
To put down the digits
*So he dropped math and took **up** divinity*

--Anonymous

MOVING?

BE SURE TO LET THE JOURNAL KNOW!

Send your name, old address with *zip* code and new address with *zip code* to:

David Ballew
South Dakota School of Mines
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57701

REGIONAL MEETING OF MAA

Many regional meeting of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

MATCHING PRIZE FUND

If your chapter presents awards for ***outstanding mathematical papers*** or ***student acheivement in mathematics,*** you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of ***mathematical films.*** To apply, or for more information, write to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

MATHEMATICAL
RESEARCH AND DEVELOPMENT

by *Pad J. Nahin*
University of New Hampshire

The lecture hall was packed, and the air buzzed with whispered excitement. Wild speculations on what was about to happen flew from mouth to ear. The great, Full Professor of Mathematics, Dr. Oliver Osgood, head researcher on five simultaneous government grants, was to speak that day at the graduate seminar on his newest, most astounding discovery.

Not since von Neumann proved the Fundamental Theorem of Games, not even since Fermat scribbled his Last Theorem in a margin, nay, not since Archimedes bounded pi, had such a penetrating intellect laid bare the most intimate details of the Queen of the Sciences. Professor Osgood's insight had once again cast bright light into a hitherto dark corner of mathematics. They were all here to learn what new, incredible secret was now in humanity's possession.

The door burst open(one could almost hear trumpets), and the Great Man strode to the podium. Tall, thin, head awash in a mass of white, unruly hair, eyebrows thickly bushy, he placed a single sheet of paper on the wooden surface before him. He stared at it intently. Impressed, the audience exploded into applause. He acknowledged their acclaim with a boyish (yet mature) smile, then held his arms high to hush the excited crowd.

"Thank you, thank you. You are generous, most kind, indeed. It gives me enormous pleasure to be able to share with you the latest fruits of my research. The Government, which sponsors my work, has agreed to the seminar being unclassified, and I wish to thank those enlightened people for the wisdom of their decision."

"Now, let me get right to it, with no further delay. I have found a new number!"

The audience stared, dumbfounded, at Professor Osgood. This was just incredible.

"Yes. between two and three, somewhat closer to the two than the three, is a previously unsuspected number. Like a lost penny it has remained hidden through the centuries among the rationals, secreted away beneath the irrationals, but most deeply covered by the transcendentals. Yet, I have found it! With the help, of course, of forty-three graduate students, and hundreds of hours on the university computer system. I thank them all."

A mere associate professor jumped to his feet. "But Professor, this absolutely marvelous! What *is* this number? Will you write it on the blackboard for us?" A hundred pencils leaped into the hands of the audience, quivering in the excited anticipation of recording the new number.

Professor Osgood shook his head in a charmingly rueful way. "No, I'm sorry, I can't. As you all know, there are two kinds of mathematical proof. The first kind, a constructive proof, would give us the actual value of the number. But my proof is of the second kind, an existence proof. It shows the number exists, but doesn't tell us *where* exactly it is. But we are working on that! The Government has graciously allocated ten million dollars more for my research in the coming fiscal year."

The audience thundered its approval. Those were big time bucks! There would be summer research money for all!

"Well then, Professor, can you tell us more about your proof?"

"Sorry, that's classified DEEP SECRET. Besides myself, only the Secretary of Defense, the Chairman of the Joint Chiefs of Staff, and the President have seen it. I assure you the secret is safe with *them*. I *can* tell you it is based on the fact that three is odd, and two is even, but beyond that, I'm sure you understand my reluctance to say more."

The mere associate professor nodded his head to show his understanding, and asked, "Tell us, Professor, does that mean this new number has *military* significance?"

Professor Osgood smiled good naturedly, and replied, "We're sure of it. The integrity of our NATO forces in Europe, of our nuclear submarine fleet, of our SAC bases, of our ICBMs, all will be immensely improved by knowledge of this number." He leaned forward toward the spellbound

audience.

"I *will* tell you this much. If the new number is squared, then added to itself, and the result finally multiplied by the cube root of twenty-nine, the theory indicates we *should* have the absolute, total power to..." Before Professor Osgood could finish, a short, fat, heavily sweating man pushed forward from the rear of the room. "No, no, Professor, that's classified DEEP SECRET, too! You must say no more!"

Professor Osgood smiled gratefully at the excited man. "Ladies and gentlemen, my apologies for this lapse. This is Colonel Stanley, CIA, and I thank him for setting me straight." The sweaty, almost dripping agent sat down with a plop and a wet squish, relieved at having held back a major leak.

The Professor continued. "Our work for the coming year has the highest urgency. Naturally, we *must* beat the Russians to this new number. The whole strategic balance of technological parity could be unbalanced if we lose this race. There *must* not be a Number Gap!" He paused for a dramatic moment as his eyes blazed, his brow wrinkled, and his fists clenched. He cut a mighty impressive figure, he sure did. The room burst into light from the flash camera of the local news photographer. What a picture it would make on the front page of the late edition!

"And finally, ladies and gentlemen, knowledge of this new number will give us an additional bargaining chip at the upcoming arms control talks! The very future of East-West detente may hinge on our work over the next twelve months!" Professor Osgood then picked up the single sheet of paper before him (having by now decided to go along with his broker's recommendation to dump soybean futures and to buy Acme Gaskets), and strode from the room.

The lecture hall rocked with the thunderous applause of the audience. They knew the worth of what they had heard. With scientists like Professor Osgood, the Free World would never have to worry.

Because, as everybody knew, somewhere in the Soviet Union the Russians probably had a Professor Osgood, too.

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NEW CHAPTER INSTALLATIONS

1978--1979

Massachusetts Epsilon	Boston University
Illinois Eta	Augustana College
Delaware Beta	Deleware State College
Minnesota Epsilon	St. Cloud State University
Minnesota Delta	St. John's University
Kentucky Delta	University of Louisville
New Yohk Alpha Alpha	Queens College
Oklahoma Gamma	Cameron University
Georgia Epsilon	Valdosta State College
South Carolina Delta	Furman University

1977--1978

Missouri Epsilon	Northwest Missouri State University
North Carolina Theta	Univ. of North Carolina at Charlotte
Georgia Delta	Atlanta University Center
Wisconsin Gamma	Univ. of Wisconsin-Parkside
New Yohk Omega	Saint Bonaventure University
Mississippi Gamma	Jackson State University
Tennessee Delta	Univ. of Tennessee at Knoxville
Machachusetts Delta	University of Lowell
California Lambda	University of California

There are now 222 Chapter of Pi Mu Epsilon and in 1978 through 1979 2385 new members were initiated. Welcome to all of the new Chapters and new members.

CLOSE ENCOUNTERS OF THE MATHEMATICAL KIND

Eileen L. Poiani
Saint Peter's College

Nothing can be more absurd than the practice which prevails in our country of men and women not following the same pursuits with all their strength and with one wind, for thus the state, instead of being a whole, is reduced to a half.

--Plato (429-347 B.C.)

One pursuit in which women have continually fallen behind men is the study of mathematics and in particular of calculus.

Today, nearly three hundred years after the discovery of the calculus, sociologist Lucy Sells, [6], identifies mathenatics as the "critical filter" in career access for both college and non-college bound students. Evidence shows that young women with aptitude have traditionally taken little more than the minimum high school mathematics requirement and consequently have been inadequately prepared to take college calculus, thereby closing the doors to all but a handful of major fields. (The analogous situation for minority students is even more discouraging.)

Perhaps the data most frequently cited to support this claim is that of the freshman class entering Berkeley in 1972, [4]. Fifty-seven percent of the men had taken four full years of high school mathematics, but only 8% of the women. Thus 92% of the entering women did not qualify to enter the mathematics track required of every major at Berkeley except librarianship, education, social sciences, and humanities--all traditionally female career areas.

In a more recent study at the University of Maryland in Fall, 1977, 63% of the entering white men had had $3\frac{1}{2}$ years or more of pre-calculus in high school, compared with 31% of white women, 27% of black men, and 19% of black women. Progress in equalizing the study of mathematics continues to be very slow.

Where do women turn without enough close encounters of the mathematical kind? Women now make up nearly 52% of the U.S. population, but according to observations by the Lawrence Hall of Science at Berkeley, they are:

1. Only 1% of the U.S. Engineering force and 3% of the physicists,
2. 99% of the nurses and secretaries,
3. 4% of the lawyers,
4. 9% of the doctors and dentists,
5. 2% of the secondary school principals and 1% of the school superintendents, although 73% of the women in college in 1977 were majoring in education in one form or another,
6. Only 8% of library directors, but over 90% of the professional library staff, and
7. Barely 3% of college and university presidents.

Average starting salaries for bachelor's degree holders show dramatic differences based on career demand for a strong undergraduate mathematics program. Statistics from a nationwide sample, adapted from the College Placement Council Salary Survey of March, 1978, show the following average yearly salaries:

Humanities	$9,948	Business and Management	$12,668
Social Sciences	10,056	Sciences	13,668
Economics	11,400	Engineering	16,668

The differential of over $6,700 is staggering. The continuing inverse relationship between the highest paid areas and those most heavily populated by women is even more unsettling.

Lest we become too pessimistic, however, we should make note of a countervailing trend both at the bachelor's and master's degree levels., [1] and [2]. Here the representation of women has increased most in those fields where women have traditionally been least represented. But according to studies by the National Center for Educational Statistics, this trend has not yet reached the doctoral level, [3].

Mathematics plays the critical filter role not only at the college level, but also in the work force. In the 1978-79 edition of Occupational Outlook Handbook, the U.S. Bureau of Labor Statistics forecasts that the vast majority of the 46 million job openings through 1985 will require fewer than four years college training. Major growth areas include computers, transportation, health services, secretaries, bookkeepers, cashiers, mechanics, and police offecers. These areas require good basic arithmetic skills and in the future will demand an ability to handle data, understand

statistics, read computer printouts and graphs. Advancement in any one of these fields may well depend on a person's mastery of algebra and geometry. In fact, in the November 1977 issue of Today's Secretary, it was pointed out that:

> *"if you want to advance in a secretarial career, math may be one of the most important hurdles to overcome to be successful. Today, the highest paid secretaries are in the technical fields."*

The ability to handle budgets for the boss is a desirable asset when seeking advancement to the boss's position.

