PI MU EPSILON JOURNAL

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CONTENTS

PI MU EPSILON JOURNAL
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TAXICAB GEOMETRY

by Barbara E. Reynolds
St. Louis University and
Cardinal Stritch College

Introduction:

Around the turn of the century, Hermann Minkowski [4] published a whole family of "metrics" -- that is, examples of spaces in which a way of measuring distance has been defined so as to fulfill the axioms of a metric space. Among these metrics is one which is referred to as the "taxicab metric" because of the way it mimics the distances that a taxicab would have to drive in an ideally laid-out city in which all streets run due north/south or east/west. In 1975, Eugene Krause [3] made the comment that "apparently no one has yet set *up* a full geometry based on the taxicab metric. It would seem that the time has *come* to do so."

Definitions and Background:

A metric space [1] is a mathematical structure which consists of a set of points and a rule (or function) for measuring distance between any two points in the set. In general we require that this distance function have three properties:

1). The distance between any two points is always non-negative; $d(A,B) \ge 0$. (And if $d(A,B) = 0$, then $A = B$.)

2). The distance from point A to point B is always the same as the distance from point B to point A; $d(A,B) = d(B,A)$.

3). The distance from point A to point B plus the distance from C to A is greater than or equal to the distance from A to B; that is, $d(A,B) \le d(A,C) + d(C,B)$.

The usual (2-dimensional) Euclidean space, E_2, consists of points from R_2 which can be represented graphically in the coordinate plane, or analytically as ordered pairs of real numbers. For example, the point A = (2,3) can be represented graphically as in Figure 1.

The Euclidean distance defined on any two points A, B in R_2 is defined consistent with the Pythagorean Theorem:

$$d_E(A,B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} .$$

With respect to the "real world" this distance measure could be referred to as the "as the crow flies" distance.

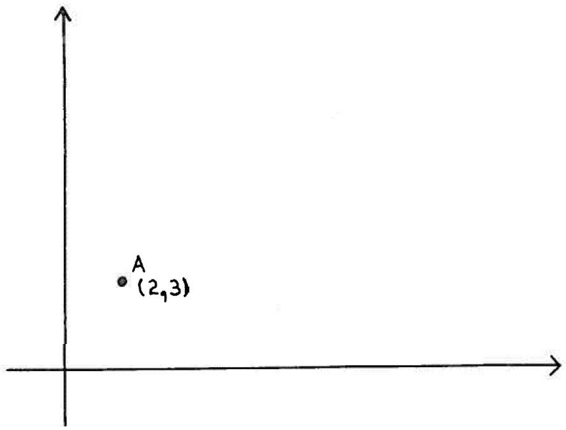

FIGURE 1

The (2-dimensional) <u>taxicab space</u>, T_2, consists of the same point set as (2-dimensional) Euclidean space. The taxicab distance function is defined for each pair of points A,B in R_2 by

$$d_T(A,B) = |a_1 - b_1| + |a_2 - b_2| .$$

As an example, let $A = (1,1)$ and $B = (3,4)$. Then A and B can be represented graphically as in Figure 2.

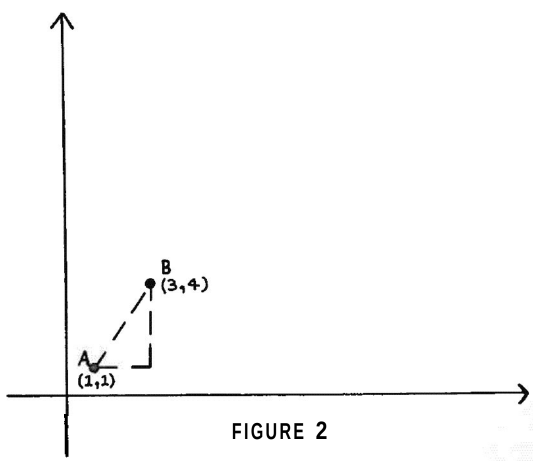

FIGURE 2

In this example, the Euclidean distance from A to B would be $\sqrt{13}$, while the taxicab distance would be 5. Notice that if we think of A and B as being street corners in a carefully laid-out city, there are a number of

different paths we could take in driving (or walking) from A to B. Two such paths are shown in Figure 3. As long as we must stay on the streets and cannot cut diagonally across any block, the shortest paths from A to B are each five blocks.

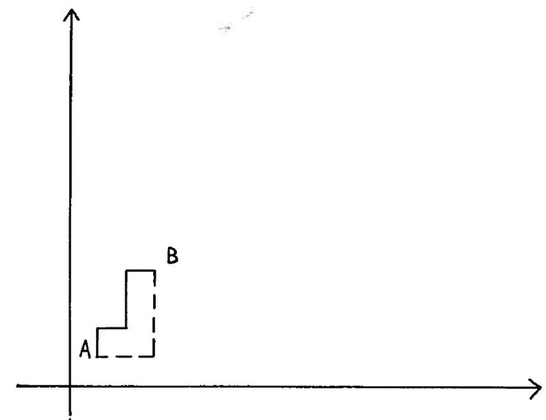

FIGURE 3

One way to get a feel for the effect that a certain way of measuring distance has on a space is to look at various familiar geometric figures -- for instance, circles, ellipses, hyperbolas, and parabolas.

<u>Circles</u>:

In analytic geometry, we define a <u>circle</u> as the set of points in R_2 at a constant distance from a given fixed point. If we use the Euclidean distance measure, circles are "round." We might be initially surprised as we plot points which are three units from the point $A = (4,3)$ using the taxicab distance measure. (See Figure 4).

Analytically, the taxi-circle with center $C = (h,k)$ and radius r is the collection of points

$$\{P = (p_1,p_2) \in R_2 : |p_1 - h| + |p_2 - k| = r\}.$$

This describes a taxi-circle as the union of four straight line segments with vertices at $(h,k\pm r)$ and $(h\pm r,k)$. Each of these line segments has a slope of ± 1. The complete taxi-circle is shown in Figure 5.

It is interesting to observe that if the mathematical contant Π_T is defined in the usual way as the ratio of the circumference of a circle to its diameter, then the value of Π_T is 4. It is also worth commenting

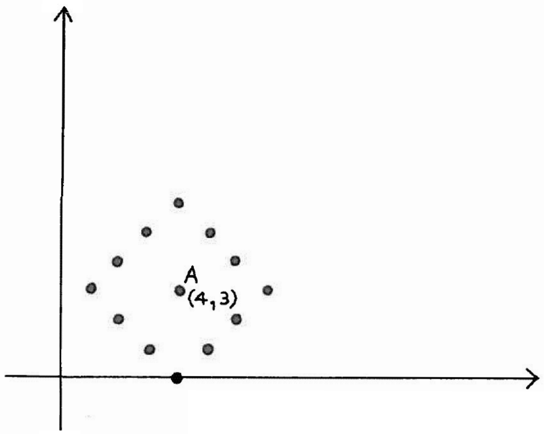

FIGURE 4

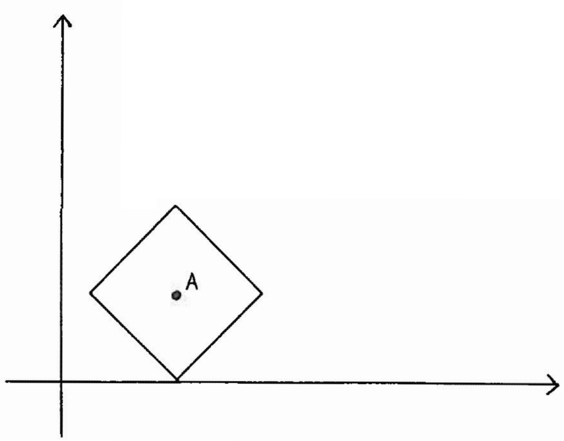

FIGURE 5

that while two distinct Euclidean circles may intersect in at most two points, two distinct taxi-circles may additionally intersect along one or two straight line segments (see Figures 6 and 7.)

Ellipse:

An ellipse may be defined as the set of all points P in R_2, the sum of whose distances from two fixed points, A and B, is constant. In other words, if A and B are fixed points in R_2, then an ellipse is the collection of points

$$\{P \in R_2 : \; d(P,A) + d(P,B) = c\},$$

where c is constant. The fixed points A and B are called the foci of the ellipse. Our experience with taxi-circles may lead us to suspect that the shape of the ellipse depends on whether we use the Euclidean distance measure or the taxicab distance measure.

There is an excellent discussion of ellipses of the Euclidean kind in Jacobs [2] with an interesting experiment in drawing ellipses with the aid of a loop of string and a couple of thumb tacks. Measuring distance by means of a tautly stretched string is essentially a Euclidean method, since this measures distance along "straight" paths. Just as we cannot use a compass to draw a taxi-circle, we cannot draw a taxi-ellipse by stretching a loop of string around two thumb tacks. Krause [3] suggests a method for drawing taxi-ellipses.

Suppose we are given fixed points A = (1,3) and B = (5,3), and are asked to draw the taxi-ellipse

$$\{P \in R_2 : d_T(P,A) + d_T(P,B) = c\},$$

where $c = 6$. Since $6 = 4 + 2$, we could draw a circle of radius 4 with A as center, and a circle of radius 2 with B as center. (See Figure 6.)

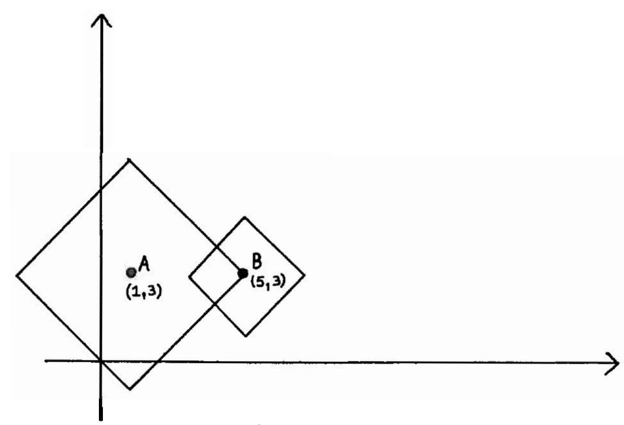

FIGURE 6

Each of the points where these two circles intersect will be at a distance of 4 from A and at a distance of 2 from B and therefore will be on the desired taxi-ellipse.

We can continue this process using circles of different radii, always choosing pairs of circles whose radii add up to 6. But a curious thing happens when we choose circles of radii 1 and 5. (See Figure 7.) These circles intersect not in one or two points but along two whole sides of the smaller circle.

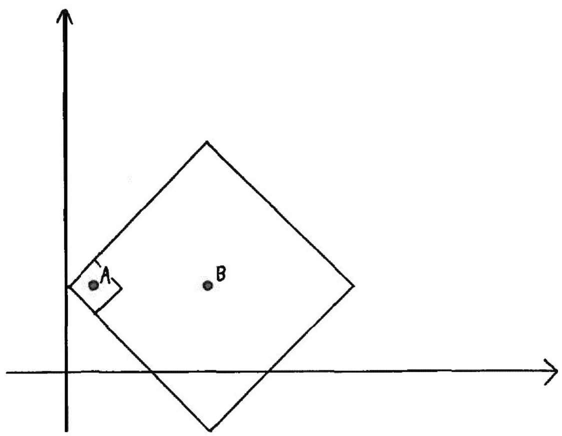

FIGURE 7

By experimenting with various pairs of circles, we find that the complete taxi-ellipse with foci $A = (1,3)$, $B = (5,3)$ and $c = 6$ is as shown in Figure 8.

Lest we rush too hastily to the conclusion that all taxi-ellipses are some kind of hexagons, a little further experimentation will show that the ellipse with foci at $A = (1,3)$, $B = (3,1)$, and $c = 6$ is octagonal in shape. (See Figure 9.)

A familiar result from analytic geometry is that as the foci of an Euclidean ellipse move closer together, the ellipse becomes more circular [2].

Observe that on the coordinate plane any two points either are opposite vertices of a rectangle with sides parallel to the axes (Figure 10-a), or lie on a straight line segment parallel to the x- or y-axis (Figure 10-b.) (And we could say that a line segment is a rectangle whose width is zero).

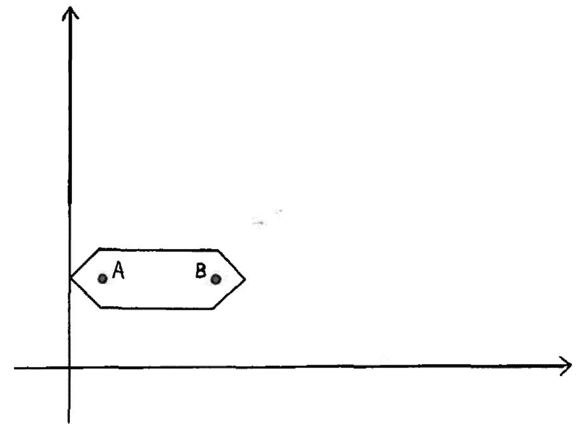

FIGURE 8

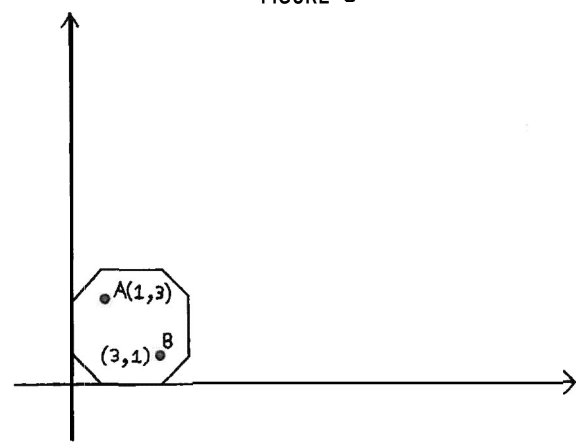

FIGURE 9

Now as A and B move closer together, this rectangle becomes smaller, until finally, when A and B are the same point, both the length and the width of this rectangle are zero.

Look again at Figures 8 and 9 and observe the relationship between the horizontal and vertical sides of the ellipse and the litte rectangle determined by the foci. If the foci move closer together, the rectangle will become smaller, and the horizontal and vertical sides of the ellipse will become shorter until finally the foci merge into a single point and the ellipse becomes a perfect taxi-circle.

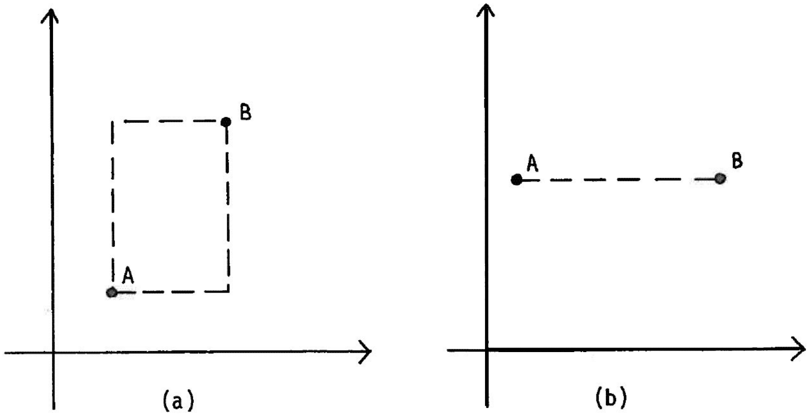

FIGURE 10

Hyperbola:

A hyperbola may be defined in a manner similar to the ellipse by replacing the word "sum" with "difference." Thus a <u>hyperbola</u> is the set of all points P in R_2 the difference of whose distances from two fixed points (<u>foci</u>), A and B, is constant. That is, if A and B are fixed points in R_2, a hyperbola is

$$\{P \in R_2 : |d(P,A) - d(P,B)| = c\},$$

where c is constant.

Since the constant, a, is the result after taking the absolute value, the smallest possible value for c is zero. If $c = 0$, then

$$\{P \in R_2 : |d(P,A) - d(P,B)| = c\} = \{P \in R_2 : d(P,A) = d(P,B)\} .$$

If we are in E_2, this is usually called the "perpendicular bisector" of the line segment AB. However in T_2, the "bisector" may not even be a straight line. (See Figure 11.) (I leave a question for my reader: Under what conditions on the fixed points A and B will this "bisector" be a straight line in T_2?)

To investigate taxi-hyperbolas we can use a method similar to the one we used for drawing ellipses; that is, we can find the intersection of pairs of circles centered at A and B whose radii are r_A and r_B, respectively, where $|r_A - r_B| = c$. However, we quickly observe that when c is greater than $d(A,B)$, the circles will fail to intersect (Figure 12),

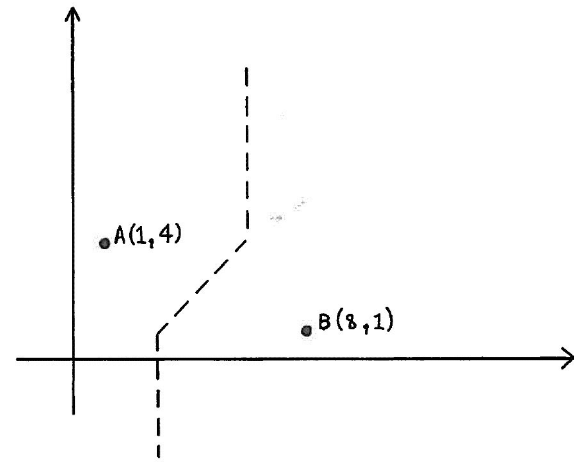

FIGURE 11

and hence the hyperbola for any value of $c > d(A,B)$ will be the null set. (This result holds whether we use d_T or d_E.) So the constant, c, is bounded between zero and $d(A,B)$: $0 \le c \le d(A,B)$.

