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FRENET FORMULAS IN N-DIMENSIONS AND SOME APPLICATIONS

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Discussion. During our freshman calculus course, we encounter briefly the notion of the curvature function of a space curve. curvature, as defined in most standard calculus texts, is the rate at which the direction of the unit velocity vector $\overline{T}$ of a particle changes or turns as it moves along the arc of a curve. In mathematical notation, we write $||d\overline{T}/ds|| = k(s)$, s the arc length parameter of the curve. From the fact that $<\overline{T}, d\overline{T}/ds> = 0$, we have (i) &/& $= k(s)\overline{N}$, where $\overline{N}$ is the unit normal vector of the curve. Since $<\overline{N}, d\overline{N}/ds> = 0$ and knowing that $0 = d(<T,N>)/ds = <\overline{T}\cdot d\overline{N}/ds> + <d\overline{T}/ds, \overline{N}>$, we see that (ii) $d\overline{N}/ds = -k(s)\overline{T}$. (i) and (ii) are called the Frenet equations (named after Jean-Frederic Frenet (1816–1900) French mathematician) for plane curves. The Frenet formulas are of great importance in the study of curves in differential geometry and in this paper, we give a generalization of the formulas to higher dimensional Euclidean space curves, and use them to prove two interesting theorems in Euclidean differential geometry.

Theorem 1. $I \subset R^1$ $X: I \to R^n$ a C^∞ map with arc length parametrization. We assume that at each point of $X(s)$, the vectors

$$X'(s),\ X''(s), \ldots, X^{[n-1]}(s)$$

are linearly independent. Then we can find an orthonormal set of linearly independent vectors, called the Frenet n-frame

$$V_1(s), V_2(s), \ldots, V_n(s)$$

satisfying the relations:

$$V'_1 = k_1(s)V_2$$

$$V'_i = -k_{i-1}(s)V_{i-1} + k_i(s)V_{i+1} \qquad 2 \le i \le n-1$$

$$V'_n = -k_{n-1}(s)V_{n-1}$$

We call the above relations the generalized Frenet formulas and $k_i(s)$ denote the ith curvature of $X(s)$. In the special case $n = 3, k_2$ is also known as the torsion of $X(s)$.

Proof. We apply the Gram-Schmidt orthogonalization process to vectors $X^{[r]}(s)$ $1 \le r \le n-1$.

Thus

$$E_1(s) = X'(s)\ ,\quad V_1(s) = E_1(s)/||E_2(s)||$$

$$E_i(s) = X^{[i]}(s) - \sum_{k<i} \left[\langle X^{[i]}(s),\ V_k(s)\rangle\right]\ V_k(s)$$

$$V_i(s) = E_i(s)/||E_i(s)|| \qquad 2 \le i \le n-1$$

$$V_n(s) = \text{the unique vector satisfying } \langle V_n, V_i\rangle = 0$$

$$1 \le i \le n-1\ .$$

It follows that $\langle V_i, V_j\rangle = \delta_{ij}$, the delta function.

Hence $$\langle V_i, V'_j\rangle + \langle V_i, V_j\rangle = 0 \qquad (*)$$

Also, each V_i is a linear combination of $V_1,\ldots,V_n$ as can be seen by taking $E'_i(s)$ and noting that $X^{[i+1]}(s)$ is a linear combination of $V_1,\ldots,V_{i+1}$ $1 \le i \le n-1$.

So let $V'_i = \sum_{s=1}^{i+1} a_{is}V_s$, $V'_j = \sum_{t=1}^{j+1} a_{jt}V_t$.

From (*), we get

$$0 = \langle (\sum_{s=1}^{i+1} a_{is}V_s),\ V_j\rangle + \langle (\sum_{t=1}^{j+1} a_{jt}V_t),\ V_i\rangle$$

which imply that

$$a_{ij} + a_{ji} = 0 \qquad \text{for all } i,j$$
$$a_{ij} = 0 \qquad j < i-1$$
$$a_{ji} = 0 \qquad j > i+1$$

Hence

$$V'_i = a_{ii-1}V_{i-1} + a_{ii+1}V_{i+1}$$
$$= -k_{i-1}(s)V_{i-1} + k_i(s)V_{i+1}$$

$$k_i(s) = a_{ii+1} = -a_{i+1i} \qquad 1 \le i \le n-1$$

Finally, V'_n = linear combination of $V_1,\ldots,V_n$ implies that $V'_n = -k_{n-1}(s)V_{n-1}$.

For our first application, we will show that curvature can be interpreted as the rate of turning of osculating hyperplanes. We saw before that in the two-dimensional case, there is only one curvature, and it is determined by the rate of turning of the tangent vector, the hyperplane of one one dimension. To generalize this, we first need some definitions and lemmas.

Definition 1. A p-plane, denoted Π, is a subspace spanned by p linearly independent vectors in R^n. In particular, if the p linearly independent vectors are $V_1(s),\ldots,V_p(s)$ then it is termed the osculating p-plane of $X(s)$ at s.

In the following lemma, we arrive at a definition of the angle between two p-planes.

Lemma 1. M^p, N^p two p-planes. Let $\{u_1,\ldots,u_p\}$ and $\{v_1,\ldots,v_p\}$ be bases for M^p and N^p respectively. Consider the p-vectors $u = u_1\wedge u_2\wedge \ldots \wedge u_p$ a $v = v_1\wedge\ldots\wedge v_p$, where $\wedge$ denote the exterior product in R^n. Then the angle between M^p and N^p is equal to the unique angle ϕ between u and v, satisfying

$$\cos\phi = \frac{\langle\langle u,v\rangle\rangle}{\sqrt{\langle\langle u,u\rangle\rangle}\sqrt{\langle\langle v,v\rangle\rangle}}$$

$$\frac{|\det(\langle u_i,v_j\rangle)|}{\sqrt{\det(\langle u_i,u_j\rangle)}\ \sqrt{\det(\langle v_i,v_j\rangle)}}$$

where $\det(\langle u_i,v_j\rangle)$ denote the determinant whose entries in the (i,j) cell is $\langle u_i,v_j\rangle$.

Observe that this is a natural generalization of the well-known cases $p = 1$ and $p = 2$, where $\langle\langle u,v\rangle\rangle$ is the usual Euclidean dot product and cross product respectively. The proof involves the notion of a reduction factor for p-dimensional measure under orthogonal projection between two p-planes, details of which the reader is referred to [2] pp. 1051-3.

Lemma 2. Let $S = [s_o,s] \subset I$, $T:S \to R^n$, a C^1 map such that $||T(s)|| = 1$ for each $s \in S$. Let $\theta(s)$ denote the angle between $T(s_o)$ and $T(s)$, then $\theta'(s_o) = ||T'(s_o)||$. (In taking the derivative, we always choose the direction of increasing θ, hence $\theta'(s) \ge 0$).

Proof.

$$<T(s) - T(s_o), T(s) - T(s_o)> = 2 - 2<T(s), T(s_o)>$$

$$= 2 - 2\cos\theta(s) = 4\left(\sin\frac{\theta}{2}\right)^2$$

Hence

$$||T(s) - T(s_o)|| = 2\sin\frac{\theta}{2}$$

Futhermore

$$||T'(s_o)|| = \lim_{s_o\to s} \left|\left|\frac{T(s) - T(s_o)}{s - s_o}\right|\right| = \lim_{s_o\to s} \frac{2\sin\frac{\theta(s)}{2}}{s - s_o}$$

$$= \lim_{s_0\to s} \frac{6(s)}{s - s_o}\frac{\sin\frac{\theta(s)}{2}}{2} = \theta'(s_o)$$

Theorem 2. Let $\theta(s)$ denote the angle between the osculating p-planes $M^p(s)$, $N^p(s)$. Then $\theta'(s) = k_p(s)$, the pth curvature of $X(s)$.

Proof.

Consider the p-vector

$$T_p(s) = V_1(x)\wedge\ldots\wedge V_p(s)$$

$||T_p(s)|| = 1$ since $\ll T(s), T(s)\gg = \det(<v_i, v_j>) = \det(pxp$ identity matrix$) = 1$.

Using the fact that the product rule for differentiation holds for the exterior product in R^n, we have

$$T'_p(s) = \sum_{i=1}^{p} V_1\wedge\ldots\wedge V_{i-1}\wedge V'_i\wedge V_{i+1}\ldots\wedge V_p$$

Substituting the Frenet formulas for V'_i and applying the anti-symmetry axiom $V_i\wedge V_j = -V_j\wedge V_i$,

$$T'_p(s) = V_1\wedge\ldots\wedge V_{p-1}\wedge V'_p$$

$$= k_p(s)\, V_1\wedge\ldots\wedge V_{p-1}\wedge V_{p+1}$$

hence by lemma 1 and 2, $\theta'(s) = ||T'_p(s)|| = k_p(s)$.

For our final application of theorem 1, we will prove the n-dimensional analog of Fenchel's theorem (Wernel Fenchel (1905-) German mathematician), which states that if $X(s)$ is a closed unit speed curve of length L in R^3, then $\int_0^L|k_1(s)|ds \geq 2\pi$ with equality if and only if $X(s)$ is a convex plane curve. The remarkable fact is that the same theorem is

true with R^3 replaced by R^n, and this is the content of our next theorem.

Theorem 3. Let $X(s)$ be a C^∞ closed curve of length L in R^n parametrized by arclength. Then $\int_0^L |k_1(s)|\,ds \geq 2\pi$ where $k_1(s)$ is the first curvature of $X(s)$. Equality occurs if and only if $X(s)$ is a convex planar curve.

Here we divert our attention for a while and consider the Gauss map $G:S\to S^{n-1}(1)$, the unit hypersphere of dimension n-1. Then $G(s) = V_1(s)$, the tangent vector to $X(s)$. The image curve, called the tangent indicatrix, has length

$$\lambda = \int_0^L||\,V'_1(s)||\,ds \quad .$$

But $V'_1(s) = k_1(s)V_2$, hence $\lambda = \int_0^L |\,k_1(s)|\,ds$

Thus we have to show that $\lambda \geq 2$. We will prove this using integral geometry and to do so, we need to n-dimensionalized an integral formula due to Crofton [4].

Lemma 3. Let g denote a great circle and A a curve on S^{n-1}. Then

$$\int n(\Delta\cap g)\,dg = \frac{\lambda}{\pi}A(S^{n-1})$$

where λ is the length of A, $A(S^{n-1})$ the surface content of S^{n-1}, and $n(\Delta\cap g)$ the number of intersections of A with g.

Proof

Let $e_1(\sigma),\ldots,e_n(\sigma)$ be a Frenet n-frame for A with arclength σ. Then $d\sigma = k_1(s)ds$, s the arclength of g, and the Frenet formulas for A are:

$$de_1/d\sigma = e_2 \quad (\text{note } k_1(\sigma) = 1)$$

$$de_i/d\sigma = -k_{i-1}(\sigma)e_{i-1} + K_1(\sigma)e_{i+1} \quad 1\leq i\leq n-1$$

$$de_n/d\sigma = -k_{n-1}(\sigma)e_{n-1}$$

Next, using generalized polar coordinates [3], we can describe the pole of a great circle g as

$$P = \sum_{i=2}^{n}\cos\alpha_{n-2}\cos\alpha_{n-1}\cdots\cos\alpha_{i-1}\sin\alpha_{i-2}e_i = \sum_{i=2}^{n}a_ie_i$$

$$0\leq\alpha_1\leq 2\pi,\quad 0\leq\alpha_i\leq\pi,\quad 2\leq i\leq n-2 .$$

Observe that

$$\frac{\partial P}{\partial x_i} = \cos\alpha_{n-2}\cdots\cos\alpha_{i+1}(\cos\alpha_i e_{i+2} - \sin\alpha_i \sin\alpha_{i-1} e_{i+1}\cdots - \sin\alpha_i\cdots\cos\alpha_1 e_2.$$

$$= \cos\alpha_{n-2}\cdots\cos a_{i+1} W_i \qquad 1 \le i \le n-2 .$$

Clearly, the W. form an orthonormal set of vectors as i varies from 1 to $n-2$. We wish to find the element of area $n(\Delta \cap g)dg$ of the great circles. Differentiating P

$$dP = \sum_{i=1}^{n-2} \frac{\partial P}{\partial x_i} d\alpha_i + \sum_{i=3}^{n} a_i de_i + \cos\alpha_{n-2}\cdots\cos\alpha_1(-e_1 + k_2(\sigma)e_3)d\sigma$$

and rearranging the terms, we can express dP into a linear combination of orthogonal set of vectors V_i such that

$$dP = -\cos\alpha_{n-2}\cdots\cos\alpha_1 d\sigma e_1 + \sum_{i=1}^{n-2} \cos\alpha_{n-2}\cdots\cos\alpha_{i+1} d\alpha_i V_i$$

It follows that

$$n(\Delta \cap g)dg = \left|\text{product of coefficients of } e_1, V_1, \ldots, V_{n-2}\right|$$

$$= \left|\prod_{i=1}^{n-2} \cos^i a_i \; d\alpha_i d\sigma\right|$$

Using the known formulas $\displaystyle\int_0^\pi |\cos^i\theta \mid d\theta = \sqrt{\pi}\, \frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i}{2}+1\right)}$

and $\displaystyle A(S^{n-1}) = \frac{\sigma\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$, $\Gamma(x)$ denote the gamma function, we integrate the above expression and obtain

$$\int n(\Delta \cap g)dg = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^\lambda d\sigma \left|\prod_{i=1}^{n-2} \cos^i\alpha_i d\alpha_i\right| = \frac{\lambda}{\pi} A(S^{n-1}) .$$

Proof of Theorem 3.

Let A be a fixed unit vector in R^n . Consider the height function

$$H(s) = \langle A, X(s)\rangle$$

Since $H(s)$ is continuous and bounded, $H'(s) = \langle A, X'(s)\rangle = (A, V(s)) = 0$ has at least two solutions. But A determines one or more great circles

g in S^{n-1}, hence $n(\Delta \cap g) \ge 2$ and

$$\lambda = \frac{\pi}{A(S^{n-1})} \int n(\Delta \cap g)dg \ge \frac{2\pi}{A(S^{n-1})} \int dg = 2\pi .$$

Corollary. If A is a great circle of S^n , then $\lambda = 2\pi$ for all $n \ge 1$.

Proof. $n(\Delta \cap g) = 2$ for all g except for certain sets of measure zero which have no effect on the integral.

Next, suppose $X(s)$ is a convex curve lying on a hyperplane of dimension $m<n$. Then the Gauss nap of $X(s)$ is one-one and traces out a great circle on S^{n-1}, hence $\int |k_1|\, dS = 2\pi$.