A study, [4], of 1300 children aged 2 through 12 showed no sex difference in the liking of mathematics. These children were asked to rank the subjects of English, math, science, and social studies from the most to least favorite. More than 54% ranked math as first or second favorite, and there was no difference in preferences between boys and girls. Many different reasons have been advanced to explain why the disproportionately small number of women in mathematics and related areas has persisted over the years.

A growing wealth of research points to societal conditioning rather than lack of ability in the infrequent encounters between women and mathematics. The masculine stereotype of mathematicians; overt discouragement of young women from studying mathematics - by parents, teachers, and advisers; and the traditionally unreceptive climate toward the education of women often thwarted all but the most determined women from mathematical pursuits.

But women did make significant contributions to mathematics although their Who's Who list is little known. The list is too long to enumerate here, but the reader is referred to Lynn Osen's book, *Women in Mathematics*, [5], for assistance in completing the following quiz. The unfamiliarity of their names reflects the historical exclusion of these really notable women from the mainstream of mathematical development. Answers to the quiz appear at the end of the article.

Matching Some Notable

Women Mathematicians **with their** **Major Accomplishments**

___ 1. *Hypatia (370-415)* A. *Translator and Analyzer of Newton's Principia.*

___ 2. *Emilie de Breteuil* (1706 - 1749)	B. *Expositor par excellence of the physical sciences.*
___ 3. *Maria Gastana Agnesi* (1718 - 1799)	C. *Pioneer of the modem computer compiling system and programing languages.*
___ 4. *Caroline Herschel* (1750 - 1848)	D. *Commentator on the conics and Diophantine equations; inventor of the hydroscope and other apparatus for astronomy.*
___ 5. *Sophie Germain* (1776 - 1831)	E. *Creator of modem abstract algebra.*
___ 6. *Mary Fairfax Somerville* (1780 - 1872)	F. *First woman author of a major textbook on calculus.*
___ 7. *Sonya Corvin - Krukovsky Kovalevsky* (1850 - 1891)	G. *Expert in detailed mathematical calculations associated with astronomy; the first woman to ever see a cornet.*
___ 8. *Emmy Noether* (1882 - 1935)	H. *One of the founders of mathematical physics, with several papers focusing on the theory of elasticity.*
___ 9. *Grace Murray Hopper* (1906 -)	I. *Researcher in such diverse subjects as partial differential equations, theory of Abelian functions, and application of analysis to the theory of numbers.*

To help smash the erroneous stereotype that women do not belong in mathematics and to convince them that they do, WAM (Women and Mathematics) was created. WAM is a secondary school lectureship program created in 1975 by the Mathematical Association of America under a grant from IBM. By conservative estimates made in the summer of 1978, WAM visits had reached more than 260 schools, 25,000 students, 2,500 parents, teachers, and guidance counselors.

The idea for WAM was sparked when IBM representatives hosted a reception for top scorers in the U.S.A. Mathematics Olympiad and noticed no women were among the winners. The U.S.A. Olympiad is an annual contest for invited high school students who have excelled in previous mathematics competitions. From among the top scorers in the U.S.A. Olympiad, a special

team is chosen for participation in the International Olympiad held each summer. For the first time in 1978, two young women scored highly on the U.S.A. Olympiad and they participated in the training session for the international competition.

The absence of women from the Olympiad symbolized their abscence from all those fields previously mentioned in this article. To encourage 9th and 10th graders to keep career doors open by taking more than the minimum mathematics requirement, IBM agreed to support WAM. School visits by lecturers from a variety of career fields are planned for half to one full day at no cost to the school. The visit includes a formal talk on how the speaker uses mathematics and an informal session for students, counselors, and teachers.

Since many factors influence attitudes towards mathematics, WAM also arranges presentations to professional societies, guidance counselors, elementary school teachers, math teachers, civic organizations, parents associations, and legislative leaders. By conservative estimates as of April 1979, WAM visits had reached more than 442 schools, 38,600 students, and 3.600 teachers, guidance counselors and parents.

Further information about WAM can be obtained through the MAA headquarters, 1529 Eighteenth Street, N.W., Washington, D.C. 20036 or from the National Director, Dr. Eileen L. Poiani, Saint Peter's College, Jersey City, N.J. 07306.

Answers to ***Quiz:***
d,a,f,g,h,b,i,e,c

Eileen L. Poiani is one of the four National **Pi Mu** Epsilon councillors.

FRATERNITY KEY-PINS

Gold Clad key-pins are available at the National Office (the Univ. of Maryland, Department of Mathematics) at the special price of $8.00.

Be sure to indicate the chapter into which you were initiated and the approximate date of initiation.

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (HENDRIX COLLEGE) had a variety of programs and papers during the year. These included: *Julie Anderson*, "A Report On The Summer Meetings In Providence"; *Dr. Frank Hudson* (University of Central Arkansas), "A Pure Mathematician In An Applied Situation"; *Janet Dillahunty* (Southwestern Bell Telephone), "Employment Opportunities For Math Graduates"; *Dr. Glenn Webb* (Vanderbilt University), "Mathematical Models of Epidemics"; *Dr. William Summers* (University of Arkansas, Fayetteville), "The Cantor Set: Homeomorphisms and Fixed Points"; *Dr. Robert C. Walls* (University of Arkansas Medical Center), "Biometry and Its Applications"; *Gene Web@*, 'The Symmetric Derivative"; *David Sutherland*, "Algebraic Properties of the Laplace Transform"; *Sandy Scrimshire*, "Computer Applications in Population Genetics"; *Lisa Orton*, "Characterizations of Convex Sets"; *Agnes Tulio*, "Uniform Limits of Step Functions"; *Dr. Thomas Fomby* (Southern Methodist University), "A Microeconomic Model of the U. S. Economy"; *Dr. John Churchill*, "Mathematicians in the History of Philosophy".

Integral to the Chapter's activities was the annual spring travel program. During this period several students presented their undergraduate papers to various colleges and universities in Arkansas and the neighboring states. Talks were given at the Oklahoma-Arkansas MAA meeting at Oklahoma State University, The Arkansas Academy of Science, The University of Arkansas at Fayetteville, North Texas State University, and at Southwestern College in Memphis. The latter was a joint symposium for Undergraduate Mathematical Research, with students from the University of the South at Sewanee, Southwestern at Memphis, and Hendrix participating.

MINNESOTA EPSILON (ST. CLOUD STATE UNIVERSITY) had Chapter activities which included a number of presentations by guest speakers, participation in a free student tutoring service, organization of both Fall and Spring mathematics and computer science student-faculty picnics, sponsorship of a scholarship program, and cosponsorship of the annual Spring SCSU Mathematics Contest for area junior high and senior high students. This years speakers and their topics were as follows: *Jerry Lenz* (St. John's University), "Nonstandard Analysis"; *Mahmoud A. Kishta*, "The Golden Section"; *Eric Nummela*, "An Axiomatic Approach to Set Theory"; *Joseph Gallian* (University of Minnesota, Duluth), "Weird Dice"; *Jack Anderson*, "Placement Files and the Job Situation"; *Tom Haigh* (St. John's University), "What I Wish Every Student Had Learned About Graphing In High School"; *Eric Nummela*, "Everything You Always Wanted to Know About Mobius Strips"; *Mike Sweeney* (The St. Paul Companies), "What is an Actuary?"; *John Rongitsch* (Rongitsch Incorporated), "Experience in Establishing My Own Computer Services Company"; *Richard Jarvinen* (St. Mary's College), "Vector Spaces"; *David Boyer*, "A Digestible Introduction to Nonstandard Real Number Theory"; *Robert Earles*, "Combinatorial Mathematics, An Interesting Problem".

MISSOURI GAMMA (ST. LOUIS UNIVERSITY) hosted the Fifth Annual Pi Mu Epsilon Regional Student Conference which was held Nov. 3-4, 1978. The first day was devoted to area High School Students. One hundred and fifty-five students and faculty from fifteen high schools attended. The second day consisted of three parts: Undergraduate Presentations, Graduate Presentations, and two invited lectures. The two invited lectures were *Dr. David Ballew* (South Dakota School of Mines and Technology), "Approximations to Pi", and *Sister Juliana Lucey* (Marquette University), "Some Applications of Numerical Analysis To Differential Equations". Seventy people were in attendance, representing fourteen universities and colleges. Papers given included: *Edwin Eigel Jr.* (SLU), "Negative Fish"; *Charles Fold* (SLU), "Crystals and Symmetry"; *Barbara Reynolds*, (SLU), "Taxicab Geometry"; *Nick Sortal* (SIU-C), "Mathematical Problem Solving"; *Jo Fiene* (SIU-C), 'Women in Mathematics"; *Joseph Boor* (SIU-C), 'Determining Computationally Whether or Not a Polygon is Either Simply Connected or Convex"; *Anita Zettler* (Murray State University), "The Ring of Continuous Functions on the Unit Interval, etc."; *Camile Stelzer* (Maryville College), "Practical Situations Involving the Use of Discrete Bayesian Statistics"; *Ghegohq Battle* (Washington University), "Numerical Treatment of Meteorological Data"; *Barney Smith* (SLU), "The Trisection Problem--Various Solutions"; *Martin Franck* (Milliken Univ.), "The Geostrophic Wind As a Sum of the Corioles Vector and the Pressure Gradient Vector".

MISSOURI DELTA (WESTMINSTER COLLEGE) heard a talk by former professor *Geohge Hinkle* entitled "Problem Solving".

MONTANA ALPHA (UNIVERSITY OF MONTANA) hosted several "lemonade parties" for undergraduates during the Fall quarter. Talks and papers presented to the Chapter included: *Rick Demarinis*, "What's a Novelist Doing Here Talking to Mathematicians", (Rick first spoke to the Chapter in 1962 on "Waring's Problem", but since then he has become a successful novelist after working in industry as a mathematician.); *Howard Reinhardt*, "Exercises, Puzzles and Problems"; *Robert Stevens*, "Photographs of Some Famous Mathematicians--I'll trade you my G. H. Hardy for your Herman Weyl". In addition two movies, "Turning the Sphere Inside Out" and 'Space Filling Curves", were seen.