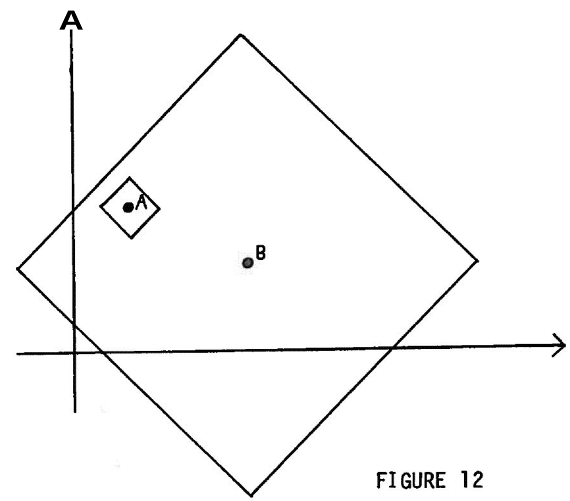

FIGURE 12

While the shape of the ellipse depended only on the shape of the little rectangle determined by its foci, A and B, the shape of the hyperbola is also dependent on the magnitude of the constant, c, relative to the difference in the lengths of the sides of the rectangle. In other words, if we define $k = \left||a_1 - b_1| - |a_2 - b_2|\right|$, the shape of the hyperbola depends partly on whether c is less than, equal to, or greater than k. (See Figure 13.)

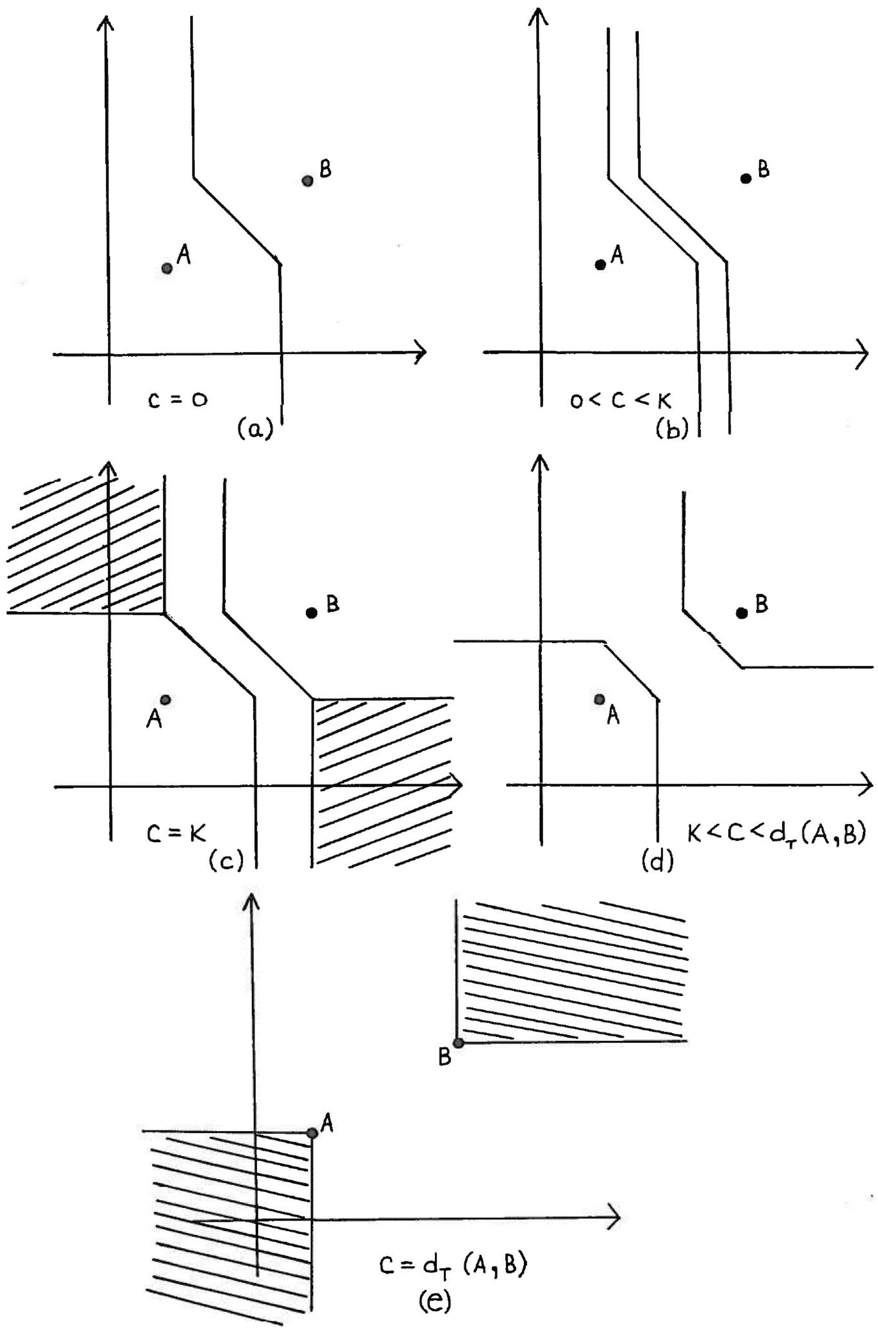

FIGURE 13

An interesting thing happens if c is equal to k or to $d_T(A,B)$. Any pair of circles centered at A, B, respectively, whose radii differ by either k or $d_T(A,B)$ intersect, not in one or two points, but in a line segment. And so in either case the hyperbola is not linear and the shaded areas are part of the hyperbola. (Look again at Figure 13-c, e.) Furthermore, A and B can be chosen so that k = 0. Then the "bisector" of AB is not linear (Figure 14.)

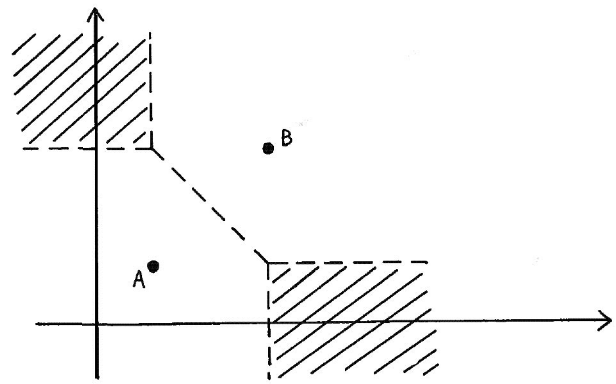

FIGURE 14

Parabola:

So far we have considered circles, ellipses, and hyperbolas. Next we might want to consider parabolas in this taxicab space. A parabola may be defined as the set of all points which are equidistant from a given point, F, called the focus, and a given line, D, called the directrix. That is, given a fixed point, F, and a fixed line, D, in R_2, a parabola is the collection of points

$$\{P \in R_2 : d(P,F) = d(P,D)\}.$$

But this raises the question of just exactly how do we measure the distance from a point to a line. In E_2, we sometimes speak of "dropping a perpendicular" from the point P to the line D. But in T_2 we have already discovered that the "perpendicular bisector" of two points is usually not straight, and may not even be a line. So we might be a little suspicious of the "perpendicular" from a point to a line in T_2. We would like to define the distance from a point P to a line D as the length of the shortest path from P to D, or as the distance from P to the point of D which is closest to P. Is there a systematic way of finding the point on D which is closest to P? In Figure 15, which of A, B, C is closest to P? Is there any point of D which is closer to P than these?

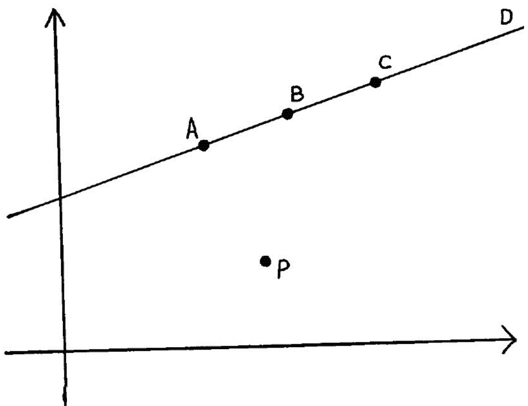

FIGURE 15

Conclusion:

In this article I have defined the taxicab space, T_2, and have given some examples of ideas which are familiar to us in "ordinary" Euclidean space, E_2, but which are quite different in T_2. I leave the reader now with two big questions to explore:

1). How do we measure the distance from a point to a line in T_2?

2). What do parabolas look like in T_2?

And, just perhaps, in looking for the answers to these questions, you will discover a number of other interesting properties of taxicab geometry...

This paper was written while Dr. Reynolds was a graduate student at St. Louis University. Currently she is on the faculty at Cardinal Stritch College, Milwaukee, Wisconsin.

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THE ASYMPTOTES OF AN OBLIQUE HYPERBOLA

by Stewart Venit
California State University

Once the asymptotes of a given hyperbola have been computed, it is a simple matter to determine the extent of its graph and to quickly sketch it. Now if the hyperbola is positioned so that its transverse axis is horizontal or vertical, one can relatively easily compute the asymptotes from its equation - this requires little more than a translation of axes via a "completing the square" process. However, if the transverse axis is oblique (neither horizontal nor vertical), the standard procedure for finding the asymptotes takes considerably more work. It necessitates a cumbersome rotation of axes in addition to the translation. In this note we will provide a simpler technique for determining the asymptotes of an oblique hyperbola - one that requires only the solution of a pair of linear equations and a single quadratic one.

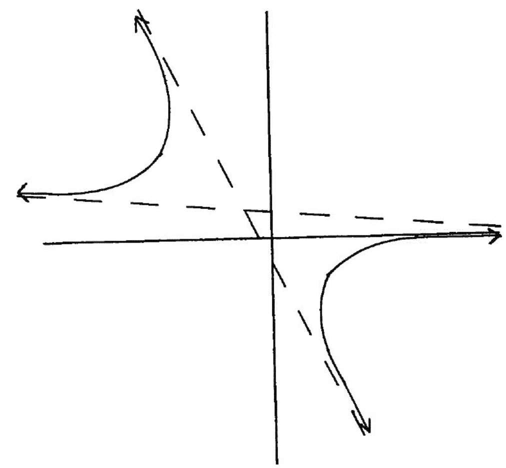

FIGURE 1

The general equation of a hyperbola may be written

(1) $$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

where A, B, C, D, E and F are real numbers with $B^2 - 4AC > 0$. To determine the asymptotes we employ the following theorem together with the point-slope form of the equation of a straight line (as illustrated in the example below). The proof of this theorem will be deferred until the end of the note.

Theorem. Consider the hyperbola given by equation (1).

(i) Its asymptotes intersect in the point (p,q), where (p,q) is the unique solution of the linear system

$$(2)\qquad \begin{aligned} (2A)p + (B)q &= -D \\ (B)p + (2C)q &= -E. \end{aligned}$$

(ii) If $C \neq 0$, the slopes of the asymptotes are the distinct real roots of the quadratic equation

$$(3)\qquad Cm^2 + Bm + A = 0.$$

If $C = 0$, the hyperbola has one vertical asymptote and one with slope equal to $-A/B$.

Note. As we shall see in the proof of the Theorem, the hyperbola (1) is degenerate if and only if $F = Ap^2 + Bpq + Cq^2 + Dp + Eq$, where (p,q) is the solution of (2). In this case, the graph of (1) is a pair of lines whose point of intersection and slopes are given by the Theorem.

Example. Consider the hyperbola with equation $2x^2 + xy - y^2 - 7x + 5y = 0$. Here, $A = 2$, $B = 1$, $C = -1$, $D = -7$, $E = 5$ and $F = 0$. So, the linear system (2) becomes

$$\begin{aligned} 4p + q &= 7 \\ p - 2q &= -5, \end{aligned}$$

which has the solution $p = 1$, $q = 3$. Thus, both asymptotes pass through the point $(1,3)$. Also, since $C \neq 0$, the slopes of the asymptotes are the roots of (equation (3)) $-m^2 + m + 2 = 0$. That is, one asymptote has slope -1 and the other has slope 2. Finally, employing the point-slope form, $y - q = m(x - p)$, we obtain the equations of the asymptotes

$$y - 3 = (-1)(x - 1) \qquad (\text{or } y = -x + 4)$$

and

$$y - 3 = 2(x - 1) \qquad (\text{or } y = 2x + 1).$$

Proof of the Theorem. We first translate the x- and y-axes by means of the change of variables

$$(4)\qquad x = \bar{x} + p,\ y = \bar{y} + q$$

Substituting (4) into (1), we obtain

$$A\bar{x}^2 + B\bar{x}\bar{y} + C\bar{y}^2 + (2Ap + Bq + D)\bar{x} + (Bp + 2Cq + E)\bar{y} = F',$$

where $F' = F - (Ap^2 + Bpq + Cq^2 + Dp + Eq)$. Thus, if (p,q) satisfies the system (2), equation (1) is transformed into

$$(5)\qquad A\bar{x}^2 + B\bar{x}\bar{y} + C\bar{y}^2 = F'.$$

Moreover, this transformation is always possible - the linear system (2) has a unique solution since the determinant of its coefficient matrix, $4AC - B^2$, is nonzero.

Now, equation (5) represents a hyperbola with center at $\bar{x} = 0$, $\bar{y} = 0$, so, by virtue of (4), in the xy-coordinate system its center is located at $x = p$, $y = q$. But the center of a hyperbola is also the point of intersection of its asymptotes. Thus, we have verified part (i) of the Theorem.

Before proceeding to part (ii), notice that equation (5) can be written as $(a\bar{x} + b\bar{y})(c\bar{x} + d\bar{y}) = 0$, where a, b, c and d are real, if and only if $F' = 0$. Consequently, equation (5), and hence equation (1) as well, represents a pair of lines (i.e., a "degenerate hyperbola") if and only if $F - (Ap^2 + Bpq + Cq^2 + Dp + Eq) = 0$, where (p,q) satisfies (2).

It suffices to prove our assertions of part (ii) for equation (5) (rather than (1)) since under the translation (4) both the slope of a line and the constants A, B and C remain unchanged.

If $C \neq 0$, solving equation (5) for $\bar{y}$ we obtain

$$\bar{y} = \left(-B\bar{x} \pm \sqrt{(B\bar{x})^2 - 4C(A\bar{x}^2 - F')}\right)/2C$$

or

$$\bar{y} = \left(-B\bar{x} \pm \sqrt{(B^2 - 4AC)\bar{x}^2 + 4CF'}\right)/2C.$$

Now, as $|\bar{x}| \to \infty$ the term, $4CF'$ becomes negligible compared to $(B^2 - 4AC)\bar{x}^2$, so

$$\bar{y} \to \left(-B\bar{x} \pm \sqrt{(B^2 - 4AC)\bar{x}^2}\right)/2C;$$

i.e.,

$$\bar{y} \to [(-B \pm \sqrt{B^2 - 4AC})/2C]\bar{x}.$$

But the coefficients of $\bar{x}$ in the last expression are the real distinct (since $B^2 - 4AC > 0$) roots of equation (3). Thus, if $C \neq 0$, the asymptotes have the slopes claimed,

If $C = 0$, equation (5) takes the form

$$(6) \qquad A\bar{x}^2 + B\bar{x}\bar{y} = F',$$

in which B cannot be zero or else $B^2 - 4AC = 0$, a contradiction. Now, if $A = 0$, equation (6) reduces to $\bar{x}\bar{y} = F'/B$, which we know has one vertical asymptote and one of slope $0 = -A/B$, as desired. Hence, we may assume that $A \neq 0$ and solve equation (6) for $\bar{x}$ obtaining

$$\bar{x} = (-B\bar{y} \pm \sqrt{(B\bar{y})^2 + 4AF'})/2A .$$

Now, as $|\bar{y}| \to \infty$, the term $4AF'$ becomes negligible compared to $(B\bar{y})^2$ so

$$\bar{x} \to (-B\bar{y} \pm \sqrt{(B\bar{y})^2})/2A = \left[(-B \pm B)/2A\right]\bar{y} .$$

That is, as $|\bar{y}| \to \infty$, $\bar{x} \to 0$ or $\bar{x} \to (-B/A)\bar{y}$. Thus, in this case equation (5) has the asymptotes $\bar{x} = 0$ and $\bar{x} = (-B/A)\bar{y}$ (i.e., $\bar{y} = (-A/B)\bar{x}$), and the assertion is proved.

1978-79 CONTEST

The papm for the 1978-79 Student Papal Contest have. judged and the. first two prizes were announced and the papers published in the Fall 7979 Issue of this Journal. The papal beginning on the next page won Third Prize. Entries for the 1979-80 Contest are now being accepted. First prize -64 $200, Second prize -64 $100 and Third prize -64 $50. Authors are eligible if they have. not received their Master's degree at the. time they submit their papal.

Each Chapter which submits five papm creates a mini-content amoung just those papers. The but will receive $20 and all me. eligible for the National Content.

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Gold Clad key-pins are available at the National Office (The University of Maryland, Department of Mathematics) at the special price of $8.00. Be sure to indicate the Chapter into which you were initiated and the approximate date of initiation.

Third Prize Paper

A MATRIX MODEL FOR USE IN POPULATION ECOLOGY

by Michael J. Young
Portland State University

One of the most important applications of mathematics in ecology has been in models for population growth. The literature on the subject is vast and dates back from the time of Malthus in the 18th century. Even what has been termed the basic principle of population ecology is stated in mathematical language: In an unlimited, constant, and favorable environment, the number of individuals of a species will increase exponentially [Poole, 19741.

