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David Ballew
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The papers for the 1978-79 National Paper Contest have been judged. and the first two papers in this issue of the Journal were the first and second prizes ($200 and $100 respectively). The Journal encourages undergraduates and beginning graduate students to submit their work for possible publication. We are your fraternity and wish to use your papers when possible. Entries in the 1979-80 contest are now being accepted and the winners will be announced in the Fall 1980 issue.

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Cordially,

Richard V. Andree
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RVA: at

THE PROBABILITY OF GENERATING A CYCLIC GROUP

by Deborah L. Massari
University of Akron

Often, interesting results in one area of mathematics may be viewed in the light of other mathematical interests. Such is the case between group theory and probability. In a recent paper in The Mathematical Gazette, MacHale [3] considered the question of determining the probability, Pr(G), that a pair of elements of a finite non-commutative group G commute with each other. We will consider a related question; namely, determining the probability that a randomly chosen element of a finite cyclic group is a generator of the group.

Let G denote the probability that an arbitrary element chosen at random is a generator of a finite cyclic group G. We will let p and q denote distinct primes. Positive integers will be denoted by n, m, e, and f. Also n(G) = |p| is a prime divisor of |G|. C denotes a cyclic group of order n.

We shall now consider the following situations. The first case will be |G| = p. Secondly, |G| = p^n (n > 1). The third case will be |G| = p^m q. |G| = \prod_{i=1}^{k} p_i^e_i where p_i is prime, e_i is a positive integer for each i. Our objective is to prove that P(G) is independent of the order of G and that P(G) depends only on n(G).

A major result in the study of cyclic groups is if G = \langle a \rangle, |G| = n, then G = \langle a^k \rangle if and only if (n, k) = 1. A corollary to this theorem is that the number of generators of a cyclic group G, |G| = n, is \phi(n) where \phi is the Euler-phi function. \[\phi(n) = n(1 - \frac{1}{p}).\] Thus, since \phi(n) is the number of generators out of n elements, the probability that any one randomly chosen element of G is a generator is \phi(n)/n.

Let us now examine |G| = p. Then, P(G) = \phi(p)/p. Since, \phi(p)/p = (p-1)/p = 1 - \frac{1}{p}, P(G) = 1 - \frac{1}{p} = \frac{p-1}{p}. As an example, consider C_3 under addition with Cayley table:
Both 1 and 2 are generators of $C_3$. So, of three elements, two generate the group. Thus, $P(G) = 2/3$. Likewise, according to the result just established, $P(C_2) = \phi(3)/3 = 1 - 1/3 = 2/3$.

If $|G| = p^n$, $P(G) = \phi(p^n)/p^n = p^n(1 - 1/p)/p^n = 1 - 1/p$, [2, p. 29]. Hence, if $|G| = p^n$ and $|H| = p$, then $P(G) = P(H)$. Consider $C_8 = \{0, 1, 2, \ldots, 7\}$ under addition. The generators are 1, 3, 5, 7. Thus, $P(C_8) = 4/8 = 1/2$. According to the formula established $|C_8| = 2^3$,

$$P(C_8) = 1 - 1/2 = 1/2.$$ Thirdly, let $|G| = p^n q^m$.

$$P(G) = \phi(p^n q^m) / p^n q^m = \phi(p^n) / p^n = (1 - 1/p),$$

$[2, p. 28]$. As an example, consider $C_6$ under addition.

$$|C_6| = 2 \cdot 3, P(C_6) = (1 - 1/2)(1 - 1/3) = 1/3.$$ Another example is $C_{36}$ under addition.

$$C_{36} = 2^2 \cdot 3^2, P(C_{36}) = (1 - 1/2)(1 - 1/3) = 1/3.$$ Notice, $P(C_6) = P(C_{36})$.

The last case to be considered is $|G| = \prod_{i=1}^{k} p_i^{e_i}$. According to the general formula,
In a similar manner, \( P(H) = \)
\[
\frac{1}{k} \prod_{i=1}^{k} P_i^x_i = \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).
\]
Hence, \( P(G) = P(H) \)
So, if for example \( |G| = 1890 \) and \( |H| = 6300 \), then \( P(G) = P(H) \).
This follows since \( |G| = 1890 = 2 \cdot 3^2 \cdot 5 \cdot 7 \) and \( |H| = 6300 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \).
According to our result, \( P(G) \) is independent of \( |G| \) and \( P(G) \) depends only on \( \pi(G) \).

REFERENCES

This paper was awarded First Prize in the National Pi Mu Epsilon Paper Contest. The author was an undergraduate when the paper was submitted.

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NON-LINEAR ADDITIVE FUNCTIONS

by Julie D. Anderson
Hendrix College

Introduction

What functions \( f: \mathbb{R} \to \mathbb{R} \) can be found satisfying \( f(x+y) = f(x) + f(y) \) (\( \xi \)) for every \( x, y \in \mathbb{R} \)? Clearly the function \( f(x) = ax \) is one, but is every solution linear? The research generated by this question has led to some interesting results. [6]

A function which satisfies the condition (\( \xi \)) will be called additive. One for which \( f(cx) = cf(x) \) for every \( c \in \mathbb{R} \) will be called homogeneous. If a function is both additive and homogeneous, it is linear. [6]

History

A survey of the research concerning additive functions is given by Green and Gustin [4, pp. 503–505] beginning with 1821 when Cauchy demonstrated that any additive function is rationally homogeneous, or \( f(rx) = rf(x) \) for every rational number \( r \) [5]. Furthermore, he found that the only continuous additive functions are real homogeneous and thus linear, and any discontinuous additive function is continuous at no point. Therefore a non-linear additive function, if such exists, will be totally discontinuous. Further restrictions were placed on a non-linear additive function by Darboux who showed in 1875 that an additive function bounded above or below on some interval is continuous, hence linear [3].

The question of the existence of non-linear additive functions was resolved in 1905 by Hamel, who used the rational homogeneity of an additive function and a set which he constructed (now referred to as a Hamel basis) using Zermelo’s well-ordering theorem. Hamel also noted that the graph of such a function is dense in the plane [6].

In summary, the following theorem demonstrates the slight conditions which cause an additive function to be linear [9, Th. 5, p. 17]:

Theorem. If \( f: \mathbb{R} \to \mathbb{R} \) is additive then the following are equivalent:

1) \( f \) is linear.
2) $f$ is continuous at some point.

3) There exists an interval on which $f$ is bounded.

4) There exists an interval on which $f$ is monotonic.

5) The graph of $f$ is not dense in $\mathbb{R}^2$.

Construction. One begins to see that the construction of a non-linear additive function is no trivial task. At the heart of it is the concept of a Hamel basis for the real numbers, the existence of which will be demonstrated using ideas concerning vector spaces.

Let $X$ be a vector space over the field $F$ and $S$ be a subset of $X$. The subspace generated by $S$, denoted $\langle S \rangle$, is defined to be the smallest subspace of $X$ containing $S$. It can be shown that

$$\langle S \rangle = \left\{ \sum_{i=1}^{n} a_i x_i : x_i \in S, a_i \in F \right\}.$$ 

If $\langle S \rangle$ is generated by no proper subset of $S$, we say $S$ is linearly independent. A Hamel basis in $X$ is a linearly independent subset of $X$ which generates $X$. If $S$ is a Hamel basis in $X$ it can be established that every $x$ in $X$ is a unique (finite) linear combination of members of $S$. These proofs are fairly standard fare in any linear algebra text.

The next step is to demonstrate that every vector space has a Hamel basis. One lemma is necessary.

Zorn's Lemma. If $S$ is a nonempty partially ordered set such that every totally ordered subset has an upper bound in $S$, then $S$ has a maximal element $S_0$. The statement that $S_0$ is a maximal element of $S$ means that if $S \in S$ and $S \subseteq S_0$ then $S = S_0$. Zorn's lemma is equivalent to the axiom of choice and Zermelo's well-ordering principle (see [5] or [6]).

Theorem. Let $X$ be a vector space over the field $F$. Then $X$ has a Hamel basis. [6]

Proof. Let $S$ be the totality of linearly independent subsets of $X$ (partially ordered by inclusion) and $T$ be a totally ordered subset of $S$.

Claim: $T$ has an upper bound in $S$.

Consider $U = \bigcup_{T \in T} T$. Clearly $U$ is an upper bound of $T$, but is $U$ in $S$?

Suppose $U$ is not in $S$. Then $U$ is not linearly independent, or $\langle U \rangle$ is generated by a proper subset of $U$. Therefore there exists $r \in U$ such that

$$z = \sum_{i=1}^{n} a_i r_i$$

where $a_i \in F, r_i \in U, r_i \neq 0$. $T$ is totally ordered by inclusion, so there exists $R \in T$ such that $r \in R$ and $r \in R_i$ for $i = 1, 2, \ldots, n$. Then $R$ is not linearly independent, which is a contradiction. Therefore $U$ is in $S$.

Thus by Zorn's lemma, $S$ has a maximal element $S_0$. Suppose $x \in X$ such that $x \neq \langle S_0 \rangle$. Then $S_0 \cup \{x\}$ which in turn belongs to $S$. But since $S_0$ is a maximal element of $S$, this implies that $S_0 = S_0 \cup \{x\}$, which is clearly false. Thus if $x \in X$, then $x \in \langle S_0 \rangle$, and we have that $S_0$ is a Hamel basis in $X$.

Consider the vector space of the real numbers over the field of rational numbers, denoted $\mathbb{R}_Q$, and let $B$ be a Hamel basis for $\mathbb{R}_Q$. If a function $f: \mathbb{R} \to \mathbb{R}$ is additive and $x = \sum r_i b_i$ where $r_i \in \mathbb{Q}$ and $b_i \in B$, then

$$f(x) = f(\sum r_i b_i) = \sum r_i f(b_i).$$

Thus the function is completely determined by the way it is defined on the Hamel basis.

To construct a non-linear additive function, we begin by defining $f: \mathbb{B} \to \mathbb{R}$ so that $f$ is non-linear. Then the function is extended to $F: \mathbb{R} \to \mathbb{R}$ in such a way as to make $F$ additive. To insure non-linearity, consider that for a linear function, $f(x) = ax$, the ratio $f(x)/x$ is the constant $a$ for all $x \neq 0$. Therefore define $f: \mathbb{B} \to \mathbb{R}$ so there exist $b_1, b_2 \in B$ such that $f(b_1)/b_1 \neq f(b_2)/b_2$. Since every $x \in \mathbb{R}$ can be expressed as

$$x = \sum_{i=1}^{n} r_i b_i$$

where $r_i \in \mathbb{Q}$ and $b_i \in B$, extend $f$ to $F: \mathbb{R} \to \mathbb{R}$ by defining

$$F(x) = \sum_{i=1}^{n} r_i f(b_i).$$

The verification that $F$ is additive is fairly straightforward.

Example. In spite of the pathology of these functions, there are examples with some nice properties.

Example. Consider the function $f: \mathbb{B} \to \mathbb{R}$ defined by

$$f(x) = \sum_{0 \neq b}^1 x = b \neq b.$$
where \( b \) is fixed in \( \mathbb{B} \). Extend this to \( F : \mathbb{R} \rightarrow \mathbb{R} \) as described above. Then 
\[ F(b) = 1, \text{ and the graph of } F \text{ restricted to rational multiples of } b \text{ is}
\[ \text{dense in the line } \{(x, y): y = \frac{1}{b} x\}. \]
Where \( \sigma \in \mathbb{B} \) and \( \sigma \neq b \), \( F(\sigma) = 0 \) and the graph of \( F \) restricted to rational multiples of \( a \) is dense in the horizontal axis. For \( \nu \in \mathbb{Q} \), consider 
\[ F(b + \nu a) = f(b) + \nu f(a) = 1. \]
The graph of \( F \) restricted to rational multiples of \( b + \nu a \) is dense in the line 
\[ \{(x, y): y = \frac{1}{b + \nu a} x\}. \]
Since \( \{b + \nu a: \nu \in \mathbb{Q}\} \) is dense in \( \mathbb{R} \), \( \nu \mathbb{Q} \) is dense in the plane. Consequently, the graph of \( F \) is dense in the plane. See Figure 1. Notice that this function takes on only rational values.

\[ \text{FIGURE 1} \]

**Example 2.** Another example is described by Wilansky In [8, Ex. 6, p. 116]. Again, \( b \) is fixed in \( \mathbb{B} \) and \( f : \mathbb{B} \rightarrow \mathbb{R} \) and \( g : \mathbb{B} \rightarrow \mathbb{R} \) are defined by

\[ \begin{align*}
  f(x) &= \begin{cases} 
  b & x = b \\
  0 & x \neq b
  \end{cases} \\
  g(x) &= \begin{cases} 
  b & x = b \\
  0 & x \neq b
  \end{cases}
\end{align*} \]

Extend \( f \) and \( g \) to \( F \) and \( G \) respectively. Let \( \nu = \sum_{i=1}^{n} r_i b_1 + r_i b_2 + \cdots + r_i b_n \) where 
\[ b \in \{b_1, b_2, \ldots, b_n\}. \]

Then
\[ F(\nu x) = F \left[ \sum_{i=1}^{n} r_i b_i x + r_i b_i \right] \]
\[ = F \left[ \sum_{i=1}^{n} r_i b_i x + r_i b_i \right] \]
\[ = \sum_{i=1}^{n} r_i b_i f(b_i) + r_i b_i g(b_i) \]
\[ = \sum_{i=1}^{n} r_i b_i (b_i x + r_i b) \]
\[ = \nu x. \]

Similarly, \( G(F(x)) = x \). The existence of an inverse of \( F \) implies that \( F \) is one-to-one and onto \( \mathbb{R} \) -- a totally discontinuous function, dense in the plane, yet mapping to every real number exactly once.

**Remark.** It is of interest to note that the graphs of the functions in Examples 1 and 2 are disconnected. In Example 1, the function takes on only rational values, so a horizontal line through an irrational point on the vertical axis separates the graph. As for Example 2, according to a result of F. B. Jones, the connected graph of a discontinuous additive function must intersect every continuum in the plane not lying wholly in a vertical line [7, Th. 2, p. 116, 117]. Since \( F(x) \) in Example 2 is one-to-one and onto \( \mathbb{R} \), every horizontal line contains only one point of the graph. Choose a segment of a horizontal line such that the segment does not contain a point of the graph. Then this continuum, the segment, is not intersected by the graph. Thus the graph is not connected. Jones demonstrates that there does exist a discontinuous additive function whose graph is connected.