NEW JERSEY DELTA (SETON HALL UNIVERSITY) heard three papers and viewed two films. The papers and authors were: *Dr. J. W. Andrushkiw*, "Undergraduate Research Problems in Mathematics"; *Dr. E. Guerin*, "Arithmetic Functions"; *Prof. Denny Gulick* (University of Maryland), "Between the Pot of Gold and the Rainbow". The films were: "Who Killed Determinants" and "Nonstandard Analysis".

NEW YORK OMEGA (ST. BONAVENTURE) hosted the following paper: *Dr. James Peters* (SUNY at Purchase), "Applications of the Radon and X-Ray Transforms". In addition the Chapter held a film festival at which the following were shown: "Caroms", "Isometries" and "Space Filling Curves".

NORTH CAROLINA GAMMA (NORTH CAROLINA STATE UNIVERSITY) held four program meetings during the year. They were as follows: PA. *Robert Silber*, "Instant Insanity", (This is an impressive vidiotape analysis of the famous puzzle.); PA. *Robert Silber*, "Grundy Numbers and Disjunctive Compounds of Games"; *James Bergin* (Occidental Life Insurance Company of North Carolina), "The Actuarial Profession--Performance, Training and Opportunities"; *Dr. Nicholas Rose*, "Fallacies, Howlers and Paradoxes".

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLGY) heard *Dr. David Ballew* speak on "Research Projects No One Ever Told You About".

OHIO ZETA (UNIVERSITY OF DAYTON) noted that this was the last year for *Prof. Kenneth Schraut* to serve as faculty advisor. He had held this position since the Chapter's installation in 1962. The major event for the Chapter was the Nineth Biennial Alumni Seminar on Employment

Opportunities in the Mathematical Sciences. Over fourty alumni of the mathematics department participated. Two lectures were given by members of the faculty: *Prof. Gerald Shoughnessy*, "Design of Experiments" and PA. *Clay Waldrop*, "King Chicken Theorems". *Brother Ploeger, S.M.* assisted the Chapter in sponsoring a series of problem sessions in which recreational type problems were discussed.

VIRGINIA DELTA (ROANOKE COLLEGE) sponsored and participated in a variety of programs during the year. Members of the Chapter helped produce the Fall Meeting of the Blue Ridge Council of Teachers of Mathematics (BRCTM), the Meeting of the Virginia Council of Teachers of Mathematics (VCTM), and served as judges in the Mathematics Division of the Western Virginia Regional Science Fair. In addition, the following papers and presentations were heard: *Dr.* Nanny Jane *Ingram*, "The Golden Section...It Appears Everywhere"; *Elizabeth Leonard*, "Experience as an Actuary (Intern)"; and *Cheryl Ammon*, "Experience as a Student Teacher".

1980 NATIONAL PI MU EPSILON MEETING

It is time to be making plans to send an undergraduate delegate or speaker from your chapter to attend the Annual Meeting of Pi Mu Epsilon on the University of Michigan Campus at Ann Arbor in August of 1980. Each speaker who presents a paper will receive travel fund of up to $400 and each delegate, up to $200.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

We have a supply of 10 x 14-inch Fraternity Crests available. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock---------$1.50/dozen,

(2) Purple on lavendar on goldenrod---$2.00/dozen.

PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinguish material which might well be the comain of the Problem Department, but the Editor reserves the right to nuke an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate; or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota 57701. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March 1, and that for each Sping issue, the following September 15.

Mathacrostic No. 9

submitted by Joseph D. E. Konhauser
Macalester College, St. Paul, Minnesota

Like the proceeding puzzles, this puzzle (on the next page) is a keyed anagram. The 190 letters to be entered in the diagram in the numbered spaces will be identical with those in the 26 keyed words at matching numbers and the key letters have been entered in the diagram to assist in collrelation during your solution. When completed, the initial letters will give a famous author and the title of his book. The diagram will be a quotation form that book..

Smith-Jones-Robinson Problem

submitted by Vewiser Turner, Jr.
Prairie View A&M University

(This problem is adapted from *Scientific American*, Vol 200, no. 2, p. 136, Feb. 1959, with permission)

Smith, Jones, and Robinson are the engineer, brakeman, and fireman on a train, but not necessarily in that order. Riding the train are three passengers with the same three surnames, to be identified in the following premises by "Mr." before their names.

Mr. Robinson lives in Los Angeles.

The brakeman lives in Omaha.

Mr. Jones long ago forgot all the algebra he learned in high school.

The passenger whose name is the same as the brakeman's lives in Chicago.

The brakeman and one of the passengers, a distinguished mathematical physicist, attend the same church.

Smith beat the fireman at billiards.

Who is the engineer?

Maximum Number of Knights

submitted by Pierre R. Square

What is the maximum number of knights that can be placed on a chessboard so that no two attack each other?

Z 1	A 2		Y 3	F 4	D 5	U 6	J 7	M 8	G 9		Q 10	A 11	O 12
	I 13	S 14	V 15	R 16	T 17		E 18	W 19		S 20	N 21	P 22	F 23
E 24	I 25	V 26	Q 27	T 28		M 29	P 30	S 31	T 32	V 33	J 34	Q 35	F 36
	R 37	D 38		E 39	W 40	B 41	A 42	I 43	G 44	L 45		D 46	U 47
N 48		E 49	O 50		H 51	R 52	S 53		K 54	V 55	R 56	C 57	Y 58
	X 59	F 60	U 61	B 62	S 63	P 64		C 65	M 66	J 67		X 68	K 69
U 70	N 71	I 72	Y 73		V 74	S 75	O 76	D 77	A 78		G 79	Q 80	E 81
T 82		W 83	U 84	L 85		G 86	S 87	A 88	C 89	Z 90	H 91		I 92
W 93	L 94	U 95	Q 96		X 97	H 98	N 99	I 100		F 101	B 102	E 103	T 104
Z 105	K 106	S 107		B 108	W 109	U 110		M 111	L 112		Y 113	X 114	H 115
P 116	E 117	T 118	R 119	K 120	V 121		Q 122	Z 123	G 124		D 125	A 126	L 127
B 128	I 129	Y 130	Z 131	M 132	N 133	T 134	W 135		G 136	J 137		D 138	S 139
C 140		F 141	I 142	V 143	Y 144	R 145	K 146	T 147	L 148	U 149		M 150	P 151
Z 152	V 153	K 154	Q 155	X 156	A 157		T 158	D 159	R 160	S 161	F 162	I 163	
W 164	U 165		D 166	J 167	Z 168	B 169	S 170		A 171	W 172		Z 173	B 174
H 175		P 176	I 177	J 178	G 179	U 180	V 181	Y 182	A 183	T 184	O 185		P 186
D 187	F 188	N 189	H 190										

A.	transformation which maps each line into a parallel line	11 171 88 126 157 78 2 183 42
B.	U. S. pioneer in communication theory	128 174 108 102 169 62 41
C.	the last word in a chess game	89 57 65 140
D.	the problem of equipartitioning a circle	77 38 125 5 187 138 46 166 159
E.	egg-shaped with broad end up	103 117 18 81 49 24 39
F.	Greek name assumed by Holzmann, translator of Diophantus' "Arithmetica"	60 36 23 101 4 188 162 141
G.	matrix form in which the number of leading zeros in each row increases with the row number	9 86 79 124 179 136 44
H.	sometime search objective	51 98 190 175 91 115
I.	Cretan who created a paradox by saying "All Cretans are liars"	72 92 43 163 100 177 129 25 142 13
J.	aspect of change of interest in calculus (2 wds.)	7 178 34 167 67 137
K.	arise as a consequence	154 106 120 54 69 146
L.	Babbage's antecedent of the computer	85 148 45 94 127 112
M.	showy trifle; bauble	8 132 66 29 150 111
N.	odd	21 133 48 99 189 71
O.	J. H. Conway game related to problems of simulation and artificial intelligence	185 76 50 12
P.	lowest throw at dice	116 186 151 64 176 22 30
Q.	sharp repeated tapping (comp.)	35 155 10 80 122 27 96
R.	alternative to perish	16 56 37 145 119 160 52
S.	quadrirectangular tetrahedron	87 107 161 75 31 170 14 139 20 63 53
T.	in a plane, the locus of a point the product of whose distances from two points fixed 2a apart is a^2	118 17 32 28 147 158 134 184 104 82
U.	in Chinese cosmology, all that comes to be (3 wds.)	165 61 95 6 84 110 180 70 47 149
V.	Riemann surface for double-valued function (2 wds.)	153 74 15 55 121 26 33 181 143
W.	close in count (3 wds.)	83 19 172 164 135 93 109 40
X.	formulator of first truly satisfactory postulational treatment of Euclidean geometry	68 59 114 156 97
Y.	Robert Abbott's card game which requires inductive reasoning	3 58 113 144 73 182 130
Z.	beginning of many math problems (2 wds.)	152 105 90 1 173 123 168 131

Minimum Number of Knights
submitted by Pierre R. Square

What is the minimum number of knights required so that every square on the chessboard is either occupied or under attack.

SOLUTIONS

Mathacrostic No. 7 (See Fall 1979 issue).

Definitions and key:

A. Guthrie	H. Hollerith	O. Rioting	V. Entracte
B. Eclectic	I. Epsilon	P. Slam bid	W. Motif
C. Orbiform	H. Unitary	Q. Ennead	X. Icosian
D. Wantzel	K. Nutation	R. Open set	Y. Noether
E. Escapement	L. Infeld	S. Flatworms	Z. Droodle
F. Nictitate	M. Vish	T. Tethys sea	
G. Tractrix	N. Elegant	U. Hopscotch	

First letters: G E Owen The Universe of The Mind

Quotation: *It is ironic that this movement which originally focused on the necessity for experimental testing led to a complete abstraction of the problem and to a realization that concern with logical structure represented an end in itself.*

Solved by George Levine; Henry S. Lieberman, *John Hancock Mutual Life;* Robert Prielipp and John Oman, *University of Wisconsin-Oshkosh;* Louis H. Cairoli, *Kansas State University;* Victor G. Feser, *Mary College, Bismarck;* Jeanette Bickley, *Webster Groves High School, Missouri;* Robert C, Gebhardt; Sister Stephanie Sloyan, *Georgian Court College;* Richard D. Stratton.