The most common approach to modeling populuation growth has been through the use of differential equations first formulated by A. J. Lotka in 1925. However, a more versatile model using matrix algebra was developed by [Leslie; 1945, 1948, 1959]. This paper will give the development of what is now called the Leslie matrix model and explain its particular usefulness to population ecology.

Like all models of natural phenomena, the Leslie matrix approach idealized the object of study. The following assumptions are made: population changes are based solely on age dependent birth and mortality rates; there are no population changes due to immigration or emigration. The sex ratio of males to females is assumed to remain constant and only the changes in the female portion of the population are considered. Finally, both time and age are treated discretely, and the birth and death rates within each age interval remain constant; they differ from one interval to the next.

This model has been used to describe the dynamics of a wide variety of ecological populations [see p. 42 of Pielou, 1969] with a great deal of success.

Consider a population that is divided into $m + 1$ age classes where the $(m + 1)$st is the age class of the last surviving member of the population. Then at a time $t = 0$, each of the $m + 1$ age classes can be represented by an element in the column vector

$$n_0 = \begin{bmatrix} n_{00} \\ n_{10} \\ n_{20} \\ \vdots \\ n_{m0} \end{bmatrix}$$

where the first subscript denotes the age class and the second time. Thus, n_{00} represents the number of individuals in first age class at time $t = 0$, and n_{m0} the number in the $(m+1)$st age class at time $t = 0$. It is assumed that none of the females live past the age class m. The age intervals are of the same length as the time intervals.

Associated with this age distribution vector are the following age-specific statistics:

F_x = the number of daughters born per female aged x at time t, who will be alive in the initial age class at time $t + 1$.

P_x = the probability that a female aged x at time t will be alive in the age class $x + 1$ at time $t + 1$.

In matrix form, the age distribution vector at time $t = 1$ is given by

$$\begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} & F_m \\ P_0 & 0 & \cdots & 0 & 0 \\ 0 & P_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & P_{m-1} & 0 \end{bmatrix} \begin{bmatrix} n_{00} \\ n_{10} \\ n_{20} \\ \vdots \\ n_{m0} \end{bmatrix} = \begin{bmatrix} F_0 n_{00} + \cdots + F_n n_{m0} \\ P_0 n_{00} \\ P_1 n_{10} \\ \vdots \\ P_{m-1} n_{-1,0} \end{bmatrix} = \begin{bmatrix} n_{01} \\ n_{11} \\ n_{21} \\ \vdots \\ n_{m1} \end{bmatrix}$$

or

$$Mn_0 = n_1$$

where M is the Leslie (or projection) matrix.

This matrix M consists of $m + 1$ rows and $m + 1$ columns, and holds the fecundity (number of offspring) and survivorship data. Except for the first row and the sub-diagonal immediately below the main diagonal, all other entries are zero. The survivorship values and strictly between 0 and 1 and the fecundity values are non-negative.

It should be noted that since $Mn_0 = n_1$ and $Mn_1 = M^2 n_0 = n_2$ that

$$M^t n_0 = Mn_{t-1} = n_t$$

where n_t denotes the age distribution vector after t time units have elapsed.

Example

Let the matrix M and the initial age distribution vector n_0 be as follows:

$$M = \begin{bmatrix} 0 & 0 & 12 \\ 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} ; \quad n_0 = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

Then the age distribution at time $t = 1$ is

$$Mn_0 = n_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

Subsequent age distribution vectors are found in the same way, i.e., $n_2 = Mn_1 = M^2 n_0 = (36\ 0\ 2)'$, $n_3 = Mn_2 = M^3 n_0 = (24\ 12\ 0)'$, etc.

Properties of the Basic Matrix

In examining the properties of the matrix M, it is not necessary to consider the whole matrix. Unless the females in the population are reproductive until the end of their lifespan, some of the entries of the first row of the matrix M will be zero. In other words, if x = k is the last age class within which reproduction occurs, then F_k is the last F_x figure that is not equal to zero. Now the matrix M can partitioned symmetrically at this point to give

$$M = \left[\begin{array}{c|c} A & 0 \\ (k+1) \times (k+1) & (k+1) \times (m-k) \\ \hline B & C \\ (m-k) \times (k+1) & (m-k) \times (m-k) \end{array}\right]$$

Then forming the series of matrices M^2, M^3, M^4, ...,

$$M^t = \begin{bmatrix} A^t & 0 \\ f(ABC) & C^t \end{bmatrix}$$

Now C is triangular with its only nonzero elements $P_{k+1}, \ldots, P_{m-1}$ on the subdiagonal. Therefore, when $t \geq m - k$, $C^t = 0$ and M^t has zeroes in last m - k columns. This expresses the biological fact that females alive in the post-reproductive ages contribute nothing to the population when they themselves are dead after a maximum of m - k time units.

In considering only the portion of the population of reproductive age, we need only examine the matrix A.

First of all, we note that A is square of order k + 1. The matrix is nonsingular since the determinant $|A| = (-1)^{k+2}(P_0P_1 \ldots P_{k-1}F_k)$. However, more importantly for our purposes, the matrix A is non-negative (the entries are greater than or equal to zero) and irreducible (it is impossible to get from A, however one permutes the rows with each other, or the columns with each other, a matrix that can be partitioned into square submatrices one of which is 0 of order n > 1). It is these conditions that will be referenced in the next part of this paper.

The Stable Age Distribution

An important question now arises: does there ever come a time when the age distribution vector becomes stable? In other words, at a time t = s, does there exist a vector n and a scalar A such that

$$An = An .$$

This is the same as asking if a latent root and the latent vector associated with it exist for the matrix A. Since A is non-negative and irreducible, it satisfies the necessary conditions of the Perron-Frobenius Theorem: any matrix that meets these conditions has at least one positive real latent mot of multiplicity one, say λ_1, whose value is greater than or equal to the modulus of any complex root of the matrix (i.e., $\lambda_1 \geq |\lambda_i|$ for all $i \neq 1$ where the λ's are the latent roots).

To show that A_1 is the only positive root, we consider the coefficients of the characteristic equation of A which is

$$A - \lambda I = 0$$

Let $P_{(r)} = P_0P_1 \ldots P_r$, and expand the determinant to give

$$\lambda^{k+1} - F_0\lambda^k - P_{(0)}F_1\lambda^{k-1} - \ldots - P_{(r-1)}F_r\lambda^{k-r} - \ldots - P_{(k-1)}F_k = 0$$

The left-hand side has only one change of sign, and by Descartes' rule of signs, the equation has at most one positive real root.

Therefore, except for λ_1, all the roots of A are negative or complex [Bellman, 1960].

For the biologist, λ_1, called the <u>dominant latent root</u>, is of great

importance. Being real and positive, it is the only latent root that will give rise to a stable age distribution consisting of real and positive elements. The dominant latent root λ_1 is also called the finite rate of <u>increase</u>, and is related to the <u>intrinsic rate of increase</u>, r, by

$$ln\lambda_1 = r .$$

Thus, λ_1 can be thought of as the multiplicative growth factor per generation

$$N_{t+1} = \lambda_1 N_t$$

and r as the "compound interest" growth rate

$$N_t = N_0 e^{rt} ,$$

where N_t represents the population total at any time t. It must be remembered that terms λ_1 and r are only appropriate when the stable age distribution of a population exists.

Finding the Stable Age Distribution and the Dominant Latent Root

While the dominant latent root of the matrix A may be found thmugh the process of diagonal expansion, it is much more convenient to transform it from its original coordinate system to a new set of coordinates. Leslie (1948) has suggested the nonsingular transformation

$$B = HAH^{-1}$$

where H is a diagonal matrix with elements $(P_0P_1 \ldots P_{k-1})$, $(P_1P_2 \ldots P_{k-1})$, ..., $(P_{k-2}P_{k-1})$, $P_{k-1})$, 1, which are derived from the matrix A. Thus letting $P_{(r)} = P_0P \quad P_r$,

$$B = \begin{bmatrix} F_0 & P_{(0)}F_1 & P_{(1)}F_2 & \ldots & F_{(k-2)}F_{k-1} & P_{(k-1)}F_k \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

The effect of this transformation is to replace the P_x elements in the; principal subdiagonal of A by a series of units, and reduce A to the rational canonical form. In biological terms, it is equivalent to transforming the original population into one in which all the individuals live until their span of reproductive life is completed at the age of $k + 1$.

It can be noted that since the transformation is collinear, the matrices A and B have the same characteristic equation and thus the same latent roots. Then the vector v_s such that

$$Bv_s = \lambda v_s$$

is the stable age distribution of the matrix B, and is related to ns by

$$ns = H^{-1} v_s$$

Example

Referring to the previous example, we let $A = M$, and find that

$$H = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } H^{-1} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$B = HAH^{-1} = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

and the characteristic equation is found by examining the coefficients of the first row of B,

$$\begin{aligned} \lambda^3 - 3\lambda - 2 &= (\lambda - 2)(\lambda^2 + 2\lambda + 1) \\ &= (\lambda - 2)(\lambda + 1)^2 \\ &= 0 \end{aligned}$$

Then the dominant latent root $\lambda_1 = 2$, and the stable age distribution v_s is computed by letting

$$v_{0s} = \lambda_1 v_{1s},\ v_{1s} = \lambda_1 v_{2s}, \ldots, v_{k-1,s} = \lambda_1 v_{ks}$$

and arbitrarily setting v_{ks} equal to one. Then we have $v_s = (4\ 2\ 1)'$ and by $n_s = H^{-1} v_s$, we find that the stable age distribution of the matrix A is $n_s = (24\ 4\ 1)'$.

Remarks

Through the use of computers and numerical methods, matrices much

larger than the ones shown in the examples can be handled with ease. It is this "programmability" that makes this Leslie approach so popular with ecologists. Hypothetical populations or natural populations based on field data can be studied under a variety of conditions using this model.

Pennycuick, et al., (1968) consider such factors as density effects and time lag, and Darwin and Williams (1964) consider the effect of hunting on a population.

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THEORY ON INTEGRAL EQUATIONS OF THE FORM

$$A_1^d + A_2^d + \cdots + A_n^d + B^d = C^d$$

by Kevin Theall

We wish to find integral solutions (B,C) to equations of the form

$$A_1^d + A_2^d + \cdots + A_n^d + B^d = C^d$$

where $A_1, \cdots, A_n$ is a sequence of integers. When $d = 2$ or 3 we describe methods for obtaining all solutions. When $d > 3$, we show that B must be a root of a certain algebraic equation.

First we consider the case $d = 2$.

Lemma. If $M = f\cdot N$ is an integer where f and N have the same parity and $f \le N$, then M can be expressed as the sum of f consecutive odd integers.

Proof: If M is odd, consider the sum $S = N - (f-1) + N - (f-3) + \cdots + N + \cdots + N + (f-3) + N + (f-1)$. There are exactly fterms in this sum. Moreover, $S = N + (f-1) + N + (f-3) + \cdots + N + \cdots + N - (f-3) + N - (f-1)$, so $2S = 2N + 2N + \cdots + 2N + \cdots + 2N + 2N = 2N\cdot f$. Thus $S = N\cdot f = M$.

Similarly, if M is even, then $M = N - (f-1) + \cdots + (N-1) + (N+1) + \cdots + \# + (f-1)$.

Example 1.

Express 21 as a sum of one or more series of consecutive odd integers.

Solution: Factoring yields the following pairs of factors whose. products are 21 and which satisfy the conditions of the Lemma:

$$(1,21)\ ,\ (3,7).$$

In the first case let $f = 1$; then N becomes 21 and the only odd term. In the second case, let $f = 3$ and $N = 7$. We have by our Lemma,

$$21 = 5 + 7 + 9.$$

Example 2.

Express 24 as a sum of one or more series of consecutive odd integers.

Solution: The proper factor pairs whose products are 24 are:

$$(4,6)\ ,\ (2,12).$$

By letting $f = 4$ and $f = 2$ respectively, we get

$$24 = 3 + 5 + 7 + 9 \text{ and}$$
$$24 = 11 + 13.$$

Theorem 1. Let $A_1, \cdots, A_n$ be a sequence of positive integers and set $M = A_1^2 + \cdots + A_n^2$. If B and C are solutions to the equation

(1) $A_1^2 + \cdots + A_n^2 + B^2 = C^2$,

then there exists an integer f such that $B = (M - f^2)/2f$ and $C = B + f$. Here $M = f\cdot N$, where f and N have the same parity and $f \le N$. Moreover, all solutions B and C to equation (1) arise this way.

Proof. The key to the proof is to exploit the well-known formula

$$1 + 3 + \cdots + 2k - 1 = k^2.$$

First suppose $M = f\cdot N$, where $f \le N$ and f, N have the same parity. By the Lemma

$$M = N - (f-1) + N - (f-3) + \cdots + N + (f-3) + N + (f-1).$$

Now

$$1 + 3 + \cdots + N - (f+1) = \left(\frac{N-f}{2}\right)^2 = B^2,$$

where

$B = \frac{N-f}{2} = \frac{M/f-f}{2} = \frac{M-f^2}{2f}$ is an integer because $N \equiv f \pmod 2$.

Hence $M + B^2 = 1 + 3 + \cdots + N - (f+1) + N - (f-1) + \cdots + N + (f-1)$

$$= \left(\frac{N+f}{2}\right)^2 = C^2, \text{ where } C = \frac{N+f}{2} = \frac{N-f}{2} + f = B + f.$$

Suppose now B and C are a solution to equation (1). Then $B^2 = 1 + 3 + \cdots + (2B - 1)$ and $C^2 = 1 + 3 + \cdots t\ 2C - 1$. Hence $M = C^2 - B^2 = (2B + 1) + \cdots + (2C - 1) = (2C - 1) + \cdots + (2B + 1)$; so $2M = 2(B + C) + \cdots + 2(B + C) = 2(B + C)\cdot f$, where f is the number of terms. Thus $M = C^2 - B^2 = (C + B)(C - B) = (C + B)f$, making $f = C - B$. Setting $N = C + B$, we have $f \equiv \pmod 2$ and $f \le N$. Moreover, $B = (N - f)/2 = (M - f^2)/2f$ and $C = B + f$.

Example. Find all solutions (B,C) to the equation

$$12^2 + 16^2 + B^2 = C^2.$$

Here $M = 400$. Factoring M into factor pairs whose products are 400 and which satisfy the conditions of the Theorem, we get

$$(2,200),\ (4,100),\ (8,50),\ (10,40), \text{ and } (20,20).$$

It is important to realize that throughout the proofs we set $f \le N$ so that calculations are facilitated when employing the formula. Now using the main result in the Theorem we have

1. $B = \frac{400-2^2}{2(2)} = 99$; $C = 101$
2. $B = \frac{400-4^2}{2(4)} = 48$; $C = 52$
3. $B = \frac{400-8^2}{2(8)} = 21$; $C = 29$
4. $B = \frac{400-10^2}{2(10)} = 15$; $C = 25$
5. $B = \frac{400-20^2}{2(20)} = 0$; $C = 20$

Furthermore, by properly affixing minus signs, we obtain all of the solutions (B,C) as follows:

(99,101), (-99,-101), (-99,101), (99,-101);
(48,52), (-48,-52), (-48,52), (48,-52);
(21,29), (-21,-29), (-21,29), (21,-29);
(15,25), (-15,-25), (-15,25), (15,-25);
(0,20), (0,-20).

Theorem 2. Let $M = A_1^3 + \cdots t\ A_n^3$. If $A_1^3 + \cdots + A_n^3 + B^3 = C^3$ has an integral solution (B,C), then there exists a divisor f of M with $f \equiv M \pmod 2$ such that $\sqrt{12Mf - 3f^4}$ is integral. Moreover, $B = -f/2 \pm \frac{1}{6f}\sqrt{12Mf - 3f^4}$ and $C = B + f$. Conversely, if $\sqrt{12Mf - 3f^4}$ is integral, then the values of B and C described in the previous statement form an integral solution to $M + B^3 = C^3$.

Proof. If $M + B^3 = C^3$, then $M = C^3 - B^3 = (C - B)(C^2 + CB + B^2)$. Letting $f = (C - B)$, f divides M and

$$M \equiv C^3 - B^3 \equiv C - B \equiv f \pmod 2$$

since $a^3 \equiv a \pmod 2$ for any integer a.

Now

$$M = C^3 - B^3 = (B + f)^3 - B^3 = 3B^2f + 3Bf^2 + f^3.$$

Thus $3fB^2 + 3f^2B + f^3 - M = 0$ and so

$$B = \frac{-3f^2 \pm \sqrt{9f^4 - 12f^4 + 12Mf}}{6f}$$

$$= -\frac{f}{2} \pm \frac{1}{6f}\sqrt{12fM - 3f^4}.$$

Since B is integral, so is $\sqrt{12fM - 3f^4}$.