Conversely, suppose that $A = 2\pi$. Then $n(\Delta \cap g) = 2$ almost everywhere (except for sets of measure 0). Let g be a great circle which intersects A in exactly 2 points, P_1 and P_2. Claim: P_1 and P_2 bisect A into two arcs of equal length. Suppose not. Let $(P_1P_2)\Delta$ denote the longer arc of A and $(P_1P_2)g$, the shorter arc of g. Then because g is a geodesic in S^{n-1} , the curve

$$\zeta = (P_1P_2)g + (P_1P_2)\Delta$$

has length $<2\pi$. "Smoothing out" the portion of ζ at P_1 and P_2 (that is, making ζ have a continuous tanget at P_1 and P_2), we can find a closed curve ζ' in R^n whose tangent indicatrix is ζ and has length $<2\pi$, contrary to above results. Thus $(P_1P_2)\Delta = \pi = \frac{1}{2}$ (length of g). The tangents to A at P_1 and P_2 are continuous, hence A is itself a great circle. Since the Gauss map is one-one, $X(s)$ must be a convex planar curve. This completes the proof.

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DIFFERENTIABILITY AND DIRECTIONAL DERIVATIVES

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In the process of introducing the concept of differentiability for a function of several variables it is important to establish the relationships with the various notions of one-dimensional derivatives such as partial derivatives, directional derivatives or derivatives along differentiable curves. Of course differentiability implies the existence of all the one-dimensional derivatives. That the existence of the derivative along all differentiable curves passing through a point does not imply differentiability is shown by

$$(1)\qquad f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

(cf. [2], p. 240). Complete characterizations of differentiability at a point in terms of one-dimensional derivatives seem to be overshadowed by the elegant result that the existence of continuous partial derivatives in an open set is equivalent to continuous differentiability there. In order to recapture this innate simplicity in the characterization of differentiability the classical definitions need to be altered (see [1]).

The purpose of this note is to present a characterization of differentiability at a point that can be appreciated by advanced calculus students but is not mentioned by the current textbooks on the subject. For the sake of clarity we consider functions $f : \mathbb{R}^2 \to \mathbb{R}^1$ and their behavior at $\overline{0} \in \mathbb{R}^2$. The theorem given here can be established for functions on $\mathbb{R}^n$, $n > 2$, relative to an arbitrary point by simply making notational changes.

The function f given in (1) does not satisfy the chain rule formula

$$(2)\qquad (f \circ \gamma)'(0) = \gamma'(0) \cdot \nabla f(\overline{0})$$

for every straight line γ passing through $\overline{0}$. It is easy to find examples of nondifferentiable functions which satisfy (2) for every straight line passing through $\overline{0}$. However, if the existence of $(f \circ \gamma)'(0)$ for every suitable curve γ passing through $\overline{0}$ is combined with the requirement that (2) hold, then one obtains differentiability.

Theorem. A function $f : \mathbb{R}^2 \to \mathbb{R}^1$ is differentiable at $\overline{0}$ if and only if (2) holds for every curve $\gamma : \mathbb{R}^1 \to \mathbb{R}^2$ with $\gamma(0) = \overline{0}$ and $\gamma'(0)$ existing.

The necessity is a consequence of the chain rule. The proof of the sufficiency proceeds by contradiction. Assume that (2) holds for every γ of the sort under consideration but that f is not differentiable at $\overline{0}$. We first show that if T is a rotation of $\mathbb{R}^2$ then $f \circ T$ satisfies these conditions as well.

Clearly, $f \circ T$ is not differentiable. To show that $f \circ T$ satisfies (3) for every γ we note that

$$(3)\qquad \nabla(f \circ T)(\overline{0}) = T^t \nabla f(\overline{0}).$$

Indeed, the choice of $\gamma(t) = T(t\overline{i})$ in (2) gives $\frac{\partial}{\partial x}(f \circ T)(\overline{0}) = T\overline{i} \cdot \nabla f(\overline{0})$. This and the formula corresponding to the choice $\gamma(t) = T(t\overline{j})$ imply (3). If γ is one of the curves under consideration, then so is $T \circ \gamma$. Thus using (2) relative to $T \circ \gamma$ and (3) gives

$$\begin{aligned} ((f \circ T) \circ \gamma)'(0) &= (f \circ (T \circ \gamma))'(0) \\ &= (T \circ \gamma)'(0) \cdot \nabla f(\overline{0}) \\ &= (T\, \gamma'(0)) \cdot \nabla f(\overline{0}) \\ &= \gamma'(0) \cdot T^t\, \nabla f(\overline{0}) \\ &= \gamma'(0) \cdot \nabla (f \circ T)(\overline{0}). \end{aligned}$$

This establishes the claim about $f \circ T$.

The assumption that f is not differentiable at $\overline{0}$ implies that there is an $\varepsilon_0 > 0$ and a sequence of points $\{\overline{x}_n\}$ with $\lim \overline{x}_n = \overline{0}$, $\overline{x}_n \neq \overline{0}$ and

$$(4)\qquad \frac{|f(\overline{x}_n) - f(\overline{0}) - \overline{x}_n \cdot \nabla f(\overline{0})|}{\overline{x}_n} \geq \varepsilon_0,$$

for $n = 1,2,\dots$. Without loss of generality we may assume that the sector $S = \{\bar{x} = (x,y);\ 3^{-1/2} \le y/x \le 3^{1/2}\}$ contains infinitely many points of the sequence $\{\bar{x}_n\}$, since this must be true for at least one of the sectors obtained by rotating S through an angle of $\pi k/6$, $k = 1,2,\dots,11$. We choose a subsequence, again denoted by $\{\bar{x}_n\}$, of points lying in S . Set $\bar{x}_n = (x_n,y_n)$, $M_n = y_n/x_n$ and $M = \limsup M_n$. Clearly $3^{-1/2} \le M \le 3^{1/2}$. By respectedly passing to subsequences we may assume that $M = \lim M_n$ and that $\{x_n\}$ is strictly decreasing.

We define a curve $\gamma : (-1,x_1) \to \mathbb{R}^2$, $\gamma(t) = (x(t),y(t))$ by setting $x(t) = t$ and

$$y(t) = \begin{cases} Mt, & \text{if } t \in (-1,0], \\ \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}}(t - x_n) + y_n, & \text{if } t \in [x_n, x_{n-1}), \\ & n = 2,3,\dots . \end{cases}$$

We claim that γ is the sort of curve specified in the theorem. For this the only detail that needs some justification is the existence of $\gamma'(0)$. Since the derivative of y from the left is M we shall show $\lim_{t\to 0} y(t)/t = M$, as well. Note that $y(t)/t = y(t)/x(t)$ is the slope of the line joining $\bar{0}$ to $(x(t),y(t))$ and $(x(t),y(t))$ is on the line segment joining (x_{n-1},y_{n-1}) to (x_n,y_n) for a suitable n . Thus $y(t)/t$ is between the slopes M, M_{n-1} which shows that $\lim_{t\to 0^+} y(t)/t = M$.

We have $\lim_{t\to 0} (\gamma(t) - t\gamma'(0))/t = \bar{0}$ and consequently

$$(5) \qquad \lim_{t\to 0} \frac{(\gamma(t) - t\gamma'(0)) \cdot \nabla f(\bar{0})}{t} = 0$$

On the other hand, since we are assuming that f and γ satisfy (2), we have

$$(6) \qquad \lim_{t\to 0} \frac{f(\gamma(t)) - f(\bar{0}) - t\gamma'(0) \cdot \nabla f(\bar{0})}{t} = 0 .$$

Subtracting (5) from (6) gives

$$\lim_{t\to 0} \frac{f(\gamma(t)) - f(\bar{0}) - \gamma(t) \cdot \nabla f(\bar{0})}{t} = 0.$$

In particular, if we take the limit along the sequence $\{\bar{x}_n\}$, then we obtain

$$(7) \qquad \lim_{n\to\infty} \frac{|f(\bar{x}_n) - f(\bar{0}) - \bar{x}_n \cdot \nabla f(\bar{0})|}{x_n} = 0.$$

The limit in (7) continues to be 0 if x_n is replaced by the larger quantity $|\bar{x}_n|$. This contradicts (4) and completes the proof of the theorem.

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OBTAINING THE SHORTEST CONFIDENCE INTERVAL FOR AN UNKNOWN PARAMETER

by B. Crain G. Helmer, M. Hsueh, R. Jaffe and T. Kim

In this article we look at several examples of confidence interval estimation of an unknown parameter. In each example we determine how to make the interval as short as possible, and hence maximize the precision of the estimate.

Example 1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(\mu, \sigma^2)$. Assume that μ *is unknown* and σ^2 *is known.* Define the sample mean $\bar{X}$ by

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i .$$

Then it is well known that $\bar{X}$ is normally distributed with mean μ and variance σ^2/n, i.e., $\bar{X}$ is $N(\mu, \sigma^2/n)$. It follows that

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

is $N(0,1)$.

Now suppose we would like to obtain a 95 percent confidence interval for μ. We use the following process. From standard tables of the $N(0,1)$ distribution, select numbers a and b (with $a < b$) such that $Pr(a \leq Z \leq b) = .95$. The following inequalities are then equivalent:

$$Pr(a \leq Z \leq b) = .95,$$

$$Pr(\bar{X} - \frac{b\sigma}{\sqrt{n}} < \mu < \bar{X} - \frac{a\sigma}{\sqrt{n}} = .95 .$$

The *(random)* interval $[\bar{X} - b\sigma/\sqrt{n}, \bar{X} - a\sigma/\sqrt{n}]$ is called *a* 95 percent confidence interval for μ, and it contains the unknown value of μ for 95 percent of all random samples of size n.

We are concerned here with making the width of the interval as small as possible, where

$$width = (\bar{X} - \frac{a\sigma}{\sqrt{n}}) - (\bar{X} - \frac{b\sigma}{\sqrt{n}})$$

$$= \frac{\sigma}{\sqrt{n}}(b - a) .$$

Thus we must minimize $(b - a)$, subject to the condition that $Pr(a \leq Z \leq b) = .95$, where Z is $N(0, 1)$. Let $F(Z) = Pr(Z \leq z)$ be the cumulative distribution function for z. Then

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx, \quad -\infty < z < \infty ,$$

where

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} , \quad -\infty < z < \infty ,$$

is the $N(0, 1)$ probability density function. We therefore want to minimize $(b - a)$, subject to $Pr(a \leq Z \leq b) = F(b) - F(a) = .95$.

Now thinking of a as a function of b , we have

$$\frac{d}{db}[F(b) - F(a)] = 0,$$

$$F'(b) - F'(a)\frac{da}{db} = 0,$$

$$f(b) - f(a)\frac{da}{db} = 0. \tag{1}$$

We also have

$$\frac{d}{db}(b - a) = 0,$$

$$1 - \frac{da}{db} = 0 ,$$

$$\frac{da}{db} = 1 . \tag{2}$$

Equations (1) and (2) imply that

$$f(b) = \frac{1}{\sqrt{2\pi}} e^{-b^2/2} = \frac{1}{\sqrt{2\pi}} e^{-a^2/2} = f(a) . \tag{3}$$

Thus the solution is to choose $b = -a > 0$, and this yields the shortest 95 percent confidence interval for μ .

Example 2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(\mu, \sigma^2)$. Assume that μ and σ^2 are both unknown. We would like a 95 percent confidence interval for σ^2.

We have already defined the sample mean $\overline{X}$. The sample variance s^2 is defined by

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 .$$

We use the fact that $Y = (n-1)s^2/\sigma^2$ has a chi-squared distribution with $n - 1$ degrees of freedom. From the chi-squared tables we can select a and b (with $0 < a < b < \infty$) such that $Pr(a \le Y \le b) = .95$. The following inequalities are then equivalent:

$$Pr(a \le Y \le b) = .95 ,$$

$$Pr\left(\frac{(n-1)s^2}{b} < \sigma^2 < \frac{(n-1)s^2}{a}\right) = .95 .$$

The random interval $[(n-1)s^2/b, (n-1)s^2/a]$ is a 95 percent confidence interval for the unknown value of σ^2, and 95 percent of all such intervals will contain σ^2.

Now the *width* of the interval is

$$width = \frac{(n-1)s^2}{a} - \frac{(n-1)s^2}{b}$$

$$= (n-1)s^2\left(\frac{1}{a} - \frac{1}{b}\right) .$$

Using the fact that $(n-1)s^2/\sigma^2$ is chi-squared with $n - 1$ degrees of freedom, and the expected value of a chi-square random variable is equal to its degrees of freedom, we have that

$$E[width] = E\left[(n-1)s^2\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

$$= E\left[\frac{(n-1)s^2}{\sigma^2}\right]\sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= (n-1)\,\sigma^2\left(\frac{1}{a} - \frac{1}{b}\right) .$$

We therefore wish to minimize $((1/a) - (1/b))$, subject to $Pr(a \le Y \le b) = .95$.

Now let $G(y)$ be the cumulative distribution function for Y, that is, $G(y) = Pr(Y \le y)$, and let $g(y) = G'(y)$ be the corresponding probability density function for Y. We wish to minimize $((1/a) - (1/b))$, subject to $Pr(a \le Y \le b) = G(b) - G(a) = .95$. Again, thinking of a as a function of b, we have

$$\frac{d}{db}[G(b) - G(a)] = 0 ,$$

$$G'(b) - G'(a)\frac{da}{db} = 0 ,$$

$$g(b) - g(a)\frac{da}{db} = 0 . \qquad (4)$$

In addition, we have

$$\frac{d}{db}\left(\frac{1}{a} - \frac{1}{b}\right) = 0 ,$$

$$-\frac{1}{a^2}\frac{da}{db} + \frac{1}{b^2} = 0,$$

$$\frac{da}{db} = \frac{a^2}{b^2} . \qquad (5)$$

Combining equations (4) and (5) we have

$$g(b) = \frac{a^2}{b^2}\, g(a) ,$$

$$b^2\, g(b) = a^2\, g(a) . \qquad (6)$$

Now the actual algebraic form of $g(x)$ is

$$g(x) = \frac{x^{\frac{n-1}{2} - 1}\, e^{-\frac{x}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} , \quad 0 \le x \le \infty ,$$

and zero elsewhere. Thus we should try to choose a and b such that

$$b^2 b^{\frac{n-1}{2}-1} e^{-\frac{b}{2}} = a^2 a^{\frac{n-1}{2}-1} e^{-\frac{a}{2}}, \tag{7}$$

$$b^{\frac{n+3}{2}-1} e^{-\frac{b}{2}} = a^{\frac{n+3}{2}-1} e^{-\frac{a}{2}},$$

$$\frac{b^{\frac{n+3}{2}-1} e^{-\frac{b}{2}}}{\Gamma(\frac{n+3}{2})2^{\frac{n+3}{2}}} = \frac{a^{\frac{n+3}{2}-1} e^{-\frac{a}{2}}}{\Gamma(\frac{n+3}{2})2^{\frac{n+3}{2}}} \tag{8}$$

Equation (8) indicates that we choose a and b such that $h(a) = h(b)$, where $h(x)$ is the probability density function for a chi-squared random variable with $n + 3$ degrees of freedom. Condition (7) can also be expressed as

$$b^{\frac{n+1}{2}} e^{-\frac{b}{2}} = a^{\frac{n+1}{2}} e^{-\frac{a}{2}},$$

$$b^{n+1} e^{-b} = a^{n+1} e^{-a}. \tag{9}$$

The actual determination of a and b via (9) would have to be done numerically, i.e., using the computer. This task has in fact been performed by *Lindley, East and Hamilton* (see Reference [2]). They have prepared a table giving the appropriate values of a and b for degrees of freedom varying from 1 to 100 (sample size from 2 to 101).