Mathacrostic No. 8 (See Fall 1979 issue)

The Journal *apologizes to Professor Gerald Perham of* St. Joseph's *University in Havertown, Pa who submitted this Mathacrostic but did not get credit in the Fall 1979 issue.*

Definitions and Key:

A. Lotto	I. Mathmag	P. Aeschylus	W. Nonado
B. Entity	J. Educt	Q. Net	X. Torsion
C. Branch	K. Azaleamum	R. Desarguesian	Y. Entente
D. Eisenstein	L. staffed	S. Tower of Hanoi	
E. Sumner	M. Umbo	T. Hahn-Banach	Z. Gradient
F. Gittite	N. Richmond	U. Eugenic	a. Redcoat
G. Utah	O. Ethology	V. Id	b. Awning
H. Entire			c. Latticed

First Letters: Lebesgue Measure and Integration

Quotation: *Since that remote age in which man learned to count, number has become one of the fundamental ideas that engage our thought--an idea so immediate and so clear to the understanding that in trying to analyze it, we at first succeed only in obscuring it.*

Solved by: Sister Stephanie Sloyan, *Georgian Court College;* Victor G. Feser, *Mary College, Bismarck;* Louis H. Cairoli, *Kansas State University;* Henry S. Lieberman, *John Hancock Mutual Life Insurance Company;* George Levine, *Commercial Union Insurance Company;* Joseph D. E. Konhauser, *Macalester College, St. Paul;* Jeanette Bickley, *Webster Groves High School, Missouri;* Robert Prielipp, *University of Wisconsin-Oshkosh;* Robert C. Gebhardt; Richard Stratton.

LOCAL AWARDS

If your chapter has presented or will present awards this year to either undergraduates or graduates (whether members of Pi Mu Epsilon *or not), please send the names of the recipients to the Editor for publication in the* Journal.

He who can, does.
He who cannot, teaches.
He who cannot teach, administrates.
He who cannot administrate, does research.
He who cannot do research, publishes research.
He who cannot publish research, edits research.
He who cannot edit research, gives lectures.
He who cannot give lectures, introduces lecturers.
He who cannot introduce lecturers, passes judgment on them.
He who cannot pass judgment, drops out.
He who drops out, gets hungry.
He who gets hungry, grows a garden.
He who grows, harvests.
He who harvests, cans.

LOCAL CHAPTER AWARDS WINNERS

ALABAMA DELTA (UNIVERSITY OF SOUTH ALABAMA). The PI MU EPSILON AWARD award for outstanding achievement in the field of mathematics was awarded to

Daniel Dix.

GEORGIA BETA (GEORGIA INSTITUTE OF TECHNOLOGY). The awards for outstanding graduates in mathematics were given to:

Virginia 1. Foard
Randolph C. Nicklas
James S. Tomlin
Clive M. Webster.

MISSOURI GAMMA (UNIVERSITY OF ST. LOUIS). The James W. Garneau Mathematics Award was given to:

William C. Dale III.

The Francis Regen Scholarship award was presented to:

First Place: Paul L. Sventek
Second Place: Maureen McGovern
John M. Hanson.

The Missouri Gamma Undergraduate Award was presented to:

Donald Schisler.

The Pi Mu Epsilon contest awards were won by:

Senior Contest Winner: Michael Mog
Junior Contest Winner: Alan Ho.

The John J. Andrews Graduate Service Award was given to:

Joel Baumeyer, FSC.

The Al and Shelly Beradino Fraternityship Award for active participation in the affairs of the fraternity was presented to:

Camile Stelzer
Cecile Stelzer.

MONTANA ALPHA (UNIVERSITY OF MONTANA). The John Peterson Book award for the outstanding senior graduating in mathematics education:

Melinda Williams.

NEW JERSEY BETA (RUTGERS, THE STATE UNIVERSITY). The Junior Book Award winner is:

Alice Davenport.

NEW YORK OMEGA (ST. BONAVENTURE UNIVERSITY). The New York Omega Chapter Award of academic excellence, interest in mathematics and service to the Chapter was presented to:

Paul McGuire.

OHIO NU (UNIVERSITY OF AKRON). The awards for Outstanding Science Fair projects were given to:

Karl Kaiser
Matt Wagner.

The Samuel Selby Mathematics Scholarship Award for 1979-80 was presented to:

Carl Andrews.

The Senior Teaching Assistant Award was given to:

Doug Seifert

and

Nancy Calvin

the Graduate Teaching Award.

RHODE ISLAND BETA (RHODE ISLAND COLLEGE). The Christopher Mitchell Awards were presented to:

Denise Larivee
Scott Chianese.

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY). The Award for the Outstanding Mathematics Senior was given to:

Bruce Weeks.

Mathematicians:
We're Number $-e^{-i\pi}$

DULUTH MEETING OF PI MU EPSILON
AUGUST 21-23, 1979

The following papers were presented at the Duluth Meeting

1. *On Continued Fraction Representations of Liouville Numbers* — MICHAEL FILASETA, AZ Alpha, Univ. of Arizona

2. *Behavior of the Permanent of a Special Class of Doubly Stochastic Matrices* — PHUONG ANH VU, TX Theta, Univ. of Houston

3. *Physical Models For Applications of Rational Numbers in Reduced Form* — JO ANN FIENE, IL Delta, Southern Illinois University, Carbondale

4. *Topological Properties of the Generalized long Line* — ALI ENAYAT, IA Alpha, Iowa State University

5. *Enzyme Kinetics of Phenylketonuria* — C. P. KUCHINAD, WI Alpha, Marquette University

6. *A Hard (But Interesting) Problem* — MICHAEL L. CALL, IN Gamma, Rose-Hulman Institute of Technology

7. *Time as a Dynamical Variable in Quantum Physics* — STEPHEN L. KERR, TN Delta, University of Tennessee, Knoxville

8. *Love Sonnet for a Mathematician* — MARK WALKER, SD Beta, South Dakota School of Mines & Technology

9. *Infinitesimals: Where They Come From and What They Can Do* — PROFESSOR H. JEROME KEISLER, J. SUTHERLAND FRAME, Lecturer, University of Wisconsin-Madison

10. *The Axiom of Choke* — JOHN B. VAUGHN, MO Gamma, St. Louis University

11. *The Kernel of the Laplace Transformation* — DAVID C. SUTHERLAND, AR Beta, Hendrix College

12. *The Knight's Tour Problem* — GARY RICARD, SD Beta, South Dakota School of Mines and Technology

13. *A Simple Proof of a Theorem by H. Scheffe* — PETER WESTFALL, CA Lambda, University of California at Davis

14. *Mathematics and the Boiling Points of Alkanes* — MICHAEL J. SCHELL, NC Theta, University of North Carolina at Charlotte

15. *Two Problems in Number Theory* — JANET REID, FL Eta, University of North Florida

16. *Uniform Algebras and Scattered Spaces* — ROBERT C. SMITH, AR Alpha, University of Arkansas

17. *A Case Against Computer Crime* — LONNIE EMARD, MO Epsilon, Northwest Missouri State University

18. *Measure Construction Using Cauchy Sequences* — STEPHEN W. SEMMES, GA Gamma, Armstrong State College

19. *Cyclic Numbers* — RICHARD O. GRIFFIN, MA Delta, University of Lowell

20. *The Consequences of Cauchy's Integral Theorem* — ALFRED EARL BYRUM, NC Delta, East Carolina University

21. *An Infinite Number of Magic Squares* — STEPHEN J. RUBERG, OH Delta, Miami University

AN ANALYST'S BOOKSHELF

Peter Hagis, Jr.
Temple University

The following is a recommended Reading List for all Mathematics students and instructors:

1. The Jacobians, and their Struggle for Independence.
2. A Ten-Day Diet to Improve Indeterminate Forms.
3. Cheaper by the Googol.
4. 1001 Best-Loved Double Integrals.
5. The Torus and I.
6. A Short Table of Even Primes.
7. Will Success Spoil Runga-Kutta?
8. Dining Out in Hilbert Space.
9. One Hundred Tasty Fillings for Empty Sets.
10. Life Begins at e.
11. How to Keep Condensation Points from Dripping into Open Sets.
12. A Child's Gargen of Tschebycheff Polynomials.
13. Tom Swift and His Electric Cycloid.
14. A Treasury of Matrices--Upright and Inverted.
15. Improving Lipshcitz Conditions in the Slums of New York.
16. The Decline and Fall of e^{-x}.
17. How to Prevent Rust on Riemann Surfaces.
18. The Peano Postulates Transcribed for Violin and Cello.
19. First Aid for Dedekind Cuts and Bruises.
20. A Collection of Happy Endings for Incomplete Beta Functions.

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PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable,. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on a separate sheet properly identified with name and address and mailed before the end of June 1980.

Address all communications concerning this department to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

449. *Proposed by Richard I. Hess, Palo Verdes, California.*

A fairly young man was married at the beginning of the month. At the end of the month his wife gave him a chess set for his birthday. If he was married and received the chess set on the same day of the week he was born, how old was he when he got married?

450. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

In triangle ABC, let $\angle A \le \angle B \le \angle C$. Then

$$s \gtreqless (R+r)\sqrt{3} \quad \text{if and only if} \quad \angle B \gtreqless \pi/3$$

is a well-known theorem, where s is the triangle's semiperimeter, r its inradius, and R its circumradius. Prove it.

451. *Proposed by Solomon W. Golomb, University of Southern California, Lob Angeles, California.*

Find all instances of three consecutive terms in a row of Pascal's triangle in the ratio 1:2:3.

452. *Proposed by Tom M. Apostol, California Institute of Technology.*

Given integers $m>n>0$, let

$$a = \sqrt{m} + \sqrt{n}, \qquad b = \sqrt{m} - \sqrt{n}.$$

If $m - n$ is twice an odd integer, prove that both a and b are irrational.

453. *Proposed by Jack Garfunkel, Queens College, Flushing, Nw York.*

Given two intersecting lines and a circle tangent to each of them, construct a square having two of its vertices on the circumference of the circle and the other two on the intersecting lines.