Conversely, assume that $\sqrt{12fM - 3f^4}$ is integral. Then $D = 12fM - 3f^4$ is a square. Since $f|M$, it follows that $f^2|D$. Now $D/f^2 = 12M/f - 3f^2$ is divisible by 3 and since D/f^2 is a square it must also be divisible by 9. Thus $9f^2|D$ and $3f|\sqrt{D}$. Also

$$D/9f^2 \equiv D/f^2 \equiv 12M/f - 3f^2 \equiv f (\text{mod } 2),$$

so that

$$\sqrt{D}/3f \equiv D/9f^2 \equiv f (\text{mod } 2).$$

Hence $-f \pm \sqrt{D}/3f \equiv 0 (\text{mod } 2)$ and

$$B = -f/2 \pm \frac{1}{6f}\sqrt{D} \text{ is an integer.}$$

Since $M = C^3 - B^3 = (B + f)^3 - B^3$ if and only if

$$M = 3B^2f + 3Bf^2 + f^3,$$

we need only prove the latter statement. This is done by the following computation:

$$3B^2f + 3Bf^2 + f^3 =$$

$$3(-f/2 \pm \frac{1}{6f}\sqrt{D})^2 f + 3(-f/2 \pm \frac{1}{6f}\sqrt{D})f^2 + f^3 =$$

$$3f(f^2/4 \pm \frac{1}{6}\sqrt{D} + \frac{1}{36f^2} \cdot D) + 3f^2(-f/2 \pm \frac{1}{6f}\sqrt{D}) + f^3 =$$

$$\frac{3f^3}{4} \pm \frac{f}{2}\sqrt{D} + \frac{1}{12f}D - \frac{3f^3}{2} \pm \frac{f}{2}\sqrt{D} + f^3 =$$

$$\frac{1}{4}f^3 + \frac{1}{12f}D = \frac{1}{4}f^3 + M - \frac{1}{4}f^3 = M.$$

Example. Find all integral solutions (B,C) to the equation

$$3^3 + 4^3 + B^3 = C^3.$$

First we set $M = 91$ and then factor M to obtain the following factor pairs (N,f):

$$(1,91),\ (7,13).$$

Both must be tested in the formula we just proved. By substituting

all four integers into the formula one finds that the only two which satisfy the conditions of the Theorem are $(1,7)$. Thus,

$$B = -\frac{1}{2} \pm \frac{1}{6(1)}\ 12(91)(1) - (3)(1)^4 = 5 \text{ or } -6;\ C = 6 \text{ or } -5.$$

$$B = -\frac{7}{2} \pm \frac{1}{6(7)}\ 12(91)(7) - (3)(7)^4 = -3 \text{ or } -4;\ C = 3 \text{ or } 4.$$

Theorem 3. If (B,C) is a solution to $A_1^d + \cdots + A_n^d + B^d = C^d$, then B is a root of the equation

$$a_1 f x^{d-1} + a_2 f^2 x^{d-2} + a_{d-1} f^{d-1} x + (f^d - M).$$

Here $M = A_1^d + A + \cdots + A_n^d$, f is a divisor of M with $f \equiv M$ (mod 2) and $a_k = \binom{d}{k}$ is a binomial coefficient. Moreover $C = B + f$ and $f \leq [\sqrt[d]{M}]$.

Proof. Note $M = C^d - B^d = (C - B)(C^{d-1} + C^{d-2} + \cdots + B^{d-1})$. Setting $f = C - B$, we have $C = B + f$ and $M \equiv C^d - B^d \equiv C - B \equiv f$ (mod 2) since $a^d \equiv a (\text{mod } 2)$ for any a. Now

$$M = C^d - B^d = (B + f)^d - B^d$$

$$= \sum_{k=0}^{d} \binom{d}{k} B^{d-k} f^k - B^d$$

$$= \sum_{k=1}^{d} a_k f^k B^{d-k}.$$

Thus,

$$a_1 f B^{d-1} + a_2 f^2 B^{d-2} + \cdots + a_{d-1} f^{d-1} B + f^d - M = 0$$

as desired.

Dividing the equation

$$M = a_1 f B^{d-1} + a_2 f^2 B^{d-2} + \cdots + a_{d-1} f^{d-1} B + f^d$$

by f^d gives

$$M/f^d = a_1 (B/f)^{d-1} + a_2 (B/f)^{d-2} + \cdots + a_{d-1} B/f + 1.$$

Hence

$$M/f^d \leq 1$$

or

$$M \leq f^d.$$

It follows $f \leq \sqrt[d]{M}$, and because f is an integer, that

$$f \leq [\sqrt[d]{M}].$$

CHAPTER REPORTS

California Lambda (University of California, Davis). The report is late because of romantic distractions involving the Chapter Secretary. A variety of programs were held, and several guest speakers participated: *Prof. Donald W. Crowe,* "Patterns of Primitive Art"; *Prof. David Barnette,* "How They solved the Four Color Problem"; *Prof. Alan Edelson,* "Solving Differential Equations With Topology"; *Dr. Ronald Graham,* (Bell Labs), "Problems on Progressions"; *Mr. Darrell Winn,* (Dramatic Arts Department, UC Davis), "Additive Mixing of Colored Light--White Light is a Myth".

Minnesota Delta (St. John's University). A regional Pi Mu Epsilon Conference was held on April 10 and 11. *Professor Mary Ellen Rudin* (University of Wisconsin) gave three lectures--"What Does a Topologist Do?", "What Does Set Theory Have to Do With Mathematics?" and "Crazy Spaces". There were sessions for contributed papers and these will be listed in the Fall 1980 issue.

Montana Alpha (University of Montana). There was a film program which was very successful.

South Dakota Beta (South Dakota School of Mines and Techology). The Chapter participated in a variety of activities including a film program, The Western South Dakota Career Fair, The Western South Dakota Mathematics Contest (as proctors and graders) and guest speakers. The Chapter heard: *Prof. Patrick Brown* (College of Wooster), "The Four Color Problem".

Texas Epsilon (Sam Houston State University). The Chapter heard talks by *Dr. Harry Bohan* on teaching in secondary schools and *Dr. Burris* on the new telephone connected computer. *Oh. Robert Goad* gave a lecture on "Cartesian Products and the Ol' Baloney Slicing Technique".

Is your Chapter submitting material for this Section of the **Journal?** Reports *and* Lists of Chapter Awards should be sent to the Editor.

PROBABILITY OF BEING A LOSER

by Elliot A. Tanis
Hope College

1. *Introduction*

The American public is fascinated with gambling as evidenced by the fact that almost every state in the United States permits it in some form - bingo, horse racing, jai alai, casinos, or state run lotteries. Most students are also fascinated with probability questions connected with games of chance.

In this paper we determine the proportion of gamblers who are losers when playing certain games of chance. In particular, suppose that a bettor decides to continue to play until m \$1 bets have been placed. Let Y_m equal the total number of dollars "won" after placing these m bets. We shall consider the following questions for chuck-a-luck, craps, roulette, and the Michigan Daily Lottery (a numbers game):

(1) What is the value of $P(Y_m < 0)$? That is, what is the probability that the bettor is behind after placing m bets?

(2) What can be said about the distribution of Y_m?

2. *Chuck-A-Luck*

In the game of chuck-a-luck, a cage containing three dice is turned and the numbers on the dice are observed. One possible wager is for the bettor to place a \$1 bet on a particular number. The payoff is \$1 for each of the three dice that shows that number. The dollar is lost only when that number does not appear on any of the three dice.

If we let X denote the payoff for a single \$1 bet, the probability density function (*p.d.f.*) of X is defined by

$$f(x) = \begin{cases} (5/6)^3, & x = -1, \\ 3(1/6)(5/6)^2, & x = 1, \\ 3(1/6)^2(5/6), & x = 2, \\ (1/6)^3, & x = 3. \end{cases} \qquad (2.1)$$

The mean and variance of X are $\mu = -17/216 = -0.07870$ and $\sigma^2 = 57{,}815/216^2 = 1.23918$.

Suppose that m bets will be placed and let the payoffs on these bets be $X_1, X_2, \ldots, X_m$, a random sample from the distribution having p.d.f. $f(x)$. If we let $Y_m = \sum_{i=1}^{m} X_i$, then Y_m is equal to the number of dollars "won" after placing m bets..

We shall now determine the probability of being behind after placing $m = 300$ bets. Let $Y = Y_{300}$. Using the Central Limit Theorem, we obtain

$$P(Y < 0) \approx P\left(Z < \frac{-0.5 - 300(-0.07870)}{\sqrt{(300)(1.23918)}}\right)$$

$$= P(Z < 1.19859)$$

$$= 0.8846, \qquad (2.2)$$

where Z has a normal distribution with mean 0 and variance 1. That is, the probability of being behind after placing 300 bets is approximately 0.8846. It can easily be shown that the probability of being behind after placing $m = 1000$ bets is approximately 0.9868.

In Figure 1 we have depicted $P(Y_m < 0)$ for m going from 100 to 3,000.

From the Central Limit Theorem we know that the distribution of Y_m is approximately normal with mean $\mu = m(-0.07870)$ and variance $\sigma^2 = m(1.23918)$. It is interesting to illustrate this empirically. In particular we simulated $m = 300$ \$1 bets in chuck-a-luck for each of $n = 2000$ gamblers. For each of the 2000 trials, we kept track of the number of dollars "won". A histogram of these 2000 outcomes is depicted in Figure 2. Superimposed over the histogram is a normal probability density function with mean $\mu = 300(-0.07870) = -23.610$ and variance $\sigma^2 = 300(1.23918) = 371.754$. In this simulation the sample mean, $\bar{x} = -23.116$, is close to p, and the sample variance, $s^2 = 362.109$ is close to a^2. Also the proportion of gamblers who are behind is $1757/2000 = 0.8785$ which is close to $P(Y_{300} < 0) \approx 0.8846$.

3. *Roulette and Craps*

Consider games of chance in which a \$1 bet is placed and the probability of winning \$1 is p while the probability of losing \$1 is $1 - p$. If we let X denote the payoff for such a game, the probability density function of X is given by

$$f(x) = \begin{cases} 1 - p, & x = -1, \\ p, & x = 1. \end{cases} \qquad (3.1)$$

The mean and variance of X are $\mu = 2p - 1$ and $\sigma^2 = 4p(1 - p)$, respectively.

Again let $X_1, X_2, \ldots, X_m$ denote the outcomes of m bets and let $Y_m = \sum_{i=1}^{m} X_i$. Then Y_m is equal to the amount "won" after placing m bets. Using the Central Limit Theorem to determine the probability of being behind after placing m bets, we have

$$P(Y_m < 0) \approx P\left(Z < \frac{-0.5 - m(2p - 1)}{\sqrt{m4p(1 - p)}}\right), \qquad (3.2)$$

where Z has a standard normal distribution.

Two casino games will be used for illustration, namely, roulette and craps.

A possible bet in roulette is to bet on red. In this case, $p = 18/38 = 0.47368$ is the probability of winning on a particular bet. After placing $m = 300$ bets, the probability of being behind is (using equation 3.2)

$$P(Y_{300} < 0) \approx 0.8117. \qquad (3.3)$$

In Figure 3 is depicted $P(Y_m < 0)$ for m going from 100 to 3,000 for roulette in which the gambler places m \$1 bets, for each of which the probability of winning \$1 is $p = 18/38$. Examples of such bets are betting on red or betting on even.

In the game of craps, the probability of winning on a particular bet is $p = 0.49293$. After placing $m = 300$ bets, the probability of being behind is (using equation 3.2)

$$P(Y_{300} < 0) \approx 0.5856. \qquad (3.4)$$

In Figure 4 is depicted $P(Y_m < 0)$ form going from 100 to 3,000 for craps in which the gambler has placed m \$1 bets.

The distribution of Y_m could easily be illustrated empirically for

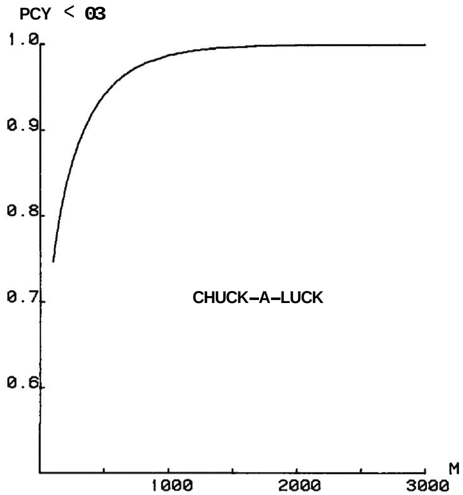

FIGURE 1

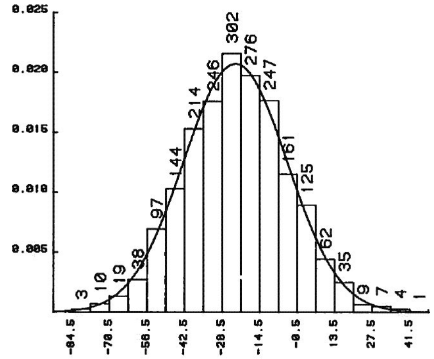

FIGURE 2

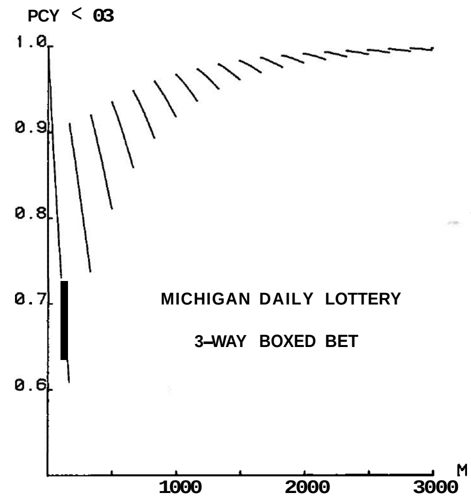

FIGURE 6

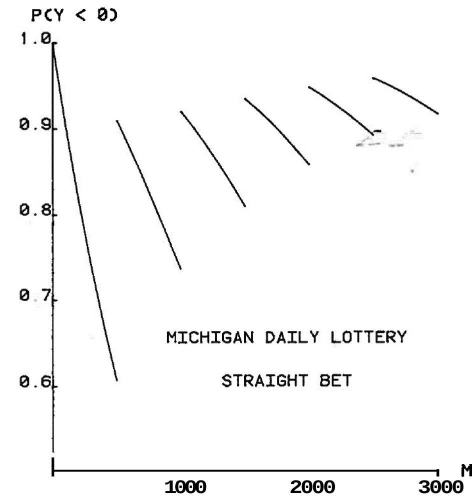

FIGURE 5

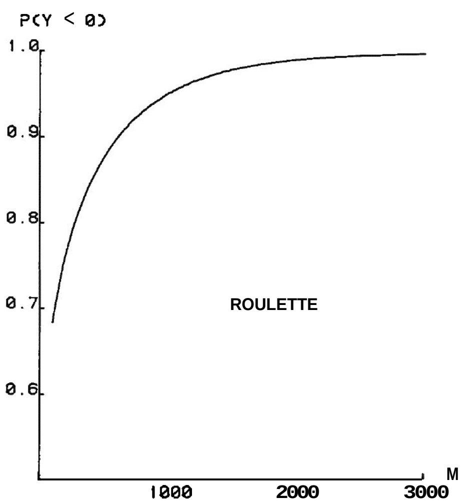

FIGURE 3

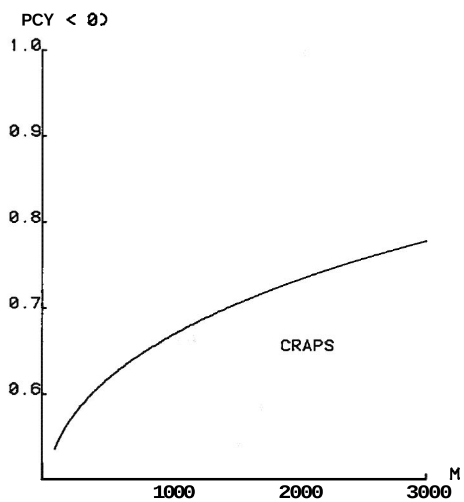

FIGURE 4

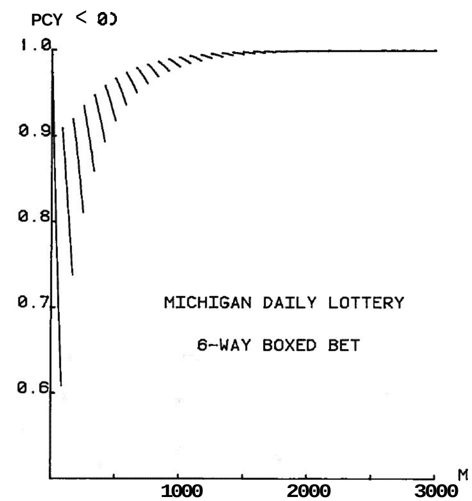

FIGURE 7

roulette and craps just as it was for chuck-a-luck.

4. *Michigan Daily Lottery*

In the Michigan Daily Lottery, a bettor may select a three digit integer from 000 to 999, inclusive, and place a \$1 bet on this number. If the state also selects this number, the prize to the bettor is \$500. Actually, the net gain to the bettor is only \$499 because the state does not return the \$1 bet. If X denotes the gain per bet, then $P(X = -1) = 0.999$ and $P(X = 499) = 0.001$.

Suppose that on m different days, a \$1 bet is placed. If we let $X_1, X_2, \ldots, X_m$ denote the payoffs on these m bets, then $Y_m = \sum_{i=1}^{m} X_i$ is the total number of dollars "won" after m bets. Before we consider $P(Y_m < 0)$ in general, we shall look at two special cases.