**Example 3.** Consider the usefulness of these functions in studying another functional equation, \( h(x + y) = h(x)h(y) \). The obvious solution is
exponential. \( h(x) = e^{ax} \), but using non-linear additive functions instead of \( ax \) yields discontinuous solutions. For example, consider \( h(x) = e^{F(x)} \) where \( F(x) \) is as defined in Example 1. The graph of \( h \) exhibits much of the pathology of the graph of \( F(x) \). Consider the fixed \( b \):

\[ h(b) = e^{F(b)} = e^b, \]

and the graph of \( h \) restricted to rational multiples of \( b \) is dense in the exponential curve \( \{(x,y) : y = \exp(b)\} \). Where \( a \in \mathbb{B} \) and of \( b \), \( h(c) = e^{a+b} \), so the graph of \( h \) restricted to rational multiples of \( c \) is dense in the line \( \{(x,y) : y = 1\} \). \( h(b+c) = e^c \), and the graph of \( h \) restricted to rational multiples of \( b+c \) is dense in the curve \( \{(x,y) : y = \exp\left(\frac{b}{b+c}\right)\} \). Evidently, the graph of \( h \) is the homeomorphic image of the graph of \( F \) under the mapping \((x,y) \mapsto (x, e^y)\).

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9. VanDenHeuvel, J., On Additive Functions Being Linear, Hendrix College, (1978). (This undergraduate research paper by a Hendrix colleague served as the starting point for my work.)

This paper was written under the direction of Professor Robert Eslinger, Hendrix College, Conway, Arkansas.

This paper was awarded second prize in the 1978-79 National Pi Mu Epsilon paper competition. The author was an undergraduate at the time the paper was submitted.

THE DOUBLE FERRIS WHEEL PROBLEM

by Al Parish
College of Charleston

In certain real-world situations, mathematical models may prove to be quite accurate and at the same time have reasonable assumptions. Granted, this case is most unusual, but we will be interested in just such a situation. The problem may be stated quite simply in the following manner: describe the equation of motion of the double Ferris wheel with arbitrary physical dimensions.

For those of you who have never had the dubious pleasure of riding the device, or for those of you who have and have attempted to forget it, a basic description is in order. A double Ferris wheel consists of a large supporting beam with a rotating beam attached at some convenient point on the supporting one. At each end of the rotating beam is a wheel with chairs attached to it for riders to sit in. Since a picture is worth a thousand words, the following one may help:
We shall refer to the rotating and supporting beams together as the central structure. As pictured in Figure 1, we shall define the following variables:

- $R_1$: radius of the central structure
- $R_2$: radius of a single wheel
- $\omega_1$: angular velocity of the rotating beam
- $\omega_2$: angular velocity of the rider about the center of the wheel
- $\theta$: angle of the rotating beam at time $t$ with the horizontal
- $U$: angle of the rider to the horizontal

The following assumptions must be made for this model:

i) There is no force of friction between the chair and the steel axis on which the chair sits.

ii) Angular velocity of both the wheel and rotating beam is uniform. This assumption eliminates initial and final "jerk" and acceleration which cause the rocking motion of the chair at start and stop of the ride.

iii) No external forces such as wind interfere with the motion.

First we consider the motion of the rotating beam about the center of the central structure. Clearly the endpoint of the beam (also the center of a wheel) rotates in a circle, which we chose to be centered at the origin, of radius $R_1$ and therefore satisfies:

$$x^2 + y^2 = R_1^2,$$

or in polar coordinates

$$x = R_1 \cos \theta, \quad y = R_1 \sin \theta,$$

where the angle $\theta$ is defined above.

Now consider the motion of the rider on the wheel. The equation of motion here is similar to that in the previous derivation. The rider rotates in a circle, centered about the endpoint of the rotating beam of radius $R_2$ and so satisfies:

$$x^2 + y^2 = R_2^2,$$

or in polar coordinates,

$$x = R_2 \cos \theta, \quad y = R_2 \sin \theta.$$

A standard translation of axes yields the following two equations:

$$x = R_1 \cos \theta + R_2 \cos \theta, \quad y = R_1 \sin \theta + R_2 \sin \theta.$$

These two parametric equations describe the equation of motion of the rider given the two angles $\theta$ and $U$. But these two angles may be difficult to find at some moment in time, so a stopwatch may be used to measure the two variables $\omega_1$ and $\omega_2$. For pure convenience, we shall assume that the rider is at point $A$ in Figure 2 at time $t = 0$. Now the angles $\theta$ and $U$ may be expressed in terms of the following simple differential equations:

$$\frac{d\theta(t)}{dt} = \omega_1, \quad \frac{dU(t)}{dt} = \omega_2.$$

Integrating with respect to time, we obtain the following solutions:

$$\theta(t) = \omega_1 t + C, \quad U(t) = \omega_2 t + E.$$

But we assumed that $\theta = U = 0$ at time $t = 0$, so that $C = E = 0$. Recall that $\omega_1$ and $\omega_2$ are angular velocities and are hence measured in revolutions per time period. Radian measure is far more desirable so we may write:

$$\omega_1 = 2\pi F_1 \quad \text{and} \quad \omega_2 = 2\pi F_2.$$

The two previous parametric equations of (5) become the following two equations:

$$x = R_1 \cos (2\pi F_1 t) + R_2 \cos (2\pi F_2 t), \quad y = R_1 \sin (2\pi F_1 t) + R_2 \sin (2\pi F_2 t).$$

These last two equations describe the motion of the rider in terms of the easily measured parameter time. We must now concern ourselves with the physical interpretation of these equations. The graph for this situation is shown in Figure 3. It could be described as a three-leaved rose with a hyperbolic triangular center.
Return to Figure 1 for a moment. The angles \( \theta \) and \( U \) are measured in the same direction, implying the beam and rider rotate in the same direction. Obviously this need not be the case. So let us replace \( U \) by \(-U\), or equivalently, applying (7) and (8), replace \( V_2 \) by \(-V_2\) and \( F_2 \) by \(-F_2\). Since the parametric equation of \( x \) in (9) involved only the even cosine function, it remains the same. However, the equation involving \( y \) contains the odd sine function so that this equation becomes:

\[
(10) \quad y = R_1 \sin (2nF_1 t) - R_2 \sin (2nF_2 t). 
\]

This apparently minor change creates an entirely new motion as depicted in Figure 4.

Let us now consider the displacement of the rider from the origin at time \( t \). Squaring the parametric equations and adding yields the following rather complicated expression:

\[
(11) \quad x^2 + y^2 = R_1^2 \cos^2 (2nF_1 t) + R_2^2 \cos^2 (2nF_2 t) + 2R_1 R_2 \cos (2nF_1 t) \\
\cos (2nF_2 t) + R_1^2 \sin^2 (2nF_1 t) + R_2^2 \sin^2 (2nF_2 t) \\
+ 2R_1 R_2 \sin (2nF_1 t) \sin (2nF_2 t).
\]

The square root of this expression is the equation for displacement we desire. Fortunately, we may simplify this expression by using the two familiar trigonometric identities:

\[
(12) \quad \cos^2 A + \sin^2 A = 1 \quad \text{and} \quad \cos (A-B) = \cos A \cos B + \sin A \sin B.
\]

Applying these to (11) yields the following:

\[
(13) \quad D = x^2 + y^2 = R_1^2 + R_2^2 + 2R_1 R_2 \cos 2n(F_1 - F_2) t.
\]

The square root of this expression is the displacement we desire. When the angles \( \theta \) and \( U \) are in opposite directions, a similar analysis yields the same displacement as above except that the final term involving cosine has \( F_1 + F_2 \).

In conclusion, if any reader still wishes to ride one of these machines, all that the writer can say is "HAVE FUN!"; as for the author, he has never been on one and now that he knows what the motion is, he will never ride one in the future!

Editor's Note: It might be of interest to consider accelerations at various points on the curves. The Journal would be interested in hearing from its readers on this subject.

The author was an undergraduate when the paper was submitted.
A common topic of recreational mathematics deals with manipulating the digits of an integer by means of some function, and then studying the results. One of the oldest, best-known, and most striking examples of this: start with the integer 153; cube its digits and add them; the result is 153 again. In this way, 153 can be called a function of its digits; such an integer is known in the literature as a narcissistic number [2], [3].

Of course, a wide variety of "functions" can be used here. Also, since the use of base 10 is a mathematical accident, one really ought to consider other bases too—this is not always done in the literature. The most fertile extension of the idea consists of generating sequences of integers; beginning with an integer $N$ in base $B$, and a function $f$, one sets $N_0$ and defines recursively:

$$N_{i+1} = f(N_i), \quad i = 0, 1, 2, \ldots$$

Thus, using the function mentioned above, and working in base 10, one may begin at random, say with $N = 76$. The resulting sequence is 76, 559, 979, 1801, 514, 190, 730, 370, 370;--. Obviously the sequence has become repeating; and this would occur at any time a duplication occurs anywhere in the sequence.

The question now is: under what conditions for $N$, $B$, and $f$ does the sequence repeat? This paper will show that the answer is—under most conditions!

Some preparatory remarks: a repeating block will be called a loop; if the loop is of length one, it will be called a self-loop. (A self-loop is a narcissistic number.) Examples will be given in various bases, but the treatment is general for base $B$. In base $B$, the integer $N$ is written:
\( a_0B^m + a_1B^{m-1} + \cdots + a_mB + a_{m+1} \), where the digits are the \( a_i \) with \( a_0 \neq 0 \), and \( N \) has \( m+1 \) digits. \( N \) will always be non-negative.

The first section of the paper presents the general method in detail with a number of remarks. The remaining sections can then be made quite brief, since the same method is used in each with obvious modifications.

**POWER-SUMS**

Given \( N \) in base \( B \), let \( k \) be a positive integer. Then the \( k \)-power-sum of \( N \), written \( P(N, B, k) \) or simply \( P(N) \); is defined:

\[
P(N) = a_k^k + a_{k-1}^k + \cdots + a_m^k + a_{m+1}^k \quad [1].
\]

The example at the beginning of this paper is a 3-power-sum. Other examples, with \( B = 7 \) and \( k = 3 \), are:

loop 466, 1096, 466, \( \cdots \), self-loop 505. Self-loops in base 10 are also called digital invariants [5], [6], [7].

**Theorem 1.** For any \( B \geq 2 \), for any \( k \geq 1 \), for any \( N \geq 0 \) the sequence defined by \( P(N) \) has a loop.

**Proof.** If \( N \) has \( m+1 \) digits, then \( N \geq B^m \). From the definition, \( P(N) = (B-1)^k(m+1) \), since the digit a \( 2 \) \( B-1 \) for each \( i \). Then \( P(N)/N = (B-1)^k(m+1)/B^m \); let the latter be \( f(m) \). We now consider \( f(x) \); where \( x \) is any real number in \( (0,1) \); after analyzing \( f(x) \), we will return to integer values of \( x \). (cf. [4], ch. 4).

Clearly \( f(x) = (B-1)^k(x+1)/B^x \rightarrow 0 \) as \( x \to 0; \) i.e., for any \( \epsilon > 0 \), there exists \( x_0 \) such that \( x > x_0 \) implies \( f(x) < \epsilon \). Choosing \( \epsilon = 1 \) and returning to integer values, we have for some integer \( Z_1 \), \( N > Z \) implies \( f(m) = 1 \), or \( P(N) < N \). So, if in the sequence \( N_1 > Z, \) then \( N_2 < N_1 \) and succeeding terms of the sequence decrease until for some \( j \) (possibly equal to \( i+1 \)) \( N_j = Z \). After this point we have no direct information on the next terms; but whenever a term appears that is greater than \( Z \), the same decreasing behavior occurs.

Therefore in the infinite sequence \( \{N_i\} \) we have infinitely many terms less than \( Z \); but all these terms are positive integers, and so a repetition must occur. Therefore the sequence has a loop. Q.E.D.

**Corollary.** Every power-sum loop contains an integer less than \( Z \).

Here is an illustration: For \( B = 10 \) and \( k = 3 \), we have \( f(x) = 729(x+1)/10 \), which is less than 1 for \( x > 4 \). This yields \( Z = 10000 \). In fact, this is far larger than necessary; the optimum value of \( Z \), it can be shown, is 2189. Note that \( P(9999) = 2189 \). Note also that the theorem does not say \( N > Z \) implies \( P(N) < Z \) (since, e.g., \( P(9999) = 2916 > Z \) but rather \( N > Z \) implies \( P(N) < Z \).

**Remarks:**

1) If \( k = 1 \), then the only loops are the trivial self-loops consisting of the one-digit integers in that base. In any base \( B \), there are the trivial self-loops 0 and 1.

2) This theorem, and the following ones, unfortunately do not guarantee the existence of loops other than trivial ones.

**SUM-POWERS**

Given \( N \) in base \( B \) and let \( k \) be a positive integer. Then the sum-\( k \)-power of \( N \), written \( S(N, B, k) \) or simply \( S(N) \), is defined:

\[
S(N) = (a_0 + a_1 + \cdots + a_m + a_{m+1})^k \quad [2].
\]

Examples in base 6, with \( k = 3 \), are: loop 13, 2212, 131, \( \cdots \), self-loop 3213. The search for sum-power self-loops is fairly easy to do empirically. since a sum-\( k \)-power is precisely a \( k \)—power, all one needs to do is look through a table of powers for the base in question.

For any base \( B \) and any \( k \), there are, of course, the trivial self-loops 0 and 1.

**Theorem 2.** For any \( B \geq 2 \), for any \( k \geq 1 \), for any \( N \geq 0 \) the sequence defined by \( S(N) \) has a loop.
Proof. We follow exactly the pattern of the proof of Theorem 1. We have \( S(N)/N \leq ((B-1)(m+1))/B^m \), and this is \( f(m) \). Then clearly \( f(x) \to 0 \) as \( x \to \infty \) (again we have a polynomial in \( x \) over an exponential in \( x \)), and the rest follows as before.

SELF-POWERS

Given \( N \) in base \( B \), the self-power of \( N \), written \( L(N,B) \) or simply \( L(N) \), is defined:

\[
L(N) = a_0 \cdot a_1^m + \ldots + a_{m-1} a_m^{m+1} [2], [3].
\]

This definition requires us to first define a value for \( a_0 \). The obvious choices are 0 and 1. Either one may be used, and in fact both choices are usually considered in discussing this function [3].

Examples in base 5 are as follows: defining \( 0^0 = 0 \), loop: 11, 2, 2011, 11, \( \cdots \); self-loops: 103 and 2024. Defining \( 0^0 = 1 \): loop 104, 2013, 113, 104, \( \cdots \); but no nontrivial self-loops exist.

Under either definition of \( 0^0 \) we have

**Theorem 2.** For any \( B \geq 2 \), for any \( N \geq 20 \): the sequence defined by \( L(N) \) has a loop.

Proof: We have \( L(N)/N \leq (m+1)(B-1)/B^m \to 0 \), etc.