We have discussed the problem of finding the shortest possible confidence interval for an unknown parameter in two specific cases. The method of this article is quite general, and should apply to other cases as well.

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TAXICAB GEOMETRY: ANOTHER LOOK AT CONIC SECTIONS

By *David Iny*
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The following article is written in response to [1] Moser and Kramer. They attempt to answer a question posed by [2] Reynolds.

The article by [2] Reynolds defines the taxicab distance between (a_1,a_2) and (b_1,b_2) by $d(A,B) = |a_1-b_1| + |a_2-b_2|$. She then deduces the nature of circles, ellipses, and hyperbolas using definitions analogous to those of Euclidean geometry.

The article by [1] Moser and Kramer defines a parabola as the locus of points equidistant from a focus (x_0,y_0) and a line (the directrix) of the form $\{(x,y) \mid Ax + By + C = 0\}$. The approach taken is not entirely satisfying since they do not attempt to justify their definition of a line.

In Euclidean geometry, a line is defined as the locus of points in a plane equidistant from two distinct points. As [2] Reynolds points out, this locus does not necessarily take the form $\{(x,y) \mid Ax + By + C = 0\}$.

For the purposes of this paper, we define a line to be the locus of points equidistant from two distinct points. We also define the distance between a point P and the line l as the minimum of $d(Q,P)$ where Q is any point on l. We define a parabola as the locus of points equidistant from a focus and a line (the directrix).

In Figure 1, the line ℓ is equidistant from $A(-2,4)$ and $B(2,2)$. The diagram pictures the parabola with focus $(5,4)$ and directrix l. Two more parabolas are shown in Figures 2 and 3. Note that in Figure 3 our definition of a line agrees with [1] Moser and Kramer. Therefore, the reader should not be surprised that of the three parabolas shown, only the one in Figure 3 is a parabola according to [1] Moser and Kramer.

In [2] Reynolds defines an ellipse by $\{P \in R_2 \mid d(P,A) + d(P,B) = c\}$ where A and B are two fixed points (foci) and a is a constant. For a description of such ellipses, the reader is referred to the article [2] quoted above.

Let us define an ellipse of the second kind with respect to a given line ℓ (the directrix), a given point F (the focus), and a given eccentricity $e (0 < e < 1)$. Then such an ellipse is defined by $\{P \in R_2 \mid d(P,F)/d(P,\ell) = e\}$ where $d(P,\ell)$ denotes the shortest distance from P to ℓ.

In Figure 4, the line ℓ is equidistant from $(-1,6)$ and $(3,-4)$. The diagram pictures the ellipse corresponding to the directrix I, focus $(1,4)$ and eccentricity $1/2$. It is left to the reader to show that this ellipse is a convex hexagon with vertices at $(1,7)$, $(-1,5)$, $(\frac{-3}{2},4)$, $(-1,11/3)$, $(1,3)$ and $(2,4)$. It is a simple matter to show that this ellipse does not have the form given by [2] Reynolds.

In this paper we started with the natural definition of a line as the locus of points equidistant from two distinct points. We showed how this affects the results obtained by [1] Moser and Kramer. Finally, we showed that an ellipse defined using a line, focus, and eccentricity is not equivalent to an ellipse using two foci. We conclude that equivalent definitions under an Euclidean norm may yield contradictory definitions when generalized to a taxicab norm.

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FIGURE 1

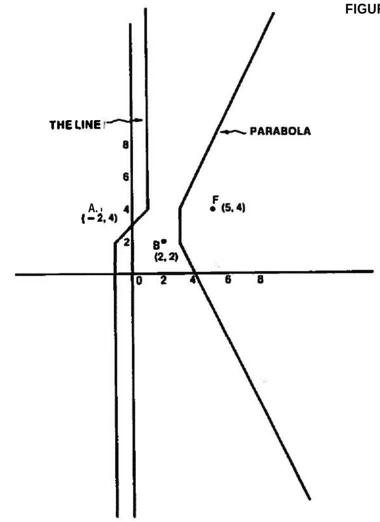
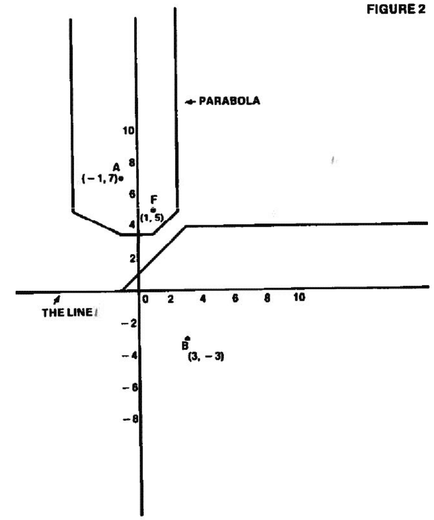


FIGURE 2



FIGURE!

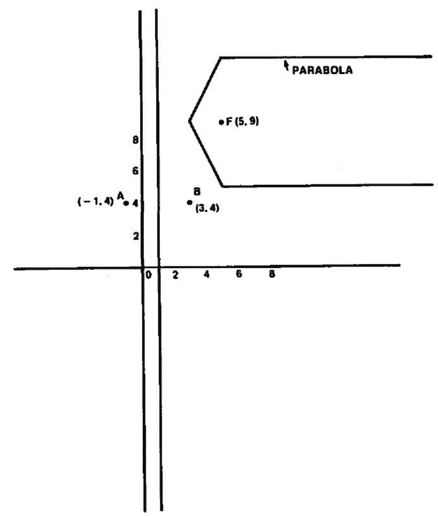
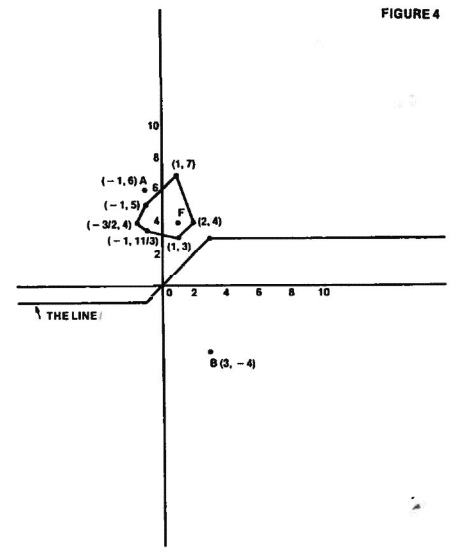


FIGURE 4



COMPLEX MATRICES AND POLYNOMIALS

by Alan C. Wilde
University of Michigan

A circulant matrix is an $n \times n$ matrix with arbitrary entries in the top row and forming each successive row by moving the entries over one place to the right from the order in the previous row. A typical circulant X is of the form:

$$(1)\quad X = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \cdots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & x_2 \cdots & x_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & x_3 & x_4 \cdots & x_0 \end{bmatrix}.$$

A particular circulant K is one with $x_1 = 1$ and $x_h = 0$ for $h \neq 1$. It is of the form:

$$(2)\quad K = \begin{bmatrix} 0 & 1 & 0 & 0 \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 \\ & & & \cdot & \\ 1 & 0 & 0 & 0 \cdots & 0 \end{bmatrix}.$$

Let $E_{h,j}$ be an $n \times n$ matrix with the $(h+1,\ j+1)$ entry equal to 1 and all the other entries equal to 0. Then the set $\{E_{h,j} \mid 0 \le h \le n-1,\ 0 \le j \le n-1\}$, is a basis for the vector space of $n \times n$ complex matrices. The $E_{h,j}$ have the property that

$$(3)\quad E_{h,j}\,E_{p,q} = \begin{cases} E_{h,q} & \text{if } j = p \\ 0 & \text{if } j \neq p\ . \end{cases}$$

By the definition of K,

$$(4)\quad K = \sum_{s=0}^{n-1} E_{s,s+1}\ .$$

Using equation (3) and mathematical induction, we find that

$$(5)\quad K^j = \sum_{s=0}^{n-1} E_{s,s+j} \qquad \text{for } 0 \le j \le n-1.$$

Thus K^j is the circulant with $x_j = 1$ and $x_h = 0$ for all $h \neq j$. Also, every circulant is a polynomial function of K, i.e.

$$(6)\quad X = \sum_{j=0}^{n-1} x_j K^j$$

where $K^n = I$. (See Davis [1]).

Let $\zeta = e^{2\pi i/n}$. Since K satisfies the polynomial equation $\lambda^n - 1 = 0$, then the solutions $1, \zeta, \zeta^2, \ldots \zeta^{n-1}$ of the equation are the eigen-values of K. (See Davis [1]). So the diagonal matrix equivalent to K is

$$(7)\quad \begin{bmatrix} 1 & & & & & 0 \\ & \zeta & & & & \\ & & \zeta^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & \cdot & \\ & & & & & \zeta^{n-1} \end{bmatrix}$$

Then

$$(8)\quad A = \sum_{s=0}^{n-1} \zeta^s E_{s,s}\ .$$

Now we state the theorem.

Theorem. Every $n\times n$ complex matrix is a complex polynomial function of K and A.

Proof. By equations (8) and (3) and induction,

$$(9)\quad A^h = \sum_{s=0}^{n-1} \zeta^{hs} E_{s,s} \text{ for } 0 \le h \le n-1.$$

Using (3), (5), and (9),

$$(10)\quad K^j A^h = \sum_{s=0}^{n-1} \zeta^{h(s+j)} E_{s,\, s+j}$$

and

$$(11)\quad A^h K^j = \sum_{s=0}^{n-1} \zeta^{hs} E_{x,\, s+j}\ .$$

So

$$(12)\quad A^h K^j = \zeta^{-hj} K^j A^h\ .$$

In particular, $AK = \zeta^{-1}KA$, so any finite product of K's and A's can reduce to the form $\zeta^s K^j A^h$.

Now since

$$\frac{1}{n}\sum_{s=0}^{n-1} \zeta^{hs} = \begin{cases} 1 \text{ if } h = 0 \\ 0 \text{ if } h \neq 0 \end{cases}$$

$$(13)\quad \frac{1}{n}\sum_{s=0}^{n-1} \zeta^{-hs} A^s = E_{h,h} \text{ for } 0 \le h \le n-1\ .$$

Also,

$$(14)\quad E_{h,h} E_{j,j} = \begin{cases} E_{h,h} \text{ if } h = j \\ 0 \text{ if } h \neq j\ , \end{cases}$$

i.e. the $E_{h,h}$'s are orthogonal idempotents and add to I.

By equation (3),

$$(15)\quad K^{-h} E_{0,0} K^j = \left(\sum_{s=0}^{n=1} E_{s,s-h}\right) E_{0,0} \left(\sum_{t=0}^{n=1} E_{t,t+j}\right)$$
$$= E_{h,0}\left(\sum_{t=0}^{n-1} E_{t,t+j}\right)$$
$$= E_{h,j}\ .$$

But by equation (13), $E_{0,0} = \frac{1}{n}\sum_{s=0}^{n-1} A^s$.

So

$$(16)\quad E_{h,j} = K^{-h}\left(\frac{1}{n}\sum_{s=0}^{n-1} A^s\right) K^j\ .$$

Finally, by equation (12),

$$(17)\quad E_{h,j} = K^{j-h}\left(\frac{1}{n}\sum_{s=0}^{n-1} \zeta^{-hs} A^s\right)\ .$$

Since every complex $n\times n$ matrix is a linear combination of $E_{h,j}$'s over $\mathbb{C}$, we have proved our theorem.

Thus our proof gives the polynomials explicitly by equation (17). In conclusion, the algebra of complex $n\times n$ matrices can be generated by exactly two matrices.

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AN UPPER BOUND ON THE ERROR FOR THE CUBIC INTERPOLATING POLYNOMIAL

by John C. Beasley
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A bound for the error in cubic interpolation is derived as a corollary of the following theorem, whose proof is found in [1]. This bound is then used to determine the spacing in a table of equally spaced values so that interpolation with a cubic polynomial will yield any desired accuracy.

Theorem 1. Let $f(x)$ be a real-valued function defined on $[a,b]$ and $n+1$ times differentiable on (a,b). If $p_n(x)$ is the polynomial of degree less than or equal to n which interpolates $f(x)$ at the $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$, then for all $x \in [a,b)$, there exists $\xi = \xi(x) \in (a,b)$ such that the error, e_n of the interpolating polynomial is:

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$

Consider the third-degree interpolating polynomial $p_3(x)$ evaluated at the equally spaced points $x_{i-2} < x_{i-1} < x_i < x_{i+1}$ with increment, h. By Theorem 1, the error in this interpolating polynomial is:

$$|e_3(x)| = |f(x) - p_3(x)| = \left|\frac{f^{(4)}(\xi)}{4!}\right| \, |(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})|$$

where $\xi = \xi(x) \in (x_{i-2}, x_{i+1})$.

Since we do not know ξ, we can merely bound $|f^{(4)}(\xi)|$ by the maximum value of the fourth derivative of $|f(x)|$, which we denote by M:

$$|f^{(4)}(\xi)| \le \max_{x_{i-2}<x<x_{i+1}} |f^{(4)}(x)| = M.$$

Needed also is the maximum value of

$$\Psi(x) = |(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})|,$$

for $x \in [x_{i-2}, x_{i+1}]$.

We let $s = \dfrac{x - x_i}{h}$ so that $x = x_i + sh$. Each factor of $\Psi(x)$ can be written

$$\begin{aligned}
x - x_{i-2} &= (x_i + sh) - (x_i - 2h) = (s + 2)h \\
x - x_{i-1} &= (x_i + sh) - (x_i - h) = (s + 1)h \\
x - x_i &= (x_i + sh) - (x_i - 0h) = sh \\
x - x_{i+1} &= (x_i + sh) - (x_i + h) = (s - 1)h
\end{aligned}$$

By making this change of variables we obtain

$$\Psi(s) = h^4(s + 2)(s + 1)(s)(s - 1) = h^4(s^4 + 2s^3 - s^2 - 2s).$$

The maximum value of this function, $|\Psi(s)|$, occurs at one of the critical numbers, which are found by solving:

$$\frac{d}{ds}[h^4(s^4 + 2s^3 - s^2 - 2s)] = h^4(4s^3 + 6s^2 - 2s - 2) = 0$$

The rational root theorem implies that possible roots of this cubic polynomial are $\pm 1, \pm 2, \pm 4, \pm\frac{1}{2}$. By synthetic division, a root of $\Psi'(s)$ is $c_1 = -\frac{1}{2}$. It follows from $(s + \frac{1}{2})(4s^2 + 4s - 4) = 0$ that $c_2 = (1 + \sqrt{5})/2$ and $c_3 = (1 - \sqrt{5})/2$ are also roots.