Suppose that m = 600 \$1 bets have been placed. Then the possible values for $Y = Y_{600}$ are -600, -100, 400, 900, 1400, ... Let W_m denote the number of winning tickets out of m bets. Then $P(W_{600} = k) = \binom{600}{k}(0.001)^k(0.999)^{600-k}$, for k = 0, 1, 2, ..., 600. That is, $W = W_{600}$ has a binomial distribution with parameters n = 600 and $p = 0.001$. After placing 600 bets, the probability that a bettor is behind is given by

$$\begin{aligned} P(Y_{600} < 0) &= P(Y = -600) + P(Y = -100) \\ &= P(W = 0) + P(W = 1) \\ &= (0.999)^{600} + 600(0.001)(0.999)^{599} \\ &= 0.87816. \end{aligned} \tag{4.1}$$

Suppose now that a bettor has purchased $m = 3{,}200$ tickets. Letting W_{3200} equal the number of winning tickets, we see that this bettor is a loser if $W_{3200} \le 6$ and a winner if $W_{3200} \ge 7$. We can again use the fact that W_{3200} has a binomial distribution with n = 3200 and $p = 0.001$. A Poisson approximation of the binomial probabilities with $\lambda = (3200)(.001) = 3.2$ makes the calculations very easy. We have

$$P(Y_{3200} < 0) = P(W_{3200}\ 5\ 6) \approx 0.955 \tag{4.2}$$

using a Poisson probability table. If $m = 3400$ tickets are purchased,

$$P(Y_{3400} < 0) = P(W_{3400} \le 6) \approx 0.942, \tag{4.3}$$

using a Poisson approximation with $\lambda = 3.4$. If an additional 200 tickets is purchased, we have

$$P(Y_{3600} < 0) = P(W_{3600} \le 7) \simeq 0.969, \tag{4.4}$$

using a Poisson approximation with $\lambda = 3.6$ Thus, we see that $P(Y_m < 0)$ does not increase monotonically as m increases, which was the case with the casino games of chance.

In general, if $500k < m\ 5\ 500(k+1)$, a bettor is a loser if $W_m \le k$. In Figure 5 we have depicted $P(Y_m < 0)$ for $1\ 5\ m\ 5\ 3000$. These bets are called straight bets.

Two other types of bets are possible in the Michigan Daily Lottery. These are called boxed bets. In a 3-way box, the bettor may box a number like 355 and win if the state selects either 355, 535, or 553. The payoff for winning is \$166. The gain to the bettor would be \$165 because again the state does not return the \$1 bet. If X denotes the gain per bet to the bettor, $P(X = -1) = 0.997$ and $P(X = 165) = 0.003$. If W_m denotes the number of winning tickets in m bets, W_m has a binomial distribution with parameters $n = m$ and $p = 0.003$.

For the 3-way boxed bet, if $166k < m \le 166(k+1)$, a bettor is a loser if $W_m \le k$. In Figure 6 we have depicted $P(Y_m < 0)$ for $1 \le m \le 3000$ where Y_m is the number of dollars "won" after m bets have been placed.

The other type of boxed bet is a 6-way box. A bettor may box a number like 678 and win if any of the 6 permutations of 678 is selected. Of course, the payoff is reduced and is equal to \$83. If X denotes the gain per bet to the bettor, $P(X = -1) = 0.994$ and $P(X = 82) = 0.006$.

For the 6-way boxed bet, if $83k < m$ a $83(k+1)$, a bettor is a loser if $W_m \le k$ where W_m is the number of winning tickets in m bets. In Figure 7 we have depicted $P(Y_m < 0)$ for $1 \le m \le 3000$ where Y_m is the number of dollars "won" after m bets have been placed.

In comparing these three different types of bets in the Michigan Daily Lottery, it is interesting to note that, although the 6-way boxed bet gives the largest probability of winning a prize (0.006), the probability of being behind after placing m bets increases to one the most rapidly with the 6-way boxed bet.

5. *Summary*

The examples in this paper give illustrations of approximating probabilities using the Central Limit Theorem and also of approximating binomial probabilities using the Poisson distribution. The long range expectation of being a loser was demonstrated for several games of chance.

The reader is encouraged to use simulation techniques on the computer to estimate $p = P(Y_m < 0)$ for any one of the games and for different values of m . He or she should decide how many times, say n , that the simulation should be repeated to give the confidence level and maximum error of the estimate desired.

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SOME CONDITIONS FOR ONE-TO-ONE-NESS

by Richard K. Williams
Southern Methodist University

If $f(z)$ is analytic in a convex domain D, a well-known sufficient condition for $f(z)$ to be one-to-one in D is that there exists a complex constant c such that $Re\{cf'(z)\} > 0$ in D. (See [2, p. 582].) The following theorem is an easy generalization of this condition.

Theorem 1. If $f(z)$ is analytic in a domain D, and if $\phi(z)$ is a one-to-one analytic function which maps D onto a convex domain, and if c is a complex constant such that $Re\left\{c\frac{f'(z)}{\phi'(z)}\right\} > 0$ in D, then $f(z)$ is one-to-one in D.

Proof. Let $\phi(D) = D_1$. Then if $g = f\phi^{-1}$, g is analytic in D_1, and if $w = \phi(z)$, $\text{Re}\{cg'(w)\} = Re\left\{c\frac{f'(z)}{\phi'(z)}\right\} > 0$ for each $w \in D_1$, so that g is one-to-one in D_1, and therefore f is one-to-one in D.

Theorem 1 clearly generalizes the condition mentioned in the first paragraph of this paper, as is seen by taking D to be convex and $\phi(z) \equiv z$. Theorem 1 is a slight generalization of Theorem 1 of [3], where D was taken to be the open unit disk.

The purpose of this paper is to derive three sufficient conditions for one-to-one-ness by specializing the choice of $\phi(z)$ in Theorem 1.

Theorem 2. If $f(z)$ is analytic for $|z| < 1$, if a is a real number such that $|a| \geq 1$, and if c is a complex number such that $Re\{c(z-a)^2 f'(z)\} > 0$ for $|z| < 1$, then $f(z)$ is one-to-one for $|z| < 1$.

Proof. Take $\phi(z) = -\frac{1}{z-a}$. Clearly $\phi(z)$ is analytic and one-to-one for $|z| < 1$. If $|a| > 1$, $\phi(z)$ maps $|z| < 1$ onto the open disk with center $\left(\frac{a}{a^2-1}, 0\right)$ and radius $\frac{1}{a^2-1}$. If $a = 1$, $\phi(z)$ maps $|z| < 1$ onto the half-plane $\text{Re}\{\phi(z)\} > 1/2$, and if $a = -1$, $\phi(z)$ maps $|z| < 1$ onto the half-plane $Re\{\phi(z)\} < -1/2$. Thus, in all cases, the range of $\phi(z)$ is convex. Also,

$Re\left\{c\,\frac{f'(z)}{\phi'(z)}\right\} = Re\left\{c(z-a)^2 f'(z)\right] > 0$ for $|z|<1$. By Theorem 1, $f(z)$ is one-to-one for $|z|<1$.

Theorem 3. If $f(z)$ is analytic for $|z|<1$, if a is a real number such that $|a|\geq 1$, and if c is a complex number such that $Re\{c(z-a)f'(z)\} > 0$ for $|z|<1$, then $f(z)$ is one-to-one for $|z|<1$.

Proof. Let $\phi(z) = \log(z-a)$, where the branch of the logarithm is chosen so that $\phi(z)$ will be analytic for $z<1$. (Make the cut along the negative real axis if $a<0$ and along the positive real axis if $a>0$. For each such branch, $\phi(z)$ is one-to-one. A well-known condition that a function $\phi(z)$, analytic and one-to-one for $|z|<1$, map $|z|<1$ onto a convex domain is that $Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} \geq 0$ for $|z|<1$. (See [1, p. 166].) Here $Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} = Re\left\{\frac{-a}{z-a}\right\} = \frac{-a(x-a)}{(x-a)^2 + y^2}$. If $a \geq 1$, $x-a<1-a \leq 0$, so $-a(x-a) > 0$. If $a \leq -1$, $x-a>-1-a \geq 0$, so $-a(x-a) > 0$. Hence, $\phi(z)$ maps $|z|<1$ onto a convex domain. Also $Re\left\{c\,\frac{f'(z)}{\phi'(z)}\right\} = Re\left\{c(z-a)f'(z)\right] > 0$ for $z|<1$, so that $f(z)$ is one-to-one for $|z|<1$.

Theorem 4. Let D be the domain $0 < \arg z < \frac{\pi}{n}$, where n is a non-zero integer. Let $f(z)$ be analytic in D, and let c be a complex number such that $Re\{cz^{n+1}f'(a)\} > 0$ in D. Then $f(z)$ is one-to-one in D.

Proof. Let $\phi(z) = -\frac{z^{-n}}{n}$. Clearly $\phi(z)$ is analytic in D, and it maps D in a one-to-one fashion onto $0<\arg[\phi(z)]<\pi$, whether n be positive or negative. Hence the range of ϕ is convex, and $Re\left\{c\,\frac{f'(z)}{\phi'(z)}\right\} =$ R e $^{n+1}f'(z)\}> 0$ in O, so that $f(z)$ is one-to-one in D.

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DEPENDENT EVENTS

by *Genovevo C. Lopez and Joseph M. Moser*
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The following definition of mutually independent events can be found in many texts; one of which is Feller, [1].

Definition. The events $A_1, A_2, \ldots, A_n$ are mutually independent if, for all combinations $1 \leq i < j < k < \ldots \leq n$, the multiplication rules

$$P[A_iA_j] = P[A_i]P[A_j]$$

$$P[A_iA_jA_k] = P[A_i]P[A_j]P[A_k]$$

$$P[A_1A_2 \ldots A_n] = P[A_1]P[A_2] \ldots P[A_n]$$ apply.

Bernstein gives an example of pairwise independence but not mutual independence. This example can be found, for instance, in Tucker, [2]. Wong, [3], gives an interesting example of n events such that any $n-1$ of them are mutually independent but all n of them are not.

We wish to give an example of n events where half of the events are independent and the other half are not mutually independent. After the example, we will indicate the natural extension of the ratio of independent events to $1/k$, where k > 2.

Example. Consider the integers 1, 2, 3, 4, 5, 6. Pair these integers as follows: (1,2) (3,4) (5,6). Next, we distribute these pairs into three boxes, A, B, C. It is easy to see that this can be done in 3 ways. However, box A is empty or contains one pair or two pairs or three pairs the same number of times as boxes B and C, so that we can restrict ourselves to the contents of box A without loss of generality. Also, for our purpose, we can assume that the pairs are ordered; that is, the integers increase as one reads the contents of the box from left to right.

Now let A_i be the event that the box contains the integer i, $1 \leq i \leq 6$. Clearly,

$$P[A_i] = \frac{9}{27} = \frac{1}{3} \quad .$$

Let A_iA_k be the event that the box contains two distinct integers i and k such that $1 \le i < k \le 6$ and $k - i \ge 2$. Then,

$$P[A_iA_k] = \frac{3}{27} = \frac{1}{9} = P[A_i]P[A_k].$$

Let $A_iA_kA_l$ be the event that the box contains three distinct integers i, k, l such that $1 \le i < k < l \le 6$, $l - k \ge 2$ and $k - i \ge 2$. Then,

$$P[A_iA_kA_l] = \frac{1}{27} = \frac{1}{3^3} = P[A_i]P[A_k]P[A_l].$$

Define $A_iA_kA_lA_m$ to be the event that the box contains the integers i, k, l, m with the restrictions that $1 \le i < k < l \le 6$, $k - i \ge 2$, $l - k \ge 2$ and $1 \le m \le 6$. An example: 1, 3, 5, 2. It is clear that

$$P[A_iA_kA_lA_m] = \frac{1}{3^3} \ne P[A_i]P[A_k]P[A_l]P[A_m].$$

Similarly define the events $A_iA_kA_lA_mA_r$ and $A_iA_kA_lA_mA_rA_v$ where $1 \le i < k < l \le 6$, $l - k \ge 2$, $k - i \ge 2$; $1 \le m < r < v \le 6$, $v - r \ge 2$ and $r - m \ge 2$. Then $P[A_iA_kA_lA_mA_r] = \frac{1}{3^3} \ne P[A_i]P[A_k]P[A_l]P[A_m]P[A_r]$, and the same can be said for $A_iA_kA_lA_mA_rA_v$.

We have therefore exhibited events where three events are mutually independent and three are not.

In order to construct events such that one-third of the events are mutually independent and two-thirds are not, one considers $n = 3k$ and uses the triplets (1,2,3), (4,5,6), ..., $(n - 2, n - 1, n)$ and demands that the differences be three or greater.

Generalizing, one can construct events such that $1/k$ of them are independent and $\frac{k-1}{k}$ of them are not, where $k \ge 4$.

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A COLLECTION OF MATHEMUSICALS

Geohge E. Lindamood

National Bureau of Standards

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5. The Eigenvalue Song.
6. Fiddler on the Riemann Surface.
7. Finian's Spectral Sequence.
8. Gentlemen Prefer Banach (Spaces).
9. Group Crazy (or perhaps it was Group, Quasi).
10. Guys and Duals.
11. Haar.
12. Hausdorff of Flowers.
13. Heaviside Story.
14. Hello Duality!
15. Hit the Descartes.
16. How to Succeed in Bourbaki Without Really Trying.
17. Kiss Me, Tate.
18. The Matrix Man.
19. The Most Happy Fermat.
20. My Fair Cauchy.
21. Pal Thué.
22. Paint Your (van der) Waerden.
23. Porgy and Bessel.
24. On a Regular Function You Can Differentiate Forever.
25. The Ring and I.
26. The Sound of Monoids.
27. South P-adic.
28. Trigadoon.
29. The Student Principia.

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THE AXIOM OF CHOICE

by John Vaughn
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The explicit use of the Axiom of Choice (AC) dates from 1904 and a paper by Ernest Zermelo on the well-ordering of sets. The Axiom has since become a standard fixture in many mathematics texts but not before much controversy arose over its non-constructive nature. We will explore the need for such an axiom and point out an unusual, even paradoxical, result that obtains from its adoption.

We begin by looking for the first instance where we encounter any need for some version of a choice axiom. We assume some basic ZF (Zermelo-Fraenkel) settheory: sets are equal when they contain exactly the same members, relations are sets of ordered pairs and functions are certain subsets of relations, power set axiom, etc. Suppose to a growing list of facts we wish to add the following plausible theorem:

If A, B are non-empty sets and F is a function from A onto B, then there is a function H, (a right inverse of F), such that $F \circ H = I_B$ (the identity map on B).

We cannot simply take $H = F^{-1}$ since in general F^{-1} is not a function. In fact, since F is onto, we know that for any $b \in B$ there is something in A which is b's pre-image under F. This does not suffice to define a function however. Any such $b \in B$ may have several pre-images in A and we need to select out of R "just enough" ordered pairs to construct H. Here we must simply assert the existence of our set without explicitly specifying its members. Our appeal is thus to the use of some axiom of the form: For any relation R (F^{-1} in our case) there is a function $H \subseteq R$ such that domain R = domain H. This "quirk" about our axiom, its non-explicit nature, is what sets it apart from the other axioms of set theory. Until recently it has been typical for authors to point out those proofs using any version of AC. A more familiar version of AC is that the Cartesian Product of non-empty sets is non-empty. AC occurs in a surprising number of theorems, among them: Prime Ideal Theorem, Stone Representation Theorem, and Vitali's construction of a non-Lebesgue measur-

able set. (Not to mention those set theory exercises where the student attempts cunningly to disguise his use of AC in order to prove Theorem 4.2 without it.)

One may wonder whether the other axioms of ZF imply AC, or indeed perhaps even contradict AC without our being aware of it. The late Kurt Godel showed in 1938 the consistency of AC when added to the other ZF axioms. The former possibility was settled in 1963 when Paul J. Cohen announced the independence of AC from the other axioms of ZF. Early results in this area are due to Fraenkel, Mostowski, Specker and others. Such foundational questions aside, why should we hesitate to employ this axiom, especially in the light of its apparently crucial usage in several standard theorems of mathematics? One reason may be the following "paradoxical" result stemming from its adoption.

Banach-Tarski Paradox. Given a closed ball X, there exists a decomposition of X into disjoint sets Y, Z such that X is identical in size and shape with both Y and Z. We sketch the proof:

Consider the finitely presented group G on the letters $1, \Phi, \Psi$ such that $\Phi^2 = \Psi^3 = 1$. We put all words in these letters into reduced form. We consider a ball X with unit radius and two axes of rotation L_1 and L_2 through the center $\{a\}$. We choose the angle a between L_1 and L_2 such that cosine (a) is transcendental. We now consider the rotations of 120° (Ψ) about L_1 and 180° (Φ) about L_2. Our choice of a means that distinct rotations of X correspond to distinct (reduced) words in G.

We have the following facts:

$A \cdot \Phi = B \cup C \qquad A \cdot \Psi = B \qquad A \cdot \Psi^2 = C$

and $G = A \cup B \cup C$ where $A \cap B \cap C = \Phi$

Let Q be the set of all fixed points on the unit spheres under all non-trivial rotations $\alpha \varepsilon G$. It is easy to see Q is countable. For each $x t$ S - Q look at its orbit, O_X under all $a t G$. Clearly if x and y, where x # y, are in S - Q either $\Theta = \Theta_Y$ or $O_x \cap O_Y = \Phi$. These collections, the O , form a partition of S - Q into disjoint sets. Let K be the set which contains exactly one member from each of these sets. This is our use of AC.

Let:

$A = K \cdot A \qquad B = K \cdot B \qquad C = K \cdot C$.

From our construction $A \cap B \cap C = \phi$ and more importantly:

$A \approx B \cup C$ $\quad$ $A \approx B$ $\quad$ $A \approx C$

(where $M \approx N$ means both M and N can be decomposed in the same (finite) number of disjoint pieces which are pairwise identical in size and shape.)

By definition we have:

$S = A \cup B \cup C \cup Q$ $\quad$ so $\quad$ $X = \overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{Q} \cup \{c\}$

(where for $P \subseteq S$, $\overline{P}$ is the set of points whose projection from c onto S is P).