FACTORIAL-SUMS

We now introduce a new type of function. Given \( N \) in base \( B \), the factorial-sum of \( N \), written \( F(N,B) \) or simply \( F(N) \), is defined:

\[
F(N) = a! + a_1! + \ldots + a_{m-1} a_m^{m+1}! [2], [3].
\]

In base 6, two examples of self-loops are 41 and 42. In any base, there is the trivial self-loop 1. If \( B > 2 \), there is also the trivial self-loop 2. (For this function, 0 is not a self-loop.)

**Theorem 3.** For any \( B \geq 2 \), for any \( N \geq 0 \): the sequence defined by \( F(N) \) has a loop.

Proof: We have \( F(N)/N \leq (m+1)(B-1)/B^m \to 0 \), etc.

**SUM-FACTORIALS**

Finally we present a function for which the general method does **not** work. Given \( N \) in base \( B \), the sum-factorial of \( N \), written \( G(N,B) \) or simply \( G(N) \), is defined:

\[
G(N) = (a_0 + a_1 + \ldots + a_m + a_{m+1}!)
\]

Trivial self-loops exist: 1 is such in any base \( B \); 2 is such in any base \( B > 2 \).

If we follow the pattern used so far, we set up the ratio \( G(N)/N \leq ((B-1)(m+1))/B^m \); but the latter ratio is divergent! This can be seen quickly as each term is greater than \( m!/B^m \); now this is the reciprocal of \( B^m/m! \), and the series resulting from such a sequence is convergent (it defines the number \( e \)). But then the terms \( B^m/m! \to 0 \) as \( m \to \infty \), and therefore the terms \( m!/B^m \to \infty \) as \( m \to \infty \).

Now, of course, this divergence does not settle the question of the ratio \( G(N)/N \): the terms here are term-wise less than those of a divergent sequence, and so it might still be convergent, after all. I have not been able to decide the question, but empirical evidence supports my closing conjectures:

1. The sequence \( G(N)/N \) is divergent.
2. Given \( N \geq 0 \) in base \( B \geq 2 \), the sequence defined by \( G(N) \) has no loops except the trivial ones.

**REFERENCES**

CALL FOR STUDENT PAPERS
April 11-12, 1980

The Arkansas Alpha chapter of Pi Mu Epsilon invites you to join us for a student mathematics conference. There will be sessions of the student conference on Friday afternoon and Saturday morning. Talks may be on any topic related to mathematics, statistics, or computing and may range from expository to research, to applications, problems, etc...

Presentation time should be fifteen or thirty minutes. Please send your title, presentation time, and a short abstract by March 15, 1980, to the address shown below.

For more details, write to:
Professor Debra Qualls
Department of Mathematics
University of Arkansas
Fayetteville, AR 72701

The Journal sadly announces that Professor Carl G. Townsend recently died in an automobile accident. Professor Townsend was the advisor of the Illinois Delta chapter.

LINEAR APPROXIMATIONS TO SQUARE ROOTS

by Stewart M. Venit
California State University, Los Angeles

Assume that we are given a positive number, \( a \), with \( n^2 < a \leq (n+1)^2 \), where \( n \) is a nonnegative integer. One simple means of approximating the positive square root of \( a \) is to evaluate at \( x = a \) the linear polynomial interpolating \( f(x) = \sqrt{x} \) using the points \((n^2, n)\) and \(((n+1)^2, n+1)\). A somewhat more sophisticated way is to form the linear Taylor polynomial for \( f(x) = \sqrt{x} \), expanded about either \( x = n^2 \) or \( x = (n+1)^2 \), and to evaluate it at \( x = a \). Although the latter approximation is normally used only "near" the center of expansion, we will show here that it is "usually" a better one than the former throughout the interval \((n^2, (n+1)^2)\).

First, we will more precisely define our approximations. The interpolating polynomial described above is simply the line through the points \((n^2, n)\) and \(((n+1)^2, n+1)\); and we can obtain its equation \( y = n + (x - n^2)/(2n + 1) \), by using the two-point form. Evaluating this function at \( x = a \), we obtain an approximation we shall call \( s(a) \) ("s" for "secant"):

\[(1) \quad s(a) = n + (a - n^2)/(2n + 1) \text{ if } n^2 < a \leq (n+1)^2.\]

(This equation also represents one iteration of the method of "false position," \( z_{k+2} = z_k - (z_k - z_{k+1})g(z_k)/(g(z_k) - g(z_{k+1})) \), for \( g(z) = x^2 - a \) with \( z_k = n \) and \( z_{k+1} = n + 1 \).)

The two linear Taylor polynomials, the one expanded about \( x = n^2 \) and the other about \( x = (n+1)^2 \), are given by, respectively,

\[y : n + (x - n^2)/(2n) \text{ and } y : (n + 1) + (x - (n+1)^2)/(2n + 2).\]

These lines intersect at the point \((n(n + 1), n + 1/2)\) (see Figure 1).

Thus, in the interval \((n^2, (n+1)^2)\), the error in this type of approximation will be minimized by choosing the first of these lines if \( a \) is such that \( n^2 < a \leq n(n+1) \) and choosing the second if \( n(n+1) \leq a \leq (n+1)^2 \). We thus arrive at the following procedure to find an approximation \( t(a) \) ("t" for "tangent") to \( \sqrt{a} \) based upon the
Taylor lines. When \( a > 1 \), take

\[
(2) \quad t(a) = n + (a - n^2)/(2n) \quad \text{if } n^2 < a \leq n(n + 1),
\]

\[
(3) \quad t(a) = (n + 1) + (a - (n + 1)^2)/(2n + 2) \quad \text{if } (n + 1) \leq a \leq (n + 1)^2.
\]

Since \( f'(0) \) does not exist, we use formula (3) for any \( a \) lying in the interval \( (0, 1) \). (Equations (2) and (3) also represent one iteration of Newton’s method, \( z_{k+1} = a - g(z_k)/g'(z_k) \) for the function

\( g(x) = x^2 - a \) \( \text{with } z_0 = n \) and \( a = n + 1 \), respectively.)

The errors in approximating \( \sqrt{a} \) by \( t(a) \) and \( s(a) \), \( |t(a) - \sqrt{a}| \) and \( |s(a) - \sqrt{a}| \), respectively, are obtained by subtracting \( \sqrt{a} \) from both sides of equations (1), (2), (3). If \( a > 1 \) and \( n^2 < a \leq n(n + 1) \),

\[
t(a) = \sqrt{a} = (n - \sqrt{a} + (a - n^2)/(2n) = (n - \sqrt{a})\left[1 - (\sqrt{a} + n)/(2n)\right] = (n - \sqrt{a})\left[(n - \sqrt{a})/(2n)\right].
\]

Thus,

\[
(4) \quad t(a) = \sqrt{a} = (n - \sqrt{a})^2/(2n) \quad \text{if } a > 1 \text{ and } n^2 < a \leq n(n + 1).
\]

Similarly,

\[
(5) \quad t(a) = \sqrt{a} = (n + 1 - \sqrt{a})^2/(2n + 2) \quad \text{if } 0 < a \leq 1 \text{ or } n(n + 1) \leq a \leq (n + 1)^2,
\]

\[
(6) \quad s(a) = \sqrt{a} = (n - \sqrt{a})(n + 1 - \sqrt{a})/(2n + 1) \text{ if } n^2 < a \leq (n + 1)^2.
\]

Now, \( t(a) \) is a better approximation than \( s(a) \) for those \( a \) for which \( |t(a) - \sqrt{a}| < |s(a) - \sqrt{a}| \). From equations (4), (5) and (6), or from Figure 1, we see that \( t(a) - \sqrt{a} \geq 0 \), and \( s(a) - \sqrt{a} \leq 0 \). Hence, \( t(a) \) is more accurate if \( t(a) - \sqrt{a} < s(a) - \sqrt{a} \); that is, if

\[
(t(a) - \sqrt{a}) + (s(a) - \sqrt{a}) < 0. \quad \text{Now, for } a > 1 \text{ and } n^2 < a \leq n(n + 1), \text{ equations (4) and (6) yield}
\]

\[
(t(a) - \sqrt{a}) + (s(a) - \sqrt{a}) = (n - \sqrt{a})\left[\frac{n - \sqrt{a}}{2n} + \frac{n + 1 - \sqrt{a}}{2n + 1}\right]
\]

\[
= (\sqrt{a} - n)\left[\frac{\sqrt{a}(4n + 1) - (4n^2 + 3n)}{2n(2n + 1)}\right].
\]

This expression will be negative, as desired, if \( \sqrt{a} < (4n^2 + 3n)/(4n + 1) \); that is, if \( a < (4n^2 + 3n)^2/(4n + 1)^2 \). Performing the division, we see that this will be true if \( a < n(n + 1) - n/(4n + 1)^2 \). So for \( a > 1 \) and \( n^2 < a \leq n(n + 1) \), \( t(a) \) yields a better approximation than \( s(a) \) for those \( a \) which satisfy

\[
(7) \quad a < n(n + 1) - n/(4n + 1)^2.
\]

For \( 0 < a < s1 \) or \( n(n + 1) \leq a \leq (n + 1)^2 \), a similar computation using (5) and (6) gives a negative value for \( (t(a) - \sqrt{a}) + (s(a) - \sqrt{a}) \) when

\[
(8) \quad a > n(n + 1) + (n + 1)/(4n + 3)^2.
\]

Combining the results of (7) and (8), we see that the Taylor lines \( t(a) \) will produce a more accurate approximation to \( \sqrt{a} \) than the interpolating line except when \( a \) is such that

\[
\begin{cases}
\frac{n}{(4n + 1)^2} \leq (a - n(n + 1)) \leq \frac{n + 1}{(4n + 3)^2} & \text{if } a > 1, \\
\frac{n}{(4n + 3)^2} \leq (n(n + 1) + (n + 1)/(4n + 3)^2)
\end{cases}
\]

\( a \leq 1/9 \) (from (8) with \( n = 0 \)) if \( 0 < a < s1 \).

For example, the approximation \( t(a) \) is more accurate when:

\( n = 1 \left(1 < a \leq 4\right) \), unless \( a \) lies in the interval \([2 - 1/25, 2 + 2/49] \);

\( n = 2 \left(4 < a \leq 9\right) \), unless \( a \) lies in the interval \([6 - 2/81, 6 + 3/121] \);

\( n = 9 \left(81 < a \leq 100\right) \), unless \( a \) lies in the interval \([89.993, 90.997] \).

The table below further illustrates these results.

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<th>( s(a) )</th>
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**FIGURE 1**
LARGEST PRIME FOUND TO DATE

Scientists. Harry Nelson and David Slowinski, at Lawrence Livermore have shown that $2^{44,497} - 1$ is prime. The number is 13,395 digits in length.

REFEREES FOR THIS ISSUE

The following mathematicians have served as referees for papers considered since the last issue. The Journal appreciates their help and contributions. Almost every article published is revised and improved by the referee's comments and suggestions. Many authors have requested that these contributions be specifically acknowledged.

Bruce Peterson, Middlebury College; Leroy J. Dickey, University of Waterloo; L. Carlitz, Duke University; E. A. Down, Virginia Polytechnic Institute and State University; Wallace Groman, Susquehanna University; David Roselle, Virginia Polytechnic Institute and State University; J. Sutherland Frame, Michigan State University; Patrick Lung, Old Dominion University; James Bardsdale, Western Kentucky University; Donald Bushnell, Ft. Lewis College; David Anderson, Central Washington State College; Dennis Burke, Miami University; Gary Charrand, Western Michigan University; Hudson V. Kronk, SUNY at Binghamton; Don R. Lick, Western Michigan University; Frieda Holley, Metropolitan State College; Dean C. Benson, C. A. Grimm, Dale Rogovin, Ronald Weber, Saites Sengupta, Sumedha Sengupta, Royce Opp and the Editor, all of The South Dakota School of Mines and Technology; Mary Ellen Rudin, University of Wisconsin.

AN EMBEDDING THEOREM FOR SEPARABLE METRIC SPACES

by Roger C. McCann
Mississippi State University

The concept of homeomorphism is fundamental in topology. One of the most important applications of the use of homeomorphism is in proving Urysohn's metrization theorem: a second countable, regular space $X$ is metrizable. This theorem can be proved by constructing a homeomorphism from $X$ onto a subset of infinite dimensional Euclidean space $\mathbb{R}^\omega$, (cf. page 138), i.e., onto a subset of the set of all sequences $a = (a_1, a_2, \ldots)$ of real numbers such that $\sum_{n=1}^{\infty} a_n^2 < \infty$, endowed with the metric $p(a,b) = \sum_{n=1}^{\infty} (a_n - b_n)^2$ where $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$. The proof of this theorem is not elementary. An equivalent result (The proof of the equivalence is essentially the proof of Urysohn's metrization theorem), whose proof requires only basic properties of separability and continuity of a mapping on a metric space is "A separable metric space is homeomorphic to a subset of $\mathbb{R}^\omega."$ The elementary nature of the proof of this result permits the result to be presented early in a beginning course on topology as either a theorem or modified to be an exercise.

Henceforth, $(X,d_1)$ will denote a separable metric space and $\{x_n\}$ will denote a countable dense subset of $X$. It is well known that the function $d:X \times [0,1] \to [0,1]$ defined by

$$d(x,y) = \frac{d_1(x,y)}{d_1(x,y) + 1}$$

is a metric on $X$ which is equivalent to $d_1$. Hence, without loss of generality, we may consider the metric space $(X,d)$.

We begin by defining a countable number of functions $f_n:X \to [0,1)$ by

$$f_n(x) = d(x,x_n).$$

Evidently each $f_n$ is continuous and $0 \leq f_n(x) < 1$ for every $x \in X$ and $n = 1, 2, \ldots$. A mapping $h:X \to \mathbb{R}^\omega$ may now be defined by
Lemma 1. Let $x \in X$ and $\epsilon > 0$. Then there is a positive integer $k$ such that $f_k(x) - f_k(y) \geq \frac{1}{2} \epsilon$ whenever $d(x, y) \geq \epsilon$. Hence, $h$ is one-to-one.

Proof. Let $y \in X$ be such that $d(x, y) \geq \epsilon$. Set $B = \{z | d(z, x) < \frac{1}{4} \epsilon \}$ Since $\{x_n\}$ is dense in $X$ and $B$ is a neighborhood of $x$, there is a positive integer $k$ such that $x_k \in B$. Then

$$f_k(x) = f_k(x) \leq \frac{3}{4} \epsilon \leq d(y, x) = f_k(y).$$

Hence, $f_k(y) - f_k(x) \geq \frac{1}{2} \epsilon$.