454. *Proposed by Marian Haste, Reno, Nevada.*

The point within a triangle whose combined distances to the vertices is a minimum is (or should be) known as the Fermat-Torricelli point, designated by T. In a triangle ABC, if AT, BT, CT form a geometric progression with a common ratio of 2, find the angles of the triangle.

455. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Young Leslie Morley noticed that the perimeter of a 6 x 4 rectangle equals the area of a 2 x 10 rectangle while the area of the 6 x 4 rectangle equals the perimeter of the 2 x 10 rectangle also. Show that there are an infinite number of pairs of rectangles related in the same way and find all pairs of such rectangles whose sides are integers.

456. *Proposed by Paul Erdős, Spaceship Earth.*

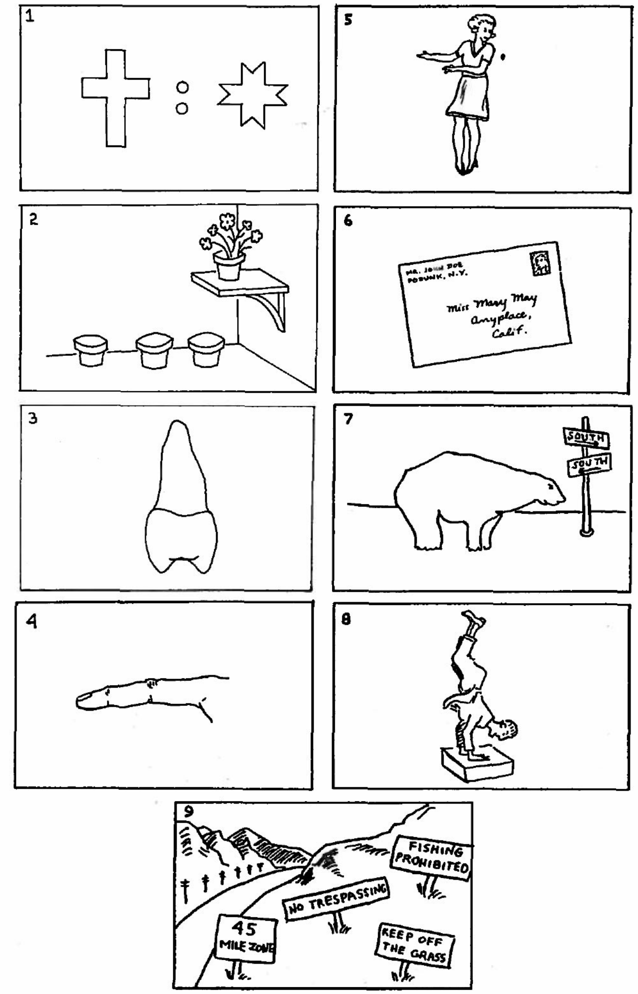
Is there an infinite path on visible lattice points avoiding all (u, v) where both u and v are primes? (The proposer offers Twenty-five Dollars for a solution.)

457. *Proposed by the late R. Robinson Rowe.*

Defining the last n digits of a square as its n-tail, what is the longest n-tail consisting of some part of the cardinal sequence 0, 1, 2, 3, ...9? What is the smallest square with that n-tail?

458. *Proposed by Charles W. Trigg, San Diego, California and Leon Bankoff, Los Angeles, California.*

Translate each of the following sketches into a mathematical term.



459. *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

Let x, y, and z be positive integers. Then (x, y, z) is a Pythagorean triangle if and only if $x^2 + y^2 = z^2$. Prove that every Pythagorean triangle where both x and z are prime numbers and $x \geq 11$ is such that 60 divides y.

460. *Proposed by Barbara Seville, University of Bologna, Italy. Dedicated to: Jean J. Pedersen, University of Santa Clara.*

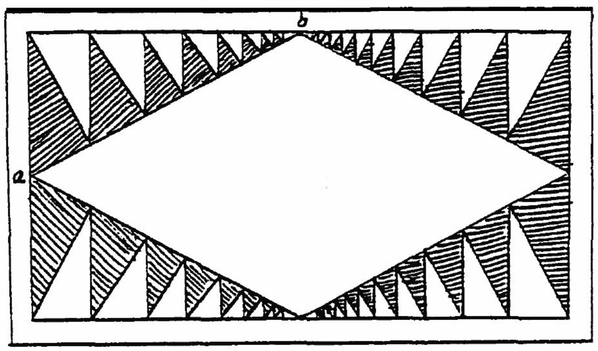
The dihedral angle of a cube is 90°. The other four Platonic solids have dihedral angles which are approximately $70^\circ 31' 43.60''$, $109^\circ 28' 16.3956''$, $116^\circ 33' 54.18''$, and $138^\circ 11' 22.866''$. How closely can these angles be constructed with straightedge and compasses? Can good approximations be accomplished by paper folding? If so, how?

461. *Proposed by David C. Kay, University of Oklahoma, Norman, Oklahoma.*

(a) A right triangle with unit hypotenuse and legs r and s is used to form a sequence of similar right triangles $T_1, T_2, T_3, \ldots$ where the sides of T_1 are r times those of the given triangle, and for $n \geq 1$ the sides of T_{n+1} are s times those of T_n. Prove that the sequence T_n will tile the given triangle.

(b) What happens if the multipliers r and s are reversed?

(c) The art of the Hopi American Indians is known for its zigzag patterns. The blanket illustrated below is made from a rectangle of (inside) dimensions $a \times b$, and the zigzag is formed by dropping perpendiculars to alternating sides of the triangle in the design. Show that the area of the design (shaded portion) is given by the formula $(a^3b + ab^3)/(2a^2 + 4b^2)$.



SOLUTIONS

425. *[Fall 1978] Proposed by Charles W. Trigg, San Diego, California.*

Without using its altitude, compute the volume of a regular tetrahedron by the prismoidal formula.

Solution by the Proposer.

In the regular tetrahedron $ABCD$ with edge e, as in Figure 1, the medians of two of the equilateral triangular faces are $MD = MC = e\sqrt{3}/2$. The bimedian, MN, is an altitude of the isosceles triangle DMC, so $MN =$

$$\left[(e\sqrt{3}/2)^2 - (e/2)^2 \right]^{1/2} \quad \text{or} \quad e/\sqrt{2}.$$

(Considering the regular tetrahedron inscribed in the cube of Figure 2, the bimedian equals the edge of the cube, so immediately is $e/\sqrt{2}$.)

The regular tetrahedron ABCD may be considered to be a prismatoid with bases AB and CD of zero area lying in parallel planes. Then the bimedian MN is the altitude of the prismatoid, which has a square midsection $EFGH$ with area $= (e/2)^2$.

Then, by the prismoidal formula, $V = h[b + 4m + b']/6$, the volume of the regular tetrahedron $ABCD$ is

$$V = (e/\sqrt{2})\left[0 + 4(e/2)^2 + 0\right]/6 \quad \text{or} \quad e^3\sqrt{2}/12$$

For other unorthodox methods of determining the volume of the regular tetrahedron see *Charles W. Trigg*, "The Volume of the Regular Tetrahedron," Eureka (Canada) [now Crux Mathematicorum], 3 (August-September 1977), 181-183.

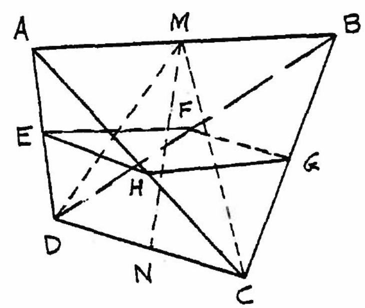


FIGURE 1

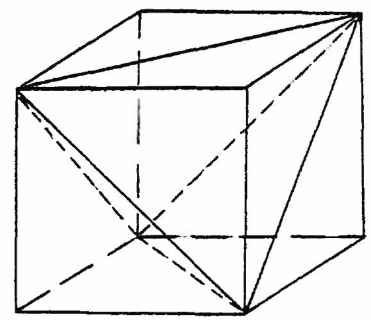


FIGURE 2

Also solved by DONALD CANARD, *Anaheim, California;* ALFRED E. NEUMAN, *Mu Alpha Delta Fraternity, New York;* BARBARA SEVILLE, *University of Bologna, Italy;* and ZELDA KATZ, *Beverly Hills, California.*

426. [Fall 1978] *Proposed by the late R. Robinson Rowe.*

With some oversimplification of an actual event, after a cold dry snow had been falling steadily for 72 hours, a niphometer showed a depth of 340 cm., compared to a reading of 175 cm. after the first 24 hours. Assuming that underlying snow had been compacted only by the weight of its snow overburden, so that the depth varied as a power of time, what would have been the depth after 12 and 48 hours?

Solution by the Proposer.

Let Z be the depth in centimeters and T the elapsed time in days. Then in the general power function, $Z = AT^b$, the given data furnish two equations, $175 = A \times 1^b = A$, and $340 = A \times 3^b$. Whence by division, $340/175 = 3^b$ and b = 0.604 544 161. Then for any other time T, $Z_T = 175\,T^b$.

After 12 hours, T = 1/2 and Z = 115.093 8512.

After 48 hours, T = 2 and Z = 266.087 1946.

The niphometer being read to the nearest centimeter, the depths were 115 and 266, respectively.

Comment: The exponential relation is approximate at best and has limitations. Density of the uncompacted snow at the surface may be as low as 0.05. Compaction under the weight of later snow begins much like elastic compression, but with the elastic modulus increasing with density. Before the density reaches 0.50, the character of the packed snow is so changed that the exponential relation is replaced by another, as enormous pressures slowly fuse the snow into ice with a density of 0.917.

The actual event occurred about 25 years ago. The niphometer was 100 feet outside the SPRR station at Norden, supposedly read every 6 hours by the stationmaster. The gage being hard to reach during a storm, he had devised an unofficial alternative. He built a shelf at sill level outside a station window. Every 6 hours he would open the window, measure the depth of snow on the shelf with a yardstick, sweep off the shelf, add this depth to a cumulating total, and record this in the gage book. After 72 hours the 12 increments had added to more than 230 inches. But at the end of the storm when he snowshoed out to the niphometer, it read only 180 inches! So he erased all that valuable record and substituted 15 inches for each increment. I wanted it later when asked to derive the maximum one-hour snowfall as a guide to design of Interstate 80 (median width for snow storage and size, type and number of snow-removal devices.) I had to rely on better data from the Snoquale Pass in Washington.