From above:

$\overline{A} \approx \overline{B} \cup \overline{C}$ $\quad$ $\overline{A} \approx \overline{B}$ $\quad$ $\overline{A} \approx \overline{C}$ $\quad$ hence $\overline{A} \approx \overline{A} \cup \overline{B} \cup \overline{C} \approx \overline{C}$

Let $H_1 = \overline{A} \cup \overline{C} \cup \{c\}$ $\quad$ then $\quad$ $H_2 = X - H_1$

Now strangely enough:

$H_1 = \overline{A} \cup \overline{Q} \cup \{c\} \approx \overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{Q} \cup \{c\} = X$ so $H_1 \approx X$

Now to show $H_2 \approx$ X, we need a new center and a corresponding piece for $\overline{Q}$ in $\overline{B} \cup \overline{C}$.

Since $\overline{C} \approx \overline{A} \cup \overline{B} \cup \overline{C}$, it can be shown that there is a set $E \subseteq C$ such that $E \approx Q$ so $\overline{E} \approx \overline{Q}$. Let $p \in \overline{C} - \overline{E}$ then:

$X \approx H_1 = \overline{A} \cup \overline{Q} \cup \{c\} \approx \overline{B} \cup \overline{E} \cup \{p\} \subseteq \overline{B} \cup \overline{C} \subseteq X \approx X$

therefore we have: $\overline{B} \cup \overline{C} = H_2 \approx X$.

This paper was written while John Vaughn was an undergraduate at St. Louis University.

REFERENCES

1. Enderton, H., *Elements of Set Theory*, Academic Press, New York, 1977.
2. Jech, T., *The Axiom of Choice*, North Holland Publishing, Amsterdam, 1973.
3. Stromberg, K., *"Banach-Tarski Paradox"*, American Mathematical Monthly, Vol. 86, no. 3, (April 1979), 151-161.

?

PUZZLE SECTION

David Ballew

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for that department.

Address all proposed puzzles and puzzle solutions to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the Fall issue will be the next February 15, and puzzles appearing in the Spring issue will be due on the next September 15.

Mathacrostic No. 10

submitted by Joseph V. E. Konhauser
Macalester College, St. Paul, Minnesota

Like the proceeding puzzles, this puzzle (on the next page) is a keyed anagram. The *224* letters to be entered in the diagram in the numbered spaces will be identical with those in the 28 keyed words at matching numbers, and the key letters have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book.

Cross-Number Puzzles

submitted by Mark Isaak
Student, University of California, Berkeley

In the cross-number puzzles (starting two pages forward), each of the letters stands for a positive, nonzero integer. The algebraic expressions evaluate out to two to five digit numbers which fit in the squares as in a normal crossword puzzle. None of the numbers in the squares have any leading zeros; i.e., if there is room for a four digit number, that number will be at least 1000, never, for example, 0999.

1 O		2 F	3 M		4 R	5 U	6 J		7 D	8 b	9 I	10 A	
11 F	12 S	13 Z	14 C		15 R		16 G	17 Y	18 T		19 H	20 P	21 W
22 E	23 Q	24 a		25 U	26 B		27 G	28 O	29 P		30 Z	31 K	32 Q
33 X	34 W		35 A	36 R	37 a		38 J	39 R		40 P	41 U	42 O	43 N
44 R	45 T		46 R		47 D	48 L	49 E	50 N		51 S	52 L		53 a
54 b	55 R	56 F		57 b	58 G	59 J	60 O		61 I	62 G	63 B	64 J	
65 V	66 I	67 Z	68 N		69 L		70 Q	71 A	72 b		73 I	74 F	75 V
76 B	77 T	78 L	79 a		80 Q	81 C		82 L	83 S	84 F		85 E	86 D
87 K	88 A	89 I	90 P	91 N	92 W		93 b	94 U	95 P		96 Y	97 J	98 M
99 A	100 Q	101 K	102 V	103 C	104 I		105 C	106 B	107 X	108 G	109 M	110 T	
111 J	112 R		113 K	114 N	115 D		116 U	117 J	118 a		119 T	120 L	121 Q
122 E		123 N	124 C	125 V	126 A	127 W	128 U	129 b		130 Z		131 S	132 U
133 C	134 D	135 F	136 W	137 P	138 b		139 B	140 X	141 H	142 V	143 I	144 G	
145 C	146 I		147 S		148 C	149 H	150 F	151 E	152 O	153 T	154 a	155 L	
156 L	157 D	158 F	159 G	160 A		161 P	162 Z	163 M	164 b		165 W	166 H	167 N
168 U	169 P	170 T	171 C	172 X		173 Y	174 H	175 b	176 D	177 L	178 R		179 O
180 C	181 L		182 I	183 K	184 X	185 C	186 Z		187 B	188 V	189 C	190 X	191 T
	192 R	193 K		194 L	195 B	196 H	197 P	198 X		199 Q	200 A	201 O	
202 V	203 D	204 B		205 a	206 W	207 F	208 P	209 G	210 O	211 Z	212 Q	213 b	214 K
215 L	216 J		217 H	218 O	219 D	220 a	221 W	222 F		223 Z	224 S		

KEY WORDS AND PHRASES

A. empirical rule for the distances of the planets from the sun (2 wds.) — 35 71 126 99 88 160 200 10

B. short and pudgy (comp.) — 195 187 63 106 139 26 204 76

C. mirror image of a non-reflexive figure — 189 81 185 103 14 124 133 105 145 171 148 180

D. offered 100000 marks in 1908 for a complete proof of Fermat's Last Theorem — 115 134 176 219 47 7 86 157 203

E. popular board game of ancient Egypt — 85 49 122 22 151

F. one of a practically indistinguish-able pair — 135 11 150 158 2 74 222 207 84 56

G. mathematical best-seller — 108 159 144 16 58 62 27 209

H. squabble — 149 174 196 141 19 166 217

I. word puzzle — 143 61 182 9 104 146 66 73 89

J. neutral group element — 111 216 59 117 6 97 38 64

K. creation of Backus and Ziller — 193 31 183 101 214 87 113

L. earth measurer (ca. 230 B.C.) — 181 155 69 82 52 156 194 120 48 78 215 177

M. in the plane, a bounded closed convex set — 3 98 163 109

N. sometimes measured in degrees — 123 91 43 68 167 114 50

O. as opposed to analytic — 42 201 60 152 28 218 179 1 210

P. solution of x*x = x — 208 95 29 40 20 90 197 137 169 161

Q. pioneer in the use of indivisibles (1602-1675) — 100 80 70 121 32 212 23 199

R. transformation of period two — 15 4 55 192 44 36 178 46 39 112

S. an investing cover — 131 83 224 147 51 12

T. projective collineation which leaves a given line fixed — 170 45 110 77 191 153 119 18

U. a contradiction — 116 94 25 168 128 5 132 41

V. pen name of Hubert Phillips, master creator of inferential-type puzzles — 188 202 65 102 142 75 125

W. two-cusped epicycloid — 206 92 21 136 221 165 127 34

X. slim margin of victory — 140 172 184 33 190 107 198

Y. a small mass — 173 17 96

Z. mechanical angle trisector — 186 211 223 13 162 130 30 67

a. inventor of the straight logarith-mic slide rule (1574-1660) — 220 205 79 53 37 24 154 118

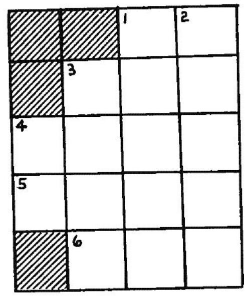
b. system of numeration based on powers of -2 — 164 213 129 54 57 175 8 93 138 72

ACROSS

1. A
3. BC/37
4. 33BD/70
5. 27D
6. (3/2)C + 13

DOWN

1. 26AB + (10D)/E + 5
2. 13AB
3. $(1504E^2)/C + 60$
4. E

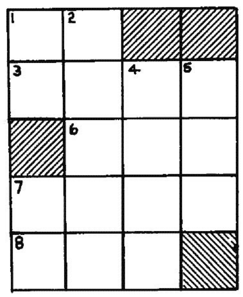


ACROSS

1. A
3. $A^2 + B^4 + 1311$
6. 11C − 11 +D/(E + 2)
7. $(F^3 + D)/9$
8. 2A + 43G − 2HJ

DOWN

1. GJ
2. 2635G + 14A
4. A + 581D + 24G
5. $G^2 - 99$
7. CD

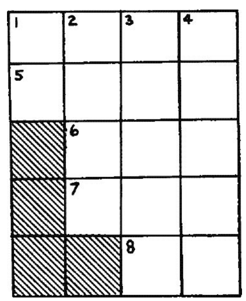


ACROSS

1. $2A^4 + 1100$
5. 241BC + 10
6. 2D + E
7. 31F
8. 12G + 14

DOWN

1. (B + 1)/4 + 14
2. 12CG + 13H
3. $(D^2 + G)/3$
4. $2G^2H + H$

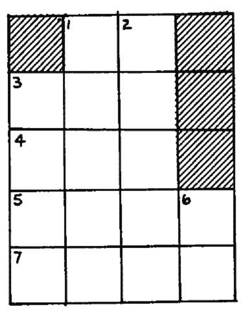


ACROSS

1. A
3. 4B + 2
4. $A^2 + 11C$
5. 3AD
7. (367E)/B + 1/12

DOWN

1. E/23 + F
2. G^3
3. 27HJ
6. J

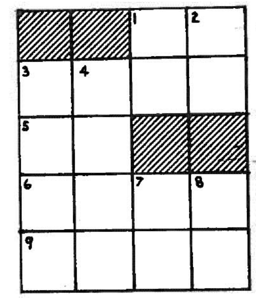


ACROSS

1. $\sqrt{A}$
3. BC − 5
5. $4\sqrt{C} - (D/28)$
6. E
9. C + 1234

DOWN

1. $\sqrt{A + 105}$
2. $\sqrt{A} + E$
3. BF + 11
4. 17D
7. $\sqrt{E}$
8. F/4

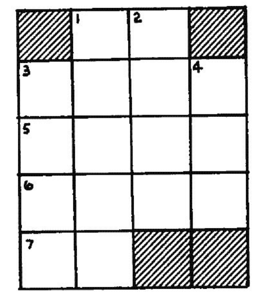


ACROSS

1. $\log_2 A$
3. A
5. 601B
6. 18C − D
7. E

DOWN

1. $120252/\log_2 A$
2. 2EF
3. D − 9C
4. 41F

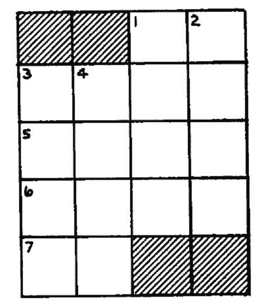


ACROSS

1. A
3. 3B
5. (C/17)((B/3)+111)
6. $(D^2 - 25/A) + 130E$
7. D − 5

DOWN

1. 101(D − 3)
2. 1000F + 252 + 10((B/16C) + E)
3. $D^2 - 25$
4. 5B + 21F

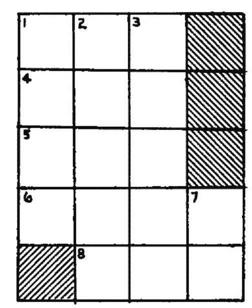


ACROSS

1. A^2
4. (4BC/45) + 4
5. $D^E + 59$
6. C^3
8. 70F + 7G + 10

DOWN

1. BH
2. ACDE + CDEH − 9AFG
3. 11BC − 20
7. D^3



SOLUTIONS

Mathacrostic No. 9 (See Fall 1979 issue) (*proposed by* J. D. E. Konhauser)

Definitions and Key:

A. Homothecy	H. Talent	O. Life	V. Two sheets
B. Shannon	I. Epimenides	P. Ambsace	W. Off by one
C. Mate	J. Rate of	Q. Rat-a-tat	X. Pasch
D. Cyclotomy	K. Result	R. Publish	Y. Eleusis
E. Obovoid	L. Engine	S. Orthoscheme	Z. Show that
F. Xylander	M. Gewgaw	T. Lemniscate	
G. Echelon	N. Uneven	U. Yin and Yang	

First Letters: HSM Coxeter Regular Polytopes

Quotation: *We... enlarge the scope of Euclidean geometry by denying one of the usual axioms ("Two planes which have one common point have another"), and we establish the consistency of the resulting abstract system by means of the analytical model.*

Solved by: Alan Wayne, *Holiday, Florida;* Joseph Testen, *Mobile, Ala;* Jeanette Bickley, *Webster Groves High School, Missouri;* Victor Feser, *Mary College, Bismarck;* Robert C. Gebhardt, *Hopatcong, NJ;* Henry Lieberman, *John Hancock Mutual Life Ins. Co.;* Patricia Tuchman, Allan Tuchman, Michael Haney, *University of Illinois;* Louis Cairoli, *Kansas State University.*

Smith-Jones-Robinson Problem (*proposed by* Vewiser Turner)

The distinguished mathematical physicist must be from Omaha, because the brakeman is from Omaha and both attend the same church.

Since Mr. Robinson lives in Los Angles, and Mr. Jones forgot all the algebra he learned in high school, neither can be the distinguished mathematical physicist.

By elimination, Mr. Smith must be the distinguished mathematical physicist from Omaha.

Since Mr. Robinson lives in Los Angeles and Mr. Smith in Omaha, Mr. Jones must live in Chicago, and the brakeman is Mr. Jones.

Because Mr. Smith beat the fireman at billiards, Mr. Smith must be either the engineer or the brakeman.

But Mr. Jones is the brakeman, so Mr. Smith must be the Engineer. (*solution by* Daniel Cousins, *Miami University, Oxford, Ohio*)

Solved by: Michael J. Lenart, *Rutgers College;* Janda Cook, *Lamar University;* Alan Wayne, *Pasco-Hernando Community College, Florida,* (who noted that a variant of this puzzle occured on page 71 of the July, 1979, Reader's Digest and that Martin Gardner had solved the problem in the Feb., 1959 issue of Scientific American); Randall J. Scheer, *SUNY-Potsdam;* Henry Lieberman, *John Hancock Mutual Life Ins. Co.;* Victor Feser, *Mary College, Bismark;* Ralph King, *St. Bonaventure University;* Louis Cairoli, *Kansas State University;* Patricia Tuchman, Allan Tuchman, Michael Haney, *University of Illinois;* Roger Kuehl, *Kansas City;* George Rainey, *UCLA;* Mark Evans, *LaMarque, Texas.*

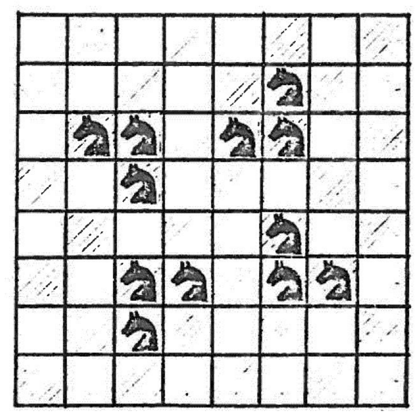
Maximum Number of Knights (See Fall 1979 issue) (*Proposed by P. Square*)

Since knights can only attackknights of an opposite color, the maximum number is 32, either on all the black or all the red.

Solved by: Janda Cook, *Lamar University;* Randall Scheer, *SUNY-Potsdam;* Victor Feser, *Mary College, Bismarck;* Michael Ecker, *Scranton, PA;* I. J. Good, *Virginia Polytechnic Institute and State University,* (who proposed the same problem in the Mathematical Gazette as Problem 3000 in Feb. 1962, p.54); Alan Levine, *McNeese State University;* Louis Cairoli, *Kansas State University;* Roger Kuehl, *Kansas City.*

Minimum Number of Knights (See Fall 1979 issue) (*Proposed by P. Square*)

Twelveknights are required so that every square is either occupied or under attack.



Solution by: Roger Kuehl, *Kansas City; Also Solved by:* Louis Cairoli, *Kansas State University;* Randall J. Scheer, *SUNY-Potsdam.*

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

Problem proposals offered for publication should be sent to Professor Clayton W. Dodge, Mathematics Department, University of Maine, Orono, Maine 04473

To facilitate consideration of solutions for publication, solvers should submit each solution on separate sheets (one side only) properly identified with name and address and mailed before November 1, 1980 to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Contributors desiring acknowledgement of their proposals and solutions are requested to enclose a stamped and self-addressed postcard or, for those outside the U.S.A., an unstamped card or mailing label.

Problems for Solution

462. *Proposed by the late R. Robinson Rowe.*

A pilot down at Aville asked a native how far it was to Btown and was told, "It's south 1500 miles, then east 1000 miles, or east 500 miles and south 1500 miles." How far was it directly?

463. *Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India.*

Let $f(n)$ be a function defined over positive integers and $\sum_{d/n} f(d) = n$. Then, prove that $f(n) = \phi(n)$, the Euler's function denoting the number of integers prime to and not greater than n.

464. *Proposed by Solomon W. Golomb, University oft Southern California, Lob Angeles.*

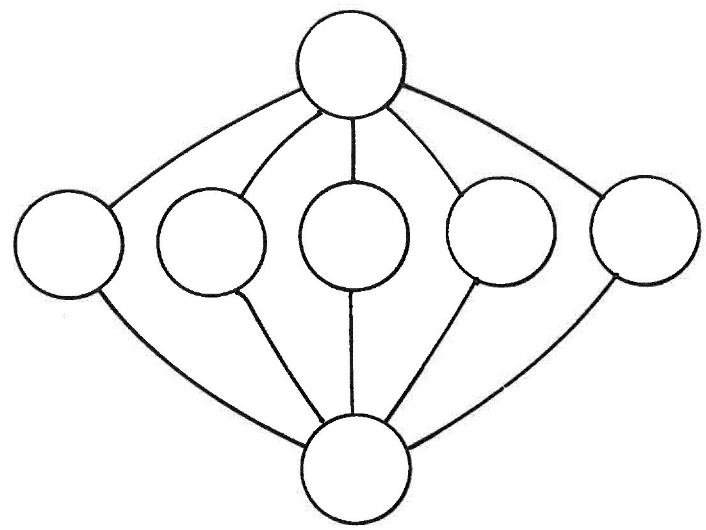
For all positive integers a and b with $1<a<b$, show that $(a!)^{b-1} < (b!)^{a-1}$.