Now, the maximum value of the function $|\Psi(s)|$ occurs at either c_1, c_2, or c_3. By calculating $|\Psi(c_1)|$, $|\Psi(c_2)|$, and $|\Psi(c_3)|$ we find that

$$\begin{aligned}
|\Psi(c_1)| &= \frac{9}{16}h^4 \\
|\Psi(c_2)| &= h^4 \\
|\Psi(c_3)| &= h^4.
\end{aligned}$$

Thus,

$$\max \left| (x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1}) \right| = h^4 .$$

It follows that for any $x \in [x_{i-2}, x_{i+1}]$, the error is,

$$| e_3(x) | = | f(x) - p_3(x) | \le \frac{Mh^4}{24}$$

Using this bound for $e_3(x)$, we are able to determine the spacing h in a table of equally spaced values of a function $f(x)$ on $[a,b]$, so that cubic interpolation yields a desired accuracy.

Example: Determine the spacing h in a table of equally spaced values of the function $f(x) = \sqrt{x}$ between 1 and 2, so that cubic interpolation will yield a desired accuracy.

Solution: Using the above formula;

$$|e_3(x)| \le \frac{Mh^4}{24} \text{ where } M = \underset{1 \le x \le 2}{\text{maximum}} | f^{(4)}(x) | = \frac{15}{16}$$

$$|e_3(x)| = \frac{5h^4}{128} .$$

For seven-place accuracy, we would choose h so that

$$\frac{5h^4}{128} < 5 \times 10^{-8}, \text{ or } h \approx 0.0336.$$

For six-place accuracy, we would choose h so that

$$\frac{5h^4}{128} < 5 \times 10^{-7}, \text{ or } h \approx 0.0598 .$$

The number of entries in the table, N, would be given by $N = (2 - 1)/h = 1/h$. To obtain seven-place accuracy, $N \approx 30$, and for six-place accuracy, $N \approx 17$.

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This work was initiated wider the direction of Dr. S. E. Sims of Louisiana Tech University.

A FAMILY OF CONVERGENT SERIES WITH SUMS

by Steven Kahan
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In a standard calculus course, little time is actually devoted to the determination of the exact value of a convergent infinite series. In fact, with the exception of some simple geometric and telescoping series, one must usually wait until power series are developed before obtaining a different source for these computational examples. We now investigate another family of such convergent series.

Theorem. For any positive integer m,

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+m}{1+m}} = \frac{1+m}{m}$$

Proof.

$$\frac{1}{\binom{k+m}{1+m}} = \frac{1}{\frac{(k+m)!}{(1+m)!(k-1)!}} = \frac{(1+m)!(k-1)!}{(k+m)!} = (1+m)\frac{m!(k-1)!}{(k+m)!} =$$

$$(1+m)\frac{m!}{k(k+1)(k+2)\cdots(k+m)} .$$

Then it suffices to show that $\displaystyle\sum_{k=1}^{\infty} \frac{m!}{k(k+1)(k+2)\cdots(k+m)} = \frac{1}{m}$. To accomplish this, we observe that $\displaystyle\frac{m!}{k(k+1)(k+2)\cdots(k+m)} = f(k) - f(k+1)$, where $\displaystyle f(k) = \sum_{i=0}^{m-1} \frac{(-1)^i\binom{m-1}{i}}{k+i}$. Thus, $\displaystyle\sum_{k=1}^{\infty} \frac{m!}{k(k+1)(k+2)\cdots(k+m)}$ is a telescoping series whose n^{th} partial sum, s_n, is given by $f(1) - f(n+1)$. That is $\displaystyle s_n = \sum_{i=0}^{m-1} \frac{(-1)^i\binom{m-1}{i}}{1+i} - \sum_{i=0}^{m-1} \frac{(-1)^i\binom{m-1}{i}}{n+1+i}$, from which we find that $\displaystyle\lim_{n\to\infty} s_n = \sum_{i=0}^{m-1} \frac{(-1)^i\binom{m-1}{i}}{1+i}$. Our result therefore follows if we can verify that this summation is equal to $\frac{1}{m}$. To do so, consider $\displaystyle\sum_{i=0}^{m-1} \binom{m-1}{i} x^i$, the binomial expansion of $(1+x)^{m-1}$. Integration produces $\sum_{i=0}^{m-1} \binom{m-1}{i}\frac{x^{1+i}}{1+i} = \frac{(1+x)^m}{m} + C$, with the choice of $x = 0$ yielding $C = -\frac{1}{m}$. Next, we select $x = -1$ to obtain $\sum_{i=0}^{m-1} \frac{(-1)^{1+i}\binom{m-1}{i}}{1+i} = -\frac{1}{m}$.

Dividing this last equation by -1 completes the argument.

This result permits us to immediately compute the sum of the reciprocals of the entries on any diagonal of Pascal's triangle beginning beneath the second row.

ANOTHER DEMONSTRATION OF THE EXISTENCE OF EULER'S CONSTANT

by Norman Schaumberger,
Bronx Community College

For all positive integers n

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \tag{1}$$

In this note we use both sides of this familiar inequality to prove that if $\gamma_n = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n$ then

(a) $0 < \gamma_n < 1$ and

(b) γ_n tends to a limit γ as $n \to \infty$.

The number γ is called Euler's constant and its decimal expansion starts $\gamma = 0.57721566\ldots$. Like e and π, γ is defined as a limit and appears often in analysis and number theory. Although it is not yet known whether γ is rational or not there are a number of ways of demonstrating its existence. The use of (1) to accomplish this purpose can serve to introduce the student to another important mathematical constant quite early in a standard calculus course.

Using $\prod_{k=1}^{n-1} \frac{k+1}{k} = n$ it follows that

$$\sum_{k=1}^{n-1} \ln\left(1 + \frac{1}{k}\right) = \ln n. \tag{2}$$

Combining (2) with the left side of (1), we get $\ln n =$ $\sum_{k=1}^{n-1} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{n-1} \frac{1}{k} \ln\left(1 + \frac{1}{k}\right)^k < \sum_{k=1}^{n-1} \frac{1}{k} \ln e = \sum_{k=1}^{n-1} \frac{1}{k}$. Hence $\gamma_n = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n > 0$.

Now using (2) and the right side of (1), we have

$$\ln n = \sum_{k=1}^{n-1} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{n-1} \frac{1}{k+1} \ln\left(1 + \frac{1}{k}\right)^{k+1} > \sum_{k=1}^{n-1} \frac{1}{k+1} \ln e = \sum_{k=1}^{n-1} \frac{1}{k+1}.$$

Hence $\ln n > \sum_{k=1}^{n-1} \frac{1}{k+1}$ and consequently $\gamma_n = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n < 1 - \frac{1}{n} < 1$.

This completes (a).

(b) now follows by observing that $\gamma_{n+1} - \gamma_n = \frac{1}{n} - \ln(n+1) + \ln n.$
Taking logs on the left side of (1), we get $\frac{1}{n} > \ln\left(1 + \frac{1}{n}\right) = \ln(n+1) - \ln n.$
Thus $\gamma_{n+1} - \gamma_n > 0$ and γ_n increases as n increases. Hence since γ_n is bounded above by 1, it tends to a limit ≤ 1 as $n \to \infty$.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

We have a supply of 10 x 14-inch Fraternity Crests available, and one in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

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Dr. Richard Good
Department of Mathematics
University of Maryland
College Park, Maryland 20742

MATCHING PRIZE FUND

If your Chapter presents awards for Outstanding Mathematical Papers or Student Achievement in Mathematics, you may apply to the National Office to match the amount spent by your Chapter. For example, $30 of awards can result in you Chapter receiving $15 reimbursement from the National Office. The maximum matching for one chapter is $50. These funds can also be used for the rental of Mathematics Films. Write to:

Dr. Richard Good
Department of Mathematics
University of Maryland
College Park, Maryland 20742

A COLLEGE GLOSSARY
DEFINITIONS OF PHRASES USED IN COLLEGES AND UNIVERSITIES WHEN ANSWERING STUDENTS' QUESTIONS

Anonymous

1. IT IS IN THE PROCESS: So wrapped up in red tape that the situation is almost hopeless.
2. WE WILL LOOK INTO IT: By the time the wheel makes a full turn we assume that you'll have forgotten all about it.
3. PROGRAM: An assignment that can't be completed by one phone call.
4. EXPEDITE: To compound confusion with commotion.
5. COORDINATOR: The guy who has a desk between two expediters.
6. CHANNELS: The trail left by intra-office memos.
7. CONSULTANT (OR EXPERT): Any ordinary guy with a briefcase more than 50 miles away from home.
8. ACTIVATE: To make carbons and add more names to the memo.
9. IMPLEMENT A PROGRAM: Hire more people and expand the office.
10. UNDER CONSIDERATION: We're looking in the damn files for it.
11. MEETING: A mass mulling of masterminds (goof offs).
12. RELIABLE SOURCE: The guy you just met.
13. INFORMED SOURCE: The guy who told the guy you just met.
14. UNIMPEACHABLE SOURCE: The guy who started the rumor in the first place.
15. CLARIFICATION: We are a little stupid on this subject, tell us more about it and we'll give you an answer.
16. WE ARE MAKING A SURVEY: We need more time to think of an answer.
17. A THOROUGH SEARCH WAS MADE OF FILES: Somebody looked in the waste basket.
18. SEE ME, OR LET'S DISCUSS: Come down to my office; we'll play a game of cribbage.
19. WE WILL ADVISE YOU IN DUE TIME: When we get it figured out we'll let you know.
20. LET'S GET TOGETHER ON THIS: I am assuming that you are as confused as I am.

21. FORWARDED FOR YOUR CONSIDERATION: You hold the bag awhile.
22. NOTED AND FORWARDED: I don't know what this damn thing is about. maybe you do.
23. EXPERT: One who knows more and more about less and less, this college's full of them.
24. NO FURTHER ACTION IS DEEMED NECESSARY: Don't confuse me with facts, my mind is made up.

FRATERNITY KEY-PINS

Gold Clad Key-Pins are available at the National Office at the Special Price of $8.00 each. Write to:

Dr. Richard Good, National Office
Department of Mathematics
University of Maryland
College Park, Maryland 20742

Please indicate the Chapter into which you were initiated and the approximate date of initiation.

1984 NATIONAL PI MU EPSILON MEETING

It is time to be making plans to send an undergraduate delegate or speaker from your Chapter to the Annual Meeting of, PI MU EPSILON *in Eugene, Oregon in August of, 1984. Each Speaker who presents a paper will receive travel benefits up to $500 and each delegate, up to $250 (only one speaker on delegate can. be funded from a single Chapter, but others can attend.)*

PUZZLE SECTION

Edited by
Joseph D.E. Konhauser

This Department is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used 'here will be sent to the Problem Editor if deemed appropriate for that Department.

Address all proposed puzzles and puzzle solutions to Prof. Joseph Konhauser, Department of Mathematics, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

Mathacrostic No. 18

Submitted by Joseph D.E. Konhauser
Macalester College, St. Paul, Minnesota

Like the preceding puzzles, this puzzle (on the following two pages) is a keyed anagram. The 234 letters to be entered in the diagram in the numbered spaces will be identical with those in the 26 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give the name of an author and the title of a book; the completed diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

1 O	2 M	3 V		4 F	5 P	6 J	7 A	8 S		9 R	10 X	11 L	12 M
13 Q	14 U	15 D	16 Y	17 E		18 F	19 W	20 C	21 N		22 I	23 H	24 U
	25 J	26 K	27 Z		28 M	29 G	30 R	31 O	32 P	33 D	34 L		35 T
36 W	37 B		38 U	39 A	40 I		41 K	42 C		43 J	44 F	45 N	46 H
47 Y	48 Q	49 U	50 E		51 F	52 P	53 M	54 G	55 I	56 L	57 J		58 X
59 W	60 K	61 V	62 B	63 Q	64 Y	65 Z	66 F	67 A	68 T		69 S	70 O	71 H
	72 K	73 Y	74 N	75 I	76 L	77 E	78 P	79 J	80 U		81 I	82 R	83 W
84 C	85 X	86 Z	87 A	88 S	89 O	90 M		91 B	92 F	93 Q	94 D	95 X	
96 K	97 J	98 V	99 F		100 A	101 L		102 R	103 P	104 Q		105 E	106 I
107 N	108 O	109 W	110 B	111 Z		112 M	113 X		114 Y	115 H	116 Q	117 K	118 D
	119 U	120 M	121 F	122 T	123 C	124 H	125 B	126 R	127 J	128 V	129 E		130 M
131 S	132 O		133 K	134 D	135 W	136 A	137 F	138 N	139 I	140 U	141 R	142 Q	
143 O	144 M		145 X	146 C		147 F	148 V	149 P	150 L	151 B	152 O	153 C	154 J
	155 Z	156 X	157 Y		158 K	159 W	160 G	161 C		162 F	163 Q	164 U	165 R
166 S		167 I	168 Z		169 H	170 D	171 V	172 K	173 B	174 P	175 N	176 U	177 S
	178 J	179 C	180 E		181 X	182 U	183 Y	184 A	185 S	186 Z	187 V	188 D	189 B
	190 X	191 K	192 S		193 K	194 F	195 C	196 X	197 E	198 S	199 I	200 P	
201 B	202 G		203 S	204 J	205 M		206 Z	207 K	208 Y	209 C	210 R		211 T
212 W		213 D	214 J	215 H	216 B	217 I	218 S	219 X	220 V		221 B	222 R	223 X
224 O	225 C	226 A	227 M	228 S	229 Q		230 N	231 W	232 P	233 L	234 V		