This interesting problem was also solved (in essentially the same manner) by DAVID DEL SESTO, *North Providence, R.I.;* CLAYTON W. DODGE, *University of Maine at Orono;* and CHARLES H. LINCOLN, *Goldsboro, N.C.* Two incorrect solutions were received.

427. [Fall 1978] *Proposed by Jackie E. Fritts, Texas A&M University, College Station, Texas.*

If a, b, c, d are integers and $u = \sqrt{a^2 + b^2}$, $v = \sqrt{(a-c)^2 + (b-d)^2}$ and $w = \sqrt{c^2 + d^2}$, then $\sqrt{(u+v+w)(u+v-w)(u-v+w)(-u+v+w)}$ is an even integer.

I. *Solution by M. S. Klamkin, University of Alberta, Canada.*

A solution follows immediately by considering a geometric interpretation, with u, v, w the lengths of the sides of a lattice triangle with coordinates $(0,0)$, (a,b), (c,d). I corresponds to 4 times the area of the triangle. Since the area is also given by

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ c & d & 1 \end{vmatrix},$$

I must be an even integer.

II. *Solution by Léo Sauvé, Algonquin College, Ottawa, Canada.*

In a coordinate plane, the area of a triangle whose vertices are the origin, $(2a, 2b)$, and $(2c, 2d)$ is easily found to be $2|ad - bc|$. On the other hand, the sides of such a triangle are $2u$, $2v$, $2w$, and its area, by Heron's formula, is precisely equal to the given expression, which is therefore equal to the even integer, $2|ad - bd|$.

Also solved by CHARLES D. ALLISON, *Huntington Beach, Calif.;* JEANETTE BICKLEY, *St. Louis, Missouri;* WALTER BLUMBERG, *New Hyde Park, Long Island, N.Y.;* DAVID DEL SESTO, *North Providence, R.I.;* CLAYTON DODGE, *University of Maine at Orono;* ROBERT FULLER, *Savannah, Georgia;* TAGHI REZAY GARACANI, *Jackson, Mississippi;* CHARLES H. LINCOLN, *Goldsboro, N.C.;* R. S. LUTHAR, *University of Wisconsin, Janesville;* JAMES A. PARSLY, *Oak Ridge, Tennessee;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* KENNETH M. WILKE, *Topeka, Kansas;* RANDALL J. SCHEER, *SUNY at Potsdam,* *N. Y.*; and the *Proposer*.

About half of the solutions offered were by straightforward, hammer-and-tongs algebraic manipulation, while the rest were essentially geometric.

428. [Fall 1978] *Proposed by Solomon W. Golomb, University of Southern California.*

One circle of radius a may be "exactly surrounded" by six circles of radius a. It may also be exactly surrounded by n circles of radius t, for any $n \geq 3$, where

$$t = a(\csc \pi/n - 1)^{-1}.$$

Suppose instead we surround it with n+1 circles, one of radius a and n of radius b (again $n \geq 3$). Find an expression for b/a as a function of n. (Note: For $n = 3$, $b/a = (3 + \sqrt{17})/2$, and of course for $n = 5$, $b/a = 1$. What about $n = 4$ and $n = 6$ as individual special cases?

Solution by the Proposer.

When a circle of radius a is exactly surrounded by n circles of radius b and one of radius a, then $\pi = \theta + (n-1)\phi$ where $\theta = \cos^{-1}\left(\frac{a}{a+b}\right)$ and $\phi = \sin^{-1}\left(\frac{b}{a+b}\right)$. Let $\frac{a}{a+b} = \alpha$ and $\frac{b}{a+b} = \beta$, with $\alpha+\beta = 1$.

Then

$$\cos\left[(n-1)\sin^{-1}\beta\right] = \cos\left[\pi - \cos^{-1}(1-\beta)\right] = \beta - 1.$$

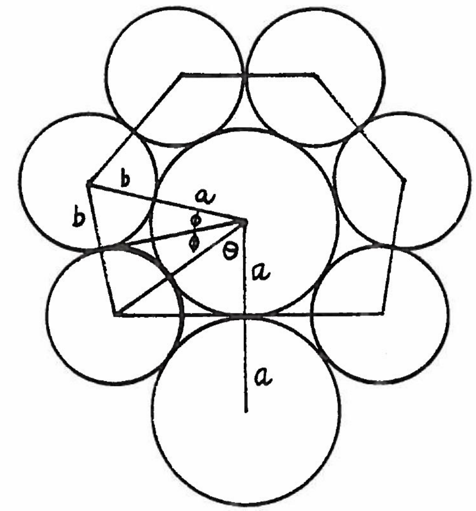
Let $\sin^{-1}\beta = z$. Then $\cos(n-1)z$ is a polynomial of degree n-1 in $\cos z = \cos(\sin^{-1}\beta) = \sqrt{1-\beta^2}$. Thus, β is a root of a polynomial $f(x)$ of degree $m \leq 2(n-1)$, for all $n \geq 3$, and $b/a = \beta/\alpha$ is a root of $(1+x)^m f\left(\frac{x}{1+x}\right)$ which also has degree $m \leq 2(n-1)$.

For $n = 3$, 4, and 6, the following table shows the polynomial $g_n(x)$ and its root which equals b/a for that value of n.

n	polynomial $g_n(x)$	root b/a
3	x^2-3x-2	3.56155...
4	$9x^4-8x^3-10x^2+1$	1.56849...
6	$25x^8-188x^7+236x^6+436x^5-2x^4-180x^3-68x^2-4x+1$	0.73403...

(See also Technology Review, vol. 81, no. 2, November, 1978, p. 84).

"Diagram for $n = 6$, with $b/a = 0.73403...$"



Three incorrect solutions were received, all based on the erroneous assumption that the common internal tangent of the unequal surrounding circles passed through the center of the inner circle.

429. [Fall 1978] *Proposed by Richard S. Field, Santa Monica, California.*

Let P denote the product of n random numbers selected from the interval 0 to 1. Question: Is the expected value of P greater or less than the expected value of the n-th power of a single number randomly selected from the interval 0 to 1?

Solution by M. S. Klamkin, University of Alberta.

Assuming the random numbers are independently and identically distributed variables, we show it is less. Let the probability density function be $\rho(x)$, then we have to show that

$$\frac{\int_0^1 x^n \rho(x)\,dx}{\int_0^1 \rho(x)\,dx} \geq \frac{\int_0^1 \cdots \int_0^1 x_1 x_2 \cdots x_n \rho(x_1)\rho(x_2)\cdots\rho(x_n)\,dx_1\,dx_2\cdots dx_n}{\int_0^1 \cdots \int_0^1 \rho(x_1)\rho(x_2)\cdots\rho(x_n)\,dx_1\,dx_2\cdots dx_n}$$

or

$$(1) \qquad \frac{\int_0^1 x^n \rho(x)\,dx}{\int_0^1 \rho(x)\,dx} \geq \left\{\frac{\int_0^1 x\rho(x)\,dx}{\int_0^1 \rho(x)\,dx}\right\}^n .$$

More generally it is known that if $u(x)$ is a convex function, then

$$E[u(X)] \geq u[E(X)]$$

(W. Feller, An Introduction to Probability Theory and its Application, J. Wiley, N.Y., 1971, p. 153) and which follows immediately from Jensen's inequality on convex functions. (1) corresponds to the special case $u(x) = x^n$ and there can be no equality if $n>1$. Also, it is known (Holder's inequality) that

$$(2) \qquad \{E(\phi^p)\}^{1/p}\{E(X^q)\}^{1/q} \geq E(\phi X)$$

where $p,q > 1$, $1/p + 1/q = 1$, $\phi,\psi \geq 0$ and the integrals exist (loc. cit., p. 155). (1) is also a special case of (2) and (2) can also be proved by means of Jensen's inequality.

Also solved by SAMUEL GUT, *Brooklyn, New York;* DONALD CANARD, *Anaheim, California;* J. WALKER, *General Hospital, Los Angeles, California;* ZELDA KATZ, *Beverly Hills, California, and the Proposer.*

Comment: If A denotes the product of n randomly selected numbers from 0 to 1 and B the n-th power of a single randomly selected number from the same interval, then

$$A: \quad E\left[\prod_1^n R_i\right] = 1/2^n$$

$$B: \quad E(R^n) = \frac{1}{x}\int_0^1 x^n dx = 1/(n+1) .$$

For n = 1, of course, the results are the same. As n grows, the difference is quite startling. For example, for n = 10, A gives 1/1024 while B gives 1/11.

430. [Fall 1978] *Proposed by John M. Howell, Littlerock, California.*

Given any rectangle. Form a new rectangle by adding a square to the long side. Repeat. What is the limit of the long side to the short side?

Solution by Clayton W. Dodge, University of Maine at Orono.

If we accept the existence of such a limit, then let the limiting rectangle have sides r and 1 with $r > 1$. Then r is the ratio we seek, and by appending a square to the long side we obtain another rectangle with the same ratio of dimensions, that is,

$$\frac{r}{1} = \frac{r+1}{r}, \quad \text{so} \quad r^2 - r - 1 = 0.$$

Thus

$$r = \frac{1 \pm \sqrt{1+4}}{2} \quad \text{and, since } r \text{ is positive,} \quad r = \frac{1+\sqrt{5}}{2},$$

the golden ratio.

Since it is not customary to accept the existence of a limit without proof, we offer a second solution.

Solution II. Let f_n denote the nth Fibonacci number. That is, $f_1 = f_2 = 1$ and $f_{n+2} = f_n + f_{n+1}$ for all positive integers n. It is well known and easily proved by mathematical induction that

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^{-n}}{2^n\sqrt{5}} \doteq \frac{(1+\sqrt{5})^n}{2^n\sqrt{5}}$$

so that we have

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2},$$

the golden ratio. It is also easy to show that if a, b, c, d are all positive and if $a/b < c/d$, then their mediant $(a+c)/(b+d)$ satisfies the inequality

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} .$$

It follows that if a/b and c/d have a common limit, then their mediant has this same limit. Now we are ready to look at the problem at hand.