465. *Proposed by Charles W. Trigg, San Diego, California.*

What is the shortest strip of equilateral triangles of side k that, while remaining intact, can be folded along the sides of the triangles so as to completely cover the surface of an octahedron with edges k?

466. *Proposed by Herbert Taylor, South Pasadena, California.*

Let the adversary put four distinct symbols in each box (node) of this graph. Prove or disprove: No matter what pattern of symbols he puts, we can choose two symbols from each box in such a way that adjacent boxes have disjoint chosen 2-sets.



467. *Proposed by Paul Erdös, Spaceship Earth, and John L. Selfridge, University of Michigan.*

Determine the greatest power which divides $n!$ Prove that for $n \geqq 6$ it is a square.

468. *Proposed by Michael W. Eckm, Pennsylvania State. University, Worthington Scranton Campus.*

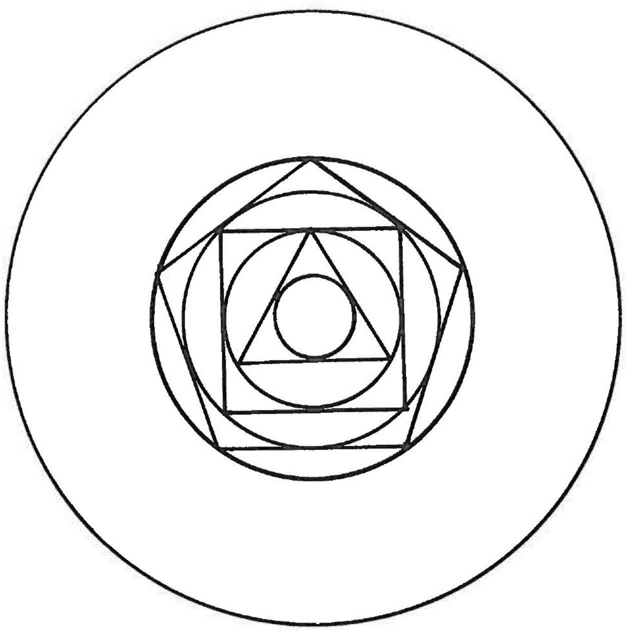
A priori, the expression a^{b^c} is ambiguous in that it would mean either $(a^b)^a$ or $a^{(bc)}$. Assuming a, b, and c are positive integers, find all triples (a, b, c) for which the two expressions are equal.

469. *Proposed by Richard 1. Hess, Palos Verdes, California.*

Start with a unit circle and circumscribe an equilateral triangle about it. Then circumscribe a circle about the triangle and a square about the circle. Continue indefinitely circumscribing circle, regular pentagon, circle, regular hexagon, etc.

a) Prove that there is a circle which contains the entire structure.

* b) Find the radius of the smallest such circle.



470. *Proposed by Tom Apostol, California Institute of Technology.*

Given integers $m > n > 0$. Let

$$\alpha = a\sqrt{m} \quad b\sqrt{n}$$
$$\beta = c\sqrt{m} \quad d\sqrt{n}$$

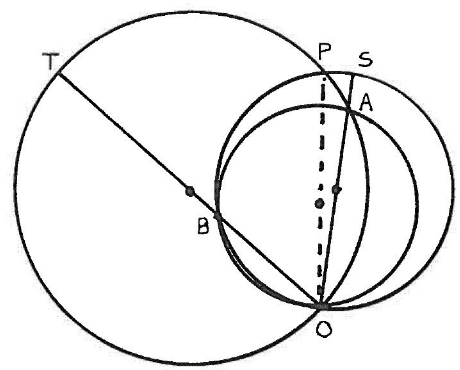
where a, b, a d are rational numbers.

(a) If ad + bc = 0 or if mn is a square, prove that both α and β are rational or both are irrational.

(b) If $m = r^2$ and $n = a^2$ for some pair of integers $r > s > 0$ then α and β are both rational. Prove that the converse is also true if ad $\neq bc$.

471. *Proposed by Clayton W. Dodge., University of Maine at Orono.*

Let two circles meet at O and P, and let the diameters OS and OT of the two circles cut the other circle at A and B. Prove that chord OP passes through the center of circle OAB.



472. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Evaluate $$\int \frac{5}{16 + 9\cos^2 x}\,dx \,,$$

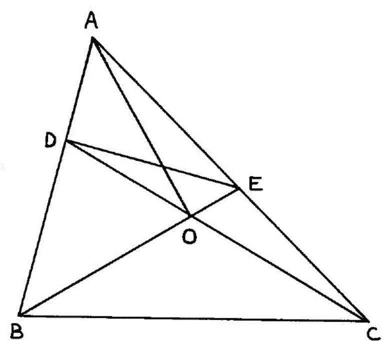
473. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

In an acute triangle ABC with angle $A = 60°$, P is a point within the triangle. D and E are the feet of the Cevians through P from C and B respectively.

a) If BD = DE = EC, prove that AP = BP = CP.

b) Conversely, if AP = BP = CP, prove that BD = DE = EC.

c) If angle PBC = angle PCB = 30°, show that BD = DE = EC.



Solutions

423. [Spring 1978; Spring 1979] *Proposed by Richard S. Field, Santa Monica, California.*

Find all solutions in positive integers of the equation $A^D - B^D = C^C$ where D is a prime number.

Solution by the Proposer.

$C = a^D - b^D$; $A = C^{\frac{C-1}{D}} a$; $B = C^{\frac{C-1}{D}} b$; where a, b are any integers obeying the restriction $a = b + Dk + 1 (k = 0, 1, 2, \ldots)$.

Proof. By substitution, $A^D - B^D = (C^{\frac{C-1}{D}} a)^D - (C^{\frac{C-1}{D}} b)^D = C^{C-1}(a^D - b^D) = C^{C-1}C = C^C$.

This proves the identity, but the further requirement that A and B be integers must be met by insuring that $(C-1)/D$ be an integer, i.e., $C \equiv 1 \bmod D$. We accomplish this by the restriction $a = b + Dk + 1$ $(k = 0, 1, 2, \ldots)$. To demonstrate, expand $C = a^D - b^D = (b + Dk + 1)^D - b^D$ into

$$C = \binom{D}{1} b^{D-1}(Dk+1) + \binom{D}{2} b^{D-2}(Dk+1)^2 + \ldots + \binom{D}{D-1} b(Dk+1)^{D-1} + (Dk+1)^D,$$

(noting that terms b^D drop out). Here all terms but the last contain binomial coefficients $\binom{D}{0<i<D}$ and, since D is prime, each term but the last is divisible by D. (The reader may easily verify that $\binom{D}{0<i<D}$ is a multiple of D by necessity if and only if D is prime). Then since obviously the last term $(Dk+1) \equiv 1 \bmod D$, then also $C \equiv 1 \bmod D$.

Footnote. We conjecture that the above process generates <u>all</u> solutions to the problem posed.

Dedicated to D. L. *Silverman* "who taught me *how*".

Also solved by MIKE CALL, *Rose-Hulman Institute of Technology, Terre Haute, Indiana 47803.* Clayton Dodge, *University of Maine at Orono,* Spenser Hurd, *University of Georgia* and Mike May, *St. Louis, Missouri.*

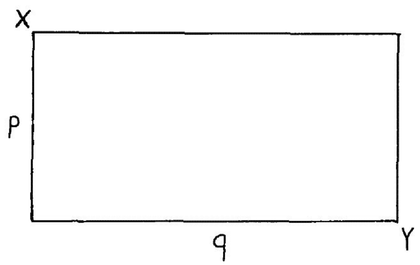
438. [Spring 1979] *Proposed by Ernst Straus, University of California at Los Angeles.*

Prove that the sum of the lengths of alternate sides of a hexagon with concurrent major diagonals inscribed in the unit circle is less than 4.

With the exception of the proposer's solution, only one response has been received for this problem, one that considered only the trivial case where the major diagonals are diameters of the circle. Readers are invited to submit general solutions.

439. [Spring 1979] *Proposed by Richard I. Hess, Palos Verdes, California.*

A bug starts at Monday noon at the upper left corner (X) of a p by q rectangle and crawls within the rectangle to the diagonally opposite corner (Y), arriving at 6 P.M. Exhausted, he sleeps till noon Tuesday. At that time he embarks for X, crawling along another path in the rectangle and arriving at X 6 P.M. Tuesday. Prove that at some time Tuesday the bug was at a point no farther than p from where he was at the same time Monday.



Solution by Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, Mass.

The problem is the same as if two bugs start at the same time, one from X and the other from Y, each to the opposite corner and along different paths. Then, at some time between noon and 6 P.M. they will both be at the same horizontal distance between the left and right sides of the rectangle. Since the paths of the bugs are both within the rectangle then at the time described above, the vertical distance between the bugs must be no greater than the side p of the rectangle.

Practically identical solutions were received from WALTER BLUMBERG, CLAYTON W. DODGE, MARK EVANS, SAMUEL GUT *and the Proposer.* Their solutions were characterized by the complete absence of mathematical symbols and mathematical jargon. While there is certainly no objection to mathematical solutions to mathematical problems, a simple word-solution intelligible to any layman is to be preferred. Some of the other submitted solutions were profuse with subscripts, coordinates, inequalities, vinculi, functional relations, intermediate value theorems, Greek symbols, graphs, continuous functions and derivatives -- all reminiscent of the sledge hammer method of swatting a fly.

Solutions were also received from CHUCK ALLISON, MIKE CALL, CAROL DIMINNIE, MICHAEL ECKER, VICTOR G. FESER, ROBERT C. GEBHARDT, SPENCER P. HURD, MICHAEL MAY, JAMES A. PARSLY, PETER SZABAGA, DANIEL WAGGONER AND WILLIAN E. WARREN.

440. [Spring 1979] *Proposed by Charles W. Trigg, San Diego, California.*

Are there any prime values of $p < 10^5$ for which the equation $x^5 - y^5 = p$ has a solution in positive integers? How about $x^5 + y^5 = p$?

Solution by Carol Diminnie and Charles Diminnie, St. Bonaventure.

A. $p = x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ implies that either $x - y = 1$ or $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1$. Since x and y are positive integers, $x - y = 1$.

Therefore, $p = (y+1)^4 + (y+1)^3y + (y+1)^2y^2 + (y+1)y^3 + y^4 =$

$$5y^4 + 10y^3 + 10y^2 + 5y + 1.$$

$y = 1$, $x = 2$, $p = 31$

$y = 2$, $x = 3$, $p = 211$

$y = 5$, $x = 6$, $p = 4651$

$y = 10$, $x = 11$, $p = 61{,}051$

For $y = 3, 4, 6, 7, 8, 9, 11$, $5y^4 + 10y^3 + 10y^2 + 5y + 1$ is not prime, while if $y > 11$, $p > 10^5$. Therefore, the only solutions to this problem are the four given above.

B. $x^5 + y^5 = p < 10^5$ and x, y positive integers implies that $x^5 < 105$ and $y^5 < 10^5$. Therefore, $x < 10$ and $y < 10$. If $p = 2$, $x = y = 1$, otherwise p is odd. Due to the symmetry of x and y in this problem, we may assume that x is odd and y is even. However, for all such values of x and y, $x^5 + y^5$ is not prime. Therefore, the only solution to this problem is $p = 2$, $x = y = 1$.

Also solved bq WALTER BLUMBERG, MIKE CALL, CLAYTON W. DODGE, MICHAEL W. ECKER, RANDY L. EKL, VICTOR G. FESER, ROBERT C. GEBHARDT, SPENCER P. HURD, THEODORE JUNGREIS, DONALD KING, HENRY S. LIEBERMAN, MICHAEL MAY, BOB PIERLIPP, EDITH E. RISEN, RANDALL J. SCHEER, DALE E. WATTS, KEENTH M. WILKE, *and the Proposer.*

ECKERT called attention to TYCMJ, Sept. 1978, problem 121, where it was shown that p cannot be a Fermat prime, 2 $+ 1$.

441. [Spring 19791 *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Prove that a self-complementary graph with an even number of vertices has no more than $2i$ vertices of degree i, and that the number of them is even.

Solution by Walter Blumberg, Flushing High School, Flushing, N.Y.

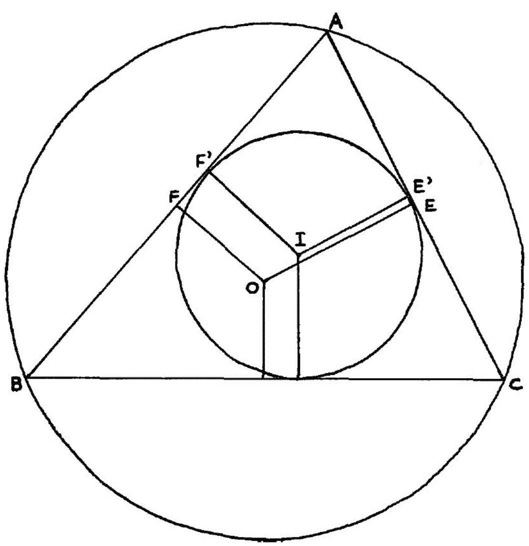
Let G be the self-complementary graph with n vertices. The complement $\overline{G}$ of G is isomorphic to G. Let $K(i)$ be the number of vertices in G of degree i. Then obviously, $K(i) = K(n-1-i)$.

If $K(i) = 0$, then in this special case, $K(i) \le 2i$, and $K(i)$ is even. Assume $K(i) > 0$. Let (A_t), $t = 1,\dots,K(i)$ be the vertices in G of degree i. Let (B_s), $s = 1,\dots,K(i)$ be the vertices in G of degree $n-1-i$. Since n is even, i and $n-1-i$ have opposite parity. Hence, the sets (A_t) and (B_s) are disjoint.

Consider the $[K(i)]^2$ ordered pairs (A_t,B_s), $t = 1,\dots,K(i)$, $s = 1,\dots,K(i)$. In $\overline{G}$, A_t is of degree $n-1-i$ and B_s is of degree i. If in G, A_t and B_s are joined (not joined) by an edge, then in $\overline{G}$, A_t and B_s are not joined (joined) by an edge. Because of the isomorphism, this means that in G there are as many combinations (A_t,B_s) which are connected by an edge as those combinations which are not. Therefore, there are $[K(i)]^2/2$ edges connecting members of (A_t) with members of (B_s). As an immediate consequence, $K(i)$ is even. Finally, one of the $K(i)$ members of (A_t) must therefore have at least $\frac{K(i)^2/2}{K(i)} = K(i)/2$ edges. Since each A_t has degree i, it follows that $K(i)/2 \le i$. Hence $K(i) \le 2i$. NOTE: This inequality can be tightened up a bit. Since now $K(i) = K(n-1-i) \le 2(n-1-i)$, we have $K(i) \le \min[2i, 2(n-1-i)]$.

Also solved by MIKE CALL *and the Proposer.*

442. [Spring 1979] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*



Show that the sum of the perpendiculars from the circumcenter of a triangle to its sides is not less than the sum of the perpendiculars drawn from the incenter to the sides of the triangle.

Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, New Jersey.

We are required to show that $OD + OE + OF \ge 3r$, where O is the circumcenter and D, E, F are the feet of the perpendiculars from O on BC, AC and AB respectively, and r is the inradius.

$OD + OE + OF = R + r$, where R is the circumradius.

(The sum of the distances of the circumcenter from the three sides of a triangle is equal to the circumradius increased by the inradius. N. Altshiller-Court, College Geometry, page 73 of the first edition, page 83 of the second edition. This is known as Carnot's Theorem).

It is also known by Euler's Theorem that $d^2 = R^2 - 2Rr$, where d is the distance between the circumcenter and the incenter. This yields the well-known relation $R \ge 2r$.

It follows that $OD + OD + OF \ge 3r$, with equality when O and I coincide, that is, when the triangle is equilateral.

Also solved by WALTER BLUMBERG, HENRY S. LIEBERMAN, *and the Proposer.*

443. [Spring 1979] *Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

If x and y are ay real numbers, prove that

$$x^2 + 5y^2 \ge 4xy.$$

Amalgam of solutions offered by WALTER BLUMBERG, *Long Island, New York,* THEODORE JUNGREIS, *Brooklyn, New York, and* EDITH E. RISEN, *Oregon City, Oregon.*

By the Arithmetic-Geometric Mean Inequality,

$$x^2 + 5y^2 \ge 2\sqrt{5x^2y^2} = 2\sqrt{5}\,|xy|.$$

This inequality is sharper than the one proposed and clearly shows that equality holds only when $x = y = 0$.

Editor's Comment.

This extremely easy problem was withdrawn by the proposer too late to be deleted from copy already in press. His original purpose was to show its relationship to another simple problem, number 431, (Fall 1978; Fall 1979). My objective in using this proposal was to encourage an increase in reader response and to observe the variety of methods of solution, a subject dear to the heart of all problem editors. Furthermore, as an inequality addict, I was interested in seeing how many solvers would notice the stronger result shown above. All of these objectives were attained. Only three of the 35 solvers noticed the stronger inequality; this is not to be interpreted as a reflection on the many different methods of solution that yielded only the result requested. Most surprising were the various attacks on the problem. In addition to the typical algebraic methods, solutions were submitted that involved calculus, critical points, relative minima, partial derivatives, generalizations, analytic geometry using polar coordinates, indirect proofs, discriminants of quadratic equations, geometric representations and proofs by contradiction -- a vertitable salad of approaches to the solution of a problem.