	Words	Definitions
A.	87 39 226 7 184 100 136 67	abstractions of information processing devices
B.	125 216 151 221 91 62 110 189 37 201 173	a problem most difficult of solution (2 wds.)
C.	153 20 225 179 209 84 42 161 123 195 146	that property of real numbers which asserts that no real number is an upper bound for the integers
D.	213 170 118 94 15 134 188 33	kind of translation known as a screw displacement
E.	50 77 17 180 105 129 197	England's greatest inventor of mathematical puzzles (1847-1930)
F.	147 194 51 4 92 44 121 162 18 137 66 99	lying on; passing through (2 wds.)
G.	202 160 29 54	free from admixture or dilution
H.	46 124 215 23 71 115 169	tending in probability to a limiting form which is independent of the initial position
I.	217 75 81 139 167 55 40 199 106 22	fermenting agent for proof analysis
J.	214 127 25 57 97 178 43 204 6 79 154	concisely (3 wds.)
K.	60 117 133 41 191 26 207 172 72 158 96 193	______ of ______ (1401-1464), cardinal who believed that "the Divine, being infinite, is inaccessible to the mind of man." (2 wds.)
L.	76 56 233 150 101 34 11	stratagem in which one appears to decline an advantage
M.	227 28 112 12 130 2 205 90 53 120 144	"corresponds to water as the cube does to earth" Plato, Timaeus
N.	138 21 175 107 74 230 45	geometry in which no parallel postulate is assumed
O.	143 70 132 224 31 1 108 152 89	kind of reasoning which suggests conclusions
P.	149 103 174 52 200 32 232 78 5	the place or point of entering or beginning
Q.	104 229 48 163 142 63 93 116 [illegible]	truncated syllogism in which one of the propositions, usually the premise, is understood but not stated
R.	9 82 30 165 210 222 141 102 126	a dance figure in which one or more couples dance round and round with hands joined
S.	192 218 198 166 203 177 69 8 88 228 131 185	its trilingual inscription furnished the key to ancient Egyptian (2 wds.)
T.	211 122 35 68	in combinatorial geometry, a bounded, closed convex set
U.	119 80 38 164 182 14 49 24 140 176	relating to the study of control and communication in the animal and the machine
V.	234 171 220 98 187 148 128 3 61	established or settled firmly
W.	159 59 135 212 231 109 36 19 83	its combination of clockwise and counterclockwise spirals consists of successive Fibonacci numbers
X.	95 196 219 181 190 145 156 223 10 85 58 113	bridge stratagem (3 wds.)
Y.	16 208 47 183 114 73 157 64	to scale
Z.	111 65 168 186 206 155 86 27	increase, accumulate, expand, or multiply rapidly

SOLUTIONS

Mathacrostic No. 17. (See Fall 1983 Issue) *(Proposed by Joseph D.E. Konhauser, Macalester College, St. Paul, Minnesota)*

Words:

A. Rosolio	K. Insolation	U. Dithyramb
B. Umpteenth	L. Noggin	V. Tychonoff
C. Diesis	M. Fata Morgana	W. Halophytic
D. Yeshiva	N. Isinglass	X. Eyewash
E. Ratsbane	O. Nanosecond	Y. Misology
F. Unless	P. Ichnite	Z. Inquest
G. Chiffchaff	Q. Teaser	a. Navel Point
H. Kettle of Fish	R. Yokefellow	b. Dextral
I. Equal	S. Aptronym	
J. Rakoczy	T. Nibble	

First Letters: RUDY RUCKER INFINITY AND THE MIND

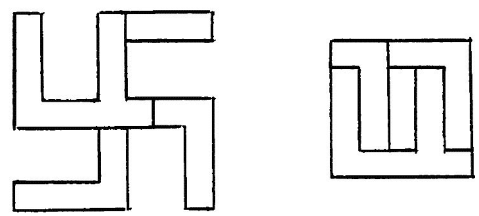
Quotation: *Fully formalized proofs have a nitpicking, obsessive quality. 'Sat by the tame token, they are satisfyingly solid and self-explanatory. Nothing is left to the imagination, and ... one can check whether or not a sequence of strings of symbols is a proof in a wholly mechanical fashion.*

Solved by: David Bahnemann, Northwest Missouri State University, Maryville, MO; Jeanette Bickley, Webster Groves High School, MO; Betsy Curtis, Meadville, PA; Victor G. Feser, Mary College, Bismarck, ND; Robert Forsberq, Lexington, MA; Robert C. Gebhardt, County College of Morris, Randolph, NJ; Joel Haack, Oklahoma State University, Stillwater, OK; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, MA; Robert Prielipp, The University of Wisconsin, Oshkosh, WI; Sister Stephanie Sloyan, Georgian Court College, Lakewood, NJ; Michael J. Taylor, Indianapolis Power and Light Co., Indianapolis, IN; Patricia A. and Allan M. Tuchman, University of Illinois, Champaign, IL.

Puzzle Editor's Note: The names of Robert Forsberg, Lexington, MA and Robert C. Gebhardt, Hopatcong, NJ were inadvertently omitted from the list of solvers of Mathacrostic No. 16.

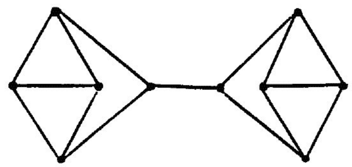
COMMENTS ON PUZZLES 1 - 7 (See Fall 1983 Issue)

Three readers responded to ***Puzzle #1.*** Victor G. Feser, John H. Scott and Richard A. Wilson proved that the construction is impossible by three-coloring the $8\times8\times8 = 512$ subcubes in such a way that each 1×3 block contains one subcube of each color. The two removed corner cubes are of the same color. An easy counting argument now can be used to settle the question. The proposer, I. J. Good, provided an equivalent argument without introducing a coloring. Ten answers were submitted for ***Puzzle #2.*** Victor G. Feser and Alan Hinkle, F.S.A., submitted $7-\sqrt{7/.\bar{7}}$. Robert W. Prielipp submitted $\sqrt{7/.\bar{7}+7}$. Jointly, Tommy Leavelle and David Sutherland submitted $[\sqrt{7+7+7}]$. Steve White, F.S.A., and Victor G. Feser submitted other solutions involving the greatest integer function. ***Puzzle #3*** drew six responses. Five of the answers were asked-for five-piece dissections of the swastika. Emil Slowinski discovered a four-piece dissection whose existence was totally unsuspected by the proposer. It is shown below.



Eleven responses were received for ***Puzzle #4.*** Solutions varied from several particular trios of numbers for a, b and c to infinite families of solutions. Robert W. Prielipp and Patrick Costello suggested taking $a=s(s^3+t^3)$, $b=t(s^3+t^3)$ and $c=s^3 t t^3$, where s and t are unequal positive integers greater than unity. David E. Penney established the generalization that if m and n are relatively prime, then the equation $a^m+b^m=c^n$ has infinitely many solutions in distinct positive integers a, b, and c.

Puzzle #5 attracted solutions from four readers. The solution shown below, which avoids a pair of nodes joined by two distinct arcs, was submitted by **John H. Scott.**



Using three colors, it is impossible to color the arcs so that no two arcs of the same color terminate at the same node. Six solutions were received for *Puzzle #6*. One reader claimed that six marks were needed. The other five produced the five-mark solution with marks separated by successive distances 1, 3, 10, 2, and 5. In responding to *Puzzle #7*, seven readers did the obvious and subtracted 1 from each of the eight given numbers to obtain the set (1, 2, 3, 5, 8, 13, 21, 30) with sum 83. The set (1, 2, 3, 5, 9, 15, 20, 25) with lower sum 80 was completely overlooked.

<u>List of Solvers</u>: **Maureen J. Brennan** (*3, 4, 7*), **Paul Buis** (*4*), **Patrick Costello** (*4*), **Victor G. Feser** (*1, 2, 6*), **Alan Hinkle** (*2*), **Tommy Leavelle** (*2, 4*), **Marijo LeVan** (*7*), **Glen E. Mills** (*3, 4, 6, 7*), **Patricia A. Mills** (*I*), **David E. Penney** (*4*), **Robert W. Prielipp** (*2, 4*), **John H. Scott** (*1, I, 3, 4, 5, 6, 7*), **Emil Slowinski** (*2, 3, 4, 5, 6, 7*), **David Sutherland** (*2, 4*), **Michael J. Taylor** (*2, 3, 4, 5, 6, 7*), **Steve White** (*2*), and **Richard A. Wilson** (*1, 3, 5, 6, 7*).

<u>Late Solutions</u>: **Susan Sadofsky** (*Puzzles #1 and #3*, Spring 1983 Issue)

PUZZLES FOR SOLUTION

1. *Proposed by Joseph Konhauser, Macalester College, St. Pad, Minnesota.*

With just one sphere and one cube, what is the largest number of pieces into which three-space can be divided?

2. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

Using each of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 exactly one time, write two fractions whose sum is unity.

3. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

Can every odd integer which is a multiple of three be written as a sum of four perfect cubes? For example,

$$21 = (1)^3 + (1)^3 + (-2)^3 + (3)^3.$$

4. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

Using just two colors, in how many distinguishable ways can one color the edges of a regular tetrahedron?

5. *Proposed by Joseph Konhauser, Macalester College, St. Pad, Minnesota.*

A certain light signal goes on at precisely 12:00 noon. Thereafter the light goes off and on at equal time intervals each lasting a whole number of minutes. If the light is off at 12:09 p.m., on at 12:17 p.m. and on at 12:58 p.m., is the light on or off at 2:00 p.m.?

6. ***Proposed by** Joseph **Konhauser, Macalester College,** St. **Paul, Minnesota.***

By means of an example, show that three colors are not sufficient for coloring all the points of the plane so that no two points spaced one unit apart are colored alike. [Hadwiger, H., Ungeloste Probleme, Nr. 40, Elemente der Math. 16(1961), 103-104.]

7. ***Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.***

In a certain country postage stamps are available in four denominations. All postages from 1 through 24 require at most three stamps. What are the four denominations?

LETTER TO THE EDITOR -----------

Dear Editor:

I saw the article $\tan^{-1}1 + \tan^{-1}2 + \tan^{1}3 = \pi$ *in the Fall 1983, No. 9 Issue of the Pi Mu Epsilon Journal. The above result is a particular case of the following general theorem:*

If a 1, b 1, then

$$\tan^{-1}a + \tan^{1}b + \tan^{-1}\frac{a+b}{ab^{-1}} = \pi$$

This general result is not difficult to prove.

Sincerely,

R. S. Luthar
The University i f Wisconsin
Janesville, Wisconsin 53545

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this Section. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should, be addressed to C. W. Dodge, Math Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1984.

Corrections

554. [Fall 19831 ***Proposed by Charles W. Trigg,** San **Diego, California.***

The	*S.P.F.A.*	(Society for Persecution of Feline Animals)
established a	*P U R R.*	
	F R E E	
	A R E A	at its headquarters.

In the word square each letter uniquely represents a decimal digit, and each word and acronym represents a square integer. What are these squares?

In problem 557 [Fall 1983] - the limits on the integral were reversed; the lower limit should be 0, the upper limit 1. Also the figures for problems 558 and 560 [Fall 1983] were reversed; that for problem 558 appears on page 616 while that for problem 560 is on page 615.

Problems for Solution

561. ***Proposed by I. Don, Guiva Dam, California.***

For what values of n does $n!$ have 6 for its last nonzero digit?

562. ***Proposed by Walter Blumberg, Coral Springs, Florida.***

Prove that $\tan 1^o \tan 61^o = \tan 3^o \tan 31^o$.

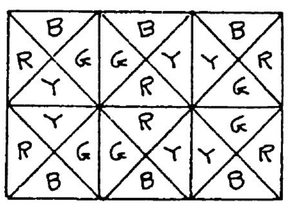
563. ***Proposed by Morris Katz, Macwahoc, Maine.***

There is a unique solution to this odd alphametric when *TEN* is divisible by 9 and when *TEN* is taken either odd or even (I've forgotten which).

TWELVE
TEN
TEN
THIRTY

564. ***Proposed by Charles W. Trigg, San Diego, California***

A tetrachromatic square is a square in which each of the four triangles formed by drawing the diagonals has a different color. With four specific different colors, six distinct tetrachromatic squares can be formed, not counting rotations. The six distinct tetrachromatic unit squares can be assembled into a 2-by-3 rectangle with matching colors on the edges that come into contact. The rectangle then contains seven solidly colored squares. This may be done in a variety of ways, one of which is shown in the figure.



Show that in any matched-edge assembly:

a) There are never only two colors of solidly colored squares;

b) The assembly can never have central symmetry; and

c) The perimeter of the rectangle can never consist of unit segments of just two alternating colors.

(For a related problem, see problem 282 [Fall 1973, pp. 480-1].)

565. ***Proposed by Walter Blumberg, Coral Springs, Florida.***

Let *ABCD* be a square and choose point *E* on segment *AB* and point *F* on segment *BC* such that angles *AED* and *DEF* are equal. Prove that $EF = AE + FC$.

566. ***Proposed by N. J. Kuenzi, University of Wisconsin-Oshkosh.***

If $\{p_n\}$ is a sequence of probabilities generated by the recurrence relation

$$p_{n+1} = p_n - \frac{1}{2} p_n^2 \text{ for } n \geq 0,$$

for which initial probabilities p_0 does $\lim_{n\to\infty} p_n$ exist?

567. ***Proposed by R. S. Luthar, University of Wisconsin-Janesville.***

Find the exact value of $\sin 20^o \sin 40^o \sin 80^o$.

568. ***Proposed by Robert C. Gebhardt, Hopeitcong, Nw Jersey.***

Find a simple expression for the power series

$$1 + \frac{x}{2!} - \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} + \frac{x^5}{6!} - \frac{x^6}{7!} - \frac{x^7}{8!} + \cdots.$$

569. ***Proposed by Robert C. Gebhardt, Hopatcong, N w Jersey.***

a) Find the largest regular tetrahedron that can be folded from a square piece of paper (without cutting)..

b) Prove whether it is possible to fold a regular tetrahedron from a square piece of paper without overlapping or cutting.

570. ***Proposed by Richard I. Hess, Rancho Palos Verdes, California.***

The natural logarithm of a complex number $z = re^{i\theta}$ is defined by

$$\ln z = se^{i\lambda}$$

where

$$s = ((\ln r)^2 + \theta^2)^{1/2}, \quad \lambda = \tan^{-1} (\theta/\ln r),$$

and

$$0 \leq \lambda \leq \pi/2 \text{ for } r \geq 1 \quad \text{or} \quad \pi/2 < \lambda < \pi \text{ for } 0 < r < 1.$$

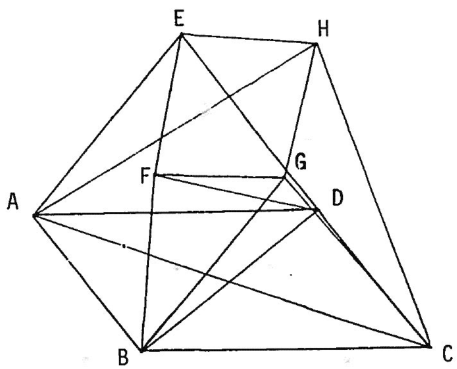
Find a number z_0 such that $\ln z_0 = z_0$.

571. ***Proposed by Chuck Allison, Huntington Beach, California.***

Assume a pegboard with one line of holes numbered 1 through n . Find the probability of picking correspondingly numbered pegs one at a time at random and placing them in their corresponding holes contiguously. That is, if peg number k is chosen first, then the second peg must be next to it, either number $k-1$ or number $k+1$. If pegs p, $p+1$, $p+2$, $\cdots$, q have already been chosen, the next peg must be either $p-1$ or $q+1$, so that no gaps ever appear between pegs.