Let a_1 and a_2 with $a_1 < a_2$ be the sides of the given rectange R_1. The next rectangle R_2 has sides a_2 and $a_3 = a_1 + a_2$. The third rectangle R_3 has sides a_3 and $a_4 = a_2 + a_3$, and so forth. We see that

$$a_3 = a_1 + a_2,$$

$$a_4 = a_2 + a_3 = a_1 + 2a_2 ,$$

and so forth. In general, it is easy to show by mathematical induction that

$$a_n = f_{n-2}a_1 + f_{n-1}a_2 \cdot$$

Now we have

$$\frac{a_{n+1}}{a_n} = \frac{f_{n-1}a_1 + f_n a_2}{f_{n-2}a_1 + f_{n-1}a_2} \, .$$

Since we have shown that f_{n-1}/f_{n-2} and f_n/f_{n-1} each have the same limit $(1+\sqrt{5})/2$ as $n \to \infty$, then their mediant a_{n+1}/a_n has this same limit.

Also solved by CHARLES D. ALLISON, *San Pedro, California* (Two *Solutions*); JEANETTE BICKLEY, *St. Louis, Missouri;* WALTER BLUMBERG, *N w Uqde Park, N w York;* LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas;* MICHAEL W. ECKER, *Scranton, Pennsylvania;* VICTOR G. FESER, *Mary College, Bismarck, N.V.;* JACKIE E. FRITTS, *Texas A&M University;* PETER A. LINDSTROM, *Genesee C.C., Batavia, New York;* CHARLES H. LINCOLN, *Goldsboio, N.C.;* LÉO SAUVÉ, *Algonquin College, Ottawa, Canada;* ALBERT WHITE, *St. Bonadventure University;* KENNETH M. WILKE, *Topeka, Kansas;* and *the Proposer.*

The solutions offered by Blumberg and by Feser were based on continued fractions rather than the customary Fibonaaci Sequence.

431. [Fall 1978] *Proposed bq Jack Garfunkel, Forest Hills High School, Flushing, N w York.*

In a right triangle ABC, with sides a, b, and hypotenuse c, show that $4(ac + b^2) \le 5c^2$.

Solution by Charles H. Lincoln, Goldsboro, N.C.

$(2a - c)^2 \ge 0$ implies $4a^2 + c^2 \ge 4ac$.

Using the Pythagorean Theorem, $4(c^2 - b^2) + c^2 \ge 4ac$, or $5c^2 \ge 4ac + 4b^2 = 4(ac + b^2)$. Equality occurs when $2a = c$, that is, when angle $A = 30^\circ$.

Also solved bq CHUCK ALLISON, *San Pedro, California;* JEANETTE BICKLEY, *Webster Groves High School, Missouri;* WALTER BLUMBERG, *Flushing High School, Flushing, N.Y. York;* BILL BURNS, *Seton Hall University, South Orange, New Jersey;* LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas;* CHARLES R. DIMINNIE, *St. Bonadventure University, N w York;*

CLAYTON W. DODGE, *University of Maine at Orono;* MICHAEL W. ECKER, *Pennsylvania State. University, Scranton, Pennsylvania;* MARK EVANS, *LaMargue, Texas;* ROBERT FULLER, *Armstrong State University, Savannah, Georgia;* TAGHI REZAY GARACONI, *Jackson State University, Jackson, Mississippi,* SAMUEL GUT, *Brooklyn, N.Y. York;* SUSAN HOFFMAN, *Iona College, Larchmont, New Yolk,* R. S. LUTHAR, *University of, Wisconsin, Janesville, Wisconsin;* JAMES A. PARSLY, *Oak. Ridge, Tennessee;* BOB PRIELIPP, *The. University of Wisconsin-Oshkosh;* KENNETH M. WILKE, *Topeka, Kansas;* and *the Proposer.*

Comment: To encourage participation in the Problem Department, I intentionally chose to offer this extremely simple problem. From time to time, similar "loss leaders" will appear in order to entice new customers into the store. The *simplicity* of a problem proposal is not to be construed as a reflection on the mathematical calibre of the proposer or the editor.

432. [Fall 1978] *Proposed by Erwin Just, Bronx Community College, of CUNY, Bronx, N w York.*

Does there exist an integer n for which the equation

$$\sum_{i=0}^{m} 3^{ix} = 7^y$$

has a solution in positive integers?

Solution by Walter Blumberg, Flushing High School, Flushing, New York.

Suppose for some integer m the equation has positive integeral solutions x and y. Obviously $m \ne 0$. Hence $m \ge 1$, in which case the trivial inequality $(m + 1) \le 2m$ follows. From the equation we have $3^{mx} < 7^y < 9^y = 3^{2y}$. Therefore $mx < 2y$. Now $(3^x - 1)\sum_{i=0}^{m} 3^{ix} = 3^{x(m+1)} - 1 \equiv 0 \pmod{7^y}$. Since 3 is a primitive root mod 7^y, it follows that $x(m + 1) = 0 \pmod{6 \cdot 7^{y-1}}$, where $6 \cdot 7^{y-1}$ is the totient of 7^y. Consequently, $6 \cdot 7^{y-1} \le x(m+1)$. But $7^{y-1} \ge 1 + 6(y-1) = 6y - 5$.

Recalling some previous inequalities, $(m + 1) \le 2m$ and $mx < 2y$, we now have the following:

$6(6y - 5) \le 6 \cdot 7^{y-1} \le x(m + 1) \le 2mx < 4y$. Thus $6(6y - 5) < 4y$, which leads easily to the contradiction $y < 15/16 < 1$. Hence the given equation has no solutions in positive integers.

A similar solution was offered by the *Proposer*, ERWIN JUST. DALE WATTS, *Colorado Springs, Colorado*, obtained the same result by taking $\log_7$ of both sides, arriving at the equation

$$x\left\{\sum_{i=0}^{m} i\right\}\log_7 3 = y.$$

Then since $\log_7 3 = \log_{10}3/\log_{10}7$, he ultimately achieved the contradiction $10^y = 3$, which has no integral solutions for $y > 0$.

433. [Fall 1978] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Pay this bill for four. That is, solve for BILL, which is divisible by 4.

$$\begin{array}{r} \text{PAY} \\ \underline{\text{MY}} \\ \text{BILL} \end{array}$$

I. *Solution by Charles W. Trigg, San Diego, California.*

Immediately, $P = 9$, $I = 0$, and $B = 1$. Then

$$2Y = L + k\text{, where } k = 0 \text{ or } 1\text{, and}$$
$$A + M + k = L + 10.$$

Now if BILL is divisible by 4, so is LL. Hence L = 4 or 8. But, if L = 8, then Y or A or M = 9, a duplication. Hence L = 4 and Y = 2 with $(A,M) = (6, 8)$, or Y = 7 with $(A, M) = (5, 8)$. Thus there are four solutions:

$$\begin{array}{rrrr} 962 & 982 & 957 & 987 \\ \underline{82} & \underline{62} & \underline{87} & \underline{57} \\ 1044 & 1044 & 1044 & 1044 \end{array}$$

but only one value of the BILL.

II. *Computer Solution by Jeanette Bickley, St. Louis, Missouri.*

Editor's Note. This program and print-out was submitted as a gracious response to *my* request. Ms. Bickley used a Digital Equipment Corporation PDP 11/70 computer, with the program written in Basic-Plus.

Ms Bickley writes: "We know that P = 9, B = 1 and I = 0. The program considers all other possible digit choices for each of the other letters. For those choices that satisfy the addition, it checks whether the sum is divisible by 4. If all conditions of the problem are satisfied, the BILL is printed."


```
READY

LIST
BILL     08:06 AM        17-JAN-79
10 FOR M=2 to 8
20 FOR A=2 to 8
30 FOR Y=2 to 8
40 FOR L=2 to 8
50 IF M=A THEN 170
60 IF M=Y THEN 170
70 IF M=L THEN 170
80 IF A=Y THEN 170
90 IF A=L THEN 170
100 IF Y=L THEN 170
110 X=1000+10*L+L
120 IF X=900+10*A+Y+10*M+Y THEN 140
130 GOTO 170
140 IF X/4=INT(X/4) THEN 160
150 GOTO 170
160 PRINT "BILL IS 1   0";L;L"FOR M="M",A="A",Y="Y",L="L
170 NEXT L
180 NEXT Y
190 NEXT A
200 NEXT M
210 END

READY

RUN
BILL     08:07 AM        17-JAN-79
BILL IS 1  0  4   4  FOR M= 5 ,A=8 ,Y= 7 ,L= 4
BILL IS 1  0  4   4  FOR M= 6 ,A=8 ,Y= 2 ,L= 4
BILL IS 1  0  4   4  FOR M= 8 ,A=5 ,Y= 7 ,L= 4
BILL IS 1  0  4   4  FOR M= 8 ,A=6 ,Y= 2 ,L= 4

READY
```

Also solved by MARK EVANS, *LaMarque, Texas;* DOUGLAS JUNGREIS (Age 12), *Brooklyn, New York;* CHARLES H. LINCOLN, *Goldsboro, N.C.;* and the *Proposer.*

Incomplete solutions were received from BILL BURNS, *Seton Hall University, South Orange, N. J.;* LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas;* DAVID DEL SESTO, *North Providence, Rhode Island;* MICHAEL W. ECKER, *Scranton, Pennsylvania;* VICTOR FESER, *Mary College, Bismarck, N.D.;* SUSAN HOFFMAN, *Iona College, Larchmont, N.Y.;* SUSAN IWANSKI, *Greenlawn, New York,* PETER A. LINDSTROM, *Genesee Community College, Batavia, N. Y.;* ALBERT WHITE, *St. Bonadventure University, New York;* KENNETH M. WILKE, *Topeka, Kansas;* and the *Proposer.*

Lindstrom offered the soloution: $A = 0$, $B = 3$, $I = 7$, $L = 4$, $M = 1$, $P = 2$, $Y = 8$ for the multiplication version of this problem.

Editor's Note: The preponderance of incomplete solutions provides a rebuttal to the contention that cryptarithms are a trivial form of mathematical activity.