Lemma 2. $h$ is continuous.

Proof. Let $x \in X$ and $\epsilon > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, there is an $N > 1$ such that $\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\epsilon}{2}$. For each $n$, there is a $\delta_n > 0$ such that

$$|f_n(x) - f_n(y)| < \epsilon' \left\{ \begin{array}{ll}
\frac{1}{n^2} & \text{if } d(x, y) < \delta_n \\
\frac{1}{n^2} & \text{if } d(x, y) > \delta_n
\end{array} \right.$$ 

Set $\delta = \min \{\delta_1, \delta_2, \ldots, \delta_{N-1}\}$. Then

$$p(h(x), h(y)) = \sum_{n=1}^{\infty} \left[ \frac{f_n(x) - f_n(y)}{n^2} \right]^2$$

$$= \sum_{n=1}^{N-1} \left[ \frac{f_n(x) - f_n(y)}{n^2} \right]^2 + \sum_{n=N}^{\infty} \left[ \frac{f_n(x) - f_n(y)}{n^2} \right]^2$$

$$\leq \sum_{n=1}^{N-1} \left[ \epsilon' \frac{1}{n^2} \right]^2 + \sum_{n=N}^{\infty} \frac{1}{n^2}$$

$$\leq \frac{1}{n^2} + \frac{1}{n^2} = \epsilon$$

whenever $d(x, y) < \delta$. Since $x$ is an arbitrary element of $X$ and $\epsilon$ is an arbitrary positive number, $h$ must be continuous.

Lemma 3. $h^{-1}$ is continuous.

Proof. Let $y \in X$ and $\{y_n\}$ be any sequence in $X$ such that $\{h(y_n)\}$ converges to $h(y)$. We will show that $\{y_n\}$ converges to $y$. Suppose the contrary. Then there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and an $\epsilon > 0$ such that $d(y_{n_i}, y) > \epsilon$ for every $i$. By Lemma 1 there is a positive integer $k$ such that $f_k(y_{n_i}) - f_k(y) \geq \frac{1}{2} \epsilon$ for every $i$. Hence,

$$p(h(y), h(y_{n_i})) = \sum_{n=1}^{\infty} \left[ \frac{f_n(y) - f_n(y_{n_i})}{n^2} \right]^2$$

$$\geq \frac{f_n(y) - f_n(y_{n_i})}{n^2}$$

for every $n$. This is impossible because $\{h(y_{n_i})\}$ is a subsequence of $\{h(y_n)\}$ and $\{h(y_n)\}$ converges to $h(y)$. It follows that $h^{-1}$ is continuous at $h(y)$. Since $y$ is an arbitrary point of $X$, $h^{-1}$ is continuous.

Combining these three lemmas we have

Theorem. A separable metric space is homeomorphic to a subset of $R^n$.

REFERENCES


A graduate student at Trinity
Computed the square of infinity
But it gave him the fidgets
To put down the digits
So he dropped math and took up divinity

--Anonymous
MOVING?

BE SURE TO LET THE JOURNAL KNOW!
Send your name, old address with zip code and new address with zip code to:

David Ballew
South Dakota School of Mines and Technology
Rapid City, South Dakota 57701

REGIONAL MEETING OF MAA
Many regional meeting of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary - Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

MATCHING PRIZE FUND
If your chapter presents awards for outstanding mathematical papers or student achievement in mathematics, you may apply to the National Office to match the amount spent by your chapter. For example, $30 of awards can result in the chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of mathematical films.
To apply, or for more information, write to:

Dr. Richard A. Good
Secretary - Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

MATHEMATICAL RESEARCH AND DEVELOPMENT
by Paul J. Nahin
University of New Hampshire

The lecture hall was packed, and the air buzzed with whispered excitement. Wild speculations on what was about to happen flew from mouth to ear. The great, Full Professor of Mathematics, Dr. Oliver Osgood, head researcher on five simultaneous government grants, was to speak that day at the graduate seminar on his newest, most astounding discovery.

Not since von Neumann proved the Fundamental Theorem of Games, not even since Fermat scribbled his Last Theorem in a margin, nay, not since Archimedes bounded pi, had such a penetrating intellect laid bare the most intimate details of the Queen of the Sciences. Professor Osgood's insight had once again cast bright light into a hitherto dark corner of mathematics. They were all here to learn what new, incredible secret was now in humanity's possession.

The door burst open (one could almost hear trumpets), and the Great Man strode to the podium. Tall, thin, head awash in a mass of white, unruly hair, eyebrows thickly bushy, he placed a single sheet of paper on the wooden surface before him. He stared at it intently. Impressed, the audience exploded into applause. He acknowledged their acclaim with a boyish (yet mature) smile, then held his arms high to hush the excited crowd.

"Thank you, thank you. You are generous, most kind, indeed. It gives me enormous pleasure to be able to share with you the latest fruits of my research. The Government, which sponsors my work, has agreed to the seminar being unclassified, and I wish to thank those enlightened people for the wisdom of their decision."

"Now, let me get right to it, with no further delay. I have found a new number!"

The audience stared, dumbfounded, at Professor Osgood. This was just incredible.

"Yes. Between two and three, somewhat closer to the two than the
three, is a previously unsuspected number. Like a lost penny it has remained hidden through the centuries among the rationals, secreted away beneath the irrationals, but most deeply covered by the transcendentals. Yet, I have found it! With the help, of course, of forty-three graduate students, and hundreds of hours on the university computer system. I thank them all."

A mere associate professor jumped to his feet. "But Professor, this absolutely marvelous! What is this number? Will you write it on the blackboard for us?" A hundred pencils leaped into the hands of the audience, quivering in the excited anticipation of recording the new number.

Professor Osgood shook his head in a charmingly rueful way. "No, I'm sorry, I can't. As you all know, there are two kinds of mathematical proof. The first kind, a constructive proof, would give us the actual value of the number. But my proof is of the second kind, an existence proof. It shows the number exists, but doesn't tell us where exactly it is. But we are working on that! The Government has graciously allocated ten million dollars more for my research in the coming fiscal year."

The audience thundered its approval. Those were big time bucks! There would be summer research money for all!

"Well then, Professor, can you tell us more about your proof?"

"Sorry, that's classified DEEP SECRET. Besides myself, only the Secretary of Defense, the Chairman of the Joint Chiefs of Staff, and the President have seen it. I assure you the secret is safe with them. I can tell you it is based on the fact that three is odd, and two is even, but beyond that, I'm sure you understand my reluctance to say more."

The mere associate professor nodded his head to show his understanding, and asked, "Tell us, Professor, does that mean this new number has military significance?"

Professor Osgood smiled good-naturedly, and replied, "We're sure of it. The integrity of our NATO forces in Europe, of our nuclear submarine fleet, of our SAC bases, of our ICBMs, all will be immensely improved by knowledge of this number." He leaned forward toward the spellbound audience.

"I will tell you this much. If the new number is squared, then added to itself, and the result finally multiplied by the cube root of twenty-nine, the theory indicates we should have the absolute, total power to..." Before Professor Osgood could finish, a short, fat, heavily sweating man pushed forward from the rear of the room. "No, no, Professor, that's classified DEEP SECRET, too! You must say no more!"

Professor Osgood smiled gratefully at the excited man. "Ladies and gentlemen, my apologies for this lapse. This is Colonel Stanley, CIA, and I thank him for setting me straight." The sweaty, almost dripping agent sat down with a plop and a wet squish, relieved at having held back a major leak.

The Professor continued. "Our work for the coming year has the highest urgency. Naturally, we must beat the Russians to this new number. The whole strategic balance of technological parity could be unbalanced if we lose this race. There must not be a Number Gap! He paused for a dramatic moment as his eyes blazed, his brow wrinkled, and his fists clenched. He cut a mighty impressive figure, he sure did. The room burst into light from the flash camera of the local news photographer. What a picture it would make on the front page of the late edition!

"And finally, ladies and gentlemen, knowledge of this new number will give us an additional bargaining chip at the upcoming arms control talks! The very future of East-West detente may hinge on our work over the next twelve months!" Professor Osgood then picked up the single sheet of paper before him (having by now decided to go along with his broker's recommendation to dump soybean futures and to buy Acme Gaskets), and strode from the room.

The lecture hall rocked with the thunderous applause of the audience. They knew the worth of what they had heard. With scientists like Professor Osgood, the Free World would never have to worry.

Because, as everybody knew, somewhere in the Soviet Union the Russians probably had a Professor Osgood, too.

NEW CHAPTER INSTALLATIONS

1978--1979

Massachusetts Epsilon
Illinois Eta
Delaware Beta
Minnesota Epsilon
Minnesota Delta
Kentucky Delta
New York Alpha Alpha
Oklahoma Gamma
Georgia Epsilon
South Carolina Delta

Boston University
Auguata College
Delaware State College
St. Cloud State University
St. John’s University
University of Louisville
Queens College
Cameron University
Valdosta State College
Furman University

1977--1978

Missouri Epsilon
North Carolina Theta
Georgia Delta
Wisconsin Gamma
New York Omega
Mississippi Gamma
Tennessee Delta
Massachussetts Delta
California Lambda

Northwest Missouri State University
Univ. of North Carolina at Charlotte
Atlanta University Center
Univ. of Wisconsin-Parkside
Saint Bonaventure University
Jackson State University
Univ. of Tennessee at Knoxville
University of Lowell
University of California

There are now 222 Chapter of Pi Mu Epsilon and in 1978 through 1979 2385 new members were initiated. Welcome to all of the new Chapters and new members.

CLOSE ENCOUNTERS OF THE MATHEMATICAL KIND

Elleen L. Poiani
Saint Peter’s College

Nothing can be more absurd than the practice which prevails in our country of men and women not following the same pursuits with all their strength and with one wind, for thus the state, instead of being a whole, is reduced to a half.

—Plato (428–347 B.C.)

One pursuit in which women have continually fallen behind men is the study of mathematics and in particular of calculus.

Today, nearly three hundred years after the discovery of the calculus, sociologist Lucy Sells, [6], identifies mathematics as the “critical filter” in career access for both college and non-college bound students. Evidence shows that young women with aptitude have traditionally taken little more than the minimum high school mathematics requirement and consequently have been inadequately prepared to take college calculus, thereby closing the doors to all but a handful of major fields. (The analogous situation for minority students is even more discouraging.)

Perhaps the data most frequently cited to support this claim is that of the freshman class entering Berkeley in 1972, [5]. Fifty-seven percent of the men had taken four full years of high school mathematics, but only 8% of the women. Thus 92% of the entering women did not qualify to enter the mathematics track required of every major at Berkeley except librarianship, education, social sciences, and humanities—all traditionally female career areas.

In a more recent study at the University of Maryland in Fall, 1977, 63% of the entering white men had had 3½ years or more of pre-calculus in high school, compared with 31% of white women, 27% of black men, and 19% of black women. Progress in equalizing the study of mathematics continues to be very slow.

Where do women turn without enough close encounters of the mathematical kind? Women now make up nearly 52% of the U.S. population, but according to observations by the Lawrence Hall of Science at Berkeley,
they are:

1. Only 1% of the U.S. Engineering force and 3% of the physicists,
2. 99% of the nurses and secretaries,
3. 4% of the lawyers,
4. 9% of the doctors and dentists,
5. 2% of the secondary school principals and 1% of the school superintendents, although 73% of the women in college in 1977 were majoring in education in one form or another,
6. Only 8% of library directors, but over 90% of the professional library staff, and
7. Barely 3% of college and university presidents.

Average starting salaries for bachelor's degree holders show dramatic differences based on career demand for a strong undergraduate mathematics program. Statistics from a nationwide sample, adapted from the College Placement Council Salary Survey of March, 1978, show the following average yearly salaries:

<table>
<thead>
<tr>
<th>Field</th>
<th>Average Salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Humanities</td>
<td>$9,948</td>
</tr>
<tr>
<td>Business and Management</td>
<td>$12,668</td>
</tr>
<tr>
<td>Social Sciences</td>
<td>$10,056</td>
</tr>
<tr>
<td>Sciences</td>
<td>$13,668</td>
</tr>
<tr>
<td>Economics</td>
<td>$11,400</td>
</tr>
<tr>
<td>Engineering</td>
<td>$16,668</td>
</tr>
</tbody>
</table>

The differential of over $6,700 is staggering. The continuing inverse relationship between the highest paid areas and those most heavily populated by women is even more unsettling.

Lest we become too pessimistic, however, we should make note of a countervailing trend both at the bachelor's and master's degree levels. [1] and [2]. Here the representation of women has increased most in those fields where women have traditionally been least represented. But according to studies by the National Center for Educational Statistics, this trend has not yet reached the doctoral level. [3].

Mathematics plays the critical filter role not only at the college level, but also in the work force. In the 1978-79 edition of Occupational Outlook Handbook, the U.S. Bureau of Labor Statistics forecasts that the vast majority of the 46 million job openings through 1985 will require fewer than four years college training. Major growth areas include computers, transportation, health services, secretaries, bookkeepers, cashiers, mechanics, and police officers. These areas require good basic arithmetic skills and in the future will demand an ability to handle data, understand statistics, read computer printouts and graphs. Advancement in any one of these fields may well depend on a person's mastery of algebra and geometry. In fact, in the November 1977 issue of Today's Secretary, it was pointed out that:

"If you want to advance in a secretarial career, math may be one of the most important hurdles to overcome to be successful. Today, the highest paid secretaries are in the technical fields."

The ability to handle budgets for the boss is a desirable asset when seeking advancement to the boss's position.

A study, [4], of 1300 children aged 2 through 12 showed no sex difference in the liking of mathematics. These children were asked to rank the subjects of English, math, science, and social studies from the most to least favorite. More than 54% ranked math as first or second favorite, and there was no difference in preferences between boys and girls. Many different reasons have been advanced to explain why the disproportionately small number of women in mathematics and related areas has persisted over the years.

A growing wealth of research points to societal conditioning rather than lack of ability in the infrequent encounters between women and mathematics. The masculine stereotype of mathematicians; overt discouragement of young women from studying mathematics - by parents, teachers, and advisers; and the traditionally unreceptive climate toward the education of women often thwarted all but the most determined women from mathematical pursuits.

But women did make significant contributions to mathematics although their Who’s Who list is little known. The list is too long to enumerate here, but the reader is referred to Lynn Osen’s book, Women in Mathematics, [5], for assistance in completing the following quiz. The unfamiliarity of their names reflects the historical exclusion of these really notable women from the mainstream of mathematical development. Answers to the quiz appear at the end of the article.