572. *Proposed by Jack Garfunkel, Flushing, New York.*

Let ABCD be a parallelogram and construct directly similar triangles on sides AD, BC, and diagonals AC and BD. See the figure, in which triangles ADE, ACH, BDF, and BCG are the directly similar triangles. What restrictions on the appended triangles are necessary for EFGH to be a rhombus?



573. *Proposed by William S. Cariens, Lorain County Community College, Elyria, Ohio.*

Prove that when any parabola of the form

(1) $$y = x^2 + ax + b$$

is intersected by a straight line

(2) $$y = px + q,$$

then the sum of the derivatives of equation (1) at the two points of intersection is always twice the slope of the straight line.

Solutions

213. [Spring 1969, Spring 1979] *Proposed by Gregory Wulczyn, Bucknell University.*

II. Comment by Léo Sauvé, Editor, Crux *Mathematicorum* **9 (1983) 181,** *Ottawa, Ontario, Canada..*

The theorem is false. This was proved recently by O. Bottema and J. T. Groenman in *Nieuw Tijdschrift voor wiskunde*, **70** (1983) 143-151. Readers are invited to try and find the error in the Wulczyn solution. If they find it, they should inform the Problem Editor of *PMEJ*. If they don't find it, then it may be that Bottema and Groenman are "in Dutch."

III. Disproof by Leroy F. Meyers, Ohio State University, Columbus, Ohio.

In response to Léo's prodding (see II, above), the error is that the length of the exsymmedian x_a (which is the tangent at A to the circumcircle of triangle ABC) should have absolute value bars:

$$x_a = \frac{b \sin C}{|\sin(B-C)|},$$

and similarly, of course, for x_b and x .

If the triangle is isosceles, that is, if $a=b$, then $A=B$ and $x_a = x_b$ follows readily.

If $x_a = x_b$ and

$$\frac{b \sin C}{\sin(B-C)} = \frac{a \sin C}{\sin(A-C)},$$

then the proposer's solution holds and the triangle is isosceles.

So suppose that $x_a = x_b$ and we have

$$\frac{b \sin C}{\sin(B-C)} = -\frac{a \sin C}{\sin(A-C)}$$

Then we have that

$$b \sin A \cos C - b \cos A \sin C = -a \sin B \cos C + a \cos B \sin C$$

and

$$\frac{b \sin A \cos C}{\sin C} - b \cos A = -\frac{a \sin B \cos C}{\sin C} + a \cos B.$$

By the law of sines we make the replacements $(\sin A)/(\sin C) = a/c$ and $(\sin B)/(\sin C) = b/c$. Also using the law of cosines we replace $\cos C = (a^2 + b^2 - c^2)/2ab$ and similarly for $\cos A$ and $\cos B$. We then have

$$\frac{ab}{c}\cdot\frac{a^2+b^2-c^2}{2ab}-\frac{b^2+c^2-a^2}{2c}=-\frac{ab}{c}\cdot\frac{a^2+b^2-c^2}{2ab}+\frac{a^2+c^2-b^2}{2c}$$

$$a^2+b^2-c^2-b^2-c^2+a^2=-a^2-b^2+c^2+a^2+c^2-b^2,$$

and finally,

$$a^2+b^2=2c^2 .$$

All the above steps are reversible.

Editorial note. Triangles ABC having $a^2+b^2=2c^2$, called &-isosceles, or automedian, or RMS triangles, have been discussed at length in *Crux Mathematicorum:* problem 210 [1977: 10, 160, 196; 1978: 13, 193], problem 309 [1978: 12, 200], and problem 313 [1978: 35, 207], where other references are given.

518. [Spring 1982, Spring 1983] *Proposed by Michael U. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A baseball player gets a hit and observes that his batting average rises by exactly 10 points, i.e., by .010, and no rounding is necessary at all, where batting average is ratio of number of hits to times at bat (excluding walks, etc.). If this is not the player's first hit, how many hits does he now have?

II. Comment and Solution by the Proposer.

We mathematicians ought to give preference to mathematics (over computers) in our problem columns whenever there exists an elegant mathematical solution. In particular, a computer solution should not be allowed in this unnecessary case.

Let h and b be the numbers of hits and at bats before the last time at bat. Then the condition requires that

$$\frac{h+1}{b+1}-\frac{h}{b}=0.010=\frac{1}{100},$$

which can be rewritten in the form

$$h=\frac{b(99-b)}{100},$$

so $0<b<99$ (since $h>0$). Since we seek positive integral solutions, then $b(99-b)$ must be divisible by 100. Since b and $99-b$ have opposite parity, we must factor 100 into one odd and one even factor and then b must be divisible by one factor and $99-b$ by the other. Clearly $100=1\cdot 100$ is unusable and $100=5\cdot 20$ requires that both b and $99-b$ be divisible by 5, which is impossible since 99 is not divisible by 5. Thus we have $100=25\cdot 4$. If b or $99-b$ is 25, then the other is 74, which is not divisible by 4. Since the member divisible by 25 must be odd, we have b or $99-b$ equal to 75 and the other equal to 24. Either way,

$$h=\frac{75\cdot 24}{100}=18,$$

and he now has $h+1=19$ hits. The batting averages are $h/b=18/75=.240$ and $19/76=.250$ when $b=75$ and $18/24=.750$ and $19/25=.760$ when $b=24$.

Editorial note. Michael Ecker is Problem Editor for *Popular Computing.* Hence his chastisement carries double weight and this editor will do double penance by not switching on his computer for two full weeks.

534. [Spring 1983] *Proposed by Charles W. Trigg, San Diego, California.*

Find the mathematical term that is the anagram of each of the following words and phrases: (1) *RITES OF,* (2) *NILE GETS MEN,* (3) *PANTS GONE,* (4) *IRAN CLAD,* (5) *COVERT,* (6) *CLERIC,* (7) *GRABS ALE,* (8) *IRON LAD,* (9) *TRIED A VIVE,* (10) *HAG, NO SEX,* (11) *ALTERING,* (12) *RELATING.*

Amalgam of solutions submitted independently by ROBERT C. GEBHARDT, *Hopatcong, New Jersey,* RICHARD I. HESS, *Rancho Palos Verdes, California,* GLEN E. MILLS, *Pensacola Junior College, Florida,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* LEO SAUVÉ, *Algonquin College, Ottawa, Ontario, Canada, and the* PROPOSER.

We have (1) *RITES OF = FORTIES,* (2) *NILE GETS MEN = LINE SEGMENT,* (3) *PANTS GONE = PENTAGONS,* (4) *IRAN CLAD = CARDINAL,* (5) *COVERT = VECTOR,* (6) *CLERIC = CIRCLE,* (7) *GRABS ALE = ALGEBRAS,* (8) *IRON LAD = ORDINAL,* (9) *TRIED A VIVE = DERIVATIVE,* (10) *HAG, NO SEX = HEXAGONS,* (11) and (12) *ALTERING = RELATING = TRIANGLE = INTEGRAL.*

Also mostly solved by JEANETTE BICKLEY, *St. Louis, MO,* VICTOR G. FESER, *Mary College, Bismarck, ND,* ROGER KUEHL, *Kansas City, MO,* MICHAEL J. TAYLOR, *Indianapolis Power & Light Co., IN, and* KEVIN THEALL, *Laurel, MD.*

Also GEBHARDT listed (2) *NILE GETS MEN = ELEMENT SIGN.* KUEHL pointed out that *PROFESSOR CHARLES W. TRIGG = PROWLS, RECHARGES FOR*

GIST = GRASPS CLEW TO FRESH RIGOR = FIGGERS SPRAWL, SORT CHORE = OTHERS WRIGGLE FOR SCRAPS = CHARGE GIRLS, WREST PROOFS (possible advice to women mathematicians). PRIELIPP stated that (1) *RITES OF = RISE OF T.* The PROPOSER mused that: The contradiction *IS SUM = MINUS* may be amusing. Also observe that in the following pairs, each is the anagram of the other: *OSE S.O.E. (not otherwise enumerated), TWO TOW, SEVEN EVENS, NEIN SIHE,* and *TEN NET.* The last pair is palindromic. Too bad that "a minima" is bad grammar.

535. [Spring 1983] *Proposed by Stanley Rabinowitz, Digital Equipment Coup., Merrimack, New Hampshire.*

In the small hamlet of Abacinia, two base systems are in common use. Also, everyone speaks the truth. One resident said, "26 people use my base, base 10, and only 22 people speak base 14." Another said, "Of the 25 residents, 13 are bilingual and 1 is illiterate." How many residents are there?

Solution by Rogm Kuehl, Kansas City, Missouri.

Let the first resident speak base b. Then the second resident speaks base $b + 4$ since in that base the total population will be represented by a smaller numeral (25) than the numeral used by the first speaker as is the case. The total population is therefore $2(b+4) + 5 = 2b + 13$. The number of people speaking base b, according to the first speaker, is $2b + 6$ and the number speaking base $b + 4$ is $2b + 2$. According to the second speaker $1(b + 4) + 3 = b + 7$ people speak both bases and 1 is illiterate. Therefore the total population is

$$(2b + 6) + (2b + 2) + 1 - (b + 7) = 3b + 2.$$

Equating this to $2b + 13$, we get that the two bases are

$$b = 11 \qquad\qquad b + 4 = 15.$$

Now the total population, $2b + 13$, is 35 (base ten).

Also solved by WALTER BLUMBERG, *Coral Springs, FL,* who interpreted the first resident's statement to mean that 26 other people also use base 10, so the total using base 10 is 27, and the total population then is 33 (base ten) people, MARK EVANS, *Louisville, KY,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* GLEN E. MILLS, *Pensacola Junior College, FL.,* VANCE E. PINCHBECK, *Valhalla, NY,* HARRY SEDINGER, *St. Bonaventure University, NY,* MICHAEL J. TAYLOR, *Indianapolis Power & Light Co., IN.,* and the PROPOSER. Three incorrect solutions were received, one of which assumed that the first speaker's 10 meant ten and hence the statement of the problem was inconsistent.

537. [Spring 1983] *Proposed by Charles W. Trigg, San Diego, California.*

Find the unique four-digit integer in the decimal system that can be converted into its equivalent in the septenary system (base 7) by interchanging the left hand the right hand digit pairs.

Solution by Kenneth M. Wilke, Topeka, Kansas.

The number sought is $abcd_{10} = cdab_7$, so $c > a$. Hence we have the equation

$$1000a + 100b + 10c + d = 343c + 49d + 7a + b$$

or

$$331a + 33b = 111c + 16d.$$

Then we see that $a \equiv d \pmod 3$ and $c + 5d \equiv a \pmod{11}$. Also we have a, b, c, d each < 7 and $a < c$. Then we have the following table:

a	b	c	d	
1	2	3	4	
2			5	($c = 9$ - impossible)
3		6	8	($b = 7$ - impossible).

Hence $1234_{10} = 3412_7$ is the unique solution.

Also solved by FRANK BATTLES, *Massachusetts Maritime Academy, Buzzards Bay, MA,* JEANETTE BICKLEY, *St. Louis, MO,* WALTER BLUMBERG, *Coral Springs, FL,* VICTOR G. FESER, *Mary College, Bismarck, ND,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* ROGER KUEHL, *Kansas City, MO,* HENRY S. LIEBERMAN, *Waban, MA,* GLEN E. MILLS, *Pensacola, FL,* WHIT MURRILL, *Baton Rouge., LA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* MICHAEL J. TAYLOR, *Indianapolis Power & Light Co., IN,* KEVIN THEALL, *Laurel, MU,* and the PROPOSER. WADE H. SHERARD, *Furman University, Greenville, SC* interpreted the problem to be $abcd_{10} = bade$. He found the solution $1431_{10} = 4113_7$. (There is a second solution $1454_{10} = 41457$ - *ed.*) Also two incorrect solutions were received.

538. [Spring 1983] *Proposed by Emmanuel, O.C. Imonitie, Northwest Missouri State. University, Maryville.*

The roots of $ax^2 + bx + c = 0$, where none of the coefficients a, b, and c is zero, are a and β. The roots of $a^2x^2 + b^2x + c^2 = 0$ are 2α and 26. Show that the equation whose roots are $n\alpha$ and $n\beta$ is $x^2 + 2nx + 4n^2 = 0$.

Solution by Russell Todd, Bristol, Rhode, Island.

Since the roots of $ax^2 + bx + c = 0$ are a and β, the equation may be written as

$$0 = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta .$$

Dividing the original equation by a and comparing coefficients with that above we have that

$$\frac{b}{a} = -(\alpha + \beta) \quad \text{and} \quad \frac{c}{a} = \alpha\beta.$$

Similarly for the second equation we get that

$$\frac{b^2}{a^2} = -2(\alpha + \beta) \quad \text{and} \quad \frac{c^2}{a^2} = 4\alpha\beta .$$

These relations may readily be solved to get that

$$\alpha + \beta = 2 \qquad \alpha\beta = 4 . \tag{1}$$

The roots of the final equation are $n\alpha$ and $n\beta$, so we have

$$0 = (x - n\alpha)(x - n\beta) = x^2 - n(\alpha + \beta) + n^2\alpha\beta ,$$

which, from equations (1), becomes the desired result

$$x^2 - 2nx + 4n^2 = 0 .$$

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* MA, JEANETTE BICKLEY, *St. Louis, Mo,* WALTER BLUMBERG, *Cowl Springs, Fl,* BRUCE EDWARDS, *University of, Florida, Gainesville,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MO, VICTOR G. FESER, *Mary College, Bismarck,* NU, JACK GARFUNKEL, *Flushing,* NY, ROBERT C. GEBHARDT, *Hopatcong,* NJ, RICHARD I. HESS, *Rancho Palos Verdes,* CA, RALPH KING, *St. Bonaventure,* NY, ROGER KUEHL, *Kansas City,* MO, HENRY S. LIEBERMAN, *Waban,* MA, BOB PRIELIPP, *University of, Wisconsin-Oshkosh,* HARRY SEDINGER, *St. Bonaventure University,* NY, WADE H. SHERARD, *Furman University, Greenville,* SC, MICHAEL J. TAYLOR, *Indianapolis Power & Light Co,* IN, KEVIN THEALL, *Looted,* MD, W. R. UTZ, *Columbia,* MO, HAO-NHIEN QUI VU, *Purdue University, West Lafayette,* IN, KENNETH M. WILKE, TOPEKA, KS, and the PROPOSER. FESER, KUEHL, and THEALL each solved equations (1), obtaining $a, \beta = -1 \pm i\sqrt{3}$ as part of their solutions.

539. [Spring 1983] *Proposed by Hao-Nhien Q. Vu, Purdue University, West Lafayette, Indiana.*

Find a quadratic equation with integral coefficients that has cos $72°$ and cos $144°$ as roots.

*Does there exist such a quadratic with roots sin $72°$ and sin $144°$?