434. [Fall 1978] *Proposed by Sidney Penner, Bronx Community College of the City University of New York.*

Consider $(2n+1)^2$ hexagons arranged in a "diamond" pattern, the k-th column from the left and also from the right consisting of k hexagons, $1 \le k \le 2n+1$. Show that if exactly one of the six hexagons adjacent to the center hexagon is deleted then it is impossible to tile the remaining hexagons by trominoes as in Figure 2. (Figure 1 illustrated the 5^2 case in which each of the hexagons adjacent to the center one is labeled A.)

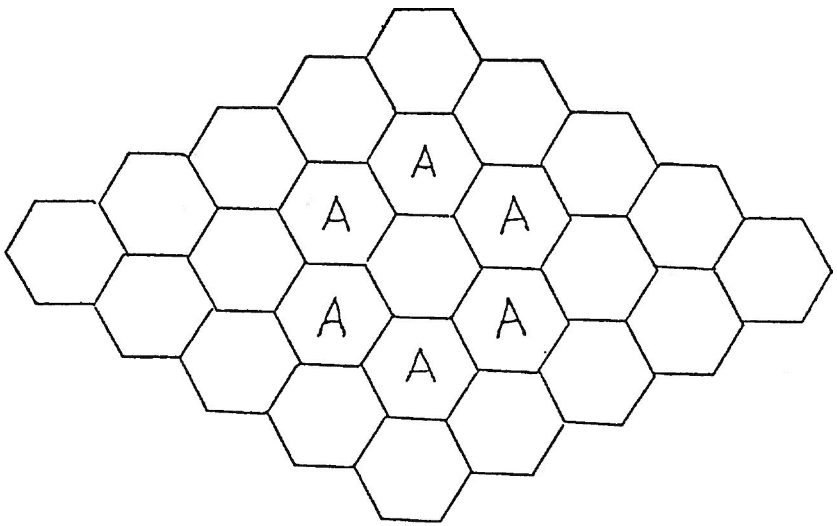


Figure 1

Figure 2

Solution by Clayton W. Dodge, University of Maine at Orono.

Color the center hexagon red (R) and then color each remaining hexagon red, green (G), or blue (B) so that no two adjacent hexagons have the same color. By the symmetry of the figure the number of B's is equal to the number of G's. That is, if the figure is reflected in its horizontal line of symmetry (through the center of the center hexagon), each G hexagon is reflected to a B one and vice versa. Also, all "A" hexagons are either B or G, so deleting one of them leaves the figure with different numbers of blue and green hexagons. Since each tromino must cover three hexagons of all three different colors, any figure exactly coverable by such trominoes must have the same number of hexagons of each color. Thus, deleting an "A" hexagon leaves the figure not coverable.

Also solved by VICTOR G. FESER, *Mary College, Bismarck, N.D.: and the. Proposer.*

435. [Fall 1978] *Proposed by David R. Simonds, Rensselaer Polytechnic Institute.*

Two non-congruent triangles are "almost congruent" if two sides and three angles of one triangle are congruent to two sides and three angles of the other triangle. Clearly two such triangles are similar. Show that the ratio of similarity k is such that $\phi^{-1} < k < \phi$, where $\phi = (1+\sqrt{5})/2$, the familiar Golden Ratio.

Editor's Note: This old problem is being reopened with the hope of eliciting fresh insights.

Solution by Clayton W. Dodge, University of Maine at Orono.

Let $a > b > c$ be the sides of the smaller triangle. Then

$$r = \frac{a}{b} = \frac{b}{c}.$$

From the triangle inequalities

$$a < b + c \quad \text{and} \quad a + b > c,$$

we obtain

$$\frac{a}{b} < \frac{b}{b} + \frac{c}{b} \quad \text{and} \quad \frac{a}{b} + \frac{b}{b} > \frac{c}{b},$$

$$r < 1 + \frac{1}{r} \quad \text{and} \quad r + 1 > \frac{1}{r},$$

and finally

$$r^2 - r - 1 < 0 \quad \text{and} \quad r^2 + r - 1 > 0.$$

Replace the inequality signs by equality signs and solve the two quadratic equations for their positive roots since r must be positive. We obtain

$$\frac{-1+\sqrt{5}}{2} = \frac{2}{1+\sqrt{5}} \quad \text{and} \quad \frac{1+\sqrt{5}}{2}.$$

The two inequalities are satisfied, for positive r, when

$$\phi^{-1} = \frac{2}{1+\sqrt{5}} < r < \frac{1+\sqrt{5}}{2} = \phi.$$

Also solved by WALTER BLUMBERG, *New Hyde Park, N. Y.*; LOUIS H. CAIROLI, *Kansas State University, Manhattan, Kansas*; MICHAEL W. ECKER, *Pennsylvania State University*; KENNETH M. WILKE, *Topeka, Kansas*; *and the Proposer.*

Editor's Note: This problem has been treated several times in the Mathematics Teacher (April 1953, pp. 295-296; December 1954, pp. 561-562; March 1977, pp. 253-257). A related problem, proposed by Victor Thébault in the May 1954 issue of the American Mathematical Monthly (Problem E 1117), considered the construction of a right triangle in which the legs and the altitude on the hypotenuse can be taken as the sides of another right triangle.

436. [Fall 1978] *Proposed by Carl Spangler and Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

P_1 and P_2 are distinct points on lines L_1 and L_2 respectively. Let L_1 and L_2 rotate about P_1 and P_2 respectively with equal angular velocities. Describe the locus of their intersection.

Solution by Clayton W. Dodge, University of Maine at Orono.

Let X be the point of intersection of the two lines and let Q be any point so that P_2 is between P_1 and Q. Denote angles QP_1X and QP_2X by α and β and angle P_1XP_2 by γ. If k is the angular velocity, then there are constants α_1 and β_1 such that

$$\alpha = kt + \alpha_1 \quad \text{and} \quad \beta = kt + \beta_1.$$

Then we have

$$\gamma = \beta - \alpha = \beta_1 - \alpha_1,$$

a constant. Hence X describes a circle through P_1 and P_2 such that arc P_1P_2 intercepts an inscribed angle of $\beta_1 - \alpha_1$. The center of the circle is easily constructed by drawing the line L through P_2 having $\beta = \beta_1 - \alpha_1$. The center is then at the intersection of the perpendicular to L at P_2 and the perpendicular bisector of P_1P_2.

Also solved by CHUCK ALLISON, *San Pedro, California*; CHARLES H. LINCOLN, *Goldsboro, N. C.*; R. S. LUTHAR, *University of Wisconsin-Janesville*; *and the Proposers.*

437. [Fall 1978] *Proposed by Zelda Katz, Beverly Hills, California.*

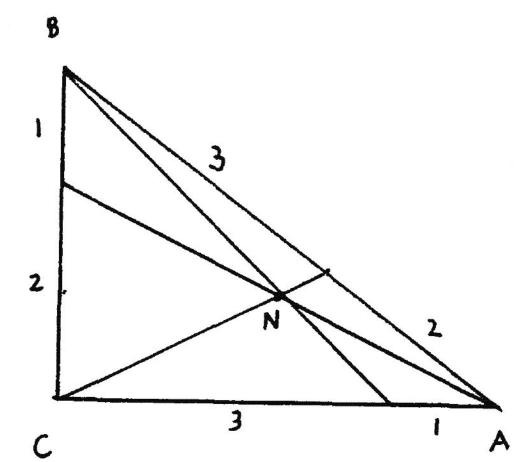
In times gone by, it was fairly well-known that N, the Nagel point of a triangle, is the intersection of the lines from the vertices to the points of contact of the opposite escribed circles. In the triangle whose sides are $AB = 5$, $BC = 3$, and $CA = 4$, show that the areas of triangles ABN, CAN, and BCN are 1, 2 and 3 respectively.

Solution by the Proposer.

Let D, E, F denote the traces of the Cevians through N, the Nagel point, on the sides BC, CA, AB of the 3:4:5 right triangle ABC. Then

$$\Delta CBN/\Delta BNA = CE/EA = (s-c)/(s-a) = 3/1.$$
$$\Delta CNA/\Delta BNA = CD/DB = (s-b)/(s-c) = 2/1.$$
$$\Delta CNA/\Delta BCN = AF/FB = (s-b)/(s-c) = 2/3.$$

The required ratios follow easily.



Also solved by WALTER BLUMBERG, *New Hyde Park*, L.I., N.Y.; LOUIS H. CAIROLI, *Kansas* State *University;* CLAYTON W. DODGE, *University of Maine at Orono;* CHARLES H. LINCOLN, *Goldsboro*, N.C.; SISTER STEPHANIE SLOYAN, *Georgian Court College*, Lakewood, N.J.; and J. WALKER, *General Hospital, Los Angeles, California*.

405. [Fall 1977, Corrected]. *Proposed* by *Norman Schaumberger, Bronx* Community *College, Bronx*, N. Y.

Locate a point P in the interior of a triangle such that the product of the three distances from p to the sides of the triangle is a maximum.

Amalgam of solutions by *Walter Blumberg* and *the Proposer*.

Let x, y, z be the distances from an interior point of the triangle to sides a, b, c respectively. Using the arithmetic mean - geometric mean inequality, we have

$\sqrt[3]{(ax)(by)(cz)} \le (ax + by + cz)/3 = 2K/3$, where K is the area of the triangle. Then xyz 5 $8K^3/27abc$, with equality if and only if $ax = by =$ as. It is known that $ax = by = cz$ only if P is the centroid of triangle ABC. Consequently this is the desired point.

Also solved by CLAYTON W. DODGE, *University* of. Marine *at Orono;* M. S. KLAMKIN, *University* of, *Alberta*, Canada; CHARLES H. LINCOLN, *Goldsboro*, N. C.; SISTER STEPHANIE SLOYAN, *Georgian Court College*, Lakewood, Nw *Jersey;* and ROD WOODBURY, *San Pedro, California*.

Two incorrect solutions were received.

Editor's Note: This triangle inequality was proposed by L. Carlitz in the January 1964 issue of the Mathematics Magazine, with a solution by J. A. Tyrrell published in September 1964 (page 279). This reference was located in Geometric Inequalities, by Bottema et al., item 12.29, page 112.

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