Matching Some Notable Women Mathematicians with their Major Accomplishments

<table>
<thead>
<tr>
<th>Women Mathematicians</th>
<th>with their Major Accomplishments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Hypatia (370-415)</td>
<td>A. Translator and Analyzer of Newton's Principia.</td>
</tr>
</tbody>
</table>
2. Émilie de Breteuil
   (1706 - 1749)

3. Maria Gaventa Agnesi
   (1718 - 1799)

4. Caroline Herschel
   (1750 - 1848)

5. Sophie Germain
   (1776 - 1831)

6. Mary Fairfax Somerville
   (1780 - 1872)

7. Sonya Kovalevsky
   (1850 - 1891)

8. Emmy Noether
   (1882 - 1955)

9. Grace Murray Hopper
   (1906 - )

B. Expositor par excellence of the physical sciences.

C. Pioneer of the modern computer compiling system and programming languages.

D. Commentator on the conics and Diophantine equations; inventor of the hydroscope and other apparatus for astronomy.

E. Creator of modern abstract algebra.

F. First woman author of a major textbook on calculus.

G. Expert in detailed mathematical calculations associated with astronomy; the first woman to ever see a comet.

H. One of the founders of mathematical physics, with several papers focusing on the theory of elasticity.

I. Researcher in such diverse subjects as partial differential equations, theory of Abelian functions, and application of analysis to the theory of numbers.

To help smash the erroneous stereotype that women do not belong in mathematics and to convince them that they do, WAM (Women and Mathematics) was created. WAM is a secondary school lectureship program created in 1975 by the Mathematical Association of America under a grant from IBM. By conservative estimates made in the summer of 1978, WAM visits had reached more than 260 schools, 25,000 students, 2,500 parents, teachers, and guidance counselors.

The idea for WAM was sparked when IBM representatives hosted a reception for top scorers in the U.S.A. Mathematics Olympiad and noticed no women were among the winners. The U.S.A. Olympiad is an annual contest for invited high school students who have excelled in previous mathematics competitions. From among the top scorers in the U.S.A. Olympiad, a special team is chosen for participation in the International Olympiad held each summer. For the first time in 1978, two young women scored highly on the U.S.A. Olympiad and they participated in the training session for the international competition.

The absence of women from the Olympiad symbolized their absence from all those fields previously mentioned in this article. To encourage 9th and 10th graders to keep career doors open by taking more than the minimum mathematics requirement, IBM agreed to support WAM. School visits by lecturers from a variety of career fields are planned for half to one full day at no cost to the school. The visit includes a formal talk on how the speaker uses mathematics and an informal session for students, counselors, and teachers.

Since many factors influence attitudes towards mathematics, WAM also arranges presentations to professional societies, guidance counselors, elementary school teachers, math teachers, civic organizations, parents associations, and legislative leaders. By conservative estimates as of April 1979, WAM visits had reached more than 442 schools, 38,600 students, and 3,600 teachers, guidance counselors and parents.

Further information about WAM can be obtained through the MAA headquarters, 1529 Eighteenth Street, N.W., Washington, D.C. 20036 or from the National Director, Dr. Eileen L. Poiani, Saint Peter's College, Jersey City, N.J. 07306.

Answers to Quiz:

Eileen L. Poiani is one of the four National Pi Mu Epsilon councillors.

FRATERNITY KEY-PINS

Gold Clad key-pins are available at the National Office (the Univ. of Maryland, Department of Mathematics) at the special price of $8.00.

Be sure to indicate the chapter into which you were initiated and the approximate date of initiation.
GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (HENDRIX COLLEGE) had a variety of programs and papers during the year. These included: Julie Anderson, "A Report On The Summer Meetings In Providence"; Dr. Frank Hudson (University of Central Arkansas), "A Pure Mathematician In An Applied Situation"; Janet Villahuruty (Southwestern Bell Telephone), "Employment Opportunities For Math Graduates"; Dr. Glenn Webb (Vanderbilt University), "Mathematical Models of Epidemics"; Dr. William Summers (University of Arkansas, Fayetteville), "The Cantor Set: Homeomorphisms and Fixed Points"; Dr. Robert C. Walls (University of Arkansas Medical Center), "Biometry and Its Applications"; Gene Web®, "The Symmetric Derivative"; David Sutherland, "Algebraic Properties of the Laplace Transform"; Sandy Scrimshire, "Computer Applications in Population Genetics"; Lisa Orton, "Characterizations of Convex Sets"; Agnes Tullio, "Uniform Limits of Step Functions"; Dr. Thomas Fomby (Southern Methodist University), "A Microeconomic Model of the U.S. Economy"; Dr. John Churchill, "Mathematicians in the History of Philosophy".

Integral to the chapter’s activities was the annual spring travel program. During this period several students presented their undergraduate papers to various colleges and universities in Arkansas and the neighboring states. Talks were given at the Oklahoma-Arkansas MAA meeting at Oklahoma State University, the Arkansas Academy of Science, the University of Arkansas at Fayetteville, North Texas State University, and at Southwestern College in Memphis. The latter was a joint symposium for undergraduate mathematical research, with students from the University of the South at Sewanee, Southwestern at Memphis, and Hendrix participating.

MINNESOTA EPSILON (ST. CLOUD STATE UNIVERSITY) had chapter activities which included a number of presentations by guest speakers, participation in a free student tutoring service, organization of both Fall and Spring mathematics and computer science student-faculty picnics, sponsorship of a scholarship program, and cosponsorship of the annual Spring SCSU Mathematics Contest for area junior high and senior high schools. This year speakers and their topics were as follows: Jerry Lenz (St. John’s University), "Nonstandard Analysis"; Mahmoud A. Kishka, "The Golden Section"; Eric Nummela, "An Axiomatic Approach To Set Theory"; Joseph Gallian (University of Minnesota, Duluth), "Weird Dice"; Jack Anderson, "Placement Files and the Job Situation"; Tim Haigh (St. John’s University), "What I Wish Every Student Had Learned About Graphing In High School"; Eric Nummela, "Everything You Always Wanted To Know About Mobius Strips"; Mike Sweeney (The St. Paul Companies), "What is an Actuary?"; John Rongitsch (Rongitsch Incorporated), "Experience in Establishing My Own Computer Services Company"; Richard Jarvinen (St. Mary’s College), "Vector Spaces"; David Boyer, "A Digestible Introduction to Nonstandard Real Number Theory"; Robert Earles, "Combinatorial Mathematics, An Interesting Problem".

MISSOURI GAMMA (ST. LOUIS UNIVERSITY) hosted the Fifth Annual Pi Mu Epsilon Regional Student Conference which was held Nov. 3-4, 1978. The first day was devoted to area High School Students. One hundred and fifty-five students and faculty from fifteen high schools attended. The second day consisted of three parts: Undergraduate Presentations, Graduate Presentations, and two invited lectures. The two invited lectures were Dr. David Ballew (South Dakota School of Mines and Technology), "Approximations to Pi", and Sister Juliana Lucey (Marquette University), "Some Applications of Numerical Analysis To Differential Equations".

Seventy people were in attendance, representing fourteen universities and colleges. Papers given included: Edwin Eitel Jr. (SLU), "Negative Fish"; Charles Fold (SLU), "Crystals and Symmetry"; Barbara Reynolds, (SLU), "Taxicab Geometry"; Nick Sortal (SLU-C), "Mathematical Problem Solving"; Jo Fiene (SLU-C), "Women in Mathematics"; Joseph Boor (SLU-C), "Determining Computationally Whether or Not a Polygon is Either Simply Connected or Convex"; Anita Zettler (Murray State University), "The Ring of Continuous Functions on the Unit Interval, etc."; Camile Stelzer (Maryville College), "Practical Situations Involving the Use of Discrete Bayesian Statistics"; Gheogory Battle (Washington University), "Numerical Treatment of Meteorological Data"; Barney Smith (SLU), "The Trisection Problem--Various Solutions"; Martin Franck (Milken Univ.), "The Geostrophic Wind As a Sum of the Coriolis Vector and the Pressure Gradient Vector".

MISSOURI DELTA (WESTMINSTER COLLEGE) heard a talk by former professor George Hinkle entitled "Problem Solving".
MONTANA ALPHA (UNIVERSITY OF MONTANA) hosted several "lemonade parties" for undergraduates during the Fall quarter. Talks and papers presented to the Chapter included: Rick Demarinis, "What's a Novelist Doing Here Talking to Mathematicians"; Howard Reinhartz, "Exercises, Puzzles and Problems"; Robert Stevens, "Photographs of Some Famous Mathematicians--I'll trade you my G. H. Hardy for your Herman Weyl". In addition two movies, "Turning the Sphere Inside Out" and 'Space Filling Curves', were seen.

NEW JERSEY DELTA (SETON HALL UNIVERSITY) heard three papers and viewed two films. The papers and authors were: Dr. J. W. Andrushkin, "Undergraduate Research Problems in Mathematics"; Dr. E. Guerin, "Arithmetic Functions"; Prof. Denny Gulick (University of Maryland), "Between the Pot of Gold and the Rainbow". The films were: "Who Killed Determinants" and "Nonstandard Analysis".

NEW YORK OMEGA (ST. BONAVENTURE) hosted the following paper: Dr. James Peters (SUNY at Purchase), "Applications of the Radon and X-Ray Transforms". In addition the Chapter held a film festival at which the following were shown: "Caroms", "Isometries" and "Space Filling Curves".

NORTH CAROLINA GAMMA (NORTH CAROLINA STATE UNIVERSITY) held four program meetings during the year. They were as follows: PA Robert Silber, "Instant Insanity"; (This is an impressive videotape analysis of the famous puzzle.); PA Robert Silber, "Grundy Numbers and Disjunctive Compounds of Games"; James Bergin (Occidental Life Insurance Company of North Carolina), "The Actuarial Profession--Performance, Training and Opportunities"; Dr. Nicholas Rose, "Fallacies, Howlers and Paradoxes".

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY) heard Dr. David Ballov speak on "Research Projects No One Ever Told You About".

OHIO ZETA (UNIVERSITY OF DAYTON) noted that this was the last year for Prof. Kenneth Schaut to serve as faculty advisor. He had held this position since the Chapter's installation in 1962. The major event for the Chapter was the Nineth Biennial Alumni Seminar on Employment Opportunities in the Mathematical Sciences. Over forty alumni of the mathematics department participated. Two lectures were given by members of the faculty: Prof. Gerald Shoughnessy, "Design of Experiments" and PA Clay Waldrop, "King Chicken Theorems". Brother Ploeger, S.M. assisted the Chapter in sponsoring a series of problem sessions in which recreational type problems were discussed.

VIRGINIA DELTA (ROANOKE COLLEGE) sponsored and participated in a variety of programs during the year. Members of the Chapter helped produce the Fall Meeting of the Blue Ridge Council of Teachers of Mathematics (BRCTM), the Meeting of the Virginia Council of Teachers of Mathematics (VCTM), and served as judges in the Mathematics Division of the Western Virginia Regional Science Fair. In addition, the following papers and presentations were heard: Dr. Nanny Jane Ingram, "The Golden Section...It Appears Everywhere"; Elizabeth Leonard, "Experience as an Actuary (Intern)"; and Cheryl Ammon, "Experience as a Student Teacher".

1980 NATIONAL PI MU EPSILON MEETING

It is time to make plans to send an undergraduate delegate or speaker from your chapter to attend the Annual Meeting of Pi Mu Epsilon on the University of Michigan Campus at Ann Arbor in August of 1980. Each speaker who presents a paper will receive travel fund of up to $400 and each delegate, up to $200.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

We have a supply of 10 x 14-inch Fraternity Crests available. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

1. Purple on goldenrod stock---------$1.50/dozen;
2. Purple on lavendar on goldenrod---$2.00/dozen.
PUZZLE SECTION

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems involving numbers, geometric figures, patterns, or logic whose solution consists of an answer immediately recognizable as correct by simple observation, and not necessitating a formal mathematical proof. Although logical reasoning of a sort must be used to solve a puzzle in this section, little or no use of algebra, geometry, or calculus will be necessary. Admittedly, this statement does not serve to precisely distinguish material which might well be the domain of the Problem Department, but the Editor reserves the right to make an occasional arbitrary decision and will publish puzzles submitted by readers when deemed suitable for this department and believed to be new or not accessible in books. Material not used here will be sent to the Problem Editor for consideration in the Problem Department, if appropriate; or returned to the author.

Address all proposed puzzles, puzzle solutions or other correspondence to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota 57701. Please do not send such material to the Problem Editor as this will delay your recognition as a contributor to this department. Deadlines for solutions of puzzles appearing in each Fall issue is the following March 1, and that for each Spring issue, the following September 15.

Mathacroistic No. 9
submitted by Joseph D. E. Konhauser
Macalester College, St. Paul, Minnesota

Like the proceeding puzzles, this puzzle (on the next page) is a keyed anagram. The 190 letters to be entered in the diagram in the numbered spaces will be identical with those in the 26 keyed words at matching numbers and the key letters have been entered in the diagram to assist in correlation during your solution. When completed, the initial letters will give a famous author and the title of his book. The diagram will be a quotation form that book.

Smith-Jones-Robinson Problem
submitted by Veniser Turner, Jr.
Prairie View A&M University

(This problem is adapted from Scientific American, Vol 200, no. 2, p. 136, Feb. 1959, with permission)

Smith, Jones, and Robinson are the engineer, brakeman, and fireman on a train, but not necessarily in that order. Riding the train are three passengers with the same three surnames, to be identified in the following premises by "Mr." before their names.

Mr. Robinson lives in Los Angeles.
The brakeman lives in Omaha.

Mr. Jones long ago forgot all the algebra he learned in high school.
The passenger whose name is the same as the brakeman's lives in Chicago.
The brakeman and one of the passengers, a distinguished mathematical physicist, attend the same church.

Smith beat the fireman at billiards.

Who is the engineer?

Maximum Number of Knights
submitted by Pierre R. Square

What is the maximum number of knights that can be placed on a chess-board so that no two attack each other?
A. transformation which maps each line into a parallel line
B. U. S. pioneer in communication theory
C. the last word in a chess game
D. the problem of equipartitioning a circle
E. egg-shaped with broad end up
F. Greek name assumed by Holzmann, translator of Diophantus' "Arithmeticat"
G. matrix form in which the number of leading zeros in each row increases with the row number
H. sometime search objective
I. Cretan who created a paradox by saying "All Cretans are liars"
J. aspect of change of interest in calculus (2 wds.)
K. arise as a consequence
L. Babbage's antecedent of the computer
M. showy trifle; bauble
N. odd
O. J. H. Conway game related to problems of simulation and artificial intelligence
P. lowest throw at dice
Q. sharp repeated tapping (comp.)
R. alternative to perish
S. quadirectangular tetrahedron
T. in a plane, the locus of a point the product of whose distances from two points fixed 2a apart is a^2
U. in Chinese cosmology, all that comes to be (3 wds.)
V. Riemann surface for double-valued function (2 wds.)
W. close in count (3 wds.)
X. formulator of first truly satisfactory postulational treatment of Euclidean geometry
Y. Robert Abbott's card game which requires inductive reasoning
Z. beginning of many math problems (2 wds.)
Minimum Number of Knights

submitted by Pierre R. Square

What is the minimum number of knights required so that every square on the chessboard is either occupied or under attack.