I. *Solution by Flank Battles and Laura Kelleher (jointly), Massachusetts Maritime Academy, Buzzards Bay.*

We replace $72°$ and $144°$ by their radian equivalents $2\pi/5$ and $4\pi/5$ and we let $z = e^{2i\pi/5} = \cos(2i\pi/5) + i\sin(2i\pi/5)$. Note that $z^5 = 1$ and that

$$\cos\frac{2\pi}{5} = \frac{z + \bar{z}}{2} = \frac{2 + 2^4}{2}$$

and

$$\cos^2\frac{2\pi}{5} = \frac{z^2 + 2z^5 + z^8}{4} = \frac{z^2 + 2 + z^3}{4}$$

Clearing of fractions and adding we obtain

$$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} = z^2 + 2 + z^3 + z + z^4$$

$$= 1 + 1 + z + z^2 + z^3 + z^4$$

$$= 1 + \frac{1 - z^5}{1 - z} = 1$$

from which we see that $\cos(2\pi/5)$ is a root of the quadratic equation

$$4x^2 + 2x - 1 = 0 .$$

A similar procedure shows that $\cos(4\pi/5) = (z^2 + \bar{z}^2)/2$ also satisfies this same quadratic. Because $\cos(2\pi/5) > 0$ and $\cos(4\pi/5) < 0$, then these quantities are the two distinct roots of the stated quadratic.

By solving the above quadratic for $\cos(2\pi/5) = (-1 + \sqrt{5})/4$ and using the relation $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$\sin\frac{2\pi}{5} = \sqrt{\frac{5 + \sqrt{5}}{8}}$$

If this value is a solution to a quadratic equation $mx^2 + nx = p$ with integer coefficients m, n, and p, which we rewrite in the form

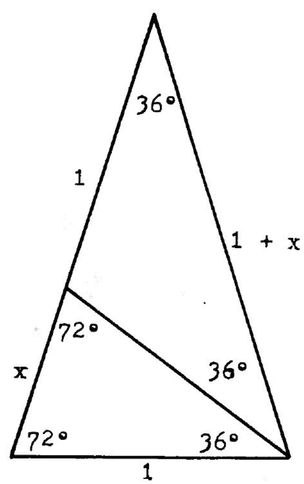
$$x = \frac{p}{n} - \frac{mx^2}{2} ,$$

then we must have

$$\sqrt{\frac{5+\sqrt{5}}{8}} = \frac{p}{n} - \frac{m}{n} \cdot \frac{5+\sqrt{5}}{8} .$$

Now square both sides of this equation and solve for $\sqrt{5}$ to get that $\sqrt{5}$ is a rational function of m, n, and p with integer coefficients, a contradiction since $\sqrt{5}$ is irrational. Thus there is no quadratic equation with integer coefficients having $\sin(2\pi/5)$ as a root, so of course there is no such equation having both $\sin(2\pi/5)$ and $\sin(4\pi/5)$ as roots.

II. Solution to the first part by Henry S. Lieberman, Waban, Massachusetts.



Consider the "golden triangle" with unit base, shown in the figure. By the law of sines we have that

$$\frac{x}{\sin 36^o} = \frac{1}{\sin 72^o} ,$$

$$x = \frac{\sin 36^o}{\sin 72} = \frac{\sin 36^o}{2 \sin 36^o \cos 36^o} = \frac{1}{2 \cos 36^o} ,$$

so $\cos 36^o = 1/2x$. But, by similar triangles, we have $x/1 = 1/(1+x)$, so $x^2 + x - 1 = 0$. The positive root of this equation is $x = (-1+\sqrt{5})/2$, so that

$$\cos 36^o = \frac{\sqrt{5}+1}{4} .$$

Therefore

$$\cos 72 = \cos(2 \cdot 36^o) = 2\cos^2 36^o - 1$$
$$= 2\left(\frac{5+2\sqrt{5}+1}{16}\right) - 1 = \frac{-1+\sqrt{5}}{4}$$

and

$$\cos 144^o = -\cos 36^o = -\frac{1+\sqrt{5}}{4} .$$

It follows that

$$\cos 72^o + \cos 144^o = \frac{-1+\sqrt{5}}{4} + \frac{-1-\sqrt{5}}{4} = \frac{-1}{2}$$

and

$$\cos 72^o \cos 144^o = \frac{-1+\sqrt{5}}{4} \cdot \frac{-1-\sqrt{5}}{4} = \frac{1-5}{16} = \frac{-1}{4} .$$

Hence the quadratic equation

$$x^2 + \frac{1}{2}x - \frac{1}{4} = 0 \quad \text{or} \quad 4x^2 + 2x - 1 = 0$$

has $\cos 72$ and $\cos 144^o$ as roots.

Also solved by JEANETTE BICKLEY, St. Louis, MO, WALTER BLUMBERG, Coral Springs, FL, BRUCE EDWARDS, Gainesville, FL, RUSSELL EULER, Northwest Missouri State University, Maryville, JACK GARFUNKEL, Flushing, NY, RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Rensselaer Polytechnic Institute, Troy, NY, RALPH KING, St. Bonaventure, NY, HENRY S. LIEBERMAN, Waban, MA, G. MAVRIGIAN, Youngstown State University, OH, GLEN E. MILLS, Pensacola Junior College, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, HARRY SEDINGER, St. Bonaventure University, NY, CHARLES W. TRIGG, San Diego, CA, W. VANCE UNDERHILL, East Texas State University, Commerce, TX, KENNETH M. WILKE, Topeka, KS, and the PROPOSER. *Solutions to the first part of the problem were also submitted by* ROGER KUEHL, Kansas City, MO, and MICHAEL J. TAYLOR, Indianapolis Power & Light Co., IN.

540. [Spring 1983] *Proposed by M. S. Klamkin, University of Alberta, Edmonton, Canada.*

If the radii r_1, r_2, r_3 of the three escribed circles of a given triangle $A_1A_2A_3$ satisfy the equation,

$$\left(\frac{r_1}{r_2} - 1\right)\left(\frac{r_1}{r_3} - 1\right) = 2,$$

determine which of the angles A_1, A_2, A_3 is the largest.

Amalgam of solutions by WALTER BLUMBERG, Coral Springs, FL, and

HENRY S. LIEBERMAN, *Waban, Massachusetts.*

Let a_i denote the length of the side opposite angle A_i, let $s = (a_1 + a_2 + a_3)/2$ and let A be the area of triangle $A_1A_2A_3$. Then $A = r_i(s - a_i)$ for $i = 1, 2, 3$, so we have

$$\frac{r_1}{r_2} = \frac{s - a_2}{s - a_1} \quad \text{and} \quad \frac{r_1}{r_3} = \frac{a - a_3}{a - a_1}$$

from which it follows that

$$\left(\frac{s - a_2}{s - a_1} - 1\right)\left(\frac{s - a_3}{s - a_1} - 1\right) = 2 .$$

Now multiply through by $(s - a_1)^2$ to clear of fractions, multiply out and simplify to arrive at

$$a_1^2 = a_2^2 + a_3^2 .$$

Hence $A_1A_2A_3$ is a right triangle with right angle at A_1, so A_1 is the largest angle of the triangle.

Also solved by DAVID INY, *Rensselaer Polytechnic Institute, Troy,* NY, MICHAEL J. TAYLOR, *Indianapolis Power & Light* Co., IN, CHARLES W. TRIGG, San *Diego,* CA, and the. PROPOSER.

541. [Spring **1983**] *Proposed* by *Stanley Rabinowitz, Digital Equipment* Cohp., *Merrimack, New Hampshire.*

A line meets the boundary of an annulus A_1 (the ring between two concentric circles) in four points P, Q, R, S with R and S between P and Q. A second annulus A_2 is constructed by drawing circles on PQ and RS as diameters. Find the relationship between the areas of A_1 and

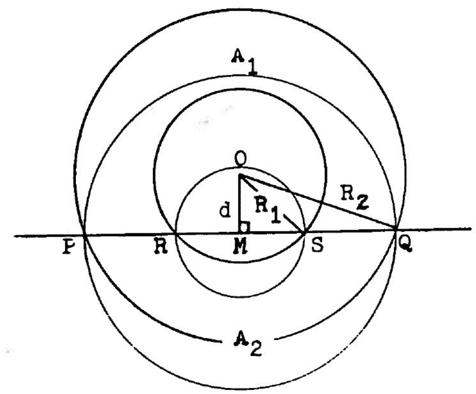
Solution by *Russell Euler, Northwest Missouri* State. *University, Maryville.*

The figure summarizes the hypotheses. Then the area of A_2 is

$$\pi(MQ^2 - MS^2) = \pi[(R_2^2 - d^2) - (R_1^2 - d^2)] = \pi(R_2^2 - R_1^2) ,$$

which is the area of A_1.

Also solved by WALTER BLUMBERG, *Coral Springs,* FL, RICHARD I. HESS, Rancho *Palos Verdes,* CA, DAVID INY, *Rensselaer Polytechnic Institute, Troy, NY,* ROGER KUEHL, *Kansas City,* **MO**, HENRY S. LIEBERMAN, *Waban,* MA,



G. MAVRIGIAN, *Youngstown* Stout-*University,* OH, HENRY SEDINGER, St. *Bonaventure University,* NY, WADE H. SHERARD, *Furman University, Greenville,* SC, MICHAEL J. TAYLOR, *Indianapolis Power & Light* Co., IN, HAO-NHIEN Q. VU, *Purdue University, West Lafayette,* IN, and the. PROPOSER.

542. [Spring **1983**] Phopobed by *Herbert* R. *Bailey, Rose Polytechnic Institute,* Terre *Haute, Indiana.*

A circle of unit radius is to be covered by three circles of equal radii. Find the minimum radius required.

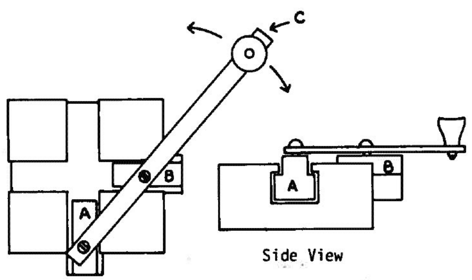
solution by *Harry Sedinger,* St. *Bonaventure University,* New *York.*

Let the three circles have radius r. Note that at least one circle must cover at least $1/3$ of the unit circle's circumference, A chord corresponding to such a third has length $\sqrt{3}$, so $r \geq \sqrt{3}/2$. Now draw three such chords by inscribing an equilateral triangle in the circle and take three circles with radii $\sqrt{3}/2$ and centers on the midpoints of the chords. It is easy to see that the three circles form the desired covering. Hence $r = \sqrt{3}/2$ is sufficient.

Also solved by RICHARD I. HESS, *Rancho Palos Verdes,* CA, JOHN M. HOWELL, *Littlerock,* CA, DAVID INY, *Rensselaer Polytechnic Institute,* Ttoy, NY, ROGER KUEHL, *Kansas City,* MO, VANCE E. PINCHBECK, *Valhalla,* NY, MICHAEL J. TAYLOR, *Indianapolis Power & Light* Co., IN, HAO-NHIEN QUI VU, *Purdue University, West Lafayette,* IN, and the. PROPOSER. G. MAVRIGIAN, *Youngstown* State. *University,* OH interpreted the problem as seeking three circles that surround the given circle. He found that the radius of n circles tangent externally to the unit circle and tangent each to its neighbors is $1/[\csc(\pi/n) - 1]$.

543. *[Spring 1983]* ***Proposed by Dominic C. Milioto, Southeastern Louisiana University, Hammond.***

A linkage device, shown in the figure, consists of a wood block with two tracks cut perpendicular to one another and crossing at the center of the block. Riding within the tracks are two small skids A and B, joined together by a long handle. As the handle is turned, the skids move within their respective tracks: A up and down and B from side to side. Describe the curve generated by point C (at the end of the handle) as the handle is turned.



I. Solution by David Iny, Rensselaer Polytechnic Institute, Troy, New York.

Introduce a Cartesian coordinate system where point A runs along the y-axis and B runs along the x-axis. Let the fixed distances between A and B be r and between A and C be R. When A is at $(0, -a)$, then B is at $(\pm\sqrt{r^2 - a^2}, 0)$ and, by similar triangles, C is at $((\pm R/r)\sqrt{r^2 - a^2}, (R/r)a - a)$. Now set x and y equal to the coordinates of C, square each equation, and eliminate a between them to obtain

$$\frac{x^2}{R^2} + \frac{y^2}{(R - r)^2} = 1 ,$$

an ellipse with center at the origin (the center of the block), semi-major axis equal to AC, and semi-minor axis equal to BC.

II. Solution by Robert C. Gebhardt, Hopatcong, New Jersey.

The curve is an ellipse. This device is well-known to mechanical engineering students, the elliptical trammel. See pp. 281-5 of Schwamb, Merrille and James, *Elements of Mechanism, 3rd ed.* (New York: John Wiley & Sons, Inc., 1921).

Also solved by WALTER BLUMBERG, *Coral Springs*, FL, RICHARD I. HESS, *Rancho Palos Verdes*, CA, RALPH KING, *St. Bonaventure University*, NY, ROGER KUEHL, *Kansas City*, MO, HENRY S. LIEBERMAN, *Waban*, MA, WHIT MURRILL, *University of Louisiana, Baton Rouge*, HARRY SEDINGER, *St. Bonaventure University*, NY, MICHAEL J. TAYLOR, *Indianapolis Power & Light Co.*, IN, KEVIN THEALL, *Laurel*, MD, RUSSELL TODD, *Bristol*, RI, HAO-NHIEN QUI VU, *Purdue University, West Lafayette*, IN, and *the* PROPOSER. KUEHL stated that he had seen this linkage described as a "do nothing" in souvenir shops in Missouri and Iowa.

544. *[Spring 1983]* ***Proposed by Jack Garfunkel, Flushing, New York, and Clifford Gardner, Austin, Texas.***

Show that a quadrilateral $ABCD$ with sides $AD = BC = s$ and $\measuredangle A + \measuredangle B = 120^o$ has maximum area if it is an isosceles trapezoid. A solution without calculus is preferred.

I. Solution by M. S. Klamkin, University of Alberta, Edmonton, Canada.

As the problem is stated, there is no maximum area since AB and CD can be arbitrarily large. Therefore we change the problem by adding the restriction that $BC + CD + DA = p$.

Then, if we reflect $ABCD$ across AB, the resulting figure $CDAD'C'B$ will be a hexagon of maximum area having a given perimeter $2p$. Hence it must be a regular hexagon, so $ABCD$ is an isosceles trapezoid with $\measuredangle A = \measuredangle B = 60^o$.

II. Solution by Henry S. Lieberman, Waban, Massachusetts.

In order that $ABCD$ have maximum area it must be cyclic. See *Theorem 3.36* in Niven: Maxima and Minima *Without Calculus*, pp. 53-4, which states: A quadrilateral inscribed in a circle has a larger area than any other quadrilateral with sides of the same lengths in the same order. (This is proven without calculus, of course).