This problem was also solved by CHUCK ALLISON, MIKE CALL, CAROL B. DIMINNIE, CLAYTON W. DODGE, MICHAEL W. ECKER, KENNETH EIGER, RANDY L. EKL, MARK EVANS, VICTOR G. FESER, HOWARD FORMAN, ROBERT A FULLER, ROBERT G. GEBHARDT, SAMUEL GUT, EDWARD HACKETT, SPECER HURD, MARTIN F. KAIL, DONALD KING, HENRY S. LIEBERMAN, PETER A. LINDSTROM, CHARLES W. TRIGG, BOB PRIELIPP, MICHAEL MAY, JAMES A. PARSLY, JOHN PUTZ, DONNA MARIE SASSANO, PETER SZABAGA, DALE E. WATTS, KENNETH M. WILKE, WILLIAM E. WARREN, *and the Proposer.*

444. [SPRING 19791 *Proposed by Peter A. Lindstrom, Genesee Community Cottage., Batavia, New York.*

In terms of n, which is the first non-zero digit of

$$\prod_{i=1}^{n/2} (i)(n-i+1) \text{ for even } n \geq 6?$$

Solution by the Proposer

Expanding $\prod_{i=1}^{n/2} (i)(n-i+1)$ term by term, we obtain

$$\prod_{i=1}^{n/2} (i)(n-i+1) = (1)(n)(2)(n-1)\ldots(n/2)(n-n/2+1),$$

$$= (1)(2)\ldots(n/2)(n/2+1)\ldots(n-1)(n),$$

$$= n!\ , \text{ as } n \text{ is even.}$$

Since the first non-zero digit is determined by the number of terminal zeros, we first have to determine the number of factors of 5 in $n!$, as $10 = 5.2$ and the number of factors of 5 in n is less than the number of factors of 2. This can be determined by the following well-known theorem:

If n is a positive integer and p is a prime, then p appears in the canonical representation of $n!$ with the exponent e, where

$$e = \sum_{i=1}^{r} \left[\frac{n}{p^i}\right],$$

where [ ] is the greatest integer function and r is determined by the inequality $p^r \leq n < p^{r+1}$.

Thus, th3 number of factors of 5 in $n!$ is given by

$$\sum_{i=1}^{r} \left[\frac{n}{5^i}\right], \text{ where } 5^r \leqq n < 5^{r+1}$$

Hence, the $\left(\sum_{i=1}^{r} \left[\frac{n}{5^i}\right] + 1\right)^{th}$ digit is the first non-zero digit of $\prod_{i=1}^{n/2} (i)(n-i+1)$ for even $n \geqq 6$.

Also solved by MIKE CALL, MARK EVANS, *and* SPENCER P. HURD.

445. [Spring 1979] *Proposed by the late Richard S. Field.*

A "Tribonacci-like" integer sequence $\{A_n\}$ is defined in which $m_1A_i + m_2A_{i+1} + m_3A_{n+2} = A_{i+3}$, ($A_0 = A = A = 1$; m_1, m_2, m are arbitrary integers).

A particular sequence of this kind is found ($m_1 = -1$, $m_2 = 5$, $m_3 = 5$) which appears to yield only perfect squares, viz.: 1, 1, 1, 9, 49, 289, 1681,...

a) Prove that for this particular sequence, the successive terms continue to be perfect squares.

b) Can other values of m_1, m_2 and m_3 be found which result in the same property, namely, a sequence of perfect squares?

Solution by Clayton W. Dodge, University of Maine at Orono.

Let us consider the sequence $\{s_n\}$ of square roots

$\pm 1, \pm 1, \pm 1, \pm 3, \pm 7, \pm 17, \pm 41, \ldots$

and assume we can find a recursion formula that will yield all plus signs, at least after the first few terms. Assume constants u and v so that

$$s_{n+2} = us_n + vs_{n+1}.$$

Then we have

$$41 = 7u + 17v \quad \text{and} \quad 17 = 3u + 7v,$$

whose common solution is $u = 1$ and $v = 2$. The resulting formula

(1) $$s_{n+2} = s_n + 2\ {}_{n+1}$$

determines the sequence

$-1, 1, 1, 3, 7, 17, 41, \ldots,$

clearly a sequence of integers. From (1) we get

$$s_{n+3} = s_{n+1} + 2s_{n+2} \quad \text{and} \quad s_n = s_{n+2} - 2s_{n+1}$$

and by squaring,

$$s_{n+3}^2 = s_{n+1}^2 + 4s_{n+1}s_{n+2} + 4s_{n+2}^2,$$

$$s_n^2 = s_{n+2}^2 - 4s_{n+1}s_{n+2} + 4s_{n+1}^2.$$

Adding these two equations, we get

$$s_{n+3}^2 = 5s_{n+1}^2 + 5s_{n+2}^2 - s_n^2,$$

the recursion formula for the given sequence, proving it to be a sequence of squares.

By assuming the form

$$a_n = \alpha^n$$

and substituting into the recursion formula (1), we get the roots

$$\alpha = 1 \pm \sqrt{2},$$

so we take

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2}.$$

Next set

$$s_n = p\alpha^n + q\beta^n$$

for some constants p and q and substitute $s_1 = -1$ and $s_2 = 1$ and solve for p and q, obtaining the result

$$s_n = \frac{(1 + 2)^{n-2} + (1 - 2)^{n-2}}{2}.$$

We have established part (a).

The technique of part (a) serves to find sequences for part (b) as well. Quite generally, let $\{t_n\}$ be the sequence of integers given by"

$$t_1,\ t_2, \text{ and } t_{n+2} = ft_n + gt_{n+1}$$

for some given integers f, g, t_1, and t_2. The technique of part (a) yields the recursion formula

$$t_{n+3}^2 = (f^2 + fg^2)t_{n+1}^2 + (f + g^2)t_{n+2}^2 - f^3t_n^2$$

which yields only squares when t_1^2, t_2^2, and $t_3^2 = (ft_1 + gt_2)^2$ are its first three terms.

For $f = -1$ and $g = 2$, the t_n form an arithmetic sequence; for f = f = $t_1 = t_2 = 1$, the t_n form the sequence 1, 1, 2, 3, 5, 8, ... of Fibonacci numbers. Finally, $f = g = 0$ yields the t_n sequence t_1, t_2, 0, 0, 0, ..., a rather trivial example.

Also solved by WALTER BLUMBERG, MIKE CALL, RALPH GARFIELD, THEODORE JUNGREIS, MICHAEL MAY, JOHN OMAN *and* BOB PRIELIPP (*jointly*), KENNETH M. WILKE, *and the Proposer.*

446. [Spring 1979] *Proposed by Clayton W. Dodge, University of Maine, Orono.*

A teacher showing the factorization of $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ emphasized that the second factor is not a square (not $[x + y]$ squared), and then chose $x = 5$ and $y = 3$ at random, obtaining

$$x^2 + xy + y^2 = 49,$$

which is a square.

a) Explain this apparent contradiction.

b) Show that the equation $x^2 + xy + y^2 = 49$ illustrates that a 3:5:7 triangle has a 120° angle.

Solution by Charles W. Trigg, San Diego, California.

a) The statement that $x^2 + xy + y^2$ is not a square means that the expression is not decomposable into two equal algebraic polynomials with real coefficients. The demonstration that, for $f(x,y) = x^2 + xy + y^2$, $f(5,3) = 49$, a square, could well have been followed by $f(3,4) = 37$, a prime, and $f(1,4) = 21$, a composite non-square integer. Thus it could have been emphasized that the factorability of a polynomial cannot be determined by substituting specific numerical values in it.

Again, $x + y$ is a square for all $x = p^2 - k$, $y = k$. And, $x^2 + y^2 = z^2$ has a two-parameter solution in integers, namely: $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$. Indeed, any desired number of integer solutions of $x^2 + xy + y^2 = z^2$ are given by

$$x = m^2 - n^2,\quad y = 2mn + n^2,\quad z = m^2 + mn + n^2.$$

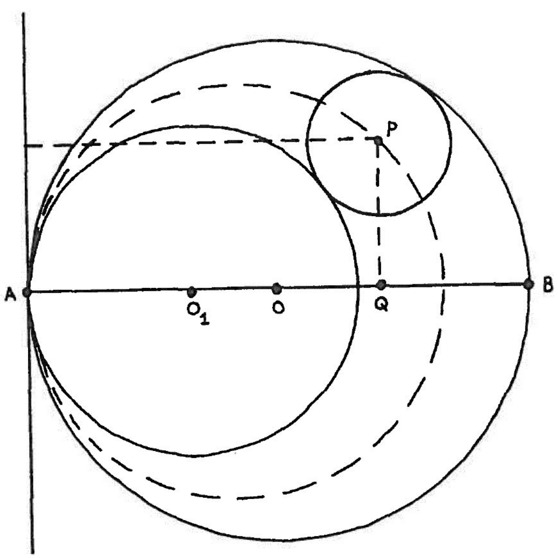
In the present case, $m = 2$, $n = 1$.

b) $z^2 = x^2 + y^2 - 2(-1/2)xy$ is the law of cosines for a triangle with sides x,y,z and a 120° angle, which has a cosine of $-1/2$, opposite z. Since $y = 3$, $x = 5$, $z = 7$ satisfies the equation, a 3:5:7 triangle has a 120° angle opposite the longest side.

Also solved by CHUCK ALLISON, MIKE CALL, MICHAEL W. ECKER, MARK EVANS, VICTOR G. FESER, ROBERT C. GEBHARDT, SAMUEL GUT, DONALD KING, JIM METZ, BOB PRIELIPP, JOHN PUTZ, PETER SZABAGA, WILLIAM E. WARREN, DALE E. WATTS, KENNETH M. WILKE, JOHN A. WINTERINK, *and the Proposer.*

447. [Spring 19791 *Proposed by Zelda Katz, Beverly Hills, California.*

A variable circle touches the circumference of two internally tangent circles, as shown in the figure.



a) Show that the center of the variable circle lies on an ellipse whose foci are the centers of the fixed circle.

b) Show that the center of the variable circle bears a constant ratio to the distances from its center to the common tangent of the fixed circles.

c) Show that this constant ratio is equal to the eccentricity of the ellipse.

Solution by Zazou Katz, Beveryly Hills, California.

Let Q denote the projection of P upon AB and let AQ = d. Let $AO_1 = r$, $AO = R$, $O_1P = r + p$, $OP = R - p$, where p is the radius of the variable circle (P).

Then,

a) $O_1P + OP = (r + p) + (R - p) = R + r$, a constant. Hence P describes an ellipse whose foci are O_1 and O.

b) and c) Since $O_1P^2 - OP^2 = O_1Q^2 - OQ^2$, we may write

$(r + p)^2 - (R - p)^2 = (AQ - r)^2 - (AQ - R)^2$, or

$(r + R)(r - R + 2p) = (2AQ - r - R)(R - r)$, whereupon

$$(r + R) = \frac{2p - (R - r)}{2d - (r + R)} = \frac{p}{d} = e,$$ the eccentricity of the ellipse, a constant, defined by the ratio of the distance between the foci to the length of the major axis.

Also solved by WALTER BLUMBERG, MIKE CALL, CLAYTON W. DODGE, MICHAEL W. ECKER, ROBERT C. GEBHARDT, HENRY S. LIEBERMAN, SISTER STEPHANIE SLOYAN, CHARLES W. TRIGG, WILLIAM E. WARREN, KENNETH M. WILKE, J. A. WINTERINK, ROGER E. KUEHL, *and the Proposer.*

Editor's Comment.

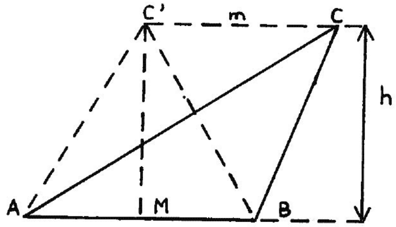
Special commendation is due to ROGER E. KUEHL, the Kansas City, Missouri traffic engineer, for his excellent solution, beautifully calligraphed and precisely drafted, which however is too lengthy for publication here. Mr. Kuehl was the proposer of problem 297 [Spring 1973; Fall 1974] which involved the construction of an S-curve with circles of equal radius, connecting two non-parallel straight roads.

448. [Spring 1979] *Proposed by the late R. Robinson Rowe.*

Analogous to the median, call a line from a vertex of a triangle to a third point of the opposite side a "tredian". Then if both tredians are drawn from each vertex, the 6 lines will intersect at 12 interior points and divide the area into 19 subareas, each a rational part of the area of the triangle. Find two triangles for which each subarea is an integer, one being a Pythagorean right triangle and the other with consecutive integers for its three sides.

Solution by the Proposer.

Consider the general triangle ABC with altitude h and m the abscissal difference between C and M at midpoint of AB. Transform this triangle by the relation $x' = x - my/h$ to the isosceles triangle ABC'. Each horizontal differential element will be invariant in length, and so will an areal aggregate of such elements be invariant in area.



The ordinates of the third points on AC and BC, $h/3$ and $2h/3$, be unchanged, so all tredians will be transformed to a symmetrical array about MC'. Hence the 19 subareas have become symmetrical, and it follows that the subareas in ABC, though not symmetrical in shape, are symmetrical in area. Then similar transformations on AC and BC as bases would prove triaxial symmetry of areas.

Thus this triaxial symmetry of areas simplifies our problem by using an equilateral triangle, which shows that there are just 5 different areas among the 19 subareas. Then for a unit-area triangle,

$$3A + 6B + 3C + 6E + F = 1 \qquad (1)$$

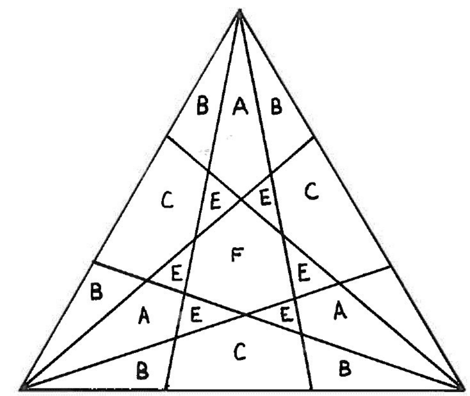
$$A + 3B + C + E = 1/3 \qquad (2)$$

$$A + C + 4E + F = 1/3 \qquad (3)$$

$$\left.\begin{array}{ll} 2B + C = 1/5 & (4) \\ A + 2B + C + E = 2/7 & (5) \end{array}\right\} \text{See Addenda}$$

Whence: A = 1/14; B = 1/21; C = 11/105; E = 1/70; F = 1/10 (6)

The least common denominator for these 5 subarea fractions is 210 = 5 x 6 x 7. Hence any triangle with integral sides and an area divisible by 210 will be fully integral for the subareas as well. The pythagorean triangle 20, 21, 29 has an area of 210 and the subareas in



order are: 15, 10, 22, 3, 21.

For the other task, it will be helpful to review the generators for triangles with consecutive integer sides. Let the sides be $m-1$, m and $m+1$ and the area A. Then in order n, beginning with the trivial $n = 0$:

n	m	A
0	2	0
1	4	6
2	14	84
3	52	1170

Generators are: $m_{n+1} = 4m_n - m_{n-1}$ (7)

$$A_{n+1} = 14A_n - 14A_{n-1} \qquad (8)$$

From (8) can be derived:

$$A_{n+2} = 195A_n - 14A_{n-1} \qquad (9)$$

$$A_{n+3} = 2716A_n - 195A_{n-1} \qquad (10)$$

From (9) it is clear that if A_n is divisible by 7, so is A_{n+2}, and since $A_2 = 84$ is divisible by 7, then A is always divisible by 7 for $n = 2a$. Likewise from (10), A is always divisible by 5 for $n = 3b$.

Finally, since A is alwaysdivisible by 6, A will be divisible by

$5 \times 6 \times 7 = 210$ when $n = 2 \times 3 = 6$, or any multiple of 6.

The least solution, then, is $n = 6$, for which the sides are 2701, 2702, 2703 and the area is $A = 3\,161\,340 = 210 \times 15054$.

Addenda. Perhaps I should explain the fractions in (4) and (5). If the altitude of a triangle is unity and AB its base, and if secants from A and B meet the opposite sides at ordinates a and b, then the ordinate of their intersection is

$$c = \frac{ab}{a + b - ab}$$

For (4), $a = b = 1/3$, the numerator is 1/9 and the denominator 5/9, so $c = 1/5$. For (5), $a = 1/3$, $b = 2/3$, numerator 2/9, denominator 7/9 and $c = 2/7$. I think this relation, in one form or another, is fairly well known.

Also solved by MIKE CALL [by *computer*], CLAYTON W. DODGE, and KENNETH M. WILKE.

REFEREES FOR THIS ISSUE

The Journal *receives ten papers for every one that is published, and many of those that are published require extensive rewriting. As soon as a paper is received by the Editor, it is sent to at least two referees. These persons do yeoman work rewriting, correcting, advising, proving* **and** *reproving. Without them the* Journal *could not survive. The Referees who have contributed services since the Fall Issue are:* Eric C. **Nummela,** New England College; Joseph Konhauser, Macalester College; Clayton Dodge, University of Maine; Robert Eslinger, Hendrix College; Donald **Bushnell,** Ft. Lewis College; J. Sutherland Frame, Michigan State; Dennis Burke, Miami University; Hudson Kronk, SUNY at Binghamton; Dean C. Benson, C. A. **Grimm,** Salias Sengupta, Roger Opp, Dale **Rognlie,** Ronald Weger, South Dakota School of Mines and Technology;

Thank you from the Journal **and** *the Editors.*

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