SOLUTIONS

Mathacrostic No. 7 (See Fall 1979 issue).

Definitions and key:

A. Guthrie  H. Hollerith  O. Rioting  V. Entracte
B. Eclectic  I. Epsilon  P. Slam bid  W. Motif
C. Orbiform  H. Unitary  Q. Ennead  X. Icosian
D. Mantzel  K. Notation  R. Open set  Y. Noether
E. Escapement  L. Infold  S. Flatworms  Z. Doodie
F. Nictitate  M. Vish  T. Tethys sea
G. Tractrix  N. Elegant  U. Hopscotch

First letters: G E Owen The Universe of The Mind

Quotation: It is ironic that this movement which originally focused on the necessity for experimental testing led to a complete abstraction of the problem and to a realization that concern with logical structure represented an end in itself.

Solved by George Levine; Henry S. Lieberman, John Hancock Mutual Life; Robert Prielipp and John Oman, University of Wisconsin-Oshkosh; Louis H. Cairoli, Kansas State University; Victor G. Feser, Mary College, Bismarck; Louis H. Cairoli, Kansas State University; Henry S. Lieberman, John Hancock Mutual Life Insurance Company; George Levine, Commercial Union Insurance Company; Joseph D. E. Konhauser, Wabash College, St. Paul; Jeanette Bickley, Webster Groves High School, Missouri; Robert Prielipp, University of Wisconsin-Oshkosh; Robert C. Gebhardt; Richard Stratton.

Mathacrostic No. 8 (See Fall 1979 issue)

The Journal apologizes to Professor Gerald Perham of St. Joseph's University in Hammond, It who submitted this Mathacrostic but did not get credit in the Fall 1979 issue.

Definitions and Key:


First Letters: Lebesgue Measure and Integration

Quotation: Since that remote age in which man learned to count, number has become one of the fundamental ideas that engage our thought—an idea so immediate and so clear to the understanding that in trying to analyze it, we at first succeed only in obscuring it.

Solved by: Sister Stephanie Sloyan, Georgian Court College; Victor G. Feser, Mary College, Bismarck; Louis H. Cairoli, Kansas State University; Henry S. Lieberman, John Hancock Mutual Life Insurance Company; George Levine, Commercial Union Insurance Company; Joseph D. E. Konhauser, Wabash College, St. Paul; Jeanette Bickley, Webster Groves High School, Missouri; Robert Prielipp, University of Wisconsin-Oshkosh; Robert C. Gebhardt; Richard Stratton.

LOCAL AWARDS

If your chapter has presented or will present awards this year to either undergraduates or graduates (whether members of Pi Mu Epsilon or not), please send the names of the recipients to the Editor for publication in the Journal.

He who can, does.
He who can't, teaches.
He who can't teach, administers.
He who can't administer, does research.
He who can't do research, publishes research.
He who can't publish research, edits research.
He who can't edit research, gives lectures.
He who can't give lectures, introduces lecturers.
He who can't introduce lecturers, passes judgment on them.
He who can't pass judgment, drops out.
He who drops out, gets hungry.
He who gets hungry, grows a garden.
He who grows, harvests.
He who harvests, cans.
LOCAL CHAPTER AWARDS WINNERS

ALABAMA DELTA (UNIVERSITY OF SOUTH ALABAMA). The PI Mu Epsilon Award award for outstanding achievement in the field of mathematics was awarded to

Daniel Dix.

GEORGIA BETA (GEORGIA INSTITUTE OF TECHNOLOGY). The awards for outstanding graduates in mathematics were given to:

Virginia J. Foard
Randolph C. Nicklas
James S. Tomlin
Clive M. Webster.

MISSOURI GAMMA (UNIVERSITY OF ST. LOUIS). The James W. Garneau Mathematics Award was given to:

William C. Dale III.

The Francis Regan Scholarship award was presented to:

First Place: Paul L. Sventek
Second Place: Maureen McGovern
John M. Hanson.

The Missouri Gamma Undergraduate Award was presented to:

Donald Schuster.

The PI Mu Epsilon contest awards were won by:

Senior Contest Winner: Michael Ming
Junior Contest Winner: Alan Ho.

The John J. Andrews Graduate Service Award was given to:

Joel Baumeyer, FSC.

The A1 and Shelly Beradino Fraternityship Award for active participation in the affairs of the fraternity was presented to:

Camille Stetzer
Cecile Stetzer.

MONTANA ALPHA (UNIVERSITY OF MONTANA). The John Peterson Book award for the outstanding senior graduating in mathematics education:

Melinda Williams.

NEW JERSEY BETA (RUTGERS. THE STATE UNIVERSITY). The Junior Book Award winner is:

Alice Davenport.

NEW YORK OMEGA (ST. BONAVENTURE UNIVERSITY). The New York Omega Chapter Award of academic excellence, interest in mathematics and service to the Chapter was presented to:

Paul McGuire.

OHIO NU (UNIVERSITY OF AKRON). The awards for Outstanding Science Fair projects were given to:

Karl Kaiser
Matt Wagner.

The Samuel Selby Mathematics Scholarship Award for 1979-80 was presented to:

Carl Andrews.

The Senior Teaching Assistant Award was given to:

Doug Seltgert
and

Nancy Calvin.

the Graduate Teaching Award.

RHODE ISLAND BETA (RHODE ISLAND COLLEGE). The Christopher Mitchell Awards were presented to:

Denise Larivee
Scott Chianese.

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY). The Award for the Outstanding Mathematics Senior was given to:

Bruce Weeks.

Mathematicians:
We're Number -e^{-i\pi}
The following papers were presented at the Duluth Meeting of Pi Mu Epsilon:

1. On Continued Fraction Representations of Liouville Numbers
   - Michael Filaseta
   - A2 Alpha
   - Univ. of Arizona

2. Behavior of the Permanent of a Special Class of Doubly Stochastic Matrices
   - Phuong Anh Vu
   - TX Theta
   - Univ. of Houston

3. Physical Models for Applications of Rational Numbers in Reduced Form
   - Jo Ann Fiene
   - IL Delta
   - Southern Illinois University, Carbondale

   - Ali Enayat
   - IA Alpha
   - Iowa State University

5. Enzyme Kinetics of Phenylketonuria
   - C. P. Kuchinad
   - WI Alpha
   - Marquette University

6. A Hard (But Interesting) Problem
   - Michael L. Call
   - IN Gamma
   - Rose-Hulman Institute of Technology

7. Time as a Dynamical Variable in Quantum Physics
   - Stephen L. Kerr
   - TN Delta
   - University of Tennessee, Knoxville

8. Love Sonnet for a Mathematician
   - Mark Walker
   - SD Beta
   - South Dakota School of Mines & Technology

9. Infinitesimals: Where They Come From and What They Can Do
   - Professor H. Jerome Keisler
   - J. Sutherland Frame
   - Lecturer
   - University of Wisconsin-Madison

10. The Axiom of Choice
    - John B. Vaughn
    - MO Gamma
    - St. Louis University

11. The Kernel of the Laplace Transformation
    - David C. Sutherland
    - AR Beta
    - Hendrix College

12. The Knight’s Tour Problem
    - Gary Ricard
    - SD Beta
    - South Dakota School of Mines and Technology

13. A Simple Proof of a Theorem by H. Schaefer
    - Peter Westfall
    - CA Lambda
    - University of California at Davis

14. Mathematics and the Boiling Points of Alkanes
    - Michael J. Schell
    - NC Theta
    - University of North Carolina at Charlotte

15. Two Problems in Number Theory
    - Janet Reid
    - FL Eta
    - University of North Florida

16. Uniform Algebras and Scattered Spaces
    - Robert C. Smith
    - AR Alpha
    - University of Arkansas

17. A Case Against Computer Crime
    - Lonnie Emard
    - MO Epsilon
    - Northwest Missouri State University

18. Measure Construction Using Cauchy Sequences
    - Stephen W. Semmes
    - GA Gamma
    - Armstrong State College

19. Cyclic Numbers
    - Richard O. Griffin
    - MA Delta
    - University of Lowell

20. The Consequences of Cauchy’s Integral Theorem
    - Alfred Earl Byrum
    - NC Delta
    - East Carolina University

21. An Infinite Number of Magic Squares
    - Stephen J. Ruberg
    - OH Delta
    - Miami University
The following is a recommended Reading List for all Mathematics students and instructors:

1. The Jacobians, and their Struggle for Independence.
2. A Ten-Day Diet to Improve Indeterminate Forms.
3. Cheaper by the Googol.
4. 1001 Best-Loved Double Integrals.
5. The Torus and I.
6. A Short Table of Even Primes.
7. Will Success Spoil Runge-Kutta?
8. Dining Out in Hilbert Space.
9. One Hundred Tasty Fillings for Empty Sets.
10. Life Begins at e.
11. How to Keep Condensation Points from Dripping into Open Sets.
13. Tom Swift and His Electric Cycloid.
15. Improving Lipschitz Conditions in the Slums of New York.
16. The Decline and Fall of e^{-x}.
17. How to Prevent Rust on Riemann Surfaces.
18. The Peano Postulates Transcribed for Violin and Cello.
19. First Aid for Dedekind Cuts and Bruises.


PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor’s evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor.

Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

To facilitate consideration of solutions for publication, solvers should submit each solution on a separate sheet properly identified with name and address and mailed before the end of June 1980.

Address all communications concerning this department to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

A fairly young man was married at the beginning of the month. At the end of the month his wife gave him a chess set for his birthday. If he was married and received the chess set on the same day of the week he was born, how old was he when he got married?

450. Proposed by Clayton W. Dodge, University of Maine at Orono.
In triangle ABC, let \( \angle A \leq \angle B \leq \angle C \). Then

\[ s > \frac{(R + r)\sqrt{3}}{2} \]

is a well-known theorem, where \( s \) is the triangle's semiperimeter, \( r \) its inradius, and \( R \) its circumradius. Prove it.
451. Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, California.

Find all instances of three consecutive terms in a row of Pascal's triangle in the ratio 1:2:3.

452. Proposed by Tom M. Apostol, California Institute of Technology.

Given integers $m > n > 0$, let

$$a = \sqrt{m} + \sqrt{n}, \quad b = \sqrt{m} - \sqrt{n}.$$ 

If $m - n$ is twice an odd integer, prove that both $a$ and $b$ are irrational.


Given two intersecting lines and a circle tangent to each of them, construct a square having two of its vertices on the circumference of the circle and the other two on the intersecting lines.


The point within a triangle whose combined distances to the vertices is a minimum is (or should be) known as the Fermat-Torricelli point, designated by $T$. In a triangle $ABC$, if $AT$, $BT$, $CT$ form a geometric progression with a common ratio of 2, find the angles of the triangle.

455. Proposed by Kenneth M. Wilke, Topeka, Kansas.

Young Leslie Morley noticed that the perimeter of a $6 \times 4$ rectangle equals the area of a $2 \times 10$ rectangle while the area of the $6 \times 4$ rectangle equals the perimeter of the $2 \times 10$ rectangle also. Show that there are an infinite number of pairs of rectangles related in the same way and find all pairs of such rectangles whose sides are integers.

456. Proposed by Paul Erdős, Spaceship Earth.

Is there an infinite path on visible lattice points avoiding all $(u, v)$ where both $u$ and $v$ are primes? (The proposer offers Twenty-five Dollars for a solution.)

457. Proposed by the late R. Robinson Rowe.

Defining the last $n$ digits of a square as its $n$-tail, what is the longest $n$-tail consisting of some part of the cardinal sequence $0, 1, 2, 3, \ldots$? What is the smallest square with that $n$-tail?

458. Proposed by Charles W. Trigg, San Diego, California and Leon Bankoff, Los Angeles, California.

Translate each of the following sketches into a mathematical term.


Let $x$, $y$, and $z$ be positive integers. Then $(x, y, z)$ is a Pythagorean triangle if and only if $x^2 + y^2 = z^2$. Prove that every Pythagorean triangle where both $x$ and $z$ are prime numbers and $x \equiv 1 \pmod{10}$ is such that 60 divides $y$. 
460. Proposed by Barbara Seville, University of Bologna, Italy. Dedicated to: Jean J. Pedersen, University of Santa Clara.

The dihedral angle of a cube is $90^\circ$. The other four Platonic solids have dihedral angles which are approximately $70^\circ 31'43.60''$, $109^\circ 28'16.3956''$, $116^\circ 33'54.18''$, and $138^\circ 11'22.866''$. How closely can these angles be constructed with straightedge and compasses? Can good approximations be accomplished by paper folding? If so, how?

461. Proposed by David C. Kay, University of Oklahoma, Norman, Oklahoma.

(a) A right triangle with unit hypotenuse and legs $r$ and $s$ is used to form a sequence of similar right triangles $T_1$, $T_2$, $T_3$, ... where the sides of $T_1$ are $r$ times those of the given triangle, and for $n \geq 1$ the sides of $T_{n+1}$ are $s$ times those of $T_n$. Prove that the sequence $T_n$ will tile the given triangle.

(b) What happens if the multipliers $r$ and $s$ are reversed?

(c) The art of the Hopi American Indians is known for its zigzag patterns. The blanket illustrated below is made from a rectangle of (inside) dimensions $a \times b$, and the zigzag is formed by dropping perpendiculars to alternating sides of the triangle in the design. Show that the area of the design (shaded portion) is given by the formula $(a^3 + ab^2)/(2a^2 + 4b^2)$.

SOLUTIONS


Without using its altitude, compute the volume of a regular tetrahedron by the prismoidal formula.

Solution by the Proposer.

In the regular tetrahedron $ABCD$ with edge $e$, as in Figure 1, the medians of two of the equilateral triangular faces are $MD = MC = e\sqrt{3}/2$. The bimedian, $MN$, is an altitude of the isosceles triangle $DMC$, so $MN = \left[ e \sqrt{3}/2 - (e/2)^2 \right]^{1/2} = e/\sqrt{2}$.

(Considering the regular tetrahedron inscribed in the cube of Figure 2, the bimedian equals the edge of the cube, so immediately is $e/\sqrt{2}$.)