So suppose we have a quadrilateral as described in the problem and inscribed in a circle. Since chords AD and BC are equal, their arcs are equal. Now angle A is measured by half the sum of arcs $BC + CD$, which equals half the sum of the arcs $CD + DA$, which measures angle B, so $\measuredangle A = \measuredangle B$ and $ABCD$ is an isosceles trapezoid.

Note that this proof is independent of the magnitude of $\measuredangle A + \measuredangle B$. In the particular case of this problem we would have, of course, $\measuredangle A = \measuredangle B = 60^o$ at maximum area.

Also solved by WALTER BLUMBERG, *Coral Springs, FL*, RALPH KING, *St. Bonaventure University, NY*, HARRY SEDINGER, *St. Bonaventure University, NY*, MICHAEL J. TAYLOR, *Indianapolis Power & Light Co., IN*, *and the* PROPOSERS.

545. [Spring 1983] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let f_n denote the nth Fibonacci number ($f_1 = 1$, $f_2 = 1$, $f_{n+2} = f_n + f_{n+1}$ for n a positive integer). Find a formula for f_{m+n} in terms of f_m and f_n (only).

I. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh.*

We shall show that

$$f_{m+n} = \frac{1}{2} f_n \sqrt{5f_m^2 + 4(-1)^m} + \frac{1}{2} f_m \sqrt{5f_n^2 + 4(-1)^n}.$$

To aid us in proving the above, we use the following well-known results, which can readily be proved by mathematical induction: If m, n, and j are positive integers and letting $f_0 = 0$, then

$$(1) \qquad f_{m+n} = f_{m-1}f_n + f_m f_{n+1}$$

and

$$(2) \qquad (f_{j-1} + f_{j+1})^2 = 5f_j^2 + 4(-1)^j.$$

From (2) we get

$$4(-1)^j + 5f_j^2 - ((f_{j+1} - f_j) + f_{j+1})^2 = 0,$$

which we multiply out and divide by 4 to get

$$f_{j+1}^2 - f_j f_{j+1} - f_j^2 - (-1)^j = 0,$$

a quadratic in f_{j+1}. Solving by the quadratic formula, we see that

$$(3) \qquad f_{j+1} = \frac{1}{2}\left(f_j + \sqrt{5f_j^2 + 4(-1)^j}\right).$$

Finally, from (1) and (3), we get that

$$\begin{aligned} f_{m+n} &= (f_{m+1} - f_m)f_n + f_m f_{n+1} \\ &= \frac{1}{2}\left(\sqrt{5f_m^2 + 4(-1)^m} - f_m\right)f_n + \frac{1}{2}\left(\sqrt{5f_n^2 + 4(-1)^n} + f_n\right)f_m \\ &= \frac{1}{2} f_n \sqrt{5f_m^2 + 4(-1)^m} + \frac{1}{2} f_m \sqrt{5f_n^2 + 4(-1)^n}. \end{aligned}$$

II. *Solution by* M. S. *Klamkin, University of Alberta, Edmonton, Canada.*

If l_n denotes the nth Lucas number ($l_1 = 1$, $l_2 = 3$, and $l_{n+2} = l_n + l_{n+1}$), then the formula of Solution I follows immediately from the following known relations:

$$(1) \qquad 2f_{m+n} = f_m l_n + f_n l_m$$

and

$$(2) \qquad l_j^2 = 5f_j^2 + 4(-1)^j.$$

Also solved by RUSSELL EULER, *Northwest Missouri State University, Maryville*, KEVIN THEALL, *Laurel, MD*, *and the* PROPOSER. EULER used the formula $f_n = [(1+\sqrt{5})^n - (1-\sqrt{5})^n]/2^n\sqrt{5}$ to write f_{m+n} in terms of f_n, f_m, $(1+\sqrt{5})$, and $(1-\sqrt{5})$. Two incorrect solutions were received.

546. [Spring 1983] *Proposed by Robert C. Gebhardt, Parsippany, New Jersey.*

Show that the square of the sum of the squares of four integers can be expressed as the sum of the squares of three integers, as in $(2^2 + 3^2 + 4^2 + 5^2)^2 = 14^2 + 28^2 + 44^2$.

I. *Solution by Victor G. Feser, Mary College, Bismarck, North Dakota.*

As stated, the problem is trivial: for any, a, b, c, d we can write

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 + c^2 + d^2)^2 + 0^2 + 0^2.$$

Unfortunately, it will not work simply to exclude zeros, since in some cases they must be allowed. For example,

$$(1^2 + 1^2 + 1^2 + 1^2)^2 = 16$$

and the only way 16 can be expressed as the sum of three squares is $4^2 + 0^2 + 0^2$. Another such example is $a = b = 1$ and $c = d = 2$.

II. *Additional comment by* M. S. *Klamkin, University of Alberta, Edmonton, Canada.*

The result is not true. A theorem of Gauss states that any natural number is either of the form $a^2 + b^2 + c^2$ or $a^2 + b^2 + 2c^2$

where a, b, c are natural numbers. Hence

$$(1 + 1 + 1 + 1)^2 = 16 = 2^2 + 2^2 + 2 \cdot 2^2$$

and

$$(2^2 + 2^2 + 1 + 1)^2 = 100 = 5^2 + 5^2 + 2 \cdot 5^2 .$$

III. *Solution by Walter Blumberg, Coral Springs, Florida.*

Using the proposer's example as a guide, we are led to the following special case: The square of the sum of the squares of four unequal positive integers in arithmetic progression can be expressed as the sum of the squares of three positive integers. To prove this statement we let a and x be positive integers, and we leave it for the reader to verify the identity

$$[a^2 + (a + x)^2 + (a + 2x)^2 + (a + 3x)^2]^2 = (4ax + 6x^2)^2 + (8ax + 12x^2)^2 + (4a^2 + 12ax + 4x^2)^2 .$$

IV. *Amalgam of solutions by* JACK GARFUNKEL, *Flushing, New York, and* DAVID INY, *Rensselaer Polytechnic Institute, Troy, New York.*

The following is an identity due to Lebesgue (see Long: *Elementary Introduction to Number Theory*, p. 143):

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac - 2bd)^2 + (2ad + 2bc)^2 .$$

Also solved by RICHARD I. HESS, *Rancho Palos Verdes, CA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER. WILKE gave the reference Carmichael, *Diophantine Analysis*, pp. 35-8, New York: Dover, 1959.

CHAPTER REPORTS

IOWA ALPHA CHAPTER (Iowa State University) The 60th Annual Initiation Banquet of the Iowa Alpha Chapter was held on May 1, 1983. Forty new members were initiated. *Professor Stephen J. Willson* of the Mathematics Department was the guest speaker and talked about *"Composing a Function with Itself."*

Pi Mu Epsilon Scholarship Awards of $50 each were presented to *Deborah Hwang* and *Craig McClanghan* who scored highest on a competitive examination. An additional award of $50 was presented to *William Somsky* for outstanding achievement on the **Putnam Exam.**

Other department awards were presented as follows:

The Dio Lewis Holl Awards to outstanding graduating senior mathematics majors; *William Somsky* and *Fred Adams.*

The Gertrude Herr Adamson Awards for demonstrated ingenuity in mathematics; *David Bachman, Rita Hanion, Kurtis Ruby* and *William Somsky.*

Other Iowa Alpha activities for the 1982-83 year included the following talks: *"Summation of Series"* by *Prof. Peter Colwell* of the Iowa State Mathematics Department; *"Shape of the Universe Through the Glasses of Finite Geometry"* by *Prof. Leslaw Szczerba*, Warsaw, Poland; *"Fractions That Go On Forever"* by *Prof. Bryan Crain* of the Iowa State Mathematics Department.

KANSAS GAMMA CHAPTER (Wichita State University) Talk by *Dr. George Milliken. "Uses and Misuses of Statistics in Sex and Race Discrimination Cases."* Banquet lecture by *Dr. Melvin Snyder, "Digression on Some Small Numbers."* Lecture by *Lewis Townsend, "On Robotics."* Lecture by *Dr. D. V. Chopra, "History of Pi Mu Epsilon."*

MASSACHUSETTS GAMMA (Bridgewater State College) The Initiation Address was a talk given by *Dr. Hugo D'Alarcao* entitled *"Platonic Solids and Brussels Sprouts."* This talk was one of a series of three talks held that week at Bridgewater State College sponsored by Massachusetts Gamma. The week was referred to as *"Euler Week"* during which we used the 200th anniversary of the year of Euler's death to learn some of the achievements of that great mind. The other talks, also given by members of the Bridgewater State College mathematics faculty were: *"Leonard Euler: His Life and Work"*, by *Prof. J. P. Scalisi;* and *"Euler and the Case of the 36 Officers"*, by *Prof. Thomas E. Moore.* In the spring term, 1984, Massachusetts Gamma will sponsor a week whose theme will be *Women and Mathematics.*

MICHIGAN DELTA CHAPTER (Hope College) The following programs and activities were sponsored by Pi Mu Epsilon and Math Club: *Mr. Roger Klassen*, Purdue University, spoke on *"The Use of Quality Control Charts in Statistical Quality Control"*; *Dr. Richard Vandervelde*, Hope College, spoke on *"Why Are Manhole Covers Round?"*; *Prof. John Van Iwaarden*, Hope College, spoke on *"An Introductory Look at Mathematical Modeling"*; *Dr. Harold Johnson*, Trinity College, spoke on *"The C.A.T. Scanner and Applications to Geology"*; *Dr. Mel Nyman*, Alma College, spoke on *"A Growth Model for the Giant Kelp"*: *Mr. John Stoughton*, University of North Carolina-Ashville, spoke on *"Derivation of the Trig Functions Using Differential Equations"*; and *Rick Meyers*, Senior Member of the Technical Staff at Apple Computer, Inc., spoke on *"Smalltalk-80"* and *"The LISA Personal Office System"*.

MISSOURI GAMMA (St. Louis University, Fontbonne College, Lindenwood College, Parks College) Presented the following awards: The James W. Garneau Mathematics Award was given to *Cherylyn Meek Claiborne*; The Francis Regan Scholarship was presented to *Daniel Deneufbourg*; The Missouri Gamma Undergraduate Award was earned by *Maureen Slattery*; The Missouri Gamma Graduate Award was given to *Leona Martens*; The winners of the Pi Mu Epsilon Contest Awards were: Senior - *Lynn Marie Nord*, Junior - *John Martin*; The John J. Andrews Graduate Service Award was presented to *M. Victoria Klamon*; and The Beradino Family Fraternityship Award was given to *Robert Quast*. The James Case Memorial Lecture was presented by *Prof. Patrick Cassens* of Central State University in Oklahoma; the title of his talk was *"Distance in Geometry as Defined by the Metric"*.

NEW YORK PHI CHAPTER (Clarkson College of Technology) *Dr. Philip Schwartau*, Brandis University, spoke on *"From Potsdam to a Ph.D."* *Marcia Borden* won the coveted Clarkson Memorial Award for the highest four year overall grade point average. This is the sixth consecutive year that this graduating senior award has been won by a member of the chapter.

NEW YORK ALPHA CHAPTER (Queens College) *Dr. Kenneth B. Kramer* of the Queens College Math Department spoke on *"Trap-Door Functions and Secret Codes"*: *Dr. Ronald I. Rothenberg* of the Queens College Math Department spoke on *"Using the Computer Language BASIC in the Math Classroom."* *Linda Hechtman* and *Hal Weinstein* were the recipients of the 1983 Pi Mu Epsilon prize for excellence in mathematics and service to the New York Alpha Chapter.

ALPHA CHAPTER (University of Pennsylvania) The Chapter conducted the following activities: Workshop, *"Job-Seeking Strategies in Mathematics-Related Fields"*; Workshop, *"Careers in Mathematics at I.B.M." by Of. John Sims* of I.B.M. ; Lecture, *"Mathematics and Econometrics" by Prof. Michael McCarthy* of the Economics Department; and Lecture, *"Soap Bubbles and Surface Areas" by Dr. Jerry Kazdan* of the Mathematics Department.

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Some persons have said that Euler's Formula is the most interesting equations in all of mathematics

$$e^{i\pi} + 1 = 0$$

It contains the five most important constants; e, π, i, 1, and 0. It contains the three fundamental arithmetic operations; addition, multiplication, and exponentiation and, finally, it contains the equality.

LETTER TO THE EDITOR ------------

Dear Editor:

I am writing in response to your "challenge to the reader" in the Fall 1983 issue of, Pi Mu Epsilon Journal. Under the. article entitled "Does $(a + ib)^{(c+id)}$ Equal a Real? you asked if the reader could find any other interesting cases of a real resulting from raising a complex number to a complex power, besides $i^i = e^{-\pi/2}$. Well, I've been working on that in my spare time as a senior physics major cut Lamar University in Beaumont, Texas. I have fairly easily compiled a list of several cases, plus a few other interesting (useless) tidbits.

These were all derived using:

$i^2 = -1$	$\sin ix = i \sinh x$
$a^2 + b^2 = r^2$	$\cos ix = \cosh x$
$\tan\theta = \frac{b}{a}$	$\tan ix = i \tanh x$
$\ln(a + ib) = \ln r + i\theta$	$\tanh ix = i \tan x$
$0 \le \text{argument} < 2\pi$	$\cosh ix = \cos x$
	$\sinh ix = i \sin x$

(Naturally, everything is in radians.)

$(a + ib)^{(c + id)}$	= Real	= Real
i^i	$e^{-\pi/2}$	0.2079
$\sqrt[i]{i}$	$e^{\pi/2}$	4.810
$i^{\ln i}$	$e^{-\pi^2/4}$	0.08480
$\left(\frac{2}{\pi}\ln i\right)^{\ln i}$	$e^{-\pi^2/4}$	0.08480
$i^{\sin i}$	$e^{-\frac{\pi}{2}\sinh 1}$	0.1579
$i^{\sinh i}$	$e^{-\frac{\pi}{2}\sin 1}$	0.2667
$i^{\tan i}$	$e^{-\frac{\pi}{2}\tan 1}$	0.3023
$i^{\tanh i}$	$e^{-\frac{\pi}{2}\tan 1}$	0.08661
$i^{\cos i}$	$e^{-\frac{\pi}{2}\cosh 1}$	0.8858
$i^{\cosh i}$	$e^{-\frac{\pi}{2}\cos 1}$	0.4280
$(\cos i)^{(\cos i)}$	$(\cosh 1)^{(\cosh 1)}$	1.953
$(\cosh i)^{(\cosh i)}$	$(\cos 1)^{(\cos 1)}$	0.7170
$(\cos i)^{(\cosh i)}$	$(\cosh 1)^{(\cos 1)}$	1.264
$(\cosh i)^{(\cos i)}$	$(\cos 1)^{(\cosh 1)}$	0.3686

Mathematically Yours,

Timothy R. Durgin
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