The regular tetrahedron $ABCD$ may be considered to be a prismatoid with bases $AB$ and $CD$ of zero area lying in parallel planes. Then the bimedian $MN$ is the altitude of the prismatoid, which has a square midsection $FGHJ$ with area $=(e/2)^2$.

Then, by the prismoidal formula, $V = h\left[ b + m + b' \right]/6$, the volume of the regular tetrahedron $ABCD$ is

$$V = (e/\sqrt{2}) \left[ 0 + 4(e/2)^2 + 0 \right]/6 = e^3 \sqrt{2}/12$$

For other unorthodox methods of determining the volume of the regular tetrahedron see Charles W. Trigg, "The Volume of the Regular Tetrahedron," Eureka (Canada) [now Crux Mathematicorum], 3 (August-September 1977), 181-183.

FIGURE 1

FIGURE 2

Also solved by DONALD CANARD, Anaheim, California; ALFRED E. NEUMAN, Mu Alpha Delta Fraternity, New York; BARBARA SEVILLE, University of Bologna, Italy; and ZELDA KATZ, Beverly Hills, California.
426. [Fall 1978] Proposed by the late R. Robinson Rowe.

With some oversimplification of an actual event, after a cold dry
snow had been falling steadily for 72 hours, a niphometer showed a depth
of 340 cm., compared to a reading of 175 cm. after the first 24 hours.
Assuming that underlying snow had been compacted only by the weight of
its snow overburden, so that the depth varied as a power of time, what
would have been the depth after 12 and 48 hours?

Solution by the Proposer.

Let \( Z \) be the depth in centimeters and \( T \) the elapsed time in days.
Then in the general power function, \( Z = AT^b \), the given data furnish two
equations, \( 175 = A \times 1^b = A \), and \( 340 = A \times 3^b \). Whence by division,
\( \frac{340}{175} = 3^b \) and \( b = 0.604 \times 544 \times 161 \). Then for any other time
\( T, Z_T = 175 \times T^b \).

After 12 hours, \( T = \frac{1}{2} \) and \( Z = 115.093 \times 8512 \).
After 48 hours, \( T = 2 \) and \( Z = 266.087 \times 1946 \).

The niphometer being read to the nearest centimeter, the depths
were 115 and 266, respectively.

Comment: The exponential relation is approximate at best and has
limitations. Density of the uncompacted snow at the surface may be as
low as 0.05. Compaction under the weight of later snow begins much like
elastic compression, but with the elastic modulus increasing with
density. Before the density reaches 0.50, the character of the packed
snow is so changed that the exponential relation is replaced by another,
as enormous pressures slowly fuse the snow into ice with a density of
0.917.

The actual event occurred about 25 years ago. The niphometer was
100 feet outside the SPPR station at Norden, supposedly read every 6
hours by the stationmaster. The gage being hard to reach during a
storm, he had devised an unofficial alternative. He built a shelf at
still level outside a station window. Every 6 hours he would open the
window, measure the depth of snow on the shelf with a yardstick, sweep
off the shelf, add this depth to a cumulating total, and record this in
the gage book. After 72 hours the 12 increments had added to more than
230 inches. But at the end of the storm when he snowshoed out to the
niphometer, it read only 180 inches! So he erased all that valuable
record and substituted 15 inches for each increment. I wanted it later
when asked to derive the maximum one-hour snowfall as a guide to design
of Interstate 80 (median width for snow storage and size, type and number
of snow-removal devices.) I had to rely on better data from the Snoquale
Pass in Washington.

This interesting problem was also solved (in essentially the same
manner) by DAVID DEL SESTO, North Providence, R.I.; CLAYTON W. DODGE, -
University of Maine at Orono; and CHARLES H. LINCOLN, Goldsboro, N.C.
Two incorrect solutions were received.

427. [Fall 1978] Proposed by Jackie E. Frick, Texas A&M
University, College Station, Texas.

If \( a, b, c, d \) are integers and \( u = \sqrt{a^2 + b^2}, v = \sqrt{(a-c)^2 + (b-d)^2} \)
and \( w = \sqrt{a^2 + d^2} \), then \( \sqrt{(u+v+w)(u-v+w)(u+v-w)(u-v-w)} \) is an even integer.

I. Solution by M. S. Klamkin, University of Alberta, Canada.

A solution follows immediately by considering a geometric inter
pretation, with \( u, v, w \) the lengths of the sides of a lattice triangle
with coordinates \((0,0), (a,b), (c,d)\). It corresponds to 4 times the area
of the triangle. Since the area is also given by

\[
\frac{1}{2} \left| \begin{array}{ccc}
0 & 0 & 1 \\
a & b & 1 \\
c & d & 1
\end{array} \right|
\]

I must be an even integer.

II. Solution by Léo Savel, Algonquin College, Ottawa, Canada.

In a coordinate plane, the area of a triangle whose vertices are
the origin, \((2a, 2b)\), and \((2c, 2d)\) is easily found to be \(2 \mid ad - bc \mid \).
On the other hand, the sides of such a triangle are \(2u, 2d, 2w\), and its
area, by Heron’s formula, is precisely equal to the given expression,
which is therefore equal to the even integer, \(2 \mid ad - bd \mid \).

Also solved by CHARLES D. ALLISON, Huntington Beach, Calif.;
JEANETTE BICKLEY, St. Louis, Missouri; WALTER BLUMBERG, New Hyde Park,
Long Island, N.Y.; DAVID DEL SESTO, North Providence, R.I.; CLAYTON DODGE,
University of Maine at Orono; ROBERT FULLER, Savannah, Georgia; TAGHI
REZAY GARACANI, Jackson, Mississippi; CHARLES H. LINCOLN, Goldsboro, N.C.;
R. S. LUTHAR, University of Wisconsin, Janesville; JAMES A. PARSLEY, Oak
Ridge, Tennessee; BOB PRIELIPP, The University of Wisconsin-Oshkosh;
KENNETH M. WILKE, Topeka, Kansas; RANDALL J. SCHEER, SUNY at Potsdam.
N. Y. and the Proposer.

About half of the solutions offered were by straightforward, hammer-and-tongs algebraic manipulation, while the rest were essentially geometric.

428. [Fall 1978] Proposed by Solomon W. Golomb, University of Southern California.

One circle of radius \(a\) may be "exactly surrounded" by six circles of radius \(a\). It may also be exactly surrounded by \(n\) circles of radius \(\bar{t}\), for any \(n\geq 3\), where

\[
t = a(\cos \frac{n\pi}{n-1})^{-1}.
\]

Suppose instead we surround it with \(n+1\) circles, one of radius \(a\) and \(n\) of radius \(\bar{b}\) (again \(n\geq 3\)). Find an expression for \(b/a\) as a function of \(n\).

What about \(n = 4\) and \(n = 6\) as individual special cases?

Solution by the Proposer.

When a circle of radius \(\bar{a}\) is exactly surrounded by \(n\) circles of radius \(\bar{t}\) and one of radius \(\bar{a}\), then \(\bar{t} = \Theta/(n-1)\bar{a}\) where \(\Theta = \cos \frac{1}{(\bar{a}+\bar{b})}\) and \(\bar{t} = \sin^{-1}(\frac{\bar{b}}{\bar{a}+\bar{b}})\). Let \(\frac{\bar{a}}{\bar{a}+\bar{b}} = \alpha\) and \(\frac{\bar{b}}{\bar{a}+\bar{b}} = \beta\), with \(\alpha + \beta = 1\).

Then

\[
\cos \left(\frac{(n-1)\sin^{-1}\beta}{\bar{t}}\right) = \cos \left[\pi - \cos^{-1}(1-\beta)\right] = \beta - 1.
\]

It is a polynomial of degree \(n-1\) in \(\cos \bar{s} = \tan \frac{1}{(n-1)}\). Thus, \(\beta\) is a root of a polynomial \(f(x)\) of degree \(m \leq 2(n-1)\), for all \(n\geq 3\), and \(b/a = \beta/a\) is a root of \((1+x)^{m} f(\frac{x}{1-x})\) which also has degree \(m \leq 2(n-1)\).

For \(n = 3, 4, \text{ and } 6\), the following table shows the polynomial \(g_{n}(x)\) and its root which equals \(b/a\) for that value of \(n\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>polynomial (g_{n}(x))</th>
<th>root (b/a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(x^{2}-3a^{2})</td>
<td>3.56155...</td>
</tr>
<tr>
<td>4</td>
<td>(9x^{4}-8x^{3}-10x^{2}+1)</td>
<td>1.56849...</td>
</tr>
<tr>
<td>6</td>
<td>(25x^{8}-188x^{7}+236x^{6}+436x^{5}-2x^{4}-180x^{3}-68x^{2}+4x+1)</td>
<td>0.73403...</td>
</tr>
</tbody>
</table>

(See also Technology Review, vol. 81, no. 2, November, 1978, p. 84).

Three incorrect solutions were received: all based on the erroneous assumption that the common internal tangent of the unequal surrounding circles passed through the center of the inner circle.

429. [Fall 1978] Proposed by Richard S. Field, Santa Monica, California.

Let \(P\) denote the product of \(n\) random numbers selected from the interval \(0 \text{ to } 1\). Question: Is the expected value of \(P\) greater or less than the expected value of the \(n\)-th power of a single number randomly selected from the interval \(0 \text{ to } 1\)?

Solution by M. S. Klamkin, University of Alberta.

Assuming the random numbers are independently and identically distributed variables, we show it is less. Let the probability density function be \(p(x)\), then we have to show that

\[
\int_{0}^{1}p(x)\,dx \leq \left(\int_{0}^{1}p(x)\,dx\right)^{n}
\]

\[
\int_{0}^{1}p(x)\,dx \leq \left(\int_{0}^{1}p(x)\,dx\right)^{n}
\]

\[
\int_{0}^{1}p(x)\,dx \leq \left(\int_{0}^{1}p(x)\,dx\right)^{n}
\]

"Diagram for \(n = 6\), with \(b/a = 0.73403...\)"
More generally it is known that if \( u(x) \) is a convex function, then

\[
E[u(X)] \geq u(E(X))
\]

(W. Feller, *An Introduction to Probability Theory and its Application*, J. Wiley, N.Y., 1971, p. 153) and which follows immediately from Jensen's inequality on convex functions. (1) corresponds to the special case \( u(x) = x^n \) and there can be no equality if \( n > 1 \). Also, it is known (Holder's inequality) that

\[
\left( \frac{1}{p} \int |X|^p \, dx \right)^{1/p} \left( \frac{1}{q} \int |X|^q \, dx \right)^{1/q} \geq \int |X| \, dx
\]

where \( p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \), and the integrals exist (loc. cit., p. 155). (1) is also a special case of (2) and (2) can also be proved by means of Jensen's inequality.

Also solved by SAMUEL GUT, Brooklyn, New York; DONALD CANARD, Anchorage, California; J. WALKER, General Hospital, Los Angeles, California; ZELDA KATZ, Beverly Hills, California, and the Proposer.

**Comment:** If \( A \) denotes the product of \( n \) randomly selected numbers from 0 to 1 and \( B \) the \( n \)-th power of a single randomly selected number from the same interval, then

\[
A: \quad E \left[ \prod_{k=1}^{n} X_k \right] = 1/2^n
\]

\[
B: \quad E(X^n) = \frac{1}{n} \int_0^1 x^n \, dx = 1/(n+1)
\]

For \( n = 1 \), of course, the results are the same. As \( n \) grows, the difference is quite startling. For example, for \( n = 10 \), \( A \) gives 1/1024 while \( B \) gives 1/11.


Given any rectangle. Form a new rectangle by adding a square to the long side. Repeat. What is the limit of the long side to the short side?

Solution by Clayton W. Dodge, University of Maine at Orono.

If we accept the existence of such a limit, then let the limiting rectangle have sides \( r \) and 1 with \( r > 1 \). Then \( r \) is the ratio we seek, and by appending a square to the long side we obtain another rectangle with the same ratio of dimensions, that is,

\[
\frac{r}{1} = \frac{r + 1}{r}, \quad \text{so} \quad r^2 - r - 1 = 0.
\]

Thus

\[
r = 1 + \frac{1}{2} \sqrt{5} \quad \text{and, since } r \text{ is positive, } r = \frac{1 + \sqrt{5}}{2}.
\]

the golden ratio.

Since it is not customary to accept the existence of a limit without proof, we offer a second solution.

**solution II.** Let \( f_n \) denote the \( n \)-th Fibonacci number. That is,

\[
f_1 = f_2 = 1 \quad \text{and} \quad f_{n+2} = f_n + f_{n+1} \quad \text{for all positive integers } n.
\]

It is well known and easily proved by mathematical induction that

\[
f_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \quad \text{and} \quad f_n = \frac{(1 + \sqrt{5})^n}{2^n \sqrt{5}}
\]

so that we have

\[
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \frac{1 + \sqrt{5}}{2},
\]

the golden ratio. It is also easy to show that if \( a, b, c, d \) are all positive and if \( a/b < c/d \), then their mediant \((a + c)/(b + d)\) satisfies the inequality

\[
\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}.
\]

It follows that if \( a/b \) and \( c/d \) have a common limit, then their mediant has this same limit. Now we are ready to look at the problem at hand.

Let \( a_1 \) and \( a_2 \) with \( a_1 < a_2 \) be the sides of the given rectangle \( R_1 \).

The next rectangle \( R_2 \) has sides \( a_2 \) and \( a_3 = a_1 + a_2 \). The third rectangle \( R_3 \) has sides \( a_3 \) and \( a_4 = a_2 + a_3 \), and so forth. We see that

\[
a_3 = a_1 + a_2,
\]

\[
a_4 = a_2 + a_3 = a_1 + 2a_2,
\]

and

...
and so forth. In general, it is easy to show by mathematical induction that

\[ a_n = f_{n-1}a_1 + f_{n-2}a_2. \]

Now we have

\[ \frac{a_{n+1}}{a_n} = \frac{f_{n-1}a_1 + f_{n-2}a_2}{f_{n-2}a_1 + f_{n-1}a_2}. \]

Since we have shown that \( f_{n-1}/f_{n-2} \) and \( f_{n-2}/f_{n-1} \) each have the same limit \((1 + \sqrt{5})/2\) as \( n \to \infty \), then their mediant \( a_{n+1}/a_n \) has this same limit.

Also solved by CHARLES D. ALLISON, San Pedro, California (Two Solutions); JEANETTE BICKLEY, St. Louis, Missouri; WALTER BLUMBERG, New Ugo Park, New York; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; MICHAEL W. ECKER, Scranton, Pennsylvania; VICTOR G. FESER, Mary College, Bismarck, N.V.; JACKIE E. FRITTS, Texas A&M University; PETER A. LINDSTROM, Geneva C.C., Batavia, New York; CHARLES H. LINCOLN, Goldsboro, N.C.; LÉO SAUVÉ, Algonquin College, Ottawa, Canada; ALBERT WHITE, St. Bonaventure University; KENNETH M. WILKE, Topeka, Kansas; and the Proposer.

The solutions offered by Blumberg and by Feser were based on continued fractions rather than the customary Fibonacci sequence.


In a right triangle \( \triangle ABC \), with sides \( a \), \( b \), and hypotenuse \( c \), show that \( 4(ac + b^2) \leq 5c^2 \).

Solution by Charles H. Lincoln, Goldsboro, N.C.

\[(2a - c)^2 \geq 0 \implies 4a^2 + c^2 \geq 4ac.\]

Using the Pythagorean Theorem, \( 4(c^2 - b^2) + c^2 \geq 4ac \), or

\[ 5c^2 \geq 4ac + 4b^2 = 4(ac + b^2). \]

Equality occurs when \( 2a = c \), that is, when angle \( A = 90^\circ \).

Also solved by CHUCK ALLISON, San Pedro, California; JEANETTE BICKLEY, Webster Groves High School, Missouri; WALTER BLUMBERG, Forest Hills, New York; BILL BURNS, Seton Hall University, South Orange, New Jersey; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CHARLES R. DIMINNIE, St. Bonaventure University, New York; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Scranton, Pennsylvania; MARK EVANS, LaHague, Texas; ROBERT FULLER, Armstrong State University, Savannah, Georgia; TAGHIZ REZAY GARACONI, Jackson State University, Jackson, Mississippi; SAMUEL GUT, Brooklyn, New York; SUSAN HOFFMAN, Iona College, Larchmont, New York; R. S. LUTHAR, University of Wisconsin, Janesville, Wisconsin; JAMES A. PARSLY, Oak Ridge, Tennessee; BOB PRIELIPP, The University of Wisconsin-Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the Proposer.

Comment: To encourage participation in the Problem Department, I intentionally chose to offer this extremely simple problem. From time to time, similar "loss leaders" will appear in order to entice new customers into the store. The simplicity of a problem proposal is not to be construed as a reflection on the mathematical calibre of the proposer or the editor.


Does there exist an integer \( n \) for which the equation

\[ \sum_{x=0}^{m} 3^{mx} = n^x \]

has a solution in positive integers?

Solution by WALTER BLUMBERG, Forest Hills High School, Forest Hills, New York.

Suppose for some integer \( m \) the equation has positive integral solutions \( x \) and \( y \). Obviously \( m \neq 0 \). Hence \( m \geq 1 \), in which case the trivial inequality \( (m + 1) \leq 2m \) follows. From the equation we have \( 3^m x < n^y \leq 3^y \). Therefore \( m < 2y \). Now \( (3^{2y} - 1) \sum_{x=0}^{m} 3^{mx} \equiv 0 \pmod{3^{2y}} \). Since 3 is a primitive root \( \pmod{3^{2y}} \), it follows that \( x(m + 1) = 0 \pmod{6(y - 1)} \), where \( 6(y - 1) \) is the totient of \( 3^{2y} \). Consequently, \( 6(y - 1) \leq x(m + 1) \). But \( 6(y - 1) \geq 1 + 6(y - 1) = 6y - 5 \).

Recalling some previous inequalities, \( (m + 1) \leq 2m \) and \( m < 2y \), we now have the following:

\[ 6(6y - 5) \leq 6(y - 1) \leq x(m + 1) \leq 2mx < 4y. \]

Thus \( 6(6y - 5) < 4y \), which leads easily to the contradiction \( y < 15/16 < 1 \). Hence the given
equation has no solutions in positive integers.

A similar solution was offered by the Proposer, ERWIN JUST. DALE WATTS, Colorado Springs, Colorado, obtained the same result by taking
\[ \log_z y = x. \]
Then since \( \log_7 3 = \log_{10} 3 / \log_{10} 7 \), he ultimately achieved the contradiction 10^2 = 3, which has no integral solutions for \( y > 0 \).

433. [Fall 1978] Proposed by Clayton W. Dodge, University of Maine at Orono.

Pay this bill for four. That is, solve for BILL, which is divisible by 4.

PAY
\[ \text{BILL} \]

1. Solution by Charles W. Trigg, San Diego, California.

Immediately, \( P = 9, T = 0, \) and \( B = 1. \) Then
\[ 2Y = L + k, \]
where \( k = 0 \) or \( 1 \), and
\[ A + M + k = L = 10. \]

Now if BILL is divisible by 4, so is \( LL \). Hence \( L = 4 \) or 8.

But, if \( L = 8 \), then \( Y \) or \( A \) or \( M \) = 9, a duplication. Hence \( L = 4 \) and \( Y = 2 \) with \( (A, M) = (6, 8) \), or \( Y = 7 \) with \( (A, M) = (5, 8) \). Thus there are four solutions:

<table>
<thead>
<tr>
<th>962</th>
<th>982</th>
<th>957</th>
<th>987</th>
</tr>
</thead>
<tbody>
<tr>
<td>82</td>
<td>62</td>
<td>87</td>
<td>57</td>
</tr>
<tr>
<td>1044</td>
<td>1044</td>
<td>1044</td>
<td>1044</td>
</tr>
</tbody>
</table>

but only one value of the BILL.

11. Computer Solution by Jeanette Bickley, St. Louis, Missouri.

Editor's Note. This program and print-out was submitted as a gracious response to my request. Ms. Bickley used a Digital Equipment Corporation PDP 11/70 computer, with the program written in Basic-Plus.

Ms. Bickley writes: "We know that \( P = 9, B = 1 \) and \( L = 0. \) The program considers all other possible digit choices for each of the other letters. For those choices that satisfy the addition, it checks whether the sum is divisible by 4. If all conditions of the problem are satisfied, the BILL is printed."

Also solved by MARK EVANS, LaMarque, Texas; DOUGLAS JUNGREIS (Age 121), Brooklyn, New York; CHARLES H. LINCOLN, Goldsboro, N.C.; and the Proposer.

Incomplete solutions were received from BILL BURNS, Seton Hall University, South Orange, N. J.; LOUIS H. CAIRLOI, Kansas State University, Manhattan, Kansas; DAVID DEL SESTO, North Providence, Rhode Island; MICHAEL W. ECKER, Scranton, Pennsylvania; VICTOR FESER, Mary College, Bismarck, N.D.; SUSAN HOFFMAN, Iona College, Larchmont, N.Y.; SUSAN IWANSKI, Greenlawn, New York; PETER A. LINDESTROM, Genesee Community College, Batavia, N. Y.; ALBERT WHITE, St. Bonaventure University, Nv York; KENNETH M. WILKE, Topeka, Kansas; and the Proposer.
Editor's Note: The preponderance of incomplete solutions provides a rebuttal to the contention that cryptarithms are a trivial form of mathematical activity.

434. [Fall 1978] Proposed by Sidney Penner, Bronx Community College of the City University of New York.

Consider \((2n + 1)^2\) hexagons arranged in a "diamond" pattern, the \(k\)-th column from the left and also from the right consisting of \(k\) hexagons, \(1 \leq k \leq 2n + 1\). Show that if exactly one of the six hexagons adjacent to the center hexagon is deleted then it is impossible to tile the remaining hexagons by trominoes as in Figure 2. (Figure 1 illustrated the \(5^2\) case in which each of the hexagons adjacent to the center one is labeled \(A\).)

Solution by Clayton W. Dodge, University of Maine at Orono.

Color the center hexagon red (\(R\)) and then color each remaining hexagon red, green (\(G\)), or blue (\(B\)) so that no two adjacent hexagons have the same color. By the symmetry of the figure the number of \(B\)'s is equal to the number of \(G\)'s. That is, if the figure is reflected in its horizontal line of symmetry (through the center of the center hexagon), each \(G\) hexagon is reflected to a \(B\) one and vice versa. Also, all "A" hexagons are either \(B\) or \(G\), so deleting one of them leaves the figure with different numbers of blue and green hexagons. Since each tromino must cover three hexagons of all three different colors, any figure exactly coverable by such trominoes must have the same number of hexagons of each color. Thus, deleting an "A" hexagon leaves the figure not coverable.

Also solved by VICTOR G. FESER, Mary College, Bismarck, N.D.; and the Proposer.


Two non-congruent triangles are "almost congruent" if two sides and three angles of one triangle are congruent to two sides and three angles of the other triangle. Clearly two such triangles are similar. Show that the ratio of similarity \(k\) is such that \(\phi^{-1} < k < \phi\), where \(0 \equiv (1 + \sqrt{5})/2\), the familiar Golden Ratio.

Editor's Note: This old problem is being reopened with the hope of eliciting fresh insights.

Solution by Clayton W. Dodge, University of Maine at Orono.

Let \(a > b > c\) be the sides of the smaller triangle. Then

\[ r = \frac{a}{b} = \frac{b}{c}. \]

From the triangle inequalities

\[ a < b + c \quad \text{and} \quad a + b > c, \]

we obtain

\[ \frac{a}{b} < \frac{b}{c} \quad \text{and} \quad \frac{a}{b} + \frac{b}{c} > \phi, \]

\[ r < 1 + \frac{1}{r} \quad \text{and} \quad r + 1 > \frac{1}{r}, \]

and finally

\[ r^2 - r - 1 < 0 \quad \text{and} \quad r^2 + r - 1 > 0. \]
Replace the inequality signs by equality signs and solve the two quadratic equations for their positive roots since \( r \) must be positive.

We obtain

\[
\frac{-1 + \sqrt{5}}{2} = \frac{2}{1 + \sqrt{5}} \quad \text{and} \quad \frac{1 + \sqrt{5}}{2} = \phi.
\]

The two inequalities are satisfied, for positive \( r \), when

\[
\frac{1}{2} < r < \frac{1 + \sqrt{5}}{2} = \phi.
\]

Also solved by WALTER BLUMBERG, New Hyde Park, N. Y.; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; MICHAEL W. ECKER, Pennsylvania State University; KENNETH M. WILKE, Topeka, Kansas; and the Proposer.

Editor's Note: This problem has been treated several times in the Mathematics Teacher (April 1953, pp. 295-296; December 1954, pp. 561-562; March 1977, pp. 253-257). A related problem, proposed by Victor Thébault in the May 1954 issue of the American Mathematical Monthly (Problem E 1117), considered the construction of a right triangle in which the legs and the altitude on the hypotenuse can be taken as the sides of another right triangle.


\( P_1 \) and \( P_2 \) are distinct points on lines \( L_1 \) and \( L_2 \) respectively. Let \( L_1 \) and \( L_2 \) rotate about \( P_1 \) and \( P_2 \) respectively with equal angular velocities. Describe the locus of their intersection.

Solution by CLAYTON W. DODGE, University of Maine at Orono.

Let \( X \) be the point of intersection of the two lines and let \( Q \) be any point so that \( P_2 \) is between \( P_1 \) and \( Q \). Denote angles \( \angle QP_1 X \) and \( \angle QP_2 X \) by \( \alpha \) and \( \beta \) and angle \( \angle XP_1 P_2 \) by \( \gamma \). If \( k \) is the angular velocity, then there are constants \( \alpha_1 \) and \( \beta_1 \) such that

\[
\alpha = kt + \alpha_1 \quad \text{and} \quad \beta = kt + \beta_1.
\]

Then we have

\[
\gamma = \beta - \alpha = \beta_1 - \alpha_1,
\]

a constant. Hence \( X \) describes a circle through \( P_1 \) and \( P_2 \) such that \( \angle XP_1 P_2 \) intercepts an inscribed angle of \( \beta_1 - \alpha_1 \). The center of the circle is easily constructed by drawing the line \( L \) through \( P_2 \) having \( \beta = \beta_1 - \alpha_1 \). The center is then at the intersection of the perpendicular to \( L \) at \( P_2 \) and the perpendicular bisector of \( P_1 P_2 \).

Also solved by CHUCK ALLISON, San Pedro, California; CHARLES H. LINCOLN, Goldsboro, N. C.; R. S. LUTHAR, University of Wisconsin-Janesville; and the Proposer.


In times gone by, it was fairly well-known that \( N \), the Nagel point of a triangle, is the intersection of the lines from the vertices to the points of contact of the opposite escribed circles. In the triangle whose sides are \( AB = 5 \), \( BC = 3 \), and \( CA = 4 \), show that the areas of triangles \( ABN \), \( CAN \), and \( BCN \) are 1, 2 and 3 respectively.

Solution by the Proposer.

Let \( D \), \( E \), \( F \) denote the traces of the Cevians through \( N \), the Nagel point, on the sides \( BC \), \( CA \), \( AB \) of the 3:4:5 right triangle \( ABC \). Then

\[
\frac{\triangle ABN}{\triangle ABC} = \frac{CE}{EA} = \frac{(s-c)/(s-a)}{3/1}.
\]

\[
\frac{\triangle CAN}{\triangle ABC} = \frac{CD/DB}{(s-b)/(s-c)} = \frac{2/1}{2/3}.
\]

\[
\frac{\triangle BCN}{\triangle ABC} = \frac{AF/FB}{(s-b)/(s-a)} = \frac{2/3}{2/3}.
\]

The required ratios follow easily.
Locate a point \( P \) in the interior of a triangle such that the product of the three distances from \( P \) to the sides of the triangle is a maximum.

Amalgam of solutions by Walter Blumberg and the Proposer.

Let \( x, y, z \) be the distances from an interior point of the triangle to sides \( a, b, c \) respectively. Using the arithmetic mean–geometric mean inequality, we have

\[
3 \sqrt{(ax)(by)(cz)} \leq \frac{(ax + by + cz)}{3} = \frac{2K}{3},
\]

where \( K \) is the area of the triangle. Then \( xyz \leq \frac{8K^3}{27abc} \), with equality if and only if \( ax = by = cz \). It is known that \( ax = by = cz \) only if \( P \) is the centroid of triangle \( ABC \). Consequently this is the desired point.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; M. S. KLAMKIN, University of Alberta, Canada; CHARLES H. LINCOLN, Goldsboro, N. C.; SISTER STEPHANIE SLOYAN, Georgian Court College, Lakewood, N. J.; and ROD WOODBURY, San Pedro, California.

Two incorrect solutions were received.

Editor’s Note: This triangle inequality was proposed by L. Carlitz in the January 1964 issue of the Mathematics Magazine, with a solution by J. A. Tyrrell published in September 1964 (page 279). This reference was located in Geometric Inequalities, by Bottema et al., item 12.29, page 112.